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# A global bifurcation result for a second order singular equation ${ }^{1}$ 

Anna Capietto, Walter Dambrosio<br>and Duccio Papini<br>Dedicated, with gratefulness and friendship, to Professor Fabio Zanolin on the occasion of his 60th birthday


#### Abstract

We deal with a boundary value problem associated to a second order singular equation in the open interval $(0,1]$. We first study the eigenvalue problem in the linear case and discuss the nodal properties of the eigenfunctions. We then give a global bifurcation result for nonlinear problems.


Keywords: self-adjoint singular operator, spectrum, nodal properties, global bifurcation MS Classification 2010: 34C23, 34B09, 35P05

## 1. Introduction

We are concerned with a second order ODE of the form

$$
\begin{equation*}
-u^{\prime \prime}+q(x) u=\lambda u+g(x, u) u, \lambda \in \mathbb{R}, x \in(0,1] \tag{1}
\end{equation*}
$$

where $q \in C((0,1])$ satisfies

$$
\begin{equation*}
\lim _{x \rightarrow 0^{+}} \frac{q(x)}{l / x^{\alpha}}=1 \tag{2}
\end{equation*}
$$

for some $l>0$ and $\alpha \in(0,5 / 4)$, and $g \in C([0,1] \times \mathbb{R})$ is such that

$$
\begin{equation*}
\lim _{u \rightarrow 0} g(x, u)=0, \quad \text { uniformly in } x \in(0,1] . \tag{3}
\end{equation*}
$$

The constant $5 / 4$ arises in a rather straightforward manner in the study of the differential operator in the left-hand side of (1) (cf. [17, p. 287-288]); details are given in Remark 2.3 below.

[^0]We will look for solutions $u$ of (1) such that $u \in H_{0}^{2}(0,1)$.
When the $x$-variable belongs to a compact interval, problems of the form (1) have been very widely studied. A more limited number of contributions is available in the literature when the $x$-variable belongs to a (semi)-open interval, as it is the case in the present paper, or to an unbounded interval $[7,8]$.

We treat (1) in the framework of bifurcation theory. For this reason, we first discuss in Section 2 the eigenvalue problem

$$
\begin{equation*}
-u^{\prime \prime}+q(x) u=\lambda u, \quad x \in(0,1], \lambda \in \mathbb{R} . \tag{4}
\end{equation*}
$$

For such singular problems, the well-known embedding of (4) (by an elementary application of the integration by parts rule, together with the boundary condition $u(0)=0=u(1))$ in the setting of eigenvalue problems for compact self-adjoint operators cannot be performed. Thus, the questions of the existence of eigenvalues and of the nodal properties of the associated eigenfunctions have various delicate features. For a comprehensive account on the spectral properties of the Schrödinger operator we refer to the books [12] and [10]; for more specific results on singular problems in $(0,1)$ we refer, among many others, to $[5,14]$.

However, the linear spectral theory for singular problems is well-established and can be found, among others, in the classical book by Coddington and Levinson [4] and in the (relatively) more recent text by Weidmann [17]. The former monograph focuses on a generalization of the so-called "expansion theorem" valid for functions in $L^{2}([0,1])$ and, by doing this, a sort of "generalized shooting method" is performed. On the other hand, in [17] the singular problem is tackled from an abstract point of view; more precisely, it is considered the general question of the existence of a self-adjoint realization of the formal differential expression $\tau u=-u^{\prime \prime}+q(x) u$ and the important Weyl alternative theorem [17, Theorem 5.6] is used. It is interesting to observe that the approach in [4] (based on more elementary ODE techniques) and the abstract one in [17] lead in different ways to the important concepts of "limit point case" and "limit circle case". The knowledge of one (or the other) case is ensured by suitable assumptions on $q$ and leads to information on the boundary conditions to be added to (4) in order to have a self-adjoint realization of $\tau$.

