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RESEARCH ARTICLE

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A soft introduction to algebraic entropy

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Abstract The goal of this mainly expository paper is to develop the theory of the algebraic entropy in the basic setting of vector spaces V over a field K. Many complications encountered in more general settings do not appear at this first level. We will prove the basic properties of the algebraic entropy of linear transformations $\phi : V \to V$ of vector spaces and its characterization as the rank of V viewed as module over the polynomial ring $K[X]$ through the action of ϕ. The two main theorems on the algebraic entropy, namely, the Addition Theorem and the Uniqueness Theorem, whose proofs require many efforts in more general settings, are easily deduced from the above characterization. The adjoint algebraic entropy of a linear transformation, its connection with the algebraic entropy of the adjoint map of the dual space and the dichotomy of its behavior are also illustrated.

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الملخص

هدف هذه الورقة البحثية التفسيرية في المقام الأول هو تطوير نظرية للإنتروبيا الجبرية في الإطار الأساس لفضاءات متجهة V على حقل K. لا تظهر العديد من التعقيدات التي تُواجَه في أطر أعم عند هذا المستوى الأول. سوف نثبت الخصائص الأساس للإنتروبيا الجبرية للتحويلات الخطية $\phi : V \to V$ للفضاءات المتجهة، وتمييزَها كرتبة لـِ V عند النظر له كحلقي على حلقة كثيرات الحدود $K[X]$ عبر التأثير ϕ. باستخدام التمييز أعلاه، يتم بسهولة استنتاج المبرهنتين الأساس عن الإنتروبيا الجبرية، أي مبرهنة الجمع ومبرهنة الوحدانية واللتين يتطلب إثباتهما جهوداً كبيرة في أطر أعم. يتم أيضاً توضيح الإنتروبيا الجبرية القرينة لتحويل خطي، وارتباطها بالإنتروبيا الجبرية للراسم القرين للفضاء الثِّنْوِي، والتفرع الثنائي لسلوكها.

1 Introduction

The following questions, even if formulated in a vague way, should excite curiosity in people with a basic mathematical education:

Question 1.1 *Given a linear transformation $\phi : V \to V$ of a vector space V over a fixed field K, how chaotic is the iterated action of ϕ in V? Can we measure the dynamical behavior of the linear transformation ϕ?*

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The attempt to give a precise answer to the above questions, not in the setting of vector spaces, but in the less elementary setting of Abelian groups, originated the theory of the algebraic entropy (see [1,6,14,20]). Nowadays, this theory is extended to R-modules over arbitrary rings R (see [2,15,16,18,21]), and also to topological groups (see [9]). As usual, the more general the considered ring R is, the less specific are the results obtainable for R-modules. Nevertheless, general deep results also for modules over arbitrary rings have been recently obtained for the algebraic entropy.

The goal of this paper is to develop the theory of the algebraic entropy in the simplest possible case, that is, in the setting of vector spaces over commutative fields. This soft approach to the subject should be accessible to people with a basic knowledge of linear algebra and of the theory of modules over PID's. Many complications arising even for Abelian groups, or for modules over more general rings, do not appear at this first level. However, the basic properties and the two main theorems, namely, the Addition Theorem and the Uniqueness Theorem, have their full significance even for the algebraic entropy of linear transformations of vector spaces.

As motivation for our goal, it is worthwhile to remark that the general study of the algebraic entropy reduces in many different cases to linear transformations of vector spaces: for instance, if $\phi : G \to G$ is an endomorphism of an Abelian group, the study of $\mathrm{ent}(\phi)$, the algebraic entropy of ϕ, reduces to that of $\mathrm{ent}(\overline{\phi})$, where $\overline{\phi} : G/pG \to G/pG$ (p a prime number) is the linear transformation induced by ϕ on the vector space G/pG over the field with p elements (see [6]). Furthermore, in the investigation of the rank entropy, which is associated with the rank of modules over an integral domain R, one can reduce to linear transformations of vector spaces over the field of quotients K of R (see [16]).

The plan of the paper is as follows. In Sect. 2, we will give the definition of algebraic entropy of a linear transformation, which is based on the computation of the limit of a sequence of positive real numbers; furthermore, we will show how the algebraic structure of a vector space allows one to avoid the limit calculation in computing the value of the algebraic entropy.

In Sect. 3, we will prove the basic properties of the algebraic entropy and in Sect. 4 we will discuss the structure of $K[X]$-module induced on the K-vector space V by a linear transformation ϕ; we will denote such a module by V_ϕ. This matter reflects a classical point of view in linear algebra (see Chapter 2 in Kaplansky's monograph [12] or the book by Warner [19, pp. 659–674]), and is of essential importance in dealing with algebraic entropies. Moreover, we will show the characterization of the linear transformations $\phi : V \to V$ with finite algebraic entropy, interpreted as properties of $K[X]$-modules. From this characterization, we will derive the main formula: $\mathrm{ent}(\phi) = \mathrm{rk}_{K[X]}(V_\phi)$.

Section 5 will be devoted to the proof of the Addition Theorem and the Uniqueness Theorem, two fundamental results in the theory of algebraic entropy. Unlike in more general settings, where the two theorems have long and complicated proofs, in our case they follow quite easily from the formula $\mathrm{ent}(\phi) = \mathrm{rk}_{K[X]}(V_\phi)$. In fact, the Addition Theorem is just a corollary of the fact that the rank is an additive function, and the Uniqueness Theorem is a consequence of a result on length functions proved by Northcott and Reufel [13].

In Sect. 6, we investigate the adjoint algebraic entropy of a linear transformation $\phi : V \to V$, that is, $\mathrm{ent}^\star(\phi)$, which was introduced for endomorphisms of Abelian groups in [5] and studied also in [11]. After providing its basic properties, we will prove the main formula $\mathrm{ent}^\star(\phi) = \mathrm{ent}(\phi^*)$, where $\phi^* : V^* \to V^*$ is the adjoint linear transformation of the dual space V^* of V. A relevant difference between the algebraic entropy ent and its adjoint version $\mathrm{ent}^\star$ is that the latter presents a dichotomy in its behavior, since it takes only values 0 and ∞. The proof of this dichotomy furnished here for vector spaces, similar to the analogous proof for Abelian groups given in [5], makes an essential use of some structure results of modules over PID's. In the setting of Abelian groups, the adjoint algebraic entropy does not satisfy the Addition Theorem, except when one considers only bounded groups; in the present setting of vector spaces, we will prove the Addition Theorem for $\mathrm{ent}^\star$ in full generality, as an easy consequence of the dichotomy of $\mathrm{ent}^\star$.

2 Measuring the dynamical behavior of linear transformations

We can specify Question 1.1 in the Introduction by asking how complicated are the sets $F + \phi F + \phi^2 F + \cdots + \phi^{n-1} F$ ($n \geq 1$) of the sums of the iterated images of a finite subset F of V. We could consider this set theoretical question but, since V has an algebraic structure, it is reasonable to formulate the above question not just for finite subsets of V, but for the subspaces they generate in V (see the next Remark 2.5, which explains the main differences between taking finite subsets or finite dimensional subspaces of V). So, we are led to consider a finite dimensional subspace F of V and its iterated images: $F, \phi F, \phi^2 F, \ldots, \phi^n F, \ldots$. If we take the subspace of V generated by the first n of these subspaces, that is,

$$F + \phi F + \phi^2 F + \cdots + \phi^{n-1} F,$$

we obtain again a finite dimensional subspace of V, called the *n-th partial ϕ-trajectory* of F in V, and denoted by $T_n(\phi, F)$. Note that $T_n(\phi, F) \leq T_{n+1}(\phi, F)$ for all n. Adding all the subspaces $\phi^n F$ we get:

$$\bigcup_n T_n(\phi, F) = F + \phi F + \phi^2 F + \cdots + \phi^n F + \cdots$$

which is a subspace of V no longer of finite dimension, in general; it is denoted by $T(\phi, F)$ and it is called the *ϕ-trajectory* of F in V. The ϕ-trajectory $T(\phi, F)$ is said to be *cyclic* if $\dim(F) = 1$.

Everybody should agree that infinite dimensional subspaces are less easy to handle than those of finite dimension, so Question 1.1 could be reformulated more precisely as follows:

Question 2.1 *Given a linear transformation $\phi : V \to V$ of a vector space V over a field K, and a finite dimensional subspace F of V, when is the ϕ-trajectory $T(\phi, F)$ infinite dimensional? If this occurs, can we estimate how fast the partial ϕ-trajectories $T_n(\phi, F)$ grow?*

From an intuitive point of view, we could say that ϕ creates chaos in V if there exist ϕ-trajectories $T(\phi, F)$ of infinite dimension, and that the chaos is bigger when the growth of these trajectories is faster.

Some easy examples could give some hints for a possible answer to our questions.

Example 2.2 Let K be a field and $V = \bigoplus_{n \geq 0} K x_n$ a K-vector space with countable basis $\{x_0, x_1, x_2, \ldots\}$. Let $\beta : V \to V$ be the *right Bernoulli shift*, that is the linear transformation defined by the assignment $\beta(x_n) = x_{n+1}$ for each $n \geq 0$. We compare the chaos created by β with the chaos created by other linear transformations of V.

Obviously, β is more chaotic than the identity map 1_V of V. Less trivial is the comparison with the linear transformation $\phi : V \to V$ defined by the assignments:

$$x_0 \mapsto x_0, \quad x_1 \mapsto x_2 \mapsto x_1, \quad x_3 \mapsto x_4 \mapsto x_5 \mapsto x_3, \quad x_6 \mapsto x_7 \mapsto x_8 \mapsto x_9 \mapsto x_6, \quad \ldots.$$

Since for all $x \in V$ there exists $k > 0$ such that $\phi^k(x) = x$, one can easily deduce that for every finite dimensional subspace F of V, the ϕ-trajectory of F has finite dimension, while, for instance, the β-trajectory of $K x_0$ is the whole space V. We can conclude that β is more chaotic than ϕ.

Example 2.3 We compare now the right Bernoulli shift β of Example 2.2 with its square β^2, which sends x_n to x_{n+2}. Take the finite dimensional subspace $F = \bigoplus_{0 \leq i \leq r} K x_i$ of V, for a suitable $r \geq 1$; for every $n \geq 0$ we have

$$\dim(F + \beta F + \cdots + \beta^n F) = r + n$$

and

$$\dim(F + \beta^2 F + \beta^4 F + \beta^6 F + \cdots + \beta^{2n} F) = r + 2n.$$

Hence, both the β-trajectory and the β^2-trajectory of F have infinite dimension, but the growth of their partial trajectories are different: the growth of $T_n(\beta^2, F)$ is faster than the growth of $T_n(\beta, F)$. If we compute the asymptotic average growth of F under the action of β and β^2, we get:

$$\lim_{n\to\infty} \frac{\dim(F + \beta F + \beta^2 F + \cdots + \beta^n F)}{n+1} = \lim_{n\to\infty} \frac{r+n}{n+1} = 1$$

and

$$\lim_{n\to\infty} \frac{\dim(F + \beta^2 F + \beta^4 F + \cdots + \beta^{2n} F)}{n+1} = \lim_{n\to\infty} \frac{r+2n}{n+1} = 2.$$

This suggests that the chaos created by β^2 doubles the chaos created by β.

