Measurable Stochastics for Brane Calculus[*]

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We give a stochastic extension of the Brane Calculus, along the lines of recent work by Cardelli and Mardare [8]. In this presentation, the semantics of a Brane process is a measure of the stochastic distribution of possible derivations. To this end, we first introduce a labelled transition system for Brane Calculus, proving its adequacy w.r.t. the usual reduction semantics. Then, brane systems are presented as Markov processes over the measurable space generated by terms up-to syntactic congruence, and where the measures are indexed by the actions of this new LTS. Finally, we provide a SOS presentation of this stochastic semantics, which is compositional and syntax-driven.

1 Introduction

The Brane Calculus (BC) [7] is a calculus of mobile processes designed for modeling membrane interactions within a cell. A process of this calculus represents a system of nested membranes, carrying their active components *on* membranes, not inside them. Membranes interact according to three reaction rules, corresponding to phagocytosis, endo/exocytosis, and pinocytosis.

In the original definition, reaction rules do not consider quantitative aspects like rates, volumes, etc. However, it is important to address these aspects, e.g. for implementing stochastic simulations, or for connecting Brane Calculus with quantitative models at lower abstraction levels (such as stochastic π-calculus and κ-calculus for protein interactions).

In this paper, we introduce a stochastic semantics for the Brane Calculus. Clearly, a stochastic calculus could be obtained just by adding rates to reaction rules; however, the resulting "pointwise" rated reduction semantics is not fully satisfactory for several reasons. First, it is not compositional, i.e., reaction rates of a process are not given in terms of the rates of its components. Secondly, stochastic reaction rules are not easy to deal with in presence of large populations of agents (as it is often the case in biological systems), because we have to count large number of occurrences for calculating the effective reaction rates. Third, it does not generalize easily to other quantitative (e.g. geometric) aspects.

To overcome these issues, we adopt a novel approach recently introduced by Cardelli and Mardare [8], which is particularly suited when a measure of similarity of behaviours is important (similar ideas have been proposed for probabilistic automata [10, 14], and Markov processes [9, 5, 12]). The main point of this approach is that the semantics of a process is a *measure* of the stochastic distribution of the possible outcomes. Thus, processes form a *measurable space*, and each process is given an action-indexed family of measures on this space. For an action a, the measure μ_a associated to a process P specifies for each measurable set S of processes, the rate $\mu_a(S) \in \mathbb{R}^+$ of a-transitions from P to (elements of) S. The resulting structures, called *Markov processes* (MPs), are not continuous-time Markov chains because each transition is from a state to a possibly infinite class of states (closed to the congruence relation over processes) and consequently cannot be described in a pointwise style. An advantage of this approach is that we can apply results from measure theory for solving otherwise difficult issues,

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like instance-counting problems; moreover, process measures are defined compositionally, and can be characterized also by means of operational semantics in GSOS form. Finally, other measurable aspects of processes (e.g., volumes) can be dealt with along the same lines.

However, the approach of [8] has been applied to CCS only, and in order to adapt it to BC, we have to solve some problems. First, in order to define systems of BC as Markov processes, we need to define *actions* for brane systems, which corresponds to define a *labelled transition system* (LTS) for BC. Defining correctly a labelled transition system for a complex calculus like BC is notoriously a difficult task, because labels should describe precisely how a system can interact with the surrounding environment. In our case, we have been inspired by the so-called *IPO construction* [11], following the approach of [13, 3] for the case of Mobile Ambients. As a result, in Section 3 we introduce the first labelled transition system for BC, and prove its adequacy with respect to the original reduction semantics.

A peculiar feature of this LTS is that after a transition, a system can yield a *higher-order* term, i.e., a system with "holes" (like π-calculus' "abstractions"). This has several consequences on our work. First, we have to define a suitable syntax for these higher-order terms; for this reason, in Section 2 we introduce a *simply typed* version of brane calculus from the beginning, extended with metavariables, λ-abstractions and applications. Well-formedness of terms is guaranteed by a suitable typing system; e.g., membranes and systems are represented by ground terms of type mem and sys, respectively. Secondly, the bisimulation definition has to be accommodated in order to deal with transitions yielding higher-order terms (Section 3). Finally, also the definition of Markov kernel is affected, as we will see in Section 4: the kernel cannot be defined simply on the space of brane systems, but must consider also these higher-order terms—in fact, the measure will be defined over all well-typed terms, also those higher-order.

After that these issues have been addressed and the Markov kernel has been defined, we can look for a simpler presentation of the semantics of Markov processes. In Section 5 we present a SOS system for processes, capturing the Markov kernel over processes: the stochastic bisimilarity induced by this SOS semantics corresponds to the Markov bisimilarity defined in Section 4. Therefore, this semantics can be fruitfully used for simulations, or for verifying system equivalences.

Some concluding remarks and directions for further work are in Section 6.

2   Brane Calculus

In this section we recall Cardelli's Brane Calculus [7] focusing on its basic and finite version (without communication primitives, molecular complexes and replication). Here we adopt an alternative presentation of the calculus: instead of the traditional term grammar of [7], we introduce an unstructured version but equipped with a type system. Although typed terms may seem unnecessary now, they will be useful in Section 3, where a labelled transition semantics will be introduced.

Syntax   The grammars for terms and types are specified below.

$$
\begin{array}{ll}
(\textit{Terms}) & M ::= \mathbf{0} \mid \diamond \mid X \mid \alpha.M \mid M|M \mid M \circ M \mid M \langle\!\langle M \rangle\!\rangle \\
& \alpha ::= \circlearrowright_n \mid \circlearrowright_n^{\perp}(M) \mid \circlearrowleft_n \mid \circlearrowleft_n^{\perp} \mid \circledcirc_n(M) \\
(\textit{Types}) & t ::= \mathsf{sys} \mid \mathsf{mem} \mid \mathsf{act}
\end{array}
$$

The subscripted names n are taken from a countable set Λ, while term variables X are taken from a countable set $\mathscr{X}$, assumed to be disjoint from each other.

$$\frac{\Gamma(X) = t}{\Gamma \vdash X : t} \ (\text{var}) \qquad \frac{a \in \{\mho_n, \circlearrowleft_n, \mho_n^\perp\}}{\Gamma \vdash a : \mathsf{act}} \ (\text{act}) \qquad \frac{a \in \{\mho_n^\perp, \circledcirc_n\} \quad \Gamma \vdash M : \mathsf{mem}}{\Gamma \vdash a(M) : \mathsf{act}} \ (\text{act-arg})$$

$$\frac{}{\Gamma \vdash \mathbf{0} : \mathsf{mem}} \ (\text{zero}) \qquad \frac{\Gamma_1 \vdash \alpha : \mathsf{act} \quad \Gamma_2 \vdash M : \mathsf{mem}}{\Gamma_1, \Gamma_2 \vdash \alpha.M : \mathsf{mem}} \ (\alpha\text{-pref}) \qquad \frac{\Gamma_1 \vdash M : \mathsf{mem} \quad \Gamma_2 \vdash N : \mathsf{mem}}{\Gamma_1, \Gamma_2 \vdash M|N : \mathsf{mem}} \ (\text{par})$$

$$\frac{}{\Gamma \vdash \diamond : \mathsf{sys}} \ (\text{void}) \qquad \frac{\Gamma_1 \vdash M : \mathsf{mem} \quad \Gamma_2 \vdash N : \mathsf{sys}}{\Gamma_1, \Gamma_2 \vdash M \llparenthesis N \rrparenthesis : \mathsf{sys}} \ (\text{loc}) \qquad \frac{\Gamma_1 \vdash M : \mathsf{sys} \quad \Gamma_2 \vdash N : \mathsf{sys}}{\Gamma_1, \Gamma_2 \vdash M \circ N : \mathsf{sys}} \ (\text{comp})$$

Table 1: Typing system for Brane Calculus

Types are assigned to terms as usual. A type *environment* Γ is a finite map from term variables to types. If Γ is an environment and X a variable not in the domain of Γ, we denote by $\Gamma, X : t$ the environment which assigns type $\Gamma(Y)$ to each variable $Y \in dom(\Gamma)$ and type t to X; if Γ_1 and Γ_2 have disjoint domains, Γ_1, Γ_2 denotes the environment which assigns type $\Gamma_1(X)$ to variables $X \in dom(\Gamma_1)$ and $\Gamma_2(Y)$ to variables $Y \in dom(\Gamma_2)$. The type inference rules are given in Table 1. Notice that this type system admits only *linear* terms, that is, each variable can occur at most once. Indeed, in rules (α-pref), (par), (loc), and (comp), the environment extension Γ_1, Γ_2 is defined only when Γ_1 and Γ_2 have disjoint domains.

In the rest of the paper we assume to work only with well-typed terms. The set of well-typed terms will be denoted as $\mathbb{T}$, while $\mathbb{P}$ and $\mathbb{M}$ denote the set of terms of type sys and mem, respectively. By convention we shall use x, y ... for variables of type mem, and X, Y ... for variables of type sys. A similar convention is used for base type terms: σ, τ, ... are terms in $\mathbb{M}$, while P, Q, ... are terms in $\mathbb{P}$.

Systems can be rearranged according to a structural congruence relation; the intended meaning is that two congruent terms actually denote the same system. Structural congruence, $\equiv$, is the smallest equivalence relation on (possibly open) terms of the language which contains the axioms and rules listed below. We write $\Gamma \vdash M \equiv N$ as shorthand for $\Gamma \vdash M : t$ and $\Gamma \vdash N : t$ and $M \equiv N$. Notice that this notation implicitly assumes that structural equivalent terms must be of the same type.

$$\Gamma \vdash P \circ Q \equiv Q \circ P \qquad \Gamma \vdash P \circ (Q \circ R) \equiv (P \circ Q) \circ R \qquad \Gamma \vdash P \circ \diamond \equiv P \qquad \Gamma \vdash \mathbf{0} \llparenthesis \diamond \rrparenthesis \equiv \diamond$$

$$\Gamma \vdash \sigma | \tau \equiv \tau | \sigma \qquad \Gamma \vdash \sigma | (\tau | \rho) \equiv (\sigma | \tau) | \rho \qquad \Gamma \vdash \sigma | \mathbf{0} \equiv \sigma$$

$$\frac{\Gamma \vdash P \equiv Q}{\Gamma \vdash P \circ R \equiv Q \circ R} \qquad \frac{\Gamma \vdash \sigma \equiv \tau}{\Gamma \vdash \sigma | \rho \equiv \tau | \rho} \qquad \frac{\Gamma \vdash P \equiv Q \quad \Gamma \vdash \sigma \equiv \tau}{\Gamma \vdash \sigma \llparenthesis P \rrparenthesis \equiv \tau \llparenthesis Q \rrparenthesis}$$

$$\frac{\Gamma \vdash \alpha \equiv \beta \quad \Gamma \vdash \sigma \equiv \tau}{\Gamma \vdash \alpha.\sigma \equiv \beta.\tau} \qquad \frac{\Gamma \vdash \sigma \equiv \tau}{\Gamma \vdash \circledcirc(\sigma) \equiv \circledcirc(\tau)} \qquad \frac{\Gamma \vdash \sigma \equiv \tau}{\Gamma \vdash \mho_n^\perp(\sigma) \equiv \mho_n^\perp(\tau)}$$

With respect to the structural congruence of [7], we have added the possibility of rearranging the sub-membranes contained in co-phago and pino actions (the last three rules of the table above).