In the setting of the present paper, the operator $\tau$ is regular at $x=1$; this implies that it is in the limit circle case. Moreover, under assumption (2), from [17, Theorem 6.4] it follows that $\tau$ is in the limit circle case also in $x=0$. Thus, the differential operator $A: u \mapsto \tau u$ with

$$
\begin{array}{r}
D(A)=\left\{u \in L^{2}(0,1): u, u^{\prime} \in A C(0,1), \tau u \in L^{2}(0,1),\right. \\
\left.\lim _{x \rightarrow 0^{+}}\left(x u^{\prime}(x)-u(x)\right)=0=u(1)\right\}
\end{array}
$$

is a self-adjoint realization of $\tau$ ([17, p. 287-288]). We prove in Proposition 2.2 that in fact $D(A)=H_{0}^{2}(0,1)$; to do this, we need some knowledge of the behaviour of the solutions of (4) near zero. These estimates are developed in Proposition 2.1 by means of the classical Levinson theorem [6, Theorem 1.8.1]. Finally, at the end of Section 2 we focus on the nodal properties of a solution to (4); more precisely, in Proposition 2.4 we prove that (4) is non-oscillatory and conclude in Proposition 2.5 that the spectrum of $A$ is purely discrete and that, for every $n \in \mathbb{N}$, the eigenfunction associated to the eigenvalue $\lambda_{n}$ has $(n-1)$ simple zeros in $(0,1)$.

Section 3 contains a global bifurcation result (Theorem 3.2) which follows in a rather straightforward manner as an application of the celebrated Rabinowitz theorem in [11].

In order to exclude alternative (2) in Theorem 3.2, we use a technique that we already applied for Hamiltonian systems in $\mathbb{R}^{2 N}$ in [2] and for planar Diractype systems in [3]. More precisely, we introduce a continuous integer-valued functional defined on the set of solutions to (1). Due to the singularity at $x=0$, some care is necessary in order to prove its continuity; this is the content of Proposition 3.4. We can then state and prove our main result (Theorem 3.5).

In what follows, for a given function $p$ we write $p(x) \sim \frac{m}{x^{a}}, x \rightarrow 0^{+}$, when

$$
\begin{equation*}
\lim _{x \rightarrow 0^{+}} \frac{p(x)}{m / x^{a}}=1 \tag{5}
\end{equation*}
$$

for some $m, a \in \mathbb{R}^{+}$.
Finally, we write

$$
H_{0}^{2}(0,1)=\left\{u \in H^{2}(0,1): u(0)=0=u(1)\right\}
$$

equipped with the norm defined by

$$
\|u\|^{2}=\|u\|_{L^{2}(0,1)}^{2}+\left\|u^{\prime \prime}\right\|_{L^{2}(0,1)}^{2}, \quad \forall u \in H_{0}^{2}(0,1) .
$$

## 2. The linear equation

In this section we study a linear second order equation of the form

$$
\begin{equation*}
-u^{\prime \prime}+q(x) u=\lambda u, \quad x \in(0,1], \lambda \in \mathbb{R} \tag{6}
\end{equation*}
$$

We will assume that $q \in C((0,1])$ and that

$$
\begin{equation*}
q(x) \sim \frac{l}{x^{\alpha}}, \quad x \rightarrow 0^{+} \tag{7}
\end{equation*}
$$

for some $l>0$ and $\alpha \in(0,5 / 4)$. Without loss of generality we may suppose that

$$
\begin{equation*}
q(x)>0, \quad \forall x \in(0,1] . \tag{8}
\end{equation*}
$$

For every $u:(0,1] \rightarrow \mathbb{R}$ we denote by $\tau u$ the formal expression

$$
\tau u=-u^{\prime \prime}+q(x) u
$$

First of all, we study the asymptotic behaviour of solutions of (6) when $x \rightarrow 0^{+}$; to this aim, let us introduce the change of variables $t=-\log x$ and let

$$
w(t)=u\left(e^{-t}\right), \quad \forall t>0 .
$$

From the relations

$$
\begin{align*}
w^{\prime}(t) & =-e^{-t} u^{\prime}\left(e^{-t}\right) \\
w^{\prime \prime}(t) & =e^{-t} u^{\prime}\left(e^{-t}\right)+e^{-2 t} u^{\prime \prime}\left(e^{-t}\right) \tag{9}
\end{align*}
$$

we deduce that $u$ is a solution of $(6)$ on $(0,1)$ if and only if $w$ is a solution of

$$
\begin{equation*}
-w^{\prime \prime}-w^{\prime}+e^{-2 t} q\left(e^{-t}\right) w=\lambda e^{-2 t} w \tag{10}
\end{equation*}
$$

