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Some Characterization Results for Permutation Algebras[⋆]

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Abstract

In recent years, many general presentations (*meta*models) for calculi with name-passing, either operational or denotational in flavour, have been proposed. In this paper, we investigate the connections among some of these proposals, namely *permutation algebras*, *named sets* and *sheaf categories*, with the aim of establishing a bridge between different approaches to the abstract specification of nominal calculi.

Keywords: Semantics of programming languages; name-passing calculi; categorical and algebraic metamodels of languages.

Introduction

Since the introduction of π-calculus, the notion of *name* has been recognized as central in models for concurrency, mobility, staged computation, metaprogramming, memory region allocation, etc. In recent years, several approaches have been proposed as general frameworks (*meta*models) for streamlining the development of these models featuring name passing and/or allocation.

One of the most common approaches is to consider *categories of functors* over the category $\mathbb{I}$ of finite sets and injective functions, such as presheaves $\mathbf{Set}^{\mathbb{I}}$; see e.g. Moggi, Stark, Hofmann, Fiore and Turi, among others [13, 16,

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8, 5]. Presheaves represent "staged computations", indexed by the (finite) sets of names currently allocated. In these categories, the classical results for definining initial algebra/final coalgebra can be extended to deal with names, and thus are well suited for interpreting specifications given by polynomial functors. A variation considers only a subcategory of *sheaves*, leading to models supporting classical logic; see e.g. Stark, Hofmann, and others [16,8,2].

An alternative approach, based on the Fraenkel-Mostowski permutation model of set theory with atoms (FM-sets), is proposed by Gabbay and Pitts [6]. A different theory of sets with permutations, *named sets*, has been introduced as a basis for an operational model of *History Dependent automata* [14]; while for the development of a structured coalgebraic theory in that formalism also *permutation algebras*, which are models of suitable algebraic specifications of permutations, have been considered by Ferrari, Montanari, Pistore [4].

It comes as no surprise that there are so many approaches: despite all ultimately cope with the same issues, they are inspired by different aims and perspective, leading to different solutions and choices. It is therefore important to investigate the relationships between these metamodels. First, this will point out similarities and differences between metamodels. Possibly, apparently peculiar idiosyncrasies are either justified, or revealed to be inessential. Moreover, these interconnections allow for transferring properties, techniques and constructions among metamodels, thus cross-fertilizing each other. In fact, this formal comparison allows for highlighting weak points of some metamodel, and possibly for suggesting improvements.

However, these approaches are not always easily comparable, also because they dwell in different meta-logical settings (category theory, (non-standard) set theory, algebraic specifications, automata theory…). So far we know that the model of FM-sets with finite support used in [6] is equivalent to the category of sheaves used in [8, 2], which, of course, is a full reflective subcategory of $\mathbf{Set}^{\mathbb{I}}$ used in [8,5]. However, the big picture is still incomplete, since the connections with other approaches, and in particular with those rooted on permutation algebras, are still unclear.

This is indeed the aim of this work: we study the connections between *permutation algebras*, *named sets* and *sheaf categories*. Permutation algebras are algebras over permutation signatures. Usually we are interested in permutation algebras whose elements are *finitely supported*—i.e., we rule out processes and terms with infinite free names at once. One of the results of this paper is that permutation algebras with finite support are equivalent to the category of pullback preserving functors $\mathbb{I} \to \mathbf{Set}$, i.e., the so-called *Schanuel topos*.[1]

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[1] A stronger result, stating the equivalence between the Schanuel topos and the category of permutation algebras over the signature containing only *finite kernel* premutations, has

On the other hand, named sets have been introduced as a building block of History Depended Automata, an operational model whose states are equipped with a set of names and name bijections. To some extent, named sets have been intended to be an implementation of permutation algebras. In this paper, we make this connection precise: It turns out that named sets form a category which is equivalent to the category of finite kernel algebras with finite support.

These results confirm that permutation algebras with finite support (and named sets) are a good metamodel for formalisms dealing with names, as much as $\mathrm{Sh}(\mathbb{I}^{op})$ and the FM-sets are.

Synopsis.

In Section 1 we recall the basic definitions about (finite kernel) permutations, permutation algebras, and finite support. In Section 2 we show that permutation algebras can be seen as *continuous G-sets*, and therefore that permutation algebras with finite support ultimately correspond to the Schanuel topos. In Section 3 we consider *named sets*, and we show that they also form a category which is equivalent to the category of finite kernel permutation algebras with finite support. Finally, some conclusions are drawn in Section 4.

1 Permutation algebras

This section recalls the main definitions on *permutation algebras:* They are mostly drawn from [14], with some additional references to the literature.

Definition 1.1 (permutation group) *Given a set A, a* permutation *on A is a bijective endofunction on A. The set of all such permutations is denoted by* $\mathrm{Aut}(A)$*, and it forms a group, called the* permutation group *of A, where the operation is function composition: For all $\pi_1, \pi_2 \in \mathrm{Aut}(A)$, $\pi_1\pi_2 \triangleq \pi_1 \circ \pi_2$.*

On sets, permutations coincide with *automorphisms* (because there is no structure to preserve), hence the notation denoting the permutation group. We stick however to permutations since now this is almost the standard usage in theoretical computer science, and it is the term used in our main references: See [14, Section 2.1] and the initial paragraphs of [6, Section 3].

Definition 1.2 (finite kernel permutations) *Let $\pi \in \mathrm{Aut}(A)$ be a permutation on A. The* kernel *of π is defined as $\ker(\pi) \triangleq \{a \in A \mid \pi(a) \neq a\}$. The set $\mathrm{Aut}^{fk}(A)$ of* finite kernel *permutations forms a subgroup of $\mathrm{Aut}(A)$.*

Let us now fix A as $\omega = \{0, 1, 2, \ldots\}$, the set of natural numbers. In the paper we will restrict our attention to permutations on ω, i.e., belonging to

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been omitted from this work due to lack of space; we refer then the reader to [7].

$\mathrm{Aut}(\omega)$, even if our definitions and remarks could apply in full generality.

Definition 1.3 (permutation signature and algebras) *The permutation signature* Σ_π *is given by the set of unary operators* $\{\widehat{\pi} \mid \pi \in \mathrm{Aut}(\omega)\}$, *together with the pair of axioms schemata* $\widehat{id}(x) = x$ *and* $\widehat{\pi}_1(\widehat{\pi}_2(x)) = \widehat{\pi_1\pi_2}(x)$.

A permutation algebra $\mathcal{A} = (A, \{\widehat{\pi}_A\})$ *is an algebra for* Σ_π. *A permutation morphism* $\sigma : \mathcal{A} \to \mathcal{B}$ *is an algebra morphism, i.e., a function* $\sigma : A \to B$ *such that* $\sigma(\widehat{\pi}_A(x)) = \widehat{\pi}_B(\sigma(x))$. *Finally,* $Alg(\Sigma_\pi)$ *(often shortened as* Alg_π*) denotes the category of permutation algebras and their morphisms.*

For example, the same set of permutations $\mathrm{Aut}(\omega)$ forms the carrier of a permutation algebra. An interesting example is given by the permutation algebra for the π-calculus: The carrier contains all the processes, up-to structural congruence, and the interpretation of a permutation is the associated name substitution (see also [14, Definition 15 and Section 3]).

