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Minimal pseudocompact group topologies on free abelian groups

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Dedicated to 60th birthday of Bob Lowen

Abstract

Let κ be a cardinal and let F_κ denote the free abelian group with κ many generators. If F_κ admits a pseudocompact group topology, then $\kappa \geq \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum. We show that the existence of a minimal pseudocompact group topology on $F_\mathfrak{c}$ is equivalent to the Lusin's hypothesis $2^{\omega_1} = \mathfrak{c}$. For $\kappa > \mathfrak{c}$, we prove that F_κ admits a minimal pseudocompact group topology if and only if F_κ has both a minimal group topology and a pseudocompact group topology. If G is an infinite minimal abelian group, then either $|G| = 2^\sigma$ for some cardinal σ, or $w(G) = \min\{\sigma : |G| \leq 2^\sigma\}$, where $w(G)$ is the weight of G. Moreover, we show that the equality $|G| = 2^{w(G)}$ holds true whenever $\mathrm{cf}(w(G)) > \omega$.

Throughout this paper all topological groups are Hausdorff. We denote by $\mathbb{Z}$, $\mathbb{P}$ and $\mathbb{N}$ respectively the set of integers, the set of primes and the set of natural numbers. Moreover $\mathbb{Q}$ denotes the set of rationals and $\mathbb{R}$ the set of reals. For $p \in \mathbb{P}$ the symbol $\mathbb{Z}_p$ is used for the group of p-adic integers and $\mathbb{Z}(p^\infty)$ denotes Prüfer's group. For a cardinal κ we use F_κ to denote the free abelian group with κ many generators. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum. For a topological group G the symbol $w(G)$ stands for the weight of G, $\widetilde{G}$ denotes the completion of G, the Pontryagin dual of a topological abelian group G is denoted by $\widehat{G}$. For undefined terms see [16, 17].

1 Introduction

The following notion was introduced independently by Choquet (see Doïtchinov [14]) and Stephenson [24].

Definition 1.1. A Hausdorff group topology τ on a group G is called *minimal* provided that every Hausdorff group topology τ' on G such that $\tau' \subseteq \tau$ satisfies $\tau' = \tau$. Equivalently, a Hausdorff topological group G is minimal if every continuous isomorphism $f : G \to H$ between G and a Hausdorff topological group H is a topological isomorphism.

There exist abelian groups which admit no minimal group topologies at all, e.g., $\mathbb{Q}$ [22] or $\mathbb{Z}(p^\infty)$ [11]. This suggests the general problem to determine the algebraic structure of the minimal abelian groups, or equivalently, the following:

Problem 1.2. *Describe the abelian groups that admit minimal group topologies.*

Prodanov solved Problem 1.2 first for all free abelian groups of finite rank [21] and later on he improved this result, extending it to all cardinals $\leq \mathfrak{c}$ [22]:

Theorem 1.3. *(a)* [21] *For every $n \in \mathbb{N}$, F_n admits minimal group topologies.*

(b) [22] *For a cardinal $\kappa \leq \mathfrak{c}$, F_κ admits minimal group topologies.*

Since $|F_\kappa| = \kappa$ for uncountable free abelian groups, these groups are determined up to isomorphism by their cardinality. This imposes the problem of characterizing the cardinality of minimal abelian groups. The following set-theoretic definition is ultimately relevant to this problem.

Definition 1.4. (i) For infinite cardinals κ and σ we will use the notation $\mathbf{Min}(\kappa, \sigma)$ to denote the following statement: there exists a sequence of cardinals $\{\sigma_n : n \in \mathbb{N}\}$ such that

$$\sigma = \sup_{n\in\mathbb{N}} \sigma_n \text{ and} \sup_{n\in\mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^{\sigma}. \tag{1}$$

We will also say that the sequence $\{\sigma_n : n \in \mathbb{N}\}$ as above *witnesses* $\mathbf{Min}(\kappa, \sigma)$.

(ii) An infinite cardinal number κ satisfying $\mathbf{Min}(\kappa, \sigma)$ for some infinite cardinal σ will be called a *Stoyanov* cardinal.

(iii) For the sake of convenience, we add to the class of Stoyanov cardinals also all finite cardinals.

Cardinals from items (ii) in the above definition were first introduced by Stoyanov in [25] under the name "permissible cardinals". Their importance is evident from the following fundamental result of Stoyanov [25] providing a complete characterization of the possible cardinalities of minimal abelian groups and in this way solving Problem 1.2 for all free abelian groups.

Theorem 1.5. [25]

(a) If G is a minimal abelian group, then $|G|$ is a Stoyanov cardinal.

(b) For a cardinal κ, F_κ admits minimal group topologies if and only if κ is a Stoyanov cardinal.

The non-abelian case has a completely different flavor compared to item (b) of the above theorem:

Theorem 1.6. [23] *Every free group admits a minimal group topology.*

A topological group G is *pseudocompact* if every continuous real-valued function of G is bounded [18]. In the line of Theorem 1.5 characterizing the free abelian groups admitting *minimal* topologies, one can characterize the free abelian groups that admit *pseudocompact* topologies ([5, 13], see Theorem 4.5). The aim of this article is the *simultaneous* minimal and pseudocompact topologization of free abelian groups. To achieve this goal, we need a very careful alternative description of Stoyanov cardinals (Proposition 3.5) as well as a more precise form of Theorem 1.5 (see Theoem 2.1).

The following two facts will be frequently used in the sequel. The first one concerns a restriction on the size of pseudocompact groups due to van Douwen.

Theorem 1.7. [26] *If G is an infinite pseudocompact group, then $|G| \geq \mathfrak{c}$.*

The second one is the "minimality criterion", due to Prodanov and Stephenson [21, 24], describing the dense minimal subgroups of compact groups. A subgroup H of a topological group G is *essential* if H non-trivially intersects every non-trivial closed normal subgroup of G [21, 24].

Theorem 1.8. [10, 12, 21, 24] *A dense subgroup H of a compact group G is minimal if and only if H is essential in G.*

2 Main results

2.1 Cardinality and weight of minimal abelian groups

We start with a sharper version of Theorem 1.5, showing that the fact that the cardinality of an infinite minimal abelian group is a Stoyanov cardinal is witnessed by the weight of the group:

Theorem 2.1. *If G is an infinite minimal abelian group, then $\mathbf{Min}(|G|, w(G))$ holds.*

This theorem, along with the complete "internal" characterization of the Stoyanov cardinals obtained in §3 (see Proposition 3.5), permits us to establish some new important relations between the cardinality and the weight of an arbitrary minimal abelian group.

Theorem 2.2. *If κ is a cardinal with* $\mathrm{cf}(\kappa) > \omega$ *and G is a minimal abelian group with $w(G) \geq \kappa$. Then $|G| \geq 2^{\kappa}$.*

Let us recall that $|G| = 2^{w(G)}$ hold for every compact group G [3]. Taking $\kappa = w(G)$ in Theorem 2.2 we extend the following extension of this property to all minimal abelian groups.