on $(0,+\infty)$. Equation (10) can be written in the form

$$
\begin{equation*}
Y^{\prime}=(C+R(t)) Y \tag{11}
\end{equation*}
$$

where $Y=(w, z)^{T}$ and

$$
C=\left(\begin{array}{cc}
0 & 1  \tag{12}\\
0 & -1
\end{array}\right), \quad R(t)=\left(\begin{array}{cc}
0 & 0 \\
e^{-2 t} q\left(e^{-t}\right)-\lambda e^{-2 t} & 0
\end{array}\right), \quad \forall t>0
$$

Now, let us observe that $C$ has eigenvalues $\lambda_{1}=0, \lambda_{2}=-1$ and corresponding eigenvectors $u_{1}=(1,0), u_{2}=(1,-1)$ and that $R \in L^{1}(0,+\infty)$; therefore, an application of [6, Theorem 1.8.1] implies that (11) has two linearly independent solutions $Y_{1}, Y_{2}$ such that

$$
\begin{align*}
& Y_{1}(t)=u_{1}+o(1), \quad t \rightarrow+\infty \\
& Y_{2}(t)=\left(u_{2}+o(1)\right) e^{-t}, \quad t \rightarrow+\infty \tag{13}
\end{align*}
$$

As a consequence, we obtain the following result:

Proposition 2.1. For every $\lambda \in \mathbb{R}$ the equation (6) has two linearly independent solutions $u_{1, \lambda}, u_{2, \lambda}$ such that

$$
\begin{align*}
& u_{1, \lambda}(x)=1+o(1), u_{1, \lambda}^{\prime}(x)=o\left(\frac{1}{x}\right) \quad x \rightarrow 0^{+}  \tag{14}\\
& u_{2, \lambda}(x)=x+o(x), u_{2, \lambda}^{\prime}(x)=1+o(1), \quad x \rightarrow 0^{+}
\end{align*}
$$

and $u_{2, \lambda} \in H^{2}(0,1)$.
For every $f \in L^{2}(0,1)$ the solutions of $\tau u=f$ are given by

$$
\begin{equation*}
u(x)=c_{1} u_{1,0}(x)+c_{2} u_{2,0}(x)+u_{f}(x), \quad \forall x \in(0,1), c_{1}, c_{2} \in \mathbb{R} \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
& u_{f}(x)=\int_{0}^{x} G(x, t) f(t) d t, \quad \forall x \in(0,1) \\
& G(x, t)=u_{1,0}(t) u_{2,0}(x)-u_{2,0}(t) u_{1,0}(x), \quad \forall x \in(0,1), t \in(0,1) \tag{16}
\end{align*}
$$

fulfill $G \in L^{\infty}\left((0,1)^{2}\right)$, $u_{f}(0)=0=u_{f}^{\prime}(0)$ and $u_{f} \in H^{2}(0,1)$.
Proof. The estimates in (14) follow from (9) and (13), while (16) is the usual variation of constants formula. Moreover, from (14) we obtain that $u_{2, \lambda}, u_{2, \lambda}^{\prime} \in$ $L^{2}(0,1)$. On the other hand we have

$$
\begin{equation*}
q(x) u_{2, \lambda}(x) \sim x^{1-\alpha}, \quad x \rightarrow 0^{+} \tag{17}
\end{equation*}
$$

which implies that $q u_{2, \lambda} \in L^{2}(0,1)$, since $\alpha<5 / 4$ (cf. Remark 2.3 for comments on this restriction); using the fact that $\tau u_{2, \lambda}=\lambda u_{2, \lambda}$, we deduce that

$$
u_{2, \lambda}^{\prime \prime}=\lambda u_{2, \lambda}-q u_{2, \lambda} \in L^{2}(0,1)
$$

