PSEUDOCOMPACT TOTALLY DENSE SUBGROUPS

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ABSTRACT. It was shown in [9] that if a compact abelian group K admits a proper totally dense pseudocompact subgroup, then K cannot have a torsion closed G_δ-subgroup; moreover this condition was shown to be also sufficient under LH. We prove in ZFC that this condition ensures actually the existence of a proper totally dense subgroup H of K that contains an ω-bounded dense subgroup of K (such an H is necessarily pseudocompact). This answers two questions from [9].

1. INTRODUCTION

A subgroup H of a topological abelian group G is said to be *totally dense* if H densely intersects every closed subgroup of G [15]. The interest in the totally dense subgroups of a compact abelian group K stems from the fact that they are precisely those dense subgroups of K, that satisfy the open mapping theorem [13, 7, 8]. In other words the totally dense subgroups of a compact abelian group K have a compactness-like property. Therefore it is natural to expect that in presence of other compactness-like properties they may turn out to be actually compact, hence coincide with K. Indeed no compact abelian group K has a proper totally dense countably compact subgroup [9, Theorem 1.4]. We remind here that a topological group G is *countably compact* if every countable open cover of G has a finite subcover, G is *pseudocompact* if every continuous real-valued function of G is bounded and G is *ω-bounded* if every countable subset of G is contained in a compact subgroup of G. The following chain of implications holds:

$$\text{compact} \Rightarrow \omega\text{-bounded} \Rightarrow \text{countably compact} \Rightarrow \text{pseudocompact}.$$

Comfort and Soundararajan [4] studied for the first time the question of when a compact abelian group K admits a proper totally dense *pseudocompact* subgroup and provided a solution in the case when K is connected (iff K is not metrizable). Later Comfort and Robertson [2] studied the compact abelian groups K that admit a totally dense pseudocompact subgroup of size $< |K|$. They showed that this question depends only on the cardinal $\alpha = w(K)$ and *ZFC cannot decide whether there exists a compact abelian group K with totally dense pseudocompact subgroups of size $< |K|$* [2, Theorem 6.3].

For the sake of brevity let us give the following definition (see [5, Definition 1.2]).

Definition 1.1. A compact abelian group K *has the property TD_ω* if there exists a proper totally dense subgroup H of K that contains an ω-bounded dense subgroup.

Since the subgroup H must be pseudocompact, the property TD_ω implies that K has a proper totally dense pseudocompact subgroup. In these terms the following theorem was proved extending the ideas from [4] (a self-contained proof of this theorem is given in §2 for reader's convenience).

Theorem 1.2. [9, Theorem 1.9] *Every compact abelian group with non-metrizable connected component of zero has the property TD_ω.*

The following *necessary* condition for the existence of proper totally dense pseudocompact subgroups of compact abelian groups was given in [9, Theorem 1.7].

Theorem 1.3. *Let G be an abelian topological group which has a torsion closed G_δ-subgroup N. Then G contains no proper totally dense pseudocompact subgroups.*

Note that an abelian topological group which has no torsion closed G_δ-subgroup is non-metrizable.

Moreover, under the additional set-theoretic assumption $2^{\omega_1} = \mathfrak{c}$, known under the name of *Lusin's Hypothesis* (LH), this necessary condition was shown to be the unique constraint for the existence of proper totally dense pseudocompact subgroups of compact abelian groups:

Theorem 1.4. [9, Theorem 1.8] *Under LH a compact abelian group K contains a proper totally dense pseudocompact subgroup if and only if K has no torsion closed G_δ-subgroups.*

One of the implications of Theorem 1.4 is Theorem 1.3. To prove the other one in [9] the authors first proved it in case the group has cardinality $\mathfrak{c}$. In the general case, supposing that K has no torsion closed G_δ-subgroups, they constructed a continuous surjective homomorphism from K to a compact abelian group H with the same property and with $w(H) = \omega_1$. Under LH $|H| = 2^{\omega_1} = \mathfrak{c}$ and so H has a proper totally dense pseudocompact subgroup by the first part. The inverse image of this subgroup of H is a proper totally dense pseudocompact subgroup of K.

One of the open problems left in [9, Problem 1.11] was whether the assumption of LH in Theorem 1.4 can be removed. Another open problem left there [9, Problem 1.12] was whether a compact abelian group without torsion closed G_δ-subgroups has the property TD_ω. The main result of our paper (exposed in the next theorem) resolves positively both problems providing, as a by-product, also *independent* and shorter proofs of Theorems 1.2 and 1.4 as well as of the conjecture made in [5, Remark 3.20].