Reduction Semantics The dynamic behaviour of Brane Calculus is specified by means of a reduction semantics, defined by means of a reduction relation ("reaction") $\longrightarrow \subseteq \mathbb{P} \times \mathbb{P}$, whose rules are listed in Table 2. It is easy to see that subject reduction holds. Note that the presence of (red-equiv) makes this not a structural presentation, since term structure can change according with $\equiv$.

$$\overline{\eth_n^\perp(\rho).\tau|\tau_0(\!(Q)\!) \circ \eth_n.\sigma|\sigma_0(\!(P)\!) \Longrightarrow \tau|\tau_0(\!(\rho(\!(\sigma|\sigma_0(\!(P)\!))\!) \circ Q)\!)}} \text{ (red-phago)}$$

$$\overline{\eth_n^\perp.\tau|\tau_0(\!(\eth_n.\sigma|\sigma_0(\!(P)\!) \circ Q)\!) \Longrightarrow \sigma|\sigma_0|\tau|\tau_0(\!(Q)\!) \circ P} \text{ (red-exo)}$$

$$\overline{\circledcirc(\rho).\sigma|\sigma_0(\!(P)\!) \Longrightarrow \sigma|\sigma_0(\!(\rho(\!(\diamond)\!) \circ P)\!)} \text{ (red-pino)}$$

$$\frac{P \Longrightarrow Q}{\sigma(\!(P)\!) \Longrightarrow \sigma(\!(Q)\!)} \text{ (red-loc)} \qquad \frac{P \Longrightarrow Q}{P \circ R \Longrightarrow Q \circ R} \text{ (red-comp)} \qquad \frac{P \equiv P' \quad P' \Longrightarrow Q' \quad Q' \equiv Q}{P \Longrightarrow Q} \text{ (red-equiv)}$$

Table 2: Reduction semantics for the Brane Calculus.

3 A Labelled Transition Semantics for Brane Calculus

In this section we introduce a labeled transition system (LTS) for the Brane calculus, along the lines of [13] where a LTS is outlined in the case of Mobile Ambients. A SOS style presentation of the semantics will help in the definition of the stochastic semantics of Section 4.

We shall use a meta-syntax for simple syntactic manipulation of terms. The meta-syntax is a simply typed λ-calculus and can be thought as a primitive system of higher order abstract syntax. We extend the base types with function types, in order to type terms in the meta-syntax. We also add to the term signature the λ-abstraction and application constructs, which should not be considered as a language extension: their function is to allow for a structural definition of a labelled transition system, and as such they have no computational meaning.

$$M ::= \dots \mid \lambda X : t.\, M \mid M(M) \qquad\qquad t ::= \dots \mid t \to t$$

The λ-abstraction binds variables. Terms are taken up-to α-equivalence on bound variables.

Terms in the meta-language are typed using the standard typing rules for typed λ-terms, and the two rules below are added to the set of of type rules of Table 1.

$$\frac{\Gamma, X : t \vdash M : t'}{\Gamma \vdash \lambda X : t.\, M : t \to t'} \text{ (lambda)} \qquad\qquad \frac{\Gamma \vdash M : t \to t' \quad \Gamma \vdash N : t}{\Gamma \vdash M(N) : t'} \text{ (app)}$$

We consider terms to be syntactically equal up-to $\beta\eta$-equivalence, i.e., the smallest congruence that contains the two axioms $(\lambda X : t.\, M)(N) = M\{N/X\}$ and $\lambda X : t.\, M(X) = M$. Thus, for instance, $(\lambda X : \text{sys}.\, \sigma(\!(X)\!))(P)$ and $\sigma(\!(P)\!)$ are the same term. In the remainder of the paper, when we write a meta-syntax term of base type (sys, mem or act), we mean the term obtained by complete $\beta\eta$-reduction (which exists because our metalanguage is a simply typed-λ calculus [2]). This assumption will ease the description avoiding many technicalities which are out of the scope of the paper.

We extend structural congruence to meta-syntactic terms by adding the following rule.

$$\frac{\Gamma, X : t \vdash P \equiv Q}{\Gamma \vdash \lambda X : t.\, P \equiv \lambda X : t.\, Q}$$

Structural congruence is compatible with λ-terms evaluation in the following sense:

Lemma 3.1. *If $M \equiv N : t \to t'$ and $T \equiv S : t$ then it follows that $M(T) \equiv N(S) : t'$.*

$$\frac{}{\mho_n.\sigma \xrightarrow{\mho_n} \sigma}\ (\text{℧-pref}) \qquad \frac{}{\mho_n^\perp(\rho).\sigma \xrightarrow{\mho_n^\perp(\rho)} \sigma}\ (\text{℧}^\perp\text{-pref}) \qquad \frac{}{\circledcirc_n(\rho).\sigma \xrightarrow{\circledcirc_n(\rho)} \sigma}\ (\text{◎-pref})$$

$$\frac{}{\mho_n.\sigma \xrightarrow{\mho_n} \sigma}\ (\text{℧-pref}) \qquad \frac{}{\mho_n^\perp.\sigma \xrightarrow{\mho_n^\perp} \sigma}\ (\text{℧}^\perp\text{-pref}) \qquad \frac{\sigma \xrightarrow{\alpha} \sigma'}{\sigma|\tau \xrightarrow{\alpha} \sigma'|\tau}\ (\text{L-par}) \qquad \frac{\sigma \xrightarrow{\alpha} \sigma'}{\tau|\sigma \xrightarrow{\alpha} \tau|\sigma'}\ (\text{R-par})$$

$$\frac{\sigma \xrightarrow{\mho_n} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\mathsf{phago}_n} \lambda Z.\, Z(\sigma'(\!|P|\!))}\ (\text{phago}) \qquad \frac{P \xrightarrow{\mathsf{phago}_n} F}{P \circ Q \xrightarrow{\mathsf{phago}_n} \lambda Z.\, (F(Z) \circ Q)}\ (\text{L◦-phago})$$

$$\frac{\sigma \xrightarrow{\mho_n^\perp(\rho)} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\overline{\mathsf{phago}_n}} \lambda X.\, \sigma'(\!|\rho(\!|X|\!) \circ P|\!)}\ (\overline{\text{phago}}) \qquad \frac{P \xrightarrow{\overline{\mathsf{phago}_n}} A}{P \circ Q \xrightarrow{\overline{\mathsf{phago}_n}} \lambda X.\, (A(X) \circ Q)}\ (\text{L◦-}\overline{\text{phago}})$$

$$\frac{\sigma \xrightarrow{\mho_n} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\mathsf{exo}_n} \lambda X y.\, \sigma'|y(\!|X|\!) \circ P}\ (\text{exo}) \qquad \frac{P \xrightarrow{\mathsf{exo}_n} S}{P \circ Q \xrightarrow{\mathsf{exo}_n} \lambda X y.\, S(X \circ Q)(y)}\ (\text{L◦-exo})$$

$$\frac{P \xrightarrow{\mathsf{phago}_n} F \quad Q \xrightarrow{\overline{\mathsf{phago}_n}} A}{P \circ Q \xrightarrow{id} F(A)}\ (\text{id-phago-L}) \qquad \frac{P \xrightarrow{\mathsf{exo}_n} S \quad \sigma \xrightarrow{\mho_n^\perp} \sigma'}{\sigma(\!|P|\!) \xrightarrow{id} S(\diamond)(\sigma')}\ (\text{id-exo}) \qquad \frac{\sigma \xrightarrow{\circledcirc_n(\rho)} \sigma'}{\sigma(\!|P|\!) \xrightarrow{id} \sigma'(\!|\rho(\!|\diamond|\!) \circ P|\!)}\ (\text{id-pino})$$

$$\frac{P \xrightarrow{id} P'}{\sigma(\!|P|\!) \xrightarrow{id} \sigma(\!|P'|\!)}\ (\text{id-loc}) \qquad \frac{P \xrightarrow{id} P'}{P \circ Q \xrightarrow{id} P' \circ Q}\ (\text{L◦-id})$$

Table 3: Labelled transition system (LTS) for Brane Calculus. By convention $F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys}$, $A : \mathsf{sys} \to \mathsf{sys}$ and $S : \mathsf{sys} \to \mathsf{mem} \to \mathsf{sys}$. Symmetric rules (R◦-phago), (R◦-$\overline{\text{phago}}$), (R◦-exo), (R◦-id), and (id-phago-R) are omitted.

We can now define a labelled transition system for the Brane Calculus, following [13]. This system will be structural (i.e., in SOS format) and finitary branching; these features will allow to define smoothly the stochastic semantics in Section 4. To our knowledge this is the first finitary structural operational semantics for the Brane Calculus. (An LTS for Brane Calculus has been given in [1], but it was not structural nor finitary branching.) The rules of our LTS are given in Table 3, and are organized into two parts: rules for membrane terms and for system terms. In the definition we have omitted types since the upper/lower case notation for sys/mem typed variables makes the presentation clearer. The only exception to this notation is in (phago) and (L/R◦-phago) rules, where the variable Z has type $(\mathsf{sys} \to \mathsf{sys})$.

An example of how labelled transitions are derived using the rules in Table 3, is shown in Figure 1: the derivation leads to the (red-phago) reaction of Table 2; note, in particular, how the (id-phago) rule is applied. Similar derivations hold for all rules in Table 2 except for (red-equiv) (which makes the reduction semantics "not structural"). Hence, it is easy to see that $\xrightarrow{id} \subseteq \Longrightarrow$. The converse follows by an inductive analysis of the structure of the source processes of an *id*-transition.

Proposition 3.2. *If* $P \xrightarrow{id} Q$ *then* $P \Longrightarrow Q$. *If* $P \Longrightarrow Q$ *then* $P \xrightarrow{id} Q'$ *for some* $Q' \equiv Q$.

Labelled transitions are compatible with structural congruence in the following sense:

Lemma 3.3. *If* $P \xrightarrow{\alpha} P'$ *and* $P \equiv Q$ *then there exists* Q' *such that* $Q' \equiv P'$ *and* $Q \xrightarrow{\alpha} Q'$.