From now on, we will indicate $u_{i}=u_{i, 0}, i=1,2$. The fact that the function $G$ defined in (16) belongs to the space $L^{\infty}\left((0,1)^{2}\right)$ is a consequence of the asymptotic estimates (14). Moreover, from (16) we also deduce that $u_{f}(0)=0$ and that

$$
\begin{equation*}
u_{f}^{\prime}(x)=\int_{0}^{x}\left(u_{1}(t) u_{2}^{\prime}(x)-u_{2}(t) u_{1}^{\prime}(x)\right) f(t) d t, \quad \forall x \in(0,1), \tag{18}
\end{equation*}
$$

which implies $u_{f}^{\prime}(0)=0$.
Finally, the condition $u_{f}(0)=0=u_{f}^{\prime}(0)$ guarantees that $u_{f}, u_{f}^{\prime} \in L^{2}(0,1)$; as far as the second derivative of $u_{f}$ is concerned, let us observe that we have

$$
\tau u_{f}=f
$$

and so

$$
\begin{equation*}
u_{f}^{\prime \prime}=f-q u_{f} \tag{19}
\end{equation*}
$$

Using the fact that $u_{f}(0)=0=u_{f}^{\prime}(0)$ and (7), it follows that $q u_{f} \in L^{2}(0,1)$; hence $u_{f} \in H^{2}(0,1)$.

In what follows, we study the spectral properties of suitable self-adjoint realizations of $\tau$; to this aim, let us first observe that the differential operator $\tau$ is regular at $x=1$. As a consequence, it is in the limit circle case at $x=1$; moreover, from (7), according to [17, Theorem 6.4], $\tau$ is in the limit circle case also in $x=0$.

The differential operator $A$ defined by

$$
\begin{aligned}
& D(A)=\left\{u \in L^{2}(0,1): u, u^{\prime} \in A C(0,1), \tau u \in L^{2}(0,1)\right. \\
& \left.\qquad \lim _{x \rightarrow 0^{+}}\left(x u^{\prime}(x)-u(x)\right)=0=u(1)\right\} \\
& A u=\tau u, \quad \forall u \in D(A),
\end{aligned}
$$

is then a self-adjoint realization of $\tau$ ([17, p. 287-288]). We can show the validity of the following Proposition:

Proposition 2.2. The relation

$$
D(A)=H_{0}^{2}(0,1)
$$

holds true. Moreover, $A$ has a bounded inverse $A^{-1}: L^{2}(0,1) \rightarrow H_{0}^{2}(0,1)$.
Proof. 1. Let us start proving that $H_{0}^{2}(0,1) \subset D(A)$. It is well known that $H_{0}^{2}(0,1) \subset C^{1}(0,1)$; hence, for every $u \in H_{0}^{2}(0,1)$ we have $u, u^{\prime} \in A C(0,1)$. Moreover, using the fact that $u(0)=0$ we deduce that

$$
u(x)=u^{\prime}(0) x+o(x), \quad x \rightarrow 0^{+}
$$

and

$$
q(x) u(x)=u^{\prime}(0) x^{1-\alpha}+o\left(x^{1-\alpha}\right), \quad x \rightarrow 0^{+} ;
$$

the condition $\alpha<5 / 4$ guarantees again that $q u \in L^{2}(0,1)$ and therefore $\tau u=$ $-u^{\prime \prime}+q u \in L^{2}(0,1)$. Finally, the regularity of $u$ and $u^{\prime}$ imply that

$$
\lim _{x \rightarrow 0^{+}}\left(x u^{\prime}(x)-u(x)\right)=0
$$

and so also the boundary condition in the definition of $D(A)$ is satisfied.
Now, let us prove that $D(A) \subset H_{0}^{2}(0,1)$; for every $u \in D(A)$ let $f=\tau u \in$ $L^{2}(0,1)$. From (15) we deduce that $u$ can be written as

$$
\begin{equation*}
u=c_{1} u_{1}+c_{2} u_{2}+u_{f} \tag{20}
\end{equation*}
$$

for some $c_{1}, c_{2} \in \mathbb{R}$; it is easy to see that the function $u_{1}$ does not satisfy the boundary condition given in $x=0$ in the definition of $D(A)$, while $u_{2}$ and $u_{f}$ do. Hence $u \in D(A)$ if and only if $c_{1}=0$; the last statement of Proposition 2.1 implies then that $u \in H^{2}(0,1)$. As in the first part of the proof, the regularity
of $u$ allows to conclude that the boundary condition in $x=0$ given in $D(A)$ reduces to $u(0)=0$.
2. Let us study the invertibility of $A$; the existence of a bounded inverse of $A$ is equivalent to the fact that $0 \in \rho_{A}$, being $\rho_{A}$ the resolvent of $A$. Since $A$ is self-adjoint on $H_{0}^{2}(0,1)$, this follows from the surjectivity of $A$ (cf. [16, Theorem 5.24]); hence, it is sufficient to prove that $A$ is surjective.

