EPIMORPHISMS BETWEEN LINEAR ORDERS

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Abstract. We study the relation on linear orders induced by order preserving surjections. In particular we show that its restriction to countable orders is a bqo.

Contents

1. Some generalities and the questions

Fraïssé ([Fra48]) conjectured that the class of countable linear orders was a well-quasi-order (wqo for short) under order preserving injections (also called embeddings). Laver ([Lav71]) proved that this class is in fact a better-quasi-order (bqo for short), which is much stronger.

We are interested in the somehow dual quasi-order (qo for short) induced by order preserving surjections, or *epimorphisms*, between linear orders, in particular the countable ones. What are the combinatorial properties of this qo?

In Section 2 we present the basic definitions and facts about linear orders, bqos and wqos, and epimorphisms. The main result of Section 3

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states that order preserving surjections induce a bqo on countable linear orders; this is a consequence of a theorem of van Engelen, Miller and Steel ([EMS87]) stating that the class of countable linear orders with continuous order preserving injections preserves bqos. We also look at the stronger notion of preserving bqos and show how to adapt it to epimorphisms in order to keep its validity in this setting. In Section 4 we apply the tools that we have developed earlier to describe explicitly the relation of epimorphism on some restricted classes of linear orders, such as ordinals.

2. Background

2.1. **On linear orders.**

Definitions 1.

- A *linear order* is a reflexive, transitive, antisymmetric and total relation on a non-empty set K; we usually denote it with $\leq_K$ (or $\leq$ when no confusion arises) and we say (abusively) that K is a linear order. We also write $<_K$ or $<$ for the strict part of the order.
- Given a linear order K, a *suborder* of K is a subset of K along with the induced order on it.
- A linear order K is *dense* when, given any elements x, y in K, if $x < y$ holds then there is a $z \in K$ so that $x < z < y$ holds. Denote η the unique, up to isomorphism, dense countable linear order without end-points, i.e. the order of the rationals.
- A linear order is *scattered* when η is not among its suborders.
- A linear order is *σ-scattered* if it is a countable union of scattered suborders.
- Denote by Lin (resp. LIN) the class of all countable (resp. all) linear orders, and by $Scat$ the class of *scattered countable* linear orders.
- An order is *complete* when all its non-empty upper bounded subsets have least upper bound.
- An order is *bounded* if it has both a maximum and a minimum.

Fact 2. *Every linear order K can be completed using Dedekind cuts (see* [Ros82, Definition 2.22]*).*

Notation. If x, y are elements of an order L with $x < y$, square bracket notation will be used to denote the interval they determine. For example $[x, y] = \{z \mid x \leq z \leq y\}$ denotes the closed interval between x and y. Other similar notations such as $]x, y[$ or $[x, \rightarrow[$ are self-explaining.

The closed interval notation $[x, y]$ will be sometimes used regardless of how x and y are ordered, meaning in any case the set of all elements between them.

The following is a useful description of order preserving functions that are continuous with respect to the order topologies. Recall that a basis for the order topology consists of the open intervals, including those of the form $]\leftarrow, x[$ and $]x, \rightarrow[$.

Lemma 3. *Let $K, L \in LIN$ and let $f : K \to L$ be order preserving. Then f is continuous if and only if for every non-empty subset A of K, if the supremum (or the infimum) of A exists, then the same holds for $f(A)$ and $f(\sup A) = \sup f(A)$ (or $f(\inf A) = \inf f(A)$).*

Proof. Let f be continuous and suppose that $\sup A$ exists, in order to show that $f(\sup A) = \sup f(A)$ holds. Since f is order preserving, $f(\sup A)$ is an upper bound for $f(A)$. If there were some upper bound b of $f(A)$ with $b < f(\sup A)$, then $f^{-1}(]b, \rightarrow[)$ would contain $\sup A$ but no elements of A. Since any neighbourhood of $\sup A$ contains elements of A, this contradicts the continuity of f. Similarly for the greatest lower bound.

Conversely, assume the condition on extrema. Take $b, b' \in L$, with $b < b'$, to show that $f^{-1}(]b, b'[)$ is open. Let $a \in K$ be such that $b < f(a) < b'$ holds. It is enough to prove that if a is not the least element of K and does not have an immediate predecessor, then there is $c < a$ such that $b < f(c)$ (and similarly if a is not the last element of K and does not have an immediate successor). But then letting $A =]\leftarrow, a[$ one has $a = \sup A$, so that $f(a) = \sup f(A)$ holds, which implies the claim. A similar argument shows that the preimages of other kinds of basic open sets are open. □

We recall for convenience the definition of the backwards, the sum and the product of linear orders.

Definitions 4.

- Given $K \in LIN$ we call *backwards* or *reversal* of K and we denote $K^\star$ the order that has the same domain as K and such that $x \leq_{K^\star} y$ holds if and only if $y \leq_K x$ does.
- Let K be in LIN and for every $i \in K$ take $L_i \in LIN$. The sum $\sum_{i \in K} L_i$ is the set $\{(i, l) \mid i \in K \,\&\, l \in L_i\}$ ordered lexicographically. As a particular case we shall write finite sums as $L_1 + \cdots + L_n$.

