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# A Fibonacci Approach to Weighted Majority Games

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**Abstract** In this paper we intend to investigate the relationship between game theory and Fibonacci numbers. We call Fibonacci games the subset of constant sum homogeneous weighted majority games whose increasing sequence of all type weights and of the minimal winning quota is a string of consecutive Fibonacci numbers. Exploiting key properties of the Fibonacci sequence, we obtain closed form results able to provide a simple and insightful classification of such games. In detail: we show that the numerousness of Fibonacci games with $t$ types is $\lfloor (t+1)/2 \rfloor$; we describe unequivocally a Fibonacci game on the basis of its profile as a function of $t$ and of a proper index $z=1,\ldots,\lfloor (t+1)/2 \rfloor$; we provide rules concerning the behaviour of the total number $n(t,z)$ of non-dummy players in a Fibonacci game. It turns out that there are two kinds of Fibonacci games, associated respectively with z=1 (Fibonacci-Isbell games) and z>1.

**Keywords** Weighted majority games, Homogeneous representation, Minimal winning coalition, Type weight vector, Satellite games, Fibonacci numbers

## 1. Introduction

Homogeneous weighted majority games have been introduced at the origins of modern game theory by Von Neumann-Morgenstern [12] and, since then, extensively studied because of their capability to give an insightful formal framework able to analyse formation of coalitions and payoff division both in theory and in real world situations. Subsequent treatments of outstanding importance have been given by Ostmann [6], who gave the proof that any homogeneous weighted majority game (including non-constant sum ones) has a unique minimal homogeneous representation, and by Rosenmüller ([9] and [10]), who provided an analysis of the structure of such games based on the concept of characters of types and the role of satellite games.

In the particular case of constant sum homogenous weighted majority games, the weights of the minimal homogeneous representation are able to capture the power of the players. Such ability is revealed by the close connections between (properly normalized) those weights and some of the outstanding ideas of "solution" of a game. Examples of such connections are: a) the stable set of imputations of the main simple solution a la Von Neumann-Morgenstern; b) the nucleolus (Peleg [7], Schmeidler [11]); c) the outcome of the Montero [5] bargaining protocol (which modifies the one proposed by Baron-Ferejohn [1]) in which both the expected payoffs and actual payoff division are proportional to the voting weights.

In this paper we wish to explore the connections between constant sum homogeneous weighted majority games and Fibonacci numbers.

A bridge between these two topics has been built by Isbell [2] in a vintage paper going back to the first steps of cooperative game theory. The bridge was summarized in a couple of propositions. In the first ([2], Cor. [6], p. 185), the author showed that, weakly ordering players of a $n$ person constant sum homogeneous weighted majority game from bottom to top, the individual weight (in the minimal homogeneous representation) of the player number $i$ could not exceed the corresponding Fibonacci number (more formally $w_i \le f_i$ for any $i=1,\ldots,n$). The second proposition ([2], third indent, p. 185) claimed that a game whose individual weights satisfied $w_i = f_i$ for any $i=1,\ldots,n-2$, $w_{n-1} = f_{n-2}, w_n = f_{n-1}$, described, for any $n>2$, a constant sum homogeneous weighted majority game with minimal winning quota $q = f_n$. Hence, in such games all the first $n$ Fibonacci numbers are associated with all individual weights and the winning quota. On the basis of this connection we suggest to call Fibonacci-Isbell the set of games defined by the second proposition.

Thinking in terms of type weights rather than of individual ones and keeping account that in any $n$ person Fibonacci-Isbell game there are $t=n-2$ types of players, we argue that such games satisfy another slightly different "bridge" property: the bottom-top (strictly) ordered $n-1$ (or $t+1$) dimension vector of type weights and winning quota is a string of $n-1$ consecutive Fibonacci numbers starting from $f_2 = 1$. In order to generalize this property to other feasible

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*(n,t)* combinations, we suggest to define Fibonacci games as the subset of constant sum homogeneous weighted majority games whose bottom-top increasing sequence of type weights and winning quota (in their minimal homogeneous representation) is a string of consecutive Fibonacci numbers[1]. Henceforth, it is convenient to use the (first order) "delayed" Fibonacci sequence **g** defined, for any natural *n*, by the relation $g_n = f_{n+1}$, and delayed Fibonacci subsequences

$$\mathbf{g}^{odd} = \mathbf{g}^{o} = (g_1, g_3, \dots, g_{2m-1}, \dots)$$

$$\mathbf{g}^{even} = \mathbf{g}^{e} = (g_2, g_4, \dots, g_{2m}, \dots)$$

Moreover, following Rosenmüller ([10], p. 311), we take into consideration the "profile" of a constant sum homogeneous weighted majority game with *t* types denoted by the ordered vector $\mathbf{k} = (k_1, \dots, k_j, \dots, k_t)$, whose component $k_j$ is the number of players of type *j* in the game.

After that, it becomes clear that Fibonacci games are unequivocally described by feasible triplets $(\mathbf{k}; \mathbf{g}; g_{t+1})$ in which the profile **k** and the delayed Fibonacci sequence **g** share the dimension *t*, $g_j$ is the weight of players of type *j*, $g_{t+1}$ is the winning quota and a triplet is feasible if it meets the homogeneity conditions (see [7], Theor. 3.5 and the "test" of homogeneity[2] [9], Theor.1.4).

In our paper, we give a simple but insightful characterization of the feasible profiles **k** of Fibonacci games with *t* types. The proof of such a characterization largely exploits fundamental properties of the delayed Fibonacci sequence (as for the necessary conditions) and the satellite game approach (in the sufficiency part).