To this aim, let us first observe that condition (8) guarantees that 0 cannot be an eigenvalue of $A$. Now, let us fix $f \in L^{2}(0,1)$ and let us prove that there exists $u \in H_{0}^{2}(0,1)$ such that $A u=f$, i.e. $\tau u=f$; by applying Proposition 2.1 we deduce again that (20) holds true and the same argument of the first part of the proof implies that $c_{1}=0$.

Hence we obtain $u=c_{2} u_{2}+u_{f}$; from Proposition 2.1 we deduce that this function belongs to $H^{2}(0,1)$ and satisfies the boundary condition $u(0)=0$. In order to prove that the missing condition $u(1)=0$ is fulfilled for every $f \in$ $L^{2}(0,1)$, let us observe that $u_{2}(1) \neq 0$, otherwise $u_{2}$ would be an eigenfunction of $A$ associated to the zero eigenvalue. Therefore, $u(1)=0$ is satisfied if

$$
c_{2}=-\frac{u_{f}(1)}{u_{2}(1)},
$$

for every $f \in L^{2}(0,1)$.
Remark 2.3. As for the restriction $\alpha<5 / 4$, we observe that for the proofs of Proposition 2.1 and Proposition 2.2 it is sufficient to require the milder condition $\alpha<3 / 2$. The fact that $\alpha<5 / 4$ is used (cf. [17, p. 287-288]) in order to obtain that $D(A)$ is the one described above. Finally, we observe that in the particular case when $\alpha<1$ the problem is regular (cf., among others, [9]).

The spectral properties of $A$ are related to the oscillatory behaviour of solutions of (6). We first recall the following definition:

Definition 2.4. The differential equation (6) is oscillatory if every solution u has infinitely many zeros in $(0,1)$. It is non-oscillatory when it is not oscillatory.

We observe that the regularity assumptions on $q$ imply that solutions of (6) have a finite number of zeros in any interval of the form [a,1), for every $0<a<$ 1. Moreover, from (7) we infer that for every $\lambda \in \mathbb{R}$ there exists $c(\lambda) \in(0,1]$ such that

$$
\lambda-q(x)<0, \quad \forall x \in(0, c(\lambda))
$$

An application of the Sturm comparison theorem proves that every solution of $(6)$ has at most one zero in $(0, c(\lambda))$; as a consequence, we obtain the following result:

Proposition 2.5. For every $\lambda \in \mathbb{R}$ the differential equation (6) is non-oscillatory.

Once Proposition 2.5 is obtained, we can provide in a straightforward way some useful information on the spectral properties of $A$; more precisely, denoting by $\sigma_{\text {ess }}$ the essential spectrum of a given operator, we have:

Proposition 2.6. ([17, Theorem 14.3, Theorem 14.6 and Theorem 14.9], [12, Theorem XIII.1]) The differential operator $A$ is bounded-below and satisfies

$$
\sigma_{e s s}(A)=\varnothing
$$

Moreover, there exists a sequence $\left\{\lambda_{n}\right\}_{n \in \mathbb{N}}$ of simple eigenvalues of $A$ such that

$$
\lim _{n \rightarrow+\infty} \lambda_{n}=+\infty
$$

and for every $n \in \mathbb{N}$ the eigenfunction $u_{n}$ of $A$ associated to the eigenvalue $\lambda_{n}$ has $(n-1)$ simple zeros in $(0,1)$.