Corollary 2.3. *Let G be a minimal abelian group with* $\mathrm{cf}(w(G)) > \omega$*. Then $|G| = 2^{w(G)}$.*

Easy examples show that neither $\mathrm{cf}(w(G)) > \omega$ nor "abelian" can be removed in Corollary 2.3.

With $\kappa = \omega$ in Theorem 2.2 one obtains the following surprising metrizability criterion for "small" minimal abelian groups.

Corollary 2.4. *A minimal abelian group of size $< 2^{\omega_1}$ is metrizable.*

The condition $\mathrm{cf}(w(G)) > \omega$ plays a prominent role in the above results. In particular, Theorem 2.2 implies that $\mathrm{cf}(w(G)) = \omega$ for a minimal abelian group with $|G| < 2^{w(G)}$. Our next theorem gives a more precise information in this direction.

We say that a cardinal τ is *exponential* if $\tau = 2^{\kappa}$ for some cardinal κ, and we call τ *non-exponential* otherwise. For a cardinal κ, $\log \kappa = \min\{\lambda : 2^{\lambda} \geq \kappa\}$.

Theorem 2.5. *Let G be an infinite minimal abelian group such that $|G|$ is a non-exponential cardinal. Then $w(G) = \log |G|$ and* $\mathrm{cf}(w(G)) = \omega$.

Under the assumption of GCH, the equality $w(G) = \log |G|$ holds true for every compact group. The above theorems establishes this property in ZFC for all minimal abelian groups of non-exponential size. Let us note that the restraint "non-exponential" cannot be omitted, even in the compact case. Indeed, the equality $w(G) = \log |G|$ may fail for compact abelian groups: under Lusin's Hypothesis $2^{\omega_1} = \mathfrak{c}$, the group $G = \mathbb{Z}(2)^{\omega_1}$ has weight $\omega_1 \neq \log |G| = \log \mathfrak{c} = \omega$.[1]

2.2 Minimal pseudocompact group topologies on free abelian groups

Since pseudocompact metric spaces are compact, we immediately get the following from Corollary 2.4:

Corollary 2.6. *Let G be a abelian group such that $|G| < 2^{\omega_1}$. Then G admits a minimal pseudocompact group topology if and only if G admits a compact metric group topology.*

By Theorem 1.7 this corollary is vacuously true under Lusin's Hypothesis $2^{\omega_1} = \mathfrak{c}$. It shows that for abelian groups of "small size" minimal and pseudocompact topologizations are connected in some sense by compactness.

The next theorem discovers the surprising possibility of "simultaneous topologization" with a topology which is both minimal and pseudocompact for a free group that admits both a minimal group topology and a pseudocompact group topology. Moreover, it turns out that this topology can be chosen to be also zero-dimensional.

Theorem 2.7. *For every cardinal $\kappa > \mathfrak{c}$ the following conditions are equivalent:*

(a) F_{κ} admits both a minimal group topology and a pseudocompact group topology;

(b) F_{κ} admits a minimal pseudocompact group topology;

(c) F_{κ} admits a zero-dimensional minimal pseudocompact group topology.

Our next theorem shows that ZFC cannot decide whether the free abelian group $F_{\mathfrak{c}}$ of cardinality $\mathfrak{c}$ admits a minimal pseudocompact group topology (note that in ZFC $F_{\mathfrak{c}}$ admits a minimal group topology (Theorem 1.3) and a pseudocompact group topology [13]).

Theorem 2.8. *The following conditions are equivalent:*

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[1] Using the cardinals κ and σ from Ex. 3.4 on can find also a consistent example of a compact abelian group G (namely $G = \mathbb{Z}(2)^{\sigma}$), such that $\mathrm{cf}(w(G)) = \omega$ and still $\log |G| = \log \kappa < w(G) = \sigma$. – this will be written more decently.

(a) $F_{\mathfrak{c}}$ *admits a minimal pseudocompact group topology;*

(b) $F_{\mathfrak{c}}$ *admits a connected minimal pseudocompact group topology;*

(c) $F_{\mathfrak{c}}$ *admits a zero-dimensional minimal pseudocompact group topology;*

(d) the Lusin's Hypothesis $2^{\omega_1} = \mathfrak{c}$ *holds.*

Since every infinite pseudocompact group has cardinality $\geq \mathfrak{c}$ (Theorem 1.7), these two theorems provide a complete description of free abelian groups that have a minimal (zero-dimensional) pseudocompact group topology. The equivalence of (a) and (b) in Theorem 2.7 (resp., (a) and (d) in Theorem 2.8) was announced without proof in [9, Theorem 4.11].

Motivated by Theorem 2.7(c) and Theorem 2.8(c), where the minimal pseudocompact topology can be additionally zero-dimensional (or connected, in Theorem 2.8(b)), we conclude with the following question.

Question 2.9. *If* κ *is a cardinal, the free abelian group* F_κ *admits a minimal group topology* τ_1*,* F_κ *admits a pseudocompact group topology* τ_2 *and one of these topologies is connected, does the group* F_κ *admit a connected minimal and pseudocompact group topology?*

Theorem 2.8 answers Question 2.9 in the case of $F_{\mathfrak{c}}$. The next theorem gives an answer for $\kappa > \mathfrak{c}$, showing a symmmetric behavior, as far as connectedness is concerned (this should be compared with the equivalent items in Theorem 2.8 where item (a) contains no restriction beyond minimality and pseudocompactness, whereas item (c) contains "zero-dimensional").

Theorem 2.10. *Let* κ *and* σ *be infinite cardinals with* $\kappa > \mathfrak{c}$*. The following conditions are equivalent:*

(a) F_κ *admits a connected minimal pseudocompact group topology (of weight* σ*);*

(b) F_κ *admits a connected minimal group topology (of weight* σ*);*

(c) κ *is exponential (*$\kappa = 2^\sigma$*).*

The paper is organized as follows. In Section 3 we give some properties of Stoyanov cardinals, while Section 4 contains all necessary facts concerning pseudocompact topologization. Section 5 prepares the remaining necessary tools for the proof of the main results, deferred to Section 6. Finally, in Section 7 we discuss the counterpart of the simultaneous minimal and pseudocompact topologization for other classes of abelian groups as divisible groups, torsion-free groups and torsion groups.

3 Properties of Stoyanov cardinals

We start with an example of small Stoyanov cardinals.

Example 3.1. If $\omega \leq \kappa \leq \mathfrak{c}$, then $\mathbf{Min}(\kappa, \omega)$.

In our next example we discuss the connection between $\mathbf{Min}(\kappa, \sigma)$ and the property of κ to be exponential.

Example 3.2. Let κ be an infinite cardinal.

(a) If κ is exponential, then κ is Stoyanov. More precisely, $\mathbf{Min}(\kappa, \sigma)$ holds true for every cardinal σ with $\kappa = 2^\sigma$.

