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Abstract

A topological abelian group G is w-divisible if G has uncountable weight and the subgroup $mG = \{mx : x \in G\}$ has the same weight of G for each positive integer m. In order to "measure" w-divisibility we introduce a cardinal invariant (divisible weight) which allows for a precise description of various phenomena related to the subgroups of the compact abelian groups. We give several applications of these results.

1 Introduction

A Tychonoff topological space X is:

- *pseudocompact* if every real valued function of X is bounded [24],
- *countably compact* if every countable open cover of X has a finite subcover,
- *strongly pseudocompact* if X contains a dense countably compact subspace [2],
- *ω-bounded* if every countable subset of X is contained in a compact subset of X.

All topological groups in this paper are Hausdorff. A topological group G is:

- *precompact* if its completion $\widetilde{G}$ is compact.

For a topological group the following sequence of implications holds:

ω-bounded $\Rightarrow$ countably compact $\Rightarrow$ strongly pseudocompact $\Rightarrow$ pseudocompact $\Rightarrow$ precompact.

A subgroup of a topological abelian group G is *totally dense* in G if it densely intersects each closed subgroup of G [29]. This property is related to the open mapping theorem as follows: a dense subgroup of a compact abelian group is totally dense if and only if it satisfies the open mapping theorem [14, 15, 23].

One of the motivations of this paper is the problem of the description of the compact abelian groups admitting proper totally dense subgroups with some of the compactness-like properties from the above list. This problem has been studied by various authors — see [7, 9, 11, 12, 16]. It became clear that the first two properties have to be ruled out, as no compact abelian group can contain a proper totally dense countably compact subgroup [7, 16] (see also [11] for stronger results). So one has to limit the compactness-like property within (strong) pseudocompactness. It was proved in [16] that the compact abelian groups K with non-metrizable connected component have the following stronger property TD_ω relaxing countable compactness: there exists a proper totally dense subgroup H of K that contains an ω-bounded dense subgroup of K. Obviously such an H is strongly pseudocompact, but need not be countably compact. The final solution of the problem of when a compact abelian group admits proper totally dense pseudocompact subgroups is given in the next theorem. (A topological group G is *singular* if mG is metrizable for some positive integer m [13, Definition 1.2].)

Theorem 1.1. [12, Theorem 5.2] *For a compact abelian group K the following conditions are equivalent:*

(a) K has a proper totally dense pseudocompact subgroup;

(b) K has no closed torsion G_δ-subgroup;

(c) K is non-singular;

(d) there exists a continuous surjective homomorphism of K onto S^{ω_1}, where S is a compact non-torsion abelian group;

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(e) K has the property TD_ω.

In this paper we first generalize this theorem and then, using the new results, we answer a question from [17]. More precisely we introduce appropriate notions generalizing non-singularity and the property TD_ω and we consider also how to extend the properties involved in items (a) and (b) of Theorem 1.1.

In what follows we give first some notations and preliminary results. Then in §1.3 we expose the main results of the paper.

Notation and terminology

We denote by $\mathbb{Z}$, $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_+$ respectively the set of integers, the set of primes, the set of natural numbers and the set of positive integers. For $m \in \mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m. The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of the reals $\mathbb{R}$ and carries its usual topology. For $p \in \mathbb{P}$ the symbol $\mathbb{Z}_p$ is used for the group of p-adic integers.

Let G be an abelian group. The subgroup of torsion elements of G is $t(G)$ and $G[m] = \{x \in G : mx = 0\}$. We say that G is non-torsion if it does not coincide with $t(G)$. For a cardinal α we denote by $G^{(\alpha)}$ the direct sum of α many copies of G, that is $\bigoplus_\alpha G$. If $G = H^\alpha$, where H is a group and α an uncountable cardinal, ΣG is the Σ-product centered at 0 of G, that is the set of all elements of K with countable support; moreover $\Delta G = \{\mathbf{x} = (x_i) \in G : x_i = x_j \text{ for every } i, j < \alpha\}$ is the diagonal subgroup of G. We denote by $r_0(G)$ the free-rank of G (i.e., the cardinality of a maximal independent subset of G) and, for $p \in \mathbb{P}$, by $r_p(G)$ the p-rank of G (i.e., the dimension of $G[p]$ as a vector space over the field of p elements $\mathbb{F}_p$). The symbol $\mathfrak{c}$ stands for the cardinality of the continuum.

For a topological group G we denote by $c(G)$ the connected component of the identity e_G in G. If $c(G)$ is trivial, the group G is said to be *totally disconnected*. If M is a subset of G then $\langle M \rangle$ is the smallest subgroup of G containing M, and $\overline{M}$ is the closure of M in G. The symbol $w(G)$ stands for the weight of G. The Pontryagin dual of a topological abelian group G is denoted by $\widehat{G}$.

For undefined terms see [21, 22].

General properties of compact abelian groups

In the following fact we remind some general properties of compact abelian groups, which are applied in the paper (without giving explicit references).

Fact 1.2. [25, 26] *Let K be a compact abelian group. Then:*

(a) $|K| = 2^{w(K)}$ and $w(K) = |\widehat{K}|$;

(b) K is connected if and only if it is divisible;

(c) either $r_0(K) > \mathfrak{c}$ or K is bounded torsion.

As in [15], for a prime p, the *topological p-component* K_p of K is

$$K_p = \{x \in K : p^n x \to 0 \text{ in } K, \text{ where } n \in \mathbb{N}\}.$$

For $p \in \mathbb{P}$ and for a subset $\pi \subseteq \mathbb{P}$ consider the metrizable compact abelian groups

$$\mathbb{G}_p = \prod \{\mathbb{Z}(p^n) : n \in \mathbb{N}_+\} \quad \text{and} \quad S_\pi = \prod \{\mathbb{Z}(q) : q \in \pi\}.$$

Clearly, $\mathbb{G}_p$ is non-torsion, while S_π is non-torsion if and only if π is infinite.

For the sake of easier reference we recall here the following useful and well known property of the totally disconnected compact abelian groups.

Remark 1.3. Let K be a totally disconnected compact abelian group. Then:

(a) [4], [15, Proposition 3.5.9] $K = \prod_{p \in \mathbb{P}} K_p$ and every closed subgroup N of K is of the form $N = \prod_{p \in \mathbb{P}} N_p$, where each N_p is a closed subgroup of K_p;

(b) [15, Proposition 4.1.5] if L is a totally disconnected abelian group and $f : K \to L$ is a continuous homomorphism, then $f(K_p) = L_p$ for every $p \in \mathbb{P}$.

We recall that the totally disconnected compact abelian groups are precisely the profinite abelian groups [28]. The *profinite* groups are topological groups isomorphic to inverse limits of an inverse system of finite groups. For a prime p, a group G is a *pro-p group* if it is the inverse limit of an inverse system of finite p-groups. Equivalently a pro-p group is a profinite group G such that G/N is a finite p-group for every open normal subgroup N of G.

1.1 Measuring compactness

In the next definition we consider properties that generalize those from items (a) and (b) of Theorem 1.1. The property in item (a) is equivalent to the definition given in [27].

Definition 1.4. *Let X be a Tychonoff topological space and κ an infinite cardinal.*

(a) The space X is κ-pseudocompact *if for every continuous function $f : X \to Y$, where Y is a topological space of weight $\leq \kappa$, $f(X)$ is compact.*

(b) A subset Y of X is a G_κ-set *of X if $Y = \bigcap_{i<\kappa} O_i$, where O_i is open in X for all $i < \kappa$.*

Note that ω-pseudocompactness coincides with pseudocompactness [27, Theorem 2.1], while the G_κ-sets for $\kappa = \omega$ are the well known G_δ-sets. Moreover a κ-pseudocompact space of weight $\leq \kappa$ is compact.

Now we want also to give generalizations of ω-boundedness.

Definition 1.5. *Let κ be an infinite cardinal. A Tychonoff topological space X is:*

- weakly κ-bounded *if every subset of X of cardinality $< \kappa$ is contained in a compact subset of X;*
- κ-bounded *if every subset of X of cardinality at most κ is contained in a compact subset of X.*

Obviously every group is weakly ω-bounded. Moreover these two notions are related as follows: weakly κ-bounded coincides with the conjunction of λ-bounded for all $\lambda < \kappa$ (in particular, κ-bounded coincides with weakly κ^+-bounded).

A weakly κ-bounded group G with $w(G) < \kappa$ is necessarily compact (see Lemma 2.3), in particular every $w(G)$-bounded group G is compact. Examples of non-compact weakly κ-bounded groups of weight κ are given in Example 2.2.

In analogy with the property TD_ω, for every infinite cardinal κ we can say that a compact abelian group K:

- *has the property TD_κ* (briefly, $K \in TD_\kappa$) if K has a proper totally dense subgroup H that contains a dense κ-bounded subgroup;
- *has the property TD^κ* (briefly, $K \in TD^\kappa$) if K has a proper totally dense subgroup H that contains a dense weakly κ-bounded subgroup.

The conditions TD_κ and TD^κ have properties analogous to those of TD_ω. Obviously TD_κ coincides with TD^{κ^+}, in particular $TD_\kappa \Rightarrow TD^\kappa$ and TD^{ω_1} coincides with TD_ω. Nevertheless, for a limit cardinal κ the property TD^κ need not coincide with the conjunction of all TD_λ for $\lambda < \kappa$ (see the comments after Theorem 1.8).

1.2 Measuring non-singularity

Our next aim is to introduce a notion generalizing non-singularity.

The following definition is justified by the fact that an abelian group G is *divisible* if and only if $G = mG$ for every integer $m > 0$.

Definition 1.6. *A topological abelian group G is* w-divisible *if $w(mG) = w(G) > \omega$ for every $m \in \mathbb{N}_+$.*

In this definition we exclude the case of countable weight, because we want w-divisible groups to be non-singular and non-singular groups are necessarily non-metrizable.

Obviously every divisible topological abelian group of uncountable weight is w-divisible and every w-divisible abelian group is non-singular. For example every uncountable product $\prod_{i\in I} K_i$, where each K_i is compact, metrizable and non-torsion, is w-divisible. If G is a dense subgroup of a group K, mG is dense in mK for every $m \in \mathbb{N}_+$ and so G is w-divisible whenever K is w-divisible. Consequently, connected precompact groups G of uncountable weight are w-divisible; in fact, $\widetilde{G}$ is connected and so divisible (being compact), hence w-divisible and G is dense in $\widetilde{G}$. In general a connected abelian group need not be w-divisible. Actually, there exist connected abelian groups of every prime exponent [3].

We shall see that every w-divisible compact abelian group K of regular weight contains a proper dense weakly $w(K)$-bounded subgroup (so K contains a proper dense κ-bounded subgroup for every $\kappa < w(K)$).

The cardinal invariant of compact abelian groups K introduced in the following definition will measure w-divisibility (and singularity) of K.

Definition 1.7. *Let G be a topological abelian group. The* divisible weight *(or shortly,* d-weight*) of G is*

$$w_d(G) = \inf\{w(m!G) : m \in \mathbb{N}_+\}.$$

Indeed, a topological abelian group G is w-divisible if and only if $w(G) = w_d(G) > \omega$, whereas a non-torsion topological abelian group G is singular if and only if $w_d(G) = \omega$. We shall see in §3 that every topological abelian group G has a w-divisible subgroup H of the form $H = m!G$ for some $m \in \mathbb{N}_+$, such that $w_d(G) = w(H) = w_d(H)$, and that there exists a minimal $m \in \mathbb{N}_+$ with this property. We denote it by

$$m_d(G) = \min\{m \in \mathbb{N}_+ : w_d(G) = w(m!G)\}.$$

1.3 Main Results

Our main theorem is a complete generalization of Theorem 1.1 (indeed, to get Theorem 1.1 it suffices to take $\lambda = \omega$).