  Notice that, since according to our definition a linear order is non-empty, whenever we consider a sum it is tacitly assumed that both its index set and all of its summands are non-empty.
- Given $K, L \in LIN$ the product $K \cdot L$ or simply KL is the set $K \times L$ ordered antilexicographically.

2.2. **On better-quasi-orders.** We recall here the definitions of well-quasi-order and of better-quasi-order.

Definitions 5.

- A *quasi-order*, or qo, is a transitive reflexive relation on some set Q. We typically write $\leq_Q$ for a qo on Q.

- An infinite sequence (q_n) of elements of Q is *bad* if for all n, m in ω such that $n < m$ we have $q_n \not\leq_Q q_m$.
- A qo $(Q, \leq_Q)$ is *well-quasi-ordered*, or wqo, if there are no bad sequences.
- We let $[\omega]^\omega$ denote the set of infinite subsets of ω with the topology induced by the topology on 2^ω under the identification of a set with its characteristic function. For $X \in [\omega]^\omega$, we let $[X]^\omega$ be the set of infinite subsets of X.
- If Q is a set, a *Q-array* is a function f with domain $[X_0]^\omega$ for some $X_0 \in [\omega]^\omega$ and values in Q such that $f^{-1}(\{y\})$ is open for all $y \in Q$.
- If $(Q, \leq_Q)$ is a qo, a Q-array f is *bad* if for all $X \in \mathrm{dom}(f)$ we have $f(X) \not\leq_Q f(X^+)$, where $X^+ = X \setminus \{\min X\}$.
- A qo $(Q, \leq_Q)$ is a *better-quasi-order* (bqo) if there are no bad Q-arrays.

The original definition of bqo is due to Nash-Williams ([NW65]). The equivalent definition we gave is due to Simpson ([Sim85]). For more concerning bqos, see [EMS87, LSR90, Mar94].

Facts 6.

(1) Every bad sequence $(q_n)_{n\in\omega}$ in Q induces a bad Q-array defined by $f(X) = q_{\min X}$, so that every bqo is indeed a wqo.

(2) A straightforward application of the Galvin-Prikry theorem shows that every finite union of bqos (and in particular any finite qo) is a bqo.

(3) The Galvin-Prikry theorem implies also that every finite product of bqos is a bqo.

Definitions 7. Given K and L in LIN, we write

- $K \leq_i L$ if there is an order preserving injection from K into L.
- $K \leq_c L$ if there is an order preserving continuous injection from K into L.

It is obvious that $K \leq_c L$ implies $K \leq_i L$.

Definitions 8 ([LSR90])**.** Let $\mathcal{C}$ be a class of structures and morphisms between them such that the identities are $\mathcal{C}$-morphisms and $\mathcal{C}$-morphisms are closed under composition.

- Given a qo Q, set

$$Q^{\mathcal{C}} = \{f \mid f \text{ is a function, } \mathrm{dom}(f) \text{ is a } \mathcal{C}\text{-structure, } \mathrm{im}(f) \subseteq Q\},$$

quasi-ordered as follows

$$f_0 \leq f_1 \Leftrightarrow \exists \mathcal{C}\text{-morphism } g : \mathrm{dom}(f_0) \to \mathrm{dom}(f_1) \text{ such that } \forall x \in \mathrm{dom}(f_0)\ f_0(x) \leq_Q f_1(g(x)).$$

- $\mathcal{C}$ *preserves bqos* if for all bqo Q the class $Q^{\mathcal{C}}$ is still bqo.

Facts 9.

(1) *If a class $\mathcal{C}$ of structures preserves bqos then $\mathcal{C}$ is a bqo under the qo induced by $\mathcal{C}$-morphisms.*
(2) *If $\mathcal{C}$ preserves bqo then so does any subclass of $\mathcal{C}$.*

Using this terminology, Laver's theorem ([Lav71]) can be stated as follows.

Theorem 10. *The class of σ-scattered linear orders under $\leq_i$ preserves bqos. In particular $(Lin, \leq_i)$ preserves bqos.*

This result was strengthened by van Engelen, Miller and Steel ([EMS87, Theorem 3.5]).

Theorem 11. *The class $(Lin, \leq_c)$ preserves bqos.*

2.3. **Epimorphisms: definition and first properties.** Our main object of interest is introduced in the next definition.

Definition 12. Let $K, L \in LIN$. We write $K \leq_s L$ if there is an order preserving surjection, also called an *epimorphism*, from L onto K. Thus a witness to the fact that $K \leq_s L$ holds is a surjective function $g : L \to K$ such that for all $a, b \in L$ we have $(a \leq_L b \Rightarrow g(a) \leq_K g(b))$.

Denote by $\equiv$ the induced equivalence relation and by $[K]$ the equivalence class of K under $\equiv$. We still use $\leq_s$ for the partial order induced on equivalence classes.