(b) If σ is a cardinal number such that $\sigma = \sup_{n\in\mathbb{N}} \sigma_n$, for some cardinals σ_n and $\sigma = \sigma_n$ for some $n \in \mathbb{N}$, then $\mathbf{Min}(\kappa, \sigma)$ if and only if $\kappa = 2^\sigma$. Indeed, $\mathbf{Min}(\kappa, \sigma)$ yields $2^\sigma \geq \kappa \geq \sup_{n\in\mathbb{N}} 2^{\sigma_n} = 2^{\sigma_n} = 2^\sigma$ and so $\kappa = 2^\sigma$.

(c) *If* $\mathrm{cf}(\sigma) > \omega$*, then* $\mathbf{Min}(\kappa, \sigma)$ *if and only if* $\kappa = 2^\sigma$. If $\kappa = 2^\sigma$, then $\mathbf{Min}(\kappa, \sigma)$ by item (a). Assume $\mathbf{Min}(\kappa, \sigma)$. Then there exists a sequence of cardinals $\{\sigma_n : n \in \mathbb{N}\}$ satisfying (1), that is, such that $\sigma = \sup_{n\in\mathbb{N}} \sigma_n$ and $\sup_{n\in\mathbb{N}} 2^{\sigma_n} \leq \kappa \leq 2^\sigma$. By $\mathrm{cf}(\sigma) > \omega$ there exists $n \in \mathbb{N}$ with $\sigma = \sigma_n$. By item (b) this implies that $\kappa = 2^\sigma$.

Clearly, $\mathbf{Min}(\kappa,\sigma)$ implies $\sigma \geq \log\kappa$. We show now that this inequality becomes an equality in case κ is non-exponential.

Lemma 3.3. *Let κ be a non-exponential infinite cardinal.*

(a) $\mathbf{Min}(\kappa,\sigma)$ *if and only if* $\mathrm{cf}(\sigma)=\omega$ *and* $\log\kappa=\sigma$.

(b) $\mathbf{Min}(\kappa,\log\kappa)$ *if and only if* $\mathrm{cf}(\log\kappa)=\omega$.

Proof. (a) Assume that $\mathbf{Min}(\kappa,\sigma)$ holds. Then there exists a sequence of cardinals $\{\sigma_n : n\in\mathbb{N}\}$ satisfying (1). By Example 3.2(c) and our hypothesis, $\mathrm{cf}(\sigma)=\omega$. As mentioned above, $\mathbf{Min}(\kappa,\sigma)$ implies $\sigma\geq\log\kappa$. Assume $\sigma>\log\kappa$. Then $\sigma_n\geq\log\kappa$ for some $n\in\mathbb{N}$. Therefore $2^{\log\kappa}\leq 2^{\sigma_n}\leq\sup_{n\in\mathbb{N}}2^{\sigma_n}\leq\kappa$. Since κ is non-exponential, $2^{\log\kappa}<\kappa$, a contradiction. This proves that $\sigma=\log\kappa$.

Now assume that $\mathrm{cf}(\sigma)=\omega$ and $\log\kappa=\sigma$. Then $\kappa\leq 2^{\sigma}$, and there exists a sequence of cardinals $\{\sigma_n : n\in\mathbb{N}\}$ such that $\sigma=\sup_{n\in\mathbb{N}}\sigma_n$ and $\sigma_n<\sigma=\log\kappa$ for every $n\in\mathbb{N}$. Therefore $2^{\sigma_n}<\kappa$ for every $n\in\mathbb{N}$. Consequently $\sup_{n\in\mathbb{N}}2^{\sigma_n}\leq\kappa$ and hence $\sup_{n\in\mathbb{N}}2^{\sigma_n}\leq\kappa\leq 2^{\sigma}$, that is, $\mathbf{Min}(\kappa,\sigma)$ holds.

(b) Follows form item (a). □

Example 3.4. Let κ and σ be cardinals. According to Example 3.2(a), $\mathbf{Min}(\kappa,\sigma)$ does not imply $\mathrm{cf}(\sigma)=\omega$ in case κ is exponential (it suffices to take $\kappa=2^{\sigma}$ with $\mathrm{cf}(\sigma)>\omega$).

Let us show that the condition "κ non-exponential" of Lemma 3.3(a) is necessary (to prove that $\mathbf{Min}(\kappa,\sigma)$ implies $\log\kappa=\sigma$) even in the case $\mathrm{cf}(\sigma)=\omega$. To this end, use an appropriate Easton model [15] satisfying

$$2^{\omega_n}=\omega_{\omega+2} \text{ for all } n\in\mathbb{N} \text{ and } 2^{\omega_{\omega+1}}=\omega_{\omega+2}.$$

Let $\kappa=\omega_{\omega+2}$ and $\sigma=\omega_\omega$. Then $2^{\sigma}=\kappa$ (as $2^{\omega_{\omega+1}}=2^{\omega_n}=\kappa$ for every $n\in\mathbb{N}$). So $\mathbf{Min}(\kappa,\sigma)$ holds by Example 3.2(a). Moreover $\mathrm{cf}(\sigma)=\omega$ and $\log\kappa<\sigma$.

The next proposition describes the Stoyanov cardinals

Proposition 3.5. *If κ is a non-exponential infinite cardinal satisfying* $\mathbf{Min}(\kappa,\sigma)$ *for some cardinal σ, then* $\sigma=\log\kappa$ *and* $\mathrm{cf}(\log\kappa)=\omega$.

Proof. Since $\mathbf{Min}(\kappa,\sigma)$ holds, $\sigma=\log\kappa$ and $\mathrm{cf}(\sigma)=\omega$ by Lemma 3.3(a). □

4 Cardinal invariants related to pseudocompact groups

The following theorem describes pseudocompact groups in terms of their completion.

Theorem 4.1. [7, Theorem 4.1] *A precompact group G is pseudocompact if and only if G is G_δ-dense in $\widetilde{G}$.*

If X is a non-empty set and σ is an infinite cardinal, then a set $F\subseteq X^{\sigma}$ is *ω-dense* in X^{σ}, provided that for every countable set $A\subseteq\sigma$ and each function $\varphi\in X^{A}$ there exists $f\in F$ such that $f(\alpha)=\varphi(\alpha)$ for all $\alpha\in A$ [2] (see also [13, Definition 2.6]).

Definition 4.2. [2] (see also [13, Definition 2.6]) If κ and $\sigma\geq\omega$ are cardinals, then $\mathbf{Ps}(\kappa,\sigma)$ abbreviates the sentence "there exists an ω-dense set $F\subseteq\{0,1\}^{\sigma}$ with $|F|=\kappa$".

For a given infinite cardinal κ, the set

$$A_\kappa=\{\sigma \text{ infinite cardinal} : \mathbf{Ps}(\kappa,\sigma) \text{ holds}\}$$

is not empty because $2^{\kappa}\in A_\kappa$. Then, for the properties of cardinal numbers, A_κ admits a minimal element. So we can give the following definition of a cardinal function strictly related to $\mathbf{Ps}(-,-)$.

Definition 4.3. [2] Let σ be an infinite cardinal. Then $\delta(\sigma)$ is the minimal cardinal κ such that $\mathbf{Ps}(\kappa,\sigma)$ holds.