Theorem 1.5. *For a compact abelian group K the following conditions are equivalent:*
  (a) *K has a proper totally dense pseudocompact subgroup;*
  (b) *K has no closed torsion G_δ-subgroups;*
  (c) *K has the property TD_ω.*

The proof of Theorem 1.5 is quite different from that of Theorem 1.4 given in [9]. In fact, following [5] we add a third condition, that is K has the property TD_ω, which seems to be stronger than the others, but it turns out to be equivalent to them and it helps to prove the theorem in its full generality.

The plan of the proof is the following. It is easy to see that all these conditions imply the non-metrizability of K. The groups K with non-metrizable connected

component $c(K)$ are settled by Theorem 1.2. In case $c(K)$ is metrizable one can consider the totally disconnected quotient $K_1 = K/c(K)$ that is again non-metrizable, provided K is non-metrizable. Now every proper totally dense pseudocompact subgroup H of K_1 defines (via inverse image) a proper totally dense pseudocompact subgroup of K (see Lemma 2.5). In this way the main difficulty is confined in the totally disconnected case. In §4 we provide various results on the structure of totally disconnected compact abelian groups; this allows us to resolve the problem in the "local case" (i.e., for abelian pro-p-groups) in Lemma 5.1. Then this is applied in the proof of the general case given in §5.

Example 1.6. Assuming LH, there exists a compact abelian group that has a totally dense pseudocompact subgroup which does not contain any non-trivial dense ω-bounded subgroup (we give the construction of such an example in §5).

Every totally dense subgroup H of K is dense *essential* in K (i.e. H non-trivially intersects each non-trivial closed subgroup of K). In a forthcoming paper [12] the second named author will characterize compact abelian groups which admit proper dense essential pseudocompact subgroups in a theorem analogous to Theorem 1.5.

Notation and terminology. We denote by $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ respectively the set of integers, the set of primes, the set of natural numbers and the set of positive integers. For $m \in \mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual topology. For a prime $p \in \mathbb{P}$ the symbol $\mathbb{Z}_p$ is used for the group of p-adic integers.

Let G be an abelian group. The subgroup of torsion elements of G is $t(G)$ and $G[m]$ is the subset of all elements x of G such that $mx = 0$. We denote by $r_0(G)$ the free-rank of G. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum.

Throughout this paper all topological groups are Hausdorff. For a topological group G we use $w(G)$ to indicate the weight of G, that is the minimal cardinality of a base for the topology of G. We denote by $c(G)$ the connected component of 0 in G. If $c(G)$ is trivial, the group G is said to be *totally disconnected*. If M is a subset of G then $\langle M \rangle$ is the smallest subgroup of G containing M and $\overline{M}$ is the closure of M in G. The Pontryagin dual of a topological abelian group G is denoted by $\widehat{G}$. For a product K^σ of a group K with $\sigma > \omega$ we denote by ΣK^σ the Σ-product centered at 0 of K^σ, that is the set of all elements with support of countable cardinality. For undefined terms see [10].

2. Groups with the property TD_ω

Let us give an example of a group that has the property TD_ω.

Example 2.1. [5, Example 3.15] Let $K_p = \mathbb{Z}(p)^{\omega_1}$ and $K = \prod_{p \in \mathbb{P}} K_p$. Note that $K = S^{\omega_1}$ with $S = \prod_{p \in \mathbb{P}} \mathbb{Z}(p)$. The subgroup $H = \Sigma S^{\omega_1} + t(K)$ is a proper totally dense pseudocompact subgroup of K since $t(K) = \bigoplus_{p \in \mathbb{P}} K_p$ is totally dense in K, while ΣS^{ω_1} is a dense ω-bounded (hence pseudocompact) subgroup of K.

In this example works the idea from [2] to construct the totally dense pseudocompact subgroups as sums of two subgroups: one totally dense and the other dense pseudocompact. The problem in general is to ensure that such a sum is a proper subgroup. Lemma 2.3 offers a completely different solution.

Lemma 2.2. *Let K be an abelian group, C a torsion free subgroup of K and B a subgroup of K maximal with the property $B \cap C = \{0\}$. Then $B \cap mN = m(B \cap N)$ for every $0 \neq m \in \mathbb{Z}$ and for every subgroup N of K. In particular, if $p \in \mathbb{P}$, then $B \cap N \nsubseteq pN$ for every subgroup N of K isomorphic to $\mathbb{Z}_p$, whenever $r_0(C) < \mathfrak{c}$.*

Proof. If $x \in B \cap mN$, then $x = mz$, with $z \in N$. Assume that $z \notin B$. It follows that B is a proper subgroup of $B_1 = B + \langle z \rangle$. So there exists $y \in B_1 \cap C$, $y \neq 0$, that is $y = b + kz \in C$, where $b \in B$ and $k \in \mathbb{Z}$. Then $my = mb + kmz \in B \cap C = \{0\}$ and hence $my = 0$. As C is torsion free, we conclude that $y = 0$, finding a contradiction.