The labelled transition systems, as usual, induces a bisimulation relation on terms. Due to the generalization to meta-syntactic terms, the canonical bisimulation considers also λ-terms. Since in Brane

$$\frac{\dfrac{\rule{2cm}{0.4pt}}{\wp_n.\sigma \xrightarrow{\wp_n} \sigma}\text{($\wp$-pref)}}{\dfrac{\wp_n.\sigma|\sigma_0 \xrightarrow{\wp_n} \sigma|\sigma_0}{\wp_n.\sigma|\sigma_0⦅P⦆ \xrightarrow{phago_n} \lambda Z.\, Z(\sigma|\sigma_0⦅P⦆)}\text{(L-par)}}\text{(phago)} \qquad \frac{\dfrac{\dfrac{\rule{2cm}{0.4pt}}{\wp_n^\perp(\rho).\tau \xrightarrow{\wp_n^\perp(\rho)} \tau}\text{($\wp^\perp$-pref)}}{\wp_n^\perp(\rho).\tau|\tau_0 \xrightarrow{\wp_n^\perp(\rho)} \tau|\tau_0}\text{(L-par)}}{\wp_n^\perp(\rho).\tau|\tau_0⦅Q⦆ \xrightarrow{\overline{phago_n}} \lambda X.\, \tau|\tau_0⦅\rho⦅X⦆ \circ Q⦆}\text{(\overline{phago})}$$

$$\wp_n.\sigma|\sigma_0⦅P⦆ \circ \wp_n^\perp(\rho).\tau|\tau_0⦅Q⦆ \xrightarrow{id} \tau|\tau_0⦅\rho⦅\sigma|\sigma_0⦅P⦆⦆ \circ Q⦆ \quad \text{(id-phago-L)}$$

Figure 1: Derivation for the (red-phago) reaction.

Calculus the dynamics can happen only at the level of system terms $P \in \mathbb{P}$, we specialize the definition of bisimulation considering only labels in $\{id, phago_n, \overline{phago_n}, exo_n \mid n \in \Lambda\}$ as follows:

Definition 3.4 (Strong bisimulation). *A* bisimulation on Brane Calculus systems *is an equivalence relation $\mathscr{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$, $(P,Q) \in \mathscr{R}$ if and only if*

- *if $P \xrightarrow{id} P'$ then $\exists Q'$ such that $Q \xrightarrow{id} Q'$ and $(P',Q') \in \mathscr{R}$;*

- *if $P \xrightarrow{phago} F$ then $\exists G$ such that $Q \xrightarrow{phago_n} G$ and $(F(R), G(R)) \in \mathscr{R}$ for all σ, ρ : mem and R' : sys, where $R = \lambda X.\ \sigma⦅\rho⦅X⦆ \circ R'⦆$;*

- *if $P \xrightarrow{\overline{phago}} A$ then $\exists B$ such that $Q \xrightarrow{\overline{phago_n}} B$ and $(A(R), B(R)) \in \mathscr{R}$ for all R : sys;*

- *if $P \xrightarrow{exo} T$ then $\exists S$ such that $Q \xrightarrow{exo_n} S$ and $(T(R)(\rho), S(R)(\rho)) \in \mathscr{R}$ for all R : sys and ρ : mem;*

Two systems $P, Q \in \mathbb{P}$ are said bisimilar*, written $P \sim Q$, iff there exists a bisimulation relation $\mathscr{R}$ such that $(P,Q) \in \mathscr{R}$.*

In the definition above, when a transition yields a λ-abstracted term M, this term is instantiated according to its form. The idea is to recover a bisimulation on only system terms by sufficiently instantiating the λ-abstraction in order to recover a sys-typed term, hence a term in $\mathbb{P}$. This definition is slightly different from that in [13] for Ambient Calculus, where Rathke and Sobocinski prefer to add to the LTS derivation rules for explicit instantiation. Although this design choice is equivalent from the point of view of the resulting bisimulation relation, it leads to an infinitely-branching LTS, since they chose to endow labels with the instantiation parameters. In the definition of the stochastic semantics for the Brane Calculus we need the LTS to be finitely-branching, hence we do not add instantiation rules.

The next lemma follows directly from Lemma 3.3.

Lemma 3.5. *For arbitrary $P, Q \in \mathbb{P}$, if $P \equiv Q$ then $P \sim Q$.*

This ensures that the structural equivalence $\equiv$ is contained in $\sim$, but we can prove also a stronger result, that is $\equiv \subsetneq \sim$. Let $P = \mathbf{0}⦅\wp_n⦅⦆⦆$; it is easy to see that $P \sim \diamond$, however $P \not\equiv \diamond$, hence structural equivalence does not coincide with strong bisimulation, and in particular $\sim$ equates more terms than $\equiv$.

4 A Stochastic Semantics for Brane Calculus

In this section we present a stochastic semantics for the Brane calculus, following the construction of [8]. We assume the reader to be familiar with basic notions from measure theory (see Appendix A for a brief summary of the used definitions). We start introducing the notation used hereafter and recalling the definition of *Markov process* (MP) and stochastic bisimulation on them (more details are in [8]).

For arbitrary sets A and B, 2^A denotes the powerset of A, and both $[A \to B]$ and B^A will be used to denote the class of functions from A to B. As usual $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of natural, rational, and real numbers, and $\mathbb{Q}^+$ and $\mathbb{R}^+$ the sets of positive rational and real numbers (with zero), respectively. Given a measurable space (M, Σ), the elements of Σ are called *measurable sets* and M the *support-set*. Let $\Delta(M, \Sigma)$ be the class of measures $\mu : \Sigma \to \mathbb{R}^+$ on (M, Σ). From $\Delta(M, \Sigma)$ we distinguish two measures: the *null measure* ω for which $\omega(A) = 0$ for all $A \in \Sigma$, and, for a fixed $r \in \mathbb{R}^+$, the *r-Dirac measure on* N, $D(r, N)$, defined by $D(r, N)(\bigcup_{i \in I} N_i) = \sum_{i \in I} f_N(N_i)$, for N and $N_{i \in I}$ elements of a base for (M, Σ) (thus $\bigcup_{i \in I} N_i \in \Sigma$), where $f_N(N') = r$ if $N' = N$, otherwise 0. Given two measurable spaces (M, Σ) and (N, Θ), we use $[\![M \to N]\!]$ to denote the class of measurable functions $f : M \to N$ from (M, Σ) to (N, Θ).

Definition 4.1 (Markov kernels and Markov processes). *Let* (M, Σ) *be an measurable space, and A a denumerable set of labels. An A-Markov kernel is a tuple* $\mathscr{M} = (M, \Sigma, \theta)$*, with*

$$\theta : A \to [\![M \to \Delta(M, \Sigma)]\!].$$

An A-Markov process of $\mathscr{M}$ with $m \in M$ as initial state*, written $(\mathscr{M}, m)$, is the tuple (M, Σ, θ, m).*

A MP involves a set A of labels which represent all possible interactions with the environment. If $\alpha \in A$ is a label, m is the current state of the system, and N is a measurable set of states, the function $\theta(\alpha)(m)$ is a measure on the state space and $\theta(\alpha)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable characterizing the duration of the α-transition from m to arbitrary $n \in N$.

Given a binary relation $\mathscr{R} \subseteq M \times M$, we call a subset $N \subseteq M$ $\mathscr{R}$-closed iff $\mathscr{R} \cap (N \times M) \subseteq N \times N$. Given (M, Σ) a measurable space and $\mathscr{R} \subseteq M \times M$ a binary relation over M, with $\Sigma(\mathscr{R})$ we denote the set of measurable $\mathscr{R}$-closed subsets of M.

Definition 4.2 (Stochastic bisimulation). *For an A-Markov kernel $\mathscr{M} = (M, \Sigma, \theta)$ a* rate-bisimulation relation *is an equivalence relation $\mathscr{R} \subseteq M \times M$ such that $(m, n) \in \mathscr{R}$ iff for any $C \in \Sigma(\mathscr{R})$ and $\alpha \in A$,*

$$\theta(\alpha)(m)(C) = \theta(\alpha)(n)(C).$$

Two Markov processes $(\mathscr{M}, m)$ and $(\mathscr{M}, n)$ are stochastic bisimilar*, written $m \sim_{\mathscr{M}} n$, if m and n are related by a rate-bisimulation relation.*

In the rest of the section we define the analytic space of terms, and we show how it can be organized as an A-Markov kernel. This will implicitly give a stochastic structural operational semantics such that the canonical behavioral equivalence coincides with the bisimulation of MPs.

Let us define the measurable space of terms. The construction takes place at the level of the meta-syntactic terms, hence we will assume to work with terms in $\mathbb{T}$. Let $\mathbb{T}/_\equiv$ be the set of $\equiv$-equivalence classes on $\mathbb{T}$. For arbitrary $M \in \mathbb{T}$, we denote by $[M]_\equiv$ the $\equiv$-equivalence class of M. Note that $\mathbb{T}/_\equiv$ is a denumerable partition of $\mathbb{T}$, hence it is a generator for a σ-algebra on $\mathbb{T}$.

Definition 4.3 (Measurable space of terms). *The* measurable space of terms $(\mathbb{T}, \Sigma)$ *is the measurable space on $\mathbb{T}$ where Σ is the σ-algebra on $\mathbb{T}$ generated by $\mathbb{T}/_\equiv$.*

The measurable sets are (possibly denumerable) reunions of $\equiv$-equivalence classes on $\mathbb{T}$. In the following we use $\mathcal{T}$, $\mathcal{S}$, ... to denote arbitrary measurable sets of Σ.

In order to define an A-Markov kernel on $(\mathbb{T}, \Sigma)$ we first need to define the set of labels.

Definition 4.4 (Transition labels). *The set of* transition labels *for (meta-syntactic) terms is given by the pair $(\mathbb{A}, \iota)$, where $\mathbb{A} \triangleq \mathbb{A}_{\mathsf{sys}} \cup \mathbb{A}_{\mathsf{mem}}$ and*

(SYSTEM LABELS)	$\mathbb{A}_{\mathsf{sys}} \triangleq \{phago_n, \overline{phago}_n, exo_n \mid n \in \Lambda\}$
(MEMBRANE LABELS)	$\mathbb{A}_{\mathsf{mem}} \triangleq \{\eth_n, \eth_n^\perp(\rho), \circlearrowright_n, \circlearrowright_n^\perp, \circledcirc_n(\rho) \mid n \in \Lambda \text{ and } \rho \in \mathbb{M}\}$

The *internal action* label *id* is not in $\mathbb{A}$; we extend $\mathbb{A}$ by defining $\mathbb{A}^+ \triangleq \mathbb{A} \cup \{id\}$. It is no accident that the chosen labels are the same of the LTS of Section 3; indeed, the construction of the Markov kernel will be guided by the derivation rules listed in Table 3.