Remark 2.7. According to [17], operators of the form $\tau$ (defined on functions whose domain is $(0,+\infty)$ ) arise when the time independent Schrödinger equation with spherically symmetric potential

$$
\begin{equation*}
-\Delta u(x)+V(|x|) u(x)=\lambda u(x), \quad u \in L^{2}\left(\mathbb{R}^{m}\right) \tag{21}
\end{equation*}
$$

is reduced to an infinite system of eigenvalue problems associated to the ordinary differential operators in $L^{2}(0,+\infty)$

$$
\tau_{i}=-\frac{d^{2}}{d r^{2}}+\frac{1}{r^{2}}\left[i(i+m-2)+\frac{1}{4}(m-1)(m-3)\right]+V(r)
$$

$(i \in \mathbb{N})$. In Appendix 17.F of [17] it is treated the case of a potential $V$ satisfying assumptions (which enable to consider Coulomb potentials) that lead to (7). More precisely, it is shown that for $m=3, i=0$ the operator is in the limit circle case at zero and self-adjoint extensions of $\tau_{0}$ are described.

## 3. The main result

In this section we are interested in proving a global bifurcation result for a nonlinear eigenvalue problem of the form

$$
\begin{equation*}
-u^{\prime \prime}+q(x) u=\lambda u+g(x, u) u, \lambda \in \mathbb{R}, x \in(0,1] \tag{22}
\end{equation*}
$$

where $q \in C((0,1])$ satisfies (7) and $g \in C([0,1] \times \mathbb{R})$ is such that

$$
\begin{equation*}
\lim _{u \rightarrow 0} g(x, u)=0, \quad \text { uniformly in } x \in[0,1] . \tag{23}
\end{equation*}
$$

We will look for solutions $u$ of (22) such that $u \in H_{0}^{2}(0,1)$. To this aim, let $\Sigma$ denote the set of nontrivial solutions of $(22)$ in $H_{0}^{2}(0,1) \times \mathbb{R}$ and let $\Sigma^{\prime}=\Sigma \cup\left\{(0, \lambda) \in H_{0}^{2}(0,1) \times \mathbb{R}: \lambda\right.$ is an eigenvalue of $\left.A\right\}$, where $A$ is as in Section 2.

Let $M$ denote the Nemitskii operator associated to $g$, given by

$$
M(u)(x)=g(x, u(x)) u(x), \quad \forall x \in[0,1]
$$

for every $u \in H_{0}^{2}(0,1)$. We can show the validity of the following:
Proposition 3.1. Assume $g \in C([0,1] \times \mathbb{R})$ and (23). Then $M: H_{0}^{2}(0,1) \longrightarrow$ $L^{2}(0,1)$ is a continuous map and satisfies

$$
\begin{equation*}
M(u)=o(\|u\|), \quad u \rightarrow 0 \tag{24}
\end{equation*}
$$

Proof. 1. We first show that $M u \in L^{2}(0,1)$ when $u \in H_{0}^{2}(0,1)$. When this condition holds, $u \in L^{\infty}(0,1)$ and the continuity of $g$ implies that there exists $C_{u}>0$ such that

$$
|g(x, u(x)) u(x)| \leq C_{u}, \quad \forall x \in[0,1]
$$

As a consequence we obtain $M u \in L^{\infty}(0,1) \subset L^{2}(0,1)$.
2. Let us prove that $M$ is continuous. Let us fix $u_{0} \in X$ and let $u_{n} \in X$ such that $u_{n} \rightarrow u_{0}$ when $n \rightarrow+\infty$; the continuous embedding

$$
H_{0}^{2}(0,1) \subset L^{\infty}(0,1)
$$

and the uniform continuity of $g$ on compact subsets of $[0,1] \times \mathbb{R}$ ensure that

$$
\begin{equation*}
g\left(x, u_{n}(x)\right) \rightarrow g\left(x, u_{0}(x)\right) \quad \text { in } \quad L^{\infty}(0,1) \tag{25}
\end{equation*}
$$

This is sufficient to conclude that $M u_{n} \rightarrow M u_{0}$ in $L^{\infty}(0,1)$ and hence $M u_{n} \rightarrow$ $M u_{0}$ in $L^{2}(0,1)$.
3. Finally, let us prove (24): using again the fact that $H_{0}^{2}(0,1) \subset L^{\infty}(0,1)$, we have

$$
\|M u\|_{L^{2}(0,1)} \leq\|g(x, u(x))\|_{L^{\infty}(0,1)}\|u\|_{L^{2}(0,1)} \leq\|g(x, u(x))\|_{L^{\infty}(0,1)}\|u\|
$$

for all $u \in H_{0}^{2}(0,1)$; hence, we deduce that

$$
\frac{\|M u\|_{L^{2}(0,1)}}{\|u\|} \leq\|g(x, u(x))\|_{L^{\infty}(0,1)}, \quad \forall u \in H_{0}^{2}(0,1), u \neq 0
$$

Therefore the result follows from (23) and (25).