As a consequence, for any positive integer *t*, there are altogether[3] $\lfloor (t+1)/2 \rfloor$ Fibonacci games, i.e. the Fibonacci-Isbell one and other $\lfloor (t-1)/2 \rfloor$ games. Moreover, it turns out that any two Fibonacci games with the same *t* have a different number *n* of non-dummy players. Hence, it seems logical to associate, for any *t*, the Fibonacci games with the set of integers $z = 1, \dots, \lfloor (t+1)/2 \rfloor$ in such a way that the number *n(t,z)* of players (in the Fibonacci game with *t* types and index *z*) is an increasing function of *z*.

A closed form description of the behavior of the function *n(t,z)* is also provided. It is surprising to verify that the derived function $\Delta(t,z) = n(t,z+1) - n(t,z)$ is resumed by a matrix whose columns are the entire delayed Fibonacci sequence (whose starting point is properly shifted down with *z*), while the rows are, in backward order, the delayed Fibonacci subsequences coherent with the parity of *t* (properly truncated, so as the row *t* has $\lfloor (t-1)/2 \rfloor$ components).

We are well aware of the existence of a vast body of literature concerning applications of homogeneous weighted majority games (constant as well as non-constant sum) to the analysis of voting power and committees interactions. Besides already cited papers, other examples may be found e.g. in Kalandrakis [3], Le Breton et al [4]. Yet we do not discuss applications of the Fibonacci games here in this or in other fields. Anyway, we anticipate that we have some preliminary evidence that interesting applications to the weighted voting systems in parliamentary elections may be obtained.

Yet we do not discuss applications of the Fibonacci games here in this or in other fields, but we anticipate that we have some (not yet published) preliminary evidence that interesting applications to the weighted voting systems in parliamentary elections may be obtained.

The plan of the paper is as follows: section 2 gives a short description of the basic notations used in the paper and recalls well known concepts of homogeneous weighted majority games; section 3 defines Fibonacci games and resumes the main results of the paper; section 4 provides an explicit description of the profiles of Fibonacci games for some small values of *t*; the behaviour of the *n(t,z)* and of $\Delta(t,z)$ functions and their connections with the delayed Fibonacci (sub)sequences are presented and discussed in section 5; all the proofs are grouped in sections 6 and 7; conclusions follow in the final section 8.

## 2. Notations

Let $\Omega = \{1, \dots, n\}$ denote the set of non-dummy players of a simple constant sum game in characteristic function form. A simple game is a mapping $v : \mathcal{P}(\Omega) \to \{0,1\}$ such that $v(\varnothing) = 0$ and $v(\Omega) = 1$, and a coalition $S \in \mathcal{P}(\Omega)$ is winning if its payoff $v(S) = 1$ and losing otherwise. A simple game is constant sum if $v(S) + v(\Omega / S) = 1$ for any *S*. Moreover, *S* is minimal winning if, for any player $i \in S$, $v(S / i) = 0$.

A simple weighted majority game is described by the pair $(\overline{\mathbf{w}}_n; q)$ where[4] $\overline{\mathbf{w}}_n = \{\overline{w}_1, \dots, \overline{w}_n\}$ is the (weakly)

[1] Of course other definitions, based on more or less restrictive conditions on Fibonacci numbers, could have been chosen. An alternative approach on this line has been proposed by Gambarelli, Gnocchi and Pressacco in an unpublished communication presented in the section "Power indices" at the conference: Models of collusion, games and decisions for applications to judging, selling and voting. Oldofredi Castle, Monte Isola 18-19 June 2012.

[2] Roughly speaking, the test of homogeneity requires that, given a candidate homogeneous representation of a game, the measure of the largest (when collecting players according to rank) minimal winning coalition must exactly hit the winning quota and that each player *i* of this coalition induces (recursively) a (satellite) game whose winning quota is given by her weight $\overline{w}_i$ and either is exactly hit by the largest minimal winning coalition obtained by remaining players of smaller weight or the global weight of all these players is lower than $\overline{w}_i$.

[3] As usual $\lfloor x \rfloor$ denotes the floor*(x)*. In particular, for *t* integer even (odd), $\lfloor (t+1)/2 \rfloor = t/2$ (or *(t+1)/2)*).

[4] Henceforth the subscript of the vector is its dimension, not to be confused with the subscript of a scalar that denotes a component of the vector.

ordered vector ( $\overline{w_i} \le \overline{w_j}$ for $i<j$) of individual weights and $q$ the winning quota of the game. Thus the weight of $S$ is $\overline{w}(S) = \sum_{i \in S} \overline{w_i}$ and $\overline{w}(S) \ge q \Leftrightarrow v(S) = 1$.

A simple constant sum homogeneous weighted majority game is described by its minimal homogeneous representation, that is the ordered vector $\overline{\mathbf{h}} = (\overline{\mathbf{w}}_n ; q)$ which meets the homogeneity conditions, that is $\overline{\mathbf{h}} \in \mathsf{N}^{n+1}$, $w_1=1$, $q = (1+\overline{w}(\Omega))/2$ and $\overline{w}(S) = q$ for any minimal winning coalition.

The vector $\overline{\mathbf{w}}_n$ of the minimal homogeneous representation induces a decomposition of $\Omega$ in equivalence classes $K_1,\dots,K_j,\dots,K_t$. Each class groups all players of the same type, sharing the same individual weight and the corresponding (strongly) ordered type weight vector is:

$$\mathbf{w}_t = \{w_1,\dots,w_j,\dots,w_t\}$$

with $w_1 = 1$.

Coherently, $\mathbf{h} = (\mathbf{w}_t ; q) \in \mathsf{N}^{t+1}$ , but to unequivocally describe a game it is necessary also to introduce also the profile $\mathbf{k}_t = (k_1,\dots,k_j,\dots,k_t)$ with $k_j$ the number of players of type $j$ in the game. Thus, a constant sum homogeneous weighted majority game is described by $(\mathbf{k}_t ; \mathbf{w}_t ; q) \in \mathsf{N}^{2t+1}$.