The preceding examples lead naturally to the following:

Definition 2.4 Given a linear transformation $\phi : V \to V$ of a vector space V over a field K, and a non-zero finite dimensional subspace F of V,

(a) the *algebraic entropy of* ϕ *with respect to* F, denoted by $H(\phi, F)$, is the asymptotic average growth of the partial ϕ-trajectories of F, that is

$$H(\phi, F) = \lim_{n\to\infty} \frac{\dim(T_n(\phi, F))}{n};$$

(b) the *algebraic entropy* of ϕ, denoted by $\mathrm{ent}(\phi)$, is the supremum of the algebraic entropies of ϕ with respect to F, ranging F over the set of all finite dimensional subspaces of V, that is

$$\mathrm{ent}(\phi) = \sup_F H(\phi, F).$$

Remark 2.5 The above definitions make sense even if F is not assumed to be a finite dimensional subspace, but only a finite subset. In this case, the partial ϕ-trajectories $T_n(\phi, F)$ are no longer subspaces of V, but only finite subsets; so their size, that cannot be computed by the dimension, is computed by $\log|T_n(\phi, F)|$ (log is necessary to make the invariant additive). The next proposition, which ensures that the limit defining $H(\phi, F)$ exists and is finite, is applicable also to this different version of the algebraic entropy, which is called here Peters entropy. If the base field K is finite, then every finite dimensional subspace of V is a finite subset; hence, the Peters entropy essentially coincides with the algebraic entropy (up to the multiplicative factor $\log|K|$). But if K is infinite, the two notions are different. Actually, since the definition of the Peters entropy involves only sums of elements and not their multiplication by scalars, one can consider only the structure of Abelian group of V, disregarding that of K-vector space. We refer to [3,4,14] for many interesting results on Peters entropy in the Abelian groups setting.

Remark 2.6 As another less interesting variation of the definition of algebraic entropy, we can consider finite dimensional subspaces F of V as above, but we measure the size of the partial ϕ-trajectories by $\log|T_n(\phi, F)|$, and not by means of the dimension. Again, if K is a finite field, we do not find anything essentially new, but, if K is infinite, $\log|T_n(\phi, F)| = \infty$ for every non-zero subspace F. Therefore, in the latter case we obtain a trivial notion of entropy.

Remark 2.7 If in Definition 2.4 we replace the linear transformation $\phi : V \to V$ by an endomorphism $\psi : G \to G$ of an Abelian group G, and the invariant “dimension” by the invariant “rank”, we obtain the notion of rank-entropy, investigated in [16]. But, even if the algebraic entropy of linear transformations of $\mathbb{Q}$-vector spaces and the rank-entropy of endomorphisms of Abelian groups are calculated in the same way (recall that $\mathrm{rk}_{\mathbb{Z}}(G) = \dim_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$), there is a deep difference between the two settings. For instance, every vector space of infinite dimension has linear transformations of arbitrarily large algebraic entropy, as we will see below; on the converse, it is possible to construct torsion-free Abelian groups G of rank $2^{\aleph_0}$ such that every endomorphism of G has rank-entropy zero (see [10]). This obviously reflects the fact that endomorphism rings of Abelian groups have a much more complex structure than the endomorphism rings of vector spaces.

The next proposition shows that Definition 2.4(a) makes sense; recall that a sequence of real numbers $\{a_n\}_n$ is subadditive if $a_{n+m} \le a_n + a_m$ for all n and m.

Proposition 2.8 *The sequence* $\{\dim(T_n(\phi, F))\}_n$ *is subadditive, hence the limit*

$$H(\phi, F) = \lim_{n\to\infty} \frac{\dim(T_n(\phi, F))}{n}$$

does exist and equals $\inf_{n\ge 0} \frac{\dim(T_n(\phi,F))}{n}$.

Proof We have $T_{n+m}(\phi, F) = T_n(\phi, F) + \phi^n T_m(\phi, F)$ for every $n, m > 0$. Then for all $n, m > 0$,

$$\dim(T_{n+m}(\phi, F)) \le \dim(T_n(\phi, F)) + \dim(\phi^n(T_m(\phi, F)) \le \dim(T_n(\phi, F)) + \dim(T_m(\phi, F)).$$

This proves that $\{\dim(T_n(\phi, F))\}_n$ is subadditive. The rest of the statement follows by a well-known result of calculus, due to Fekete [7], which completes the proof. □

The next two results allow one to avoid the limit calculation in the computation of the algebraic entropy. This fact, which has no analogy for topological or metric entropies, essentially depends on the additivity of the invariant dim. The crucial fact is provided by the following:

Lemma 2.9 *For every $n > 0$ let*

$$\alpha_n = \dim\left(\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)}\right). \tag{2.1}$$

The sequence of non-negative integers $\{\alpha_n\}_n$ is decreasing, and hence stationary.

Proof Let $n > 1$. Since $T_{n+1}(\phi, F) = T_n(\phi, F) + \phi^n F$ and since $\phi T_{n-1}(\phi, F) \subseteq T_n(\phi, F)$, it follows that

$$\frac{T_{n+1}(\phi, F)}{T_n(\phi, F)} \cong \frac{\phi^n F}{T_n(\phi, F) \cap \phi^n F}$$

is a quotient of

$$B_n = \frac{\phi^n F}{\phi T_{n-1}(\phi, F) \cap \phi^n F}.$$

Therefore $\alpha_n \leq \dim B_n$. Furthermore, since $\phi T_n(\phi, F) = \phi T_{n-1}(\phi, F) + \phi^n F$, we have:

$$B_n \cong \frac{\phi T_{n-1}(\phi, F) + \phi^n F}{\phi T_{n-1}(\phi, F)} = \frac{\phi T_n(\phi, F)}{\phi T_{n-1}(\phi, F)} \cong \frac{T_n(\phi, F)}{T_{n-1}(\phi, F) + (T_n(\phi, F) \cap \ker \phi)}$$

which is a quotient of $T_n(\phi, F)/T_{n-1}(\phi, F)$, so $\dim B_n \leq \alpha_{n-1}$. Hence $\alpha_n \leq \alpha_{n-1}$, as desired. □

By means of Lemma 2.9, we can easily show how to determine the algebraic entropy of the linear transformation ϕ with respect to the subspace F, avoiding the limit calculation.

Proposition 2.10 *Let $\phi : V \to V$ be a linear transformation of the vector space V, and F a finite dimensional subspace of V. Then $H(\phi, F) = \alpha$, where α is the value of the stationary sequence $\{\alpha_n\}_n$ for n large enough. In particular, $H(\phi, F) = 0$ precisely when the sequence $\{\dim(T_n(\phi, F))\}_n$ becomes stationary, equivalently, when $\alpha_n = 0$ for every n large enough.*

Proof For every $n > 0$, in view of the definition of the α_n given in (2.1),

$$\alpha_n = \dim(T_{n+1}(\phi, F)) - \dim(T_n(\phi, F)). \tag{2.2}$$

By Lemma 2.9, the decreasing sequence $\{\alpha_n\}_n$ is stationary, so there exist $n_0 > 0$ and $\alpha \geq 0$ such that $\alpha_n = \alpha$ for every $n \geq n_0$.

Then, $\alpha = 0$ if and only if $\dim(T_{n+1}(\phi, F)) = \dim(T_n(\phi, F))$ for every $n \geq n_0$; in this case, $\dim(T(\phi, F)) = \dim(T_n(\phi, F))$ for every $n \geq n_0$. If $\alpha > 0$, since by (2.2) $\dim(T_{n_0+n}(\phi, F)) = n\alpha + \dim(T_{n_0}(\phi, F))$ for every $n \geq 0$, we have

$$H(\phi, F) = \lim_{n\to\infty} \frac{\dim(T_{n_0+n}(\phi, F))}{n_0 + n} = \lim_{n\to\infty} \frac{\dim(T_{n_0}(\phi, F)) + n\alpha}{n_0 + n} = \alpha.$$

This concludes the proof. □

From Proposition 2.10, we derive that the value of the algebraic entropy $\mathrm{ent}(\phi)$ of a linear transformation ϕ is either a non-negative integer or ∞.

We consider now the two examples of the right and the left Bernoulli shifts on a countably dimensional vector space, showing that their algebraic entropies are, respectively, 1 and 0. The example of the right Bernoulli shift is fundamental, since it plays a crucial role in the Uniqueness Theorem, discussed in Sect. 5.

Example 2.11 (a) Reconsidering Example 2.3, every finite dimensional subspace of $V = \bigoplus_{n\geq 0} K x_n$ is contained in a subspace of the form $F = \bigoplus_{0\leq i\leq r} K x_i$ for a suitable $r \geq 0$. We have shown in Example 2.3 that $H(\beta, F) = 1$, independently of the dimension of F, so $\mathrm{ent}(\beta) = 1$. Note that, using the notation of Proposition 2.10, the sequence $\{\alpha_n\}_n$ is constantly equal to 1.

(b) By means of the same argument, one can show that $\mathrm{ent}(\beta^2) = 2$. Note that $V = T(\beta^2, x_1) \oplus T(\beta^2, x_2)$, and that β^2 acts as the right Bernoulli shift on the two cyclic trajectories $T_1 = T(\beta^2, x_1)$ and $T_2 = T(\beta^2, x_2)$. Thus $\mathrm{ent}(\beta^2 \restriction_{T_i}) = 1$ for $i = 1, 2$ and $\mathrm{ent}(\beta^2) = \mathrm{ent}(\beta^2 \restriction_{T_1}) + \mathrm{ent}(\beta^2 \restriction_{T_2})$. This gives a foretaste to the Addition Theorem, to be proved in Sect. 5.
(c) Slightly modifying the previous argument, one can show that, if $W = \bigoplus_{n \geq 0} V_n$ with $V_n = V$ for all $n \geq 0$, setting $\beta(v_0, v_1, v_2, \dots) = (0, v_0, v_1, v_2, \dots)$, then $\mathrm{ent}(\beta) = \dim(V)$.
(d) Reversing the arrows, we can consider the *left Bernoulli shift* $\lambda : W \to W$, defined by $\lambda(v_0, v_1, v_2, \dots) = (v_1, v_2, v_3, \dots)$. For a finite dimensional subspace F of W, we get that $\lambda^r(F) = 0$ for a suitable positive integer r; from this one easily deduces that $H(\lambda, F) = 0$, independently of F, so $\mathrm{ent}(\lambda) = 0$ (again using the notation of Proposition 2.10, the sequence $\{\alpha_n\}_n$ is equal to 0 for $n > r$).

3 Basic properties of the algebraic entropy

In this section, we prove some basic properties of the algebraic entropy, starting with the announced fact that the algebraic entropy is stable under conjugated linear transformations.

Proposition 3.1 *Let $\phi : V \to V$ be a linear transformation and $\alpha : V \to W$ an isomorphism of vector spaces. Then* $\mathrm{ent}(\phi) = \mathrm{ent}(\alpha\phi\alpha^{-1})$.

Proof Let F be a finite dimensional subspace of W. For every $n > 0$ we have $T_n(\alpha\phi\alpha^{-1}, F) = \alpha(T_n(\phi, \alpha^{-1}F))$. Therefore,

$$H(\alpha\phi\alpha^{-1}, F) = \lim_{n\to\infty} \frac{\dim(T_n(\alpha\phi\alpha^{-1}, F))}{n} = \lim_{n\to\infty} \frac{\dim(\alpha T_n(\phi, \alpha^{-1}F))}{n} = H(\phi, \alpha^{-1}F).$$

Since F is a finite dimensional subspace of W if and only if $\alpha^{-1}F$ is a finite dimensional subspace of V, we can conclude that $\mathrm{ent}(\alpha\phi\alpha^{-1}) = \mathrm{ent}(\phi)$. □

Proposition 3.1 says that the algebraic entropy is an invariant of the category $\mathrm{Mod}(K[X])$; the fact that it takes integer values (plus ∞) is expressed by saying that it is a *discrete* invariant.

The following lemma is one of the inequalities of the Addition Theorem. In particular, it shows that the algebraic entropy is monotone under restrictions to subspaces and quotients.