Since we are defining the semantics of a stochastic calculus, we equip the set of actions $\text{Act} = \{\eth_n, \eth_n^\perp, \eth_n, \eth_n^\perp, \circledcirc_n \mid n \in \Lambda\}$ with a weight function $\iota \colon \text{Act} \to \mathbb{Q}^+ \setminus \{0\}$, which assigns to each action the rate of an exponentially distributed random variable that characterizes the duration of the transition induced by the execution of that particular action. The weight function is such that $\iota(\eth_n) = \iota(\eth_n^\perp)$ and $\iota(\eth_n) = \iota(\eth_n^\perp)$; this characterizes the fact that two cooperating actions have the same execution rate.

Now, we aim to define the function $\theta \colon \mathbb{A}^+ \to [\![\mathbb{T} \to \Delta(\mathbb{T}, \Sigma)]\!]$, which will conclude the construction of an $\mathbb{A}^+$-Markov process for $(\mathbb{T}, \Sigma)$. To this end, it is useful to give some operations on measurable sets, which will ease the exposition of the inductive construction of θ. For arbitrary $\mathcal{T}, \mathcal{S} \in \Sigma$ and $M \in \mathbb{T}$,

$$
\mathcal{T}(\mathcal{S}) \triangleq \bigcup_{\substack{M:t\to t'\in\mathcal{T}\\N:t\in\mathcal{S}}} [M(N)]_\equiv \qquad \mathcal{T}_{|M} \triangleq \bigcup_{N|M\in\mathcal{T}} [N]_\equiv \qquad \mathcal{T}_{\circ M} \triangleq \bigcup_{N\circ M\in\mathcal{T}} [N]_\equiv \qquad \mathcal{T}_{M\langle\!\langle\rangle\!\rangle} \triangleq \bigcup_{M\langle\!\langle N\rangle\!\rangle\in\mathcal{T}} [N]_\equiv
$$

The next definition constructs the function $\theta \colon \mathbb{A}^+ \to [\![\mathbb{T} \to \Delta(\mathbb{T}, \Sigma)]\!]$ by induction on the structure of terms. The intuition is that for arbitrary $M \in \mathbb{T}$, $\mathcal{M} \in \Sigma$ and $\alpha \in \mathbb{A}^+$, $\theta(\alpha)(M)(\mathcal{M})$ represents the total rate of the α actions from M to (elements of) $\mathcal{M}$.

Definition 4.5. *Let* $\theta \colon \mathbb{A}^+ \to [\![\mathbb{T} \to \Delta(\mathbb{T}, \Sigma)]\!]$ *be defined by induction on the structure of terms:*

Case 0: *For any* $a \in \mathbb{A}^+$, *let* $\theta(a)(\mathbf{0}) = \omega$.

Case $\alpha.M$: *For any* $a \in \mathbb{A}^+$ *let* $\theta(a)(\alpha.M) = D(\iota(\text{act}(\alpha)), [M]_\equiv)$ *if* $a = \alpha$, $\theta(a)(\alpha.M) = \omega$ *otherwise.*

Case $M|N$: *For any* $a \in \mathbb{A}^+$ *let* $\theta(a)(M|N)(\mathcal{T}) = \theta(a)(N)(\mathcal{T}_{|M}) + \theta(a)(M)(\mathcal{T}_{|N})$.

Case $\lambda X.M$: *For any* $a \in \mathbb{A}^+$, *let* $\theta(a)(\lambda X.M) = \omega$.

Case $\diamond$: *For any* $a \in \mathbb{A}^+$, *let* $\theta(a)(\diamond) = \omega$.

Case $M\langle\!\langle N\rangle\!\rangle$: *For any* $a \in \mathbb{A}_{\text{mem}}$ *let* $\theta(a)(M\langle\!\langle N\rangle\!\rangle) = \omega$. *For all other labels in* $\mathbb{A}^+ \setminus \mathbb{A}_{\text{mem}}$

$$
\theta(phago_n)(M\langle\!\langle N\rangle\!\rangle)(\mathcal{T}) = \theta(\eth_n)(M)([\{\sigma \in \mathbb{M} \mid \lambda Z \colon \text{sys} \to \text{sys}.\ Z(\sigma\langle\!\langle N\rangle\!\rangle) \in \mathcal{T}\}]_\equiv)
$$

$$
\theta(\overline{phago_n})(M\langle\!\langle N\rangle\!\rangle)(\mathcal{T}) = \sum_{\rho\in\mathbb{M}} \theta(\eth_n^\perp(\rho))(M)(\{\sigma \in \mathbb{M} \mid \lambda X.\ \sigma\langle\!\langle\rho\langle\!\langle X\rangle\!\rangle \circ N\rangle\!\rangle \in \mathcal{T}\}/_\equiv)
$$

$$
\theta(exo_n)(M\langle\!\langle N\rangle\!\rangle)(\mathcal{T}) = \theta(\eth_n)(M)(\{\sigma \in \mathbb{M} \mid \lambda Xy.\ \sigma|y\langle\!\langle X\rangle\!\rangle \circ N \in \mathcal{T}\}/_\equiv)
$$

$$
\theta(id)(M\langle\!\langle N\rangle\!\rangle)(\mathcal{T}) = \theta(id)(N)(\mathcal{T}_{M\langle\!\langle\rangle\!\rangle}) +
$$

$$
\sum_{\substack{\rho\in\mathbb{M}}}^{n\in\Lambda} \theta(\circledcirc_n(\rho))(M)(\{\sigma \in \mathbb{M} \mid \sigma\langle\!\langle\rho\langle\!\langle\diamond\rangle\!\rangle \circ N\rangle\!\rangle \in \mathcal{T}\}/_\equiv) +
$$

$$
\sum_{\mathcal{S}([\diamond]_\equiv)(\mathcal{T}')\subseteq\mathcal{T}} \frac{\theta(exo_n)(N)(\mathcal{S}) \cdot \theta(\eth_n^\perp)(M)(\mathcal{T}')}{\iota(\eth_n^\perp)}
$$

Case $M \circ N$: *For any $a \in \mathbb{A}_{\mathsf{mem}}$ let $\theta(a)(M \circ N) = \omega$. For all other labels in $\mathbb{A}^{+} \setminus \mathbb{A}_{\mathsf{mem}}$*

$$\theta(phago_n)(M \circ N)(\mathcal{T}) = \theta(phago_n)(N)(\{F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys} \mid \lambda Z.\ (F(Z) \circ M) \in \mathcal{T}\}/_{\equiv}) +$$
$$\theta(phago_n)(M)(\{F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys} \mid \lambda Z.\ (F(Z) \circ N) \in \mathcal{T}\}/_{\equiv})$$
$$\theta(\overline{phago_n})(M \circ N)(\mathcal{T}) = \theta(\overline{phago_n})(N)(\{A : \mathsf{sys} \to \mathsf{sys} \mid \lambda X.\ (A(X) \circ M) \in \mathcal{T}\}/_{\equiv}) +$$
$$\theta(\overline{phago_n})(M)(\{A : \mathsf{sys} \to \mathsf{sys} \mid \lambda X.\ (A(X) \circ N) \in \mathcal{T}\}/_{\equiv})$$
$$\theta(exo_n)(M \circ N)(\mathcal{T}) = \theta(exo_n)(N)(\{T : \mathsf{sys} \to \mathsf{mem} \to \mathsf{sys} \mid \lambda Xy.\ T(X \circ M)(y) \in \mathcal{T}\}/_{\equiv}) +$$
$$\theta(exo_n)(M)(\{T : \mathsf{sys} \to \mathsf{mem} \to \mathsf{sys} \mid \lambda Xy.\ T(X \circ N)(y) \in \mathcal{T}\}/_{\equiv})$$
$$\theta(id)(M \circ N)(\mathcal{T}) = \theta(id)(N)(\mathcal{T}_{\circ M}) + \theta(id)(M)(\mathcal{T}_{\circ N}) +$$
$$\sum_{\substack{\mathcal{F}(\mathcal{A}) \subseteq \mathcal{T}}}^{n \in \Lambda} \frac{\theta(phago_n)(M)(\mathcal{F}) \cdot \theta(\overline{phago_n})(N)(\mathcal{A})}{\iota(\otimes_n)} +$$
$$\sum_{\substack{\mathcal{F}(\mathcal{A}) \subseteq \mathcal{T}}}^{n \in \Lambda} \frac{\theta(phago_n)(N)(\mathcal{F}) \cdot \theta(\overline{phago_n})(M)(\mathcal{A})}{\iota(\otimes_n)}$$

The intuition behind this definition is that each summand corresponds to a derivation rule of the LTS of Section 3. For example, in $\theta(id)(M \circ N)(\mathcal{T})$ the last summand corresponds to (id-phago-R) rule. Similarly, if there are no *a*-transitions for a term M in the LTS, $\theta(a)(M) = \omega$; this is the case of $\mathbf{0}$, $\diamond$, and $\lambda X.\ M$. Note that for each $\sigma \in \mathbb{M}$, $\theta(a)(\sigma) \neq \omega$ iff $a \in \mathsf{act}(\sigma)$. Consequently each infinitary sum involved in Definition 4.5 has a finite number of non-zero summands.

In particular we have a correspondence between the LTS and the function θ in the following sense.

Proposition 4.6. *For arbitrary $M \in T$ and $\alpha \in \mathbb{A}_{\mathsf{sys}}^{+}$ the following statements hold*

1. *if $\theta(\alpha)(M)(\mathcal{T}) > 0$ then there exists $M' \in \mathcal{T}$ such that $M \xrightarrow{\alpha} M'$,*

2. *if $M \xrightarrow{\alpha} M'$ then there exists $\mathcal{M} \in \Pi$ such that $M' \in \mathcal{T}$ and $\theta(\alpha)(M)(\mathcal{T}) > 0$.*

In the proposition above, (1) can be proven by induction on the structure of the term M (assumed to be well-typed); while the prove for (2) is by induction on the derivation of $M \xrightarrow{\alpha} M'$. Note that Proposition 4.6 reflects the similarity between Definition 4.5 and the definition of the LTS of Section 3.

A direct consequence of Proposition 4.6 is the following.

Corollary 4.7. $M \xrightarrow{\alpha} M'$ *iff* $\theta(\alpha)(M)([M']_{\equiv}) > 0$.