We give now some additional definitions, concerning the *finite kernel* property, again drawn from [14, Section 2.1].

Definition 1.4 (algebras for finite kernel) *The finite kernel permutation signature* Σ_π^{fk} *is obtained as the subsignature of* Σ_π *restricted to those unary operators induced by finite kernel permutations.*

The associated category of algebras is $Alg(\Sigma_\pi^{fk})$*, shortened as* Alg_π^{fk}.

Of course, finite kernel does not imply finite carrier, since each algebra in Alg_π belongs also to Alg_π^{fk}, thus the former is a subcategory of the latter. However, Alg_π^{fk} has a countable set of operators and axioms, and thus it is more amenable to the standard results out of the algebraic specification mold.

An example of algebra with finite kernel and infinite carrier is the permutation algebra for the π-calculus with bound parallelism, i.e., limited to those recursive processes whose unfolding can generate only a finite number of names (see [14, Definition 46]).

We provide now a final list of definitions, concerning the *finite support* property. They rephrase those definitions in [14, Section 2.1], according to [6, Definition 3.3], and to our needs in the following sections.

Definition 1.5 (finite support algebras) *Let* $\mathcal{A}$ *be a permutation algebra, and let* $a \in A$. *We denote as* $fix_A(a)$ *the set of permutations fixing* a *in* $\mathcal{A}$, *i.e., those permutations* π *such that* $\widehat{\pi}_A(a) = a$.

Moreover, let $X \subseteq \omega$ *be a set. We denote as* $fix(X)$ *the set of permutations fixing* X *(i.e., those permutations* π *such that* $\pi(k) = k$ *for all* $k \in X$*), and we say that the set* X supports *the element* a *if all permutations fixing* X *also fix* a *in* $\mathcal{A}$ *(i.e., if* $fix(X) \subseteq fix_A(a)$*).*

An algebra $\mathcal{A}$ *is* finitely supported *if for each element of its carrier there* *exists a finite set supporting it. The category of all finitely supported algebras is denoted by* $FSAlg(\Sigma_\pi)$*, shortened as* $FSAlg_\pi$*.*

It is important to remark that not all the algebras in Alg_π are finitely supported (hence, neither those in Alg_π^{fk}). For example, let us consider the algebra $(A, \{\hat{\pi} \mid \pi \in \mathrm{Aut}^{fk}(\omega)\})$, where A contains id and the following permutation

$$\rho(i) = \begin{cases} i-1 & \text{if } i = 2k+1 \\ i+1 & \text{if } i = 2k \end{cases} = (1,0,3,2,5,4,7,6\dots)$$

and it is closed under precomposition with finite kernel permutations. Let $\hat{\pi}(\rho) \triangleq \pi\rho$: This algebra is in Alg_π, but it is not finitely supported. Indeed, for any $X \subset \omega$ finite, we can choose $\pi \in \mathrm{Aut}^{fk}(\omega)$ such that $\pi(x) = x$ for all $x \in X$, but which swaps $\max(X)+1$ and $\max(X)+2$; then $\hat{\pi}(\rho) = \pi\rho \neq \rho$.

In general, an element of the carrier of an algebra may have different sets supporting it. The following proposition, mirroring [6, Proposition 3.4], ensures that a minimal support does exist.

Proposition 1.6 *Let* $\mathcal{A}$ *be a permutation algebra, and let* $a \in A$*. If* a *is finitely supported, then there exists a least finite subset of* ω *supporting it.*

Given an algebra $\mathcal{A}$, and a finitely supported element $a \in A$, we call *support* of a the (necessarily unique) least subset supporting it, denoted by $supp_A(a)$.

It is easy to see that $fix_A(a)$ always forms a group. Furthermore, the permutations fixing an element have a strong link to its support. We tighten up this section with a technical lemma relating a simple result, which is needed later on, concerning permutations preserving the support.

Lemma 1.7 (preserving supports) *Let* $\mathcal{A}$ *be a permutation algebra, and let* $a \in A$ *be a finitely supported element. Moreover, let* $sp_A(a)$ *be the set of permutations preserving the support of* a *(i.e.,* $sp_A(a) \triangleq \{\pi \mid \pi(supp_A(a)) = supp_A(a)\}$*). Then,* $sp_A(a)$ *is a group and* $fix_A(a) \subseteq sp_A(a)$*.*

2 Permutation algebras and continuous G-sets

In this section we show that the categories of algebras Alg_π and Alg_π^{fk} considered above are strictly related to a well-known notion of algebraic topology, namely that of *(continuous) G-sets.* This will allow for taking advantage of a large and well-established theory, which will be used in the next section.

Definition 2.1 (G-sets) *Let* G *be a group. A* G-set *is a pair* $(X, \cdot_X)$ *where* X *is a set and* $\cdot_X : X \times G \to X$ *is a* right G-action*, that is*

$$x \cdot_X id = x \qquad (x \cdot_X g_1) \cdot_X g_2 = x \cdot_X (g_1 g_2)$$

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ *between* G*-sets is a function* $f : X \to Y$ *such that* $f(x \cdot_X g) = f(x) \cdot_Y g$ *for all* $x \in X$*.*

The G*-sets and their morphisms form a category denoted by* $\mathbf{B}G^\delta$*.*

Let us consider the G-sets when G is either $\mathrm{Aut}(\omega)$ or $\mathrm{Aut}^{\mathit{fk}}(\omega)$. Clearly, every $\mathrm{Aut}(\omega)$-set is also a $\mathrm{Aut}^{\mathit{fk}}(\omega)$-set (just by restricting the action to the finite kernel permutations), mimicking the correspondence between Alg_π and Alg_π^{fk}. In fact, a stronger equivalence holds between the formalisms, as it is put in evidence by the result proved below.

Proposition 2.2 $Alg_\pi \cong \mathbf{B}\mathrm{Aut}(\omega)^\delta$ *and* $Alg_\pi^{\mathit{fk}} \cong \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega)^\delta$*.*

Proof. Let $\mathcal{A}$ a permutation algebra. We define a corresponding $\mathrm{Aut}(\omega)$-set $G(\mathcal{A}) = (A, \cdot_{G(A)})$ where $a \cdot_{G(A)} \pi \triangleq \widehat{\pi}_A(a)$ for all $a \in A$. On the other hand, if $(X, \cdot_X)$ is a $\mathrm{Aut}(\omega)$-set, the corresponding algebra $\mathcal{X} = (X, \{\pi_X\})$ is defined by taking $\widehat{\pi}_X(x) \triangleq x \cdot_X \pi$ for $\pi \in \mathrm{Aut}(\omega)$.