Theorem 1.8. *Let K be a compact abelian group and λ an infinite cardinal. The following conditions are equivalent:*

- *(a) K has a proper totally dense λ-pseudocompact subgroup;*
- *(b) K has no closed torsion G_λ-subgroup;*
- *(c) $w_d(K) > \lambda$;*
- *(d) there exists a continuous surjective homomorphism of K onto S^I, where S is a metrizable compact non-torsion abelian group and $|I| > \lambda$;*
- *(e) K has the property TD_λ.*

Moreover, $K \in TD^{w_d(K)}$ if and only if $w_d(K)$ is an uncountable regular cardinal.

This theorem will be deduced from Theorem 1.9 given below. Both proofs (as well as those of Theorems 1.10, 1.15 and 1.18) are given in §5.

Several comments are in order here. The equivalence of (c) and (e) in Theorem 1.8 implies

$$w_d(K) = \sup\{\kappa : K \in TD^\kappa\},$$

but leaves open the question of when $K \in TD^{w_d(K)}$ holds true. This motivates the final part of the theorem that settles completely this issue. Consequently, for an infinite cardinal κ the property TD^κ coincides for compact abelian groups with the conjunction of all TD_λ for $\lambda < \kappa$ precisely when κ is regular.

Analogously, the equivalence of (c) and (d) yields:

$$w_d(K) = \sup\{\kappa : \text{ there exists a compact non-torsion metrizable abelian group } S$$
$$\text{and a surjective continuous homomorphism } K \to S^\kappa\},$$

but leaves open the extreme case:

(P) when does there exist a surjective continuous homomorphism $f : K \to S^{w_d(K)}$ with a metrizable compact non-torsion abelian group S?

Example 2.8 shows a w-divisible group K of non-regular weight having no proper dense weakly $w(K)$-bounded subgroup at all (so in particular $K \notin TD^{w(K)}$). Moreover K does not satisfy the property in (P). We completely answer (P) in Theorem 1.9, showing that for the compact abelian groups K admitting a continuous surjective homomorphism onto $S^{w_d(K)}$ there is a remarkable trichotomy.

Theorem 1.9. *A compact abelian group K admits a continuous surjective homomorphism onto $S^{w_d(K)}$ for some metrizable compact non-torsion abelian group S precisely when some of the following occurs:*

- *(a) there exists a continuous surjective homomorphism $f : K \to \mathbb{T}^{w_d(K)}$ if and only if $w_d(K) = w(c(K))$;*
- *(b) for some $p \in \mathbb{P}$, there exists a continuous surjective homomorphism $f : K \to \mathbb{G}_p^{w_d(K)}$ if and only if $w_d(K) = w_d((K/c(K))_p)$;*
- *(c) if π is an infinite subset of $\{p \in \mathbb{P} : p > m_d(K)\}$, then there exists a continuous surjective homomorphism $f : K \to S_\pi^{w_d(K)}$ if and only if $w_d(K) = w((K/c(K))_p)$ for every prime $p \in \pi$.*

Moreover every compact abelian group K with $\mathrm{cf}(w_d(K)) > \omega$ admits a continuous surjective homomorphism of K onto $S^{w_d(K)}$, where S is a metrizable compact non-torsion abelian group.

Let us now consider the weaker version of (P) where the power is replaced by a product of eventually distinct metrizable compact non-torsion abelian groups. It is well known that for every non-metrizable compact abelian group K there exists a continuous surjective homomorphism of K onto a product $\prod_{i\in I} K_i$ of non-trivial metrizable compact abelian groups with $|I| = w(K)$ (e.g., a standard application of the Pontryagin duality to [1, Theorem 1.1] can produce such a surjective homomorphism). Therefore, inspired by (P), one can look for such a homomorphism asking that the groups K_i have to be also non-torsion. Since such a product $\prod_{i\in I} K_i$ is w-divisible (see Proposition 2.6(a)) and the divisible weight is monotone under continuous surjective homomorphisms of compact abelian groups (see Claim 3.2), we obtain the restriction $|I| \leq w_d(K)$ for such a surjective homomorphism. The next theorem shows that this necessary condition is also sufficient.

Theorem 1.10. *Let K be a non-singular compact abelian group. There exists a continuous surjective homomorphism of K onto $\prod_{i\in I} K_i$, where each K_i is metrizable, compact and non-torsion if and only if $|I| \leq w_d(K)$.*

Corollary 1.11. *A compact abelian group K is w-divisible if and only if there exists a continuous surjective homomorphism of K onto $\prod_{i\in I} K_i$, where each K_i is compact, metrizable and non-torsion, and $|I| = w(K) > \omega$.*

Even if these results may give the impression to be somewhat technical, they are quite useful. The remaining part of the paper is dedicated to a relevant application of Corollary 1.11.

Following [5] (see also [17, Definition 2.6]), if X is a non-empty set and σ is an infinite cardinal, then a set $F \subseteq X^\sigma$ is *ω-dense* in X^σ, provided that for every countable set $A \subseteq \sigma$ and each function $\varphi \in X^A$ there exists $f \in F$ such that $f(\alpha) = \varphi(\alpha)$ for all $\alpha \in A$.

Definition 1.12. [17, Definition 2.6] *If τ and $\sigma \geq \omega$ are cardinals, then $Ps(\tau, \sigma)$ abbreviates the sentence "there exists an ω-dense set $F \subseteq \{0,1\}^\sigma$ with $|F| = \tau$".*
*Moreover **Ps**(τ) denotes the sentence "$Ps(\tau, \sigma)$ holds for some infinite cardinal σ".*

This set-theoretic condition is closely related to the pseudocompact group topologies:

Theorem 1.13. [17, Fact 2.12 and Theorem 3.3(i)] *Let τ and $\sigma \geq \omega$ be cardinals. Then $Ps(\tau, \sigma)$ holds if and only if there exists a group G of cardinality τ which admits a pseudocompact group topology of weight σ.*

Hence obviously, if G is a pseudocompact abelian group, then $Ps(|G|, w(G))$ holds. But what about the free-rank $r_0(G)$ of G? Does $Ps(r_0(G))$ holds whenever G is a pseudocompact group? In [17] the authors proved the following theorem and left open the problem in general.

Theorem 1.14. [17, Theorem 3.21] *If G is a non-trivial connected pseudocompact abelian group, then $Ps(r_0(G), w(G))$ holds.*

One can ask also whether connectedness is a necessary condition in order that $Ps(r_0(G), w(G))$ holds. Applying Theorem 1.10 we prove the following result, that generalizes Theorem 1.14 to w-divisible groups, which are far from being connected (while connected pseudocompact groups are w-divisible).

Theorem 1.15. *If G is a w-divisible pseudocompact abelian group, then $Ps(r_0(G), w(G))$ holds.*

Note that for a pseudocompact abelian group G

$$Ps(r_0(G), w(G)) \Rightarrow w(G) \leq 2^{2^{w_d(G)}}$$

and there is an example of a singular pseudocompact abelian group G for which $Ps(r_0(G), w(G))$ and $w(G) = 2^{2^{w_d(G)}}$ hold — see Lemma 5.3 and Example 5.4. So this example shows also that the converse implication of Theorem 1.15 does not hold. This means that w-divisibility (and so also connectedness) is not a necessary condition in order that $Ps(r_0(G), w(G))$ holds. Nevertheless, the next result provides the missing equivalence at a different level (namely that of pseudocompact topologization).

Corollary 1.16. *For an infinite abelian group G and a cardinal $\sigma \geq \omega_1$ the following conditions are equivalent:*

(a) G admits a connected pseudocompact group topology of weight σ;

(b) G admits a w-divisible pseudocompact group topology of weight σ;

(c) $Ps(r_0(G), \sigma)$ and $|G| \leq 2^\sigma$ hold.

Proof. (a)⇔(c) is proved in [17, Theorem 7.1], (a)⇒(b) is obvious and (b)⇒(c) follows from Theorem 1.15. □

The characterization of the abelian groups admitting pseudocompact group topologies is still a hard open question [17, Problem 0.2] (see also [6, Problem 856] and [19, Problem 892]). The following problem seems to be important for its solution, as mentioned in [17].

Problem 1.17. [17, Problem 9.11] *Is* $\boldsymbol{Ps}(r_0(G))$ *a necessary condition for the existence of a pseudocompact group topology on a non-torsion abelian group* G*?*

Note that if G is torsion, the problem makes no sense because $r_0(G) = 0$ and $Ps(\tau, \sigma)$ is defined for infinite σ.

The following result can be easily deduced from Theorem 1.15.

Theorem 1.18. *Let* G *be a pseudocompact non-torsion abelian group. Then* $Ps(r_0(G), w_d(G))$ *holds.*

The following immediate corollary of Theorem 1.18 is precisely the answer to Problem 1.17. A completely different proof of this fact is given in [20].

Corollary 1.19. *If* G *is a pseudocompact non-torsion abelian group, then* $\boldsymbol{Ps}(r_0(G))$ *holds.*

In Corollary 1.16 we considered the problem of the characterization of the abelian groups admitting pseudocompact group topologies in the case of w-divisible topologies that go closer to the connected ones. Now we conclude with the case of singular topologies closer to the "opposite end", namely the torsion pseudocompact groups (that are always zero-dimensional, hence totally disconnected). Here we offer only the following:

Conjecture 1.20. *For an infinite abelian group* G *the following conditions are equivalent:*

(a) G *admits a singular pseudocompact group topology;*

(b) there exists $m \in \mathbb{N}_+$ *such that* $G[m]$ *admits a pseudocompact group topology and* mG *admits a compact metrizable group topology.*

So this case could be reduced to those of pseudocompact group topologies on torsion abelian groups ($G[m]$) and of metrizable compact group topologies on abelian groups (mG). In [17, §6] is given a clear criterion of when a torsion abelian group admits a pseudocompact group topology, while in [18] the groups which admit a metrizable compact group topology are well characterized.

2 The properties TD_κ and TD^κ and κ-pseudocompactness

The following lemma shows that for each infinite cardinal κ the properties TD_κ and TD^κ and κ-pseudocompactness are stable under taking inverse images. It generalizes [11, Lemma 3.12] and [12, Lemma 2.6], the proof remains almost the same.

Lemma 2.1. *Let* K *be a compact abelian group that admits a continuous surjective homomorphism* f *onto a compact abelian group* L*. Let* κ *be an infinite cardinal. If* L *has the property* TD_κ *(respectively, has the property* TD^κ*, is* κ*-pseudocompact), then* K *has the property* TD_κ *(respectively, has the property* TD^κ*, is* κ*-pseudocompact) too.*

Now we shall give examples of non-compact weakly $w(G)$-bounded groups G and of non-compact λ-bounded groups G for $\lambda < w(G)$.

Example 2.2. Let κ be an uncountable cardinal, K_i a compact non-torsion abelian group for each $i < \kappa$ and $K = \prod_{i \in I} K_i$.