Lemma 3.2 *Let $\phi : V \to V$ be a linear transformation and W a ϕ-invariant subspace of V. Then* $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi \restriction_W) + \mathrm{ent}(\overline{\phi})$*, where $\overline{\phi} : V/W \to V/W$ is the linear transformation induced by ϕ.*

Proof We prove first that

$$\mathrm{ent}(\phi) \geq \max\{\mathrm{ent}(\phi \restriction_W), \mathrm{ent}(\overline{\phi})\}. \tag{3.1}$$

Since every finite dimensional subspace of W is a finite dimensional subspace of V, it follows that $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi \restriction_W)$. Let F'/W be a finite dimensional subspace of V/W. Then there exists a finite dimensional subspace F of W such that $F'/W = (F + W)/W$. For every $n > 0$,

$$T_n(\overline{\phi}, F'/W) = \frac{T_n(\phi, F) + W}{W} \cong \frac{T_n(\phi, F)}{T_n(\phi, F) \cap W},$$

that is, $T_n(\overline{\phi}, F'/W)$ is a quotient of $T_n(\phi, F)$. Passing to the limit, this gives $H(\overline{\phi}, F'/W) \leq H(\phi, F)$; hence $\mathrm{ent}(\overline{\phi}) \leq \mathrm{ent}(\phi)$. This concludes the proof of (3.1).

If $\mathrm{ent}(\phi) = \infty$, the inequality in the thesis is satisfied. So assume that $\mathrm{ent}(\phi)$ is finite. By (3.1) both $\mathrm{ent}(\phi \restriction_W)$ and $\mathrm{ent}(\overline{\phi})$ are finite. By Proposition 2.10 there exists a finite dimensional subspace F' of W and a finite dimensional subspace F'' of V such that $\mathrm{ent}(\phi \restriction_W) = H(\phi, F')$ and $\mathrm{ent}(\overline{\phi}) = H(\overline{\phi}, F'' + W/W)$. Let $F = F' + F''$. Then

$$\mathrm{ent}(\phi \restriction_W) = H(\phi, F \cap W),$$

as $H(\phi, F') \leq H(\phi, F \cap W) \leq \mathrm{ent}(\phi \restriction_W) = H(\phi, F')$, and

$$\mathrm{ent}(\overline{\phi}) = H(\overline{\phi}, F + W/W).$$

We need to prove now that

$$H(\phi, F) \geq H(\phi\restriction_W, F \cap W) + H(\overline{\phi}, (F+W)/W). \tag{3.2}$$

To this end, consider the exact sequence

$$0 \to T_n(\phi, F) \cap W \to T_n(\phi, F) \to \frac{T_n(\phi, F)}{T_n(\phi, F) \cap W} \cong \frac{T_n(\phi, F) + W}{W} \to 0.$$

Then,

$$\dim(T_n(\phi, F)) = \dim(T_n(\phi, F) \cap W) + \dim\left(\frac{T_n(\phi, F) + W}{W}\right).$$

Since, $\frac{T_n(\phi, F)+W}{W} = T_n(\overline{\phi}, \frac{F+W}{W})$,

$$\dim(T_n(\phi, F)) = \dim(T_n(\phi, F) \cap W) + \dim\left(T_n\left(\overline{\phi}, \frac{F+W}{W}\right)\right).$$

Moreover, $T_n(\phi\restriction_W, F \cap W) \subseteq T_n(\phi, F) \cap W$ and so

$$\dim(T_n(\phi, F)) \geq \dim(T_n(\phi\restriction_W, F \cap W)) + \dim\left(T_n\left(\overline{\phi}, \frac{F+W}{W}\right)\right).$$

Dividing by n and passing to the limit, we have (3.2).

Now, (3.2) implies $\mathrm{ent}(\phi) \geq H(\phi, F) \geq H(\phi\restriction_W, F \cap W) + H(\overline{\phi}, F + W/W)$, and hence $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_W) + \mathrm{ent}(\overline{\phi})$. □

From Lemma 3.2, we derive the following important property of the algebraic entropy.

Proposition 3.3 *Let $\phi : V \to V$ be a linear transformation. If V is the direct limit of ϕ-invariant subspaces V_σ, then* $\mathrm{ent}(\phi) = \sup_\sigma \mathrm{ent}(\phi\restriction_{V_\sigma})$.

Proof By Lemma 3.2, $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi\restriction_{V_\sigma})$ for every σ and so $\mathrm{ent}(\phi) \geq \sup_\sigma \mathrm{ent}(\phi\restriction_{V_\sigma})$. Let F be a finite dimensional subspace of V. There exists σ such that $F \subseteq V_\sigma$. Then $H(\phi, F) \leq \mathrm{ent}(\phi\restriction_{V_\sigma})$ and so $\mathrm{ent}(\phi) \leq \sup_\sigma \mathrm{ent}(\phi\restriction_{V_\sigma})$. □

The following property is the so-called logarithmic law for the algebraic entropy; compare it with Example 2.11.

Proposition 3.4 *Let $\phi : V \to V$ be a linear transformation. Then,* $\mathrm{ent}(\phi^k) = k \cdot \mathrm{ent}(\phi)$ *for all $k \geq 0$. If ϕ is an automorphism, then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi^{-1})$*; in particular,* $\mathrm{ent}(\phi^k) = |k| \cdot \mathrm{ent}(\phi)$ *for every integer k.*

Proof For $k = 0$, it is enough to note that $\mathrm{ent}(id_V) = 0$. So let $k > 0$ and let F be a finite dimensional subspace of V. For every $n > 0$, we have $T_{nk}(\phi, F) = T_n(\phi^k, T_k(\phi, F))$. Let $E = T_k(\phi, F)$. Then

$$k \cdot H(\phi, F) = k \cdot \lim_{n\to\infty} \frac{\dim(T_{nk}(\phi, F))}{nk} = \lim_{n\to\infty} \frac{\dim(T_n(\phi^k, E))}{n} = H(\phi^k, E) \leq \mathrm{ent}(\phi^k);$$

consequently, $k \cdot \mathrm{ent}(\phi) \leq \mathrm{ent}(\phi^k)$. Conversely,

$$\mathrm{ent}(\phi) \geq H(\phi, F) = \lim_{n\to\infty} \frac{\dim(T_{nk}(\phi, F))}{nk} = \lim_{n\to\infty} \frac{\dim(T_n(\phi^k, E))}{nk} = \frac{H(\phi^k, E)}{k}.$$

Since $T_n(\phi^k, F) \leq T_n(\phi^k, E)$, it follows that $\dim(T_n(\phi^k, F)) \leq \dim(T_n(\phi^k, E))$, and so $k \cdot \mathrm{ent}(\phi) \geq H(\phi^k, F)$. Hence $k \cdot \mathrm{ent}(\phi) \geq \mathrm{ent}(\phi^k)$.

Assume now that ϕ is invertible. For every $n > 0$, we have $T_n(\phi, F) = \phi^{n-1} T_n(\phi^{-1}, F)$, and so $H(\phi, F) = H(\phi^{-1}, F)$. Hence, $\mathrm{ent}(\phi) = \mathrm{ent}(\phi^{-1})$. □

The following is a particular case of the Addition Theorem.

Lemma 3.5 *If $V = V_1 \oplus V_2$ for some subspaces V_1, V_2 of V, and $\phi = \phi_1 \oplus \phi_2 : V \to V$ for some linear transformations $\phi_i : V_i \to V_i$, $i = 1, 2$, then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi_1) + \mathrm{ent}(\phi_2)$.

Proof Let F_1 be a finite dimensional subspace of V_1 and F_2, a finite dimensional subspace of V_2. Since $T_n(\phi, F_1 \times F_2) = T_n(\phi_1, F_1) + T_n(\phi_2, F_2)$ for every $n > 0$, it follows that $H(\phi, F_1 \times F_2) = H(\phi_1, F_1) + H(\phi_2, F_2)$. Since every finite dimensional subspace F of V is contained in $F_1 \times F_2$, where F_1 is the projection of F onto V_1 and F_2 is the projection of F onto V_2, it follows that $\mathrm{ent}(\phi) = \mathrm{ent}(\phi_1) + \mathrm{ent}(\phi_2)$. □

4 Passing to modules over polynomial rings

In this section, we connect the algebraic entropy of linear transformations of K-vector spaces with the structure of $K[X]$-modules, following the classical approach that can be found for instance in [12, Chapter 12] and [19, pp. 659–674].

Fixing a field K, we can define the category whose objects are the pairs (V, ϕ) with V a K-vector space and $\phi : V \to V$ a linear transformation. In this category, a morphism $\alpha : (V, \phi) \to (W, \psi)$ is a commutative square of the form

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\alpha"'] & V \arrow[d, "\alpha"] \\ W \arrow[r, "\psi"] & W \end{tikzcd} \tag{4.1}$$

where α is a linear transformation from V to W. This category is just isomorphic to the category $\mathrm{Mod}(K[X])$ of modules over the polynomial ring over K, which is a PID; the equivalence functor is given by $(V, \phi) \mapsto V_\phi \in \mathrm{Mod}(K[X])$, where V_ϕ as a K-vector space is just V and X acts on V_ϕ via ϕ; in detail, if $v \in V$ and $f(X) = a_0 + a_1 X + \cdots + a_m X^m$ is a polynomial in $K[X]$, then

$$f(X) \cdot v = (f(\phi))(v) = a_0 v + a_1 \phi(v) + \cdots + a_m \phi^m(v).$$

The homomorphism α in (4.1) becomes a $K[X]$-homomorphism, as it commutes with the action of X. In this way, a ϕ-invariant subspace of V is just a $K[X]$-subspace of V_ϕ; furthermore, V_ϕ and W_ψ are isomorphic as $K[X]$-modules if and only if there exists a K-isomorphism $\alpha : V \to W$ such that $\psi = \alpha\phi\alpha^{-1}$, that is, ϕ and ψ are conjugated.

Every $K[X]$-module can be viewed as a K-vector space V with the multiplication by X acting as a K-endomorphism. So, in the following, when dealing with $K[X]$-modules, we will always consider objects written in the form V_ϕ; we will sometimes abuse notation, denoting a ϕ-invariant subspace W of V with the structure of $K[X]$-module induced by the restriction of ϕ to W simply by W_ϕ.

At this point, we can interpret the algebraic entropy as a map

$$\mathrm{ent} : \mathrm{Mod}(K[X]) \to \mathbb{R}_{\geq 0} \cup \{\infty\},$$

which associates to the $K[X]$-module V_ϕ the value $\mathrm{ent}(\phi)$. Indeed, in Proposition 3.1, we have seen that two $K[X]$-modules V_ϕ and W_ψ are isomorphic exactly when ϕ and ψ are conjugated, which implies that isomorphic $K[X]$-modules have the same algebraic entropy. Therefore, ent can be viewed as an invariant of the category $\mathrm{Mod}(K[X])$ with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ (this point of view is fully developed in [15]). Moreover, as every module is the direct limit of its finitely generated submodules, Proposition 3.3 says in particular that the algebraic entropy is an upper continuous invariant of $\mathrm{Mod}(K[X])$, that is, $\mathrm{ent}(V_\phi) = \sup_W \mathrm{ent}(W)$, ranging W in the set of the finitely generated $K[X]$-submodules of V_ϕ, that is, of the ϕ-trajectories of finite dimensional subspaces of V.

In [12], one can find the description of when V_ϕ is primary (under the assumption that K is algebraically closed), or primary of bounded order, or cyclic, and some classical theorems holding for Abelian p-groups are adapted to the present situation.

We introduce now a functor from the category $\mathrm{Vect}(K)$ of K-vector spaces to the category $\mathrm{Mod}(K[X])$ of the $K[X]$-modules, which will play a crucial role in the Uniqueness Theorem 5.3.

