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Algebraic entropy of generalized shifts on direct products

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Abstract

For a set Γ, a function $\lambda : \Gamma \to \Gamma$ and a non-trivial abelian group K, the *generalized shift* $\sigma_\lambda : K^\Gamma \to K^\Gamma$ is defined by $(x_i)_{i\in\Gamma} \mapsto (x_{\lambda(i)})_{i\in\Gamma}$ [AHK]. In this paper we compute the algebraic entropy of σ_λ; it is either zero or infinite, depending exclusively on the properties of λ. This solves two problems posed in [AADGH].

1 Introduction

The general aim of this paper is to study the algebraic entropy of relevant endomorphisms of abelian groups, as generalized shifts are, in view of the recent results in [AADGH] and [DGSZ].

According to Adler, Konheim and McAndrew [AKM] and Weiss [W] the algebraic entropy is defined as follows. Let G be an abelian group and F a finite subgroup of G; for an endomorphism $\phi : G \to G$ and a positive integer n, let $T_n(\phi, F) = F + \phi(F) + \ldots + \phi^{n-1}(F)$ be the *n-th ϕ-trajectory* of F with respect to ϕ. The *algebraic entropy of ϕ with respect to F* is

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log |T_n(\phi, F)|}{n},$$

and the *algebraic entropy* of $\phi : G \to G$ is

$$\mathrm{ent}(\phi) = \sup\{H(\phi, F) : F \text{ is a finite subgroup of } G\}.$$

In Section 2 we collect the general results on the algebraic entropy that we use in this paper, including the so-called Addition Theorem from [DGSZ] (see Theorem 2.1 below).

In [AHK] the notion of generalized shift was introduced as follows.

Definition 1.1. Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function and K an abelian group. The *generalized shift* $\sigma_{\lambda,K} : K^\Gamma \to K^\Gamma$ is defined by $(x_i)_{i\in\Gamma} \mapsto (x_{\lambda(i)})_{i\in\Gamma}$. When there is no need to specify the group K, we simply write σ_λ.

In Section 2 we give basic properties of the generalized shifts.

The interest in studying the generalized shifts arises from the fact that there is a close relation between the generalized shifts and the Bernoulli shifts: let K be a non-trivial finite abelian group, and denote by $\mathbb{N}$ and $\mathbb{Z}$ respectively the set of natural numbers and the set of integers; then:

(a) the *two-sided Bernoulli shift* $\overline{\beta}_K$ of the group $K^\mathbb{Z}$ is defined by

$$\overline{\beta}_K((x_n)_{n\in\mathbb{Z}}) = (x_{n-1})_{n\in\mathbb{Z}}, \text{ for } (x_n)_{n\in\mathbb{Z}} \in K^\mathbb{Z};$$

(b) the *right Bernoulli shift* β_K and the *left Bernoulli shift* ${}_K\beta$ of the group $K^\mathbb{N}$ are defined respectively by

$$\beta_K(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots) \text{ and } {}_K\beta(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

The left Bernoulli shift ${}_K\beta$ and the two-sided Bernoulli shift $\overline{\beta}_K$ are relevant for both ergodic theory and topological dynamics and they are generalized shifts (see Example 6.4). The right Bernoulli shift β_K restricted to the direct sum $\bigoplus_{\mathbb{N}} K$ is fundamental for the algebraic entropy (see [DGSZ]). It cannot be obtained as a generalized shift from any function $\lambda : \mathbb{N} \to \mathbb{N}$; nevertheless, it can be well "approximated" by a generalized shift [AADGH] (see Example 6.4).

In [AADGH] the restriction of a generalized shift σ_λ to the direct sum $\bigoplus_\Gamma K$ was considered. Indeed, a precise formula was found for the algebraic entropy of this restriction (see (6.1) in Theorem 6.1). In particular, the algebraic entropy of $\sigma_\lambda \restriction_{\bigoplus_\Gamma K}$ depends on the combinatorial invariant that measures the number of strings of λ (see Definitions 3.1 and 3.2) and on the cardinality of K. Note that in this case λ must have finite fibers in order that $\bigoplus_\Gamma K$ is σ_λ-invariant.

Problems 6.1 and 6.2 in [AADGH] ask to calculate the algebraic entropy of $\sigma_\lambda : K^\Gamma \to K^\Gamma$ and to relate this entropy with the algebraic entropy of $\sigma_\lambda \restriction_{\bigoplus_\Gamma K} : \bigoplus_\Gamma K \to \bigoplus_\Gamma K$. We provide a complete answer to these questions. More precisely, we show that $\mathrm{ent}(\sigma_\lambda)$ depends only on the combinatorial properties of the map λ, unlike $\mathrm{ent}(\sigma_\lambda \restriction_{\bigoplus_\Gamma K})$. Indeed, Theorem 1.3 shows that $\mathrm{ent}(\sigma_\lambda) = 0$ if and only if λ is bounded (in the sense of the next Definition 1.2), otherwise $\mathrm{ent}(\sigma_\lambda)$ is infinite.

The function $\lambda : \Gamma \to \Gamma$ of a set Γ defines a preorder $\leq_\lambda$ on Γ in a natural way: $i \leq_\lambda j$ in Γ if there exists $s \in \mathbb{N}$ such that $\lambda^s(j) = i$. The preorder $\leq_\lambda$ is not an order in general: two distinct elements i and j of Γ violate the antisymmetry for $\leq_\lambda$ if and only if i and j are in the same orbit of a periodic point (which could be i or j) of λ. We say that a subset I of $(\Gamma, \leq_\lambda)$ is *totally preordered* if for every $i, j \in I$ either $i \leq_\lambda j$ or $j \leq_\lambda i$ (without asking that these elements satisfy antisymmetry).

Definition 1.2. Let Γ be a set. A function $\lambda : \Gamma \to \Gamma$ is *bounded* if there exists $N \in \mathbb{N}$ such that $|I| \leq N$ for every totally preordered subset I of $(\Gamma, \leq_\lambda)$.

In Section 3 we analyze the properties of a function λ which play a role with respect to the algebraic entropy of σ_λ and find characterizations of bounded functions. Indeed, Theorem 3.3 shows that a function is bounded if and only if it admits no strings, no infinite orbits and no ladders, and has bounded periodic orbits; here, a string of λ is an infinite increasing chain in $(\Gamma, \leq_\lambda)$, while an infinite orbit is an infinite decreasing chain in $(\Gamma, \leq_\lambda)$ and a ladder of λ is a disjoint union of infinitely many finite chains in $(\Gamma, \leq_\lambda)$ of strictly increasing length where the top element of each finite chain is a maximal element (for the precise definitions of these notions see Definition 3.1).

Theorem 3.3 proves also that for a function λ it is equivalent to be bounded or quasi-periodic (the definition is given below).

For a function $f : X \to X$ of a set X, a point $x \in X$ is said to be *quasi-periodic* if there exist $n_x < m_x$ in $\mathbb{N}$ such that $f^{n_x}(x) = f^{m_x}(x)$. The function f is *locally quasi-periodic* if every point of X is quasi-periodic, and f is *quasi-periodic* if there exist $n < m$ in $\mathbb{N}$ such that $n_x = n$ and $m_x = m$ for every $x \in X$, that is, $f^n = f^m$.

It is known from [DGSZ] that $\mathrm{ent}(\phi) = 0$ if and only if ϕ is locally quasi-periodic. The main goal of this paper is to prove the following theorem, showing in particular that for generalized shifts this "local" condition becomes "global", and that a generalized shift of finite algebraic entropy has necessarily entropy zero.

Theorem 1.3. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial finite abelian group, and $\sigma_\lambda : K^\Gamma \to K^\Gamma$ the generalized shift. The following conditions are equivalent:*

(a) $\mathrm{ent}(\sigma_\lambda) = 0$*;*

(b) $\mathrm{ent}(\sigma_\lambda)$ *is finite;*

(c) λ *is bounded;*

(d) σ_λ *is quasi-periodic;*

(e) σ_λ *is locally quasi-periodic.*

At the end of Section 2 we see that λ is quasi-periodic if and only if σ_λ is quasi-periodic. In Section 4 we prove first the equivalence of (c), (d) and (e), without involving the algebraic entropy (see Theorem 4.2), even if, as noted previously, (a)$\Leftrightarrow$(e) is already known from [DGSZ] (see Proposition 2.2(a)).

The implication (a)$\Rightarrow$(b) is obvious. The main part of the paper is dedicated to the proof of (b)$\Rightarrow$(a), that is, to prove that if $\mathrm{ent}(\sigma_\lambda)$ is positive, then $\mathrm{ent}(\sigma_\lambda)$ is infinite. Given the above equivalence (a)$\Leftrightarrow$(c), this is the same as proving (b)$\Rightarrow$(c), which is what we verify.

Section 5 contains technical lemmas, which allow the construction of large independent families of finite subgroups of $K^{\mathbb{N}}$. These families of subgroups are used in the computation of the algebraic entropy of a generalized shift σ_λ when λ admits some string or some infinite orbit.

Indeed, in Section 6 we prove that in presence of a string or of an infinite orbit of λ, the algebraic entropy of σ_λ is infinite. The same happens if λ has a ladder or if it has periodic orbits of arbitrarily large length. This shows that the entropy of σ_λ is infinite when the function λ is not bounded.

After the proof of Theorem 1.3 we explain how it solves Problems 6.1 and 6.2 in [AADGH].

As applications, in Corollary 6.5 we see that the algebraic entropy of the Bernoulli shifts considered on the direct products is infinite, and in Corollary 4.3 we strengthen a result from [DGSZ], related to the Poincaré–Birkhoff recurrence theorem of ergodic theory, in the particular case of the generalized shifts.

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2 Preliminary results

We start collecting basic results on the algebraic entropy, mainly from [DGSZ] and [W], which are applied in the sequel.

First of all, since the definition of the algebraic entropy of an endomorphism ϕ of an abelian group G is based on the finite subgroups F of G, the algebraic entropy depends only on the restriction of ϕ on $t(G)$, that is $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(G)})$. So it makes sense to consider endomorphisms of torsion abelian groups.

Let G be a torsion abelian group, $\phi : G \to G$ an endomorphism and H a ϕ-invariant subgroup of G. Denote by $\overline{\phi} : G/H \to G/H$ the endomorphism induced on the quotient by ϕ. Then $\mathrm{ent}(\phi) \geq \max\{\mathrm{ent}(\phi \restriction_H), \mathrm{ent}(\overline{\phi})\}$. Moreover, the following important result on the algebraic entropy holds true:

Theorem 2.1 (Addition Theorem)**.** [DGSZ, Theorem 3.1] *Let G be a torsion abelian group, $\phi : G \to G$ an endomorphism and H a ϕ-invariant subgroup of G. If $\overline{\phi} : G/H \to G/H$ is the endomorphism induced on the quotient by ϕ, then*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_H) + \mathrm{ent}(\overline{\phi}).$$

For an endomorphism ϕ of a torsion abelian group G and a finite subgroup F of G, the *ϕ-trajectory* of F is $T(\phi, F) = \sum_{n\in\mathbb{N}} \phi^n(F)$. Let

$$t_\phi(G) = \{x \in G : |T(\phi, \langle x\rangle)| \text{ is finite}\}$$

be the *ϕ-torsion* subgroup of G. Then $t_\phi(G)$ is the largest ϕ-invariant subgroup of G such that $\mathrm{ent}(\phi \restriction_{t_\phi(G)}) = 0$. In particular every quasi-periodic point x of ϕ in G has finite trajectory and so it is in $t_\phi(G)$.