Finally, we denote by $\mathbf{s}_t = (s_1,\dots,s_j,\dots,s_t)$, with $s_j = |S \wedge K_j| \le k_j$, the "profile" of the coalition $S$, so that $w(S) = \sum_{j=1}^{t} s_j w_j$ is an alternative formalization of the weight of $S$ and, by homogeneity, $w(\Omega) = \sum_{j=1}^{t} k_j w_j = 2q-1$.

## 3. Fibonacci Games: Main Results

The Fibonacci sequence $f$ is defined by the well known finite difference equation:

$$f_n = f_{n-1} + f_{n-2}$$

holding for any natural $n>2$ with initial conditions $f_1 = f_2 = 1$ . Henceforth, we exploit the "delayed" Fibonacci sequence $g_n = f_{n+1}$ for any $n$, and denote, for any integer $m$, by $\mathbf{g}_m \in \mathsf{N}^m$ the vector $(g_1, g_2, \dots, g_m)$. Coherently, the denumerable sequence $\mathbf{g}_\infty = (g_1,\dots,g_n,\dots)$, and $\mathbf{g}_\infty = \mathbf{g}^o \vee \mathbf{g}^e$ with

$$\mathbf{g}^o = (g_1, g_3, \dots, g_{2m-1}, \dots)$$

$$\mathbf{g}^e = (g_2, g_4, \dots, g_{2m}, \dots)$$

the two denumerable (sub)sequences obtained extracting from $\mathbf{g}_\infty$ all elements of odd or, respectively, even index.

**Definition 3.1** *The set of Fibonacci games is the subset of constant sum homogeneous weighted majority games with* $\mathbf{h} = \mathbf{g}_{t+1}$, *that is* $(\mathbf{k}_t ; \mathbf{w}_t ; q) = (\mathbf{k}_t ; \mathbf{g}_t ; g_{t+1})$.

**Remark 3.1** *Feasibility conditions require that in any Fibonacci game with t types,*

$$\sum_{j=1}^{t} k_j g_j = 2g_{t+1} - 1$$

*constant for any* $\mathbf{k}_t$*: the winning quota is the same for all Fibonacci games with the same t.*

*Proof.* Homogeneity requires that $w(\Omega) = 2q-1$, i.e.

$$\sum_{j=1}^{t} k_j g_j = 2g_{t+1} - 1.$$

**Theorem 3.1** *Both for t=1 and t=2 there is a unique Fibonacci game; their* $\mathbf{k}_t$ *are respectively* $\mathbf{k}_1$*=(3), and* $\mathbf{k}_2$*=(3,1); for any t>2, a game is a Fibonacci game if and only if its* $\mathbf{k}_t$*=*$\mathbf{k}$*(t,z) is given, for any value of the counter* $z = 1,\dots,\lfloor (t-1)/2 \rfloor$ *and with* $j_0 = t+1-2z$*, by:*

$$\mathbf{k}(t,z) = (k_1, \mathbf{k}', k_{j_0}, \mathbf{k}'') = (2 + g_{t-1} - g_{j_0}, \mathbf{1}_{j_0-2}, 2, \mathbf{1}_{2z-1}) \quad (1)$$

*or, putting* $z = \lfloor (t+1)/2 \rfloor$ *in order to have a complete indexation of the games, by:*

$$\mathbf{k}(t, \lfloor (t+1)/2 \rfloor) = (2 + g_{t-1}, \mathbf{1}_{t-1}) \quad (2)$$

**Corollary 3.1** *For any t, there are exactly*

$$\Phi(t) = \lfloor (t+1)/2 \rfloor \quad (3)$$

*Fibonacci games with t types.*

Let us denote by *n(t,z)* the total number of non-dummy players in a Fibonacci game with *t* types and index *z*.

**Theorem 3.2** *For any t,* $z_1 < z_2 \Leftrightarrow n(t,z_1) < n(t,z_2)$.

Proofs of Th. 3.1 and 3.2 follow in sections 6 and 7.

## 4. Profile of Fibonacci Games

According to the Definition 3.1, given $t$ there are no degrees of freedom in the choice of the sequence of type weights and of the winning quota. Hence, a Fibonacci game is unequivocally described by its **k***(t,z)* vector that follows the rules described in Formulas (1) and (2). Table 1 helps to understand the structure of the profile vectors of Fibonacci games for a set of small values of $t$.

We distinguish between two patterns of profiles: for $z = \lfloor (t+1)/2 \rfloor$ (the last vector of each row), all type components, except the bottom one, are 1, while this latter component plays a balancing role (according to Formula (2)); for all the other $z$, there is just one (non bottom) component equal to 2 in place $j_0 = j(t,z) = t+1-2z$, all the others (non bottom) are still 1 and, again, the bottom component plays the balancing role (according to Formula (1)).

**Table 1.** The profile **k**$(t,z)$ for a set of small values of $t$

| t | Φ(t) | **k**(t,1) | n(t,1) | **k**(t,2) | n(t,2) | **k**(t,3) | n(t,3) | **k**(t,4) | n(t,4) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | (3) | 3 | | | | | | |
| 2 | 1 | (3,1) | 4 | | | | | | |
| 3 | 2 | (2,2,1) | 5 | (4,1,1) | 6 | | | | |
| 4 | 2 | (2,1,2,1) | 6 | (5,1,1,1) | 8 | | | | |
| 5 | 3 | (2,1,1,2,1) | 7 | (5,2,1,1,1) | 10 | (7,1,1,1,1) | 11 | | |
| 6 | 3 | (2,1,1,1,2,1) | 8 | (7,1,2,1,1,1) | 13 | (10,1,1,1,1,1) | 15 | | |
| 7 | 4 | (2,1,1,1,1,2,1) | 9 | (10,1,1,2,1,1,1) | 17 | (13,2,1,1,1,1,1) | 20 | (15,1,1,1,1,1,1) | 21 |
| 8 | 4 | (2,1,1,1,1,1,2,1) | 10 | (15,1,1,1,2,1,1,1) | 23 | (20,1,2,1,1,1,1,1) | 28 | (23,1,1,1,1,1,1,1) | 30 |
| … | … | … | … | … | … | … | … | … | … |

## 5. The Behavior of *n(t,z)*

In this section we investigate the relation between the number of non-dummy players in the game and the characterization of the Fibonacci games by means of the profile vectors. In particular, the following properties give an answer to the questions: "given $n$, how many Fibonacci games are there?" and "which kind of Fibonacci games do we find?"