Definition 13. If $L \in LIN$ has no last element, the *cofinality* $cof(L)$ of L is the least ordinal which is the length of a sequence unbounded above in L. Given $K, L \in LIN$ a map $f : K \to L$ is *cofinal* when its range is unbounded above in L.

Similarly, when L has no least element, we define the *coinitiality* $coi(L)$ and *coinitial* maps.

Notice that $cof(L), coi(L) \leq |L|$ and hence if $L \in Lin$ has no last element then $cof(L) = \omega$, and similarly for $coi(L)$.

Facts 14. *Given $K, L \in LIN$, we have*
(1) *$K \leq_s L$ if and only if $L = \sum_{i \in K} L_i$.*
(2) *If K is finite and $|L| \geq |K|$ then $K \leq_s L$.*
(3) *If L has least (or last) element while K does not, then $K \not\leq_s L$.*
(4) *Let g witness that $K \leq_s L$ holds then:*
  (a) *g has a right inverse, that is an order preserving embedding of K into L and therefore $K \leq_s L$ implies $K \leq_i L$;*
  (b) *if K does not have least (or last) element, any such right inverse is coinitial (or cofinal) in L;*
  (c) *if L is a complete order, then so is K;*
  (d) *if K is dense, then g is continuous with respect to the order topology.*
(5) *If K, L are without maximum and $K \leq_s L$, then $cof(K) = cof(L)$. Similarly for the coinitiality of linear orders without minimum.*

Proof. For (4d) use Lemma 3.

(5) The existing epimorphism $g : L \to K$ grants $cof(K) \leq cof(L)$. The right inverse of g witnesses $cof(L) \leq cof(K)$. □

Lemma 15. *Let K be a complete order and L be any order. There is an order preserving surjection $g : L \to K$ if and only if one of the following cases holds:*

(1) K has minimum a, a maximum b and there is an order preserving injection $f : K \to L$;
(2) K has minimum a, no maximum and there is an order preserving cofinal injection $f : K \to L$;
(3) K has maximum b, no minimum and there exists an order preserving coinitial injection $f : K \to L$;
(4) K does not have minimum nor maximum and there exists an order preserving, coinitial and cofinal injection $f : K \to L$.

Proof. The necessity of the condition, in each of the four cases, is witnessed by any right inverse f of g. Conversely, for each of the four cases, an epimorphism $g : L \to K$ is built as follows:

(1), (2)

$$g(y) = \begin{cases} a & \text{if } y < f(a) \\ \sup\{x \in K \mid f(x) \leq y\} & \text{if } y \geq f(a) \end{cases}$$

(3)

$$g(y) = \begin{cases} \inf\{x \in K \mid f(x) \geq y\} & \text{if } y \leq f(b) \\ b & \text{if } y > f(b) \end{cases}$$

(4) Fix $c \in K$ and define

$$g(y) = \begin{cases} \inf\{x \in K \mid f(x) \geq y\} & \text{if } y < f(c) \\ \sup\{x \in K \mid f(x) \leq y\} & \text{if } y \geq f(c) \end{cases}$$

□

Cases 1 and 2 apply in particular when K is a well-order.

3. The structure of $\leq_s$

3.1. **Basic facts.** We start by proving the following three useful propositions.

Proposition 16. *For any $L \in Lin$:*

(1) $L \leq_s \eta$;
(2) $L \leq_s 1 + \eta$ if and only if L has a minimum;
(3) $L \leq_s \eta + 1$ if and only if L has a maximum;
(4) $L \leq_s 1 + \eta + 1$ if and only if L has minimum and maximum.

Moreover:

(5) $\inf([1+\eta],[\eta+1]) = [1+\eta+1]$*;*

(6) $\sup([1+\eta],[\eta+1]) = [\eta]$ *(even when the* sup *is taken in LIN).*

Proof. First notice that η is isomorphic to ηL for any $L \in Lin$, so (1)–(4) follow easily from Fact 14.1.

For (5) suppose $L \leq_s 1+\eta$ and $L \leq_s \eta+1$, so that L has both a first and a last element. The assertion then follows from (4).

It remains to prove (6) in LIN. If $1+\eta \leq_s L$ and $\eta+1 \leq_s L$, then L can be written both as a $(1+\eta)$-sum Σ and as an $(\eta+1)$-sum Σ'. Then the first summand in Σ can be written either as an η-sum or as an $(\eta+1)$-sum. The claim follows as $\eta+\eta = \eta+1+\eta = \eta$. □

Proposition 17. *Let L be a countable, non-scattered, linear order. Then exactly one of the following four possibilities holds:*

(1) $\eta \leq_s L$*;*

(2) $L = L_0 + \hat{L}$*, for some unique* L_0 *and* $\hat{L}$ *with* L_0 *scattered and* $\eta \leq_s \hat{L}$ *(in which case* $1+\eta \leq_s L$*);*

(3) $L = \hat{L} + L_1$*, for some unique* L_1 *and* $\hat{L}$ *with* L_1 *scattered and* $\eta \leq_s \hat{L}$ *(in which case* $\eta+1 \leq_s L$*);*

(4) $L = L_0 + \hat{L} + L_1$*, for some unique* L_0*,* L_1 *and* $\hat{L}$ *with* L_0 *and* L_1 *scattered and* $\eta \leq_s \hat{L}$*.*

In particular, $1+\eta+1 \leq_s L$ *for any countable, non-scattered* L*.*

Proof. The four cases are mutually exclusive because $\eta \not\leq_s K$ for every scattered K.