We collect in the following proposition the basic and well-known results on the algebraic entropy that we will use in the paper; for a proof of (a), (b) and (c) see [DGSZ] and [W], while (d) can be derived from (c), from the finite case and from the monotonicity of the algebraic entropy under taking invariant subgroups, both proved in [W].

Proposition 2.2. *Let G be a torsion abelian group and $\phi : G \to G$ an endomorphism. Then:*

(a) $\operatorname{ent}(\phi) = 0$ *if and only if* $t_\phi(G) = G$ *if and only if* ϕ *is locally quasi-periodic.*

(b) If H *is another abelian group,* $\eta : H \to H$ *an endomorphism, and there exists an isomorphism* $\xi : G \to H$ *such that* $\phi = \xi^{-1}\eta\xi$*, then* $\operatorname{ent}(\phi) = \operatorname{ent}(\eta)$.

(c) If G *is direct limit of* ϕ*-invariant subgroups* $\{G_i : i \in I\}$*, then* $\operatorname{ent}(\phi) = \sup_{i\in I} \operatorname{ent}(\phi\restriction_{G_i})$.

(d) If $G = \prod_{i\in I} G_i$*, where each* G_i *is a* ϕ*-invariant subgroup of* G*, then* $\operatorname{ent}(\phi) \geq \sum_{i\in I} \operatorname{ent}(\phi \restriction_{G_i})$.

Now we summarize the preliminary results on the generalized shifts, recalling in Proposition 2.3 some basic facts which are mostly proved in [AADGH] and [AHK].

Let Γ be a set and $\lambda : \Gamma \to \Gamma$ a function. If K is an abelian group, the support of an element $x = (x_i)_{i\in\Gamma}$ of K^Γ is $\operatorname{supp}(x) = \{i \in \Gamma : x_i \neq 0\}$. If $\Lambda \subseteq \Gamma$, we identify in the natural way K^Λ with the subgroup $\{x \in K^\Gamma : \operatorname{supp}(x) \subseteq \Lambda\}$ of K^Γ.

If H is a subgroup of K, then H^Γ is a $\sigma_{\lambda,K}$-invariant subgroup of K^Γ. Moreover, $\sigma_{\lambda,K} \restriction_{H^\Gamma}= \sigma_{\lambda,H} : H^\Gamma \to H^\Gamma$.

Proposition 2.3. *Let* Γ *be a set,* $\lambda, \mu : \Gamma \to \Gamma$ *functions,* K *a non-trivial abelian group, and consider the generalized shifts* $\sigma_\lambda, \sigma_\mu : K^\Gamma \to K^\Gamma$*. Then:*

(a) $\sigma_\lambda \circ \sigma_\mu = \sigma_{\lambda\circ\mu}$ *(hence* $\sigma_\lambda^m = \sigma_{\lambda^m}$ *for every* $m \in \mathbb{N}$*), and*

(b) λ *is injective (respectively, surjective) if and only if* σ_λ *is surjective (respectively, injective). In particular,* λ *is a bijection if and only if* σ_λ *is an automorphism; in this case,* $(\sigma_\mu)^{-1} = \sigma_{\mu^{-1}}$.

(c) If $x \in K^\Gamma$*, then* $\operatorname{supp}(\sigma_\lambda^m(x)) = \lambda^{-m}(\operatorname{supp}(x))$ *for every* $m \in \mathbb{N}$*, and so*

(d) $\sigma_\lambda = \sigma_\mu$ *if and only if* $\lambda = \mu$.

Proof. For a proof of (a) and (b) see [AADGH] and [AHK].

(c) If $y = \sigma_\lambda(x)$, then $i \in \operatorname{supp}(y)$ if and only if $y_i = x_{\lambda(i)} \neq 0$, that is $\lambda(i) \in \operatorname{supp}(x)$; this is equivalent to $i \in \lambda^{-1}(\operatorname{supp}(x))$, and so $\operatorname{supp}(y) = \lambda^{-1}(\operatorname{supp}(x))$. Proceeding by induction it is possible to prove that $\operatorname{supp}(\sigma_\lambda^m(x)) = \lambda^{-m}(\operatorname{supp}(x))$ for every $m \in \mathbb{N}$.

(d) If $\lambda = \mu$, the obviously $\sigma_\lambda = \sigma_\mu$. Assume then that $\sigma_\lambda = \sigma_\mu$. Let $j \in \Gamma$, $i = \lambda(j)$ and $x \in K^\Gamma$ be such that $\operatorname{supp}(x) = \{i\}$. By (c) and by hypothesis $\lambda^{-1}(i) = \operatorname{supp}(\sigma_\lambda(x)) = \operatorname{supp}(\sigma_\mu(x)) = \mu^{-1}(i)$. Then $j \in \lambda^{-1}(i) = \mu^{-1}(i)$ and in particular $\mu(j) = i = \lambda(j)$. □

Item (a) of next lemma gives a condition on λ equivalent to the σ_λ-invariance for the "rectangular" subgroups of K^Γ, while item (b) gives a sufficient condition for the algebraic entropy of a generalized shift to be infinite.

Lemma 2.4. *Let* Γ *be a set,* $\lambda : \Gamma \to \Gamma$ *a function and* K *a non-trivial finite abelian group.*

(a) If $\Lambda \subseteq \Gamma$*, then* K^Λ *is* σ_λ*-invariant if and only if* $\lambda^{-1}(\Lambda) \subseteq \Lambda$*. If* $\Lambda \supseteq \lambda^{-1}(\Lambda) \cup \lambda(\Lambda)$*, then* $\sigma_\lambda \restriction_{K^\Lambda}= \sigma_{\lambda\restriction_\Lambda}$.

(b) If $\{\Lambda_i\}_{i\in I}$ *is an infinite family of pairwise disjoint* λ^{-1}*-invariant subsets of* Γ*, and* $\operatorname{ent}(\sigma_\lambda \restriction_{K^{\Lambda_i}}) > 0$ *for every* $i \in I$*, then* $\operatorname{ent}(\sigma_\lambda) = \infty$.

Proof. (a) Assume that $\lambda^{-1}(\Lambda) \subseteq \Lambda$ and let $x \in K^\Lambda$. Then $\operatorname{supp}(x) \subseteq \Lambda$. By Proposition 2.3(c) $\operatorname{supp}(\sigma_\lambda(x)) = \lambda^{-1}(\operatorname{supp}(x)) \subseteq \lambda^{-1}(\Lambda)$. Then $\sigma_\lambda(x) \in K^{\lambda^{-1}(\Lambda)} \subseteq K^\Lambda$. This shows that $\sigma_\lambda(K^\Lambda) \subseteq K^\Lambda$. Suppose now that $\sigma_\lambda(K^\Lambda) \subseteq K^\Lambda$. Let $i \in \lambda^{-1}(\Lambda)$. Then $a = \lambda(i) \in \Lambda$ and, for $x \in K^\Gamma$ such that $\operatorname{supp}(x) = \{a\}$ we have $x \in K^\Lambda$. By Proposition 2.3(c) $\operatorname{supp}(\sigma_\lambda(x)) = \lambda^{-1}(a)$ and by hypothesis $\sigma_\lambda(x) \in K^\Lambda$, so that $\lambda^{-1}(a) \subseteq \Lambda$; in particular $i \in \Lambda$, and hence $\lambda^{-1}(\Lambda) \subseteq \Lambda$.

If $\lambda^{-1}(\Lambda) \cup \lambda(\Lambda) \subseteq \Lambda$, then it is possible to consider both $\sigma_\lambda \restriction_{K^\Lambda}$ and $\sigma_{\lambda\restriction_\Lambda}$. It is clear that they coincide on K^Λ.

(b) By hypothesis K^Γ contains a subgroup isomorphic to $\prod_{i\in I} K^{\Lambda_i}$. By (a) K^{Λ_i} is σ_λ-invariant for every $i \in I$ and by Proposition 2.2(d) $\operatorname{ent}(\sigma_\lambda) \geq \sum_{i\in I} \operatorname{ent}(\sigma_\lambda \restriction_{K^{\Lambda_i}})$. Since by hypothesis $\operatorname{ent}(\sigma_\lambda \restriction_{K^{\Lambda_i}}) > 0$, it follows that $\operatorname{ent}(\sigma_\lambda \restriction_{K^{\Lambda_i}}) \geq \log 2$ (as $\operatorname{ent}(-)$ has as values either ∞ or $\log n$ for some $n \in \mathbb{N}_+$). Hence $\operatorname{ent}(\sigma_\lambda) = \infty$. □

The following proposition shows that the quasi-periodicity of a function λ is equivalent to the quasi-periodicity of the generalized shift σ_λ.

Proposition 2.5. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial abelian group and $\sigma_\lambda : K^\Gamma \to K^\Gamma$ the generalized shift. The the following conditions are equivalent:*

(a) λ is quasi-periodic;

(b) σ_λ is quasi-periodic.

In case λ has finite fibers, also the following condition is equivalent to the previous ones:

(c) $\sigma_\lambda \restriction_{\bigoplus_\Gamma K}$ is quasi-periodic.

Proof. (a)$\Leftrightarrow$(b) Assume that λ is quasi-periodic, that is, there exist $n < m$ in $\mathbb{N}$ such that $\lambda^n = \lambda^m$. By Proposition 2.3(a,d) this is equivalent to $\sigma_\lambda^n = \sigma_{\lambda^n} = \sigma_{\lambda^m} = \sigma_\lambda^m$, that is to say σ_λ quasi-periodic.

Assume that $\lambda^{-1}(i)$ is finite for every $i \in \Gamma$. Then (a)$\Leftrightarrow$(c) can be proved exactly as (a)$\Leftrightarrow$(b), observing that for some $n < m$ in $\mathbb{N}$, as in Proposition 2.3(d), $\lambda^n = \lambda^m$ if and only if $\sigma_{\lambda^n} = \sigma_{\lambda^m}$. $\square$

3 Strings, infinite orbits, ladders and bounded functions

The main goal of this section is to characterize the bounded functions, proving Theorem 3.3. To this end we need the notions in the next Definition 3.1.