Suppose that $r_0(C) < \mathfrak{c}$ and assume for a contradiction that N is a subgroup of K isomorphic to $\mathbb{Z}_p$ with $B \cap N \subseteq pN$. Then $B \cap N = B \cap pN$. Applying the first part we have also $B \cap pN = p(B \cap N) = p(B \cap pN)$. By induction $B \cap pN = p^n(B \cap pN)$ for every $n \in \mathbb{N}$. Hence $B \cap pN = \bigcap_{n=1}^{\infty} p^n(B \cap pN)$, but $\bigcap_{n=1}^{\infty} p^n(B \cap N) \subseteq \bigcap_{n=1}^{\infty} p^n N = \{0\}$. So $B \cap N = \{0\}$. If φ is the natural projection of K onto K/B, this yields that $\varphi \restriction_N$ is injective and consequently $r_0(K/B) \geq r_0(N) = \mathfrak{c}$.

To get a contradiction we prove that $r_0(K/B) \leq r_0(C) < \mathfrak{c}$. Suppose that $r_0(K/B) > r_0(C)$. Then $\varphi(C)$ is a torsion free subgroup of K/B such that $r_0((K/B)/\varphi(C)) \geq 1$. So there exists an infinite cyclic subgroup C_1 of K/B such that $C_1 \cap \varphi(C) = \{0\}$. Now take $\varphi^{-1}(C_1)$ and observe that $\varphi^{-1}(C_1) \cap C = B \cap C = \{0\}$ and $\varphi^{-1}(C_1) \supsetneq \ker \varphi = B$, contradicting the maximality of B. $\square$

Now we are able to prove the following lemma, that produces totally dense subgroups containing a given subgroup. It was announced without a proof in [5, Lemma 3.16].

Lemma 2.3. *Let K be a compact abelian group that admits a subgroup B such that $r_0(K/B) \geq 1$. Then K has a proper totally dense subgroup H that contains B.*

Proof. Since $r_0(K/B) \geq 1$, there exists a cyclic infinite subgroup C of K such that $B \cap C = \{0\}$. Let $B_1 = B + t(K)$; then $B_1 \cap C = \{0\}$. By Zorn's Lemma there exists a subgroup H of K such that $H \supseteq B_1$, $H \cap C = \{0\}$ and H is maximal with respect to these two properties. It immediately follows that $H \supseteq B$ and that H is a proper subgroup of K. Moreover $t(K) \subseteq H$ and by Lemma 2.2 $H \cap N \nsubseteq pN$ holds for every subgroup N of K isomorphic to $\mathbb{Z}_p$ and for every $p \in \mathbb{P}$. Now apply the local criterion for total density [8, Theorem 4.3.7] to conclude that H is totally dense in K. $\square$

A subgroup H of a topological group G is G_δ-*dense* in G if it non-trivially intersects each G_δ-subset of G. If K is a compact group, by the Comfort and Ross theorem [3] a dense subgroup H of K is pseudocompact if and only if it is G_δ-dense in K. Therefore, if the subgroup B of K in Lemma 2.3 is dense and pseudocompact, then H is pseudocompact too. This will be used when this lemma will be applied in the following proposition, that in particular covers Example 2.1.

Proposition 2.4. [5, Example 3.17] *Let K be a non-torsion compact abelian group. Then K^{ω_1} has the property TD_ω. In particular $\mathbb{T}^{\omega_1}$ has the property TD_ω.*

Proof. In K^{ω_1} the Σ-product ΣK^{ω_1} is a dense ω-bounded subgroup which meets trivially the diagonal subgroup $\Delta_{K^{\omega_1}}$. Since $\Delta_{K^{\omega_1}} \cong K$, consequently $r_0(\Delta_{K^{\omega_1}}) \geq \mathfrak{c}$, that is $r_0(K^{\omega_1}/\Sigma K^{\omega_1}) \geq \mathfrak{c}$. Now apply Lemma 2.3 to conclude that K^{ω_1} has the property TD_ω. $\square$

Here we give a basic property of condition TD_ω, that will be used in the proof of our Main Theorem 5.2.