(a) Let $\lambda < \kappa$. The subgroup

$$\Sigma_\lambda K = \{x \in K : |\mathrm{supp}(x)| \leq \lambda\}$$

is the λ-Σ-*product* of the family $\{K_i : i < \kappa\}$ (for $\lambda = \omega$ we obtain the usual Σ-product). Let us show that $\Sigma_\lambda K$ is λ-bounded (non-compact). Take $A \subseteq \Sigma_\lambda K$ with $|A| \leq \lambda$. If $a \in A$, then $a \in \prod_{i \in L_a} K_i$, where $L_a \subseteq \lambda$ and $|L_a| \leq \lambda$. Define $L = \bigcup_{a \in A} L_a$. Thus $A \subseteq \prod_{i \in L} K_i$ and $|L| = |A| \cdot \sup |L_a| \leq \lambda$.

(b) Suppose that κ is regular and consider the following proper subgroup of K:

$$S = \bigcup_{\lambda < \kappa} \Sigma_\lambda K$$

(in other words $S = \{x \in K : |\mathrm{supp}(x)| < \kappa\}$). Clearly S is dense in K, hence S is not compact. Let us see that S is weakly κ-bounded (so λ-bounded for every cardinal $\lambda < \kappa$). Take $A \subseteq S$ with $|A| < \kappa$. If $a \in A$, then $a \in \prod_{i \in L_a} K_i$, where $L_a \subseteq \kappa$ and $|L_a| < \kappa$. Define $L = \bigcup_{a \in A} L_a$. Thus $A \subseteq \prod_{i \in L} K_i$ and $|L| = |A| \cdot \sup |L_a| < \kappa$, as κ is regular. Moreover, note that if each K_i is metrizable, then $w(S) = w(K) = \kappa$.

Lemma 2.3. *Let κ be an infinite cardinal. A weakly κ-bounded group G with $w(G) < \kappa$ is necessarily compact, so every $w(G)$-bounded group G is compact.*

Proof. Let X be a dense subset of G of size $\leq w(G)$. As $w(G) < \kappa$ and G is weakly κ-bounded, X is contained in a compact subset Z of G. Now the density of X in G yields the density of Z in G. Hence $Z = G$. This proves that G is compact. □

In the next example we explicitly construct a compact abelian group K which has the property TD_κ for an uncountable cardinal κ. Note that K is a power of a metrizable compact non-torsion abelian group S.

Example 2.4. Let κ be an uncountable cardinal, $p \in \mathbb{P}$ and $K_p = \mathbb{Z}(p)^{\kappa^+}$. Define $K = \prod_{p\in\mathbb{P}} K_p$. Note that $K = S^{\kappa^+}$ with $S = \prod_{p\in\mathbb{P}} \mathbb{Z}(p)$. The subgroup $H = \Sigma_\kappa S^{\kappa^+} + t(K)$ is a proper totally dense pseudocompact subgroup of K. Indeed, $t(K) = \bigoplus_{p\in\mathbb{P}} K_p$ is totally dense in K, because each closed subgroup N of K is of the form $N = \prod_{p\in\mathbb{P}} N_p$ by Remark 1.3(a), while $\Sigma_\kappa S^{\kappa^+}$ is a dense κ-bounded (hence pseudocompact) subgroup of K as proved in Example 2.2(a).

In Proposition 2.6, which generalizes [12, Proposition 2.4], we produce a compact abelian group with the property TD_κ for a given infinite cardinal κ. To prove it we need the following lemma, that was crucial for proving Theorem 1.1 in [12].

Lemma 2.5. [11, Lemma 3.16],[12, Lemma 3.6] *Let K be a compact abelian group that admits a subgroup B such that $r_0(K/B) \geq 1$. Then K has a proper totally dense subgroup H that contains B.*

Then next proposition generalizes [12, Proposition 2.4].

Proposition 2.6. *Let κ be an uncountable cardinal, I a set of indices of cardinality κ, K_i a metrizable compact non-torsion abelian group for each $i \in I$ and $K = \prod_{i\in I} K_i$. Then:*

(a) K is w-divisible;

(b) K has the property TD_λ for every $\omega \leq \lambda < \kappa$.

Proof. (a) is obvious.

(b) Take in K the λ-bounded subgroup of Example 2.2(a) $B = \Sigma_\lambda K$ and for every $i \in I$ a non-torsion element $c_i \in K_i$. Defining $C = \langle (c_i)_{i\in I}\rangle$ we have $B \cap C = \{0\}$ and so $r_0(K/B) \geq 1$. Now apply Lemma 2.5. □

The regularity of κ is essential in item (b) of Example 2.2. Indeed we have the following theorem characterizing the regularity of uncountable cardinals κ in terms of the topological property TD^κ (see also Example 2.9).

Theorem 2.7. *Let κ be an uncountable cardinal. Then the following conditions are equivalent:*

(a) κ is regular;

(b) for every family $\{K_i : i < \kappa\}$, where each K_i is a metrizable compact non-torsion abelian group, $K = \prod_{i<\kappa} K_i \in TD^\kappa$;

(c) there exists a compact abelian group K of weight κ such that $K \in TD^\kappa$;

(d) there exists a compact abelian group of weight κ with a proper dense weakly κ-bounded subgroup.

Proof. (a)⇒(b) Let $K = \prod_{i<\kappa} K_i$, where each K_i is a metrizable compact non-torsion abelian group. Argue as in the proof of Proposition 2.6(b), using Example 2.2(b), to prove that if κ is regular, then $K \in TD^\kappa$.

(b)⇒(c) and (c)⇒(d) are obvious.

(d)⇒(a) Let H be a proper dense weakly κ-bounded subgroup of K. Fix a point $x \in K \setminus H$ and assume for a contradiction that $\lambda = \mathrm{cf}(\kappa) < \kappa$, i.e.,

$$\kappa = \sup\{\kappa_\alpha : \alpha < \lambda\}, \text{ with } \kappa_\alpha < \kappa \text{ for all } \alpha < \lambda.$$

We can assume without loss of generality that K is a subgroup of $G = \mathbb{T}^\kappa$. Write $\mathbb{T}^\kappa = \prod_{\alpha<\lambda} T_\alpha$, where $T_\alpha \cong \mathbb{T}^{\kappa_\alpha}$ for each $\alpha < \lambda$. For $\alpha < \lambda$, let $N_\alpha = \prod_{\beta<\alpha} T_\beta$ and let $p_\alpha : G = \mathbb{T}^\kappa \to N_\alpha$ be the canonical projection. Since $w(N_\alpha) = \kappa_\alpha < \kappa$, $p_\alpha(H)$ is compact and dense in $p_\alpha(K)$, so they coincide. Then there exists a point $h_\alpha \in H$ such that

$$p_\alpha(h_\alpha) = p_\alpha(x). \qquad (*)$$

The set $A = \{h_\alpha : \alpha < \lambda\} \subseteq H$ has size $\leq \lambda < \kappa$. Hence the weak κ-boundedness of H implies that the H-closure C of A is compact. Then it is closed in K as well. On the other hand, for every neighborhood U of 0 in G there exists a projection p_α such that $\ker p_\alpha \subseteq U$. Now $(*)$ yields $h_\alpha - x \in U$, so $A \cap (x + U) \neq \emptyset$. This proves that $x \in C \subseteq H$, a contradiction. □

Let us note that our main result strengthens substantially Theorem 2.7 (for every w-divisible compact abelian group K the property TD^κ is equivalent to the weaker one: K has a proper dense weakly κ-bounded subgroup).

Example 2.8. Let $p_1, p_2, \dots, p_n, \dots$ be all primes written in increasing order. Then the group $K = \prod_{n=1}^\infty \mathbb{Z}(p_n)^{\aleph_n}$ is w-divisible of weight $\aleph_\omega$. Nevertheless, a standard application of the Pontryagin duality shows that there exists no continuous surjective homomorphism of K onto $S^{\aleph_\omega}$, where S is a non-trivial metrizable compact abelian group. From Theorem 2.7 it follows that K has no proper dense weakly $\aleph_\omega$-bounded subgroup and so $K \notin TD^{\aleph_\omega}$.

Note that the subgroup $\bigcup_{\lambda<\kappa} \Sigma_\lambda K$ of K considered in Example 2.2(b) is not even ω-bounded. In fact it is not even countably compact, as it contains a sequence (x_n) that converges to a point of $K \setminus \bigcup_{\lambda<\kappa} \Sigma_\lambda K$, although it is pseudocompact.

Example 2.9. Let κ be an infinite cardinal.

(a) Every κ-bounded Tychonoff space X is κ-pseudocompact. To see this let $f : G \to Y$ be a continuous function, where Y is a Tychonoff space of weight $\leq \kappa$. We can suppose without loss of generality that f is surjective. There exists a dense subset D of Y such that $|D| \leq \kappa$. There exists a subset D_1 of X such that $f \restriction_{D_1}: D_1 \to D$ is bijective. Then $|D_1| \leq \kappa$. Since X is κ-bounded, $\overline{D}_1$ is compact. Therefore $f(\overline{D}_1)$ is compact. But $f(\overline{D}_1) \supseteq D$, D is dense in Y, and so $f(\overline{D}_1) = Y$ is compact.

(b) Let G be a topological group and H a dense subgroup of G. If H is κ-pseudocompact, then G is κ-pseudocompact too. Let $f : G \to Y$ be a continuous function, where Y is a Tychonoff space of weight $\leq \kappa$. Since H is κ-pseudocompact, $f(H)$ is compact. Being $f(H)$ also dense in $f(G)$, $f(G) = f(H)$ is compact.

3 The divisible weight

3.1 General properties of w-divisible groups

For a topological abelian group G let $\Lambda(G)$ be the family of all closed G_δ-subgroups of G.

In [13, Definition 1.3] a topological group G was defined to be *almost connected* whenever $c(G) \in \Lambda(G)$. Almost connected pseudocompact groups are an example of w-divisible groups.

The proof of the following lemma is obvious.

Lemma 3.1. *Let $G = \prod_{i\in I} G_i$ where each G_i is a topological group. If G_i is w-divisible for every $i \in I$, then G is w-divisible.*

The quotient of a w-divisible group need not be w-divisible (e.g., for $p \in \mathbb{P}$ take $\mathbb{Z}_p^{\mathfrak{c}}$, which has $\mathbb{Z}(p)^{\mathfrak{c}}$ as a quotient). Nevertheless, we can study the behavior of the divisible weight in this respect. Obviously it is monotone under taking subgroups. The next claim shows that it is monotone also under taking continuous surjective homomorphisms of precompact groups.

Claim 3.2. *Let G and L be precompact abelian groups such there exists a continuous surjective homomorphism $f : G \to L$. Then $w_d(G) \geq w_d(L)$.*

Proof. For every $m \in \mathbb{N}_+$ there exists a continuous surjective homomorphism of $m!G$ onto $m!L$ and so $w(m!G) \geq w(m!L)$. □

Since the minimal positive integer m_0 in the following lemma is uniquely determined by the group G, we have denoted it by $m_d(G)$ in the introduction.

Lemma 3.3. *Let G be a non-singular abelian group. Then there exists a minimal positive integer $m_0 \in \mathbb{N}_+$ such that if $H = m_0!G$, then:*

(a) $w_d(H) = w(H) = w_d(G)$, in particular H is w-divisible;

(b) $r_0(H) = r_0(G)$;

(c) if G is pseudocompact, then H is pseudocompact too.