The properties, which hold for any feasible combination of $(t,z)$, are the following:

**Property 5.1**

$$n(t,1) = t + 2 \tag{4}$$

**Property 5.2**

$$\Delta(t,z) = n(t,z+1) - n(t,z) = g_{t-2z} \tag{5}$$

**Property 5.3**

$$n(t,2) - n(t-1,\lfloor t/2 \rfloor) = 2 \tag{6}$$

Proofs of these Properties are given in the section 7.

In Table 2 we show the behavior of the function $n(t,z)$.

**Table 2.** List of values $n(t,z)$

| t \ z | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 3 | | | | |
| 2 | 4 | | | | |
| 3 | 5 | 6 | | | |
| 4 | 6 | 8 | | | |
| 5 | 7 | 10 | 11 | | |
| 6 | 8 | 13 | 15 | | |
| 7 | 9 | 17 | 20 | 21 | |
| 8 | 10 | 23 | 28 | 30 | |
| 9 | 11 | 32 | 40 | 43 | 44 |
| 10 | 12 | 46 | 59 | 64 | 66 |
| ... | ... | ... | ... | ... | ... |

As it may be observed, there is not one to one correspondence between the number $n$ of players and each feasible combination of $(t,z)$. Essentially, for any value of $n>2$, there are surely at least one Fibonacci game or, at most, two of them. As a consequence, we distinguish between two kinds of Fibonacci games: the first one corresponds to $z=1$ and Formula (4) implies that there is exactly one Fibonacci game (of this kind) for any $n>2$. We recall that such Fibonacci-Isbell games have been described by Isbell ([2], p. 185) while studying the class of coalitionally Parsimonious games[5]. The second one emerges for $z>1$: precisely, there is only one Fibonacci game (of this second kind) for sparse values of $n$.

Summing up, there are values of $n$ corresponding to two Fibonacci games (one of the first and one of the second kind), while for other values of $n$ there is just one Fibonacci game of the first kind. For example, for $n = 8$, there are one Fibonacci game of the first kind ($z=1$, $t=6$), and one of the second kind ($z=2$, $t=4$);while for $n=9$, there is just one Fibonacci game of the first kind ($z=1$, $t=7$).

At first glance the values of $n$ associated with the second kind of games seem to follow a chaotic rule. Indeed, a regularity emerges once we consider the following table of the differences generated by Property 5.2 (see also Remark 7.1).

**Table 3.** Table of the function $\Delta(t,z)$

| t \ z | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 3 | 1 | | | |
| 4 | 2 | | | |
| 5 | 3 | 1 | | |
| 6 | 5 | 2 | | |
| 7 | 8 | 3 | 1 | |
| 8 | 13 | 5 | 2 | |
| 9 | 21 | 8 | 3 | 1 |
| 10 | 34 | 13 | 5 | 2 |
| ... | ... | ... | ... | ... |

Note that the rows corresponding to odd (even) values of $t$ are backward initial strings with $\lfloor (t-1)/2 \rfloor)$ elements of

[5] Parsimonious games are the subset of constant sum homogeneous weighted majority games characterized by the parsimony property to have, for any given number $n$ of non-dummy players in the game, the smallest number, i.e. exactly $n$, of minimal winning coalitions. For details see also [8].

the Fibonacci subsequence $\mathbf{g}^o$ ($\mathbf{g}^e$). In turn, the columns are nothing but the whole Fibonacci sequence $\mathbf{g}_\infty$ whose starting point is shifted at the row *t=1+2z*.

# 6. Proofs of Theorem 3.1

*Proof.*

For *t=1*:

$$q = g_2 = 2$$

$$w(\Omega) = k_1 \cdot g_1 = k_1 = 2q - 1 = 3$$

Hence there are three players with weight $g_1 = 1$, $\mathbf{k}_1 = (3)$ and $\Phi(1) = 1$.

For *t=2*:

$$q = g_3 = 3$$

$$w(\Omega) = \sum_{j=1}^{2} k_j g_j = 2q - 1 = 5$$

Hence, $k_1 \cdot g_1 + k_2 \cdot g_2 = k_1 \cdot 1 + k_2 \cdot 2 = 5$. There are two solutions $\mathbf{k}_2 = (3,1)$ and $\mathbf{k}_2 = (1,2)$ but the second is not feasible because the profile $\mathbf{s}_2 = (0,2)$ identifies a minimal winning coalition *S* with *w(S)=4>q*, incompatible with the homogeneous character. On the other side, the first solution is feasible with profiles (3,0) and (1,1) of the minimal winning coalitions, so that $\Phi(2) = 1$.

For *t>2*, we show at first that the conditions (1) or (2) are necessary for Fibonacci games and then that they are also sufficient.

### 6.1. Proof of Necessity

**Property 6.1** *In any Fibonacci game with t>1, the coalition S whose profile* $\mathbf{s}_t = (\mathbf{0}_{t-2}, \mathbf{1}_2)$*, i.e. made by a top and a last but top player, is minimal winning.*

*Proof.*

$$w(S) = s_{t-1} g_{t-1} + s_t g_t = g_{t-1} + g_t = g_{t+1} = q$$

**Property 6.2** *In any Fibonacci game with t>1,* $k_t=1$*.*

*Proof.* Otherwise the coalition $S{:}\mathbf{s}_t = (\mathbf{0}_{t-1}, 2)$ would be minimal winning with $w(S) = 2g_t > g_t + g_{t-1} = q$, a contradiction with the homogeneous character of the game.