Lemma 2.5. [5, Lemma 3.12] *Let K be a compact abelian group that admits a continuous surjective homomorphism f onto a compact abelian group L. If L has the property TD_ω, then K has the property TD_ω too.*

Proof. There exists a proper totally dense subgroup H of L such that H contains an ω-bounded dense subgroup M. So $f^{-1}(H)$ is a proper totally dense subgroup of K and it contains $f^{-1}(M)$ which is dense and ω-bounded. $\square$

Proof of Theorem 1.2. A standard application of Pontryagin duality shows that K admits a surjective continuous homomorphism onto $\mathbb{T}^{\omega_1}$. Now Lemma 2.5 can be applied, as $\mathbb{T}^{\omega_1}$ has the property TD_ω by Proposition 2.4. $\square$

3. Singular Groups

In order to find a more "algebraic" form of the condition of Theorem 1.3 and Theorem 1.4, we note that for a compact abelian group K the family $\Lambda(K)$ of all closed G_δ-subgroups of K is closed under countable intersections. Moreover $N \in \Lambda(K)$ if and only if K/N is metrizable.

Definition 3.1. A compact abelian group K is *singular* if there exists $m \in \mathbb{N}_+$ such that $K[m] \in \Lambda(K)$.

The next lemma offers an alternative form for singularity of compact abelian groups (mK is metrizable compact for some $m \in \mathbb{N}_+$). It is useful checking stability of this property under taking subgroups and quotients.

Lemma 3.2. [6, Lemma 4.1] *Let G be a topological abelian group and $m \in \mathbb{N}_+$.*
   (a) *If mG is metrizable, then $G[m] \in \Lambda(G)$.*
   (b) *If G is compact, then $G[m] \in \Lambda(G)$ implies that mG is metrizable.*

The following lemma gives a condition equivalent to the hypothesis of Theorem 1.3 in terms of singular groups.

Lemma 3.3. *Let K be a compact abelian group. Then K is singular if and only if there exists a torsion subgroup $N \in \Lambda(K)$.*

Proof. Obviously, if K is singular, $K[n]$ is torsion and $K[n] \in \Lambda(K)$ for some $n \in \mathbb{N}_+$. If $N \in \Lambda(K)$ is torsion, then by the compactness of N there exists $n \in \mathbb{N}_+$ such that $nN = \{0\}$. Therefore N is contained in the subgroup $K[n]$ of K. Consequently $K[n] \in \Lambda(K)$. $\square$

Lemma 3.4. *Let K be a compact abelian group and let N be a closed subgroup of K. Then K is singular if and only if N and K/N are singular. In particular finite products of singular compact abelian groups are singular.*

Proof. The necessity is obvious. Let us prove the sufficiency. Suppose that both N and K/N are singular and let X and Y be the Pontryagin duals of K and K/N respectively. We say that X is almost-countable if mX is countable for some $m \in \mathbb{N}_+$. By Pontryagin duality K is singular if and only if X is almost-countable. Since Y can be identified with a subgroup of X such that $X/Y \cong \widehat{N}$, both Y and X/Y are almost-countable. Thus there exists $m \in \mathbb{N}_+$ such that mY and $m(X/Y)$ are countable. Since $m(X/Y) = (mX + Y)/Y$ is countable, mX is contained in

a countable union $\bigcup_{n=1}^{\infty}(z_i + Y)$, where $z_i \in mX$. As mY is countable, m^2X is contained in the union $\bigcup_{n=1}^{\infty}(mz_i + mY)$, which is countable. So X is almost-countable. $\qquad\square$

4. Non-singular pro-p-groups

For an abelian topological group G and a prime p we denote by G_p the *topological p-component* of G, that is $G_p = \{x \in G : p^n x \to 0 \text{ for some } n \in \mathbb{N}\}$. By [1] a totally disconnected non-metrizable compact abelian group K can be decomposed as $K = \prod_p K_p$. Each K_p is an abelian pro-p-group (i.e., projective limit of finite p-groups) and at least one of the K_p is non-metrizable.

As we said above, the main difficulty of the problem is in the totally disconnected case. So in this section we consider the case when K is a totally disconnected compact abelian group in relation to singularity and metrizability of the topological p-components of K. In particular Example 2.1 shows that it may happen that each K_p is singular even if the product K is non-singular.

Lemma 4.1. *Let p be a prime and let K be an abelian pro-p-group.*

   (a) *If K/pK is finite, then $K \cong \mathbb{Z}_p^m \times F$, where $m \in \mathbb{N}$ and F is a finite p-group.*

   (b) *If K/pK is infinite, then $w(K) = w(K/pK)$.*

In particular K is metrizable whenever K/pK is metrizable.