The next theorem states that $(\mathbb{T}, \Sigma, \theta)$ is an $\mathbb{A}^{+}$-Markov kernel. Proving this will implicitly show the correctness of our construction, indeed it suffices to prove that for each $M \in \mathbb{T}$ and each $\alpha \in \mathbb{A}^{+}$, $\theta(\alpha)(P) : \Sigma \to \mathbb{R}^{+}$ is a measure on the measurable space $(\mathbb{T}, \Sigma)$.

Theorem 4.8 (Markov kernel for the measurable space of terms). $(\mathbb{T}, \Sigma, \theta)$ *is an $\mathbb{A}^{+}$-Markov kernel.*

A consequence of this theorem is that for each $M \in \mathbb{T}$, $(\mathbb{T}, \Sigma, \theta, M)$ is a Markov process, hence we can define a stochastic bisimulation for Brane Calculus meta-syntactic terms, simply as the stochastic bisimulation of Markov processes in $(\mathbb{T}, \Sigma, \theta)$.

From the measurable space $(\mathbb{T}, \Sigma)$ we can recover the measurable subspaces $(\mathbb{M}, \Theta)$ and $(\mathbb{P}, \Pi)$, respectively for membrane and system terms. Indeed, if Θ is the σ-algebra generated from the base $\mathbb{M}/_{\equiv}$, and Π the one generated from the base $\mathbb{P}/_{\equiv}$, both Θ and Π are contained in Σ, hence $(\mathbb{M}, \Theta)$ and $(\mathbb{P}, \Pi)$ are two subspaces of $(\mathbb{T}, \Sigma)$.

One may be led to think that by a suitable restriction on θ we can also recover two A-Markov kernels respectively on the sets $\mathbb{A}_{\mathsf{mem}}$ and $\mathbb{A}_{\mathsf{sys}}^+$ of labels. Of course, it is possible to define two Markov kernels by letting $\theta_{\mathsf{mem}}\colon \mathbb{A}_{\mathsf{mem}}^+ \to [\![\mathbb{M} \to \Delta(\mathbb{M},\Theta)]\!]$ and $\theta'_{\mathsf{sys}}\colon \mathbb{A}_{\mathsf{sys}}^+ \to [\![\mathbb{P} \to \Delta(\mathbb{P},\Pi)]\!]$ as

$$\theta_{\mathsf{mem}}(a)(\sigma)(\mathcal{M}) = \theta(a)(\sigma)(\mathcal{M}) \qquad\qquad \text{for all } \mathcal{M} \in \Theta$$
$$\theta'_{\mathsf{sys}}(b)(P)(\mathcal{P}) = \theta(b)(P)(\mathcal{P}) \qquad\qquad \text{for all } \mathcal{P} \in \Pi.$$

Although this definition works well for $(\mathbb{M},\Theta)$, it does not straightforwardly work for $(\mathbb{P},\Pi)$. In fact θ'_{sys} does not enjoy a result similar to Proposition 4.6, and moreover the stochastic bisimulation is not the equivalence that one may expect. For example, it is easy to see that for all $\alpha \in \mathbb{A}_{\mathsf{sys}}^+$ and $\mathcal{P} \in \Pi$, $\theta(\alpha)(\circlearrowright_n(\!(\diamond)\!))(\mathcal{P}) = 0$. The same thing happen for $\circlearrowleft_m(\!(\diamond)\!)$, that is, $\theta(\alpha)(\circlearrowleft_m(\!(\diamond)\!))(\mathcal{P}) = 0$, hence $(\mathbb{P},\Pi,\theta'_{\mathsf{sys}},\circlearrowright_n(\!(\diamond)\!))$ and $(\mathbb{P},\Pi,\theta'_{\mathsf{sys}},\circlearrowleft_m(\!(\diamond)\!))$ are stochastic bisimilar. By a simple analysis on the structure of $\circlearrowright_n(\!(\diamond)\!)$ and $\circlearrowleft_m(\!(\diamond)\!)$, it is easy to see that

$$\theta(\mathsf{exo}_n)(\circlearrowright_n(\!(\diamond)\!))([\lambda Xy.\,\mathbf{0}|y(\!(X)\!) \circ \diamond)]_\equiv) = \theta(\mathsf{exo}_n)(\circlearrowright_n(\!(\diamond)\!))([\lambda Xy.\,\mathbf{0}|y(\!(X)\!)]_\equiv) = \iota(\circlearrowright_n) > 0$$

$$\theta(\mathsf{phago}_m)(\circlearrowleft_m(\!(\diamond)\!))([\lambda Z.\,Z(\mathbf{0}(\!(\diamond)\!))]_\equiv) = \theta(\mathsf{phago}_m)(\circlearrowleft_m(\!(\diamond)\!))([\lambda Z.\,Z(\diamond)]_\equiv) = \iota(\circlearrowleft_m) > 0.$$

We can prove that $\theta(\mathsf{exo}_n)(\circlearrowright_n(\!(\diamond)\!))(\mathcal{T}) > 0$ iff $\lambda Xy.\,\mathbf{0}|y(\!(X)\!) \in \mathcal{T}$, and $\theta(\mathsf{phago}_m)(\circlearrowleft_m(\!(\diamond)\!))(\mathcal{T}) > 0$ iff $\lambda Z.\,Z(\diamond) \in \mathcal{T}$ (remember that both $\theta(\mathsf{exo}_n)(\circlearrowright_n(\!(\diamond)\!))$ and $\theta(\mathsf{phago}_m)(\circlearrowleft_m(\!(\diamond)\!))$ are measures). This example points out the problem: the two measurable spaces "are not of the right type". Indeed, in Definition 4.5 it easy to see that for labels in $\mathbb{A}_{\mathsf{sys}}$ (see cases $M(\!(N)\!)$ and $M \circ N$) θ contributes to the rate with a nonzero value only if the the measurable space given as (last) parameter contains terms of nonbase type. In particular, for label phago_n the type is $(\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys}$, for label $\overline{\mathsf{phago}}_n$ is $\mathsf{sys} \to \mathsf{sys}$, and for label exo_n is $\mathsf{sys} \to \mathsf{mem} \to \mathsf{sys}$.

This simple consideration induces the definition of another function:

Definition 4.9. *Let* $\tilde{\theta}_{\mathsf{sys}}\colon \mathbb{A}_{\mathsf{sys}}^+ \to [\![\mathbb{P} \to \Delta(\mathbb{P},\Pi)]\!]$ *be defined as follows, where* $P \in \mathbb{P}$ *and* $\mathcal{P} \in \Pi$.

$$\tilde{\theta}_{\mathsf{sys}}(id)(P)(\mathcal{P}) = \theta(id)(P)(\mathcal{P})$$

$$\tilde{\theta}_{\mathsf{sys}}(phago_n)(P)(\mathcal{P}) = \theta(phago_n)(P)(\mathcal{F})$$
$$\text{where } \mathcal{F} = \{F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys} \mid \exists A : \mathsf{sys} \to \mathsf{sys}.\, F(A) \in \mathcal{P}\}/_\equiv$$

$$\tilde{\theta}_{\mathsf{sys}}(\overline{phago}_n)(P)(\mathcal{P}) = \theta(\overline{phago}_n)(P)(\mathcal{A})$$
$$\text{where } \mathcal{A} = \{A : \mathsf{sys} \to \mathsf{sys} \mid \exists F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys}.\, F(A) \in \mathcal{P}\}/_\equiv$$

$$\tilde{\theta}_{\mathsf{sys}}(exo_n)(P)(\mathcal{P}) = \theta(exo_n)(P)(\mathcal{S})$$
$$\text{where } \mathcal{S} = \{S : \mathsf{sys} \to \mathsf{mem} \to \mathsf{sys} \mid \exists Q \in \mathbb{P}, \sigma \in \mathbb{M}.\, S(Q)(\sigma) \in \mathcal{P}\}/_\equiv$$

Again $(\mathbb{P},\Sigma,\tilde{\theta}_{\mathsf{sys}})$ is an $\mathbb{A}_{\mathsf{sys}}^+$-Markov kernel, and in particular the stochastic bisimulation on Markov processes is the one that we need. It is easy to see that $(\mathbb{P},\Pi,\tilde{\theta}_{\mathsf{sys}},\circlearrowright_n(\!(\diamond)\!))$ and $(\mathbb{P},\Pi,\tilde{\theta}_{\mathsf{sys}},\circlearrowleft_m(\!(\diamond)\!))$ are no more stochastic bisimilar (assuming $\iota(\circlearrowright)_n \neq \iota(\circlearrowleft)_n$).

However, the above definition is not defined inductively on the structure of terms. To this end, we provide another function $\theta_{\mathsf{sys}}\colon \mathbb{A}_{\mathsf{sys}}^+ \to [\![\mathbb{P} \to \Delta(\mathbb{P},\Pi)]\!]$, defined by induction on the structure of system terms, and we prove that θ_{sys} is an alternative characterization of $\tilde{\theta}_{\mathsf{sys}}$, i.e., θ_{sys} and $\tilde{\theta}_{\mathsf{sys}}$ coincide.

Definition 4.10. *Let* $\theta_{\mathsf{sys}}\colon \mathbb{A}_{\mathsf{sys}}^+ \to [\![\mathbb{P} \to \Delta(\mathbb{P},\Pi)]\!]$ *be defined on the structure of* $P \in \mathbb{P}$*, as follows.*

Case $P = \diamond$**:** *For any* $\alpha \in \mathbb{A}_{\mathsf{sys}}^+$*, let* $\theta_{\mathsf{sys}}(\alpha) = \omega$.