Let $\mathcal{A}, \mathcal{B}$ be two permutation algebras. A function $f : A \to B$ is a morphism $f : \mathcal{A} \to \mathcal{B}$ in Alg_π iff $f(\widehat{\pi}_A(a)) = \widehat{\pi}_B(f(a))$ for all permutations π and $a \in A$, which in turn holds iff $f(a \cdot_{G(A)} \pi) = f(a) \cdot_{G(B)} \pi$ for all π and a, which equivalently states that $f : (A, \cdot_{G(A)}) \to (B, \cdot_{G(B)})$ is a morphism in $\mathbf{B}\mathrm{Aut}(\omega)^\delta$. Clearly, this correspondence is full and faithful, hence the thesis.

Using the same argument, we have also that $Alg_\pi^{\mathit{fk}} \cong \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega)^\delta$. ☐

Also the subcategory of algebras with finite support (possibly over only finite kernel permutations) can be recasted in the more general setting of G-sets, but to this end we need to recall some notions from topology theory; see e.g. [10] for a presentation of these concepts in the context of general topology, and [11, Section V.9] and [12, II] in the context of category and topos theory.

Definition 2.3 (topological spaces) *A* topological space *is a pair* $(X, \mathcal{O}(X))$ *for* X *a set and* $\mathcal{O}(X) \subseteq \wp(X)$ *(the* topology *over* X*) is closed with respect to arbitrary union and finite intersection, and* $\emptyset, X \in \mathcal{O}(X)$*.*

A function $f : X \to Y$ *is a* continuous map $f : (X, \mathcal{O}(X)) \to (Y, \mathcal{O}(Y))$ *if* $f^{-1}(U) \in \mathcal{O}(X)$ *for all* $U \in \mathcal{O}(Y)$*.*

Topological spaces and continuous maps form a category, denoted **top**.

The elements of $\mathcal{O}(X)$ are referred to as the *open sets* of the topology.

Example 2.4 The smallest (i.e., coarsest) topology is $\mathcal{O}(X) = \{\emptyset, X\}$. On the other hand, the finest topology is the *discrete topology*, where $\mathcal{O}(X) = \wp(X)$. It is easy to prove that a topology is discrete if and only if $\{x\} \in \mathcal{O}(X)$ for all $x \in X$, i.e., if every point is separated from the others (hence the name). Clearly, every function is continuous with respect to the discrete topology.

Remark 2.5 (product of spaces) *The category* **top** *is complete and co-complete [11, Section V.9]. In particular, given a family of topological spaces* $(X_i, \mathcal{O}(X_i)) \in$ **top**, *indexed by* $i \in I$, *the product* $\prod_{i \in I}(X_i, \mathcal{O}(X_i))$ *is the topological space whose space is* $X = \prod_{i \in I} X_i$, *and the topology is the smallest topology such that the projections* $\pi_i : X \to X_i$ *are continuous. If* I *is finite, then* $\mathcal{O}(X) = \prod_{i \in I} \mathcal{O}(X_i)$. *This property does not hold however for* I *infinite.*

We recall now a last standard definition, which generalizes Definition 2.1.

Definition 2.6 (topological groups and continuous G-sets) *A group* G *is a* topological group *if its carrier is equipped with a topology, and its multiplication and inverse are continuous with respect to this topology.*

A G*-set* $(X, \cdot_X)$ *is* continuous *if* G *is topological and the action* $\cdot_X : X \times G \to G$ *is continuous with respect to* X *equipped with the discrete topology.*

A morphism $f : (X, \cdot_X) \to (Y, \cdot_Y)$ *between continuous* G*-sets is a function* $f : X \to Y$ *which respects the actions.*

For a given topological group G, *continuous* G*-sets and their morphisms form a category, denoted by* $\mathbf{B}G$.

Notice that for any group G, the category of *all* G-sets is the category of continuous G-sets where G is taken with the discrete topology – hence the notation $\mathbf{B}G^{\delta}$ from [12] we have used in Definition 2.1.

Remark 2.7 (permutation groups as topological spaces) *Let us consider the space* $\mathbf{N}$, *given as the set* ω *of natural numbers equipped with the discrete topology. The* Baire space *is the topological space* $\prod_{i=0}^{\infty} \mathbf{N} = \mathbf{N}^{\omega}$, *equipped with the infinite product topology. A base of this topology is given by the sets of the form* $\prod_{i=0}^{\infty} A_i$, *where* $A_i \neq \omega$ *only for* finitely many *indexes* i.

Let us now consider the groups $\mathrm{Aut}(\omega)$ *and* $\mathrm{Aut}^{fk}(\omega)$. *As described in [12, Section III.9] for* $\mathrm{Aut}(\omega)$, *the carriers of these groups can be seen as subspaces of the Baire space, where each* π *corresponds to the infinite list* $(\pi(0), \pi(1), \pi(2), \ldots)$. *Now, both* $\mathrm{Aut}(\omega)$ *and* $\mathrm{Aut}^{fk}(\omega)$ *inherit a topology from* $\mathbf{N}^{\omega}$*: Their open sets are of the form* $U \cap \mathrm{Aut}(\omega)$ *and* $U \cap \mathrm{Aut}^{fk}(\omega)$, *for* U *open set of* $\mathbf{N}^{\omega}$. *We can therefore consider the categories* $\mathbf{B}\mathrm{Aut}(\omega)$ *and* $\mathbf{B}\mathrm{Aut}^{fk}(\omega)$ *of continuous* $\mathrm{Aut}(\omega)$*-sets and continuous* $\mathrm{Aut}^{fk}(\omega)$*-sets, respectively.*

We are now ready to prove our first main result, namely, the correspondence between continuous G-sets and permutation algebras with finite support. We first state a technical lemma [12, I, Exercise 6].

Lemma 2.8 *Let* $(X, \cdot_X)$ *be a* G*-set, and for each* $x \in X$ *let* $I_x \triangleq \{g \in G \mid x \cdot_X g = x\}$ *be denoted the* isotropy group *of* x. *Then,* $(X, \cdot_X)$ *is continuous iff all its isotropy groups are open sets in* G.

Theorem 2.9 $\mathit{FSAlg}_{\pi} \cong \mathbf{B}\mathrm{Aut}(\omega)$ *and* $\mathit{FSAlg}_{\pi}^{fk} \cong \mathbf{B}\mathrm{Aut}^{fk}(\omega)$.