First we fix some notations. By $\mathbb{N}_+$ we denote the set of positive integers. Let X be a set and $f : X \to X$ a function. We say that a point $x \in X$ is *periodic* for f if there exists $n \in \mathbb{N}_+$, such that $f^n(x) = x$. The *period* of a periodic point $x \in X$ of f is the minimum positive integer n such that $f^n(x) = x$ (i.e., n is the length of the orbit of x). Let $\mathrm{Per}(f)$ be the set of all periodic points and for $n \in \mathbb{N}_+$ let $\mathrm{Per}_n(f)$ be the set of all periodic points of period at most n of f in X. The function f is *periodic* if there exists $n \in \mathbb{N}_+$ such that $f^n = id_X$, that is, $\mathrm{Per}(f) = \mathrm{Per}_m(f)$ for some $m \in \mathbb{N}_+$.

Definition 3.1. Let Γ be a set and $\lambda : \Gamma \to \Gamma$ a function.

(a) A *string* of λ (in Γ) is an infinite sequence $S = \{s_t\}_{t\in\mathbb{N}}$ of pairwise distinct elements of Γ, such that $\lambda(s_t) = s_{t-1}$ for every $t \in \mathbb{N}_+$.

(b) An *infinite orbit* of λ (in Γ) is an infinite sequence $A = \{a_t\}_{t\in\mathbb{N}}$ of pairwise distinct elements of Γ, such that $\lambda(a_t) = a_{t+1}$ for every $t \in \mathbb{N}$.

(c) A *ladder* of λ (in Γ) is a subset L of Γ such that $L = \bigcup_{m\in\mathbb{N}} L_m$ is a disjoint union of non-empty finite subsets $L_m = \{l_{m,0}, l_{m,1}, \dots, l_{m,b_m}\}$ of $\Gamma \setminus \mathrm{Per}(\lambda)$, with

  (i) $\lambda(l_{m,k}) = l_{m,k-1}$ for every $k \in \{1, \dots, b_m\}$ and $\lambda^{-1}(l_{m,b_m}) = \emptyset$.

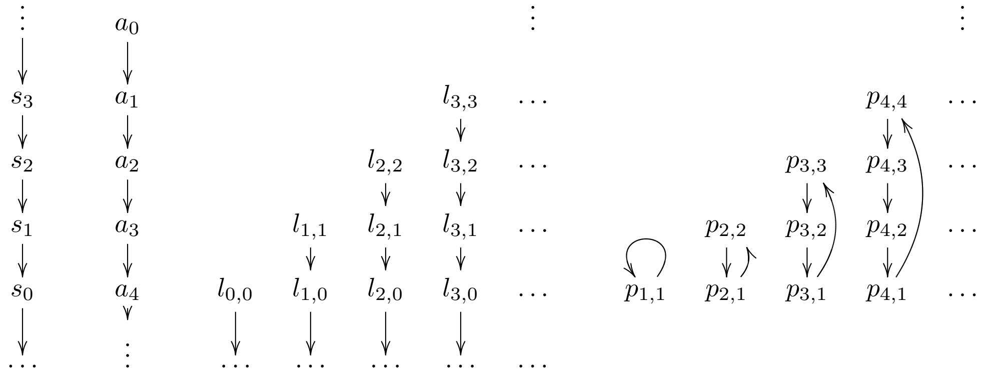
  (ii) Moreover, $b_s < b_t$ in case $s < t$ in $\mathbb{N}$.

(d) A *periodic ladder* of λ (in Γ) is a subset P of Γ such that P has a partition $P = \bigcup_{n\in\mathbb{N}_+} P_n$, where P_n is finite, $|P_n| \geq n$ and $\lambda \restriction_{P_n}: P_n \to P_n$ is a cycle of length $|P_n|$ for every $n \in \mathbb{N}_+$.

Note that by the definition of ladder, $b_m \geq m$, and so $|L_m| > m$, for every $m \in \mathbb{N}$.

The following diagram represents a string $S = \{s_t\}_{t\in\mathbb{N}}$, an infinite orbit $A = \{a_t\}_{t\in\mathbb{N}}$ of an element a_0, a ladder $L = \bigcup_{m\in\mathbb{N}}\{l_{m,0}, l_{m,1}, \dots, l_{m,m}\}$ (i.e., $b_m = m$ for every $m \in \mathbb{N}$), and a periodic ladder

$P = \bigcup_{n\in\mathbb{N}_+} P_n$ in case $P_n = \{p_{n,1}, \dots, p_{n,n}\}$ (in particular, $|P_n| = n$) for every $n \in \mathbb{N}_+$.



Following [AADGH], a string $S = \{s_t\}_{t\in\mathbb{N}}$ of λ in Γ is *acyclic* if $\lambda^n(s_0) \notin S$ for every $n \in \mathbb{N}_+$. Then an acyclic string is an ascending chain in $(\Gamma, \leq_\lambda)$. It is easy to prove that every string contains an acyclic string.

An infinite orbit A of λ is a totally ordered subset of $(\Gamma, \leq_\lambda)$, as well as each L_m in case $L = \bigcup_{m\in\mathbb{N}} L_m$ is a ladder of λ, since A and the L_m's meet trivially $\mathrm{Per}(\lambda)$. More precisely, A can be viewed also as an infinite descending chain, and each L_m is a finite chain such that its top element is a maximal element in $(\Gamma, \leq_\lambda)$ (so a ladder is disjoint union of finite chains of strictly increasing length and each finite chain ends with a maximal element of $(\Gamma, \leq_\lambda)$). Therefore a surjective λ has no ladder.

Finally note that the existence of a periodic ladder P of λ in Γ is equivalent to the existence of periodic points of arbitrarily large order, that is, to the existence of periodic orbits of arbitrarily large length, i.e., $\mathrm{Per}(\lambda) \supsetneq \mathrm{Per}_n(\lambda)$ for every $n \in \mathbb{N}_+$.

We introduce now cardinal invariants that measure respectively the number of pairwise disjoint strings, pairwise disjoint infinite orbits and pairwise disjoint ladders of a function. The first of them was already introduced in [AADGH].

Definition 3.2. Let Γ be a set and $\lambda : \Gamma \to \Gamma$ a function. Then let:

(a) $s(\lambda) = \sup\{|\mathcal{F}| : \mathcal{F}$ is a family of pairwise disjoint strings in $\Gamma\}$;

(b) $o(\lambda) = \sup\{|\mathcal{F}| : \mathcal{F}$ is a family of pairwise disjoint infinite orbits in $\Gamma\}$;

(c) $l(\lambda) = \sup\{|\mathcal{F}| : \mathcal{F}$ is a family of pairwise disjoint ladders in $\Gamma\}$;

(d) $p(\lambda) = \sup\{|\mathcal{F}| : \mathcal{F}$ is a family of pairwise disjoint periodic ladders in $\Gamma\}$.

The existence of a ladder of λ in Γ is equivalent to the existence of infinitely many pairwise disjoint ladders of λ in Γ; in other words $l(\lambda) > 0$ implies $l(\lambda) \geq \omega$. Analogously, $p(\lambda) > 0$ yields $p(\lambda) \geq \omega$.

The next result gives five equivalent characterizations of a bounded function. In particular, it shows that bounded is the same as quasi-periodic for a function; that a function is bounded if and only if it has no strings, no infinite orbits, no ladders; and the period of the periodic points is bounded by a fixed $N \in \mathbb{N}_+$.

Theorem 3.3. *Let Γ be a set and $\lambda : \Gamma \to \Gamma$ a function. The following conditions are equivalent:*

(a) λ is bounded;

(b) $s(\lambda) = o(\lambda) = l(\lambda) = p(\lambda) = 0$;

(c) there exists $N \in \mathbb{N}_+$ such that $\lambda^{-N}(\Gamma \setminus \mathrm{Per}(\lambda)) = \emptyset$ and $\mathrm{Per}(\lambda) = \mathrm{Per}_N(\lambda)$;

(d) there exists $N \in \mathbb{N}_+$ such that $\lambda^N(\Gamma) = \mathrm{Per}(\lambda)$ and $\mathrm{Per}(\lambda) = \mathrm{Per}_N(\lambda)$;

(e) λ is quasi-periodic.

Proof. (a)⇒(b) If either $s(\lambda) > 0$, or $o(\lambda) > 0$, then there exists either a string or an infinite orbit of λ in Γ, both of which are infinite totally preordered subsets of $(\Gamma, \leq_\lambda)$, and so λ is not bounded. If $l(\lambda) > 0$, then there exists a ladder $L = \bigcup_{m\in\mathbb{N}} L_m$ of λ in Γ; in particular, for every $m \in \mathbb{N}$ the set L_m is a totally ordered subset of $(\Gamma, \leq_\lambda)$ of size $> m$, and hence λ is not bounded. If $p(\lambda) > 0$, then there exists a periodic ladder $P = \bigcup_{n\in\mathbb{N}_+} P_n$; each P_n is a totally preordered subset of $(\Gamma, \leq_\lambda)$ of size $\geq n$, and so λ is not bounded.

(b)⇒(c) Suppose that for every $n \in \mathbb{N}_+$ there exists $i_n \in \Gamma\setminus\mathrm{Per}(\lambda)$ such that $\lambda^{-n}(i_n)$ is not empty. Since $p(\lambda) = 0$ is equivalent to $\mathrm{Per}(\lambda) = \mathrm{Per}_N(\lambda)$ for some $N \in \mathbb{N}_+$, and since $s(\lambda) = o(\lambda) = 0$, we have to verify that $l(\lambda) > 0$. To this end we construct a ladder of λ in Γ.

First note that, given $i \in \Gamma \setminus \mathrm{Per}(\lambda)$, since $s(\lambda) = o(\lambda) = 0$, there exist $n, m \in \mathbb{N}_+$ such that $\lambda^n(i) \in \mathrm{Per}(\lambda)$ and $\lambda^{-m}(i) = \emptyset$. So we can suppose without loss of generality (i.e., taking $\lambda^{n-1}(i)$ instead of i) that $\lambda(i) \in \mathrm{Per}(\lambda)$ and $\lambda^{-m}(i) = \emptyset$ for some $m \in \mathbb{N}_+$.