Proof. By Pontryagin duality $X = \widehat{K}$ is a p-group and $X[p]$ is the dual of the quotient K/pK.

(a) Assume that K/pK is finite. Then $X[p]$ is finite; hence $X[p]$ is isomorphic to a subgroup of $\mathbb{Z}(p^{\infty})^n$, where n is the p-rank of X (that is the dimension of $X[p]$ as a vector space over the field $\mathbb{Z}/p\mathbb{Z}$). Since $\mathbb{Z}(p^{\infty})^n$ is divisible, this immersion can be extended to $j : X \to \mathbb{Z}(p^{\infty})^n$. Now j is injective because if $x \in \ker j$ and $x \neq 0$ then we can suppose wlog that $px = 0$, that is $x \in X[p]$ and this is not possible. If $d(X)$ is the maximal divisible subgroup of X, then $d(X)$ is isomorphic to $\mathbb{Z}(p^{\infty})^m$ with $m \leq n$. Thus $X \cong \mathbb{Z}(p^{\infty})^m \oplus X_1$ where X_1 is reduced and so finite because it has finite p-rank [11]. By Pontryagin duality $K = \widehat{X} \cong \mathbb{Z}_p^m \times F$, where $F = \widehat{X_1} \cong X_1$ is finite.

(b) follows from the fact that $|X| = |X[p]|$, when $X[p]$ is infinite. $\qquad\square$

Remark 4.2. Let K be a totally disconnected compact abelian group. So $K = \prod_{p \in \mathbb{P}} K_p$. Define

$$P_s = \{p \in \mathbb{P} : K_p \text{ is singular}\} \text{ and } P_m = \{p \in \mathbb{P} : K_p \text{ is metrizable}\}$$

and observe that $P_m \subseteq P_s \subseteq \mathbb{P}$. The group K is metrizable if and only if $P_m = \mathbb{P}$.

Suppose that K is non-metrizable. Let $K_m = \prod_{p \in P_m} K_p$, $K_s = \prod_{p \in P_s \setminus P_m} K_p$ and $K_r = \prod_{p \in \mathbb{P} \setminus P_s} K_p$. If P_m (resp. $P_s \setminus P_m$, $\mathbb{P} \setminus P_m$) is empty, put $K_m = \{0\}$ (resp. $K_s = \{0\}$, $K_r = \{0\}$). Note that $K = K_m \times K_s \times K_r$ and that K_m is metrizable.

The next proposition takes care of the case when $\mathbb{P} \setminus P_m$ is infinite.

Proposition 4.3. *Let K be a totally disconnected compact abelian group such that $\mathbb{P} \setminus P_m$ is infinite. Then there exists a continuous surjective homomorphism of K onto S^{ω_1}, where S is a non-torsion metrizable compact abelian group.*

Proof. By Lemma 4.1 there exist infinitely many primes p_n ($n \in \mathbb{N}_+$) such that $K/p_n K$ is non-metrizable. Let $n \in \mathbb{N}_+$. Since $K/p_n K$ is a non-metrizable compact abelian group of exponent p_n, it is topologically isomorphic to $\mathbb{Z}(p_n)^{w(K/p_n K)}$, where $w(K/p_n K) \geq \omega_1$. This yields that for every $n \in \mathbb{N}_+$ there exists a continuous surjective homomorphism $f_n : K/p_n K \to \mathbb{Z}(p_n)^{\omega_1}$. Note that $K_q = p_n K_q \subseteq p_n K$ for every prime $q \neq p_n$. Therefore $p_n K$ coincides with the subgroup $L = p_n K_{p_n} \times \prod_{q \neq p_n} K_q$ of K. So $K/p_n K \cong K_{p_n}/p_n K_{p_n}$ and consequently f_n can be identified with $f'_n : K_{p_n}/p_n K_{p_n} \to \mathbb{Z}(p_n)^{\omega_1}$. Then $f = \prod_{n=1}^{\infty} f'_n : \prod_{n=1}^{\infty} K_{p_n}/p_n K \to \prod_{n=1}^{\infty} \mathbb{Z}(p_n)^{\omega_1} = S^{\omega_1}$, where $S = \prod_{n=1}^{\infty} \mathbb{Z}(p_n)$, is a continuous surjective homomorphism. Hence the composition of the continuous surjective homomorphism $K = \prod_{p \in \mathbb{P}} K_p \to \prod_{n=1}^{\infty} K_{p_n}/p_n K_{p_n}$ with f is a continuous surjective homomorphism of K onto S^{ω_1}. $\square$

Lemma 4.4. *Let K be a totally disconnected compact abelian group. Then K is singular if and only if $P_s = \mathbb{P}$ and $\mathbb{P} \setminus P_m$ is finite.*

Proof. Suppose that $P_s = \mathbb{P}$ and $\mathbb{P} \setminus P_m$ is finite. Then $K = K_m \times K_s$, where K_m is metrizable and K_s is singular by Lemma 3.4. So K is singular.