**Property 6.3** In any Fibonacci game with t>2, $k_{t-1} \le 2$.

*Proof.* Otherwise the coalition $S{:}\mathbf{s}_t = (\mathbf{0}_{t-2}, 3, 0)$ would be minimal winning as $2g_{t-1} < g_{t-1} + g_t = q$ with $w(S) = 3g_{t-1} > 2g_{t-1} + g_{t-2} = g_{t-1} + g_t = q$.

**Property 6.4** *In any Fibonacci game with t>3,* $k_{t-2}=1$*.*

*Proof.* Suppose on the contrary, $k_{t-2}>1$ and consider the coalition $S{:}\mathbf{s}_t = (\mathbf{0}_{t-3}, 2, 0, 1)$. *S* would be minimal winning too as $g_t + g_{t-2} < g_t + g_{t-1} = q$ with $w(S) = g_t + 2g_{t-2} > g_t + g_{t-2} + g_{t-3} = g_t + g_{t-1} = q$. A contradiction.

Properties 6.3 and 6.4 may be generalized to types *(t-h)* with *h* odd and respectively *h* even.

**Property 6.5** *In any Fibonacci game it is* $k_{t-h} \le 2$ *for h odd <t-1.*

*Proof.* Proof of Property 6.5 is based on the following lemma:

**Lemma 6.1** *In a Fibonacci game consider any player of type* $j_0=(t-h)$ *with h odd <(t-1); there exists a minimal winning coalition S in which this player is the weakest. The profile of S is given by:* $\mathbf{s}_t = (s_1, \ldots, s_j, \ldots, s_t)$ *with* $s_j = 0$ *for* $j<j_0$*;* $s_{j_0} = 1$ *;* $s_j = 1$ *for* $j>j_0$ *and* $j = (j_0+1) \bmod 2$ *and* $s_j = 0$ *for* $j>j_0$ *and* $j = j_0 \bmod 2$.

*Proof.*

$$w(S) = g_{j_0} + \sum_{j>j_0 \text{ and } j=(j_0+1) \bmod 2} g_j = g_{t-1} + g_t = g_{t+1} = q$$

Now to prove Property 6.5 suppose on the contrary $k_{j_0} > 2$ and consider the coalition $S'$ obtained from *S* by replacement of the player type $j_0 + 1$ with two additional players of type $j_0$. $S'$ would be minimal winning with $w(S') > q$, a contradiction.

**Property 6.6** *In any Fibonacci game it is* $k_{t-h} = 1$ *for h even <t-1.*

*Proof.* Proof of Property 6.6 is based on the previous Lemma 6.1 and on the following one:

**Lemma 6.2** *Suppose in a Fibonacci game there is a minimal winning coalition S and a* $j_0 = (t-h) > 2$ *such that* $s_j = 0$ *for* $j = 1, \ldots, j_0 - 1$, $s_{j_0} > 0$ *(so that a player of type* $j_0 > 2$ *is the weakest in S), then* $k_{j_0-1}=1$*.*

*Proof.* Suppose on the contrary $k_{j_0-1}>1$ and consider the coalition $S'$ obtained from S by replacement of a player of type $j_0$ with two players of type $j_0 - 1$. $S'$ would be minimal winning with $w(S') > q$, a contradiction. This Lemma holds independently from the parity of *h*.

**Lemma 6.3** *If a player of type j=t-h>2 is the weakest in a minimal winning coalition S, then there exists a minimal winning coalition* $S'$ *such that a player of type j-2 is the weakest in* $S'$.

*Proof.* In *S* replace the player of type *j* with the unique (by Lemma 6.2) player of type *j-1* and one player of type j-2. Also this Lemma holds independently from the parity of *h*.

**Lemma 6.4** *Suppose in a Fibonacci game* $k_{j_0} = 2$ *for* $j_0$*=t-h with h odd <(t-1), then: a)* $k_{j_0-1} = 1$ *(by Property 6.6) and b) there is a minimal winning coalition* $S'$ *such that the player of type* $(j_0 - 1)$ *is its weakest player.*

*Proof.* (of part b)) $S'$ is obtained from the coalition *S* in Lemma 6.1 by replacement of the player of type *t-(h-1)* with one additional player of type *t-h* and the player of type *t-(h+1)*. Clearly $S'$ is minimal winning with $w(S') = q$.

**Lemma 6.5** *If in a Fibonacci game* $k_{j_0}=2$ *for* $j_0$*=t-h with h odd <(t-1), then* $k_j=1$ *for any* $1<j<j_0$.

*Proof.* Lemma 6.4 and Lemma 6.3 imply that there exists a set of minimal winning coalitions such that each player of type *j*, $1<j\le j_0$, is the weakest in a minimal winning coalition of the set. After that Lemma 6.5 comes as an immediate corollary of Lemma 6.2.

Now, it is possible to give a more precise statement of Property 6.5 concerning the behaviour of $k_{t-h}$ for *h* odd *<(t-1)*. Indeed, Property 6.5 and Lemma 6.5 give immediately:

**Property 6.7** *In a Fibonacci game there is at most one odd h<(t-1) such that* $k_{j_0}=k_{t-h}=2$ *and* $k_j=1$ *for all j>1 and* $j\ne j_0$.