So let $l_0 \in \Gamma \setminus \mathrm{Per}(\lambda)$ be such that $\lambda(l_0) \in \mathrm{Per}(\lambda)$ and $\lambda^{-b_0-1}(l_0) = \emptyset$ where b_0 is the minimum natural number with this property. Then pick $l_{0,b_0} \in \lambda^{-b_0}(l_0)$, let $l_{0,b_0-k} = \lambda^k(l_{0,b_0})$ for every $k \in \{0, \ldots, b_0\}$ (in particular $l_{0,0} = l_0$) and define

$$L_0 = \{l_{0,0}, l_{0,1}, \ldots, l_{0,b_0}\}.$$

By our assumption there exists $l_1 \in \Gamma\setminus\mathrm{Per}(\lambda)$ such that $\lambda(l_1) \in \mathrm{Per}(\lambda)$ and $\lambda^{-b_0-1}(l_1) \neq \emptyset$. Let b_1 be the minimum natural number such that $\lambda^{-b_1-1}(l_1) = \emptyset$; in particular $b_1 > b_0$. Pick $l_{1,b_1} \in \lambda^{-b_1}(l_1)$, let $l_{1,b_1-k} = \lambda^k(l_{1,b_1})$ for every $k \in \{0, \ldots, b_1\}$ (in particular $l_{1,0} = l_1$) and define

$$L_1 = \{l_{1,0}, l_{1,1}, \ldots, l_{1,b_1}\}.$$

Proceeding by induction in this way, for every $m \in \mathbb{N}$ we have $l_m \in \Gamma\setminus\mathrm{Per}(\lambda)$ such that $\lambda(l_m) \in \mathrm{Per}(\lambda)$ and there exists a minimum natural number b_m such that $\lambda^{-b_m-1}(l_m) = \emptyset$, and $b_m > b_n$ for every $n < m$ in $\mathbb{N}$. Moreover,

$$L_m = \{l_{m,0}, l_{m,1}, \ldots, l_{m,b_m}\},$$

where $l_{m,b_m-k} = \lambda^k(l_{m,b_m})$ for every $k \in \{0, \ldots, b_m\}$ and in particular $l_{m,0} = l_m$.

By construction, for each $m \in \mathbb{N}$ we have $L_m \subseteq \Gamma \setminus \mathrm{Per}(\lambda)$, $\lambda^{-1}(l_{m,b_m}) = \emptyset$ and $b_s < b_t$ for every $s < t$ in $\mathbb{N}$. Let $L = \bigcup_{m\in\mathbb{N}} L_m$. To prove that this is a ladder it remains to verify that $L_s \cap L_t = \emptyset$ in case $s \neq t$ in $\mathbb{N}$. So let $s \leq t$ in $\mathbb{N}$ and suppose that $L_s \cap L_t$ is not empty. This means that there exist $l_{s,v} \in L_s$ and $l_{t,w} \in L_t$ such that $l_{s,v} = l_{t,w}$. If $v \leq w$, this implies $l_{s,0} = l_{t,w-v}$; since $\lambda(l_{t,w-v}) = \lambda(l_{s,0}) \in \mathrm{Per}(\lambda)$, this equality is possible only if $w - v = 0$, that is $v = w$. The same conclusion holds assuming $w \leq v$. Then $l_{s,0} = l_{t,0}$. Since $\lambda^{-b_s-1}(l_{t,0}) = \lambda^{-b_s-1}(l_{s,0}) = \emptyset$, it follows that $t \leq s$. Hence $s = t$, and $L_s = L_t$.

(c)⇒(a) Assume that there exists $N \in \mathbb{N}_+$ such that $\lambda^{-N}(\Gamma \setminus \mathrm{Per}(\lambda)) = \emptyset$ and $\mathrm{Per}(\lambda) = \mathrm{Per}_N(\lambda)$. Let I be a totally preordered subset of $(\Gamma, \leq_\lambda)$. Since $I\setminus\mathrm{Per}(\lambda)$ is totally ordered and $\lambda^{-N}(I\setminus\mathrm{Per}(\lambda)) = \emptyset$, so $I \setminus \mathrm{Per}(\lambda)$ has size at most N; moreover, $I \cap \mathrm{Per}(\lambda)$, being totally preordered, is contained in an orbit of size at most N. It follows that $|I| \leq 2N$. This proves that λ is bounded.

(c)⇔(d)⇔(e) are obvious.

□

4 Quasi-periodicity coincides with local quasi-periodicity for the generalized shifts

In this section we prove in Theorem 4.2 the equivalence of local quasi-periodicity and quasi-periodicity for a generalized shift σ_λ, and that these conditions are equivalent also to the boundedness of λ (i.e., the quasi-periodicity of λ in view of Theorem 3.3). The non-trivial part in this proof is to find a non-quasi-periodic point of σ_λ under the assumption that λ is not bounded.

For a set Γ and an abelian group K, the *diagonal* subgroup ΔK^Γ of K^Γ is $\Delta K^\Gamma = \{x = (x_i)_{i\in\Gamma} :$ for some $a \in K,\ x_i = a$ for every $i \in \Gamma\}$.

Lemma 4.1. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial abelian group, and $\sigma_\lambda : K^\Gamma \to K^\Gamma$ the generalized shift.*

(a) If λ has a ladder L in Γ, then there exists $x \in K^L$ which is not quasi-periodic for σ_λ. So $\mathrm{ent}(\sigma_\lambda) > 0$.

(b) if λ has a periodic ladder P in Γ, then there exists $x \in K^P$ which is not quasi-periodic for σ_λ. So $\mathrm{ent}(\sigma_\lambda) > 0$.

Proof. (a) Let $L = \bigcup_{m\in\mathbb{N}} L_m \subseteq \Gamma \setminus \mathrm{Per}(\lambda)$, where each $L_m = \{l_{m,0}, \dots, l_{m,b_m}\}$. Consider where $B = \{l_{m,0} : m \in \mathbb{N}\}$ and let x be a non-zero element of ΔK^B. We show that x is not quasi-periodic for σ_λ. To this end let $s < t$ in $\mathbb{N}$. By Proposition 2.3(c) $\mathrm{supp}(\sigma_\lambda^s(x)) = \lambda^{-s}(B)$ and $\mathrm{supp}(\sigma_\lambda^t(x)) = \lambda^{-t}(B)$. Then $\mathrm{supp}(\sigma_\lambda^s(x)) \cap L = \{l_{m,s} : m \in \mathbb{N}, m \geq s\}$ and $\mathrm{supp}(\sigma_\lambda^t(x)) \cap L = \{l_{m,t} : m \in \mathbb{N}, m \geq t\}$. By the definition of ladder the latter two sets have trivial intersection as $s < t$. So, since $\sigma_\lambda^s(x) \in \Delta K^{\mathrm{supp}(\sigma_\lambda^s(x))}$ and $\sigma_\lambda^t(x) \in \Delta K^{\mathrm{supp}(\sigma_\lambda^t(x))}$, it follows that $\sigma_\lambda^s(x) \neq \sigma_\lambda^t(x)$. This proves that x is not quasi-periodic.

(b) Let $P = \bigcup_{n\in\mathbb{N}_+} P_n$. By definition, for every $n \in \mathbb{N}_+$ we have $P_n \supseteq \lambda^{-1}(P_n) \cup \lambda(P_n)$. By Lemma 2.4(a) $\sigma_\lambda \restriction_{K^{P_n}} = \sigma_{\lambda_n}$, where $\lambda_n = \lambda \restriction_{P_n}$ for every $n \in \mathbb{N}_+$. Moreover, $K^P \cong \prod_{n\in\mathbb{N}_+} K^{P_n}$, and so $\sigma_\lambda \restriction_{K^P} = (\sigma_{\lambda_n})_{n\in\mathbb{N}_+}$.

For every $n \in \mathbb{N}_+$ let $x_n = (x_{n,s})_{s\in P_n} \in K^{P_n}$ be such that $x_{n,s_n} \neq 0$ for one and only one $s_n \in P_n$, that is, $\mathrm{supp}(x_n) = \{s_n\}$. Let $x = (x_n)_{n\in\mathbb{N}_+} \in \prod_{n\in\mathbb{N}_+} K^{P_n}$. We show that x is not quasi-periodic for σ_λ. Let $s < t$ in $\mathbb{N}$. We have to verify that $\sigma_\lambda^s(x) \neq \sigma_\lambda^t(x)$. Since $\sigma_\lambda^s(x) = (\sigma_{\lambda_n}^s(x_n))_{n\in\mathbb{N}_+}$ and $\sigma_\lambda^t(x) = (\sigma_{\lambda_n}^t(x_n))_{n\in\mathbb{N}_+}$, it suffices to show that there exists $n \in \mathbb{N}_+$ such that $\sigma_{\lambda_n}^s(x_n) \neq \sigma_{\lambda_n}^t(x_n)$. Take for example $n \in \mathbb{N}_+$ such that $|P_n| > t$. Then $\lambda_n^{-s}(s_n) \neq \lambda_n^{-t}(s_n)$ in view of the hypothesis that λ_n is a cycle of length $|P_n| > t > s$, and by Proposition 2.3(c) $\mathrm{supp}(\sigma_{\lambda_n}^s(x_n)) = \{\lambda_n^{-s}(s_n)\} \neq \{\lambda^{-t}(s_n)\} = \mathrm{supp}(\sigma_{\lambda_n}^t(s_n))$. Hence $\sigma_\lambda^s(x) \neq \sigma_\lambda^t(x)$.

In both (a) and (b) the existence of a non-quasi-periodic point of σ_λ implies $\mathrm{ent}(\sigma_\lambda) > 0$ in view of Proposition 2.2(a). □

Theorem 4.2. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial abelian group, and $\sigma_\lambda : K^\Gamma \to K^\Gamma$ the generalized shift. The following conditions are equivalent:*

(a) λ is bounded;

(b) σ_λ is quasi-periodic;

(c) σ_λ is locally quasi-periodic.

Proof. (a)⇔(b) By Theorem 3.3 λ is bounded if and only if λ is quasi-periodic. Then apply Proposition 2.5 to conclude that λ quasi-periodic is equivalent to σ_λ quasi-periodic.

(b)⇒(c) is obvious.

(c)⇒(a) We verify that in case λ is not bounded, then σ_λ is not locally quasi-periodic, that is, there exists $x \in K^\Gamma$ which is not quasi-periodic. By Theorem 3.3 λ non-bounded means that one of $s(\lambda)$, $o(\lambda)$, $l(\lambda)$, $p(\lambda)$ is non-zero.

Let

$$N_1 = \{n! : n \in \mathbb{N}_+\} \subseteq \mathbb{N}$$

and for every $k \in \mathbb{N}$ let

$$N_1 + k = \{n + k : n \in N_1\} \text{ and } N_1 - k = \{n - k : n \in N_1, n > k\}.$$

Suppose that $s(\lambda) > 0$. Then there exists a string $S = \{s_t\}_{t\in\mathbb{N}}$ of λ in Γ. For $k \in \mathbb{N}$ define

$$S_{1,k} = \{s_n : n \in N_1 + k\} \subseteq S.$$

Let x be a non-zero element of $\Delta K^{S_{1,0}}$. We verify that x is not quasi-periodic for σ_λ. To this aim, let $s < t$ in $\mathbb{N}$. By Proposition 2.3(c) $\mathrm{supp}(\sigma_\lambda^s(x)) = \lambda^{-s}(S_{1,0})$ and $\mathrm{supp}(\sigma_\lambda^t(x)) = \lambda^{-t}(S_{1,0})$. Then $\mathrm{supp}(\sigma_\lambda^s(x)) \cap S = S_{1,s}$ and $\mathrm{supp}(\sigma_\lambda^t(x)) \cap S = S_{1,t}$. In particular, $S_{1,s} \neq S_{1,t}$ because $N_1 + s \neq N_1 + t$, and so $\mathrm{supp}(\sigma_\lambda^s(x)) \neq \mathrm{supp}(\sigma_\lambda^t(x))$. Since $\sigma_\lambda^s(x)$ and $\sigma_\lambda^t(x)$ are elements respectively of $\Delta K^{\mathrm{supp}(\sigma_\lambda^s(x))}$ and $\Delta K^{\mathrm{supp}(\sigma_\lambda^t(x))}$, it follows that $\sigma_\lambda^s(x) \neq \sigma_\lambda^t(x)$.