By [Ros82, Theorem 4.9] L is a sum of scattered orders on a dense index set which, since L is non-scattered, is one of η, $1+\eta$, $\eta+1$ and $1+\eta+1$. Each one of the four cases corresponds to one of the cases in the statement of the proposition. It remains to prove uniqueness in the last three cases.

Take case (2) and suppose there are L_0, L_0' scattered and $\hat{L}, \hat{L}'$ above η such that $L = L_0 + \hat{L} = L_0' + \hat{L}'$ holds. Suppose $L_0 \neq L_0'$, then as both orders are tails of L one is a suborder of the other, so for instance $L_0 \subset L_0'$. Hence $L_0' = L_0 + L_0''$ for some scattered L_0'', so $\hat{L} = L_0'' + \hat{L}'$ should hold, which is impossible since we supposed that $\eta \leq_s \hat{L}$ holds. The other cases are similar. □

In cases (2)–(4) of Proposition 17 the suborder L_0 (respectively L_1) is called the *scattered initial tail* (respectively the *scattered final tail*) of L.

Proposition 18. *The relation $\leq_s$ has four minimal elements on infinite orders having countable coinitiality or a minimum, and countable cofinality or a maximum:* $[\omega], [\omega+1], [1+\omega^\star], [\omega^\star]$*.*

Proof. The four linear orders ω, $\omega+1$, $1+\omega^\star$, $\omega^\star$ are pairwise $\leq_s$-incomparable, and as they are complete we can use Lemma 15. Given

L, if L does not have a least (or a last) element, then there is a coinitial decreasing (or a cofinal increasing) sequence in L and $\omega^\star \leq_s L$ (or $\omega \leq_s L$) by Lemma 15. Otherwise $L = \{a\} + L' + \{b\}$ and L' contains either a decreasing sequence (in which case $1+\omega^\star \leq_s L$) or an increasing sequence (and then $\omega + 1 \leq_s L$), again by Lemma 15. $\square$

Lemma 15 allows the following description of the cones above the aforementioned elements, for generic orders L:

- $\omega \leq_s L$ if and only if $cof(L) = \omega$;
- $\omega+1 \leq_s L$ if and only if there is a bounded countable increasing sequence in L;
- $1+\omega^\star \leq_s L$ if and only if there is a bounded countable decreasing sequence in L;
- $\omega^\star \leq_s L$ if and only if $coi(L) = \omega$.

3.2. **The bqo $\leq_s$ on *Lin*.** By Lemma 15.1, in the very special case of complete linear orders with first and last element any order preserving injection can be reversed into an order preserving surjection. As a consequence, $\leq_s$ is indeed bqo on the fragment of *Lin* consisting of complete orders with minimum and maximum.

We are now going to extend this to all countable linear orders using the completion of any linear order K, coloring the elements of the completion according to whether they already are in K or they represent a gap of K, and making sure that the final order is bounded.

Definition 19. Given $L \in LIN$, define the *closure* of L, denoted $\overline{L}$, as the order obtained by completing L and then possibly adding a first or a last element in case L does not have them. Let the *complete coloring of* L be the map $c_L : \overline{L} \to 3$ defined by

$$c_L(x) = \begin{cases} 2 & \text{if } x \in L; \\ 1 & \text{if } x \in \{\min \overline{L}, \max \overline{L}\} \text{ and } x \notin L; \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $\leq_{col}$ the order on $3^{(LIN, \leq_c)}$ of Definition 8, where 3 is quasi-ordered by the identity.

Notice that if $L \in Lin$ is non-scattered then $\overline{L} \notin Lin$, as it contains a copy of $\mathbb{R}$.

The next lemma shows that if the colorings on the closures of two orders are comparable with respect to $\leq_{col}$, the injection can be reversed into an order preserving surjection between the original orders. This generalizes the fact we mentioned at the beginning of this section.

Lemma 20. *Given K and L in LIN, if $c_K \leq_{col} c_L$ then $K \leq_s L$.*

Proof. Fix K and L in LIN, and suppose there exists a continuous, order preserving injective map $f : \overline{K} \to \overline{L}$ such that for all $x \in \overline{K}$ we have $c_K(x) = c_L(f(x))$. In particular, $x \in K$ if and only if $f(x) \in L$.

The map f admits a canonical dual map $g : \overline{L} \longrightarrow \overline{K}$ defined by

$$g(y) = \sup\{x \in \overline{K} \mid f(x) \leq y\}$$

(this includes the case $g(y) = \min \overline{K}$ whenever $\{x \in \overline{K} \mid f(x) \leq y\} = \emptyset$). As $g(f(x)) = x$ for every $x \in \overline{K}$, the map g is a surjective order preserving map from $\overline{L}$ onto $\overline{K}$. It is now sufficient to prove that $\mathrm{im}(g|_L) = K$ holds.