Now by Remark 3.1 it is:

$$k_1 = 2g_{t+1} - 1 - \sum_{j=2}^{t} k_j g_j \tag{7}$$

and moreover[6]:

$$g_{t+2} - \sum_{j=1}^{t} g_j = 2 \tag{8}$$

Furthermore, Properties 6.6 and 6.7 imply that, for any $j=2,\dots,t$, $\mathbf{k}_t$ satisfies conditions (1) or (2). In the first case and putting $j_0=t+1-2z$, we obtain:

$$\begin{aligned} k_1 = k_1(t,z) &= 2g_{t+1} - 1 - (\sum_{j=2}^{t} g_j + g_{j_0}) = \\ &= g_{t+1} + (g_t + g_{t-1}) - \sum_{j=1}^{t} g_j - g_{j_0} = \\ &= g_{t+2} + g_{t-1} - \sum_{j=1}^{t} g_j - g_{j_0} = \\ &= 2 + g_{t-1} - g_{j_0} \end{aligned} \tag{9}$$

In particular, for *z=1*, $k_1=2$.

In the second case, we get immediately:

$$k_1 = k_1(t, \lfloor (t+1)/2 \rfloor) = 2 + g_{t-1} \tag{10}$$

To resume, by Properties 6.6 and 6.7, and by Formulas (9) and (10), necessary conditions to be satisfied by the $\mathbf{k}_t$ vector of a Fibonacci game with *t>2* are those described in Formulas (1) and (2).

### 6.2. Proof of Sufficiency

The proof is based on an adaptation to our problem of the test for homogeneity of a weighted majority game developed as Basic Lemma in [9] (Theorem 1.4) and [10] (pp. 312).

[6] By induction: indeed it is immediate to check that (8) is true for *t=1* and, if true for *t*, is true also for *t+1*.

Henceforth we use the following definitions and notations:

- $G^0=(\mathbf{k}_t^0;\mathbf{g}_t^0;\lambda^0=g_{t+1})$, "seed" game with $\mathbf{k}_t^0$ given by Formulas (1) or (2). In particular, $G^{0,z}$, seed game as a function of $z=1,\dots,\lfloor (t+1)/2 \rfloor$.
- $G^r=(\mathbf{k}_t^r;\mathbf{g}_t^r;\lambda^r)$, any game of a *r* generation[7] derived by the seed $G^0$. Note that the dimension *t* of $G^r$ depends on *r*. Whenever useful, we utilize *t(r)*. In particular, $G^{r,z}$, game of a $r$ generation derived by the seed $G^{0,z}$.
- Feasible (not feasible) game: a $G^r$ for which $\sum_{j=2}^{t} k_j^r \cdot g_j^r \ge \lambda^r$ $(<\lambda^r)$.
- For any given type index $j_0>1$, an "intermediate" player in the game $G^r$ is a player of type $j_0$, a "large" player is a player of any type $j>j_0$, a "small" player is any player of type $1<j<j_0$.
- No bottom coalition of a feasible game $G^r$: a coalition $S^r$ whose profile $\mathbf{s}^r$ has $s_1^r=0$.
- Dominant coalition[8] of a feasible game $G^r$ a coalition $S^r$ such that:
  (a) $S^r$ is no bottom
  (b) there is an index $j_0(G^r)>1$ (or shortly $j_0^r>1$) and a positive integer $c^r$, such that
  $\mathbf{s}^r=(s_j^r=0\ \forall j<j_0^r; s_j^r=c^r(1\le c^r\le k_{j_0}^r);$
  $s_j^r=k_j^r\ \forall j>j_0^r)$
  i.e. $S^r$ is made by all large players and by some (may be all) intermediate players, so that $j_0^r$ is the type index corresponding to the weakest player in the dominant coalition $S^r$.
  (c) $w(S^r)=\sum_{j=1}^{t(r)} s_j^r g_j^r=\lambda^r$
- Generation rules. Let $G^r$ a feasible game, $S^r$ its dominant coalition and $j_0^r$ the corresponding type index. To any $j_0^r\le j\le t(r)$, associate the game $G^{r+1}(j)=(\mathbf{k}^{r+1},\mathbf{g}^{r+1},\lambda^{r+1})$ in which $\mathbf{k}^{r+1}=\mathbf{k}^{r+1}(j)$:

- for $j>j_0^r$ and $k_{j_0}^r-c^r>0$:

[7] Note that the winning quota $\lambda^r$ of $G^r$ may well be greater than $\left(\sum_{j=1}^{t} k_j^r g_j r+1\right)\Big/2$, i.e. for $r>0$, $G^r$ may not be constant sum.

[8] It is the counterpart of the "largest min win" coalition (in backward lexicographic order) of a weighted majority game defined in [10], pp. 312.

$\mathbf{k}_{j_0}^{r+1} = (k_1^r, k_2^r, \ldots, k_{j_0-1}^r, k_{j_0}^r - c^r)$, the dimension of $\mathbf{g}^{r+1}$ is $j_0$ and $\lambda^{r+1} = g_j$

- for $j = j_0^r$ or $j > j_0^r$ and $k_{j_0}^r - c^r = 0$:

$\mathbf{k}_{j_0-1}^{r+1} = (k_1^r, k_2^r, \ldots, k_{j_0-1}^r)$, the dimension of $\mathbf{g}^{r+1}$ is $j_0 - 1$ and $\lambda^{r+1} = g_j$

Verbally the dominant coalition $S^r$ generates as many "satellite" games $G^{r+1}(j)$ as the number of types $j$ included in the coalition. The players of the satellite game associated with a type $j$ are all players of type $i<j$ not belonging to $S^r$; the winning quota is $g_j$. The idea is that any satellite game looks for the dominant coalition of the satellite able to replace one player of type $j$ in the generating game, while preserving the homogeneous character of the "seed" game. In turn, the dominant coalition gives rise to other satellite games of next generation and so on.

Now, our adapted version of Basic Lemma is:

**Lemma 6.6** *A seed weighted majority game $G^0$ is homogeneous if, in the set of all $G^r$ games obtained by the generation rules, do not exist feasible games lacking the dominant coalition.*

To prove the Lemma 6.6 we show that any game of the set $G^r$, $(r = 0,1,\ldots)$ generated by a seed $G^0$ coherent with formulae (1) and (2) either is not feasible or admits a dominant coalition.