Then $\delta(\sigma)$ is the minimal cardinality of an ω-dense subset of $\{0,1\}^\sigma$. The set-theoretical condition introduced in Definition 4.2 and $\delta(\sigma)$ are closely related to the pseudocompact group topologies. It was shown in [4] that $\delta(\sigma)$ coincides with the cardinal function $m(\sigma)$ defined as follows: if K is a compact group of weight σ, then $m(\sigma)$ is the minimum cardinality of a dense pseudocompact subgroup of K. In the sequel we shall use the notation $m(-)$. If K is a compact group of weight σ, then $m(\sigma)$ depends only on σ [4]; in other words:

Theorem 4.4. [4] (see also [13, Fact 2.12 and Theorem 3.3(i)]) *Let κ and $\sigma \geq \omega$ be cardinals. Then $\mathbf{Ps}(\kappa,\sigma)$ holds if and only if there exists a group G of cardinality κ which admits a pseudocompact group topology of weight σ.*

Theorem 4.5. [13, Theorem 5.10] *If κ is a cardinal, then F_κ admits pseudocompact group topologies if and only if $\mathbf{Ps}(\kappa,\sigma)$ holds for some cardinal σ.*

In the next lemma we give some properties of the cardinal function $m(-)$.

Lemma 4.6. [2] (see also [4, Theorem 2.7]) *Let σ be an infinite cardinal. Then:*

(a) $m(\sigma) \geq 2^\omega$ and $\mathrm{cf}(m(\sigma)) > \omega$;

(b) $\log\sigma \leq m(\sigma) \leq (\log\sigma)^\omega$;

(c) $m(\lambda) \leq m(\sigma)$ whenever λ is another cardinal with $\lambda \leq \sigma$.

Some useful properties of the condition $\mathbf{Ps}(\lambda,\kappa)$ are collected in the next proposition; (a) and (b) are part of [13, Lemmas 2.7 and 2.8] and (d) and (e) are particular cases of [13, Lemma 3.4(i)].

Proposition 4.7. *(a) $\mathbf{Ps}(\mathfrak{c},\omega)$ holds and moreover $m(\omega)=\mathfrak{c}$; also $\mathbf{Ps}(\mathfrak{c},\omega_1)$ holds.*

(b) If $\mathbf{Ps}(\kappa,\sigma)$ holds for some cardinals κ, $\sigma \geq \omega$, then $\kappa \geq \mathfrak{c}$ and $\mathbf{Ps}(\kappa',\sigma)$ holds for every cardinal κ' such that $\kappa \leq \kappa' \leq 2^\sigma$.

(c) For cardinals κ, $\sigma \geq \omega$, $\mathrm{Ps}(\kappa,\sigma)$ holds if and only if $m(\sigma) \leq \kappa \leq 2^\sigma$.

(d) $\mathbf{Ps}(2^\sigma,\sigma)$ and $\mathbf{Ps}\left(2^\sigma,2^{2^\sigma}\right)$ hold for every infinite cardinal σ.

(e) If σ is a cardinal such that $\sigma^\omega=\sigma$, then $\mathbf{Ps}(\sigma,2^\sigma)$ holds.

In the next lemma we show that if κ is a Stoyanov cardinal such that $\mathbf{Ps}(\kappa,-)$ holds, then for the cardinal σ that witnesses that κ is Stoyanov $\mathbf{Ps}(\kappa,\sigma)$ holds as well.

Lemma 4.8. *Let κ and σ be infinite cardinals satisfying $\mathbf{Min}(\kappa,\sigma)$. If $\mathbf{Ps}(\kappa,\lambda)$ holds for some infinite cardinal λ, then $\mathbf{Ps}(\kappa,\sigma)$ holds as well.*

Proof. Let $\{\sigma_n : n \in \mathbb{N}\}$ be a sequence of cardinals witnessing $\mathbf{Min}(\kappa,\sigma)$. If $\sigma=\sigma_n$ for some $n \in \mathbb{N}$, then $\sup_{n\in\mathbb{N}} 2^{\sigma_n} = 2^{\sigma_n} = 2^\sigma$ and so $\kappa = 2^\sigma$. Moreover $\mathbf{Ps}(2^\sigma,\sigma)$ holds true by Proposition 4.7(d).

Suppose that $\sigma > \sigma_n$ for every $n \in \mathbb{N}$. Then $\mathrm{cf}(\sigma)=\omega$ and $\sup_{n\in\mathbb{N}} 2^{\sigma_n} = 2^{<\sigma}$; consequently

$$2^{<\sigma} \leq \kappa \leq 2^\sigma.$$

Assume that σ is a strong limit cardinal. By hypothesis $\mathbf{Ps}(\kappa,\lambda)$ holds true; equivalently, $m(\lambda) \leq \kappa \leq 2^\lambda$ by Proposition 4.7(c). If $\lambda < \sigma$, then $2^\lambda < \sigma$ and so $\kappa < \sigma$, which is not possible. Hence $\sigma \leq \lambda$. By Lemma 4.6(c) $m(\sigma) \leq m(\lambda) \leq \kappa$. Moreover $\kappa \leq 2^\sigma$. By Proposition 4.7(c) $\mathbf{Ps}(\kappa,\sigma)$ holds.

Assume that σ is not a strong limit cardinal. Then there exists $n \in \mathbb{N}$ such that $2^{\sigma_n} \geq \sigma$. Then $\sigma_n \geq \log\sigma$ and by Lemma 4.6(b)

$$m(\sigma) \leq (\log\sigma)^\omega \leq \sigma_n^\omega \leq 2^{\sigma_n} \leq 2^{<\sigma}.$$

Hence $m(\sigma) \leq \kappa \leq 2^\sigma$ and so $\mathbf{Ps}(\kappa,\sigma)$ holds by Proposition 4.7(c). □

Corollary 4.9. *Let κ be a cardinal $\geq \mathfrak{c}$. If F_κ admits a minimal group topology of weight σ, that is not a strong limit cardinal, then ($\mathbf{Min}(\kappa,\sigma)$ holds and) $\mathbf{Ps}(\kappa,\sigma)$ holds true.*

5 Technical lemmas

A *variety of groups* $\mathcal{V}$ is a class of abstract groups closed under subgroups, quotients and products. For a variety $\mathcal{V}$ and $G \in \mathcal{V}$ a subset X of G is *$\mathcal{V}$-independent* if $\langle X \rangle \in \mathcal{V}$ and for each map $f : X \to H \in \mathcal{V}$ there exists a unique homomorphism $\overline{f} : \langle X \rangle \to H$ extending f. Moreover, the *$\mathcal{V}$-rank of G* is

$$r_{\mathcal{V}}(G) := \sup\{|X| : X \text{ is a } \mathcal{V}\text{-independent subset of } G\}.$$

In particular, if $\mathcal{A}$ is the variety of all abelian groups the $\mathcal{A}$-rank is the usual free rank $r(-)$ and for $\mathcal{A}_p$ the variety of all abelian p-groups for a prime p the $\mathcal{A}_p$-rank is the usual p-rank $r_p(-)$.