The Bernoulli functor $B : \mathrm{Vect}(K) \to \mathrm{Mod}(K[X])$ associates with a K-vector space V the direct sum $\bigoplus_{n\geq 0} V_n$, with $V_n = V$ for all n, endowed with a right Bernoulli shift β defined in Example 2.11. If $\alpha : V \to W$ is a K-linear transformation, then

$$B(\alpha) : B(V) \to B(W) \text{ is defined by } B(\alpha)(v_0, v_1, v_2, \dots) = (\alpha v_0, \alpha v_1, \alpha v_2, \dots).$$

It is easy to check that the Bernoulli functor is equivalent to the functor $- \otimes_K K[X]$, that is, $B(V)$ is naturally isomorphic to $V \otimes_K K[X]$. Thus, Example 2.11 can be reformulated by saying that, for all vector spaces V, the following equality holds:

$$\mathrm{ent}(V \otimes_K K[X])) = \dim(V). \tag{4.2}$$

Recall that the linear transformation $\phi : V \to V$ is *locally algebraic* if for every $v \in V$ there exists a non-zero polynomial $f(X) \in K[X]$ such that $f(\phi)(v) = 0$, and ϕ is *algebraic* if the polynomial $f(X)$ is independent of $v \in V$.

We give now an example of a locally algebraic linear transformation which fails to be algebraic.

Example 4.1 The linear transformation $\phi : V \to V$ considered in Example 2.2, defined by the assignments:

$$x_0 \mapsto x_0, \quad x_1 \mapsto x_2 \mapsto x_1, \quad x_3 \mapsto x_4 \mapsto x_5 \mapsto x_3, \quad x_6 \mapsto x_7 \mapsto x_8 \mapsto x_9 \mapsto x_6, \ \ldots$$

is locally algebraic, since for all $v \in V$ there exists $k > 0$ such that $\phi^k(v) = v$; hence, v is annihilated by the polynomial $X^k - 1$. One can easily deduce that for every finite dimensional subspace F of V the ϕ-trajectory of F has finite dimension; therefore, $\mathrm{ent}(\phi) = 0$, according to the next Theorem 4.3. Clearly, ϕ is not algebraic, since for each $k > 0$ there exist cyclic trajectories of dimension bigger than k.

In the following, we will assume the reader to be familiar with basic notions on modules over PIDs, as developed for instance in [12]. If R is such a domain and M is an R-module, we will denote as usual by $t(M)$ the torsion part of M.

Theorem 4.3 characterizes the linear transformations of zero algebraic entropy. To prove it, we need the following

Lemma 4.2 *Let $\phi : V \to V$ be a linear transformation, W a ϕ-invariant subspace of V, and $\overline{\phi} : V/W \to V/W$ the linear transformation induced by ϕ.*

(a) *If* $\mathrm{ent}(\overline{\phi}) = 0$*, then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_W)$*.*
(b) *If* $\mathrm{ent}(\phi \restriction_W) = 0$ *and* $\dim(V/W)$ *is finite, then* $\mathrm{ent}(\phi) = 0$*.*

Proof (a) By Lemma 3.2 we have $\mathrm{ent}(\phi) \geq \mathrm{ent}(\phi \restriction_W)$. To prove the converse inequality, it suffices to show that $H(\phi, F) \leq \mathrm{ent}(\phi \restriction_W)$ for every finite dimensional subspace F of V. So, let F be a finite dimensional subspace of V and $\overline{F} = F + W/W$. By Proposition 2.10, there exists $m > 0$ such that $T_{m+n}(\overline{\phi}, \overline{F}) = T_m(\overline{\phi}, \overline{F})$ for every $n \geq 0$. In particular, $\overline{\phi}^m \overline{F} \subseteq T_m(\overline{\phi}, \overline{F})$, and so $\phi^m F \subseteq T_m(\phi, F) + W$. Since $\phi^m F$ is finite dimensional, there exists a finite dimensional subspace E of W, such that $\phi^m F \subseteq T_m(\phi, F) + E$. It is possible to prove by induction that $\phi^m T_n(\phi, F) \subseteq T_m(\phi, F) + T_n(\phi, E)$ for every $n > 0$. Consequently,

$$T_{m+n}(\phi, F) = T_m(\phi, F) + \phi^m T_n(\phi, F) \subseteq T_m(\phi, F) + T_n(\phi, E)$$

for every $n > 0$. Then

$$\dim(T_{m+n}(\phi, F)) \leq \dim(T_m(\phi, F)) + \dim(T_n(\phi, E)),$$

so that dividing by $m + n$ and passing to the limit with respect to n, we obtain $H(\phi, F) \leq H(\phi, E) \leq \mathrm{ent}(\phi \restriction_W)$.

(b) The hypothesis ensures that $V = F_0 \oplus W$ for a finite dimensional subspace F_0 of V. If F is an arbitrary finite dimensional subspace of V, then $F \leq F_0 \oplus F_1$ for a finite dimensional subspace F_1 of W. In order to show that $H(\phi, F) = 0$, it is enough to prove that $H(\phi, F_0) = 0$, as $H(\phi, F_1) = H(\phi \restriction_W, F_1) = 0$, by hypothesis. Let now $F_0 + \phi F_0 = F_0 \oplus W_1$, where W_1 is a finite dimensional subspace of W. For every $n \geq 2$, we have that $T_n(\phi, F_0) \leq F_0 \oplus T_{n-1}(\phi, W_1)$. Since $H(\phi \restriction_W, W_1) = 0$, the conclusion easily follows. □

Theorem 4.3 *Let $\phi : V \to V$ be a linear transformation. The following conditions are equivalent:*

(a) $\mathrm{ent}(\phi) = 0$*;*
(b) *every ϕ-trajectory $T(\phi, F)$ of a finite dimensional subspace F of V is finite dimensional;*
(c) *V_ϕ is the union of a smooth ascending chain of $K[X]$-submodules V_σ ($\sigma < \lambda$) such that* $\dim(V_{\sigma+1}/V_\sigma)$ *is finite for all σ and $V_0 = 0$;*
(d) *ϕ is locally algebraic;*
(e) *V_ϕ is a torsion $K[X]$-module.*

Proof (a) $\Rightarrow$ (b) The condition $\mathrm{ent}(\phi) = 0$ implies that $H(\phi, F) = 0$ for every finite dimensional subspace F of V. By Proposition 2.10, there exists $k > 0$ such that $\dim(T_n(\phi, F)) = \dim(T_k(\phi, F))$ for every $n \geq k$. In particular, $\dim(T(\phi, F)) = \dim(T_k(\phi, F))$ as well, and so $\dim(T(\phi, F))$ is finite.

(b) $\Rightarrow$ (c) We construct the V_σ by transfinite induction. Let $V_0 = 0$. Suppose that $\sigma = \beta + 1$ for some β and assume that $V_\beta \neq V$; let $x + V_\beta$ be a non-zero element of V/V_β. Let $V_\sigma = V_\beta + T(\phi, Kx)$. Then, V_σ is ϕ-invariant as $T(\phi, Kx)$ and V_β are ϕ-invariant, respectively, by definition and by inductive hypothesis. By hypothesis, $\dim(T(\phi, Kx))$ is finite, moreover $\frac{V_\beta + T(\phi, Kx)}{V_\beta} \simeq \frac{T(\phi, Kx)}{V_\beta \cap T(\phi, Kx)}$, so V_σ/V_β has finite dimension. For σ limit, let $V_\sigma = \cup_{\beta<\sigma} V_\beta$. Then, $V = \bigcup_{\sigma<\lambda} V_\sigma$ for some λ.

(c) $\Rightarrow$ (a) Let $\phi_\sigma = \phi \restriction_{V_\sigma}$ for every σ. By Proposition 3.3, it suffices to verify that $\mathrm{ent}(\phi_\sigma) = 0$ for every σ. We proceed by transfinite induction on σ. If $\sigma = 0$, then $V_0 = 0$ and in particular $\mathrm{ent}(\phi_0) = 0$. If $\sigma = \beta + 1$ for some β, then Lemma 4.2 ensures that $\mathrm{ent}(\phi_\sigma) = 0$, since $\dim(V_\sigma/V_\beta)$ is finite and $\mathrm{ent}(\phi_\beta) = 0$ by inductive hypothesis. If σ is a limit ordinal, then $\mathrm{ent}(\phi_\sigma) = \sup_{\beta<\sigma} \mathrm{ent}(\phi_\beta) = 0$, in view of Proposition 3.3 and the inductive hypothesis.

(b) $\Leftrightarrow$ (d) Let $x \in V$. Item (b) implies that $\dim(T(\phi, Kx))$ is finite. Then, $T(\phi, Kx) = T_n(\phi, Kx)$ for some $n > 0$, and so $\phi^n(x) = a_0 x + a_1\phi(x) + \cdots + a_{n-1}\phi^{n-1}(x)$ for some $a_0, \ldots, a_{n-1} \in K$. Then, $f(X) = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} - X^n$ is a polynomial such that $f(\phi)(x) = 0$. Therefore, (b) implies (d). The argument can be reversed thus: as for a finite dimensional subspace $F = \bigoplus_{i\leq r} Kx_i$ of V, we have that $T(\phi, F) = \sum_{i\leq r} T(\phi, Kx_i)$, the converse implication holds too.

(d) $\Leftrightarrow$ (e) Look at the definition of ϕ that is locally algebraic. □

In the proof of Theorem 4.3, we used a special kind of trajectories, namely, those of the form $T(\phi, Kx)$, which we call *cyclic trajectories*. These trajectories are of great importance in the characterization of the endomorphisms of finite algebraic entropy given in the next Lemma 4.4 and Theorem 4.7.

The rest of this section is devoted to prove the characterization of linear transformations of vector spaces having finite algebraic entropy. The proof will be done by induction, the basic step being furnished by the following:

Lemma 4.4 *Let $\phi : V \to V$ be a linear transformation. Then, $\mathrm{ent}(\phi) = 1$ if and only if there exists a non-zero element x in V such that $T = T(\phi, Kx)$ is infinite dimensional and V_ϕ/T is a torsion $K[X]$-module.*

Proof Assume $\mathrm{ent}(\phi) = 1$. By Theorem 4.3, there exists an element $x \in V$ such that $T(\phi, Kx)$ has infinite dimension. It is immediate that this implies that $T(\phi, Kx) = \bigoplus_{n\geq 0} K\phi^n(x)$; clearly ϕ acts on $T = T(\phi, Kx)$ as the right Bernoulli shift, hence $\mathrm{ent}(\phi \restriction_T) = 1$ by Example 2.11. By Lemma 3.2 we get that $\mathrm{ent}(\overline{\phi}) = 0$, where $\overline{\phi}$ is the linear transformation induced by ϕ on V/W, and hence the conclusion follows by Theorem 4.3. Conversely, since $\mathrm{ent}(\overline{\phi}) = 0$ and $\mathrm{ent}(\phi \restriction_T) = 1$, the conclusion derives from Lemma 4.2(a). □

We give now an example, which is an application of Lemma 4.4 showing that the "bilateral" right shift has the same algebraic entropy as the right Bernoulli shift.

Example 4.5 Let $V = \bigoplus_{n\in\mathbb{Z}} Kx_n$ and β the right shift. Then, β restricted to $T(\beta, x_0)$ is the usual right Bernoulli shift, and $V_\beta/T(\beta, x_0)$ is a torsion $K[X]$-module, because $\beta^r(x_{-r}) \in T(\beta, x_0)$ for all $r \geq 0$. Hence, $\mathrm{ent}(\phi) = 1$ by Lemma 4.4.