Proof. In order to prove $\mathit{FSAlg}^{\mathit{fk}}_{\pi} \cong \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega)$, we show that the functor G of Proposition 2.2 maps algebras with finite support and finite kernel to continuous $\mathrm{Aut}^{\mathit{fk}}(\omega)$-sets, and *vice versa.*

Let $\mathcal{A} = (A, \{\hat{\pi}_A\})$ be an algebra in $\mathit{FSAlg}^{\mathit{fk}}_{\pi}$; the corresponding $\mathrm{Aut}^{\mathit{fk}}(\omega)$-set is $(A, \cdot_{G(A)})$, where $a \cdot_{G(A)} \pi \triangleq \hat{\pi}_A(a)$ for all $a \in A$. For Lemma 2.8, $G(\mathcal{A})$ is continuous if and only if for all $a \in A$, I_a is open. Indeed:

$$\begin{aligned}
I_a &= \bigcup_{\pi \in I_a} \prod_{i=0}^{\infty} \{\pi(i)\} \\
&= \bigcup_{\pi \in I_a} \left(\prod_{i=0}^{\infty} A_i^{\pi} \right) \cap \mathrm{Aut}^{\mathit{fk}}(\omega) \qquad \text{where } A_i^{\pi} \triangleq \begin{cases} \{\pi(i)\} & \text{if } i \in \mathit{supp}(a) \\ \omega & \text{otherwise} \end{cases} \\
&= \left(\bigcup_{\pi \in I_a} \prod_{i=0}^{\infty} A_i^{\pi} \right) \cap \mathrm{Aut}^{\mathit{fk}}(\omega) \qquad (1)
\end{aligned}$$

which is open in $\mathrm{Aut}^{\mathit{fk}}(\omega)$ because each $\prod_{i=0}^{\infty} A_i^{\pi}$ is open in $\mathbf{N}^{\omega}$ since $\mathit{supp}_A(a)$ is finite and thus only finitely many A_i^{π}'s are different from ω (see Remark 2.7).[2]

On the other hand, let $(X, \cdot_X)$ be a continuous $\mathrm{Aut}^{\mathit{fk}}(\omega)$-set; we prove that $\mathcal{X} = (X, \{\widehat{\pi}_X\})$ is in $\mathit{FSAlg}^{\mathit{fk}}_{\pi}$. Clearly $\mathcal{X}$ is a finite kernel permutation algebra. By Lemma 2.8, for any $x \in X$, I_x is an open set of $\mathrm{Aut}^{\mathit{fk}}(\omega)$, hence $I_x = U \cap \mathrm{Aut}^{\mathit{fk}}(\omega)$ for some U open set of $\mathbf{N}^{\omega}$. More explicitly, I_x can be written as follows

$$I_x = \left(\bigcup_{i \in I} \prod_{j=0}^{\infty} X_{ij} \right) \cap \mathrm{Aut}^{\mathit{fk}}(\omega)$$

for some family of indexes I, and where for each $i \in I$ there exists a finite $J_i \subset \omega$ such that $X_{ij} \neq \omega$ only for $j \in J_i$. Since $id \in I_x$ (it is a group), there exists $i_0 \in I$ such that $id \in \prod_{j=0}^{\infty} X_{i_0 j}$. We prove that the finite set $J \triangleq J_{i_0}$ supports x. Let $\pi \in \mathrm{Aut}^{\mathit{fk}}(\omega)$ fixing J. For all $j \in \omega$, if $j \in J$ then $\pi(j) = j \in X_{i_0 j}$, otherwise $X_{i_0 j} = \omega$. In both cases, $\pi(j) \in X_{i_0 j}$. So $\pi \in \prod_{j=0}^{\infty} X_{i_0 j}$, and therefore $\pi \in I_x$, i.e. $\hat{\pi}(x) = x \cdot_X \pi = x$, hence the thesis.

For proving $\mathit{FSAlg} \cong \mathbf{B}\mathrm{Aut}(\omega)$ we can reply the argument above, just replacing $\mathrm{Aut}^{\mathit{fk}}(\omega)$ with $\mathrm{Aut}(\omega)$. □

Using this result, we can take advantage of a well-established theory about continuous G-sets for proving properties about categories of permutation algebras with finite support. In particular, we establish easily a connection with

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[2] We can prove the equivalence (1) also directly. Obviously, $I_a \subseteq \left(\bigcup_{\pi \in I_a} \prod_{i=0}^{\infty} A_i^{\pi}\right) \cap \mathrm{Aut}^{\mathit{fk}}(\omega)$. Let $\pi \in \left(\bigcup_{\pi \in I_a} \prod_{i=0}^{\infty} A_i^{\pi}\right) \cap \mathrm{Aut}^{\mathit{fk}}(\omega)$; then, there exists $\rho \in I_a$ such that for all $i \in \mathit{supp}_A(a) : \pi(i) = \rho(i)$. Since $\rho(a) = a$, also $\pi(a) = a$, thus $\pi \in I_a$.

presheaf categories. Recall that the category of *presheaves* over a small category $\mathbf{C}$ is the category of functors $\mathbf{Set}^{\mathbf{C}^{op}}$, and natural transformations among them. Many authors used the category $\mathbf{Set}^{\mathbb{I}}$, where $\mathbb{I}$ is the category of finite subsets of ω and *injective* maps, for modeling the computational notion of dynamic allocation of names or locations; see e.g. [13,16,8,5]. For our purposes, we consider a particular subcategory of $\mathbf{Set}^{\mathbb{I}}$, namely the category of *sheaves with respect to the atomic topology*, denoted by $\mathrm{Sh}(\mathbb{I}^{op})$. This subcategory is conveniently characterized by the property below [9, Example 2.1.11(h)].

Proposition 2.10 $\mathrm{Sh}(\mathbb{I}^{op})$ *is the full subcategory of* $\mathbf{Set}^{\mathbb{I}}$ *of pullback preserving functors.*

The category $\mathrm{Sh}(\mathbb{I}^{op})$ is often called the *Schanuel topos.* It features the same important properties of $\mathbf{Set}^{\mathbb{I}}$ above: It is a topos (and hence it is cartesian closed), the functor $N = \mathbf{y}(\underline{1})$ is a sheaf, and initial algebras and final coalgebras of polynomial functors are pullback preserving. Therefore, $\mathrm{Sh}(\mathbb{I}^{op})$ can be used in place of $\mathbf{Set}^{\mathbb{I}}$ for giving the semantics of languages with dynamic name allocations, as in [16, 17, 8, 2] and ultimately also in [6] (being the Fraenkel-Mostowsky set theory essentially equivalent to $\mathrm{Sh}(\mathbb{I}^{op})$). The main difference between $\mathbf{Set}^{\mathbb{I}}$ and $\mathrm{Sh}(\mathbb{I}^{op})$ is that the latter is a Boolean topos [12, Section III.8, p. 150], while the former is not. Hence, $\mathrm{Sh}(\mathbb{I}^{op})$ is a model for *classical* logic, instead of the usual intuitionistic (extensional) higher order logic of topoi.

Now, for [12, Section III.9, Corollary 3], we know that $\mathbf{B}\mathrm{Aut}(\omega) \cong \mathrm{Sh}(\mathbb{I}^{op})$. Thus, as an immediate corollary of Theorem 2.9 we have that the category of permutation algebras with finite support, is equivalent to the Schanuel topos.

Corollary 2.11 $\mathit{FSAlg}_\pi \cong \mathrm{Sh}(\mathbb{I}^{op})$.