Now, let us observe that the search of solutions $u \in H_{0}^{2}(0,1)$ of $(22)$ is equivalent to the search of solutions of the abstract equation

$$
\begin{equation*}
A u=\lambda u+M(u), \quad(u, \lambda) \in H_{0}^{2}(0,1) \times \mathbb{R} \tag{26}
\end{equation*}
$$

on the other hand, (26) can be written in the form

$$
\begin{equation*}
w=\lambda R w+M(R w), \quad(w, \lambda) \in L^{2}(0,1) \times \mathbb{R} \tag{27}
\end{equation*}
$$

where $R: L^{2}(0,1) \rightarrow H_{0}^{2}(0,1)$ is the inverse of $A$ (cf. Proposition 2.2).
Now, from [17, Theorem 7.10] we deduce that $R$ is compact; this fact and the continuity of $M$ guarantee that the operator $M R: L^{2}(0,1) \rightarrow H_{0}^{2}(0,1)$ is compact. Moreover, the condition

$$
\begin{equation*}
M(R w)=o\left(\|w\|_{L^{2}(0,1)}\right), \quad w \rightarrow 0 \tag{28}
\end{equation*}
$$

is a consequence of (24). From an application of the global bifurcation result of Rabinowitz (cfr. [11]) to (27) we then obtain the following result:

Theorem 3.2. Assume (7) and (23). Then, for every eigenvalue $\lambda_{n}$ of $A$ there exists a continuum $C_{n}$ of nontrivial solutions of (22) in $H_{0}^{2}(0,1) \times \mathbb{R}$ bifurcating from $\left(0, \lambda_{n}\right)$ and such that one of the following conditions holds true:
(1) $C_{n}$ is unbounded in $H_{0}^{2}(0,1) \times \mathbb{R}$;
(2) $C_{n}$ contains $\left(0, \lambda_{n^{\prime}}\right) \in \Sigma^{\prime}$, with $n^{\prime} \neq n$.

Now, let us observe that a more precise description of the bifurcating branch, eventually leading to exclude condition (2), can be obtained when there exists a continuous functional $j: \Sigma^{\prime} \rightarrow \mathbb{N}$ (cf. [2, Pr. 2.1]). In order to define such a functional, we will use the fact that nontrivial solutions of (22) have a finite number of zeros in $(0,1)$; this will be a consequence of our next result.

For every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_{0}^{2}(0,1)$ of (22) let us define $q_{u, \lambda}:(0,1] \rightarrow \mathbb{R}$ by $q_{u, \lambda}(x)=q(x)-\lambda-g(x, u(x))$, for every $x \in(0,1]$. The following Lemma holds true:

Lemma 3.3. For every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_{0}^{2}(0,1)$ of (22) there exists a neighborhood $U \subset H_{0}^{2}(0,1) \times \mathbb{R}$ of $(u, \lambda)$ and $x_{u, \lambda} \in(0,1)$ such that

$$
\begin{equation*}
q_{v, \mu}(x)>0, \quad \forall(v, \mu) \in U, x \in\left(0, x_{u, \lambda}\right] . \tag{29}
\end{equation*}
$$

Proof. Let $(u, \lambda) \in H_{0}^{2}(0,1) \times \mathbb{R}, u \not \equiv 0$, be fixed and let $U$ be the neighborhood of radius 1 of $(u, \lambda)$ in $H_{0}^{2}(0,1) \times \mathbb{R}$; from the continuous embedding $L^{\infty}(0,1) \subset$ $H_{0}^{2}(0,1)$ we deduce that if $(w, \mu) \in \Sigma \cap U_{1}$ then

$$
\|w\|_{L^{\infty}(0,1)} \leq 1+\|u\|_{L^{\infty}(0,1)}, \quad|\mu| \leq 1+|\lambda|
$$

and

$$
q(x)-\mu-g(x, w(x)) \geq q(x)-|\lambda|-1-\max _{\substack{x \in[0,1],|s| \leq 1+| | u \|_{L}^{\infty}(0,1)}}|g(x, s)|, \quad \forall x \in(0,1)
$$