Lemma 4.6 *Let $\phi : V \to V$ be a linear transformation, W a ϕ-invariant subspace of V and $\overline{\phi} : V/W \to V/W$ the linear transformation induced by ϕ. If $\mathrm{ent}(\overline{\phi}) > 0$, then there exists $x \in V$ such that the elements in $\{\phi^n x\}_{n\geq 0}$ are independent and $\bigoplus_{n\geq 0} K\phi^n x \cap W = 0$.*

Proof By Theorem 4.3, there exists $\overline{x} = x + W \in V/W$ such that $T(\overline{\phi}, \overline{x})$ has countable dimension. Therefore, $T_n(\overline{\phi}, \overline{x})$ has dimension n for each $n > 0$, and so $T(\overline{\phi}, \overline{x}) = (T(\phi, x) + W)/W = \bigoplus_{n\geq 0} K\overline{\phi}^n\overline{x}$, where $\overline{\phi}^n\overline{x} \neq 0$ for all $n \geq 0$. This implies the thesis. □

Theorem 4.7 *Let $\phi : V \to V$ be a linear transformation. The following conditions are equivalent:*

(a) $\mathrm{ent}(\phi) = k$, *where* $k \geq 1$*;*
(b) *V is the union of an ascending chain of $K[X]$-submodules*

$$t(V_\phi) = V_0 < V_1 < \cdots < V_k = V,$$

such that, for all $0 \leq i \leq k$, $\mathrm{ent}(\phi_i) = 1$, where $\phi_i : V_i/V_{i-1} \to V_i/V_{i-1}$ is the linear transformation induced by ϕ, and $\mathrm{ent}(\phi \restriction_{V_i}) = i$;

(c) *there exist k independent cyclic trajectories* $T_i = T(\phi, Kx_i)$ $(i \leq k)$ *which are infinite dimensional, such that* $V_\phi / \bigoplus_{1\leq i\leq k} T_i$ *is a torsion* $K[X]$*-module.*

Proof (a) ⇒ (b) We construct the subspaces V_i by induction on $i \geq 0$. For $i = 0$, there is nothing to prove in view of Theorem 4.3. Assume that $i > 0$ and that V_{i-1} is already constructed. Then the linear transformation $\overline{\phi}_{i-1} : V/V_{i-1} \to V/V_{i-1}$ induced by ϕ has $\mathrm{ent}(\overline{\phi}_{i-1}) > 0$. In fact, if $\mathrm{ent}(\overline{\phi}_{i-1}) = 0$, then Lemma 4.2(a) would imply $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_{V_{i-1}}) = i - 1 < k$, against the hypothesis. By Lemma 4.6, there exists $x_i \in V$ such that $T_i = T(\phi, x_i)$ has countable dimension and $T_i + V_{i-1} = T_i \oplus V_{i-1}$, where T_i and V_{i-1} are ϕ-invariant subspaces of V. Since ϕ acts as the right Bernoulli shift on T_i, we have $\mathrm{ent}(\phi\restriction_{T_i}) = 1$ by Example 2.11. Then, Lemma 3.5 gives

$$\mathrm{ent}(\phi\restriction_{T_i \oplus V_{i-1}}) = \mathrm{ent}(\phi\restriction_{T_i}) + \mathrm{ent}(\phi\restriction_{V_{i-1}}) = i. \tag{4.3}$$

Let V_i be the subspace of V, containing $T_i \oplus V_{i-1}$ such that

$$V_i/T_i \oplus V_{i-1} = t((V/(T_i \oplus V_{i-1})_\phi). \tag{4.4}$$

We show that $\mathrm{ent}(\phi_i) = 1$. To this end, consider the short exact sequence

$$0 \to T_i \cong (T_i \oplus V_{i-1})/V_{i-1} \to V_i/V_{i-1} \to V_i/(T_i \oplus V_{i-1}) \to 0.$$

By (4.4), Theorem 4.3, Lemma 4.2(a) and Proposition 3.1, it follows that $\mathrm{ent}(\phi_i) = \mathrm{ent}(\phi\restriction_{T_i}) = 1$.

Consider now the short exact sequence

$$0 \to T_i \oplus V_{i-1} \to V_i \to V_i/(T_i \oplus V_{i-1}) \to 0.$$

By (4.4), Theorem 4.3, Lemma 4.2(a) and (4.3), we have $\mathrm{ent}(\phi\restriction_{V_i}) = \mathrm{ent}(\phi\restriction_{T_i \oplus V_{i-1}}) = i$.

By construction, $\mathrm{ent}(\phi\restriction_{T_k \oplus V_{k-1}}) = k = \mathrm{ent}(\phi)$. Consequently, $(V/(T_k \oplus V_{k-1}))_\phi$ is torsion in view of Lemma 3.2 and Theorem 4.3, hence $V = V_k$.

(b) ⇒ (a) is obvious.

(a) ⇒ (c) Let $m > 0$ be such that there exist $x_1, \ldots, x_m \in V$ with each $T_i = T(\phi, x_i) = \bigoplus_{n\geq 0} K\phi^n x$ of countable dimension and independent, that is,

$$T = \sum_{1\leq i\leq m} T_i = \bigoplus_{1\leq i\leq m} T_i. \tag{4.5}$$

By Lemma 4.4 at least the case $m = 1$ is possible. Moreover, $\mathrm{ent}(\phi\restriction_{T_i}) = 1$ for every $1 \leq i \leq m$, by Example 2.11 since ϕ acts as a right Bernoulli shift on each T_i, and so Lemma 3.5 yields

$$\mathrm{ent}(\phi\restriction_T) = m. \tag{4.6}$$

Then, $m \leq k$ by Lemma 3.2. Suppose that m is maximum with respect to (4.5); we verify that $m = k$. Assume looking for a contradiction that $m < k$. By (4.6) and Lemma 4.2(a) $\mathrm{ent}(\overline{\phi}) > 0$, where $\overline{\phi} : V/T \to V/T$ is the linear transformation induced by ϕ. In view of Lemma 4.6, there exists $x_{m+1} \in V$ such that $T(\phi, x_{m+1})$ has countable dimension and $T \cap T(\phi, x_{m+1}) = 0$; this contradicts the maximality of m. Thus, $m = k$ and $(V/T)_\phi$ are torsion.

(c) ⇒ (a) Let $W = \bigoplus_{1\leq i\leq k} T_i$. Since $(V/W)_\phi$ is torsion, Theorem 4.3 and Lemma 4.2(a) give $\mathrm{ent}(\phi) = \mathrm{ent}(\phi\restriction_W)$. By Lemma 3.5 $\mathrm{ent}(\phi) = \sum_{1\leq i\leq k} \mathrm{ent}(\phi\restriction_{T(\phi,x_i)})$; since ϕ acts on each $T_i(\phi, x_i)$ as a right Bernoulli shift, by Example 2.11 we can conclude that $\mathrm{ent}(\phi) = k$. □

The following is the most important consequence of Theorem 4.7, which extends to arbitrary $K[X]$-modules the formula proved in (4.2).

Corollary 4.8 *Let* $\phi : V \to V$ *be a linear transformation. Then* $\mathrm{ent}(\phi) = \mathrm{rk}_{K[X]}(V_\phi)$.

Proof It is enough to prove the equality when $\mathrm{ent}(\phi)$ is finite, say equal to k. By Theorem 4.7, V contains $\bigoplus_{1\leq i\leq k} T_i$, where $T_i = T(\phi, Kx_i)$ is isomorphic to $K[X]$ as $K[X]$-modules for all i, hence $\mathrm{rk}_{K[X]}(T_i) = 1$. Taking care that $V_\phi/(\bigoplus_{1\leq i\leq k} T_i)$ is a torsion $K[X]$-module, we get

$$\mathrm{rk}_{K[X]}(V_\phi) = \mathrm{rk}_{K[X]}\left(\bigoplus_{1\leq i\leq k} T_i\right) = k,$$

which concludes the proof. □

5 Addition Theorem, Uniqueness Theorem and their consequences

The Addition Theorem for the algebraic entropy of endomorphisms ϕ of an algebraic structure (however defined) says that, given a ϕ-invariant substructure, the algebraic entropy of ϕ is the sum of the algebraic entropy of the restriction of ϕ to the substructure and of the algebraic entropy of the induced map on the quotient structure. It is a very useful tool, since it allows one to reduce the computation of the algebraic entropy to simpler substructures.

The Addition Theorem was one of the main achievements in the study of the algebraic entropy of endomorphisms of Abelian groups in [6]. In that paper, the theorem was proved to hold for the subcategory of torsion groups, and it was proved to fail for the whole category of Abelian groups. Its demonstration was quite eleborate, reducing the proof to bounded p-groups and then inducting on the exponent of the group.

The proof of the Addition Theorem for the rank-entropy given in [16] was much simpler. Very recently, using ideas from both the above proofs and borrowing techniques typical of p-groups and of torsion-free groups, a very general Addition Theorem has been proved in [15] for suitable subcategories of modules over arbitrary rings, dealing with the algebraic entropies associated with length functions (see their definition below in this section).

In our present setting of vector spaces, the Addition Theorem loses its complexity and becomes an easy consequence of the formula proved in Corollary 4.8.

Theorem 5.1 (Addition Theorem) *Let V be a vector space over the field K and $\phi : V \to V$ a linear transformation. If W is a ϕ-invariant subspace of V, then*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_W) + \mathrm{ent}(\overline{\phi})$$

where $\overline{\phi} : V/W \to V/W$ is the linear transformation induced by ϕ.

Proof Look at the algebraic entropy as a discrete invariant on the category $\mathrm{Mod}(K[X])$. Corollary 4.8 ensures that this invariant coincides with $\mathrm{rk}_{K[X]}$, which is obviously additive. So

$$\mathrm{rk}_{K[X]}(V_\phi) = \mathrm{rk}_{K[X]}(W_{\phi\restriction_W}) + \mathrm{rk}_{K[X]}((V/W)_{\overline{\phi}})$$

from which the conclusion immediately follows. □

As a consequence of the Addition Theorem, we prove now the counterpart of the Grassmann formula for the algebraic entropy:

Corollary 5.2 *Let $\phi : V \to V$ be a linear transformation, and let U and W be ϕ-invariant subspaces of V such that $V = U + W$. Then* $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_U) + \mathrm{ent}(\phi \restriction_W) - \mathrm{ent}(\phi \restriction_{U\cap W})$.