In the proof we will exploit the following relation[9] concerning Fibonacci numbers:

$$g_{t-2} + 2g_{t-1} + \sum_{j=0}^{h} g_{t+j} = g_{t+h+2} \qquad (11)$$

*Proof.* Let us distinguish three cases:

Case $\alpha$): $z = \lfloor (t+1)/2 \rfloor$

Let us denote $G^{0,\lfloor (t+1)/2 \rfloor} = G^0 = (\mathbf{k}_t^0; \mathbf{g}_t^0; g_{t+1})$ any seed game with $\mathbf{k}_t^0 = (k_1^0, \mathbf{1}_{t-1})$. The profile of the dominant coalition $S^0$ is $\mathbf{s}_t^0 = (\mathbf{0}_{t-2}, 1_2)$ with $k_{t-1}^0 - c = 0$. Hence the satellite games associated with type *t-1* and *t* are

$$G^{1,a} = G^1(t-1) = (k_1, \mathbf{1}_{t-3}; \mathbf{g}_{t-2}; g_{t-1})$$

and

$$G^{1,b} = G^1(t) = (k_1, \mathbf{1}_{t-3}; \mathbf{g}_{t-2}; g_t)$$

By Formula (8), $G^{1,b}$ is not feasible, while $G^{1,a}$ is feasible for any $t = t(0) > 4$. Let us write briefly $G^{1,a} = G^1$; it is immediate to check that $G^1$ mimics the structure of $G^0$: only the dimension is different: it is $t = t(0)$ for $G^0$ and $t = t(1) = t(0) - 2$ for $G^1$. The profile of the dominant coalition $S^1$ of $G^1$ is $\mathbf{s}^1 = (\mathbf{0}_{t(1)-2}, \mathbf{1}_2) = (\mathbf{0}_{t-4}, \mathbf{1}_2)$ and the procedure may be recursively repeated until the integer *r* for which also $G^{1,r}$ is no longer feasible. This is resumed by the following

**Property 6.8** *Let $G^0$ be a seed game of type $\alpha$ with t=t(0)>2. For any generation $r = 0,1,\ldots,\lfloor (t-3)/2 \rfloor$ there is a unique feasible $G^r = (\mathbf{k}_t^r; \mathbf{g}_t^r; \lambda^r)$. The $\mathbf{k}_t^r$ of such game is $\mathbf{k}_t^0$ truncated at dimension $t^r$=t(0)-2r, which is coherently the dimension of $\mathbf{g}_t^r$, while $\lambda^r = g_{t(r)+1}$. Feasible $G^r$ games have dominant coalition $S^r$ whose $\mathbf{s}_{t(0)-2r}^r = (\mathbf{0}_{t(0)-2(r+1)}, \mathbf{1}_2)$.*

Case $\beta$): $z = 1$

Let $G^{0,z} = G^{0,1} = (\mathbf{k}_t^0; \mathbf{g}_{t+1}^0; g_{t+1})$ any seed game with $\mathbf{k}_t^0 = (k_1^0, \mathbf{1}_{t-3}, 2, 1)$. The profile of the dominant coalition $S^0$ is still $\mathbf{s}_t^0 = (\mathbf{0}_{t-2}, \mathbf{1}_2)$ with $k_{t-1}^0 - c = 1$. Hence, the satellite games associated to type *t-1* and *t* are

$$G^{1,a} = G^1(t-1) = (k_1, \mathbf{1}_{t-3}; \mathbf{g}_{t-2}; g_{t-1})$$

and

$$G^{1,b} = G^1(t) = (k_1, \mathbf{1}_{t-2}; \mathbf{g}_{t-1}; g_t)$$

Both games share the properties of $G^0$ of case $\alpha$), that is behave as seed games and generate for *r=2,…* sequences $G^{r,a}$ and $G^{r,b}$ according to Property 6.8. Of course the dimension of $G^{1,a} = t(1,a) = t(0) - 2$, and of $G^{1,b} = t(1,b) = t(0) - 1$.

Case $\gamma$): $1 < z < \lfloor (t-1)/2 \rfloor$

Let us write $G^{0,z} = (k_1^0, \mathbf{1}_{t-1-2z}, 2, \mathbf{1}_{2z-1})$. The profile of the dominant coalition $S^{0,z}$ is $\mathbf{s}_t^{0,z} = (\mathbf{0}_{t-2}, \mathbf{1}_2)$ with $k_{t-1}^0 - c = 0$. Let us denote the two satellite games by

$$G^{1z,a} = G^{1z}(t-1) = (\mathbf{k}_t^{1z,a}; \mathbf{g}_t^{1z,a}; \lambda^{1z,a})$$

and

$$G^{1z,b} = G^{1z}(t) = (\mathbf{k}_t^{1z,b}; \mathbf{g}_t^{1z,b}; \lambda^{1z,b}).$$

It is $\mathbf{k}_{t-2}^{1z,a} = \mathbf{k}_{t-2}^{1z,b} = (k_1^0, \mathbf{1}_{t-1-2z}, 2, \mathbf{1}_{2z-3})$, $\lambda^{1z,a} = g_{t-1}$, and $\lambda^{1z,b} = g_t$. Note that $G^{1z,a}$ mimics $G^{0,(z-1)}$; hence recursively we go back to $G^{0,1}$, i.e. the seed game of case $\beta$).

In turn, as by Formula (11), $\sum_j s_j^{1z,b} g_j = g_t$, $G^{1z,b}$ has dominant coalition with profile. $\mathbf{s}_{t-2}^{1z,b} = (\mathbf{0}_{t-2z-1}, 1, 2, \mathbf{1}_{2z-3})$

[9] By induction: it is true for *j=0*, and if true for *j* it is true for *j+1*.

Hence $G^{1z,b}$ gives rise to *2z-1* satellite games $G^{2z,b}(j)$ for $j = t(0)-2z,\ldots,t(0)-2$.