If G is compact, then H has also the following property:

(d) $G \in TD_\lambda$ ($G \in TD^\lambda$) for some infinite cardinal λ if and only if $H \in TD_\lambda$ ($H \in TD^\lambda$).

Proof. Since
$$G \geq n!G \geq (n+1)!G \text{ for every } n \in \mathbb{N}_+,$$
the sequence of weights is decreasing too:

$$w(G) \geq w(n!G) \geq w((n+1)!G) \text{ for every } n \in \mathbb{N}_+.$$

So it stabilizes in view of the property of cardinal numbers. This means that there exists $m_0 \in \mathbb{N}_+$ such that $w_d(G) = w(m_0!G)$ and $w(m_0!G) \leq w(kG)$ for every $k \in \mathbb{N}_+$, because $w(m_0!G) \leq w(k!G) \leq w(kG)$. Let $H = m_0!G$. Then $w(H) = w_d(G)$. Moreover H is w-divisible, because $w(nH) = w(nm_0!G) = w(m_0!G) = w(H)$ for every $n \in \mathbb{N}_+$ and so $w_d(H) = w(H) > \omega$. Obviously $r_0(H) = r_0(G)$ and if G is pseudocompact, then H is pseudocompact as well being a continuous image of G. □

Note that in this lemma we ask the group G to be non-singular, otherwise H would be metrizable and hence not w-divisible.

Fact 3.4. [10, 17] *If G is a pseudocompact non-torsion abelian group, then $r_0(G) \geq \mathfrak{c}$.*

Remark 3.5. Let G be a topological abelian group and let $K = \widetilde{G}$ be the completion of G. Observe that $w(mG) = w(mK)$ for every $m \in \mathbb{N}$, because the homomorphism $K \to mK$, defined by the multiplication by m, is continuous and so mG is dense in mK for every $m \in \mathbb{N}$. Then G is w-divisible if and only if K is w-divisible. More precisely the sequences $\{w(m!G) : m \in \mathbb{N}_+\}$ and $\{w(m!K) : m \in \mathbb{N}_+\}$ stabilize at the same point, that is $m_d(G) = m_d(K)$. In particular $w_d(G) = w_d(K)$.

Claim 3.6. *If $n \in \mathbb{N}_+$, $G_1, \ldots, G_n$ are topological abelian groups and $G = G_1 \times \ldots \times G_n$, then*

$$w_d(G) = \max\{w_d(G_1), \ldots, w_d(G_n)\}.$$

In particular G is w-divisible if and only if G_i is w-divisible for all $i = 1, \ldots, n$ and $w(G) = w(G_i)$.

Remark 3.7. The counterpart of Claim 3.6 for $m_d(-)$ fails to be true. To see this, let $G_1 = \mathbb{Z}(2)^{\mathfrak{c}^+} \times \mathbb{T}^{\mathfrak{c}}$ and $G_2 = \mathbb{Z}(3)^{\mathfrak{c}} \times \mathbb{T}$. Then $2 = m_d(G_1) = m_d(G_1 \times G_2) < \max\{m_d(G_1), m_d(G_2)\} = m_d(G_2) = 3$.

It is easy to see that $m_d(G) \geq \min\{m_d(G_1), \ldots, m_d(G_n)\}$, where $n \in \mathbb{N}_+$, $G_1, \ldots, G_n$ are topological abelian groups and $G = G_1 \times \ldots \times G_n$.

Claim 3.6 works with a finite number of groups, but it fails to be true in general taking infinitely many groups:

Remark 3.8. Consider the group $K = \prod_{p \in \mathbb{P}} \mathbb{Z}(p)^{\omega_1}$ and observe that $K_p = \mathbb{Z}(p)^{\omega_1}$ for each $p \in \mathbb{P}$. Then $w_d(K) = w(K)$, although $w_d(K_p) = 1$ for every $p \in \mathbb{P}$.

3.2 w-Divisible compact abelian groups

It is important to observe that for a compact abelian group K

$$w_d(K) \geq w(c(K)),$$

because $c(K)$ is divisible, being compact and connected, and so $c(K) \leq m!K$ for every $m \in \mathbb{N}_+$.

Here is a corollary of Lemma 3.3 and Theorem 1.10. It will not be used further in our proofs, but we give it in order to emphasize the analogy between w-divisible and connected/divisible compact groups, since $r_0(K) = |K| = 2^{w(K)}$ for every divisible compact abelian group K [15].

Corollary 3.9. *If K is a non-singular compact abelian group, then $r_0(K) = 2^{w_d(K)}$. In particular $r_0(K) = 2^{w(K)}$, whenever K is w-divisible.*

Proof. Let $\sigma = w_d(K)$ and $H = m_d(K)!K$. By Lemma 3.3 $\sigma = w(H)$ and H is w-divisible. By Theorem 1.10 there exists a continuous surjective homomorphism $H \to \prod_{i \in I} K_i$, where each K_i is a metrizable compact non-torsion abelian group and $|I| = \sigma$. By Fact 3.4 and since $|K_i| = \mathfrak{c}$, $r_0(K_i) = \mathfrak{c}$. Consequently $r_0(\prod_{i \in I} K_i) = 2^\sigma$ and $r_0(H) \geq 2^\sigma$. But $|H| = 2^\sigma$ and so $r_0(H) = 2^\sigma$. Hence $r_0(K) = r_0(H) = 2^\sigma$. □

Observe that, for a $p \in \mathbb{P}$, compact $\mathbb{Z}_p$-modules are precisely the abelian pro-p groups.

Lemma 3.10. *Let $p \in \mathbb{P}$. If K is a compact $\mathbb{Z}_p$-module, then*

$$w_d(K) = \inf\{w(p^n K) : n \in \mathbb{N}\}.$$

Indeed, if $m = p^k m_1$, where m_1 is coprime to p, then $mK = p^k K$.

The following fact can be obtained by a standard application of the Pontryagin duality.

Fact 3.11. *For a topological abelian group K which is either compact or discrete, and for $m \in \mathbb{N}_+$, $\widehat{mK} \cong m\widehat{K}$.*

Remark 3.12. Let $p \in \mathbb{P}$. In [30] (see [22, §35]) the *final rank* of an abelian p-group X was defined as

$$\mathrm{fin}r(X) = \inf_{n\in\mathbb{N}} r_p(p^n X).$$

Note that $\mathrm{fin}r(X) = w_d(X)$ in case $r_p(p^n X)$ is infinite for every $n \in \mathbb{N}_+$.

If K is a compact $\mathbb{Z}_p$-module with $w_d(K) \geq \omega$ and $X = \widehat{K}$, then

$$\mathrm{fin}r(X) = w_d(X) = w_d(K).$$

Since by Fact 3.11 $\widehat{p^nK} \cong p^n X$ for every $n \in \mathbb{N}$, $w(p^n K) = |p^n X| = w(p^n X)$ for every $n \in \mathbb{N}$. Hence $w_d(K) = w_d(X)$. Being $w_d(K) \geq \omega$, $|p^n X|$ is infinite for each $n \in \mathbb{N}_+$ and by the previous part of the remark $\mathrm{fin}r(X) = w_d(X)$.

Since the divisible weight coincides with the final rank for discrete abelian p-groups ($p \in \mathbb{P}$), it can be viewed as a natural generalization of the final rank to all abelian (topological) groups.

The equality $w_d(K) = w_d(X)$ of Remark 3.12 can be proved in general for a compact abelian group:

Theorem 3.13. *Let K be a topological abelian group which is either compact or discrete. Then $w_d(K) = w_d(\widehat{K})$.*

Proof. By Fact 3.11 $\widehat{m!K} \cong m!\widehat{K}$, for every $m \in \mathbb{N}$. Then

$$w(m!K) = w(\widehat{m!K}) = w(m!\widehat{K})$$

for every $m \in \mathbb{N}$. This shows that $w_d(K) = w_d(\widehat{K})$. □

Corollary 3.14. *Let K be a compact abelian group. Then K is w-divisible if and only if $\widehat{K}$ is w-divisible (in other words $|m\widehat{K}| = |\widehat{K}| > \omega$ for every $m \in \mathbb{N}_+$).*

Proof. As $w(K) = w(\widehat{K})$, the above theorem applies. Moreover $w(m\widehat{K}) = |m\widehat{K}|$ for every $m \in \mathbb{N}$. □

Lemma 3.15. *Let K be a compact abelian group. Then $w_d(K) = \max\{w(c(K)), w_d(K/c(K))\}$.*

Proof. The condition $w_d(K) > w(c(K))$ is equivalent to $w(m!K) > w(c(K))$ for every $m \in \mathbb{N}_+$. Since $c(K)$ is connected and compact, $c(K)$ is divisible and so $c(K) \leq m!K$ for every $m \in \mathbb{N}_+$. Then $w(m!K) = w(m!K/c(K)) \cdot w(c(K))$ and it follows that $w(m!K) = w((m!K)/c(K))$ for every $m \in \mathbb{N}_+$. Now note that $(m!K)/c(K)$ is isomorphic to $m!(K/c(K))$. Since $w_d(K) = \inf\{w(m!K) : m \in \mathbb{N}_+\}$, this yields the equalities

$$w_d(K) = \inf\{w((m!K)/c(K)) : m \in \mathbb{N}_+\} = \inf\{w(m!(K/c(K))) : m \in \mathbb{N}_+\} = w_d(K/c(K)),$$

which complete the proof. □

This lemma splits the study of $w_d(K)$ in two cases. The connected case is trivial as connected groups are already w-divisible. The more complicated totally disconnected case will be analyzed in the next section.

4 w-Divisibility of profinite abelian groups

For a totally disconnected compact abelian group $K = \prod_{p\in\mathbb{P}} K_p$ (see Remark 1.3(a)) the equality $w_d(K) = \sup_{p\in\mathbb{P}} w_d(K_p)$ does not hold in general by Remark 3.8.

4.1 The stable weight

Let K be a totally disconnected compact abelian group. Then $K = \prod_{p\in\mathbb{P}} K_p$ by Remark 1.3(a). Let $\alpha_p = w(K_p)$ for each $p \in \mathbb{P}$. Define the *stable weight* of K as $w_s(K) = \omega$, when $\mathbb{P}\setminus P_m(K)$ is finite, otherwise let

$$w_s(K) = \inf_{n\in\mathbb{N}} w\left(\prod_{p\in\mathbb{P},p>n} K_p\right).$$

Then $w_s(K) = \inf_{n\in\mathbb{N}} \sup_{p\in\mathbb{P},p>n} \alpha_p > \omega$. Since $\sup_{p\in\mathbb{P},p>n} \alpha_p$ is a decreasing sequence of cardinals, it stabilizes and so there exists $n_0 \in \mathbb{N}$ such that

$$w_s(K) = w\left(\prod_{p\in\mathbb{P},p>n_0} K_p\right) = \sup_{p\in\mathbb{P},p>n_0} \alpha_p.$$

Definition 4.1. *A totally disconnected compact abelian group is* stable *if* $w(K) = w_s(K) > \omega$.