Our first lemma is a generalization of [13, Lemma 4.1] that is in fact equivalent to [13, Lemma 4.1] (as can be seen from its proof below).

Lemma 5.1. *Let $\mathcal{V}$ be a variety of groups and I an infinite set. For every $i \in I$ let H_i be a group such that $r_{\mathcal{V}}(H_i) \geq \omega$. Then $r_{\mathcal{V}}\left(\prod_{i\in I} H_i\right) \geq 2^{|I|}$.*

Proof. Let H be the free group in the variety $\mathcal{V}$ with countably many generators. For every $i \in I$ the assumption of our lemma allows us to fix a monomorphism $f_i : H \to H_i$. Then the map $f : H^I \to \prod_{i\in I} H_i$ defined by $f(h) = \{f_i(h(i))\}_{i\in I}$ for $h \in H^I$, is a monomorphism. Therefore, $r_{\mathcal{V}}\left(\prod_{i\in I} H_i\right) \geq r_{\mathcal{V}}(f(H^I)) = r_{\mathcal{V}}(H^I) \geq 2^{|I|}$, where the last inequality has been proved in [13, Lemma 4.1]. $\square$

Lemma 5.2. *Suppose that I is an infinite set and H_i is a separable metric space for every $i \in I$. If* $\mathbf{Ps}(\kappa, |I|)$ *holds, then $\prod_{i\in I} H_i$ contains a G_δ-dense subset of size at most κ.*

Proof. Let $i \in I$. Since H_i is a separable metric space, $|H_i| \leq \mathfrak{c}$, and so we can fix a surjection $f_i : \mathbb{R} \to H_i$.

Let $\theta : \mathbb{R}^I \to \prod_{i\in I} H_i$ be the map defined by $\theta(g) = \{f_i(g(i))\}_{i\in I} \in \prod_{i\in I} H_i$ for every $g \in \mathbb{R}^I$. Since $\mathbf{Ps}(\kappa, |I|)$ holds, [13, Lemma 2.9] allows us to conclude that $\mathbb{R}^I$ contains an ω-dense subset X of size κ. Define $Y = \theta(X)$. Then $|Y| \leq |X| = \kappa$. It remains only to show that Y is G_δ-dense in $\prod_{i\in I} H_i$. Indeed, let E be a non-empty G_δ-subset of $\prod_{i\in I} H_i$. Then there exist a countable subset J of I and $h \in \prod_{j\in J} H_j$ such that $\{h\} \times \prod_{i\in I\setminus J} H_i \subseteq E$. For every $j \in J$ select $r_j \in \mathbb{R}$ such that $f_j(r_j) = h(j)$. Since X is ω-dense in $\mathbb{R}^I$, there exists $x \in X$ such that $x(j) = r_j$ for every $j \in J$. Now

$$\theta(x) = \{f_i(x(i))\}_{i\in I} = \{f_j(x(j))\}_{j\in J} \times \{f_i(x(i))\}_{i\in I\setminus J} = \{h(j)\}_{j\in J} \times \{f_i(x(i))\}_{i\in I\setminus J} \in \{h\} \times \prod_{i\in I\setminus J} H_i \subseteq E.$$

Therefore, $\theta(x) \in Y \cap E \neq \emptyset$. $\square$

Lemma 5.3. *Let $\kappa \geq \omega_1$ be a cardinal and G and H be topological groups in a variety $\mathcal{V}$ such that:*

(i) $r_{\mathcal{V}}(H) \geq \kappa$,

(ii) H^ω has a G_δ-dense subset of size at most κ,

(iii) G has a G_δ-dense subset of size at most κ.

Then $G \times H^{\omega_1}$ contains a G_δ-dense $\mathcal{V}$-independent subset of size κ.

Proof. Since $\kappa \geq \omega_1$, we have $|\kappa \times \omega_1| = \kappa$, and so we can use item (i) to fix a faithfully indexed $\mathcal{V}$-independent subset $X = \{x_{\alpha\beta} : \alpha \in \kappa, \beta \in \omega_1\}$ of H. For every $\beta \in \omega_1 \setminus \omega$ the topological groups $G \times H^\omega$ and $G \times H^\beta$ are isomorphic, so we can use items (ii) and (iii) to fix $\{g_{\alpha\beta} : \alpha \in \kappa\} \subseteq G$ and $\{y_{\alpha\beta} : \alpha \in \kappa\} \subseteq H^\beta$ such that $Y_\beta = \{(g_{\alpha\beta}, y_{\alpha\beta}) : \alpha \in \kappa\}$ is a G_δ-dense subset of $G \times H^\beta$.

For $\alpha \in \kappa$ and $\beta \in \omega_1 \setminus \omega$ define $z_{\alpha\beta} \in H^{\omega_1}$ by

$$z_{\alpha\beta}(\gamma) = \begin{cases} y_{\alpha\beta}(\gamma), & \text{for } \gamma \in \beta \\ x_{\alpha\beta}(\gamma), & \text{for } \gamma \in \omega_1 \setminus \beta \end{cases} \qquad \text{for } \gamma \in \omega_1. \tag{2}$$

Finally, define

$$Z = \{(g_{\alpha\beta}, z_{\alpha\beta}) : \alpha \in \kappa, \beta \in \omega_1 \setminus \omega\} \subseteq G \times H^{\omega_1}.$$

Claim 5.4. *Z is G_δ-dense in $G \times H^{\omega_1}$.*

Proof. Let E be a non-empty G_δ-subset of $G \times H^{\omega_1}$. Then there exist $\beta \in \omega_1 \setminus \omega$ and a non-empty G_δ-subset E' of $G \times H^\beta$ such that

$$E' \times H^{\omega_1 \setminus \beta} \subseteq E. \tag{3}$$

Since Y_β is G_δ-dense in $G \times H^\beta$, there exists $\alpha \in \kappa$ such that $(g_{\alpha\beta}, y_{\alpha\beta}) \in E'$. From (2) it follows that $z_{\alpha\beta} \restriction_\beta = y_{\alpha\beta}$. Combining this with (3), we conclude that $(g_{\alpha\beta}, z_{\alpha\beta}) \in E$. Thus $(g_{\alpha\beta}, z_{\alpha\beta}) \in E \cap Z \neq \emptyset$. ☐

Claim 5.5. *Z is $\mathcal{V}$-independent.*

Proof. Let F be a non-empty finite subset of $\kappa \times (\omega_1 \setminus \omega)$. Define

$$\gamma = \max\{\beta \in \omega_1 \setminus \omega : \exists\ \alpha \in \kappa\ (\alpha, \beta) \in F\}. \tag{4}$$

From (2) and (4) it follows that $z_{\alpha\beta}(\gamma) = x_{\alpha\beta}(\gamma)$ for all $(\alpha, \beta) \in F$. Therefore,

$$\{z_{\alpha\beta}(\gamma) : (\alpha, \beta) \in F\} = \{x_{\alpha\beta}(\gamma) : (\alpha, \beta) \in F\} \subseteq X.$$