Now we prove the converse implication. If $\mathbb{P} \setminus P_m$ is infinite, by Proposition 4.3 there exists a continuous surjective homomorphism of K onto S^{ω_1}, where S is a non-torsion metrizable compact abelian group. Since S^{ω_1} is non-singular, then K is non-singular by Lemma 3.4. If $\mathbb{P} \neq P_s$ then Lemma 3.4 implies that K is non-singular. $\square$

Lemma 4.5. *Let σ be a cardinal, K an abelian pro-p-group and N a closed subgroup of K isomorphic to $\mathbb{Z}_p^{\sigma}$ such that K/N is singular. Then K is non-singular if and only if $\sigma > \omega$.*

Proof. If $\sigma > \omega$ then N is non-singular and so again Lemma 3.4 implies that K is non-singular too. Suppose that $\sigma \leq \omega$. Then N is metrizable and in particular N is singular. By Lemma 3.4 this implies that K is singular. $\square$

In the sequel we denote by $\mathbb{L}_p$ the group $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)$.

Proposition 4.6. *Let K be an abelian pro-p-group. Then K has a closed subgroup N such that*

(a) *$N \cong \mathbb{Z}_p^{\sigma}$ for some cardinal σ;*
(b) *the quotient $L = K/N$ is isomorphic to $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{\alpha_n}$, for some cardinals α_n;*
(c) *if φ is the canonical projection of K onto L, then $\varphi(t(K)) = t(L)$.*

Moreover, if K is non-singular, then:

(1) *there exists a continuous surjective homomorphism of K onto $\mathbb{L}_p^{\omega_1}$, in case L is non-singular;*
(2) *$\sigma = w(N) > \omega$, in case L is singular.*

Proof. Let $X = \widehat{K}$. For a subset T of K the annihilator of T in X is $T^{\perp} = \{\chi \in X : \chi(T) = \{0\}\}$.

(a),(b) By [11, Theorem 32.3] X contains a pure subgroup B (i.e., $p^n X \cap B = p^n B$ for every $n \in \mathbb{N}$) such that $B \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$ and $X/B \cong \mathbb{Z}(p^{\infty})^{(\sigma)}$ for some cardinals σ, α_n. By Pontryagin duality K has a closed subgroup $N \cong \mathbb{Z}_p^{\sigma}$

(namely the annihilator of B) such that the quotient $L = K/N$ is isomorphic to $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{\alpha_n}$.

(c) To prove that $\varphi(t(K)) = t(L)$ we show that $\varphi(K[p^n]) = L[p^n]$ holds for every $n \in \mathbb{N}_+$. It is sufficient to see that $W = \{x \in K : p^n x \in N\} = N + K[p^n]$. Since $W = \varphi^{-1}(L[p^n])$, obviously W contains $N + K[p^n]$. Since W and $N + K[p^n]$ are closed subgroups, to prove that they coincide it suffices to prove that their annihilators coincide. From $W \supseteq N + K[p^n]$, it follows $W^{\perp} \subseteq N^{\perp} \cap K[p^n]^{\perp} = B \cap p^n X = p^n B$. Observe that the last equality follows from the purity of B. But we have also

$$W^{\perp} = \{\chi \in X : (\forall x \in K) p^n x \in N \Rightarrow \chi(x) = 0\} \supseteq p^n(N^{\perp}) = p^n B.$$

Therefore $W^{\perp} = N^{\perp} \cap K[p^n]^{\perp}$.

Suppose now that K is non-singular.

(1) If L is non-singular, $p^n L$ is non-metrizable for every $n \in \mathbb{N}_+$. Since $p^n L \cong \prod_{m=n+1}^{\infty} \mathbb{Z}(p^{m-n})^{\alpha_m}$, there are infinitely many $\alpha_n > \omega$. Hence there exists a continuous surjective homomorphism of L onto $\prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{\omega_1} \cong \mathbb{L}_p^{\omega_1}$. Combining it with $\varphi : K \to L$, we find a continuous surjective homomorphism of K onto $\mathbb{L}_p^{\omega_1}$.

(2) follows from Lemma 4.5. $\qquad\square$

It follows from [11, Theorem 35.2] that the quotient $L = K/N$ with the properties (a)–(c) from the above proposition is uniquely determined up to topological isomorphisms.