Proof Consider the short exact sequence

$$0 \to U \cap W \xrightarrow{f} U \oplus W \xrightarrow{g} U + W = V \to 0,$$

where $f : U \cap W \to U \oplus W$ is defined by $f(x) = (x, -x)$, and $g : U \oplus W \to U + W = V$ is defined by $g(x, y) = x + y$. Let $\psi = \phi \restriction_U \oplus \phi \restriction_W : U \oplus W \to U \oplus W$. Then the subspace

$$D = \ker g = \{(x, -x) \in U \oplus W : x \in U \cap W\}$$

of $U \oplus W$ is ψ-invariant and $(U \oplus W)/D \cong V$. Moreover, the induced linear transformation $\overline{\psi} : (U \oplus W)/D \to (U \oplus W)/D$ is conjugate to ϕ and so $\mathrm{ent}(\phi) = \mathrm{ent}(\overline{\psi})$ by Proposition 3.1. Analogously, it is possible to prove that $\mathrm{ent}(\psi \restriction_D) = \mathrm{ent}(\phi \restriction_{U\cap W})$. Moreover, $\mathrm{ent}(\psi) = \mathrm{ent}(\phi \restriction_U) + \mathrm{ent}(\phi \restriction_W)$ by Lemma 3.5. Hence, applying Theorem 5.1, we have

$$\mathrm{ent}(\phi \restriction_U) + \mathrm{ent}(\phi \restriction_V) = \mathrm{ent}(\psi) = \mathrm{ent}(\psi \restriction_D) + \mathrm{ent}(\overline{\psi}) = \mathrm{ent}(\phi \restriction_{U\cap W}) + \mathrm{ent}(\phi),$$

which concludes the proof. □

As we saw above, the algebraic entropy of linear transformations of K-vector spaces can be viewed as a discrete invariant for the category $\mathrm{Mod}(K[X])$. Proposition 3.3 says that it is upper continuous, and the Addition Theorem says that it is also additive. Now, upper continuous additive invariants for $\mathrm{Mod}(R)$ with values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$, where R is a commutative integral domain, have been investigated by Northcott and Reufel in [13] under the name of *length functions* (see also [17]). In particular, Theorem 2 in [13] states that a length function $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where R is an integral domain, coincides with $L(R) \cdot \mathrm{rk}_R$ provided that $L(R) < \infty$. From that theorem, we immediately derive the following

Theorem 5.3 (Uniqueness Theorem) *Given an arbitrary field K, the algebraic entropy* ent *is the unique length function $L : \mathrm{Mod}(K[X]) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for every finite dimensional vector space V, $L(B(V)) = \dim(V)$.*

Proof Let $L : \mathrm{Mod}(K[X]) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a length function such that, for every finite dimensional vector space V, $L(B(V)) = \dim(V)$; then $L(K[X]) = L(B(K)) = \dim(K) = 1$. Therefore, $L = \mathrm{rk}_{K[X]}$ by Theorem 2 in [13]. Hence the claim follows by Corollary 4.8. □

As an application of the above results, we will prove now that the algebraic entropy of the right and the left Bernoulli shift of the direct product is ∞ (see also [8]); let us denote respectively by $\hat{\beta} : \prod_{n\geq 0} Kx_n \to \prod_{n\geq 0} Kx_n$ and $\hat{\lambda} : \prod_{n\geq 0} Kx_n \to \prod_{n\geq 0} Kx_n$ such shifts.

Proposition 5.4 $\mathrm{ent}(\hat{\beta}) = \infty$ *and* $\mathrm{ent}(\hat{\lambda}) = \infty$.

Proof The canonical isomorphism of vector spaces $\prod_{n\geq 0} Kx_n \cong K[[X]]$, given by $(k_n x_n)_n \mapsto \sum_{n\geq 0} k_n X^n$, is also an isomorphism of $K[X]$-modules between $(\prod_{n\geq 0} Kx_n)_{\hat{\beta}}$ and $K[[X]]$. It is well known that $\mathrm{rk}_{K[X]}(K[[X]])$ is infinite and so Corollary 4.8 gives $\mathrm{ent}(\prod_{n\geq 0} Kx_n)_{\hat{\beta}}) = \infty$.

To see that $\mathrm{ent}(\hat{\lambda}) = \infty$, since $\mathrm{ent}(\prod_{n\geq 0} Kx_n)_{\hat{\lambda}}) = \mathrm{rk}_{K[X]}(\prod_{n\geq 0} Kx_n)_{\hat{\lambda}})$ by Corollary 4.8, we verify now that $\mathrm{rk}_{K[X]}(\prod_{n\geq 0} Kx_n)_{\hat{\lambda}})$ is infinite. To this end, for every $m \geq 2$, we provide m many independent elements of $(\prod_{n\geq 0} Kx_n)_{\hat{\lambda}}$. We consider first the case $m = 2$. Let $v^{(1)} = (v_n^{(1)})_n$ be defined by

$$v_n^{(1)} = \begin{cases} 1 & \text{if } n = (2k)! \text{ for some } k > 0, \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad v_n^{(2)} = \begin{cases} 1 & \text{if } n = (2k+1)! \text{ for some } k > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathrm{supp}(v^{(1)}) = \{(2k)! : k > 0\}$ and $\mathrm{supp}(v^{(2)}) = \{(2k+1)! : k > 0\}$. Now, let $f_1(X), f_2(X) \in K[X]$ and assume that $d_1 = \deg f_1(X) > 0$ and $d_2 = \deg f_2(X) > 0$. We have to show that $f_1(X)v^{(1)} + f_2(X)v^{(2)} \neq 0$. To this end, it suffices to find $h > 0$ such that $h \in \mathrm{supp}(f_1(X)v^{(1)}) \setminus \mathrm{supp}(f_2(X)v^{(2)})$. Let $d = \max\{d_1, d_2\}$ and $k > d$. Then

$$(2k-1)! < (2k)! - d < (2k)! < (2k+1)! - d < (2k+1)!.$$

Therefore, $(2k)! - d_1 \in \mathrm{supp}(f_1(X)v^{(1)}) \setminus \mathrm{supp}(f_2(X)v^{(2)})$, that concludes the proof for the case $m = 2$. It is possible to extend in an easy way this argument for an arbitrary $m > 0$. However, we leave it to the reader. □

We provide a sketch of a more structural proof of the equality $\mathrm{ent}(\hat{\lambda}) = \infty$ given in Proposition 5.4. Let F be the prime subfield of the field K. Let $M = \prod_{n\geq 0} Fx_n$ and consider M as an $F[X]$-module using the action of the left shift $\hat{\lambda}$. First, one has to prove that the torsion part of M as $F[X]$-module is countable: in fact, since $F[X]$ is countable, it is enough to prove that the submodule $M[f(X)]$ of M consisting of the elements annihilated by a fixed non-zero polynomial $f(X) \in F[X]$ is countable. To prove this fact, one can see that an element in $M[f(X)]$ has the coordinates of index $\geq \deg f(X)$ determined by the coefficients of $f(X)$ and the preceding coordinates; thus, there exists countably many of such elements. Thus, the rank of M as $F[X]$-module is the continuum; hence, for each natural number n, M contains a free submodule of rank n and, once tensored by K, this submodule becomes a submodule of the $K[X]$-module $\prod_{n\geq 0} Kx_n$, where the action of the indeterminate X is still that of the left shift $\hat{\lambda}$. This shows that $\mathrm{ent}(\hat{\lambda}) = \infty$.

6 Adjoint algebraic entropy

In this section, we introduce the adjoint algebraic entropy for linear transformations of vector spaces. In particular, first we give its definition and some basic examples, then we study its fundamental properties and lastly we prove the main theorems.

6.1 Definition

Let W be a subspace of V. The *codimension* of W in V is $\dim(V/W)$. If V is finite dimensional, then $\dim(V/W) = \dim V - \dim W$. Let $\mathcal{C}(V)$ be the family of all subspaces of V of finite codimension. We start giving basic properties of $\mathcal{C}(V)$.

Lemma 6.1 *Let $\phi : V \to V$ be a linear transformation. Then:*

(a) *$\mathcal{C}(V)$ is closed under finite intersections;*
(b) *$\mathcal{C}(V)$ is closed under counterimages, that is, $\phi^{-1}N \in \mathcal{C}(V)$ for every $N \in \mathcal{C}(V)$;*
(c) *if $\dim V$ is infinite, then $|\mathcal{C}(V)| = |K|^{\dim V}$.*

Proof (a) First, we show that if $N_1, N_2 \in \mathcal{C}(V)$, then $N_1 \cap N_2 \in \mathcal{C}(V)$. In fact, consider the linear transformation $\psi : V \to V/N_1 \oplus V/N_2$ defined by $v \mapsto (v + N_1, v + N_2)$. Since $\ker \psi = N_1 \cap N_2$, it follows that $V/(N_1 \cap N_2)$ is isomorphic to a subspace of $V/N_1 \oplus V/N_2$, and so $\dim(V/(N_1 \cap N_2)) \leq \dim(V/N_1 \oplus V/N_2) = \dim(V/N_1) + \dim(V/N_2)$, which is finite by hypothesis. Hence, $N_1 \cap N_2 \in \mathcal{C}(V)$. Proceeding by induction, it is possible to prove that each finite intersection of elements of $\mathcal{C}(V)$ is still in $\mathcal{C}(V)$.
(b) Let $N \in \mathcal{C}(V)$. Consider the linear transformation $\overline{\phi} : V/\phi^{-1}N \to V/N$ induced by ϕ; then $\overline{\phi}$ is injective and so $\dim(V/\phi^{-1}N)$ is finite, as $\dim(V/N)$ is finite. In other words, $\phi^{-1}N \in \mathcal{C}(V)$.
(c) It is well known that there is a bijection between the set $\mathcal{C}(V)$ and the set of the finite dimensional subspaces of the dual space V^*, and that this set has cardinality $\max(|K|, \dim V^*)$. Since $\dim V^* = |K|^{\dim V}$, we get that $\mathcal{C}(V) = |K|^{\dim V}$. □

To introduce the adjoint algebraic entropy, we start with some definitions. Let $N \in \mathcal{C}(V)$. For a linear transformation $\phi : V \to V$ and a positive integer n, let

$$B_n(\phi, N) = N \cap \phi^{-1}N \cap \cdots \cap \phi^{-n+1}N$$

and let

$$C_n(\phi, N) = \frac{V}{B_n(\phi, N)};$$

$C_n(\phi, N)$ is called the *n-th ϕ-cotrajectory* of N in V. Lemma 6.1 implies that $B_n(\phi, N)$ belongs to $\mathcal{C}(V)$ for every $n > 0$, hence $\dim(C_n(\phi, N))$ is finite for every $n > 0$. Let

$$B(\phi, N) = \bigcap_{n\geq 0} \phi^{-n}N \quad \text{and} \quad C(\phi, N) = \frac{V}{B(\phi, N)};$$

$C(\phi, N)$ is called the *ϕ-cotrajectory* of N in V. It is easy to check that $B(\phi, N)$ is the maximum ϕ-invariant subspace of N.

The *adjoint algebraic entropy of ϕ with respect to N* is

$$H^\star(\phi, N) = \lim_{n\to\infty} \frac{\dim(C_n(\phi, N))}{n}. \tag{6.1}$$

We will show now that this limit exists and is finite.

Lemma 6.2 *For every $n > 0$, let*

$$\gamma_n = \dim\left(\frac{C_{n+1}(\phi, N)}{C_n(\phi, N)}\right). \tag{6.2}$$

Then $\gamma_n = \dim\left(\frac{B_n(\phi,N)}{B_{n+1}(\phi,N)}\right)$ and the sequence of non-negative integers $\{\gamma_n\}_n$ is decreasing, and hence stationary.

Proof Since for every $n > 0$, $B_{n+1}(\phi, N)$ is a subspace of $B_n(\phi, N)$, it follows that

$$C_n(\phi, N) = \frac{V}{B_n(\phi, N)} \cong \frac{\frac{V}{B_{n+1}(\phi,N)}}{\frac{B_n(\phi,N)}{B_{n+1}(\phi,N)}} = \frac{C_{n+1}(\phi, N)}{\frac{B_n(\phi,N)}{B_{n+1}(\phi,N)}}.$$

Let $n > 1$. We intend to prove that $\frac{B_n(\phi,N)}{B_{n+1}(\phi,N)}$ is isomorphic to a subspace of $\frac{B_{n-1}(\phi,N)}{B_n(\phi,N)}$.