If $x \in K$ then $f(x) \in L$ and we have $g(f(x)) = x$, so that $K \subseteq \mathrm{im}(g|_L)$ holds.

Let $y \in L$ and suppose towards a contradiction that $g(y) \notin K$. There are three possible cases:

(a) there are non-empty sets $A, B \subseteq K$ such that $g(y) = \sup A = \inf B$;
(b) $g(y) = \min \overline{K}$;
(c) $g(y) = \max \overline{K}$.

(a) Notice that $f(a) \leq y$ for every $a \in A$ and hence

$$\begin{aligned} f(g(y)) &= f(\sup A) \\ &= \sup f(A) \leq y \end{aligned}$$

by Lemma 3.

On the other hand $f(b) > y$ for every $b \in B$ and hence, using again Lemma 3,

$$\begin{aligned} f(g(y)) &= f(\inf B) \\ &= \inf f(B) \geq y. \end{aligned}$$

Thus $f(g(y)) = y$ holds, against $c_K(g(y)) = 0$ and $c_L(y) = 2$.

(b) In this case we have $f(x) > y$ for every $x \in \overline{K} \setminus \{\min \overline{K}\}$. Since $\min \overline{K} = \inf(\overline{K} \setminus \{\min \overline{K}\})$ Lemma 3 implies that $f(\min \overline{K}) \geq y$. But then, since $c_K(\min \overline{K}) = 1$, we must have $f(\min \overline{K}) = \max \overline{L}$ which is impossible as $\overline{K}$ has more than one element.

(c) In this case we have $f(x) \leq y$ for every $x \in \overline{K} \setminus \{\max \overline{K}\}$ and, arguing as in (b), we obtain $f(\max \overline{K}) = \min \overline{L}$, which is also a contradiction. □

We can now prove our main result.

Theorem 21. *The qos $(Scat, \leq_s)$ and $(Lin, \leq_s)$ are bqos.*

Proof. Recall that if $L \in Lin$ then L is scattered if and only if L has countably many initial intervals ([Fra00, §6.7]). Hence if $L \in Scat$ then $\overline{L}$ is countable and complete, so that $\overline{L} \in Scat$. By Lemma 20 the map $\Phi : Scat \to 3^{Scat}, K \mapsto c_K$ satisfies $\Phi(K) \leq_{col} \Phi(L) \Rightarrow K \leq_s L$. But using Theorem 11, $(3^{(Scat, \leq_c)}, \leq_{col})$ is bqo, and finally so is $(Scat, \leq_s)$.

Now it will be shown that each of the classes of linear orders corresponding to the four cases of Proposition 17 is a bqo under $\leq_s$. The linear orders falling in case (1) constitute a unique $\equiv$-class, so they form a bqo. For the orders in case (2), assign to each such L its scattered initial tail L_0. So, for L, M in this class, one has $L_0 \leq_s M_0 \Rightarrow L \leq_s M$; since we already proved that $(Scat, \leq_s)$ is a bqo, this shows that this class is a bqo. Similarly for case (3). Finally, to each L satisfying case (4), assign the pair (L_0, L_1) of its scattered initial and final tails. So, for L, M in this class, $L_0 \leq_s M_0 \wedge L_1 \leq_s M_1 \Rightarrow L \leq_s M$; since $(Scat, \leq_s)$ is a bqo and bqos are closed under finite products, this establishes that $\leq_s$ is a bqo for the orders in case (4) too.

Since bqos are closed under finite unions, this allows to conclude that $(Lin, \leq_s)$ is a bqo. □

3.3. **Preserving bqos.** Next, one could ask if $\leq_s$ preserves bqos. Notice that to be meaningful, Definition 8 cannot be taken verbatim, since otherwise any $\leq_s$-strictly increasing sequence would provide a decreasing sequence in Q^{Lin}, for any qo Q. In any reasonable adaptation of the definition, the roles of f_0 and f_1 should be switched and the existence of a surjection $g : \mathrm{dom}(f_1) \to \mathrm{dom}(f_0)$ should be required. The first definition that comes to mind is probably the following.

Definition 22. Given a qo Q the class $Q^{(LIN, \leq_s)}$ is quasi-ordered by setting $f_0 \leq'_s f_1$ if and only if there exists an order preserving surjection $g : \mathrm{dom}(f_1) \to \mathrm{dom}(f_0)$ such that $\forall y \in \mathrm{dom}(f_1)\, f_0(g(y)) \leq_Q f_1(y)$.

However, even finite orders do not preserve bqos for this notion.