It is $G^{2z,b}(j) = (k_1^0, \mathbf{1}_{t-2z-2}; g_{t-2z-1}; g_j)$. By Formula (8), for all *j*, except $j^* = t(0)-2z$, the $G^{2z,b}(j)$ are not feasible; the survival feasible $G^{2z,b}(j^*)$ mimics the behavior of the $G_0$ seed of the case $\alpha$).

Summing up, in all three cases all $G^r$ games either are not feasible, or if feasible admit a dominant coalition and the conditions of the Basic Lemma are verified.

## 7. Proofs of Properties of Section 5

By definition:

$$n(t,z) = k_1(t,z) + \sum_{j=2}^{t} k_j(t,z) \qquad (12)$$

*Proof.* Proof of Property 5.1.

For $z = \lfloor (t+1)/2 \rfloor$ it is $\sum_{j=2}^{t} k_j(t,z) = t-1$, by Formula
(2) and $k_1 = 2 + g_{t-1}$ by Formula (10) so that:

$$n(t,z) = t+1+g_{t-1} \qquad (13)$$

For $z < \lfloor (t+1)/2 \rfloor$ it is $\sum_{j=2}^{t} k_j(t,z) = t$, by Formula (1)
and $k_1 = 2 + g_{t-1} - g_{t+1-2z}$ by Formula (9) so that:

$$n(t,z) = t+2+g_{t-1} - g_{t+1-2z} \qquad (14)$$

and in particular

$$n(t,1) = t+2$$

*Proof.* Proof of Property 5.2.

For $(z+1) < \lfloor (t+1)/2 \rfloor$ by Formula (14):

$$n(t,z+1) - n(t,z) =$$
$$= (t+2+g_{t-1} - g_{t+1-2(z+1)}) - (t+2+g_{t-1} - g_{t+1-2z}) =$$
$$= g_{t-2z+1} - g_{t-2z-1} = g_{t-2z} \qquad (15)$$

For $(z+1) = \lfloor (t+1)/2 \rfloor$ by Formulas (13) and (14):

$$n(t,z+1) - n(t,z) =$$
$$= (t+1+g_{t-1}) - (t+2+g_{t-1} - g_{t+1-2z}) =$$
$$= g_{t+1-2z} - 1 \qquad (16)$$

but $z = \lfloor (t-1)/2 \rfloor = (t/2)-1$ for *t* even and $(t-1)/2$ for *t* odd so that:

*t+1-2z=3* for *t* even and *t+1-2z=2* for *t* odd

which also means, on one side:

*t-2z=2* for *t* even and *t-2z=1* for *t* odd

and on the other one:

$g_{t+1-2z} = 3$ for *t* even and $g_{t+1-2z} = 2$ for *t* odd

and finally

$g_{t+1-2z} - 1 = 2$ for *t* even and $g_{t+1-2z} - 1 = 1$ for *t* odd

so that in both cases still

$$n(t,z+1) - n(t,z) = g_{t-2z} \qquad (17)$$

which completes the proof of Property 5.2. Note that this property implies the strict monotony of $n(t,z)$ for any *t* and then, as a by product, gives a proof of Theorem 3.2.

*Proof.* Proof of Property 5.3.

Preliminarily note that putting $t' = t-1$ it is $\lfloor (t'+1)/2 \rfloor = \lfloor (t/2) \rfloor$ so that by Formula (13):

$$n(t', \lfloor (t'+1)/2 \rfloor) = t'+1+g_{t'-1} \qquad (18)$$

equivalent to

$$n(t-1, \lfloor (t/2) \rfloor) = t + g_{t-2}$$

Moreover, it is by Formula (14) for *t>4*, or by Formula (13) for *t=3* and 4 [10]

$$n(t,2) = t+2+g_{t-1} - g_{t-3} = t+2+g_{t-2}$$

and immediately:

$$n(t,2) - n(t-1, \lfloor t/2 \rfloor) = 2 \qquad (19)$$

**Remark 7.1** *Jointly Prop. 5.2 and 5.3 imply that, putting in lexicographic order the set* $\Xi$ *of values of n(t,z), z>1, a strictly monotone sequence is obtained. By Formulas (15), (17) and (19), all members of the sequence of first differences* $\Delta$ *of the* $\Xi$ *sequence are Fibonacci numbers (see Table 3).*

## 8. Conclusions

In this paper we introduce the class of Fibonacci games. They are the subset of constant sum homogeneous weighted majority games whose sequence of all type weights and the minimal winning quota is a string of consecutive Fibonacci numbers. Exploiting properties of the Fibonacci sequence, we give closed form results able to provide a simple and insightful characterization of such games. In more detail, we compute the total number of Fibonacci games for any given value of *t* (type of players in the game); we describe, for any *(t,z)* with *z* a proper counter, the profile of any game, i.e. the vector whose components are the number of non dummy players of each type in the game; we provide a recursive rule to compute the function *n(t,z)* which gives the overall

[10] Note that for *t=3* or 4 it is $1+g_{t-2} = g_{t-1}$. Hence for such *t*: $t+2+g_{t-2} = t+1+g_{t-1}$ so that both (13) and (14) describe $n(t, \lfloor (t+1)/2 \rfloor)$.

number of non dummy players in the *(t,z)* game and we underline that such rule may be summarized through a matrix whose columns and rows are exactly sequences (or subsequences for rows) of Fibonacci numbers.

Compared to other weighted majority games, the Fibonacci games combine two specific characteristics: the presence of some, or perhaps many, "peones" (players with minimum weight), along with an almost total ranking (with one tie at most) of all the other players, whose individual power grows at the speed of the Fibonacci sequence. It seems, by some preliminary analysis, that this feature might be useful in applications to weighted voting systems in parliamentary elections. In particular, it might be worthwhile investigating the consequences coming from the assignment of a wide proposal power besides the voting rights to the strongest player. This would be the object of further research.

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