In other words, permutation algebras with finite support correspond to pullback-preserving functors $\mathbb{I} \to \mathbf{Set}$, and thus they form a Boolean topos with enough structure for defining the semantics of languages with dynamic name allocation, such as π-calculus, mobile ambients, etc. [3]

We summarize the relationships we have established among permutation algebras and G-sets in the diagram in Figure 1.

It is interesting to notice that the inclusion functors $\mathbf{B}\mathrm{Aut}(\omega) \hookrightarrow \mathbf{B}\mathrm{Aut}(\omega)^\delta$ and $\mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega) \hookrightarrow \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega)^\delta$ have right adjoints; the latter is e.g. defined on the objects as follows

$$r : \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega)^\delta \to \mathbf{B}\mathrm{Aut}^{\mathit{fk}}(\omega) \qquad (X, \cdot_X) \mapsto (\{x \in X \mid I_x \text{ open for } \mathrm{Aut}^{\mathit{fk}}(\omega)\}, \cdot_X)$$

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[3] Actually, we can prove also the stronger equivalence $\mathit{FSAlg}_\pi^{\mathit{fk}} \cong \mathrm{Sh}(\mathbb{I}^{op})$. The proof of this fact is quite technical and lengthy, so due to lack of space we refer the reader to [7].

$$\begin{tikzcd}
Alg_{\pi} \arrow[rr, hook] \arrow[dr, "\cong"] & & Alg_{\pi}^{fk} \arrow[dr, "\cong"] & \\
& \mathbf{B}\mathrm{Aut}(\omega)^{\delta} \arrow[rr, hook] \arrow[dd, shift left] \arrow[dd, phantom, "\dashv"] & & \mathbf{B}\mathrm{Aut}^{fk}(\omega)^{\delta} \arrow[dd, shift left] \arrow[dd, phantom, "\dashv"] \\
FSAlg_{\pi} \arrow[uu, hook] \arrow[rr, hook] \arrow[dr, "\cong"] & & FSAlg_{\pi}^{fk} \arrow[uu, hook] \arrow[dr, "\cong"] & \\
& \mathbf{B}\mathrm{Aut}(\omega) \cong \mathrm{Sh}(\mathbb{I}^{op}) \arrow[rr, hook] \arrow[uu, hook, shift left] & & \mathbf{B}\mathrm{Aut}^{fk}(\omega) \arrow[uu, hook, shift left]
\end{tikzcd}$$

Fig. 1. The Permutation Algebra Cube, first version.

and it is the restriction on morphisms. Therefore, r maps every $\mathbf{B}\mathrm{Aut}^{fk}(\omega)^{\delta}$-set to the largest continuous $\mathbf{B}\mathrm{Aut}^{fk}(\omega)$-set contained in it. Translating r to permutation algebras along the equivalences, this means that there exists

$$r' : Alg_{\pi}^{fk} \to FSAlg_{\pi}^{fk} \qquad (A, \{\hat{\pi}_A\}) \mapsto (B, \{\hat{\pi}_A|_B\})$$

where $B \triangleq \{a \in A \mid fix_A(a) \text{ open for } \mathrm{Aut}^{fk}(\omega)\}$. Now, $fix_A(a)$ is open if there exists a finite $J \subset \omega$ such that for $fix(J) \cap \mathrm{Aut}^{fk}(\omega) \subseteq fix_A(a)$ (see the proof of Theorem 2.9). This corresponds exactly to say that a has finite support, hence we can define directly $r'(A) = \{a \in A \mid supp_A(a) \text{ finite}\}$.

3 Permutation algebras and named sets

Named sets are the building blocks of HD-automata, the implementation counterpart of permutation algebras. The definitions below are lifted from [4, Section 3.1], and simplified according to our needs.

Definition 3.1 (named sets) *A* named set N *is a triple*

$$N = \langle Q_N, \|\cdot\|_N : Q_N \to \omega, G_N : \textstyle\prod_{q \in Q_N} \wp(\mathrm{Aut}(\|q\|_N))\rangle$$

where Q_N *is a set of* states*;* $\|\cdot\|_N$ *is the* enumerating *function; and for all* $q \in Q_N$*, the set* $G_N(q)$ *is a subgroup of* $\mathrm{Aut}(\|q\|_N)$ *(hence, closed with respect to inverse and identity), and it is called the* permutation group *of* q*.*

In this definition, and also in the following, we adopt the usual "set-theoretic" convention of representing finite ordinals by natural numbers, thus $0 = \emptyset$ and $n = \{0, \ldots, n-1\}$. Therefore, $\mathrm{Aut}(\|q\|_N) = \mathrm{Aut}(\{0, \ldots, \|q\|_N - 1\})$.

Intuitively, a state in Q_N represents a process, and thus the function $\|\cdot\|_N$ assigns to each state the number of variables possibly occurring free in it; in other words, it denotes a canonical choice of its free variables. Finally, G_N

denotes for each state the group of renamings under which it is preserved, i.e., those permutations on names that do not interfere with its possible behavior. Note also that $G_N(q) = \{id\}$ if $\|q\|_N = 0$.

Definition 3.2 (category of named sets) *Let N, M be named sets. A* named function $H : N \to M$ *is a pair*

$$H = \langle h : Q_N \to Q_M, \Lambda_h : \textstyle\prod_{q \in Q_N} \wp(\mathbb{I}(\|h(q)\|_M, \|q\|_N))\rangle$$

for h a function and $\Lambda_h(q)$ a set of injections from $\|h(q)\|_M$ to $\|q\|_N$, satisfying the additional condition

$$G_N(q) \circ \lambda \subseteq \Lambda_h(q) = \lambda \circ G_M(h(q)) \quad \forall \lambda \in \Lambda_h(q)$$

Finally, **NSet** *denotes the category of named sets and their morphisms.*

So, a named function is a state function, equipped with a set of injective renamings for each $q \in Q_N$, which are somewhat compatible with the permutations in $G_N(q)$ and $G_M(h(q))$ (and such that $\lambda_h(q) = \emptyset$ if $\|h(q)\|_M = 0$). In other words, "the whole set of $\Lambda_h(q)$ must be generated by saturating any of its elements by the permutation group of $h(q)$, and the result must be invariant with respect to the permutation group of q" [4, Section 3.1]. In particular, the identity on N is $\langle id, \mathrm{Aut}(\| \cdot \|_N)\rangle$, and composition is defined as expected.

Example 3.3 Let us consider a few simple examples. Since $1 = \{0\}$ is the singleton set, both $N_1 = \langle 1, \|0\| = 1, \mathrm{Aut}(1) = \{id\}\rangle$ and $N_2^p = \langle 1, \|0\| = 2, \mathrm{Aut}(2) = \{id, (1, 0)\}\rangle$ are named sets: same set of states, different enumerating functions. Instead, $N_2^i = \langle 1, \|0\| = 2, \{id\} \subseteq \mathrm{Aut}(2)\rangle$ is a named set with the same set of states and the same enumerating function of N_2^p, but with a different permutation group.