Proposition 23. *The qo* $2^{(\omega, \leq'_s)}$ *(where the elements* 0 *and* 1 *of* 2 *are incomparable) admits an infinite antichain.*

Proof. For $n > 0$ let s_n be the sequence that alternates 0's and 1's of length $2n$. Take m, n two integers with $0 < m < n$, then $s_n \not\leq'_s s_m$ since $m < n$. Fix any order preserving surjection $g : n \to m$, as $m < n$ there is an integer $i < n$ such that $g(i) = g(i+1)$, but $s_n(i) \neq s_n(i+1)$ so g cannot witness that $s_m \leq'_s s_n$. Consequently $(s_n)_{n \in \omega}$ is an infinite antichain. □

To find a better definition observe that $f_0 \leq'_s f_1$ if and only if for every $x \in \mathrm{dom}(f_0)$

$$\forall y \in \mathrm{dom}(f_1)(g(y) = x \implies f_0(x) \leq_Q f_1(y)).$$

Now notice that the displayed formula is equivalent to $\{f_0(x)\} \leq^\sharp_Q f_1(g^{-1}(x))$ where $f_1(g^{-1}(x)) = \{f_1(y) \mid y \in \mathrm{dom}(f_1) \wedge g(y) = x\}$ and $\leq^\sharp_Q$ is sometimes called the Smyth quasi-order: for $A, B \in \mathcal{P}(Q)$, $A \leq^\sharp_Q B$ if and only if $\forall b \in B\, \exists a \in A\, a \leq_Q b$.

There is another natural quasi-order on $\mathcal{P}(Q)$, which is variously known as the domination quasi-order, the Egli-Milner quasi-order, or the Hoare quasi-order: for $A, B \in \mathcal{P}(Q)$, $A \leq^\flat_Q B$ if and only if $\forall a \in A\, \exists b \in B\, a \leq_Q b$ (both $\leq^\sharp_Q$ and $\leq^\flat_Q$ have been studied from the viewpoint of wqo and bqo theory in [Mar01]). If we ask that

$\{f_0(x)\} \leq_Q^\flat f_1(g^{-1}(x))$ for every $x \in \mathrm{dom}(f_0)$ we obtain the following definition, which we will show makes *Lin* with surjections preserve bqos.

Definition 24. Given a qo Q the class $Q^{(LIN,\leq_s)}$ is quasi-ordered by setting $f_0 \leq_s^Q f_1$ if and only if there exists an order preserving surjection $g : \mathrm{dom}(f_1) \to \mathrm{dom}(f_0)$ such that

$$\forall x \in \mathrm{dom}(f_0)\, \exists y \in \mathrm{dom}(f_1)(g(y) = x \wedge f_0(x) \leq_Q f_1(y)).$$

When $f \in Q^{(LIN,\leq_s)}$ has domain L, it will be often convenient to stress this fact by denoting $f = (L, f)$.

Definition 25. Let Q be a qo and let $\overline{Q}$ be the disjoint union of Q with two mutually incomparable elements 0 and 1. For any $f = (L, f) \in Q^{(LIN,\leq_s)}$ we define the *closure* of f, denoted $\overline{f} = (\overline{L}, \overline{f})$, as the element of $\overline{Q}^{(LIN,\leq_s)}$ defined as follows. The order $\overline{L}$ is obtained by completing L and then possibly adding a first or a last element in case L does not have them. The coloring $\overline{f}$ of $\overline{L}$ is defined by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in L; \\ 1 & \text{if } x \in \{\min \overline{L}, \max \overline{L}\} \text{ and } x \notin L; \\ 0 & \text{ otherwise.} \end{cases}$$

Let us denote by $\leq_{cQ}$ the order on $\overline{Q}^{(LIN,\leq_c)}$ of Definition 8.

Lemma 26. *Given* (K, f_0) *and* (L, f_1) *in* $Q^{(LIN,\leq_s)}$*, if* $(\overline{K}, \overline{f_0}) \leq_{cQ} (\overline{L}, \overline{f_1})$ *then* $(K, f_0) \leq_s^Q (L, f_1)$.

Proof. The proof is essentially the same as the proof of Lemma 20. Given f witnessing $(\overline{K}, \overline{f_0}) \leq_{cQ} (\overline{L}, \overline{f_1})$, we define g and prove that $g|_L$ is an order preserving surjection onto K exactly as before. Since $g(f(x)) = x$ and $f_0(x) \leq_Q f_1(f(x))$ for every $x \in K$ we have $(K, f_0) \leq_s^Q (L, f_1)$. □

The theorem we obtain from Lemma 26 could be used to obtain the first part of Theorem 21 as a corollary.

Theorem 27. *The class* $(Scat, \leq_s)$ *preserves bqos, i.e. if* Q *is a qo then so is* $Q^{(Scat,\leq_s)}$ *under* $\leq_s^Q$.

Proof. Exactly as the first part of the proof of Theorem 21, using Lemma 26 in place of Lemma 20. □

Notice however that the second part of the proof of Theorem 21 (dealing with *Lin* in place of *Scat*) does not go through in this case. We do not know whether the result can be extended to *Lin*.

4. Description of $\leq_s$ on some special classes of orders

4.1. **$\leq_s$ on ordinals.** When restricted to ordinal numbers, the structure of relation $\leq_s$ admits a neat description.

Proposition 28.