Suppose that $o(\lambda) > 0$. Then there exists an infinite orbit $A = \{a_t\}_{t\in\mathbb{N}}$ of λ in Γ. For $k \in \mathbb{N}$ define

$$A_{1,k} = \{a_n : n \in N_1 - k\} \subseteq A.$$

Let x be a non-zero element of $\Delta K^{A_{1,0}}$. We verify that x is not quasi-periodic for σ_λ. To this aim, let $s < t$ in $\mathbb{N}$. By Proposition 2.3(c) $\mathrm{supp}(\sigma_\lambda^s(x)) = \lambda^{-s}(A_{1,0})$ and $\mathrm{supp}(\sigma_\lambda^t(x)) = \lambda^{-t}(A_{1,0})$. Then $\mathrm{supp}(\sigma_\lambda^s(x)) \cap A = A_{1,s}$ and $\mathrm{supp}(\sigma_\lambda^t(x)) \cap A = A_{1,t}$. In particular, $A_{1,s} \neq A_{1,t}$ because $N_1 - s \neq N_1 - t$, and so $\mathrm{supp}(\sigma_\lambda^s(x)) \neq \mathrm{supp}(\sigma_\lambda^t(x))$. Since $\sigma_\lambda^s(x)$ and $\sigma_\lambda^t(x)$ are elements respectively of $\Delta K^{\mathrm{supp}(\sigma_\lambda^s(x))}$ and $\Delta K^{\mathrm{supp}(\sigma_\lambda^t(x))}$, it follows that $\sigma_\lambda^s(x) \neq \sigma_\lambda^t(x)$.

If $l(\lambda) > 0$, apply Lemma 4.1(a), and if $p(\lambda) > 0$, apply Lemma 4.1(b). □

According to [DGSZ], a function $f : X \to X$ is *strongly recurrent* if it is locally periodic. In [DGSZ] an analogue of the Poincaré – Birkhoff recurrence theorem of ergodic theory was proved:

> *For ϕ a monomorphism of a torsion abelian group, ϕ is locally periodic (i.e., strongly recurrent) if and only if* $\mathrm{ent}(\phi) = 0$.

Similarly to the situation in Theorem 1.3, for injective generalized shifts σ_λ the "local" condition becomes "global":

Corollary 4.3. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial finite abelian group and $\sigma_\lambda : K^\Gamma \to K^\Gamma$ an injective generalized shift. Then the following conditions are equivalent:*

(a) σ_λ is locally periodic (i.e., strongly recurrent);

(b) σ_λ is periodic;

(c) $\mathrm{ent}(\sigma_\lambda) = 0$.

Proof. (a)⇔(c) was proved in [DGSZ], and (b)⇒(a) is clear.

(c)⇒(b) Assume that $\mathrm{ent}(\sigma_\lambda) = 0$. By Proposition 2.2(a) σ_λ is locally quasi-periodic and by Theorem 4.2 σ_λ is quasi-periodic. Since it is injective, σ_λ is periodic. □

5 Independent subgroups of $K^{\mathbb{N}}$

Let us give the following definition, which will help in explaining the content of this section.

Definition 5.1. Let G be an abelian group. A family $\{H_i : i \in I\}$ of subgroups of G is *independent* if for any finite subset $J = \{j_1, \ldots, j_n\}$ of I and any $j_0 \in I \setminus J$ then $H_{j_0} \cap (H_{j_1} + \ldots + H_{j_n}) = \{0\}$.

In particular, the H_i's in this definition are pairwise with trivial intersection. Observe that a family $\{H_n : n \in \mathbb{N}\}$ of subgroups of G is independent if and only if $H_{n+1} \cap (H_0 + \ldots + H_n) = \{0\}$ for every $n \in \mathbb{N}$.

The subsets $N_1 + k$ and $N_1 - k$ of $\mathbb{N}$ in the proof of Theorem 4.2 help in finding a non-quasi-periodic point of σ_λ when λ admits either a string or an infinite orbit. By Proposition 2.2(a) this is equivalent to say that $\mathrm{ent}(\sigma_\lambda) > 0$. But to prove Theorem 1.3 we have to show that this entropy is infinite and so we have to improve the use of the subsets $N_1 + k$ and $N_1 - k$ of $\mathbb{N}$.

With this aim, we consider in this section similar subsets of $\mathbb{N}$ defined through the use of the factorial of natural numbers. The properties of these subsets help in finding in Lemmas 5.5 and 5.6 specific independent families of finite subgroups of $K^{\mathbb{N}}$. These subgroups are "sufficiently many" with respect to the calculation of the algebraic entropy of a generalized shift σ_λ and are useful to prove Lemma 6.3, in which we see that the algebraic entropy of σ_λ is infinite in case λ admits either a string or an infinite orbit.

For every $m, n \in \mathbb{N}_+$, let

$$n!^{(m)} = n\underbrace{!\ldots!}_{m} \text{ and } N_m = \{n!^{(m)} : n \in \mathbb{N}_+\}.$$

These subsets of $\mathbb{N}$ form a (rapidly) strictly decreasing sequence

$$N_1 \supset N_2 \supset \ldots \supset N_m \supset N_{m+1} \supset \ldots;$$

indeed, $N_m \setminus N_{m+1}$ is infinite for every $m \in \mathbb{N}_+$. For $m \in \mathbb{N}_+$ and $k \in \mathbb{N}$ let

$$N_m + k = \{n + k : n \in N_m\} \text{ and } N_m - k = \{n - k : n \in N_m,\ n > k\}. \tag{5.1}$$

We collect here some useful properties of these subsets N_m of $\mathbb{N}$.

Lemma 5.2. *For every $m, k \in \mathbb{N}_+$,*

(a) $(N_m + k) \setminus (N_{m+1} + k) \not\subseteq N_1 \cup (N_1 + 1) \cup \ldots \cup (N_1 + (k-1))$, and

(b) $(N_m - k) \setminus (N_{m+1} - k) \not\subseteq N_1 \cup (N_1 - 1) \cup \ldots \cup (N_1 - (k-1))$.

Proof. (a) Let $m, k \in \mathbb{N}_+$. We have to prove that there exists $n_0 \in \mathbb{N}_+ \setminus N_1$ such that $n_0!^{(m)} + k \neq n! + h$ (i.e., $n_0!^{(m)} + (k-h) \neq n!$) for every $n \in \mathbb{N}_+$ and $h \in \{0, \ldots, k-1\}$.

Pick $n_0 \in \mathbb{N}_+$ such that $k < M \cdot M!$, where $M = n_0!^{(m-1)}$ and so $M! = n_0!^{(m)}$ (it suffices for example that $n_0 > k$). In particular $n_0 > 1$ and for every $h \in \{0, \ldots, k-1\}$

$$k - h < M \cdot M!.$$

Consequently,

$$M! + (k-h) < M! + M \cdot M! = (M+1)!.$$

Then for every $h \in \{0, \ldots, k-1\}$,

$$M! < M! + (k-h) < (M+1)!. \tag{5.2}$$

Since $(M+1)!$ is the smallest factorial bigger than $M!$, it follows that $M! + (k-h) \neq n!$ for every $n \in \mathbb{N}$.

If $n_0 = n_1!$ for some $n_1 \in \mathbb{N}_+$ (i.e., $n_0 \in N_1$), then take $n_0 + 1$ and $M = (n_0+1)!^{(m-1)}$, which satisfies the same condition (5.2) but $n_0 + 1 \notin N_1$.

(b) Let $m, k \in \mathbb{N}_+$. We have to prove that there exists $n_0 \in \mathbb{N}_+ \setminus N_1$ such that $n_0!^{(m)} - k \neq n! - h$ (i.e., $n_0!^{(m)} - (k-h) \neq n!$) for every $n \in \mathbb{N}_+$ and $h \in \{0, \ldots, k-1\}$.

Pick $n_0 \in \mathbb{N}_+$ such that $k < (M-1) \cdot (M-1)!$, where as before $M = n_0!^{(m-1)}$ and so $M! = n_0!^{(m)}$ (it suffices for example that $n_0 > k$). In particular $n_0 > 1$ and for every $h \in \{0, \ldots, k-1\}$ $k - h < (M-1) \cdot (M-1)!$, that is,

$$-(k-h) > (M-1) \cdot (M-1)!.$$

Consequently,

$$(M-1)! = M! - (M-1) \cdot (M-1)! < M! - (k-h).$$

Then, for every $h \in \{0, \ldots, k-1\}$,

$$(M-1)! < M! - (k-h) < M!. \tag{5.3}$$

Since $(M-1)!$ is the biggest factorial smaller than $M!$, it follows that $M! - (k-h) \neq n!$ for every $n \in \mathbb{N}$.

If $n_0 = n_1!$ for some $n_1 \in \mathbb{N}_+$ (i.e., $n_0 \in N_1$), then take $n_0 + 1$ and $M = (n_0+1)!^{(m-1)}$, which satisfies the same condition (5.3) and $n_0 + 1 \notin N_1$. □

In particular, it follows from this lemma that for every $m, k \in \mathbb{N}_+$,

$$N_m + k \not\subseteq N_1 \cup (N_1 + 1) \cup \ldots \cup (N_1 + (k-1)), \text{ and}$$
$$N_m - k \not\subseteq N_1 \cup (N_1 - 1) \cup \ldots \cup (N_1 - (k-1)).$$

Remark 5.3. Consider the group $K^{\mathbb{N}}$, where K is a non-trivial finite abelian group. Let $t \in \mathbb{N}_+$ and $k \in \mathbb{Z}$. If $x \in \Delta K^{N_1+k} + \ldots + \Delta K^{N_t+k}$ then $\operatorname{supp}(x) = Q_1 \dot\cup \ldots \dot\cup Q_t$, where

$$Q_1 = \begin{cases} \text{either} & (N_1+k) \setminus (N_2+k) \\ \text{or} & \emptyset \end{cases},$$
$$\vdots$$
$$Q_{t-1} = \begin{cases} \text{either} & (N_{t-1}+k) \setminus (N_t+k) \\ \text{or} & \emptyset \end{cases},$$
$$Q_t = \begin{cases} \text{either} & N_t+k \\ \text{or} & \emptyset \end{cases}.$$

In particular, if $\operatorname{supp}(x) \cap (N_t+k)$ is not empty, then Q_t is not empty. Therefore $Q_t = N_t+k$, and hence $\operatorname{supp}(x) \supseteq N_t+k$.