First, note that $\frac{B_n(\phi,N)}{B_{n+1}(\phi,N)} \cong \frac{B_n(\phi,N)+\phi^{-n}N}{\phi^{-n}N}$. From $B_n(\phi, N) = N \cap \phi^{-1}B_{n-1}(\phi, N) \leq \phi^{-1}B_{n-1}(\phi, N)$, it follows that

$$\frac{B_n(\phi, N) + \phi^{-n}N}{\phi^{-n}N} \leq A_n = \frac{\phi^{-1}B_{n-1}(\phi, N) + \phi^{-n}N}{\phi^{-n}N}.$$

Since the linear transformation $\tilde{\phi} : \frac{V}{\phi^{-n}N} \to \frac{V}{\phi^{-n+1}N}$, induced by ϕ, is injective, also its restriction to A_n is injective, and the image of A_n is contained in $L_n = \frac{B_{n-1}(\phi,N)+\phi^{-n+1}N}{\phi^{-n+1}N}$, which is isomorphic to $\frac{B_{n-1}(\phi,N)}{B_n(\phi,N)}$. Summarizing,

$$\frac{B_n(\phi, N)}{B_{n+1}(\phi, N)} \cong \frac{B_n(\phi, N) + \phi^{-n}N}{\phi^{-n}N} \leq A_n \rightarrowtail L_n \cong \frac{B_{n-1}(\phi, N)}{B_n(\phi, N)},$$

which concludes the proof. □

The following proposition shows that indeed it is possible to avoid the calculation of the limit in the definition of the adjoint algebraic entropy, that is, in (6.1).

Proposition 6.3 *Let $\phi : V \to V$ be a linear transformation and $N \in \mathcal{C}(V)$. Then $H^\star(\phi, N) = \gamma$, where γ is the value of the stationary sequence $\{\gamma_n\}_n$ for n large enough. In particular, $H(\phi, N) = 0$ precisely when the sequence $\{\dim(C_n(\phi, N))\}_n$ becomes stationary, equivalently, when $\gamma_n = 0$ for every n large enough.*

Proof For every $n > 0$, in view of the definition of the γ_n given in (6.2),

$$\gamma_n = \dim(C_{n+1}(\phi, N)) - \dim(C_n(\phi, N)). \tag{6.3}$$

By Lemma 6.2, the decreasing sequence $\{\gamma_n\}_n$ is stationary, so there exist $n_0 > 0$ and $\gamma \geq 0$ such that $\gamma_n = \gamma$ for every $n \geq n_0$.

Then, $\gamma = 0$ if and only if $\dim(C_{n+1}(\phi, N)) = \dim(C_n(\phi, N))$ for every $n \geq n_0$; in this case, $\dim(C(\phi, N)) = \dim(C_n(\phi, N))$ for every $n \geq n_0$. If $\gamma > 0$, since by (6.3) $\dim(C_{n_0+n}(\phi, N)) = n\gamma + \dim(C_{n_0}(\phi, N))$ for every $n \geq 0$, we have

$$H^\star(\phi, N) = \lim_{n\to\infty} \frac{\dim(C_{n_0+n}(\phi, N))}{n_0 + n} = \lim_{n\to\infty} \frac{\dim(C_{n_0}(\phi, N)) + n\gamma}{n_0 + n} = \gamma.$$

This concludes the proof. □

An easy computation shows that $H^\star(\phi, M)$ is an anti-monotone function on M:

Lemma 6.4 *Let ϕ be a linear transformation and let $N, M \in \mathcal{C}(V)$. If $N \leq M$, then $B_n(\phi, N) \leq B_n(\phi, M)$ and so* $\dim(C_n(\phi, N)) \geq \dim(C_n(\phi, M))$. *Therefore, $H^\star(\phi, N) \geq H^\star(\phi, M)$.*

Now we can define the *adjoint algebraic entropy* of $\phi : V \to V$ as the quantity

$$\mathrm{ent}^\star(\phi) = \sup\{H^\star(\phi, N) : N \in \mathcal{C}(V)\}.$$

By Proposition 6.3, the value of the adjoint algebraic entropy $\mathrm{ent}^\star(\phi)$ of a linear transformation ϕ is either a non-negative integer or ∞ (we will see in Theorem 6.15 that $\mathrm{ent}^\star(\phi)$ is either zero or ∞), and $\mathrm{ent}^\star(\phi) = \infty$ if and only if there exists a countable family $\{N_k\}_{k\geq 0} \subseteq \mathcal{C}(V)$ such that $H^\star(\phi, N_k)$ converges to ∞. Now we give some easy examples.

Example 6.5 (a) If $\phi : V \to V$ is a linear transformation and $N \in \mathcal{C}(V)$ is ϕ-invariant, then $H^\star(\phi, N) = 0$. Indeed, $\phi^{-1}N \supseteq N$, so $B_n(\phi, N) = N$ for every $n > 0$ and consequently $C_n(\phi, N) = V/N$ for every $n > 0$; hence, $H^\star(\phi, N) = 0$.

(b) For V any vector space, $\mathrm{ent}^\star(id_V) = \mathrm{ent}^\star(0_V) = 0$, by a trivial application of item (a).

(c) Let $x \in K$ and $\dot{x} : V \to V$ the linear transformation of V defined by $v \mapsto xv$ for every $v \in V$. Then item (a) shows that $\mathrm{ent}^\star(\dot{x}) = 0$, since all subspaces of V are $\dot{x}$-invariant.

6.2 Basic properties

In this section we present the basic properties of the adjoint algebraic entropy, in analogy to the basic properties of the algebraic entropy discussed in Sect. 3. We omit the proofs, which are a simplified version of the proofs given in [5] and which can be verified by the reader as a straightforward application of the definition, imitating the proofs of the corresponding properties of the algebraic entropy.

Property 6.6 *Let $\phi : V \to V$ be a linear transformation and $\alpha : V \to W$ an isomorphism of vector spaces. Then,* $\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\alpha\phi\alpha^{-1})$.

The following property shows that the algebraic entropy is monotone under restrictions to subspaces and quotients.

Proposition 6.7 *Let $\phi : V \to V$ be a linear transformation, W a ϕ-invariant subspace of V and $\overline{\phi} : V/W \to V/W$ the linear transformation induced by ϕ. Then* $\mathrm{ent}^\star(\phi) \geq \max\{\mathrm{ent}^\star(\phi\restriction_W), \mathrm{ent}^\star(\overline{\phi})\}$. *If $W \in \mathcal{C}(V)$, then* $\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\phi\restriction_W)$.

The following is a logarithmic law for the adjoint algebraic entropy.

Property 6.8 *Let $\phi : V \to V$ be a linear transformation. Then* $\mathrm{ent}^\star(\phi^k) = k \cdot \mathrm{ent}^\star(\phi)$ *for every $k \geq 0$. If ϕ is an automorphism, then* $\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\phi^{-1})$*; in particular,* $\mathrm{ent}^\star(\phi^k) = |k| \cdot \mathrm{ent}^\star(\phi)$ *for every integer k.*

Property 6.8 has the next good consequence, which will be applied in the proof of Theorem 6.15.

Corollary 6.9 *Let $\phi : V \to V$ be a linear transformation. If $f(X) \in K[X]$, then* $\mathrm{ent}^\star(f(\phi)) \leq \deg f \cdot \mathrm{ent}^\star(\phi)$.

Proof Let $f = a_0 + a_1X + \cdots + a_kX^k$, where $k = \deg f$ and $a_0, a_1, \ldots, a_k \in K$. Let $n > 0$ and $N \in \mathcal{C}(V)$. Then an easy check shows that $B_n(f(\phi), N) \geq B_{kn}(\phi, N)$ and that $C_{nk}(\phi, N) = C_n(\phi^k, B_k(\phi, N))$. Consequently,

$$\dim\left(C_n(f(\phi), N)\right) \leq \dim\left(C_{kn}(\phi, N)\right) = \dim\left(C_n(\phi^k, B_k(\phi, N))\right).$$

Hence, $H^\star(f(\phi), N) \leq H^\star(\phi^k, B_k(\phi, N)) \leq \mathrm{ent}^\star(\phi^k)$, and so $\mathrm{ent}^\star(f(\phi)) \leq \mathrm{ent}^\star(\phi^k)$. By Property 6.8, $\mathrm{ent}^\star(\phi^k) = k \cdot \mathrm{ent}^\star(\phi)$. □

The following is a particular case of the Addition Theorem.

Proposition 6.10 *If $V = V_1 \oplus V_2$ for some subspaces V_1, V_2 of V, and $\phi = \phi_1 \oplus \phi_2 : V \to V$ for some linear transformations $\phi_i : V_i \to V_i$, $i = 1, 2$, then* $\mathrm{ent}^\star(\phi) = \mathrm{ent}^\star(\phi_1) + \mathrm{ent}^\star(\phi_2)$.

6.3 Duality theorem, dichotomy theorem and addition theorem

Let V^* be the dual space of the vector space V. If U is a subspace of V, we set as usual:

$$U^\perp = \{\chi \in V^* : \chi(x) = 0, \text{ for every } x \in U\}$$

which is the *annihilator* of U in V^*. Moreover, for a linear transformation $\phi : V \to V$, the adjoint linear transformation $\phi^* : V^* \to V^*$ of ϕ is defined by $\phi^*(\chi) = \chi \circ \phi$ for every $\chi \in V^*$.

We collect here some known facts concerning the dual space.

(i) If V is a finite dimensional vector space, then $V^* \cong V$.
(ii) For a family $\{V_i : i \in I\}$ of vector spaces, $(\bigoplus_{i\in I} V_i)^* \cong \prod_{i\in I} V_i^*$.
(iii) If V is a vector space and W a subspace of V, then $W^\perp \cong (V/W)^*$ and $V^*/W^\perp \cong W^*$.
(iv) If $V_1, \ldots, V_n$ are subspaces of a vector space V, then $\left(\sum_{i=1}^n V_i\right)^\perp \cong \bigcap_{i=1}^n V_i^\perp$ and $\left(\bigcap_{i=1}^n V_i\right)^\perp \cong \sum_{i=1}^n V_i^\perp$.

We recall also the following two properties that will be applied to prove Theorem 6.12.

Proposition 6.11 *Let $\phi : V \to V$ be a linear transformation and W a subspace of V. Then:*

(a) *W is ϕ-invariant if and only if $W^\perp$ is ϕ^*-invariant;*
(b) *$(\phi^{-n}W)^\perp = (\phi^*)^n W^\perp$ for every $n > 0$.*

Proof (a) If W is ϕ-invariant, then $\phi^*(\chi)W = (\chi \circ \phi)W \subseteq \chi W = 0$ for every $\chi \in W^\perp$, that is, $W^\perp$ is ϕ^*-invariant. Suppose now that W is not ϕ-invariant, that is, $\phi W \not\subseteq W$. Let $x \in \phi W \setminus W$ and define $\chi : V \to K$ so that $\chi W = 0$ and $\chi x \neq 0$. Then $\phi^*(\chi)W = \chi \circ \phi W \neq 0$, and so $W^\perp$ is not ϕ^*-invariant.

(b) We prove the result for $n = 1$, that is, $(\phi^{-1}W)^\perp = \phi^* W^\perp$. Indeed, the general case easily follows as $(\phi^*)^n = (\phi^n)^*$ for every $n > 0$. Let $\pi' : V \to V/\phi^{-1}W$ and $\pi : V \to V/W$ be the canonical projections. Let $\widetilde{\phi} : V/\phi^{-1}W \to V/W$ be the linear transformation induced by ϕ, and note that $\widetilde{\phi}$ is injective. Let us consider the following diagram:

$$\begin{tikzcd}
V \arrow[r, "\pi'"] \arrow[rr, bend left, "\chi"] \arrow[d, "\phi"'] & \frac{V}{\phi^{-1}W} \arrow[r, "\eta"] \arrow[d, "\widetilde{\phi}"] & K \\
V \arrow[r, "\pi"] \arrow[rru, bend right, "\theta"'] & \frac{V}{W} \arrow[ru, "\xi"'] &
\end{tikzcd}$$

where the square commutes. If we take a $\chi \in (\phi^{-1}W)^\perp$, then $\chi = \eta \circ \pi'$ for a suitable $\eta : V/\phi^{-1}W \to K$. Since $\widetilde{\phi}$ is injective, η can be extended to $\xi : V/W \to K$, i.e., $\eta = \xi \circ \widetilde{\phi}$. Therefore,

$$\chi = \eta \circ \pi' = \xi \circ \widetilde{\phi} \circ \pi' = \xi \circ \pi \circ \phi,$$

which shows that $\chi = \phi^*(\theta)$, where $\theta = \xi \circ \pi \in W^\perp$. This proves the inclusion $(\phi^{-1}W)^\perp \subseteq \phi^* W^\perp$. Now let $\chi \in \phi^* W^\perp$. Then $\chi = \phi^*(\theta) = \theta \circ \phi$, where $\theta \in W^\perp$. So $\theta = \xi \circ \pi$, for some $\xi : V/W \to K$. Take $\eta = \xi \circ \widetilde{\phi}$ (since $\widetilde{\phi}$ is injective we can think that $\eta = \xi \restriction_{V/\phi^{-1}W}$). Therefore,

$$\chi = \theta \circ \phi = \xi \circ \pi \circ \phi = \xi \circ \widetilde{\phi} \circ \pi' = \eta \circ \pi' \in (\phi^{-1}W)^\perp.$$

This proves the inclusion $(\phi^{-1}W)^\perp \supseteq \phi^* W^\perp$ and concludes the proof. □

It is now possible to prove the main theorems on the adjoint algebraic entropy, starting from the following result connecting it with the algebraic entropy.