Claim 4.7. *Let K be an abelian pro-p-group and let N be a closed subgroup of K isomorphic to $\mathbb{Z}_p^{\sigma}$, where σ is a cardinal $> \omega$. Then there exists a continuous surjective homomorphism of K onto $\mathbb{L}_p^{\sigma}$.*

Proof. Let $N = \prod_{n=1}^{\infty} N_n$, where each $N_n \cong \mathbb{Z}_p^{\sigma}$, and $M = \prod_{n=1}^{\infty} p^n N_n$. Then $N/M \cong \prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{\sigma} = \mathbb{L}_p^{\sigma}$. Let $K_0 = K/M$. Then $K_0[p^n] \supseteq (N/M)[p^n]$ and the last group contains a subgroup isomorphic to $\mathbb{Z}(p^n)^{\sigma}$ for every $n \in \mathbb{N}_+$. Hence

$$w(p^{n-1} K_0[p^n]) \geq \sigma. \tag{1}$$

By Proposition 4.6 there exists a closed subgroup N_0 of K_0 such that $N_0 \cong \mathbb{Z}_p^{\sigma_1}$, $L_0 = K_0/N_0 \cong \prod_{n=1}^{\infty} \mathbb{Z}(p^n)^{\beta_n}$ for appropriate cardinals σ_1, β_n and the canonical projection $\varphi : K_0 \to L_0$ satisfies $\varphi(t(K_0)) = t(L_0)$. Since $K_0[p^n]$ is compact and trivially meets $N_0 = \ker \varphi$, so $\varphi\restriction_{K_0[p^n]} : K_0[p^n] \to L_0[p^n]$ is a topological isomorphism. Consequently $p^{n-1} L_0[p^n]$ is topologically isomorphic to $p^{n-1} K_0[p^n]$ and hence $w(p^{n-1} L_0[p^n]) \geq \sigma$ for every $n \in \mathbb{N}_+$ by (1). Therefore $\sup_{n \geq m} \beta_n \geq \sigma$ for every $m \in \mathbb{N}_+$.

Let us prove that there exists a continuous surjective homomorphism $f : L_0 \to \mathbb{L}_p^{\sigma}$. Then combining it with φ and with the canonical projection of K onto $K_0 = K/M$ we are done. Note that infinitely many β_n are infinite. So it is possible to suppose without loss of generality that all β_n are infinite. If there are infinitely many β_n such that $\beta_n \geq \sigma$, it is possible to find the required continuous surjective homomorphism $f : L_0 \to \mathbb{L}_p^{\sigma}$. Otherwise there exists $n_0 \in \mathbb{N}_+$ such that $\omega \leq \beta_n < \sigma$ for every $n \geq n_0$, with $\sup_{n \geq n_0} \beta_n = \sigma$. Take an increasing subsequence $\{\beta_{n_k}\}_k$ of $\{\beta_n\}_n$ such that $\sup_{n_k \geq n_0} \beta_{n_k} = \sigma$. Observe that $\prod_{k=1}^{\infty} \mathbb{Z}(p^{n_k})^{\beta_{n_k}} = \prod_{k=1}^{\infty} S_k^{\beta_{n_k}}$, where $S_k = \prod_{i=k}^{\infty} \mathbb{Z}(p^{n_i})$ is a non-torsion metrizable compact abelian group. For every $k \in \mathbb{N}_+$ there exists a continuous surjective homomorphism of S_k onto $\mathbb{L}_p$ and so the desidered continuous surjective homomorphism $f : L_0 \to \mathbb{L}_p^{\sigma}$. $\qquad\square$

5. Proof of the Main Theorem and Example 1.6

The next lemma will be used in the proof of the Main Theorem 5.2. It settles the problem in the "local case", that is when K is an abelian pro-p-group.

Lemma 5.1. *Let K be an abelian pro-p-group. Then the following properties are equivalent:*

(a) *K is non-singular;*
(b) *K has the property TD_ω;*
(c) *there exists a continuous surjective homomorphism of K onto $\mathbb{L}_p^{\omega_1}$.*

Proof. (a)$\Rightarrow$(c) follows from Proposition 4.6 and Claim 4.7 and (c)$\Rightarrow$(b) from Proposition 2.4 and Lemma 2.5.

(b)$\Rightarrow$(a) The property TD_ω implies that K has a proper totally dense pseudocompact subgroup. Now apply Theorem 1.3 and Lemma 3.3 to conclude that K is non-singular. $\square$

Now it is possible to prove our Main Theorem. For technical convenience we add the equivalent conditions (c) and (e) that may have also independent interest.