Lemma 5.4. *Let $t \in \mathbb{N}_+$, $k \in \mathbb{Z}$ and let $x \in \Delta K^{N_1+k} + \ldots + \Delta K^{N_t+k}$. If $\operatorname{supp}(x) \subsetneq N_t+k$, then $x = 0$.*

Proof. By Remark 5.3, if $\operatorname{supp}(x) \cap (N_t+k) \neq \emptyset$, it follows that $\operatorname{supp}(x) \supseteq N_t+k$. Then $\operatorname{supp}(x) = \operatorname{supp}(x) \cap (N_t+k) = \emptyset$, that is, $x = 0$. □

The following result shows that for every $k \in \mathbb{N}$ the family $\{\Delta K^{N_t+k} : t \in \mathbb{N}_+\}$ of finite subgroups of $K^{\mathbb{N}}$ is independent.

Lemma 5.5. *Consider the group $K^{\mathbb{N}}$, where K is a non-trivial finite abelian group. If $k \in \mathbb{N}$ is fixed, then for every $t \in \mathbb{N}_+$,*

(a) $\Delta K^{N_1+k} + \ldots + \Delta K^{N_t+k} = \Delta K^{N_1+k} \oplus \ldots \oplus \Delta K^{N_t+k}$;

(b) $\Delta K^{N_1-k} + \ldots + \Delta K^{N_t-k} = \Delta K^{N_1-k} \oplus \ldots \oplus \Delta K^{N_t-k}$.

Proof. (a) We proceed by induction. Let $t = 2$. Since $N_1+k \supsetneq N_2+k$, it follows that $\Delta K^{N_1+k} \cap \Delta K^{N_2+k} = \{0\}$. Assume now that for $t \geq 2$, $\Delta K^{N_1+k} + \ldots + \Delta K^{N_t+k} = \Delta K^{N_1+k} \oplus \ldots \oplus \Delta K^{N_t+k}$; we prove that

$$(\Delta K^{N_1+k} \oplus \ldots \oplus \Delta K^{N_t+k}) \cap \Delta K^{N_{t+1}+k} = \{0\}.$$

To this end let $x \in \Delta K^{N_{t+1}+k}$. Then $\operatorname{supp}(x)$ is either empty or $N_{t+1}+k$. Since $N_{t+1}+k \subsetneq N_t+k$, and in particular $\operatorname{supp}(x) \subsetneq N_t+k$, by Lemma 5.4 $x \in \Delta K^{N_1+k} \oplus \ldots \oplus \Delta K^{N_t+k}$ yields $x = 0$. This concludes the proof.

(b) is analogous to (a). □

Lemma 5.6. *Consider $K^{\mathbb{N}}$, where K is a non-trivial finite abelian group. For $t \in \mathbb{N}_+$, $l \in \mathbb{Z}$, let $\Delta_{t,l} = \Delta K^{N_1+l} \oplus \ldots \oplus \Delta K^{N_t+l}$. Then, for a fixed $t \in \mathbb{N}_+$, and for every $k \in \mathbb{N}$,*

(a) $\Delta_{t,0} + \Delta_{t,1} + \ldots + \Delta_{t,k} = \Delta_{t,0} \oplus \Delta_{t,1} \oplus \ldots \oplus \Delta_{t,k}$*; and*

(b) $\Delta_{t,0} + \Delta_{t,-1} + \ldots + \Delta_{t,-k} = \Delta_{t,0} \oplus \Delta_{t,-1} \oplus \ldots \oplus \Delta_{t,-k}$.

Proof. (a) We proceed by induction. For $k = 1$, we have to prove that $\Delta_{t,0} \cap \Delta_{t,1} = \{0\}$. Assume that $x \in \Delta_{t,1} = \Delta K^{N_1+1} \oplus \ldots \oplus \Delta K^{N_t+1}$. By Remark 5.3 $\operatorname{supp}(x) = Q_1 \dot\cup \ldots \dot\cup Q_t$, where

$$Q_1 = \begin{cases} \text{either} & (N_1+1) \setminus (N_2+1) \\ \text{or} & \emptyset \end{cases},$$
$$\vdots$$
$$Q_{t-1} = \begin{cases} \text{either} & (N_{t-1}+1) \setminus (N_t+1) \\ \text{or} & \emptyset \end{cases},$$
$$Q_t = \begin{cases} \text{either} & N_t+1 \\ \text{or} & \emptyset \end{cases}.$$

If also $x \in \Delta_{t,0} = \Delta K^{N_1} \oplus \ldots \oplus \Delta K^{N_t}$, then $\mathrm{supp}(x) \subseteq N_1$ and so, by Lemma 5.2(a), $Q_i = \emptyset$ for every $i \in \{1, \ldots, t\}$, that is $x = 0$.

Suppose now that $k \geq 2$ and that $\Delta_{t,0} + \Delta_{t,1} + \ldots + \Delta_{t,k} = \Delta_{t,0} \oplus \Delta_{t,1} \oplus \ldots \oplus \Delta_{t,k}$. We have to prove that $(\Delta_{t,0} \oplus \Delta_{t,1} \oplus \ldots \oplus \Delta_{t,k}) \cap \Delta_{t,k+1} = \{0\}$. Let $x \in \Delta_{t,k+1} = \Delta K^{N_1+(k+1)} \oplus \ldots \oplus \Delta K^{N_t+(k+1)}$. Then $\mathrm{supp}(x) = Q_1 \dot\cup \ldots \dot\cup Q_t$, where

$$Q_1 = \begin{cases} \text{either} & (N_1 + (k+1)) \setminus (N_2 + (k+1)) \\ \text{or} & \emptyset \end{cases},$$

$$\vdots$$

$$Q_{t-1} = \begin{cases} \text{either} & (N_{t-1} + (k+1)) \setminus (N_t + (k+1)) \\ \text{or} & \emptyset \end{cases},$$

$$Q_t = \begin{cases} \text{either} & N_t + (k+1) \\ \text{or} & \emptyset \end{cases}.$$

If also $x \in \Delta_{t,0} \oplus \Delta_{t,1} \oplus \ldots \oplus \Delta_{t,k} = (\Delta K^{N_1} \oplus \ldots \oplus \Delta K^{N_t}) \oplus (\Delta K^{N_1+1} \oplus \ldots \oplus \Delta K^{N_t+1}) \oplus \ldots \oplus (\Delta K^{N_1+k} \oplus \ldots \oplus \Delta K^{N_t+k})$, then $\mathrm{supp}(x) \subseteq N_1 \cup (N_1+1) \cup \ldots \cup N_1 + k$ and so, by Lemma 5.2(a), $Q_i = \emptyset$ for every $i \in \{1, \ldots, t\}$, that is $x = 0$. This concludes the proof.

(b) is analogous to (a). □

This proves that for every $t \in \mathbb{N}_+$ the families $\{\Delta_{t,k} : k \in \mathbb{N}\}$ and $\{\Delta_{t,-k} : k \in \mathbb{N}\}$ of finite subgroups of $K^{\mathbb{N}}$ are independent.

6 Proof of Theorem 1.3

In [AADGH] the algebraic entropy of a generalized shift $\sigma_\lambda : K^\Gamma \to K^\Gamma$ restricted to the direct sum $\bigoplus_\Gamma K$ was computed precisely; we recall this result in Theorem 6.1 below. As noted in the introduction, in this case we have to require that λ has finite fibers, because this is equivalent to $\bigoplus_\Gamma K$ being a σ_λ-invariant subgroup of K^Γ.

In (6.1) below the algebraic entropy of $\sigma_\lambda \restriction_{\bigoplus_\Gamma K}$ is expressed as the product of the string number $s(\lambda)$ of λ with the logarithm of the cardinality of the finite abelian group K. But while the algebraic entropy $\mathrm{ent}(-)$ is either a real number or the symbol ∞, the string number $s(-)$ is either a finite natural number or an infinite cardinal. Then for a self-map $\lambda : \Gamma \to \Gamma$ we introduce $s(\lambda)^*$ defined by $s(\lambda)^* = s(\lambda)$ if $s(\lambda)$ is finite and $s(\lambda)^* = \infty$ in case $s(\lambda)$ is infinite.

Theorem 6.1. [AADGH, Theorem 4.14] *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function such that $\lambda^{-1}(i)$ is finite for every $i \in \Gamma$, and K a non-trivial finite abelian group. Then*

$$\mathrm{ent}(\sigma_\lambda \restriction_{\bigoplus_\Gamma K}) = s(\lambda)^* \cdot \log |K|. \tag{6.1}$$

This theorem gives the idea of using strings also in the case of the calculation of the algebraic entropy of $\sigma_\lambda : K^\Gamma \to K^\Gamma$. Moreover, one of the main tools in proving this theorem was Remark 4.8 in [AADGH]; the following proposition is its counterpart for $\sigma_\lambda : K^\Gamma \to K^\Gamma$.

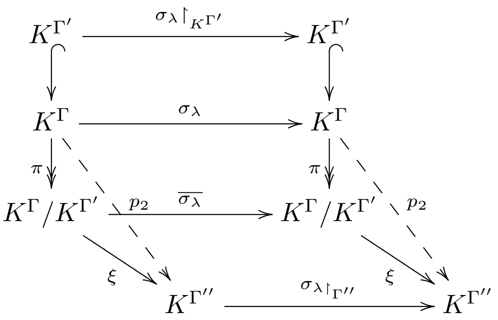
Proposition 6.2. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, and K a non-trivial finite abelian group. Suppose that $\Gamma = \Gamma' \cup \Gamma''$ a partition of Γ and that $\lambda^{-1}(\Gamma') \subseteq \Gamma'$ (i.e., $\lambda(\Gamma'') \subseteq \Gamma''$). Then*

$$\mathrm{ent}(\sigma_\lambda) = \mathrm{ent}(\sigma_\lambda \restriction_{K^{\Gamma'}}) + \mathrm{ent}(\sigma_{\lambda\restriction_{\Gamma''}}).$$

In particular, if Λ is a λ-invariant subset of Γ, then $\mathrm{ent}(\sigma_\lambda) \geq \mathrm{ent}(\sigma_{\lambda\restriction_\Lambda})$.