Notice that there is no named function from N_2^p to N_1, since any injection λ, when post-composed with $\mathrm{Aut}(2)$, generates the whole $\mathbb{I}(1, 2)$. Instead, denoting by I_j, for $j = 0, 1$, the set containing the injection mapping 0 to j, then $\langle id, I_j\rangle$ is a named function from N_2^i to N_1, while $\langle id, I_0 \cup I_1 = \mathbb{I}(1, 2)\rangle$ is not.

Similarly, there is no named function from N_2^p to N_2^i, while $\langle id, \mathrm{Aut}(2)\rangle : N_2^i \to N_2^p$ (and it does not exist for any other choice of the set of injections $\Lambda(1) \subseteq \mathrm{Aut}(2)$). In fact, it is easy to see that, given named sets $\langle Q, \| \cdot \|, G_1\rangle$ and $\langle Q, \| \cdot \|, G_2\rangle$ (i.e., same state set and enumerating function, different permutation groups), with $G_1(q)$ a subgroup of $G_2(q)$ for all $q \in Q$, then $\langle id, G_2\rangle$ is a well-defined named function from the former named set to the latter.

In the remaining of this section we relate $FSAlg_\pi^{fk}$, the category of finitely supported, finite kernel permutation algebras and their morphisms, and **NSet**, the category of named sets. We plan to sharpen and make more concise some of the results presented in [14, Section 6].

Summarizing, Proposition 3.4 and Proposition 3.7 (and the "canonical" version of the latter, Proposition 3.13: See later) prove the existence of suitable functors between the underlying categories, generalizing the functions on objects presented as Definition 49 and Definition 50, respectively, in [14, Section 6]; while Theorem 3.14 extends to a categorical equivalence the correspondence on objects proved in Theorem 51 of the same paper.

3.1 From named sets to permutation algebras

The functor from named sets to permutation algebras is obtained by a free construction, (apparently) analogous to the standard correspondence between sets and algebras. We need to introduce some notation. For $\pi \in \mathrm{Aut}(n)$ and $\pi' \in \mathrm{Aut}^{fk}(\omega)$, for $n \in \omega$, let us denote by $[\pi, \pi'] \in \mathrm{Aut}^{fk}(\omega)$ the *completion* of π with π', defined as $[\pi, \pi'](i) \triangleq \begin{cases} \pi(i) & \text{if } i < n \\ \pi'(i-n)+n & \text{otherwise} \end{cases}$

Proposition 3.4 (from sets to algebras) *Let F_O be the function mapping each named set N to the finite kernel permutation algebra freely generated from the elements of Q_N (considered as new constants), modulo the equivalence $\equiv_N$ induced by set of axioms associated to the permutations in G_N, that is, $\widehat{[\pi, \pi']}_{F(N)}(q) \equiv_N q$ (i.e., a suitable completion of π) if $\pi \in G_N(q)$.*

Moreover, given a named function $H : N \to M$, for each $q \in Q_N$ let us choose an injection $\lambda_q \in \Lambda_h(q)$, and a permutation $\widehat{\lambda_q} \in \mathrm{Aut}(\|q\|_N)$ extending λ_q. Let us denote by $H_\lambda : Q_N \to Q_M$ the function $H_\lambda(q) = [\widehat{\lambda_q}, id](h(q))$ for all $q \in Q$. Then, let F_A be the function associating to each named function H the free extension of the function H_λ.

The pair $F = \langle F_O, F_A \rangle$ defines a functor from **NSet** *to $FSAlg_\pi^{fk}$.*

Proof. The carrier of $F_O(N)$ is $\{\pi(q) \mid q \in Q_N, \pi \in \mathrm{Aut}^{fk}(\omega)\}/{\equiv_N}$. Thus, it is easy to see that the resulting algebra has finite support, proving that each element $[\pi(q)]_N$ is supported by the set $\pi(\{0, \dots, \|q\|_N - 1\})$. In order to prove this, we must show that each permutation π' fixing $\pi(\{0, \dots, \|q\|_N - 1\})$ also fixes $\hat{\pi}_{F(N)}(q)$. Then we have that

$$\begin{aligned}
\forall k' \in \pi(\|q\|_N) : \pi'(k') = k' \implies & \quad \forall k < \|q\|_N : \pi'(\pi(k)) = \pi(k) \\
\implies & \forall k < \|q\|_N : \pi^{-1}(\pi'(\pi(k))) = k \\
\implies & \quad \widehat{(\pi^{-1}\pi'\pi)}_{F(N)}(q) \equiv_N q \\
\implies & \quad \widehat{\pi}^{-1}_{F(N)}(\widehat{\pi}'_{F(N)}(\widehat{\pi}_{F(N)}(q))) \equiv_N q \\
\implies & \quad \widehat{\pi}'_{F(N)}(\widehat{\pi}_{F(N)}(q)) \equiv_N \widehat{\pi}_{F(N)}(q)
\end{aligned}$$

Let us now consider a named set function $H : N \to M$. The function H_λ can be lifted to an algebra homomorphism from the free algebra $T_{\Sigma_\pi^{fk}}(Q_N)$ to the free algebra $T_{\Sigma_\pi^{fk}}(Q_M)$. Moreover, it preserves the axioms on identity and composition: We must then prove that this holds also for the additional axioms arisen from the permutation group. This is equivalent to prove that $H_\lambda([\pi, \pi']_{F(N)}(q)) \equiv_M H_\lambda(q)$ for all $\pi \in G_N(q)$. By construction, we have that $H_\lambda([\pi, \pi']_{F(N)}(q)) \triangleq [\pi, \pi']_{F(\mathcal{M})}([\widehat{\lambda}_a, id]_{F(M)}(h(q)))$. Now, remember that there exists a $\overline{\pi} \in G_M(h(q))$ such that $\pi \circ \lambda_a = \lambda_a \circ \overline{\pi}$, and then that for a suitable $\overline{\pi}'$ we have $[\pi, \pi'] \circ [\widehat{\lambda}_a, id] = [\widehat{\lambda}_a, id] \circ [\overline{\pi}, \overline{\pi}']$: This implies that $H_\lambda([\pi, \pi']_{F(N)}(q))$ coincides with $[\widehat{\lambda}_a, id]_{F(M)}([\overline{\pi}, \overline{\pi}']_{F(\mathcal{M})}(h(q)))$, which is equivalent to $[\widehat{\lambda}_a, id]_{F(M)}(h(q))$, hence the result.

The identities $\langle id, G_N \rangle$ are clearly preserved. Concerning composition, it is enough to show that the result of the functor is independent with respect to the choice of the injection, i.e, that given a named function $H : N \to M$, then for any $\lambda, \lambda' \in \Lambda_h(q)$ the equality $[\widehat{\lambda}, id](h(q)) \equiv_M [\widehat{\lambda'}, id](h(q))$ holds. To prove the latter, note that the conditions on $\Lambda_H(q)$ ensure on the existence of a permutation $\pi \in G_M(h(q))$ such that $\lambda \circ \pi = \lambda'$, hence the equality follows.□

3.2 From permutation algebras to named sets

We first define some additional structure on supports.