Since X is a $\mathcal{V}$-independent subset of H, we conclude that $\{z_{\alpha\beta} : (\alpha, \beta) \in F\}$ is a $\mathcal{V}$-independent subset of H^{ω_1}. Thus, $\{(g_{\alpha\beta}, z_{\alpha\beta}) : (\alpha, \beta) \in F\}$ is a $\mathcal{V}$-independent subset of $G \times H^{\omega_1}$. Since F was taken arbitrary, it follows that Z is $\mathcal{V}$-independent. ☐

For the last claim we conclude that $|Z| = |\kappa \times (\omega_1 \setminus \omega)| = \kappa$. ☐

Lemma 5.6. *Assume that $\kappa \geq \omega_1$ is a cardinal, $\{H_n : n \in \mathbb{N}\}$ is a family of separable metric groups in a variety $\mathcal{V}$ and $\{\sigma_n : n \in \mathbb{N}\}$ is a sequence of cardinals such that:*

- *(i) $r_{\mathcal{V}}(H_n) \geq \omega$ for every $n \in \mathbb{N}$,*
- *(ii) $\sigma = \sup\{\sigma_n : n \in \mathbb{N}\} \geq \omega_1$,*
- *(iii) $\mathbf{Ps}(\kappa, \sigma)$ holds.*

Then $\prod_{n\in\mathbb{N}} H_n^{\sigma_n}$ has a G_δ-dense $\mathcal{V}$-independent subset of size κ.

Proof. Define $S = \{n \in \mathbb{N} : \sigma_n \geq \omega_1\}$, $G = \prod_{n\in\mathbb{N}\setminus S} H_n^{\sigma_n}$ and $H = \prod_{n\in S} H_n^{\sigma_n}$. Since $|\sigma_n \times \omega_1| = \sigma_n$ for every $n \in S$, we have

$$H^{\omega_1} \cong \prod_{n\in S} (H_n^{\sigma_n})^{\omega_1} \cong \prod_{n\in S} H_n^{\sigma_n \times \omega_1} \cong \prod_{n\in S} H_n^{\sigma_n} \cong H,$$

where $\cong$ denotes the isomorphism between topological groups. In particular, $\prod_{n\in\mathbb{N}} H_n^{\sigma_n} = G \times H \cong G \times H^{\omega_1}$. Therefore, the conclusion of our lemma would follow from Lemma 5.3 so long as we prove that G and H satisfy the assumptions of Lemma 5.3.

Let us check that the assumption of item (i) of Lemma 5.3 holds. From items (i) and (ii) of our lemma it follows that $H \cong \prod_{i\in I} H'_i$, where $|I| = \sigma$ and each H'_i is a separable metric group satisfying $r_{\mathcal{V}}(H_i) \geq \omega$. Now Lemma 5.1 yields $r_{\mathcal{V}}(H) \geq 2^\sigma$. Since $\mathbf{Ps}(\kappa, \sigma)$ holds, we have $\kappa \leq 2^\sigma$ by Proposition 4.7(c), and so $r_{\mathcal{V}}(H) \geq 2^\sigma \geq \kappa$.

Let us check that the assumption of item (ii) of Lemma 5.3 holds. Again, in view of items (i) and (ii), we have $H^\omega \cong \prod_{i\in I} H''_i$, where $|I| = \sigma$ and each H''_i is a separable metric group. Since $\mathbf{Ps}(\kappa, \sigma)$ holds by item (iii), Lemma 5.2 allows us to conclude that H^ω has G_δ-dense subset of size at most κ.

Let us check that the assumption of item (iii) of Lemma 5.3 holds. Since $\sigma_n \leq \omega$ for every $n \in \mathbb{N} \setminus S$, $G = \prod_{n\in\mathbb{N}\setminus S} H_n^{\sigma_n}$ is a separable metric group, and so $|G| \leq \mathfrak{c}$. Since $\mathbf{Ps}(\kappa, \sigma)$ holds, $\mathfrak{c} \leq \kappa$ by Proposition 4.7(b), and so G itself is a G_δ-dense subset of G of size at most κ. ☐

Corollary 5.7. *Let $\mathbb{P}$ be the set of prime numbers and $\{\sigma_p : p \in \mathbb{P}\}$ a sequence of cardinals such that $\sigma = \sup\{\sigma_p : p \in \mathbb{P}\} \geq \omega_1$. If $\kappa \geq \omega_1$ is a cardinal such that $\mathbf{Ps}(\kappa, \sigma)$ holds, then $\prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p}$ contains a G_δ-dense free subgroup F such that $|F| = \kappa$.*

Proof. Since $r(\mathbb{Z}_p) \geq \omega$ for every $p \in \mathbb{P}$, applying Lemma 5.6 for $\mathcal{V} = \mathcal{A}$ we can find a G_δ-dense $\mathcal{A}$-independent subset X of $G = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p}$ of size κ. Then the smallest subgroup F of G generated by X is free (since $\mathcal{A}$-independence coincides with the usual independence for abelian groups) and satisfies $|F| = \kappa$. Since $X \subseteq F \subseteq G$ and X is G_δ-dense in G, so is F. ☐

6 Proofs of the Main Theorems

Proof of Theorem 2.1. Let K be the compact completion $\widetilde{G}$ of G. By Theorem 1.8, G is essential in K.

We consider first the case when G is torsion-free. Then K is torsion-free as well. Therefore, since the Pontryagin dual of K is divisible, $K = \widehat{\mathbb{Q}}^{\sigma_0} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p}$, for appropriate cardinals σ_p ($p \in \mathbb{P} \cup \{0\}$) [19, Theorem 25.8]. Define $\sigma = \sup_{p\in\{0\}\cup\mathbb{P}} \sigma_p$. Clearly, $\sigma = w(K) = w(G)$ and $|G| \le |K| = 2^\sigma$. Since G is both dense and essential in K, from [1, Theorems 3.12 and 3.14] we get $\sup_{p\in\{0\}\cup\mathbb{P}} 2^{\sigma_p} \le |G|$. Therefore $\mathbf{Min}(|G|, \sigma)$ holds. Since $\sigma = w(G)$, we are done.