(1) *Let $\alpha = \omega^{\gamma_0} n_0 + \ldots + \omega^{\gamma_k} n_k$, $\beta = \omega^{\delta_0} m_0 + \ldots + \omega^{\delta_h} m_h$ be limit ordinals (that is $\gamma_k, \delta_h > 0$) in Cantor normal form. Then*

$$\alpha \leq_s \beta \Leftrightarrow \alpha \leq \beta \wedge cof(\alpha) = cof(\beta) \wedge \gamma_k \leq \delta_h.$$

(2) *If α is a successor ordinal and β is any ordinal, then $\alpha \leq_s \beta \Leftrightarrow \alpha \leq \beta$.*

(3) *If α is a limit ordinal and β is a successor ordinal, then $\alpha \not\leq_s \beta$.*

Proof.

(1) Assume that $\alpha \leq_s \beta$ holds. Then so does $\alpha \leq \beta$ as there exists an increasing, cofinal injection $f : \alpha \to \beta$. Also we have $cof(\alpha) = cof(\beta)$ by Fact 14.5. Moreover f maps the last occurrence of ω^{γ_k} in the Cantor normal form of α cofinally into β. This implies that a final interval of this ω^{γ_k} is mapped increasingly into ω^{δ_h}. By indecomposability ω^{γ_k} embeds into ω^{δ_h}, so $\gamma_k \leq \delta_h$.

Conversely, assume that $\alpha \leq \beta$, $cof(\alpha) = cof(\beta)$ and $\gamma_k \leq \delta_h$ hold. Then we have $\alpha = \alpha' + \omega^{\gamma_k}$ and $\beta = \alpha' + \beta' + \omega^{\delta_h}$ for some ordinals α', β'. To apply Lemma 15.2 it suffices to show that there is an increasing cofinal $f : \omega^{\gamma_k} \to \omega^{\delta_h}$. Let $\rho = cof(\alpha) = cof(\beta) = cof(\omega^{\gamma_k}) = cof(\omega^{\delta_h})$ and let $\varphi : \rho \to \omega^{\gamma_k}$, $\psi : \rho \to \omega^{\delta_h}$ be cofinal and increasing, such that $\varphi(0) = 0$ holds and φ is continuous at limits. Now define inductively a new increasing and cofinal function $\psi' : \rho \to \omega^{\delta_h}$ by letting

- $\psi'(0) = 0$,
- ψ' continuous at limits, and
- $\psi'(\xi + 1) = \max(\psi(\xi + 1), \psi'(\xi) + (\varphi(\xi + 1) - \varphi(\xi)))$,

where for σ, τ ordinals with $\tau \leq \sigma$, their difference $\sigma - \tau$ is defined as the unique ordinal λ such that $\tau + \lambda = \sigma$. This is possible, since $\varphi(\xi + 1) - \varphi(\xi) < \omega^{\gamma_k} \leq \omega^{\delta_h}$. For each $\zeta < \omega^{\gamma_k}$ there exist uniquely determined $\xi < \rho$ and $\tau < \varphi(\xi + 1) - \varphi(\xi)$ such that $\zeta = \varphi(\xi) + \tau$ holds. Set $f(\zeta) = \psi'(\xi) + \tau$.

(2) By Lemma 15.1.

(3) By Fact 14.3. □

Corollary 29. *Let β be an ordinal. Then $\alpha \leq_s \beta$ for every non-null $\alpha \leq \beta$ if and only if β is countable and a finite multiple of an indecomposable ordinal.*

Proof. Let β be countable and finite multiple of an indecomposable ordinal, that is $\beta = \omega^{\delta} m$ for some $m > 0$. Then every non-null $\alpha \leq \beta$ is either a successor ordinal or it has countable cofinality and it has

Cantor normal form $\alpha = \omega^{\gamma_0} n_0 + \ldots + \omega^{\gamma_h} n_h$, with $\gamma_h \leq \delta$. Now apply Proposition 28.

On the other hand, if β is uncountable there are limit ordinals less than β with different cofinalities, so there cannot be an epimorphism of β onto each of them. Finally, if β is not finite multiple of an indecomposable ordinal, then it has Cantor normal form $\beta = \omega^{\delta_0} m_0 + \ldots + \omega^{\delta_h} m_h$ with $h \geq 1$ and, by Proposition 28 there cannot be an epimorphism from β onto $\omega^{\delta_0} m_0$. □

4.2. **Exploiting completeness.** Some of the ideas used in previous sections can be employed to find an explicit description of $\leq_s$ on some other classes of linear orders.

Definition 30. According to [Ros82], if $L \in \mathit{Lin}$ and $x \in L$ let $c(x) = \{y \in L \mid [x, y] \text{ is finite}\}$ be the *condensation* of x. Let also $L^1 = \{c(x)\}_{x \in L}$ with the natural order. This is the *condensation* of L.

We first consider the class of complete bounded σ-scattered linear orders. Given such an L, define a coloured linear order (L', φ_L) on the set of colours $\{1, 2, 3, \ldots, \leftarrow, \rightarrow\}$. These colours are ordered by $\sqsubseteq$ which is the usual order relation on $\{1, 2, 3, \ldots\}$, is such that $n \sqsubseteq \leftarrow$ and $n \sqsubseteq \rightarrow$, while $\leftarrow$ and $\rightarrow$ are incomparable.