Case $P = \sigma \langle Q \rangle$: *For any $\mathcal{P} \in \Pi$,*

$$\theta_{\text{sys}}(phago_n)(\sigma\langle Q\rangle)(\mathcal{P}) = \theta_{\text{mem}}(\eth_n)(\sigma)(\{\sigma' \mid \tau\langle\rho\langle\sigma'\langle Q\rangle\rangle \circ R\rangle \in \mathcal{P}\}/_{\equiv})$$

$$\theta_{\text{sys}}(\overline{phago_n})(\sigma\langle Q\rangle)(\mathcal{P}) = \sum_{\rho \in \mathbb{M}} \theta_{\text{mem}}(\eth_n^{\perp}(\rho))(\sigma)(\{\sigma' \mid \sigma'\langle\rho\langle\tau\langle R\rangle\rangle \circ Q\rangle \in \mathcal{P}\}/_{\equiv})$$

$$\theta_{\text{sys}}(exo_n)(\sigma\langle Q\rangle)(\mathcal{P}) = \theta_{\text{mem}}(\eth_n)(\sigma)(\{\sigma' \mid \sigma'|\tau\langle R\rangle \circ Q \in \mathcal{P}\}/_{\equiv})$$

$$\theta_{\text{sys}}(id)(\sigma\langle Q\rangle)(\mathcal{P}) = \theta_{\text{sys}}(id)(Q)(\mathcal{P}_{\sigma\langle\rangle}) +$$

$$\sum_{\rho \in \mathbb{M}}^{n \in \Lambda} \theta_{\text{mem}}(\circledcirc_n(\rho))(\sigma)(\{\sigma' \mid \sigma'\langle\rho\langle\diamond\rangle \circ Q\rangle \in \mathcal{P}\}/_{\equiv}) +$$

$$\sum_{n \in \Lambda} \frac{\theta_{\text{mem}}(\eth_n^{\perp})(\sigma)(\{\sigma' \mid \sigma'|\tau\langle Q\rangle \circ R \in \mathcal{P}\}/_{\equiv}) \cdot \theta_{\text{sys}}(exo_n)(Q)(\mathcal{P})}{\iota(\eth_n)}$$

Case $P = Q \circ R$: *For all $\alpha \in \mathbb{A}_{\text{sys}}$ and $\mathcal{P} \in \Pi$*

$$\theta_{\text{sys}}(\alpha)(Q \circ R)(\mathcal{P}) = \theta_{\text{sys}}(\alpha)(R)(\mathcal{P}_{\circ Q}) + \theta_{\text{sys}}(\alpha)(Q)(\mathcal{P}_{\circ R})$$

$$\theta_{\text{sys}}(id)(Q \circ R)(\mathcal{P}) = \theta_{\text{sys}}(id)(R)(\mathcal{P}_{\circ Q}) + \theta_{\text{sys}}(id)(Q)(\mathcal{P}_{\circ R}) +$$

$$\sum_{n \in \Lambda} \frac{\theta_{\text{sys}}(phago_n)(R)(\mathcal{P}) \cdot \theta_{\text{sys}}(\overline{phago_n})(Q)(\mathcal{P})}{\iota(\eth_n)} +$$

$$\sum_{n \in \Lambda} \frac{\theta_{\text{sys}}(phago_n)(Q)(\mathcal{P}) \cdot \theta_{\text{sys}}(\overline{phago_n})(R)(\mathcal{P})}{\iota(\eth_n)}$$

The next theorem states that θ_{sys} precisely characterizes the function $\tilde{\theta}_{\text{sys}}$ defined before.

Theorem 4.11. *For all $\alpha \in \mathbb{A}_{\text{sys}}^+$, $P \in \mathbb{P}$, and $\mathcal{P} \in \Pi$, $\theta_{\text{sys}}(\alpha)(P)(\mathcal{P}) = \tilde{\theta}_{\text{sys}}(\alpha)(P)(\mathcal{P})$.*

A consequence of the previous theorem is that $(\mathbb{P}, \Pi, \theta_{\text{sys}})$ is an $\mathbb{A}_{\text{sys}}^+$-Markov kernel, and that the stochastic bisimulation on Markov processes coincides with that induced by $(\mathbb{P}, \Pi, \tilde{\theta}_{\text{sys}})$.

5 Stochastic Structural Operational Semantics and Bisimulation

In this section we introduce the stochastic structural operational semantics for the Brane Calculus, with the aim of defining a behavioral equivalence on system terms that coincides with their bisimulation as Markov processes on $(\mathbb{P}, \Pi, \theta_{\text{sys}})$. Notably, it is directly induced from the definition of the function θ_{sys} (Definition 4.10), following the pattern of [8]. Since this is an unusual construction for a structural operational semantics, some preliminary discussions are needed. Typically, a structural operational semantics for a stochastic process algebra associates a rate *r* with transitions: $P \xrightarrow{\alpha}_r P'$. Instead, in [8] transitions are not between two processes, but from a process to an infinite measurable set of processes. In order to maintain "the spirit" of process algebras Cardelli and Mardare replace the classic rules of the form $P \xrightarrow{\alpha} P'$ with rules of the form $P \to \mu$, where μ is a function from action labels to measures on the measurable space of processes. Let us see how this construction can be applied to the Brane Calculus.

For simplifying the rules of the operational semantics, we first define some operations on the functions in $\Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ and $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\text{sys}}^+}$, and analyze their properties. We say that a function $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\text{sys}}^+}$ has *finite support* if $\mathbb{A}_{\text{sys}}^+ \setminus \mu^{-1}(\omega)$ is finite or empty (recall that ω is the null measure).

Definition 5.1. *Consider the following constants and operations on $\Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ defined as follows.*

- $\omega^{\text{mem}} \colon \mathbb{A}_{\text{mem}} \to \Delta(\mathbb{M}, \Theta)$ *defined as* $\omega^{\text{mem}}(\alpha) = \omega$, *for arbitrary* $\alpha \in \mathbb{A}_{\text{mem}}$;

- *For arbitrary* $a \in \mathbb{A}_{\text{mem}}$ *and* $\sigma, \rho \in \mathbb{M}$, *let* $\alpha_\sigma \colon \mathbb{A}_{\text{mem}} \to \Delta(\mathbb{M}, \Theta)$ *be defined by*

$$[\varepsilon]_\sigma(a) = \begin{cases} D(\iota(\varepsilon), [\sigma]_\equiv) & \text{if } a = \varepsilon \\ \omega & \text{if } a \neq \varepsilon \end{cases} \qquad (\text{for } \varepsilon \in \{\eth_n, \circlearrowleft_n, \eth_n^\perp \mid n \in \Lambda\})$$

$$[\varepsilon(\rho)]_\sigma(a) = \begin{cases} D(\iota(\varepsilon), [\sigma]_\equiv) & \text{if } a = \varepsilon(\rho) \\ \omega & \text{if } a \neq \varepsilon(\rho) \end{cases} \qquad (\text{for } \varepsilon \in \{\eth_n^\perp, \circledcirc_n \mid n \in \Lambda\})$$

- *For arbitrary* $\mu, \mu' \in \Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ *with finite support,* $a \in \mathbb{A}_{\text{mem}}$, $\sigma, \tau \in \mathbb{M}$, *and* $\mathcal{M} \in \Theta$, *let the function* $\mu_\sigma \oplus_\tau \mu' \mathbb{A}_{\text{mem}} \to \Delta(\mathbb{M}, \Theta)$ *be defined by*

$$(\mu_\sigma \oplus_\tau \mu')(a)(\mathcal{M}) = \mu(a)(\mathcal{M}_\tau) + \mu'(a)(\mathcal{M}_\sigma)$$

Consider the following constants and operations on $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\text{sys}}^+}$ *and* $\Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ *defined as follows.*

- $\omega^{\text{sys}} \colon \mathbb{A}_{\text{sys}}^+ \to \Delta(\mathbb{P}, \Pi)$ *defined as* $\omega^{\text{sys}}(\alpha) = \omega$, *for arbitrary* $\alpha \in \mathbb{A}_{\text{sys}}^+$;

- *For arbitrary* $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\text{sys}}^+}$, $\mu' \in \Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ *with finite support,* $\sigma \in \mathbb{M}$, $P \in \mathbb{P}$, *and* $\mathcal{P} \in \Pi$ *let the function* $\mu @_P^\sigma \mu' \colon \mathbb{A}_{\text{sys}}^+ \to \Delta(\mathbb{P}, \Pi)$ *be defined by:*

$$(\mu @_P^\sigma \mu')(phago_n)(\mathcal{P}) = \mu'(\eth_n)(\{\sigma' \mid \tau \langle \rho \langle \sigma' \langle P \rangle \rangle \circ Q \rangle \in \mathcal{P}\}/_\equiv)$$

$$(\mu @_P^\sigma \mu')(\overline{phago}_n)(\mathcal{P}) = \sum_{\rho \in \mathbb{M}} \mu'(\eth_n^\perp(\rho))(\{\sigma' \mid \sigma' \langle \rho \langle \tau \langle Q \rangle \rangle \circ P \rangle \in \mathcal{P}\}/_\equiv)$$

$$(\mu @_P^\sigma \mu')(exo_n)(\mathcal{P}) = \mu'(\circlearrowleft_n)(\{\sigma' \mid \sigma' | \tau \langle Q \rangle \circ P \in \mathcal{P}\}/_\equiv)$$

$$(\mu @_P^\sigma \mu')(id)(\mathcal{P}) = \mu(id)(\mathcal{P}_{\sigma\langle\rangle}) + \sum_{\rho \in \mathbb{M}}^{n \in \Lambda} \mu'(\circledcirc_n(\rho))(\{\sigma' \mid \sigma' \langle \rho \langle \diamond \rangle \circ Q \rangle \in \mathcal{P}\}/_\equiv) +$$

$$\sum_{n \in \Lambda} \frac{\mu'(\eth_n^\perp)(\{\sigma' \mid \sigma' | \tau \langle P \rangle \circ Q \in \mathcal{P}\}/_\equiv) \cdot \mu(exo_n)(\mathcal{P})}{\iota(\eth_n)}$$

- *For arbitrary* $\mu, \mu' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\text{sys}}^+}$ *with finite support,* $a \in \mathbb{A}_{\text{sys}}$, $P, Q \in \mathbb{P}$, *and* $\mathcal{P} \in \Pi$, *let the function* $\mu_P \otimes_Q \mu' \colon \mathbb{A}_{\text{sys}}^+ \to \Delta(\mathbb{P}, \Pi)$ *be defined by*

$$(\mu_P \otimes_Q \mu')(a)(\mathcal{P}) = \mu(a)(\mathcal{P}_Q) + \mu'(a)(\mathcal{P}_P)$$

$$(\mu_P \otimes_Q \mu')(id)(\mathcal{P}) = \mu(id)(\mathcal{P}_Q) + \mu'(id)(\mathcal{P}_P) +$$

$$\sum_{n \in \Lambda} \frac{\left(\mu(phago_n)(\mathcal{P}) \cdot \mu'(\overline{phago}_n)(\mathcal{P})\right) + \left(\mu(\overline{phago}_n)(\mathcal{P}) \cdot \mu'(phago_n)(\mathcal{P})\right)}{\iota(\eth_n)}$$

Notice that since μ and μ' have finite support, each infinite sum involved in Definition 5.1 has a finite number of nonzero summands. The next two lemmata prove that the definitions of $_\sigma \oplus_\tau$, $_P \otimes_Q$, and $@_P^\sigma$, for arbitrary $\sigma, \tau \in \mathbb{M}$ and $P, Q \in \mathbb{P}$ are correct; they also state some basic properties of these operators.