Definition 3.5 (on finite supports) *Let $\mathcal{A}$ be a permutation algebra, and let $a \in A$ be a finitely supported element. Then, $norm_A(a) \in \mathbb{I}(|supp_A(a)|, \omega)$ denotes the (necessarily unique) order-preserving injection covering the support. Formally, $norm_A(a)(i) < norm_A(a)(i+1)$ for all $i < |supp_A(a)|$ and $norm_A(a)(\{0, \ldots, |supp_A(a)| - 1\}) = supp_A(a)$.*

Now an easy technical lemma, relating the support of two algebras.

Lemma 3.6 (mapping supports) *Let $\sigma : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism, and let $a \in A$ be finitely supported. Then, $supp_B(\sigma(a)) \subseteq supp_A(a)$.*

Proof. Let us prove that any $K \subseteq \omega$ supporting $a \in A$, supports also $\sigma(a) \in B$. Let $\pi \in \mathrm{Aut}(\omega)$ such that for all $i \in K : \pi(i) = i$. Then, by hypothesis $\hat{\pi}_A(a) = a$, and hence $\hat{\pi}_B(\sigma(a)) = \sigma(\hat{\pi}_A(a)) = \sigma(a)$. □

In other words, the lemma above implies that for each morphism σ the element $\sigma(a)$ is finitely supported if a is; and it allows for defining a functor I from finite kernel permutation algebras to named sets.

Proposition 3.7 (from algebras to sets) *Let I_O be the function mapping each $\mathcal{A} \in FSAlg_\pi^{fk}$ to the named set $\langle A, |supp_A(\cdot)|, G_{I(\mathcal{A})} \rangle$, for $G_{I(\mathcal{A})}(a)$ the set* *of permutations given by*

$$\{\pi \in \mathrm{Aut}(|supp_A(a)|) \mid \exists \pi' \in fix_A(a) : \; norm_A(a) \circ \pi = \pi' \circ norm_A(a)\}.$$

Let $\sigma : \mathcal{A} \to \mathcal{B}$*, and let* $in_\sigma(a) : |supp_B(\sigma(a))| \to |supp_A(a)|$ *the uniquely induced arrow (thanks to Lemma 3.6) such that* $norm_A(a) \circ in_\sigma(a) = norm_B(\sigma(a))$*. Hence, let* I_A *be the function associating to* σ *the named function* $\langle h_\sigma, \Lambda_\sigma \rangle$ *given by the obvious function from* A *to* B *and by the set of injections* $\Lambda_\sigma(a) = in_\sigma(a) \circ G_{I(\mathcal{B})}(\sigma(a))$ *for all* $a \in A$*.*

The pair $I = \langle I_O, I_A \rangle$ *defines a functor from* $FSAlg_\pi^{fk}$ *to* **NSet***.*

Proof. It is easy to check that $G_{I(\mathcal{A})}(a)$ is a group, since $fix_A(a)$ is so.

Concerning Λ_σ, it is clear that the condition $\lambda \circ G_{I(\mathcal{B})}(\sigma(a)) = \Lambda_\sigma(a)$ holds for all $\lambda \in \Lambda_\sigma(a)$, since λ is of the shape $in_A(a) \circ \pi$, for $\pi \in G_{I(\mathcal{B})}(\sigma(a))$.

We must now prove that $G_{I(\mathcal{A})}(a) \circ \lambda \subseteq \Lambda_\sigma(a)$ for all $\lambda \in \Lambda_\sigma(a)$. This is equivalent to ask that for all $\pi \in G_{I(\mathcal{A})}(a)$ there exists a $\overline{\pi} \in G_{I(\mathcal{B})}(\sigma(a))$ such that $\pi \circ in_\sigma(a) = in_\sigma(a) \circ \overline{\pi}$. By definition we have $norm_A(a) \circ in_\sigma(a) = norm_B(\sigma(a))$, so that for $\pi' \in fix_A(a)$ corresponding to π, we have

$$\begin{aligned} norm_A(a) \circ in_\sigma(a) = norm_B(\sigma(a)) \implies & \\ \implies \pi' \circ norm_A(a) \circ in_\sigma(a) &= \pi' \circ norm_B(\sigma(a)) \\ \implies norm_A(a) \circ \pi \circ in_\sigma(a) &= \pi' \circ norm_B(\sigma(a)) \end{aligned}$$

since $fix_A(a) \subseteq fix_B(\sigma(a)) \subseteq sp_B(\sigma(a))$, there exists $\overline{\pi} \in \mathrm{Aut}(|supp_B(\sigma(a))|)$

$$\begin{aligned} \implies norm_A(a) \circ \pi \circ in_\sigma(a) &= norm_B(\sigma(a)) \circ \overline{\pi} \\ \implies norm_A(a) \circ \pi \circ in_\sigma(a) &= norm_A(a) \circ in_\sigma(a) \circ \overline{\pi} \end{aligned}$$

and finally, since $norm_A(a)$ is injective,

$$\implies \pi \circ in_\sigma(a) = in_\sigma(a) \circ \overline{\pi}$$

The identities are clearly preserved. Concerning composition, note that the choice of arrow $in\_$ is preserved by it, in the sense that $in_\sigma(a) \circ in_{\sigma'}(\sigma(a))$ coincides with $in_{\sigma;\sigma'}(\sigma'(\sigma(a)))$. Then, we have that

$$\begin{aligned} \Lambda_{\sigma;\sigma'}(a) = \quad & in_{\sigma;\sigma'}(a) \circ G_{\mathcal{C}}(\sigma'(\sigma(a))) = in_\sigma(a) \circ in_{\sigma'}(\sigma(a)) \circ G_{\mathcal{C}}(\sigma'(\sigma(a))) \\ = \quad & in_\sigma(a) \circ G_{\mathcal{B}}(\sigma(a)) \circ in_{\sigma'}(\sigma(a)) \circ G_{\mathcal{C}}(\sigma'(\sigma(a))) = \Lambda_\sigma(a) \circ \Lambda_{\sigma'}(\sigma(a)) \end{aligned}$$

and thus compositionality holds. □

3.3 Adjunction between named sets and permutation algebras

We first give a look at the structure of the algebras obtained *via* functor F.

Lemma 3.8 *Let N be a named set, and let $q \in Q_N$. Then, the equivalence class $[q]_{\equiv_N}$ is finitely supported, and $supp_{F(N)}([q]_{\equiv_N}) = \{0, \ldots, \|q\|_N - 1\}$; furthermore, $fix_{F(\mathcal{N})}([q]_{\equiv_N}) = \{[\pi, \pi'] \mid \pi \in G_N(q), \pi' \in \mathrm{Aut}^{fk}(\omega)\}$.*

Proof. Clearly, each permutation $\pi \in fix(\{0, \ldots, \|q\|_N - 1\})$ fixes $[q]_{\equiv_N}$, since it can be written as $[id, \pi']$ (see also the proof of Proposition 3.4). Now, let us assume a $k < \|q\|_N$ such that $k \notin supp_{F(N)}([q]_N)$, and let π_k be the permutation exchanging k with $\|q\|_N + 1$, and fixing the rest. Now, we have $\pi_k(q) = q$ in $F_O(N)$, but the equivalence can not be obtained by $\equiv_N$, since the latter is generated by the permutations in $G_N(q)$. This proves the first half.