Proof. By Lemma 2.4(a) $K^{\Gamma'}$ is σ_λ-invariant. Moreover, it is possible to consider $\lambda \restriction_{\Gamma''}: \Gamma'' \to \Gamma''$. Let $p_2 : K^\Gamma = K^{\Gamma'} \oplus K^{\Gamma''} \to K^{\Gamma''}$ and $\pi : K^\Gamma \to K^\Gamma / K^{\Gamma'}$ be the canonical projections. Denote by $\xi : K^\Gamma / K^{\Gamma'} \to K^{\Gamma''}$ the (unique) isomorphism such that $p_2 = \xi \circ \pi$. Finally, let $\overline{\sigma_\lambda} : K^\Gamma / K^{\Gamma'} \to K^\Gamma / K^{\Gamma'}$

be the homomorphism induced by σ_λ. Then $\overline{\sigma_\lambda} = \xi^{-1}\sigma_{\lambda\restriction_{\Gamma''}}\xi$. The following diagram explains the situation.



By Proposition 2.2(b) $\mathrm{ent}(\overline{\sigma_\lambda}) = \mathrm{ent}(\sigma_{\lambda\restriction_{\Gamma''}})$. Applying this equality and Theorem 2.1 we have the wanted equality. □

Applying this result in the following two lemmas, we see in particular that in case a function λ is not bounded, then the algebraic entropy of the generalized shift σ_λ is necessarily infinite.

The next proposition shows that the algebraic entropy of the generalized shift σ_λ is infinite in case λ admits either a string or an infinite orbit. The proofs of (a) and (b) are similar and in both we apply the technical lemmas of Section 5.

Lemma 6.3. *Let Γ be a set and $\lambda : \Gamma \to \Gamma$ a function. Let K be a non-trivial finite abelian group and consider the generalized shift $\sigma_\lambda : K^\Gamma \to K^\Gamma$.*

(a) If $s(\lambda) > 0$, then $\mathrm{ent}(\sigma_\lambda) = \infty$.

(b) If $o(\lambda) > 0$, then $\mathrm{ent}(\sigma_\lambda) = \infty$.

Proof. (a) Let $S = \{s_n : n \in \mathbb{N}\}$ be a string of λ in Γ; we can suppose without loss of generality that it is acyclic. Let $\Lambda = S \cup \{\lambda^n(s_0) : n \in \mathbb{N}_+\}$. Then $\lambda(\Lambda) \subseteq \Lambda$. So let $\psi = \lambda\restriction_\Lambda$. By Proposition 6.2 $\mathrm{ent}(\sigma_\lambda) \geq \mathrm{ent}(\sigma_\psi)$, where $\sigma_\psi : K^\Lambda \to K^\Lambda$, and so it suffices to prove that $\mathrm{ent}(\sigma_\psi) = \infty$. For $m \in \mathbb{N}_+$ and $k \in \mathbb{N}$ let

$$S_{m,k} = \{s_n : n \in N_m + k\}, \text{ where } N_m + k \text{ is defined in (5.1).}$$

Fix $t \in \mathbb{N}_+$, and let $F_t = \Delta K^{S_{1,0}} + \ldots + \Delta K^{S_{t,0}}$. For every $k \in \mathbb{N}$, by the definition of string and of ψ, and by Proposition 2.3(c), $\sigma_\psi^k(\Delta K^{S_{m,0}}) = \Delta K^{S_{m,k}}$ for every $m \in \mathbb{N}_+$, and so

$$\sigma_\psi^k(F_t) = \sigma_\psi^k(\Delta K^{S_{1,0}}) + \ldots + \sigma_\psi^k(\Delta K^{S_{t,0}}) = \Delta K^{S_{1,k}} + \ldots + \Delta K^{S_{t,k}}.$$

By Lemma 5.5(a) this sum is direct, that is, $\sigma_\psi^k(F_t) = \Delta K^{S_{1,k}} \oplus \ldots \oplus \Delta K^{S_{t,k}} \cong K^t$ for every $k \in \mathbb{N}$. By Lemma 5.6(a) for every $k \in \mathbb{N}_+$ the sum $T_k(\sigma_\psi, F_t) = F_t + \sigma_\psi(F_t) + \ldots + \sigma_\psi^{k-1}(F_t)$ is direct, that is,

$$T_k(\sigma_\psi, F_t) = F_t \oplus \sigma_\psi(F_t) \oplus \ldots \oplus \sigma_\psi^{k-1}(F_t) \cong K^{kt}.$$

Then $|T_k(\sigma_\psi, F_t)| = |K|^{kt}$ for every $k \in \mathbb{N}_+$ and so $H(\sigma_\psi, F_t) = t \log |K|$. Since this can be done for every $t \in \mathbb{N}_+$, it follows that $\mathrm{ent}(\sigma_\psi) = \infty$.

(b) Let $A = \{a_n : n \in \mathbb{N}\}$ be an infinite orbit of λ. Then $\lambda(A) \subseteq A$; so let $\alpha = \lambda\restriction_A$. By Proposition 6.2 $\mathrm{ent}(\sigma_\lambda) \geq \mathrm{ent}(\sigma_\alpha)$, where $\sigma_\alpha : K^A \to K^A$, and so it suffices to prove that $\mathrm{ent}(\sigma_\alpha) = \infty$. For $m \in \mathbb{N}_+$ and $k \in \mathbb{N}$ let

$$A_{m,k} = \{a_n : n \in N_m - k\}, \text{ where } N_m - k \text{ is defined in (5.1).}$$

Fix $t \in \mathbb{N}_+$, and let $F_t = \Delta K^{A_{1,0}} + \ldots + \Delta K^{A_{t,0}}$. For every $k \in \mathbb{N}$, by the definition of infinite orbit and of α, and by Proposition 2.3(c), $\sigma_\alpha^k(\Delta K^{A_{m,0}}) = \Delta K^{A_{m,k}}$ for every $m \in \mathbb{N}_+$, and so

$$\sigma_\alpha^k(F_t) = \sigma_\alpha^k(\Delta K^{A_{1,0}}) + \ldots + \sigma_\alpha^k(\Delta K^{A_{t,0}}) = \Delta K^{A_{1,k}} + \ldots + \Delta K^{A_{t,k}}.$$

By Lemma 5.5(b) this sum is direct, that is, $\sigma_\alpha^k(F_t) = \Delta K^{A_{1,k}} \oplus \ldots \oplus \Delta K^{A_{t,k}} \cong K^t$ for every $k \in \mathbb{N}$. By Lemma 5.6(a) for every $k \in \mathbb{N}_+$ the sum $T_k(\sigma_\alpha, F_t) = F_t + \sigma_\alpha(F_t) + \ldots + \sigma_\alpha^{k-1}(F_t)$ is direct, that is,

$$T_k(\sigma_\alpha, F_t) = F_t \oplus \sigma_\alpha(F_t) \oplus \ldots \oplus \sigma_\alpha^{k-1}(F_t) \cong K^{kt}.$$

Then $|T_k(\sigma_\alpha, F_t)| = |K|^{kt}$ for every $k \in \mathbb{N}_+$ and so $H(\sigma_\alpha, F_t) = t \log |K|$. Since this can be done for every $t \in \mathbb{N}_+$, it follows that $\mathrm{ent}(\sigma_\alpha) = \infty$. □

It is worthwhile noting that in [DGSZ] the algebraic entropy of the Bernoulli shifts restricted to the direct sums was calculated, and in [AADGH] it was described how the left Bernoulli shift ${}_K\beta$ and the two-sided Bernoulli shift $\overline{\beta}_K$ are generalized shifts, and how the right Bernoulli shift β_K can be "approximated" by a generalized shift with the same algebraic entropy:

Example 6.4. Let K be a non-trivial finite abelian group, and consider the Bernoulli shifts $\beta_K,\ {}_K\beta : K^{\mathbb{N}} \to K^{\mathbb{N}}$ and $\overline{\beta}_K : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ (defined in the introduction).

(a) Then:

(a_1) ${}_K\beta = \sigma_{\lambda_1}$, with $\lambda_1 : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto n+1$ for every $n \in \mathbb{N}$;

(a_2) $\overline{\beta}_K = \sigma_{\lambda_2}$, with $\lambda_2 : \mathbb{Z} \to \mathbb{Z}$ defined by $n \mapsto n-1$ for every $n \in \mathbb{Z}$;

(a_3) $\mathrm{ent}(\beta_K) = \mathrm{ent}(\sigma_{\lambda_3})$, where $\lambda_3 : \mathbb{N} \to \mathbb{N}$ is defined by $n \mapsto n-1$ for every $n \in \mathbb{N}_+$ and $0 \mapsto 0$, since $\beta_K \restriction_{K^{\mathbb{N}_+}} = \sigma_{\lambda_3} \restriction_{K^{\mathbb{N}_+}}$ and $K^{\mathbb{N}}/K^{\mathbb{N}_+} \cong K$ is finite — so it is possible to apply Theorem 2.1.

Note that $s(\lambda_1) = 0$ and $o(\lambda_1) = 1$, $s(\lambda_2) = o(\lambda_2) = 1$, $s(\lambda_3) = 1$ and $o(\lambda_3) = 0$.

(b) It can be seen as a consequence of item (a) and Theorem 6.1 that

$$\mathrm{ent}(\beta_K \restriction_{\bigoplus_{\mathbb{N}} K}) = \mathrm{ent}(\overline{\beta}_K \restriction_{\bigoplus_{\mathbb{Z}} K}) = \log |K|$$

and

$$\mathrm{ent}({}_K\beta \restriction_{\bigoplus_{\mathbb{N}} K}) = 0.$$

Lemma 6.3, together with this example, gives as a corollary the value of the algebraic entropy of the Bernoulli shifts considered on the direct products:

Corollary 6.5. *Let K be a non-trivial finite abelian group, and consider the Bernoulli shifts $\beta_K,\ {}_K\beta : K^{\mathbb{N}} \to K^{\mathbb{N}}$ and $\overline{\beta}_K : K^{\mathbb{Z}} \to K^{\mathbb{Z}}$. Then*

$$\mathrm{ent}(\beta_K) = \mathrm{ent}({}_K\beta) = \mathrm{ent}(\overline{\beta}_K) = \infty.$$

Now we show that the algebraic entropy of a generalized shift σ_λ is infinite also in case λ has a ladder and in case λ has a periodic ladder, that is, periodic orbits of arbitrarily large length. The technique used in the proof of this result is different from that used in the proof of Lemma 6.3, and this is why we give them separately.

Lemma 6.6. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial finite abelian group and consider the generalized shift $\sigma_\lambda : K^\Gamma \to K^\Gamma$.*

(a) If $l(\lambda) > 0$, then $\mathrm{ent}(\sigma_\lambda) = \infty$.

(b) If $p(\lambda) > 0$, then $\mathrm{ent}(\sigma_\lambda) = \infty$.