Lemma 5.2. *The following statements hold.*

1. *For arbitrary* $\sigma, \tau, \rho \in \mathbb{M}$ *and* $\mu', \mu'', \mu''' \in \Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{\text{mem}}}$ *with finite support*

    (a) $\mu'_{\ \sigma} \oplus_\tau \mu'' = \mu''_{\ \tau} \oplus_\sigma \mu'$,

$$\frac{}{\mathbf{0} \to \omega^{\mathsf{mem}}} \text{ (zero)} \qquad \frac{\varepsilon \in \{\eth_n, \mho_n, \mho_n^{\perp}\}}{\varepsilon.\sigma \to [\varepsilon]_\sigma} \text{ (pref)} \qquad \frac{\varepsilon \in \{\mho_n^{\perp}, \circledcirc_n\}}{\varepsilon(\rho).\sigma \to [\varepsilon(\rho)]_\sigma} \text{ (pref-arg)} \qquad \frac{\sigma \to \mu' \qquad \tau \to \mu''}{\sigma|\tau \to \mu'_\sigma \mathbb{O}_\tau \mu''} \text{ (par)}$$

$$\frac{}{\diamond \to \omega^{\mathsf{sys}}} \text{ (void)} \qquad \frac{\sigma \to \nu \qquad P \to \mu}{\sigma(\!|P|\!) \to \mu @_P^\sigma \nu} \text{ (loc)} \qquad \frac{P \to \mu' \qquad Q \to \mu''}{P \circ Q \to \mu'_P \otimes_Q \mu''} \text{ (comp)}$$

Table 4: Structural operational semantics for Brane Calculus

   (b) $(\mu'_\sigma\mathbb{O}_\tau\,\mu'')_{\sigma|\tau}\mathbb{O}_\rho\,\mu''' = \mu'_\sigma\mathbb{O}_{\tau|\rho}\,(\mu''_\tau\mathbb{O}_\rho\,\mu''')$,

   (c) $\mu'_\sigma\mathbb{O}_{\boldsymbol{0}}\,\omega^{\mathsf{mem}} = \mu'$.

2. *For arbitrary $P, Q, R \in \mathbb{P}$ and $\mu', \mu'', \mu''' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\mathsf{sys}}^+}$ with finite support*

   (a) $\mu'_P\otimes_Q\mu'' = \mu''_Q\otimes_P\mu'$,

   (b) $(\mu'_P\otimes_Q\mu'')_{P|Q}\otimes_R\mu''' = \mu'_P\otimes_{Q|R}(\mu''_Q\otimes_R\mu''')$,

   (c) $\mu'_P\otimes_\diamond\omega^{\mathsf{sys}} = \mu'$.

3. $\omega^{\mathsf{sys}}@_\diamond^{\boldsymbol{0}}\omega^{\mathsf{mem}} = \omega^{\mathsf{sys}}$.

Lemma 5.3. *The following statements hold.*

1. *For arbitrary $\sigma, \sigma', \tau, \tau' \in \mathbb{M}$ and $\mu', \mu'' : \mathbb{A}_{\mathsf{mem}} \to \Delta(\mathbb{M}, \Theta)$ with finite support*

   (a) *for $\varepsilon \in \{\eth_n, \mho_n, \mho_n^\perp \mid n \in \Lambda\}$, $\sigma \equiv \tau$ implies $[\varepsilon]_\sigma = [\varepsilon]_\tau$,*

   (b) *for $\varepsilon \in \{\mho_n^\perp, \circledcirc_n \mid n \in \Lambda\}$, $\sigma \equiv \tau$ and $\rho \equiv \rho'$ imply $[\varepsilon(\rho)]_\sigma = [\varepsilon(\rho')]_\tau$,*

   (c) *$\sigma \equiv \sigma'$ and $\tau \equiv \tau'$ imply $\mu'_\sigma\mathbb{O}_\tau\mu'' = \mu'_{\sigma'}\mathbb{O}_{\tau'}\mu''$.*

2. *For arbitrary $P, P', Q, Q' \in \mathbb{P}$, $\sigma, \tau \in \mathbb{M}$, $\mu, \mu', \mu'' : \mathbb{A}_{\mathsf{sys}}^+ \to \Delta(\mathbb{P}, \Pi)$, and $\nu : \mathbb{A}_{\mathsf{mem}} \to \Delta(\mathbb{M}, \Theta)$ with finite support*

   (a) *$P \equiv Q$ and $\sigma \equiv \tau$ imply $\mu@_P^\sigma\nu = \mu@_Q^\tau\nu$,*

   (b) *$P \equiv P'$ and $Q \equiv Q'$ imply $\mu'_P\otimes_Q\mu'' = \mu'_{P'}\otimes_{Q'}\mu''$.*

    The rules of the operational semantics are listed in Table 4. The operational semantics associates with each membrane σ a mapping $\nu \in \Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{mem}}$, and with each system $P \in \mathbb{P}$ a mapping $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\mathsf{sys}}^+}$. For each $\equiv$-closed set $\mathcal{M} \in \Theta$ and each label $\alpha \in \mathbb{A}_{mem}$, $\nu(\alpha)(\mathcal{M}) \in \mathbb{R}^+$ represents the total rate of an α-transition of σ to some arbitrary element in $\mathcal{M}$; and for each set $\mathcal{P} \in \Pi$ and label $\alpha' \in \mathbb{A}_{\mathsf{sys}}^+$, similarly $\mu(\alpha')(\mathcal{P}) \in \mathbb{R}^+$ represents the total rate of an α'-transition of P to some arbitrary element in $\mathcal{P}$.

    The next lemma guarantees the consistency of the stochastic transition relation $\to$, and as a consequence the consistency of the operational semantics.

Lemma 5.4 (Uniqueness of the measure). *For each $\sigma \in \mathbb{M}$ and $P \in \mathbb{P}$:*

1. *there exists a unique $\mu \in \Delta(\mathbb{M}, \Theta)^{\mathbb{A}_{mem}}$ such that $\sigma \to \mu$; moreover, μ has finite support;*

2. *there exists a unique $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\mathsf{sys}}}$ such that $P \to \mu$; moreover, μ has finite support.*

This operational semantics can be further used to define various pointwise semantics as, e.g.:

$$P \xrightarrow{\alpha, r} Q \qquad \text{iff} \qquad P \to \mu \text{ and } \mu(\alpha)([Q]_\equiv) = r$$

Example 5.5. *Suppose $\iota(\eth_n) = r$, and $\tau, \rho \in \mathbb{M}$, $Q \in \mathbb{P}$; it is easy to see that*

1. $\eth_n.\sigma(\!|P|\!) \xrightarrow{phago_n,r} \tau(\!|\rho(\!|\sigma(\!|P|\!)|\!) \circ Q|\!)$

2. $\eth_n.\sigma(\!|P|\!) \circ \eth_n.\sigma(\!|P|\!) \xrightarrow{phago_n,2r} \tau(\!|\rho(\!|\sigma(\!|P|\!)|\!) \circ Q|\!) \circ \eth_n.\sigma(\!|P|\!)$

3. $\eth_n.\sigma(\!|P|\!) \circ \eth_n(\!|\diamond|\!) \xrightarrow{phago_n,r} \tau(\!|\rho(\!|\sigma(\!|P|\!)|\!) \circ Q|\!) \circ \eth_n(\!|\diamond|\!)$

4. $\eth_n.\sigma(\!|P|\!) \circ \eth_n(\!|\diamond|\!) \xrightarrow{phago_n,r} \eth_n.\sigma(\!|P|\!) \circ \tau(\!|\rho(\!|\diamond|\!) \circ Q|\!)$

5. $\eth_n.\sigma(\!|P|\!) \circ \eth_n^{\perp}(\rho).\tau(\!|Q|\!) \xrightarrow{id,r} \tau(\!|\rho(\!|\sigma(\!|P|\!)|\!) \circ Q|\!)$

Note that in (1) the chosen τ, ρ, and Q are not relevant, indeed the "same" transition holds also for different choices of them. At first sight (1) seems curious, but the intuition behind it is that it is not important where a cell goes once it is phagocytized, but that eventually it could be phagocytized. Examples 5.5 $(2-4)$ show that the total rate is correctly summed up, while (5) exhibits an internal action and how it influences the total rate.

The next lemma states that operational semantics does not distinguish structurally equivalent terms:

Lemma 5.6. *Stochastic transitions are up-to structural equivalence:*

1. *if $\sigma \equiv \tau$ and $\sigma \to \mu$ then $\tau \to \mu$,*

2. *if $P \equiv Q$ and $P \to \mu$ then $Q \to \mu$.*

Notice that the converse does not hold in general, that is, if for some $P, Q \in \mathbb{P}$, $P \to \mu'$ and $Q \to \mu''$, and $\mu' = \mu''$, this does not imply that $P \equiv Q$. For example, let $P = \mathbf{0}(\!|\eth_n(\!|\mathbb{1}|\!)|\!)$ and $Q = \diamond$, then

$$P \to \left(\mu_1 = (\omega_{\mathsf{sys}}@_{\diamond}^{\eth_n}[\eth_n]_{\mathbf{0}})@_{\eth_n(\!|\mathbb{1}}^{\mathbf{0}}\omega^{\mathsf{mem}}\right)$$
$$Q \to \left(\mu_2 = \omega^{\mathsf{sys}}\right)$$

It is trivial to verify that $\mu_1 = \mu_2$, however $P \not\equiv Q$. In fact for each $\alpha \in \mathbb{A}_{\mathsf{sys}}$, $\mu_1(\alpha)(\mathcal{P}) = 0$ (for all $\mathcal{P}$), since μ_1 is of the form $\mu_1'@_R^{\rho}\omega^{\mathsf{mem}}$ and its result depends only on ω^{mem}. In order to prove $\mu_1(id)(\mathcal{P}) = 0$ it suffices to verify that $(\omega_{\mathsf{sys}}@_{\diamond}^{\eth_n}[\eth_n]_{\mathbf{0}})(id)(\mathcal{P}_{\mathbf{0}(\!|\mathbb{1}}) = 0$, but again it is easy because the result depends only on ω^{sys}, hence $\mu_1 = \mu_2$.

Now, we introduce the stochastic bisimulation for the Brane Calculus as the stochastic bisimulation on the Markov kernel $(\mathbb{P}, \Pi, \theta_{\mathsf{sys}})$. We show that systems which are associated with the same function by our SOS are bisimilar, and that the bisimulation extends the structural equivalence.

Lemma 5.4 shows that the operational semantics induces a function $\vartheta: \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\mathsf{sys}}^+}$ defined by

$$\vartheta(P) = \mu \qquad \text{iff} \qquad P \to \mu$$

The next lemma shows that there is a relation between ϑ and the function θ_{sys} that organizes $\mathbb{P}$ as a Markov kernel. Indeed, it reflects the similarity between Definition 4.10 and Definition 5.1.