Lemma 4.2. *If K is a totally disconnected compact abelian group, then $w_s(K) \le w_d(K)$. In particular, K stable implies K w-divisible.*

Proof. As noted in the foregoing part of this section, there exists $n_0 \in \mathbb{N}$ such that $w_s(K) = \sup_{p\in\mathbb{P},p>n_0} w(K_p)$. By Lemma 3.3 $w_d(K) = w(H)$, where $H = m_d(K)!$. Moreover take a $n_1 \in \mathbb{N}$ such that $n_1 \ge \max\{n_0, m_d(K)\}$. Consequently $\prod_{p\in\mathbb{P},p>n_1} K_p \le H$ and so $w_s(K) = w(\prod_{p\in\mathbb{P},p>n_1} K_p) \le w(H) = w_d(K)$. □

Note that the inequality of this lemma can be strict only in case $w_d(K) = w_d(K_p)$ for some $p \in \mathbb{P}, p < n_0$: if $w_d(K) > w_s(K)$, then $w_d(K) = w(m_d(K)!K) = w(m_d(K)!\prod_{p\in\mathbb{P},p<n_0} K_p)$ by Claim 3.6; since this is a finite product, $w_d(K) = w_d(K_p)$ for some prime $p \le n_0$ again by Claim 3.6.

Let K be a totally disconnected compact abelian group. Clearly, $w_s(K) = \omega$ when K is metrizable or $\mathbb{P} \setminus P_m(K)$ is finite. For a better understanding of $w_s(K)$ in the remaining case assume that $\mathbb{P} \setminus P_m(K)$ is infinite and define the *d-spectrum* of K as

$$\Pi(K) = \{p \in \mathbb{P} \setminus P_m(K) : \alpha_p \le w_s(K)\} = \{p \in \mathbb{P} : \omega < \alpha_p \le w_s(K)\}.$$

Then the complement of $\Pi(K)$ in $\mathbb{P} \setminus P_m(K)$ is the finite set

$$\pi_f(K) = \{p \in \mathbb{P} \setminus P_m(K) : \alpha_p > w_s(K)\},$$

so $\Pi(K)$ is infinite. Moreover we have the following partition:

$$\Pi(K) = \pi^*(K) \cup \pi(K),$$

where

$$\pi^*(K) = \{p \in \mathbb{P} \setminus P_m(K) : \alpha_p < w_s(K)\} \quad \text{and} \quad \pi(K) = \{p \in \mathbb{P} \setminus P_m(K) : \alpha_p = w_s(K)\}.$$

So we have the partition $\mathbb{P} = P_m(K) \cup \pi^*(K) \cup \pi(K) \cup \pi_f(K)$ and

$$K = \prod_{p\in P_m(K)} K_p \times \prod_{p\in\pi^*(K)} K_p \times \prod_{p\in\pi(K)} K_p \times \prod_{p\in\pi_f(K)} K_p.$$

Note that $\pi(K)$ is infinite whenever $\pi(K) \ne \emptyset$. Let

$$met(K) = \prod_{p\in P_m(K)} K_p, \quad sc(K) = \prod_{p\in\Pi(K)} K_p \quad \text{and} \quad nst(K) = \prod_{p\in\pi_f(K)} K_p.$$

Then

$$K = met(K) \times sc(K) \times nst(K),$$

$met(K)$ is metrizable, while $nst(K) = \prod_{p\in\pi_f(K)} K_p$ has no stable subgroups, because $nst(K)$ is a finite product and since every closed subgroup N of $nst(K)$ is of the form $N = \prod_{p\in\pi_f(K)} N_p$ by Remark 1.3(a). We shall see below that the subgroup $sc(K)$ (we shall refer to it as *stable core* of K) is stable when $\Pi(K) \ne \emptyset$.

For the sake of completeness, set

$$\Pi(K) = \emptyset \text{ and } \pi_f(K) = \mathbb{P} \setminus P_m(K), \quad \text{when } |\mathbb{P} \setminus P_m(K)| < \infty.$$

Claim 4.3. *A totally disconnected compact abelian group K is stable if and only if $\mathbb{P} \setminus P_m(K)$ is infinite and $\pi_f(K) = \emptyset$.*

Proof. (a)⇒(b) Suppose that $p \in \pi_f(K)$. Then $w_s(K) < \alpha_p \leq w(K)$ and so K is not stable.

(b)⇒(a) Since $\mathbb{P} \setminus P_m(K)$ is infinite and $\pi_f(K)$ is empty, $w_s(K) = \sup_{p\in\mathbb{P}\setminus P_m(K)} \alpha_p = w(K)$, i.e., K is stable. □

Lemma 4.4. *Let K be a non-singular totally disconnected compact abelian group. Then:*

(a) if $\mathbb{P} \setminus P_m(K)$ is infinite, then $sc(K)$ is stable, so $w_s(sc(K)) = w(sc(K)) = w_s(K)$;

(b) either $w_d(K) = w_d(K_p)$ for some $p \in \mathbb{P}$ or $w_d(K) = w_s(K) = \sup_{p\in\Pi(K)} \alpha_p$.

Proof. By Remark 1.3(a) $K = \prod_{p\in\mathbb{P}} K_p$.

(a) Since $\mathbb{P} \setminus P_m(K)$ is infinite and $\pi_f(sc(K)) = \emptyset$, Claim 4.3 applies.

(b) Suppose that $w_d(K) > w_d(K_p)$ for every $p \in \mathbb{P}$. Then $\mathbb{P} \setminus P_m(K)$ is infinite and so also $\Pi(K)$ is infinite.

Let $D = sc(K)$. By (a) D is stable and

$$w_s(D) = w_d(D) = w(D) = \sup_{p\in\Pi(K)} \alpha_p = w_s(K).$$

Since

$$w_d(K) = \max\{w_d\,(nst(K_p)), w_d(D)\}$$

by Claim 3.6 and $w_d(K) > w_d(K_p)$ for all $p \in \mathbb{P}$ by our hypothesis, it follows that

$$w_d(K) = w_d(D).$$

Therefore $w_d(K) = w_d(D) = w(D) = w_s(K)$. □

We shall see in the next subsection that when $w_d(K) > w_d(nst(K))$, the stable core $sc(K)$ will play the essential role as far as projections on products are concerned.

4.2 Projection onto products

Remark 4.5. Let $p \in \mathbb{P}$ and let K be a compact $\mathbb{Z}_p$-module. If $X = \widehat{K}$, according to [22, Theorem 32.3] X has a basic subgroup B_0; in other words there exist cardinals $\alpha_n, n \in \mathbb{N}_+$, such that:

$B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$,
B_0 is pure (i.e., $B_0 \cap p^n X = p^n B_0$ for every $n \in \mathbb{N}$) and
X/B_0 is divisible;

so $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ, because X/B_0 is a divisible abelian p-group [22, Theorem 23.1].

Let $m \in \mathbb{N}_+$, and let

$$B_{1,m} = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(\alpha_n)} \quad \text{and} \quad B_{2,m} = \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}.$$

Then we prove that

$$X = X_{1,m} \oplus B_{1,m},$$

where $X_{1,m} = p^m X + B_{2,m}$. Indeed, $X = p^m X + B_0$ because X/B_0 is divisible. Moreover, this is a direct sum as $X_{1,m} \cap B_{1,m} = \{0\}$; in fact, if $z \in X_{1,m} \cap B_{1,m}$, then $z = b \in B_{1,m}$ and $z = x + b'$, where $x \in X_{1,m}$, $b' \in B_{2,m}$. It follows that $x = b - b' \in B_0 \cap p^m X$. By the purity of B_0, we have $B_0 \cap p^m X = p^m B_0 \subseteq B_2$ and this yields $b = 0$. Moreover, observe that $X/B_0 \cong X_{1_m}/B_{2,m}$.

Claim 4.6. [12, Claim 4.7] *Let $p \in \mathbb{P}$ and let K be a compact $\mathbb{Z}_p$-module and N a closed subgroup of K isomorphic to $\mathbb{Z}_p^\sigma$, for some cardinal $\sigma \geq \omega$. Then there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^\sigma$.*

Lemma 4.7. *Let $p \in \mathbb{P}$ and let K be a compact $\mathbb{Z}_p$-module. Then there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^{w_d(K)}$.*

Proof. Let us reduce first the lemma to the case when K is a w-divisible group. By Lemma 3.3 the subgroup $H = m_d(K)!K$ of K is w-divisible and $w(H) = w_d(K)$. Moreover, there exists a continuous surjective homomorphism $h : K \to H$, namely the one defined by $h(x) = m_d(K)!x$ for every $x \in K$. Clearly every continuous surjective homomorphism of H onto $\mathbb{G}_p^{w_d(K)}$ composed with h gives rise to a continuous surjective homomorphism of K onto $\mathbb{G}_p^{w_d(K)}$. This is why we suppose from now on that K itself if w-divisible.

Let X be the dual group of K. Observe that $|X| = w(K) > \omega$. By Remark 4.5 there exists a subgroup B_0 of X such that $B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$ and $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma')}$; put $|X/B_0| = \sigma$ and note that $\sigma = \sigma'$ in case $\sigma > \omega$. As in Remark 4.5, for every $m \in \mathbb{N}_+$ let $B_{1,m} = \bigoplus_{n=1}^{m} \mathbb{Z}(p^n)^{(\alpha_n)}$ and $B_{2,m} = \bigoplus_{n=m+1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$. Then $X = X_{1,m} \oplus B_{1,m}$, where $X_{1,m} = p^m X + B_{2,m}$ and $X_{1,m}/B_{2,m} \cong X/B_0$.

By Corollary 3.14 we have $|X| = |mX|$ for every $m \in \mathbb{N}_+$. Moreover $|X| = |mX_{1,n}|$ for every $m, n \in \mathbb{N}_+$; indeed $|mX_{1,n}| = |mp^n X + mB_{2,n}| = |X|$ by Corollary 3.14 and our hypothesis on K. Consider now the sequence of cardinals $\beta_n = \sup\{\alpha_m : m \geq n\}$. There exists $n_0 \in \mathbb{N}_+$ such that $\beta_n = \beta_{n_0} =: \beta$ for every $n \geq n_0$. Thus $|X| = \sigma \cdot \beta$, in fact $|X| = |X_{1,n_0}| = |X_{1,n_0}/B_{2,n_0}| \cdot |B_{2,n_0}| = \sigma \cdot \beta$.

If $|X| = \sigma$, then $\sigma > \omega$ because $|X| = w(K) > \omega$. So in this case $\sigma = \sigma'$ and $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$. By the Pontryagin duality the group K has a subgroup isomorphic to $\mathbb{Z}_p^\sigma$. Claim 4.6 applies to conclude that there exists a continuous surjective homomorphism of K onto $\mathbb{G}_p^\sigma$.