Theorem 5.2. *For a compact abelian group K the following conditions are equivalent:*

(a) *K has a proper totally dense pseudocompact subgroup;*
(b) *K has no closed torsion G_δ-subgroups;*
(c) *K is non-singular;*
(d) *K has the property TD_ω;*
(e) *there exists a continuous surjective homomorphism of K onto S^{ω_1} where S is compact non-torsion.*

Proof. (e)$\Rightarrow$(d) by Proposition 2.4 and Lemma 2.5, (d)$\Rightarrow$(a) is obvious, while (a)$\Rightarrow$(b) by Theorem 1.3 and (b)$\Leftrightarrow$(c) by Lemma 3.3.

(c)$\Rightarrow$(e) If $c(K)$ is not metrizable, then Theorem 1.2 applies. Suppose that $w(c(K)) \leq \omega$. So there is a continuous surjective homomorphism of K onto the non-singular totally disconnected group $K/c(K)$. Hence we can suppose without loss of generality that K is totally disconnected and $K = \prod_{p \in \mathbb{P}} K_p$. If K_p is non-singular for some $p \in \mathbb{P}$, apply Lemma 5.1. If K_p is singular for every $p \in \mathbb{P}$, then $P_s = \mathbb{P}$ and $\mathbb{P} \setminus P_m$ has to be infinite by Lemma 4.4. So Proposition 4.3 concludes the proof. $\square$

Remark 5.3. (1) It was announced in [5] that it is possible to prove that the following condition is equivalent to those of Theorem 5.2:

(f) *K has a totally dense subgroup H that contains an ω-bounded subgroup M of K such that $\overline{M}$ is a G_δ-subgroup of K and $\overline{M} \nsubseteq H$.*

(2) For compact abelian groups this condition (f), as well as condition (a) of Theorem 5.2, is preserved under taking inverse image by continuous surjective homomorphisms.

Construction of Example 1.6. Consider the compact abelian group $K = \mathbb{Z}_p^{\omega_1}$. By LH we have $|K| = \mathfrak{c}$. Call a non-limit cardinal $\alpha = \lambda + n$, where λ is a limit cardinal and $n \in \mathbb{N}_+$, *odd* if n is odd and *even* if n is even. Let $\mathcal{N} = \{N_\lambda : \lambda \text{ odd}, \lambda < \mathfrak{c}\}$ be the set of all closed subgroup of K isomorphic to $\mathbb{Z}_p$. Let then $\mathcal{G} = \{O_\nu : \nu \text{ even}, \nu < \mathfrak{c}\}$ be the set of all cosets of G_δ-subgroups of K.

We define $F_0 = \{0\}$. For $0 < \xi < \mathfrak{c}$ suppose that for every non-limit cardinal $\eta < \xi$, if η is odd, there exists $x_\eta \in N_\eta$ such that $\overline{\langle x_\eta \rangle} = N_\eta$ and if η is even, there exists $x_\eta \in O_\eta$ (if $\eta < \xi$ is a limit cardinal put $x_\eta = 0$) and they have the property that $F_\xi = \langle x_\eta : \eta < \xi \rangle$ is free; observe that $|F_\xi| < \mathfrak{c}$ since $\xi < \mathfrak{c}$. If ξ is odd, since $|N_\xi| = \mathfrak{c}$, we can choose $x_\xi \in N_\xi$ such that $\langle x_\xi \rangle \cap F_\xi = \{0\}$ and $\overline{\langle x_\xi \rangle} = N_\xi$. Indeed, we can choose $x_\xi^* \in N_\xi$ such that $\langle x_\xi^* \rangle \cap F_\xi = \{0\}$ but it can happen that $x_\xi^* \in pN_\xi$; in this case there exists $n \in \mathbb{N}_+$ such that $x_\xi^* \notin p^n N_\xi$ and so we take a $x_\xi \in N_\xi \setminus pN_\xi$ with $x_\xi^* = p^{n-1} x_\xi$. If ξ is even, since $|O_\xi| = \mathfrak{c}$, we can choose $x_\xi \in O_\xi$ such that $\langle x_\xi \rangle \cap F_\xi = \{0\}$. In both cases $F_{\xi+1}$ is free. Finally define $F_\mathfrak{c} = \langle x_\xi : 0 < \xi < \mathfrak{c} \rangle$. Then $F_\mathfrak{c}$ is free and it is totally dense and G_δ-dense (so pseudocompact) in K. Since it is free, $F_\mathfrak{c}$ cannot contain any non-trivial ω-bounded subgroup and it is proper in K. $\qquad\square$

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