From (7) we then deduce that there exists $x_{(u, \lambda)} \in(0,1)$, depending only on $(u, \lambda)$, such that

$$
q(x)-\mu-g(x, w(x))>0, \quad \forall x \in\left(0, x_{(u, \lambda)}\right] .
$$

Now, let us observe that for every $\lambda \in \mathbb{R}$ and for every nontrivial solution $u \in H_{0}^{2}(0,1)$ of $(22)$ the function $u$ is a nontrivial solution of the linear equation

$$
\begin{equation*}
-w^{\prime \prime}+(q(x)-g(x, u(x))-\lambda) w=0 \tag{30}
\end{equation*}
$$

From Lemma 3.3, with an argument similar to the one which led to Proposition 2.5 , we deduce that all the nontrivial solutions of (30) (in particular $u$ ) have a finite number of zeros in $(0,1)$. We denote by $n(u)$ this number.

We are then allowed to define the functional $j$ by setting

$$
j(u, \lambda)= \begin{cases}n(u) & \text { if } u \not \equiv 0  \tag{31}\\ n-1 & \text { if } u \equiv 0 \text { and } \lambda=\lambda_{n}\end{cases}
$$

for every $(u, \lambda) \in \Sigma^{\prime}$. Let us observe that the definition $j\left(0, \lambda_{n}\right)=n-1$ is suggested by Proposition 2.6.

Proposition 3.4. The function $j: \Sigma^{\prime} \rightarrow \mathbb{N}$ is continuous.
Proof. 1. As for the continuity of $j$ in every point of the form $\left(0, \lambda_{n}\right), n \in \mathbb{N}$, we refer to [15, Lemma 2.5].
2. Let us now fix $\left(u_{0}, \lambda_{0}\right) \in \Sigma$ and let $(u, \lambda) \in U$, with $U$ as in Lemma 3.3; this Lemma guarantees that both $u$ and $u_{0}$ have no zeros in ( $0, x_{u_{0}, \lambda_{0}}$ ).

On the other hand, in the interval $\left[x_{u_{0}, \lambda_{0}}, 1\right]$ a standard continuous dependence argument (cf. also [11]) ensures that $u$ and $u_{0}$ have the same numbers of zeros if $(u, \lambda)$ is in a sufficiently small neighborhood of $\left(u_{0}, \lambda_{0}\right)$. As a consequence, we obtain that there exists a neighborhood $U_{0}$ of $\left(u_{0}, \lambda_{0}\right)$ such that

$$
j(u, \lambda)=j\left(u_{0}, \lambda_{0}\right), \quad \forall(u, \lambda) \in U_{0}
$$

As a consequence, from Theorem 3.2 and Proposition 3.4 we deduce the final result:

Theorem 3.5. Assume (7) and (23). Then, for every eigenvalue $\lambda_{n}$ of $A$ there exists a continuum $C_{n}$ of nontrivial solutions of (22) in $H_{0}^{2}(0,1) \times \mathbb{R}$ bifurcating from $\left(0, \lambda_{n}\right)$ and such that condition (1) of Theorem 3.2 holds true and

$$
\begin{equation*}
j(u, \lambda)=n-1, \quad \forall(u, \lambda) \in C_{n} . \tag{32}
\end{equation*}
$$

Remark 3.6. Theorem 3.2 can be proved as an application of Stuart's result [15, Theorem 1.2] as well. However, since in the situation considered in this paper the singularity at zero does not affect the compactness of the operator $R$ defined after (27), we chose to apply Rabinowitz theorem [11]. We finally mention the interesting paper [1], where global branches of solutions, with prescribed nodal properties, are obtained for a second order degenerate problem in $(0,1)$.

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Authors' addresses:
Anna Capietto
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy
E-mail: anna.capietto@unito.it
Walter Dambrosio
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy
E-mail: walter.dambrosio@unito.it
Duccio Papini
Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche,
Università di Siena
Via Roma 56, 53100 Siena, Italy
E-mail: papini@dii.unisi.it


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