If $|X| > \sigma$, then $|X| = |B_{2,n_0}| = \beta > \omega$. We shall prove that

$$B_{2,n_0} \geq \bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)^{(\beta)}. \tag{†}$$

Let $A = \{n \in \mathbb{N} : n \geq n_0, \alpha_n = \beta\}$. If A is infinite, then $\bigoplus_{n\in A} \mathbb{Z}(p^n)^{(\alpha_n)} = \bigoplus_{n\in A} \mathbb{Z}(p^n)^{(\beta)}$ contains a subgroup isomorphic to $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}(p^n)^{(\beta)}$, so (†) holds true. Assume that A is finite. Then, taking an appropriate $n_1 \geq n_0$ we can assume that $A = \emptyset$, i.e., $\beta > \alpha_n$ for every $n \geq n_1$. Note that still $\beta = \sup\{\alpha_m : m \geq n_1\}$ holds true, as X is w-divisible by Corollary 3.14 and $|X| = |B_{2,n_0}|$ by our hypothesis $|X| > \sigma$. There exists an infinite subset I of ω such that $\{\alpha_n : n \in I\}$ is strictly increasing with $\beta = \sup\{\alpha_m : m \in I\}$ and $n \geq n_1$ for all $n \in I$. Clearly $\beta = \sup\{\alpha_m : m \in I'\}$ holds true also for every infinite subset I' of I. We can write $I = \bigcup_{n\in\mathbb{N}} I_n$, where each I_n is infinite and $I_n \cap I_m = \emptyset$ for every $n \neq m$. Clearly,

$$\bigoplus_{m\in I_n} \mathbb{Z}(p^m)^{(\alpha_m)} \geq \bigoplus_{m\in I_n, m\geq n} \mathbb{Z}(p^n)^{(\alpha_m)} \cong \mathbb{Z}(p^n)^{(\beta)}$$

for every $n \in \mathbb{N}$. Since $B_{2,n_0} \geq \bigoplus_{n\geq n_1} \mathbb{Z}(p^n)^{(\alpha_n)}$, we see that B_{2,n_0} contains also $\bigoplus_{n\in I} \bigoplus_{m\in I_n} \mathbb{Z}(p^m)^{(\alpha_m)}$ and the latter group contains a subgroup isomorphic to $\bigoplus_{n\geq n_1} \mathbb{Z}(p^n)^{(\beta)}$. Thus (†) holds also in this case. By the Pontryagin duality there exists a continuous surjective homomorphism of K onto the dual of the latter set, that is $\mathbb{G}_p^\beta$. □

Remark 4.8. Let $p \in \mathbb{P}$. The following more precise property of a non-singular compact $\mathbb{Z}_p$-module K can be proved: $K \cong K_1 \times B$, where B is a bounded torsion compact group and K_1 is a w-divisible compact $\mathbb{Z}_p$-module with $w(K_1) = w_d(K)$.

The next claim is the totally disconnected case of Theorem 1.10.

Claim 4.9. *If K is a totally disconnected compact abelian group, then there exists a continuous surjective homomorphism of K onto a product C of metrizable compact non-torsion abelian groups such that $w(C) = w_d(K)$.*

Proof. As K is totally disconnected, we can write $K = \prod_{p\in\mathbb{P}} K_p$ by Remark 1.3(a). Let $\alpha_p = w(K_p)$ for every $p \in \mathbb{P}$. Note that $\mathbb{P} \setminus P_m(K) \neq \emptyset$ as K is non-metrizable.

If $\mathbb{P} \setminus P_m(K)$ is finite, then by Claim 3.6 $w_d(K) = \max\{w_d(K_p) : p \in \mathbb{P} \setminus P_m(K)\}$. So there exists $p \in \mathbb{P} \setminus P_m(K)$ such that $w_d(K) = w_d(K_p)$. We can apply Lemma 4.7 to the $\mathbb{Z}_p$-module K_p to find a continuous surjective homomorphism from K_p to $S^{w_d(K_p)} = \mathbb{G}_p{}^{w_d(K)}$ (recall that the compact group $\mathbb{G}_p$ is metrizable and non-torsion). Now take the composition of this homomorphism with the canonical projection $K \to K_p$. We can argue in the same way when $w_d(K) = w_d(K_p)$ for some prime p. Therefore from now on we assume that $w_d(K) > w_d(K_p)$ for all primes p and this implies that $\mathbb{P} \setminus P_m(K)$ is infinite. By Lemma 4.4 we have $w_d(K) = w_s(K)$.

For $p \in \mathbb{P} \setminus P_m(K)$ consider the quotient K_p/pK_p. If $X = \widehat{K_p}$, by the Pontryagin duality we know that $X[p]$ is topologically isomorphic to $\widehat{K_p/pK_p}$. Moreover, since $|X|$ is not countable, $|X| = |X[p]| = \alpha_p$. Thus K_p/pK_p is topologically isomorphic to $\mathbb{Z}(p)^{\alpha_p}$. Consequently there exists a continuous surjective homomorphism of K_p onto $\mathbb{Z}(p)^{\alpha_p}$.

Since $\mathbb{P} \setminus P_m(K)$ is infinite, $\Pi(K)$ is not empty and so infinite as well. We have two cases. If $\pi(K)$ is infinite, then there exists a continuous surjective homomorphism

$$K \to \prod_{p\in\pi(K)} \mathbb{Z}(p)^{w_d(K)} = S_{\pi(K)}^{w_d(K)},$$

as $\alpha_p = w_s(K) = w_d(K)$ for all $p \in \pi(K)$. Otherwise $\pi(K)$ is empty and $\Pi(K) = \pi^*(K)$. By Lemma 4.4 $w_s(K) = \sup_{p\in\Pi(K)} \alpha_p$. Moreover $w_d(K) = w_s(K) > \alpha_p$ for all $p \in \Pi(K)$. Order the set $\{\alpha_p : p \in \Pi(K)\}$ so that $\alpha_{p_1} < \alpha_{p_2} < \dots < \alpha_{p_n} < \dots$ and note that the inclusion $\{p_n : n \in \mathbb{N}_+\} \subseteq \Pi(K)$ could be proper. Nevertheless $w_s(K) = \sup_{n\in\mathbb{N}_+} \alpha_{p_n}$. Let $C = \prod_{n=1}^{\infty} \mathbb{Z}(p_n)^{\alpha_{p_n}}$. Then $w(C) = w_s(K) = w_d(K)$ and

$$C = \prod_{n=1}^{\infty} \mathbb{Z}(p_n)^{\alpha_{p_n}} = \prod_{n=1}^{\infty}\prod_{i=1}^{n} Z(p_n)^{\alpha_{p_i}} = \prod_{i=1}^{\infty}\prod_{n=i}^{\infty} \mathbb{Z}(p_n)^{\alpha_{p_i}} = \prod_{i=1}^{\infty} S_{\pi_i}^{\alpha_{p_i}},$$

where $\pi_i = \{p_n : n \in \mathbb{N}_+, n \geq i\}$. To end up the proof note that S_{π_i} is a metrizable compact non-torsion abelian group for every $i \in \mathbb{N}_+$, since π_i is infinite. □

The following claim, which is used to prove Theorem 1.9, analyzes when it is possible to project a totally disconnected compact abelian group onto a power of a metrizable compact non-torsion abelian group.

Claim 4.10. *Let K be a totally disconnected compact abelian group, $\alpha_p = w(K_p)$ for each $p \in \mathbb{P}$ and I an uncountable set of indexes.*

(a) If $\pi \subseteq \mathbb{P}$ and $|I| \leq \alpha_p$ for all $p \in \pi$, then there exists a continuous surjective homomorphism $f : K \to S_\pi^I$.

(b) If $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$ and there exists a continuous surjective homomorphism $f : K \to S^I$, where S is a metrizable compact non-torsion abelian group, then $|I| \leq \alpha_p$ for all $p \in \pi$ for some infinite $\pi \subseteq \mathbb{P}$.

Proof. (a) For every $p \in \pi$ the inequality $|I| \leq \alpha_p$ yields that α_p is uncountable, so $w(K_p/pK_p) = \alpha_p$. Hence K_p/pK_p is isomorphic to $\mathbb{Z}(p)^{\alpha_p}$. Moreover, there exists a continuous surjective homomorphism $\mathbb{Z}(p)^{\alpha_p} \to \mathbb{Z}(p)^I$, since $|I| \leq \alpha_p$. Therefore there exists a continuous surjective homomorphism

$$K = \prod_{p\in\mathbb{P}} K_p \to \prod_{p\in\pi} \mathbb{Z}(p)^I \cong \left(\prod_{p\in\pi} \mathbb{Z}(p)\right)^I = S_\pi^I.$$

(b) Suppose that $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$ and that there exists a continuous surjective homomorphism $f : K \to S^I$, where S is a metrizable compact non-torsion abelian group. Then S is totally disconnected and compact, so $S = \prod_{p\in\mathbb{P}} S_p$ by Remark 1.3(a). Since $w_d(K_p) < |I|$ for all $p \in \mathbb{P}$, it follows that S_p is torsion for all $p \in \mathbb{P}$. Indeed, if S_p were not torsion, then the surjective homomorphism $f_p = f \restriction_{K_p} : K_p \to S_p^I$, together with Claim 3.2, would imply that $w_d(K_p) \geq w_d(S_p^I) = |I|$. Hence S_p is a bounded p-torsion group for every $p \in \mathbb{P}$, being p-torsion and compact. Since S is non-torsion, S_p has to be non-trivial for infinitely many $p \in \mathbb{P}$, and so $r_p(S) = r_p(S_p) > 0$ for infinitely many $p \in \mathbb{P}$. Let $p \in \mathbb{P}$ be such that $r_p(S_p) > 0$. By the Pontryagin duality there exists a continuous injective homomorphism $\bigoplus_I \widehat{S_p} \to \widehat{K_p}$. Since S_p is a bounded torsion abelian p-group, $\widehat{S_p}$ is a bounded torsion abelian p-group as well. Therefore $r_p(\widehat{S_p}) > 0$ and it follows that $r_p(\widehat{K_p}) \geq |I|$. Hence $\alpha_p = w(K_p) \geq |I|$. □

5 Proofs of the main results

Proof of Theorem 1.9. Assume that K admits a continuous surjective homomorphism onto $S^{w_d(K)}$ for some metrizable compact non-torsion abelian group S. Such a group S admits as a quotient one of the following four types of groups: either $\mathbb{T}$, or $\mathbb{G}_p$ or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$ or S_π for some infinite $\pi \subseteq \mathbb{P}$. To see this let $X = \widehat{S}$. Then X is a countable discrete abelian group. If X contains an isomorphic copy of $\mathbb{Z}$, then S admits $\mathbb{T}$ as a quotient. Otherwise X is torsion. If $\pi = \{p \in \mathbb{P} : r_p(X) > 0\}$ is infinite, since X has a subgroup isomorphic to $\bigoplus_{p\in\pi} \mathbb{Z}(p)$, it follows that S admits S_π as a quotient. If π is finite, there exists $p \in \mathbb{P}$ such that the subgroup of all p-torsion elements of X is infinite. So we can assume without loss of generality that X is an infinite p-group. By Remark 4.5 there exists a basic subgroup $B_0 \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)^{(\alpha_n)}$ of X, for some cardinals $\alpha_n \leq \omega$, such that $X/B_0 \cong \mathbb{Z}(p^\infty)^{(\sigma)}$ for some cardinal σ. If there exists a sequence $\{n_k\}_k$ of positive integers such that $n_k \to \infty$ and $\alpha_{n_k} > 0$, then B_0 contains a subgroup isomorphic to $\bigoplus_{k=1}^{\infty} \mathbb{Z}(p^{n_k})$. Since the latter group obviously contains a copy of the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}(p^n)$, we conclude that in this case

X has a subgroup isomorphic to $\bigoplus_{n=1}^{\infty}\mathbb{Z}(p^n)$, hence K admits $\mathbb{G}_p$ as a quotient. If this subsequence does not exist, B_0 is bounded torsion, that is, there exists $n\in\mathbb{N}_+$ such that $p^nB_0=\{0\}$. By Remark 4.5 $X\cong p^nX\oplus B_0$, where $X/B_0\cong\mathbb{Z}(p^\infty)^{(\sigma)}$. Note that $\sigma>0$, otherwise X would be bounded torsion and consequently S would be torsion. So X contains a copy of the group $\mathbb{Z}(p^\infty)$, therefore S admits $\mathbb{Z}_p$ as a quotient.