The order L' is obtained from L^1 by replacing each condensation class of order type ζ with two consecutive elements, the members of a pair of intervals of order types $\omega^\star$ and ω, respectively, of which the class is the union.

We now need to define φ_L. Given $x \in L'$, there are various possibilities:

- if $x \in L^1$ is a finite condensation class, then $\varphi_L(x) = |x|$;
- if $x \in L^1$ is a condensation class of order type $\omega^\star$ or ω, then $\varphi_L(x)$ is $\leftarrow$ or $\rightarrow$, respectively;
- If x is one of the two intervals replacing a condensation class $x' \in L^1$ of order type ζ, let $\varphi_L(x)$ be $\leftarrow$ or $\rightarrow$ according to whether it is the first or the second of the two elements in the order L'.

Proposition 31. *Given complete bounded σ-scattered linear orders K and L, if there is an order preserving injection $g : L' \to K'$ with $\forall x \in L'\ \varphi_L(x) \sqsubseteq \varphi_K(g(x))$ then $L \leq_s K$.*

Proof. Under the given hypotheses, we are going to define an epimorphism $h : K \to L$.

Given $y \in K'$, if y is in the range of g, say $g(x) = y$, define h on y as any order preserving surjection onto x. Otherwise, there are three cases. If y is less than every element in the range of g, define h on y as the constant function with value the least element of L. Similarly, if y majorizes the range of g, let h on y be constant of value the maximum of L. Finally, suppose g takes values both smaller and bigger than y. Then h on y will be constant with value $\sup \bigcup \{x \in L' \mid g(x) < y\}$. $\square$

By Theorem 10 we obtain the following.

Corollary 32. *When restricted to complete, bounded, σ-scattered orders, the relation $\leq_s$ is bqo.*

Theorem 33. *When restricted to complete scattered linear orders with fixed coinitiality and cofinality, the relation $\leq_s$ is bqo.*

Proof. Fix regular cardinals α, β. It will be shown that each of the following classes forms a bqo.

(1) Complete scattered linear orders with minimum and cofinality β.
(2) Complete scattered linear orders with maximum and coinitiality α.
(3) Complete scattered linear orders with coinitiality α and cofinality β.

First remark that if L is a scattered ordering, then given any two points $x_1 < x_2$ in L, there are consecutive $y_1, y_2 \in L$ with $x_1 \leq y_1 < y_2 \leq x_2$.

(1) Let L be complete and scattered, with minimum and cofinality β. There exists an increasing cofinal sequence $\{\ell_\xi\}_{\xi<\beta}$ in L with $\ell_0 = \min L$ and such that if ξ a successor ordinal the element ℓ_ξ has an immediate predecessor in L, while if ξ is a limit then $\ell_\xi = \sup\{\ell_\rho\}_{\rho<\xi}$. Indeed, fix any cofinal increasing sequence $\{\ell'_\xi\}_{\xi<\beta}$; by possibly modifying it, it can be assumed $\ell'_0 = \min L$ and that, for ξ a limit, $\ell'_\xi = \sup\{\ell'_\rho\}_{\rho<\xi}$. Let $\ell_0 = \ell'_0$. Suppose $\{\ell_\xi\}_{\xi<\gamma}$ has been defined with the required properties and in such a way that $\forall \xi < \gamma\ \ell'_\xi \leq \ell_\xi$, in order to define ℓ_γ. Remark that $\{\ell_\xi\}_{\xi<\gamma}$ is bounded in L. If γ is limit, let $\ell_\gamma = \sup\{\ell_\xi\}_{\xi<\gamma}$; in particular, $\ell'_\gamma \leq \ell_\gamma$. For $\gamma = \mu+1$ successor, let $\delta < \beta$ be least such that $\ell'_\delta > \ell_\mu$. Find consecutive points $y_1, y_2 \in L$ with $\ell'_\delta \leq y_1 < y_2 \leq \ell'_{\delta+1}$ and let $\ell_\gamma = y_2$.

So L is a β-sum of complete orders L_ξ with least and last element: L_ξ has end points ℓ_ξ and the immediate predecessor of $\ell_{\xi+1}$. Let φ_L be the colouring of β which maps each $\xi < \beta$ to L_ξ.

Let L and M be complete and scattered, with minimum and cofinality β. We claim that if there is an embedding f of β into itself such that for all $\xi < \beta$ we have $L_\xi \leq_s M_{f(\xi)}$ then $L \leq_s M$. Indeed, fix epimorphisms $g_\xi : M_{f(\xi)} \to L_\xi$, and define $g : M \to L$ as follows. If $\gamma = f(\xi)$ for some $\xi < \beta$ let $g|_{M_\gamma} = g_\xi$, while if $\gamma < \beta$ is not in the range of f there is a least $\delta < \beta$ such that $\gamma < f(\delta)$: let $g|_{M_\gamma}$ be constant with value $\min L_\delta$.

Using Corollary 32 and Theorem 10 we obtain the conclusion.

(2) and (3) are proved similarly. $\square$

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