In the general case, we consider the connected component $c(K)$ and the totally disconnected quotient $K/c(K)$. Then $K/c(K) \cong \prod_{p\in\mathbb{P}} \in K_p$, where each K_p is a pro-p-group. Let $\sigma_p = w(K_p)$. Then by [1, Theorems 3.12, 3.14], one has $|G| \ge |c(K)| \sup_{p\in\mathbb{P}} 2^{\sigma_p}$. Let $\sigma_0 = w(c(K))$, so that $\sigma = w(G) = w(K) = \sup\{\sigma_p : p \in \{0\} \cup \mathbb{P}\}$. Then $\sup_{p\in\{0\}\cup\mathbb{P}} 2^{\sigma_p} \le |G| \le |K| = 2^\sigma$. Therefore $\mathbf{Min}(|G|, \sigma)$ holds. Since $\sigma = w(G)$, we are done. ☐

Proof of Theorem 2.2. Let G be a minimal abelian group with $w(G) \ge \kappa$. Then $\mathbf{Min}(|G|, w(G))$ holds (Theorem 2.1). If $\mathrm{cf}(w(G)) > \omega$, then $|G| = 2^{w(G)} \ge 2^\kappa$ holds by Example 3.2(c). Assume that $\mathrm{cf}(w(G)) = \omega$ and let $\{\sigma_n : n \in \mathbb{N}\}$ be a sequence of cardinals with $w(G) = \sup_{n\in\mathbb{N}} \sigma_n$ and $\sigma_n < w(G)$ for every $n \in \mathbb{N}$. Since $\mathrm{cf}(\kappa) > \omega$, our hypothesis $w(G) \ge \kappa$ gives $w(G) > \kappa$. Then $\sigma_n \ge \kappa$ for some $n \in \mathbb{N}$. So $2^\kappa \le 2^{\sigma_n} \le |G|$. ☐

Proof of Theorem 2.5. By Theorem 2.1, $\mathbf{Min}(|G|, w(G))$ holds. Since $|G|$ is assumed to be non-exponential, the conclusion now follows from Proposition 3.5. ☐

Lemma 6.1. *Let K be a torsion-free abelian group and let F be a free subgroup of K. Then there exists a free subgroup F_0 of K containing F as a direct summand and such that:*

(a) F_0 non-trivially meets every non-zero subgroup of K, and

(b) $|F_0| = |K|$.

Proof. Let $A := K/F$ and let $\pi : K \to A$ be the canonical projection. Let F_2 be a free subgroup of A with generators $\{g_\alpha\}_{\alpha\in I}$ such that A/F_2 is torsion. Since π is surjective, for every $\alpha \in I$ there exists $f_\alpha \in K$, such that $\pi(f_\alpha) = g_\alpha$. Consider the subgroup F_1 of K generated by $\{f_\alpha\}_{\alpha\in I}$. As $\pi(F_1) = F_2$ is free, we conclude that $F_1 \cap F = \{0\}$, so $\pi \restriction_{F_1} : F_1 \to F_2$ is an isomorphism. Let us see that the subgroup $F_0 = F + F_1 = F \oplus F_1$ has the required properties. Indeed, it is free as $F_1 \cap F = \{0\}$ and both F, F_1 are free. Moreover, $K/F_0 \cong A/F_2$ is torsion and F is a direct summand of F_0. As K/F_0 is torsion, F_0 non-trivially meets every non-zero subgroup of K, so (a) holds true. Finally, $|F_0| = r(F_0) = r(K) = |K|$ as K/F_0 is torsion and the groups K, F_0 are uncountable and torsion-free. ☐

Lemma 6.2. *Let K be a compact torsion-free abelian group and let F be a free subgroup of K. Then there exists a free essential subgroup F_0 of K with $|F_0| = |K|$, containing F as a direct summand.*

Proof. Apply Lemma 6.1. ☐

Lemma 6.3. *Suppose $\mathbf{Min}(\kappa, \sigma)$ holds, and let $\{\sigma_p : p \in \mathbb{P}\}$ be the sequence of cardinals witnessing $\mathbf{Min}(\kappa, \sigma)$. Then for every free subgroup F of the group $K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p}$ satisfying $|F| = \kappa$ there exists a free essential subgroup F' of K such that $F \subseteq F'$, $|F'| = \kappa$.*

Proof. Let

$$\mathrm{wtd}(K) = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p} \text{ and } F_* = F \cap \mathrm{wtd}(K). \tag{5}$$

Then F_* is a free subgroup of $\mathrm{wtd}(K)$, so applying Lemma 6.1 to the group $\mathrm{wtd}(K)$ and its subgroup F_* we get a free subgroup F^* of $\mathrm{wtd}(K)$ such that

(i) $F^* \supseteq F_*$ and $F^* = F_* \oplus L$ for an appropriate subgroup L of F^*;

(ii) F^* non-trivially meets every non-zero subgroup of $\mathrm{wtd}(K)$, and

(iii) $|F^*| = |\mathrm{wtd}(K)| \leq \kappa = |F|$.

Since K is torsion-free, (ii) yields that F^* is essential in $\mathrm{wtd}(K)$. As $\mathrm{wtd}(K)$ is essential in K [12], we conclude that F^* is essential in K as well. From (iii) we conclude that $F' = F + F^*$ is an essential subgroup of size κ of K containing F. Finally, from (5) and (i) we get $F' = F + L$, and

$$F \cap L = F \cap \mathrm{wtd}(K) \cap L = F_* \cap L = 0.$$

Therefore, $F' = F \oplus L$ is free. □

Proof of Theorem 2.7. (c)⇒(b) and (b)⇒(a) are obvious.

(a)⇒(c) Assume τ_1 is a minimal topology of weight σ on F_κ. Then $\sigma \geq \omega_1$ as $\kappa > \mathfrak{c}$. According to Theorem 2.1 $\mathbf{Min}(\kappa, \sigma)$ holds. Now assume τ_2 is a minimal topology of weight λ on F_κ. According to Theorem 4.4 $\mathbf{Ps}(\kappa, \lambda)$ holds. Now Lemma 4.8 yields that also $\mathbf{Ps}(\kappa, \sigma)$ holds true.

Let $\{\sigma_p : p \in \mathbb{P}\}$ be a sequence of cardinals such that $\sigma = \sup\{\sigma_p : p \in \mathbb{P}\}$. Then the group $K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^{\sigma_p}$. is compact and zero-dimensional. Since $\sigma \geq \omega_1$ and $\mathbf{Ps}(\kappa, \sigma)$ holds, by Corollary 5.7 there exists a G_δ-dense free subgroup F of K with $|F| = \kappa$. Since $\mathbf{Min}(\kappa, \sigma)$ holds, according to Lemma 6.3 there exists a free essential subgroup F' of K containing F with $|F'| = \kappa$. Obviously F' is also G_δ-dense. By Theorem 4.4 the group topology induced on F' is pseudocompact. On the other hand, by the essentiality of F' in K and Theorem 1.8, the subgroup F' is also minimal. Finally F' is zero-dimensional, as a subgroup of the zero-dimensional group K. □

Proof of Theorem 2.8. The implications (b) ⇒(a) and (c)⇒(a) are obvious.

(a)⇒(d) Suppose that $F_\mathfrak{c}$ admits a minimal pseudocompact group topology. Since $F_\mathfrak{c}$ is free, $F_\mathfrak{c}$ cannot admit any compact group topology, and so $\mathfrak{c} = |F_\mathfrak{c}| \geq 2^{\omega_1}$ by Corollary 2.6. The converse inequality $\mathfrak{c} \leq 2^{\omega_1}$ is clear.

Now assume that (d) holds, i.e., $\mathfrak{c} = 2^{\omega_1}$. Then $\mathbf{Ps}(\mathfrak{c}, \omega_1)$ holds by Proposition 4.7(a).