Depending on which of these four cases occurs we have either (a) or (b) or (c).

(a) Assume that there exists a continuous surjective homomorphism $f:K\to\mathbb{T}^{w_d(K)}$. Then the restriction of f to the connected component $c(K)$ gives rise to a surjective continuous homomorphism $f\restriction_{c(K)}: c(K)\to\mathbb{T}^{w_d(K)}$. This yields $w(c(K))\geq w_d(K)$, while the inequality $w_d(K)\geq w(c(K))$ is always available, as $c(K)$ is w-divisible being divisible. This proves the equality $w_d(K)=w(c(K))$. On the other hand, if $w_d(K)=w(c(K))$ holds true, then there exists a surjective continuous homomorphism $K\to\mathbb{T}^{w_d(K)}$. This is a folklore fact of the Pontryagin duality. In fact, if $X=\widehat{K}$, X has $X/t(X)$ as a quotient and $X/t(X)\cong\widehat{c(K)}$. Consequently $X/t(X)$ is a torsion-free group of cardinality $w(c(K))$ and so there exists an injective homomorphism $\mathbb{Z}^{(w(c(K)))}\to X/t(X)$. Therefore there exists an injective homomorphism $\mathbb{Z}^{(w(c(K)))}\to X$. By the Pontryagin duality there exists a continuous surjective homomorphism $K\to\mathbb{T}^{w(c(K))}=\mathbb{T}^{w_d(K)}$.

(b) Assume that there exits a continuous surjective homomorphism $f:K\to\mathbb{Z}_p^{w_d(K)}$. From $w_d(K)\geq\omega$, we conclude that
$\mathbb{Z}_p^{w_d(K)}$ admits $\mathbb{G}_p^{w_d(K)}$ as a quotient. So we can suppose that there exists a continuous surjective homomorphism $f:K\to\mathbb{G}_p^{w_d(K)}$ for some $p\in\mathbb{P}$. Since $\mathbb{G}_p^{w_d(K)}$ is totally disconnected, $f(c(K))=\{0\}$ and so f factorizes through the projection $q:K\to K/c(K)$. This produces a continuous surjective homomorphism $K/c(K)\to\mathbb{G}_p^{w_d(K)}$. By Remark 1.3(b) there exists a continuous surjective homomorphism $(K/c(K))_p\to\mathbb{G}_p^{w_d(K)}$, hence $w_d(K)\leq w_d((K/c(K))_p)$. The other inequality is always available by Lemma 3.2. This proves the equality $w_d(K)=w_d((K/c(K))_p)$. On the other hand, if $w_d(K)=w_d((K/c(K))_p)$ holds true, then there exists a surjective continuous homomorphism $(K/c(K))_p\to\mathbb{G}_p^{w_d(K)}$ by Lemma 4.7. It remains to compose with the projection $q:K\to(K/c(K))_p$.

(c) Assume that there exists a continuous surjective homomorphism $f:K\to S_\pi^{w_d(K)}$. Since for every $p\in\pi$ there exists a surjective continuous homomorphism $\phi_p:S_\pi\to S_\pi/pS_\pi\cong\mathbb{Z}(p)$, we obtain a surjective continuous homomorphism $f_p:K\to\mathbb{Z}(p)^{w_d(K)}$. Since f_p factorizes through the canonical projection $q:K\to(K/c(K))_p$ (as noted in the proof of item (b)) applying Remark 1.3(b), we get a surjective continuous homomorphism $l:(K/c(K))_p\to\mathbb{Z}(p)^{w_d(K)}$. This proves $w((K/c(K))_p)\geq w_d(K)$ for every $p\in\pi$. The converse inequality holds, because for every $p\in\pi$, $p>m_d(K)$ and so $m_d(K)!(K/c(K))_p=(K/c(K))_p$. Hence $w_p((K/c(K))_p)\leq w(m_d(K)!(K/c(K)))\leq w_d(K)$ for every $p\in\pi$.

Now assume that $w_d(K)=w((K/c(K))_p)$ for every prime $p\in\pi$. Since $w_d(K)\geq w_d(K/c(K))\geq w((K/c(K))_p)$ for every $p\in\pi$, we obtain $w_d(K/c(K))=w((K/c(K))_p)$ for every $p\in\pi$. Apply Claim 4.10 to the totally disconnected group $K/c(K)$ to find a continuous surjective homomorphism $g:K/c(K)\to S_\pi^{w_d(K)}$. Then take the composition of g with the canonical projection $K\to K/c(K)$.

To finish the proof we have to see that if K is a non-singular compact abelian group such that there exists no continuous surjective homomorphism $f:K\to S^{w_d(K)}$, where S is a metrizable compact non-torsion abelian group, then $\mathrm{cf}(w_d(K))=\omega$. By (a) $w_d(K)>w(c(K))$ and so $w_d(K)=w_d(K/c(K))$ by Lemma 3.15. By (b) and (c) $w_d(K)=w_d(K/c(K))>w_d((K/c(K))_p)$ for all $p\in\mathbb{P}$ and $w_d(K)>w((K/c(K))_p)$ for co-finitely many $p\in\mathbb{P}$. In view of Lemma 4.4 $w_d(K)=\sup_{p\in\Pi(K/c(K))}w((K/c(K))_p)$. This proves that $\mathrm{cf}(w_d(K))=\omega$. □

From [8, Lemma 6.1] and the results in [13, Section 2] we obtain the following lemma, needed in some of the following proofs.

Lemma 5.1. *Let G be a pseudocompact abelian group. Then:*

(a) $N\in\Lambda(G)$ if and only if G/N is metrizable;

(b) $w(N)=w(G)$ whenever $N\in\Lambda(G)$;

(c) every G_δ-set X of G such that $0\in X$ contains some $N\in\Lambda(G)$.

Proof of Theorem 1.8. (a)⇒(b) Suppose that N is a closed torsion G_λ-subgroup of K. Let us prove that $w(K/N)\leq\lambda$. Since each G_δ-set of K that contains 0 contains some $M\in\Lambda(K)$ by Lemma 5.1(c), $N\supseteq\bigcap_{i<\lambda}N_i$ where $\{N_i:i<\lambda\}\subseteq\Lambda(K)$. We can assume without loss of generality that $N=\bigcap_{i<\lambda}N_i$, because $w(K/N)\leq w(K/\bigcap_{i<\lambda}N_i)$, since $N\supseteq\bigcap_{i<\lambda}N_i$. There exists an embedding $K/N\to\prod_{i<\lambda}K/N_i$. By Lemma 5.1(a) K/N_i is metrizable for all $i<\lambda$. Therefore $w(K/N)\leq w(\prod_{i<\lambda}K/N_i)\leq\lambda$.

Let H be a proper totally dense λ-pseudocompact subgroup of K. The total density of H yields $N \leq H$ as N is torsion. In fact, a totally dense subgroup H of K contains all torsion elements of K: if $x \in t(K)$, then $\langle x \rangle$ is a finite (so closed) subgroup of K; since H is totally dense, $H \cap \langle x \rangle$ is dense in $\langle x \rangle$ and so $H \geq \langle x \rangle$. On the other hand, by the λ-pseudocompactness of H we deduce from $w(K/N) \leq \kappa$ that the image of H under the canonical projection $K \to K/N$ is compact. Since it must be also dense, we conclude that $K = N + H$. Now $N \leq H$ yields $K = H$, a contradiction.

(b)⇒(c) Assume $w_d(K) \leq \lambda$. Then there exists $m \in \mathbb{N}_+$ such that $w(mK) \leq \lambda$. So for the closed torsion subgroup $N = K[m]$ of K one has $K/N \cong mK$ and consequently $w(K/N) \leq \lambda$. So N is a closed torsion G_λ-subgroup of K: since K/N is compact of weight $\leq \lambda$, there exists a topological embedding $K/N \to \mathbb{T}^\lambda$. For each $i < \lambda$ let $p_i : \mathbb{T}^\lambda \to \mathbb{T}$ be the canonical projection and $\chi_i : K/N \to \mathbb{T}$ a continuous character. If $q : K \to K/N$ is the canonical homomorphism, then $\psi_i = \chi_i \circ q : K \to \mathbb{T}$ is a continuous character of K. Then $N_i = \ker \chi_i \in \Lambda(K/N)$ and $\bigcap_{i<\lambda} N_i = \{0\}$. Moreover $\ker \psi_i \in \Lambda(K)$ for all $i < \lambda$ and hence $N = \bigcap_{i<\lambda} \ker \psi_i$ is a closed torsion G_λ-subgroup of K. This contradicts (b).

(c)⇒(d) We have to consider two cases.

Case 1. There exists a continuous surjective homomorphism of K onto the power $S^{w_d(K)}$, where S is a metrizable compact non-torsion abelian group. To see that (d) is fulfilled it suffices to take I with $|I| = w_d(G) > \lambda$.

Case 2. Now assume that such a homomorphism is not available. By Theorem 1.9 this means $\mathrm{cf}(w_d(K)) = \omega$ and $w_d(K) > w(c(K))$. Then $w_d(K) = w_d(K/c(K))$ by Lemma 3.15. So assume without loss of generality that K is totally disconnected. According to Theorem 1.9 our hypothesis yields $w_d(K) > w_d(K_p)$ for all $p \in \mathbb{P}$ (otherwise there would exist a continuous surjective homomorphism of K onto the power $\mathbb{G}_p^I$).

Now pick any $\lambda < w_d(K)$ and a set I with $\lambda < |I| < w_d(K)$. We have to prove that there exists a continuous surjective homomorphism of K onto a power S^I of a metrizable compact non-torsion abelian group S. From $|I| < w_d(K)$ we deduce that $|I| \leq \alpha_p$ for *infinitely* many $p \in \mathbb{P}$. By Claim 4.10 there exists a continuous surjective homomorphism $f : K \to S^I$, where S is a metrizable compact non-torsion abelian group.

(d)⇒(e) Assume there exists a continuous surjective homomorphism of K onto a power S^I of a metrizable compact non-torsion abelian group S such that $|I| > \kappa$. By Proposition 2.6(b) S^I has the property TD_λ, hence also K has the property TD_λ thanks to Lemma 2.1.

(e)⇒(a) follows from Example 2.9.

Let $\kappa = w_d(K)$. Assume that it is regular. By Theorem 1.10 there exists a continuous surjective homomorphism of K onto the product of metrizable compact non-torsion abelian groups C such that $w(C) = \kappa$. By Theorem 2.7, C has the property TD^κ, hence also K has the property TD^κ thanks to Lemma 2.1. To end the proof assume $K \in TD^\kappa$. By Lemma 3.3 the subgroup $H = m_d(K)!K$ of K is w-divisible, $w(H) = \kappa$ and $H \in TD^\kappa$. Now apply again Theorem 2.7 to conclude that κ is regular. □

Proof of Theorem 1.10. Since K is non-singular, $\kappa = w_d(K) > \omega$. According to Lemma 3.15

$$w_d(K) = \max\{w(c(K)), w_d(K/c(K))\}.$$

If $w_d(K) = w(c(K))$, there exists a continuous surjective homomorphism of K onto $\mathbb{T}^\kappa = C$, which is a product of metrizable compact non-torsion abelian groups of weight κ. So it is possible to suppose that $w_d(K) > w(c(K))$ and then $w_d(K) = w_d(K/c(K))$. By Claim 4.9 applied to $K/c(K)$ there exists a continuous surjective homomorphism of $K/c(K)$ onto the product of metrizable compact non-torsion abelian groups $C = \prod_{i\in I} K_i$ such that $w(C) = \kappa$. This yields $|I| = \kappa$. It remains to take the composition $K \to K/c(K) \to C$.