Proof. (a) Let $L = \bigcup_{m \in \mathbb{N}} L_m \subseteq \Gamma \setminus \mathrm{Per}(\lambda)$ be a ladder of λ in Γ, where each $L_m = \{l_{m,0}, \ldots, l_{m,b_m}\}$. Let $\Lambda = L \cup \{\lambda^n(l_{m,0}) : m \in \mathbb{N}, n \in \mathbb{N}_+\}$, which is λ-invariant and so define $\rho = \lambda \restriction_\Lambda$. By Proposition 6.2 $\mathrm{ent}(\sigma_\lambda) \geq \mathrm{ent}(\sigma_\rho)$, where $\sigma_\rho : K^\Lambda \to K^\Lambda$, and so it suffices to prove that $\mathrm{ent}(\sigma_\rho) = \infty$. Let $\mathbb{N} = \bigcup_{i \in \mathbb{N}} N_i$ be a partition of $\mathbb{N}$ in infinitely many infinite subsets N_i of $\mathbb{N}$. For each $i \in \mathbb{N}$ let $\Lambda_i = \bigcup_{m \in N_i} L_m$. Then each Λ_i is a ladder of ρ and $L = \bigcup_{i \in \mathbb{N}} \Lambda_i$ is a partition of L; so $K^L \cong \prod_{i \in \mathbb{N}} K^{\Lambda_i}$. Since each Λ_i is ρ^{-1}-invariant, by Lemma 2.4(a) each K^{Λ_i} is a σ_ρ-invariant subgroup of K^L. By Lemma 4.1(a) $\mathrm{ent}(\sigma_\rho \restriction_{K^{\Lambda_i}}) > 0$ for every $i \in \mathbb{N}$ and so Lemma 2.4(b) implies that $\mathrm{ent}(\sigma_\lambda) = \infty$.

(b) Let $P = \bigcup_{n\in\mathbb{N}_+} P_n$ be a periodic ladder of λ in Γ. Since $\lambda(P) \subseteq P$, let $\phi = \lambda\restriction_P$; by Proposition 6.2 $\mathrm{ent}(\sigma_\lambda) \geq \mathrm{ent}(\sigma_\phi)$ and so it suffices to prove that $\sigma_\phi : K^P \to K^P$ has $\mathrm{ent}(\sigma_\phi) = \infty$. Let $\mathbb{N}_+ = \bigcup_{i=1}^\infty N_i$ be a partition of $\mathbb{N}_+$ such that each N_i is infinite, and let $\Lambda_i = \bigcup_{n\in N_i} P_n$. Consequently $P = \bigcup_{i=1}^\infty \Lambda_i$ is a partition of P, and so $K^P \cong \prod_{i=1}^\infty K^{\Lambda_i}$. For every $i \in \mathbb{N}_+$, $\phi^{-1}(\Lambda_i) \subseteq \Lambda_i$, so by Lemma 2.4(a) each K^{Λ_i} is a σ_ϕ-invariant subgroup of K^P. By Lemma 4.1(b) $\mathrm{ent}(\sigma_\rho\restriction_{K^{\Lambda_i}}) > 0$ for every $i \in \mathbb{N}$ and so Lemma 2.4(b) implies that $\mathrm{ent}(\sigma_\lambda) = \infty$. □

Thanks to the characterization of bounded functions given by Theorem 3.3, and in view of the preceding results, we can now prove Theorem 1.3.

Proof of Theorem 1.3. (c)⇔(d)⇔(e) is Theorem 4.2, while (e)⇔(a) is given by Proposition 2.2(a), and (a)⇒(b) is obvious.

(b)⇒(c) Assume that λ is not bounded. By Theorem 3.3 this happens if at least one of $s(\lambda)$, $o(\lambda)$, $l(\lambda)$, $p(\lambda)$ is non-zero. Then apply respectively (a) or (b) of Lemma 6.3, or (a) or (b) of Lemma 6.6. □

Note that among the equivalent conditions of Theorem 1.3 it is not possible to add that $\sigma_\lambda\restriction_{\bigoplus_\Gamma K}$ has entropy zero, even when σ_λ is an automorphism:

Example 6.7. Let Γ be a countably infinite set and $\lambda : \Gamma \to \Gamma$ a function such that Γ is a periodic ladder of λ. Then $\Gamma = \mathrm{Per}(\lambda)$ and λ is a bijection, which is locally periodic, non-periodic (and so non-quasi-periodic). While $\mathrm{ent}(\sigma_\lambda) = \infty$ by Lemma 6.6(b), $\mathrm{ent}(\sigma_\lambda\restriction_{\bigoplus_\Gamma K}) = 0$.

Theorem 1.3 can be generalized replacing finite abelian groups by arbitrary torsion abelian groups:

Corollary 6.8. *Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function, K a non-trivial torsion abelian group and consider the generalized shift $\sigma_\lambda : K^\Gamma \to K^\Gamma$. Then $\mathrm{ent}(\sigma_\lambda) = 0$ if and only if λ is bounded, otherwise $\mathrm{ent}(\sigma_\lambda) = \infty$.*

Proof. By Proposition 2.2(c)

$$\begin{aligned}\mathrm{ent}(\sigma_\lambda) &= \sup\{\mathrm{ent}(\sigma_\lambda\restriction_{F^{(\Gamma)}}) : F \text{ is a finite subgroup of } K\}\\ &= \sup\{\mathrm{ent}(\sigma_{\lambda,F}) : F \text{ is a finite subgroup of } K\}.\end{aligned}$$

By Theorem 1.3 $\mathrm{ent}(\sigma_{\lambda,F}) = 0$ if and only if λ is bounded and otherwise $\mathrm{ent}(\sigma_{\lambda,F}) = \infty$. □

Remark 6.9. Let Γ be a set, $\lambda : \Gamma \to \Gamma$ a function and K a non-trivial finite abelian group. In [AADGH] the set $\Gamma^+ = \bigcap_{n\in\mathbb{N}_+} \lambda^n(\Gamma)$ was defined.

(a) The set Γ^+ was useful in computing the algebraic entropy of the restriction of a generalized shift to the direct sum, that is, of $\sigma_\lambda\restriction_{\bigoplus_\Gamma K}: \bigoplus_\Gamma K \to \bigoplus_\Gamma K$. Indeed, in order to consider this restriction, λ has to have $\lambda^{-1}(i)$ finite for every $i \in \Gamma$ and in this case $\lambda\restriction_{\Gamma^+}: \Gamma^+ \to \Gamma^+$ is surjective and $\mathrm{ent}(\sigma_\lambda\restriction_{\bigoplus_\Gamma K}) = \mathrm{ent}(\sigma_{\lambda\restriction_{\Gamma^+}}\restriction_{\bigoplus_{\Gamma^+} K})$.

(b) In general for a function $\lambda : \Gamma \to \Gamma$ it is not true that its restriction to Γ^+ is surjective. Take for example $\Gamma = \{g, h\} \cup \bigcup_{n\in\mathbb{N}} \Gamma_n$, where for every $n \in \mathbb{N}$, $\Gamma_n = \{g_{n,0}, \ldots, g_{n,n}\}$, $\lambda(g_{n,l}) = g_{n,l-1}$ for every $l \in \{1, \ldots, n\}$, $\lambda(g_{n,0}) = g$, $\lambda(g) = h$ and $\lambda(h) = h$. Then $\Gamma^+ = \{g, h\}$, but $g \notin \lambda(\Gamma^+) = \{h\}$.

(c) In general it is not true that $\mathrm{ent}(\sigma_\lambda) = \mathrm{ent}(\sigma_{\lambda\restriction_{\Gamma^+}})$, because for example if λ admits a ladder L in Γ, and $\Gamma = L \cup \mathrm{Per}_n(\lambda)$ for some $n \in \mathbb{N}_+$, then $\Gamma^+ = \mathrm{Per}_n(\lambda)$, and so $\mathrm{ent}(\sigma_{\lambda\restriction_{\Gamma^+}}) = 0$, while $\mathrm{ent}(\sigma_\lambda) = \infty$ by Theorem 1.3.

(d) Observe that the function considered in (c) is not surjective. In fact, it is clear that λ is surjective if and only if $\Gamma = \Gamma^+$.

We explain now in detail how Theorem 1.3 solves Problems 6.1 and 6.2 in [AADGH]. Indeed, as asked in the first part of Problem 6.1, it gives the precise value of the algebraic entropy of a generalized shift σ_λ (in particular this answers negatively Problem 6.2(b), which asked if it was possible that $0 < \mathrm{ent}(\sigma_\lambda) < \infty$).

Moreover, Example 6.4 and Corollary 6.5 answer negatively the question in Problem 6.1, showing that in general it is not true that $\mathrm{ent}(\sigma_\lambda)$ coincides with $\mathrm{ent}(\sigma_\lambda \restriction_{\bigoplus_\Gamma K})$. Indeed, the left Bernoulli shift ${}_K\beta$ is a generalized shift and has $\mathrm{ent}({}_K\beta) = \infty$ by Corollary 6.5, while $\mathrm{ent}({}_K\beta \restriction_{\bigoplus_\mathbb{N} K}) = 0$ by Example 6.4. This shows also that it is possible that $\mathrm{ent}(\sigma_\lambda \restriction_{\bigoplus_\Gamma K}) = 0$, while $\mathrm{ent}(\sigma_\lambda) > 0$, which was asked in Problem 6.2(a).

7 Open problems

For a set Γ, a function $\lambda : \Gamma \to \Gamma$ and a non-trivial abelian group K, we can consider K^Γ endowed with the product topology of the discrete topologies on K. In this way K^Γ is a compact abelian group, and the generalized shift $\sigma_\lambda : K^\Gamma \to K^\Gamma$ is continuous. In relation to Theorem 1.3, the following question arises.

Problem 7.1. *Let K be a non-trivial finite abelian group. Does there exist a continuous endomorphism $\phi : K^\mathbb{N} \to K^\mathbb{N}$ with $0 < \mathrm{ent}(\phi) < \infty$?*

We conclude the paper by setting the following problem for infinite orbits and $o(-)$, which is similar to Problem 6.6 in [AADGH] for strings and $s(-)$.

Problem 7.2. *Let Γ be an abelian group and $\lambda : \Gamma \to \Gamma$ a group endomorphism. Calculate $o(\lambda)$. In particular, is it true that $o(\lambda) > 0$ implies $o(\lambda)$ infinite?*

References

[AKM] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319.

[AADGH] M. Akhavin, F. Ayatollah Zadeh Shirazi, D. Dikranjan, A. Giordano Bruno, A. Hosseini, *Algebraic entropy of shift endomorphisms on abelian groups*, to appear in Quaestiones Mathmaticae.

[AHK] F. Ayatollah Zadeh Shirazi, F. Heidari Ardi, N. Karami Kabir, *A note on shift theory*, Math. Pannon. **19** (2008) no. 2, 187–195.

[DGSZ] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, *Algebraic entropy for abelian groups*, Trans. Amer. Math. Soc. **361** (2009), 3401–3434.

[W] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory **8** (1974/75) no. 3, 243–248.