(d)⇒(b) Form $\mathbf{Ps}(\mathfrak{c}, \omega_1)$ one can find a G_δ-dense embedding $j : F \to K := \widehat{\mathbb{Q}}^{\omega_1}$ by [13, Lemma 4.3]. On the other hand, $|\widehat{\mathbb{Q}}^{\omega_1}| = 2^{\omega_1} = \mathfrak{c}$, and this is a torsion-free group. Then by Lemma 6.2 there exists a free essential subgroup F_0 of K containing $j(F)$ with $|F_0| = \mathfrak{c}$. Then F_0 is minimal by Theorem 1.8. On the other hand, F_0 is G_δ-dense in K, which is compact and connected. By Theorems 4.1, F_0 is pseudocompact. Moreover F_0 is connected, being G_δ-dense in the connected compact group K.

(d)⇒(c) Fix an arbitrary prime $p \in \mathbb{P}$. According to [13, Lemma 4.3], there exists a G_δ-dense embedding $j : F \to K := \mathbb{Z}_p^{\omega_1}$ due to $\mathbf{Ps}(\mathfrak{c}, \omega_1)$. The compact group K is torsion-free and $|K| = 2^{\omega_1} = \mathfrak{c}$. By Lemma 6.2 there exists an essential free subgroup F_0 of K containing $j(F)$ with $|F_0| = \mathfrak{c}$. Then F_0 is G_δ-dense and also essential in the compact group K. By Theorems 4.1 and 1.8 F_0 is minimal and pseudocompact. Moreover F_0 is zero-dimensional, being a subgroup of the zero-dimensional compact group K. □

Proof of Theorem 2.10. (a)⇒(b) is obvious.

(b)⇒(c) Assume that τ_1 is a connected minimal group topology on F_κ with $w(F_\kappa, \tau_1) = \sigma$. The completion K of (F_κ, τ_1) is a compact connected group. By Theorem 1.8, F_κ is essential in K. Since F_κ is torsion-free, this yields that K is torsion-free as well. Then its Pontryagin dual $X = \widehat{K}$ is both divisible and torsion-free (as K is connected). As $|X| = w(K) = \sigma$, this yields that $X \cong \bigoplus_\sigma \mathbb{Q}$. Therefore, $K = \widehat{\mathbb{Q}}^\sigma$. Since F_κ is both dense and essential in K by Theorem 1.8, from [1, Theorems 3.12 and 3.14] we get $2^\sigma \leq |F_\kappa| \leq |K| = 2^\sigma$. Hence $\kappa = 2^\sigma$.

(c)⇒(a) Since $\kappa = 2^\sigma$, $\mathbf{Ps}(\kappa, \sigma)$ holds by Proposition 4.7(d). By [13, Lemma 4.3] there exists a free subgroup F of the compact connected group $K := \widehat{\mathbb{Q}}^\sigma$ which is G_δ-dense and $|F| = 2^\sigma$. Since K is torsion-free, by Lemma 6.2 there exists an essential free subgroup F' of K containing F. Since G_δ-dense subgroups of compact connected abelian groups are connected, and in view of Theorems 1.8 and 4.1, the topology induced on F' is connected, minimal and pseudocompact of weight $\sigma = w(K)$. Obviously, $F' \cong F_\kappa$ as $|F'| = |F| = 2^\sigma$. □

7 Final remarks and open questions

We show here that the counterpart of the simultaneous minimal and pseudocompact topologization for divisible abelian groups is much easier than in the case of free abelian groups.

The divisible groups that admit a minimal group topology were described in [8]. Here we need only the part of the characterization for divisible groups of size $\geq \mathfrak{c}$.

Theorem 7.1. [8] *A divisible abelian group of cardinality* $\geq \mathfrak{c}$ *admits some minimal group topology precisely when it admits a compact group topology.*

The concept of pseudocompactness generalizes compactness from a different angle than that of minimality. It is therefore quite surprising that minimality and pseudocompactness *combined together* "yield" compactness in the class of divisible groups. This should be compared with Theorem 2.7, where a similar phenomenon occurs (minimal and pseudocompact topologizations imply compact topologization) on a different ground.

Theorem 7.2. *An infinite divisible abelian group admits a minimal group topology and a pseudocompact group topology if and only it admits a compact group topology.*

Proof. The necessity is obvious. Suppose that an infinite divisible group G admits a minimal group topology and a pseudocompact group topology. Then $|G| \geq \mathfrak{c}$ by Theorem 1.7. Now the conclusion follows from Theorem 7.1. $\square$

Our next example demonstrates that the restriction on the cardinality in Theorem 7.1 or the hypothesis of the existence of a pseudocompact group topology in Theorem 7.2 are needed:

Example 7.3. (a) The divisible abelian group $\mathbb{Q}/\mathbb{Z}$ admits a minimal group topology [10], but does not admit a pseudocompact group topology (Theorem 1.7).

(b) The divisible Abelian group $\mathbb{Q}^{(\mathfrak{c})} \oplus (\mathbb{Q}/\mathbb{Z})^{(\omega)}$ admits a (connected) pseudocompact group topology [13], but does not admit any minimal group topology. (The latter conclusion follows from Theorem 7.1 and the fact that this group does not admit any compact group topology [19]).

Let us briefly discuss the possibilities to extended our results for free abelian groups to the case of torsion-free abelian groups. Theorem 7.2 shows that for divisible torsion-free abelian groups the situation is in some sense similar to that of free abelian groups described in Theorem 2.7: in both cases the existence of a pseudocompact group topology and a minimal group topology is equivalent to the existence of a minimal pseudocompact (actually, compact) group topology. Nevertheless, there is a substantial difference, because free abelian groups admit no compact group topology. Another important difference between both cases is that Problem 1.2 is still open for torsion-free abelian groups [9]:

Problem 7.4. *Characterize the minimal torsion-free abelian groups.*

A quotient of a minimal group need not be minimal even in the abelian case. This justified the isolation in [10] of the smaller class of totally minimal groups:

Definition 7.5. A Hausdorff topological group G is called *totally minimal* if every Hausdorff quotient group of G is minimal. Equivalently, a Hausdorff topological group G is totally minimal if every continuous group homomorphism $f : G \to H$ of G onto a Hausdorff topological group H is open.

It is clear that compact ⇒ totally minimal ⇒ minimal.

Then, since also $F_{\mathfrak{c}}$ admits a totally minimal group topology [22] and a pseudocompact group topology [13], the next questions naturally arise.

Question 7.6. *Let* κ *be a cardinal* $> \mathfrak{c}$.

(a) When does F_κ *admit a totally minimal group topology?*

(b) When does F_κ *admit a totally minimal pseudocompact group topology?*

More specifically, one can ask

Question 7.7. *Let κ be a cardinal $> \mathfrak{c}$. Is the condition*

- *F_κ admits a zero-dimensional totally minimal pseudocompact group topology*

equivalent to those of Theorem 2.7?

Let us mention finally another class of abelian groups where both problems (Problem 1.2 for minimal group topologies [11] and its counterpart for pseudocompact group topologies [6, 13]) are completely resolved. These are the torsion abelian groups. We do not know the answer of the following question.

Question 7.8. *Let G be a torsion abelian group that admits a minimal group topology and a pseudocompact group topology. Does G admit also a minimal pseudocompact group topology?*

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