Theorem 6.12 (Duality Theorem) *Let $\phi : V \to V$ be a linear transformation. Then* $\mathrm{ent}^\star(\phi) = \mathrm{ent}(\phi^*)$.

Proof Let $N \in \mathcal{C}(V)$. Then, $F = N^\perp$ is a finite dimensional subspace of V^* by facts (iii) and (i). By Proposition 6.11(b), $(\phi^{-n}N)^\perp = (\phi^*)^n F$ for every $n \geq 0$. Hence, $B_n(\phi, N)^\perp = T_n(\phi^*, F)$ for every $n > 0$ by fact (iv). It follows that

$$\dim\left(C_n(\phi, N)\right) = \dim\left(C_n(\phi, N)^*\right) = \dim\left(B_n(\phi, N)^\perp\right) = \dim\left(T_n(\phi^*, F)\right)$$

for every $n > 0$, and this concludes the proof. □

We see now that the value of the adjoint algebraic entropy on the right and the left Bernoulli shift is ∞. A direct computation of this fact is given in [9].

Proposition 6.13 $\mathrm{ent}^\star(\beta) = \mathrm{ent}^\star(\lambda) = \infty$.

Proof Identify K^* with K, and so also $K^{\mathbb{N}}$ with $(K^{(\mathbb{N})})^*$. Let $\chi = (a_0, a_1, \ldots) \in K^{\mathbb{N}}$ and consider the i-th canonical vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$ (where 1 is in i-th position) of the canonical basis of $K^{(\mathbb{N})}$ for $i \geq 0$. Then $\chi(e_i) = a_i$ for every $i \geq 0$, and hence $\chi(x) = \sum_{i \geq 0} a_i x_i$ for every $x = (x_i)_{i \geq 0} \in K^{(\mathbb{N})}$. Therefore, $\beta^*(\chi) = \chi \circ \beta = (a_1, a_2, \ldots) = \hat{\lambda}(\chi)$, because $\chi \circ \hat{\beta}(e_0) = 0$ and $\chi \circ \beta(e_i) = a_{i+1}$ for every $i > 0$. Analogously, $\lambda^*(\chi) = \chi \circ \lambda = (0, a_0, a_1, \ldots) = \hat{\beta}(\chi)$, because $\chi \circ \hat{\lambda}(e_0) = 0$ and $\chi \circ \hat{\beta}(e_i) = a_{i-1}$ for every $i > 0$. So, we have proved that

$$\beta^* = \hat{\lambda} \quad \text{and} \quad \lambda^* = \hat{\beta}.$$

By Proposition 5.4, $\mathrm{ent}(\hat{\beta}) = \mathrm{ent}(\hat{\lambda}) = \infty$, hence $\mathrm{ent}^\star(\beta) = \mathrm{ent}(\hat{\lambda}) = \infty$ and $\mathrm{ent}^\star(\lambda) = \mathrm{ent}(\hat{\beta}) = \infty$ by Theorem 6.12. □

As a consequence of Proposition 6.13, we can prove the following result relating the algebraic entropy with the adjoint algebraic entropy, which will be applied in the proof of Theorem 6.15.

Corollary 6.14 *Let $\phi : V \to V$ be a linear transformation. If* $\mathrm{ent}(\phi) > 0$*, then* $\mathrm{ent}^\star(\phi) = \infty$.

Proof By Theorem 4.3, $\mathrm{ent}(\phi) > 0$ is equivalent to the existence of an infinite trajectory $T(\phi, x) = \bigoplus_{n\geq 0} K\phi^n x$ for some $x \in V$. Then, $\phi \restriction_{T(\phi,x)}$ is a right Bernoulli shift, and so $\mathrm{ent}^\star(\phi \restriction_{T(\phi_p,x)}) = \infty$ by Proposition 6.13. Now Property 6.7 yields $\mathrm{ent}^\star(\phi) = \infty$ as well. □

The next theorem is one of the main results of this section; it has as immediate consequence the Dichotomy Theorem.

Theorem 6.15 *Let $\phi : V \to V$ be a linear transformation. Then the following conditions are equivalent:*

(a) *ϕ is algebraic;*
(b) $\mathrm{ent}^\star(\phi) = 0$*;*
(c) $\mathrm{ent}^\star(\phi)$ *is finite.*

Proof (a) $\Rightarrow$ (b) By hypothesis there exists $f(X) \in K[X]$ of $\deg f = n > 0$ such that $f(\phi)(V) = 0$. An easy computation shows that $B(\phi, N) = B_n(\phi, N)$ for every $N \in \mathcal{C}(V)$. Consequently, $C(\phi, N) = C_n(\phi, N)$, and hence $H^\star(\phi, N) = 0$ for every $N \in \mathcal{C}(V)$, that is, $\mathrm{ent}^\star(\phi) = 0$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Assume by way of contradiction that ϕ is not algebraic, that is, V_ϕ is not bounded. We prove that $\mathrm{ent}^\star(\phi) = \infty$.

If V_ϕ is not torsion (i.e., ϕ is not locally algebraic), then $\mathrm{ent}(\phi) > 0$ by Theorem 4.3, and so Corollary 6.14 gives $\mathrm{ent}^\star(\phi) = \infty$. Thus, let us assume that V_ϕ is torsion.

First, suppose that the module V_ϕ is not reduced. Then there exist an irreducible polynomial $f(X) \in K[X]$ and an independent family of elements $\{v_n\}_{n\geq 0} \subseteq V$ such that

$$f(\phi)(v_0) = 0,\ f(\phi)(v_1) = v_0, \dots, f(\phi)(v_{n+1}) = v_n, \dots.$$

Then, $f(\phi)$ is a left Bernoulli shift on $\langle v_n : n \geq 0\rangle$, and $\mathrm{ent}^\star(f(\phi)) = \infty$ by Proposition 6.13. By Corollary 6.9, $\mathrm{ent}^\star(\phi) = \infty$ as well.

Finally, suppose that V_ϕ is a reduced torsion unbounded $K[X]$-module. Then, V_ϕ contains as $K[X]$-submodule an infinite direct sum $\bigoplus_{n\geq 0} V_n$, where either $V_n = K[X]/(f_n(X))$ for each n, with $\{f_n(X)\}_n$ a sequence of different monic irreducible polynomials, or $V_n = K[X]/(f(X)^{r_n})$ for every n, with $f(X)$ a fixed irreducible polynomial and $\{r_n\}_n$ a strictly increasing sequence of positive integers. In both cases, each V_n is a ϕ-invariant finite dimensional subspace of V and a torsion cyclic $K[X]$-module.

Let us assume, without loss of generality that $V_\phi = \bigoplus_{n\geq 0} V_n$. Let $\phi_n = \phi \restriction_{V_n}$ for every $n \geq 0$. Consider $\phi^* : V^* \to V^*$. By facts (i) and (ii), $V^* \cong \prod_{n\geq 0} V_n^*$ and $V_n^* \cong V_n$ for every $n \geq 0$. Moreover, $V_m^* \cong (\bigoplus_{n\neq m} V_n)^\perp$ and so V_m^* is ϕ^*-invariant by Proposition 6.11(a), and $\phi^* \restriction_{V_m^*} = \phi_m^*$. As ϕ_n and its adjoint map ϕ_n^* have the same minimal polynomial, the two $K[X]$-modules $(V_n)_{\phi_n}$ and $(V_n^*)_{\phi_n^*}$ are isomorphic.

Now, in both cases considered above, $V^* \cong \prod_{n\geq 0} V_n^*$ is not a torsion $K[X]$-module, being a direct product of an unbounded sequence of cyclic modules; hence, ϕ^* is not locally algebraic, that is, $\mathrm{ent}(\phi^*) > 0$, by Theorem 4.3.

We can now easily conclude that $\mathrm{ent}(\phi^*) = \infty$. In fact, there exists a partition $\mathbb{N} = \dot{\bigcup}_{i\in\mathbb{N}} N_i$ of $\mathbb{N}$, where each N_i is infinite. Then, $V^* \cong \prod_{n\geq 0} V_n^* \cong \prod_{i\geq 0} W_i$, where $W_i = \prod_{n\in N_i} V_n^*$ is ϕ^*-invariant and has the same properties of V^* for every $i \geq 0$. By the previous part of the proof, $\mathrm{ent}(\phi^* \restriction_{W_i}) \geq 1$ for every $i \geq 0$ and so $\mathrm{ent}(\phi^*) \geq \sum_{i\geq 0} \mathrm{ent}(\phi^* \restriction_{W_i}) = \infty$. By Theorem 6.12, $\mathrm{ent}^\star(\phi) = \infty$ as well. □

Corollary 6.16 (Dichotomy Theorem) *Let $\phi : V \to V$ be a linear transformation. Then either* $\mathrm{ent}^\star(\phi) = 0$ *or* $\mathrm{ent}^\star(\phi) = \infty$.

Applying Theorem 6.15, it is now possible to give a short direct proof of the Addition Theorem for the adjoint algebraic entropy. If one would avoid the use of Theorem 6.15, one can prove it by applying the above properties of the duality, the Addition Theorem for the algebraic entropy and Theorem 6.12; this proof was given for example in [5].

Theorem 6.17 (Addition Theorem) *Let $\phi : V \to V$ be a linear transformation, W a ϕ-invariant subspace of V and $\overline{\phi} : V/W \to V/W$ the linear transformation induced by ϕ. Then*

$$\mathrm{ent}^{\star}(\phi) = \mathrm{ent}^{\star}(\phi \restriction_W) + \mathrm{ent}^{\star}(\overline{\phi}).$$

Proof By Proposition 6.7 and Theorem 6.15, it suffices to prove that $\mathrm{ent}^{\star}(\phi) = 0$ when $\mathrm{ent}^{\star}(\phi \restriction_W) = \mathrm{ent}^{\star}(\overline{\phi}) = 0$. In view of Theorem 6.15, $\phi \restriction_W$ and $\overline{\phi}$ are algebraic, that is, there exist $f(X), g(X) \in K[X]$ such that $f(\phi)(W) = 0$ and $g(\overline{\phi})(V/W) = 0$, i.e., $g(\phi)(V) \subseteq W$. Let $h(X) = f(X)g(X)$; then $h(\phi)(v) = (fg)(\phi)(v) = f(\phi)(g(\phi)(v)) = 0$ for every $v \in V$. This shows that ϕ is algebraic, and hence $\mathrm{ent}^{\star}(\phi) = 0$ by Theorem 6.15. □

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