Lemma 5.7. *If $(\mathbb{P}, \Pi, \theta_{\mathsf{sys}})$ is the Markov kernel of system terms and $\vartheta: \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}_{\mathsf{sys}}^+}$ is the function induced by the SOS, then for any $P \in \mathbb{P}$, $\alpha \in \mathbb{A}_{\mathsf{sys}}^+$, and $\mathcal{P} \in \Pi$,*

$$\theta_{\mathsf{sys}}(\alpha)(P)(\mathcal{P}) = \vartheta(P)(\alpha)(\mathcal{P}).$$

A direct consequence of the previous lemma is that if our SOS assigns to different systems the same function, then they are stochastic bisimilar with respect to the bisimulation on Markov processes.

Corollary 5.8. *For arbitrary $P, Q \in \mathbb{P}$, if $P \to \mu$ and $Q \to \mu$, then $P \sim_{(\mathbb{P}, \Pi, \theta_{\mathsf{sys}})} Q$.*

This guarantees that we can safely define the stochastic bisimulation for the Brane Calculus as the stochastic bisimulation on $(\mathbb{P}, \Pi, \theta_{\mathsf{sys}})$, as we do in the next definition.

Definition 5.9 (Stochastic bisimulation on systems). *A* rate-bisimulation relation *on systems in an equivalence relation $\mathscr{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$,*

$$(P, Q) \in \mathscr{R} \qquad \textit{iff} \qquad \mu(\alpha)(C) = \mu'(\alpha)(C) \quad \textit{for any } C \in \Pi(\mathscr{R}) \textit{ and any } \alpha \in \mathbb{A}_{\mathsf{sys}}^{+}$$

Two systems $P, Q \in \mathbb{P}$ are stochastic bisimilar, written $P \approx Q$, iff there exists a rate bisimulation relation $\mathscr{R}$ such that $(P, Q) \in \mathscr{R}$.

The next theorem provides a characterization of stochastic bisimulation stating that $\approx$ is the smallest rate-bisimulation relation on $\mathbb{P}$.

Theorem 5.10. *The stochastic bisimulation relation $\approx$ is the smallest equivalence relation on $\mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$,*

$$P \approx Q \qquad \textit{iff} \qquad \mu(\alpha)(C) = \mu'(\alpha)(C) \quad \textit{for any } C \in \Pi(\approx) \textit{ and any } \alpha \in \mathbb{A}_{\mathsf{sys}}^{+}.$$

Example 5.11. *For arbitrary $\sigma \in \mathbb{M}$*

  1. $\boldsymbol{0} \langle\!(\sigma \langle\!(\diamond)\!\rangle)\!\rangle \approx \diamond$

  2. $\circledcirc_n(\sigma) | \circledcirc_n(\sigma) \langle\!(\diamond)\!\rangle \not\approx \circledcirc_n(\sigma).\circledcirc_n(\sigma) \langle\!(\diamond)\!\rangle$

  3. $\eth_n^{\perp}(\boldsymbol{0}) | \eth_m^{\perp} \langle\!(\diamond)\!\rangle \approx \eth_n^{\perp}(\boldsymbol{0}).\eth_m^{\perp} \langle\!(\diamond)\!\rangle \approx \eth_n^{\perp}(\boldsymbol{0}) \langle\!(\diamond)\!\rangle$

Examples 5.11(1) and (3) are peculiar to the Brane Calculus; (2) is interesting because in the semantics without stochastic features the two processes are bisimilar: $\circledcirc_n(\sigma) | \circledcirc_n(\sigma) \langle\!(\diamond)\!\rangle \sim \circledcirc_n(\sigma).\circledcirc_n(\sigma) \langle\!(\diamond)\!\rangle$.

The next theorem shows that our stochastic bisimulation behaves correctly with respect to structural equivalence; it is a direct consequence of Lemma 5.6 and Theorem 5.10.

Theorem 5.12. *For arbitrary $P, Q \in \mathbb{P}$, if $P \equiv Q$, then $P \approx Q$.*

In addition to to the result of the previous theorem, notice that $\approx$ is strictly larger then $\equiv$, indeed from Example 5.11(1) we have that, for any $\sigma \in \mathbb{M}$, $\boldsymbol{0} \langle\!(\sigma \langle\!( )\!\rangle)\!\rangle \approx \diamond$, however $\boldsymbol{0} \langle\!(\sigma \langle\!( )\!\rangle)\!\rangle \not\equiv \diamond$.

6   Conclusions

In this paper we have presented a stochastic extension of the Brane Calculus; brane systems are interpreted as Markov processes over the measurable space generated by terms up-to syntactic congruence, and where the measures are indexed by the actions of Brane Calculus. For finding the correct actions, we have introduced a labelled transition system for Brane Calculus. Finally, we have provided a SOS presentation of this stochastic semantics, which is compositional and syntax-driven.

Stochastic semantics for calculi of biological compartments (but not Brane Calculus) have been given in literature; see [6, 4] for stochastic versions of BioAmbients and [15] for a stochastic π-calculus with polyadic synchronisation. However, these semantics are "pointwise" and not structural, tailored for stochastic simulations using Gillepie algorithm. As shown in Section 5, a "pointwise" semantics can be readily obtained from the SOS given in this paper. An interesting future work is to investigate how these simulation algorithms and techniques can be adapted to our setting.

There are several directions for further work. First, we would like to prove that the stochastic bisimilarity (which corresponds to Markov bisimilarity) is a congruence. Then, we want to extend the theory

to cover a notion of "approximate behaviour", in order to measure how much two systems are bisimilar; this is important in biological contexts, where usually we can compare only with approximate data (e.g. coming from experiments).

We can consider also to add further constructs to the Brane Calculus, like "bind&release" and replication. For the latter, we should add rules like $\frac{P \circ !P \xrightarrow{\alpha} P'}{!P \xrightarrow{\alpha} P'}$ to the LTS of Table 3; on the stochastic side, these rules would lead to a new case in Definition 4.5, which we expect to be a fixed point equation to be solved in a suitable domain (e.g., complete metric spaces).

Finally, we would like to apply the present approach to other measurable aspects; in particular, geometric (e.g. volumes), physic (e.g. pressure, temperature) and chemical aspects are of great interest in the biological domain.

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A   Some measure theory

Given a set M, a family Σ of subsets of M is called a σ-*algebra* if it contains M and is closed under the formation of complements and (infinite) countable unions:

1. $M \in \Sigma$;

2. $A \in \Sigma$ implies $A^c \in \Sigma$, where $A^c = M \setminus A$;

3. $\{A_i\}_{i \in \mathbb{N}} \subset \Sigma$ implies $\bigcup_{i \in \mathbb{N}} A_i \in \Sigma$.

Since $M \in \Sigma$ and $M^c = \emptyset$, $\emptyset \in \Sigma$, hence Σ is nonempty by definition. A σ-algebra is closed under countable set-theoretic operations: is closed under finite unions ($A, B \in \Sigma$ implies $A \cup B = A \cup B \cup \emptyset \cup \emptyset \cup \cdots \in \Sigma$), countable intersections (by DeMorgan's law $A \cap B = (A^c \cup B^c)^c$ in its finite and inifite version), and countable subtractions ($A, B \in \Sigma$ implies $A \setminus B = A \cap B^c \in \Sigma$).

Definition A.1 (Measurable Space). *Given a set M, and a σ-algebra on M, the tuple (M, Σ) is called a* measurable space*, the elements of Σ* measurable sets*, and M the* support-set.

A set $\Omega \subseteq 2^M$ is a *generator for the σ-algebra* Σ on M if Σ is the closure of Ω under complement and countable union; we write $\sigma(\Omega) = \Sigma$ and say that Σ is generated by Ω. A generator Ω for Σ is a *base of* Σ if it has disjoin elements. Note that the σ-algebra generated by a Ω is also the smallest σ-algebra containing Ω, that is, the intersection of all σ-algebras that contain Ω. In particular it holds that a completely arbitrary intersection of σ-algebras is it self a σ-algebra. A σ-algebra generated by Ω, denoted by $\sigma(\Omega)$, is minimal in the sense that if $\Omega \subset \Sigma$ and Σ is a σ-algebra, then $\sigma(\Omega) \subset \Sigma$. Some facts about a generators are that, if Ω is a σ-algebra then obviously $\sigma(\Omega) = \Omega$; if Ω is empty or $\Omega = \{\emptyset\}$, or $\Omega = \{M\}$, then $\sigma(\Omega) = \{\emptyset, M\}$; if $\Omega \subset \Sigma$ and Σ is a σ-algebra, then $\sigma(\Omega) \subset \Sigma$.

A *measure* on a measurable space (M, Σ) is a function $\mu \colon \Sigma \to \mathbb{R}^+$ such that

1. $\mu(\emptyset) = 0$;

2. for any disjoint sequence $\{N_i\}_{i \in I} \subseteq \Sigma$ with $I \subseteq \mathbb{N}$, it holds $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$.

The triple (M, Σ, μ) is called a *measure space*. A measure space (M, Σ, μ) is called *finite* if $\mu(M)$ is a finite real number; it is called σ-finite if M can be decomposed into a countable union of measurable sets of finite measure. A set in a measure space has σ-*finite measure* if it is a countable union of sets with finite measure. Specifying a measure includes specifying its domain. If μ is a measure on a measurable space (M, Σ) and Σ' is a σ-field contained in Σ, then the restriction μ' of μ to Σ' is also a measure, and in particular a measure on (M', Σ'), for some $M' \subseteq M$ for which Σ' is a σ-algebra on M'.
A notable measure is the *Dirac measure*. If Ω is a base for (M, Σ), $N \in \Omega$ and $r \in \mathbb{R}^+$, then the function $f \colon \Omega \to \mathbb{R}^+$

$$f(N') = \begin{cases} r & \text{if } N' = N \\ 0 & \text{if } N' \neq N \end{cases}$$

can be extended, by $f(\bigcup_{i \in I} N_i) = \sum_{i \in I} f(N_i)$, to a measure on (M, Σ) denoted by $D(r, N)$ and called the *r-Dirac measure on N*.

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). It can be organized as a measurable space by considering the σ-algebra generated by the sets $N_{S,r} = \{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$, for arbitrary $S \in \Sigma$ and $r > 0$ (the support-set is $\Delta(M, \Sigma)$ and the σ-algebra is $\sigma(\bigcup_{S \in \Sigma, r \in \mathbb{R}^+} N_{S,r})$).

Given two measurable spaces (M, Σ) and (N, Θ) a mapping $f \colon M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. Measurable functions are closed under composition: given $f \colon M \to N$ and $g \colon N \to O$ measurable functions then $g \circ f \colon M \to O$ is also measurable.