To prove the opposite implication, suppose that there exists a continuous surjective homomorphism of K onto $C = \prod_{i\in I} K_i$, where each K_i is compact, metrizable and non-torsion, and $|I| \leq w_d(K)$. The group C is w-divisible by Proposition 2.6(a) and the divisible weight is monotone by Claim 3.2, so $w_d(K) \geq w_d(C) = w(C) = |I|$. □

5.1 Proof of Theorems 1.15 and 1.18

Before starting the proof of Theorems 1.15 and 1.18 we note that it is immediate to weaken the hypothesis of Theorem 1.14 from connected to almost connected to have:

If G is a non-trivial almost connected pseudocompact abelian group, then $Ps(r_0(G), w(G))$ holds.

In fact $c(G)$ is a non-trivial connected pseudocompact abelian group such that $w(c(G)) = w(G)$ by Lemma 5.1(b) and $r_0(c(G)) = r_0(G)$ because $r_0(G) = \max\{r_0(c(G)), r_0(G/c(G))\}$, where $r_0(c(G)) \geq \mathfrak{c}$ by Fact 3.4 and $r_0(G/c(G)) \leq \mathfrak{c}$, since $G/c(G)$ is metrizable by Lemma 5.1(a).

Some useful properties of the condition $Ps(\tau, \sigma)$ are collected in the next lemma.

Lemma 5.2. *(a)* [17, Lemma 2.7(d)] *$Ps(\mathfrak{c}, \omega)$ holds.*

(b) [17, Lemma 2.7 (a,ii)] *If $Ps(\tau, \sigma)$ holds for some cardinals $\tau, \sigma \geq \omega$, then $Ps(\tau', \sigma)$ holds for every cardinal τ' such that $\tau \leq \tau' \leq 2^\sigma$.*

(c) [17, Lemma 2.9] *If H is a set such that $2 \leq |H| \leq \mathfrak{c}$, then $Ps(\tau, \sigma)$ holds if and only if there exists an ω-dense set $Y \subseteq H^\sigma$ with $|Y| = \tau$.*

(d) (particular case of) [17, Lemma 3.4(i)]$Ps(2^\kappa, 2^{2^\kappa})$ *holds for every infinite cardinal κ.*

Using a technique similar to that of the proof of Theorem 1.14 and applying Theorem 1.10 we prove Theorem 1.15.

Proof of Theorem 1.15. Let $w(G) = \sigma$ and $K = \widetilde{G}$. Then K is a w-divisible compact abelian group of weight σ. By Theorem 1.10 there exists a continuous surjective homomorphism $f : K \to \prod_{i \in I} K_i$, where each K_i is a metrizable compact non-torsion abelian group and $|I| = \sigma$. Note that $\sigma > \omega$. By Fact 3.4 $r_0(G) \geq \mathfrak{c}$.

Let $\varphi : \prod_{i \in I} K_i \to \prod_{i \in I} K_i/t(K_i)$ be the canonical projection. For $A \subseteq \sigma$ let

$$\varphi_A : \prod_{i \in A} K_i \to \prod_{i \in A} K_i/t(K_i).$$

Moreover

$$\pi_A : \prod_{i \in I} K_i \to \prod_{i \in A} K_i \text{ and } \bar{\pi}_A : \prod_{i \in I} K_i/t(K_i) \to \prod_{i \in A} K_i/t(K_i)$$

are the canonical projections. Let $H = f(G) \subseteq \prod_{i \in I} K_i$ and $\bar{H} = \varphi(H) \subseteq \prod_{i \in I} K_i/t(K_i)$, while

$$i : H \to \prod_{i \in I} K_i \text{ and } \bar{i} : \bar{H} \to \prod_{i \in I} K_i/t(K_i)$$

are the inclusion maps. Finally $\tilde{\varphi} = \varphi \restriction_H : H \to \bar{H}$.

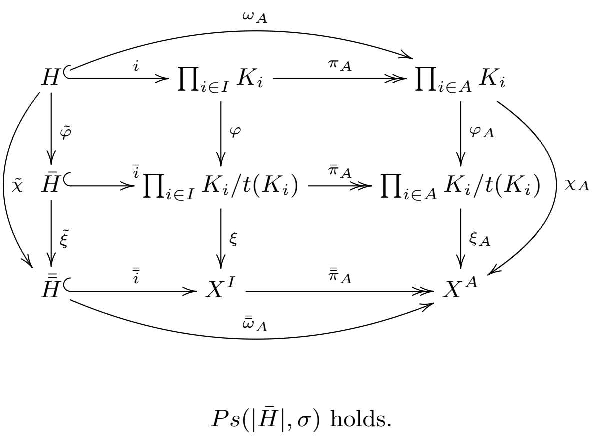
Let $i \in I$. Then $|K_i/t(K_i)| = \mathfrak{c}$, because $K_i/t(K_i)$ is torsion-free and $r_0(K_i/t(K_i)) = r_0(K_i) = \mathfrak{c}$ by Fact 3.4. Then there exists a bijection $\xi_i : K_i/t(K_i) \to X$, where X is a set of cardinality $\mathfrak{c}$. Consequently $\xi : \prod_{i \in I} K_i/t(K_i) \to X^I = X^\sigma$, defined by $\xi((k_i)_{i \in I}) = (\xi_i(k_i))_{i \in I}$ for every $(k_i)_{i \in I} \in \prod_{i \in I} K_i/t(K_i)$, is a bijection. Define $\tilde{\xi} = \xi \restriction_{\bar{H}} : \bar{H} \to \bar{\bar{H}}$, let $\bar{\bar{i}} : \bar{\bar{H}} \to X^I$ be the inclusion map and $\bar{\bar{\pi}}_A : X^I \to X^A$ the canonical projection. Moreover let

$$\tilde{\chi} = \tilde{\xi} \circ \tilde{\varphi} \quad \text{and} \quad \chi_A = \xi_A \circ \varphi_A$$

and define

$$\omega_A = \pi_A \circ i \quad \text{and} \quad \bar{\bar{\omega}}_A = \bar{\bar{\pi}}_A \circ \bar{\bar{i}}.$$

This gives the following commutative diagram:



We want to prove that

$$Ps(|\bar{H}|, \sigma) \text{ holds.}$$

To this end we prove that $\bar{\bar{H}} = \xi(\bar{H})$ is ω-dense in X^I.

Let A be a countable subset of I. Since G is a dense pseudocompact subgroup of K, H is a dense pseudocompact subgroup of $\prod_{i\in A} K_i$ by [21, Theorem 3.10.24]. Therefore $\omega_A : H \to \prod_{i\in A} K_i$ is surjective. In fact each K_i is metrizable, so $\prod_{i\in A} K_i$ is metrizable as well; since $\omega_A(H)$ is pseudocompact in the metrizable group $\prod_{i\in A} K_i$, it is compact, and being also dense, it coincides with $\prod_{i\in A} K_i$. Also χ_A is a surjection and so $\chi_A \circ \omega_A$ is surjective as well. But $\chi_A \circ \omega_A = \bar{\bar{\omega}}_A \circ \tilde{\chi}$ and hence $\bar{\bar{\omega}}_A \circ \tilde{\chi}$ is surjective; thus $\bar{\bar{\omega}}_A : \bar{\bar{H}} \to X^A$ is surjective too. Then A being an arbitrary countable subset of I proves that $\bar{\bar{H}}$ is ω-dense in X^I. Therefore $|\bar{\bar{H}}| > \omega$. Since ξ is a bijection, $|\bar{H}| = |\bar{\bar{H}}| > \omega$. This yields $Ps(|\bar{H}|, \sigma)$.

Since there exists a surjective homomorphism of G onto $\bar{H}$, $r_0(G) \geq r_0(\bar{H}) = |\bar{H}|$. (The last equality is due to $|\bar{H}| > \omega$ and the fact that $\bar{H}$ is torsion-free as a subgroup of the the torsion-free group $\prod_{i\in I} K_i/t(K_i)$.) Moreover $r_0(G) \leq |K| = 2^\sigma$. Since $Ps(|\bar{H}|, \sigma)$ holds, $Ps(r_0(G), \sigma)$ holds by Lemma 5.2(b). □

An alternative way to prove Theorem 1.18 is adopted in [20].

Proof of Theorem 1.18. If G is non-singular, by Lemma 3.3 there exists a w-divisible subgroup H of G such that $w(H) = w_d(G)$. Moreover $r_0(H) = r_0(G)$ and H is pseudocompact, since G is pseudocompact. Apply Theorem 1.15 to H to conclude that $Ps(r_0(G), w_d(G))$ holds. If G is singular, by the definition there exists $m \in \mathbb{N}_+$ such that mG is metrizable. Since G is non-torsion, $r_0(G) \geq \mathfrak{c}$ by Fact 3.4 and $w(mG) = \omega$. Hence $r_0(G) = \mathfrak{c}$ and $w_d(G) = \omega$. By Lemma 5.2(a) $Ps(\mathfrak{c}, \omega)$ holds. □

One can ask if the implication of Theorem 1.15 can be reversed. Example 5.4 shows that this is not possible. Anyway the following lemma gives a necessary condition in order that $Ps(r_0(G), w(G))$ holds for a pseudocompact abelian group G.

Lemma 5.3. *Let G be a pseudocompact abelian group. If $Ps(r_0(G), w(G))$ holds, then $w(G) \leq 2^{2^{w_d(G)}}$.*

Proof. To begin with, $Ps(r_0(G), w(G))$ yields

$$w(G) \leq 2^{r_0(G)}. \tag{1}$$

By Lemma 3.3 there exists a w-divisible pseudocompact subgroup H of G such that $w(H) = w_d(G)$ and $r_0(H) = r_0(G)$. In view of Theorem 1.15 $Ps(r_0(H), w(H))$ holds and so $r_0(H) \leq 2^{w(H)}$, that is

$$r_0(G) \leq 2^{w_d(G)}. \tag{2}$$

Equations (1) and (2) together give $w(G) \leq 2^{2^{w_d(G)}}$. □

Example 5.4. For every infinite cardinal κ there exists a compact abelian group H_κ such that:

- $w(H_\kappa) = 2^{2^{w_d(H_\kappa)}}$;
- $w_d(H_\kappa) = \kappa$;
- $r_0(H_\kappa) = 2^\kappa$;
- $Ps(r_0(H_\kappa), w(H_\kappa))$ holds.

The group $H_\kappa = \{0,1\}^{2^\kappa} \times \mathbb{T}^\kappa$ has the requested properties. Note that $Ps(r_0(H_\kappa), w(H_\kappa)) = Ps(2^\kappa, 2^{2^\kappa})$ holds because $(2^\kappa)^\omega = 2^\kappa$ by Lemma 5.2(d).

In particular every H_κ is not w-divisible and H_ω is singular.

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