Now, let us consider $\pi \in fix_{F(N)}([q]_N)$. Then, $\pi \in sp_{F(N)}([q]_N)$, so that it is of the shape $\pi = [\pi_s, \pi']$ for $\pi_s \in Aut(\|q\|_N)$. As for before, since $G_N(q)$ is a group, it follows that $\pi_s \in G_N(q)$. $\square$

Let N be a named set. By Lemma 3.8, we have that $|supp_{F(N)}([q]_{\equiv_N})| = \|q\|_N$ and $G_{I(F(N))}([q]_{\equiv_N}) = G_N(q)$, so that the pair $\eta_N = \langle in_{\equiv_N}, G_N(q)\rangle$ defines a named function from N to $I(F(N))$, for $in_{\equiv_N}$ the obvious injection mapping q to $[q]_{\equiv_N}$. Such a morphism is a strong candidate for the unit of a possible adjunction. Unfortunately, this is not the case, as explained below.

Remark 3.9 *Let $\mathcal{A} \in FSAlg_\pi$, and let us suppose that $F \dashv I$. Then, for each named function $H : N \to I(\mathcal{A})$ there exists a unique morphism $\sigma_H : F(N) \to \mathcal{A}$ such that $\eta_N; I(\sigma_H) = H$ (see [1, Definition 13.2.1]).*

Such a morphism should behave as h on Q_N, meaning that (the equivalence class) $[q]_{\equiv_N}$ has to be mapped into $h(q)$: So, this fact does constrain the choice of σ_H to be the free extension of h. To prove its existence would now be enough to show that the axiomatization is preserved, i.e., that $[\pi, \pi'](h(q)) = h(q)$ holds in $\mathcal{A}$ if $[\pi, \pi'](q) \equiv_N q$: The commutativity of the diagram follows, as well as the uniqueness of σ_H.

Let $\mathcal{W} = \langle \omega, \{\widehat{\pi}_W\}\rangle$ be the algebra such that $\widehat{\pi}_W(i) = \pi(i)$ for all $i \in \omega$. It is finitely supported, since clearly $supp_W(i) = \{i\}$ for all $i \in \omega$. Then, by construction $I(\mathcal{W}) = \langle \omega, \|i\| = 1, id\rangle$ (compare with the named sets in Example 3.3). Now, let us consider the identity on $I(\mathcal{W})$: The obvious function $\sigma_{id} : F(I(\mathcal{W})) \to \mathcal{W}$ is not *an algebra morphism.*

The problem lies on the “normalization” along the functor I, which blurs the identity of the elements of the support. We need to choose a “canonical” element for each set of elements with the same cardinality of the support.

Lemma 3.10 *Let $\mathcal{A} \in Alg_\pi$ and let $a \in A$. If a is finitely supported, then*

$supp_A(\pi(a)) = \pi(supp_A(a))$ for all $\pi \in Aut(\omega)$.

Lemma 3.11 *Let $\mathcal{A} \in Alg_\pi$, let $a \in A$ and let $Hom_A[a, a'] \triangleq \{\pi \mid \widehat{\pi}_A(a) = a'\}$. Then, $Hom_A[a, a'] \circ fix_A(a) = Hom_A[a, a'] = fix_A(a') \circ Hom_A[a, a']$.*

We now introduce a last concept, the *orbit* of an element, consisting of the family of all the other elements of an algebra which can be reached from it *via* the application of an operator of the permutation signature.

Definition 3.12 (orbits) *Let $\mathcal{A} \in Alg_\pi$ and let $a \in A$. The* orbit *of a is the set $Orb_A(a) \triangleq \{\widehat{\pi}_A(a) \mid \pi \in Aut(\omega)\}$.*

Thus, the orbit of an element a collects all the other elements that are reached from a *via* the application of a permutation, i.e., an operator of the signature. It is obvious that orbits partition a permutation algebra. Moreover, let us assume the existence for each orbit $Orb_A(a)$ of a canonical representative a_O. (We will come back on this later on.)

Proposition 3.13 (from algebras to sets, II) *Let $\widehat{I}_O$ be the function mapping each $\mathcal{A} \in FSAlg_\pi^{fk}$ to the named set $\langle\{a_O \mid a \in A\}, |supp_A(\cdot)|, G_{\widehat{I}(\mathcal{A})}\rangle$, for $G_{\widehat{I}(\mathcal{A})}(a_O)$ the set of permutations given by*

$$\{\pi \in \mathrm{Aut}(|supp_A(a_0)|) \mid \exists \pi' \in fix_A(a_0) : \ norm_A(a_O) \circ \pi = \pi' \circ norm_A(a_O)\}.$$

Let $\sigma : \mathcal{A} \to \mathcal{B}$, let $in_\sigma(a_O) : |supp_B(\sigma(a_O))| \to |supp_A(a_O)|$ be the uniquely induced arrow such that $norm_A(a_O) \circ in_\sigma(a_O) = norm_B(\sigma(a_O))$, and let $\Xi(\sigma(a)_O, \sigma(a_O)) \subseteq \mathbb{I}(|supp_A(\sigma(a)_O)|, |supp_A(\sigma(a_O))|)$ be the set of permutations given by

$$\{\pi \mid \exists \pi' \in Hom_B[\sigma(a)_O, \sigma(a_O)] : norm_B(\sigma(a_O)) \circ \pi = \pi' \circ norm_B(\sigma(a)_O)\}.$$

Hence, let $\widehat{I}_A$ be the function associating to σ the named function $\langle h_\sigma, \Lambda_\sigma\rangle$ such that $h_\sigma(a_O) = \sigma(a)_O$ and $\Lambda_\sigma(a_O) = in_\sigma(a_O) \circ \Xi(\sigma(a)_O, \sigma(a_O)) \circ G_{\widehat{I}(\mathcal{B})}(\sigma(a)_O)$ for all $a_O \in A$.

The pair $\widehat{I} = \langle \widehat{I}_O, \widehat{I}_A\rangle$ defines a functor from $FSAlg_\pi^{fk}$ to **NSet**.

Proof. The key remark for the correctness of Λ_σ is that $Hom_B[\sigma(a)_O, \sigma(a_0)] \circ fix_B(\sigma(a_O)) = fix_B(\sigma(a)_O) \circ Hom_B[\sigma(a)_O, \sigma(a_O)]$ (see Lemma 3.11 above), and equality $\Xi(\sigma(a)_O, \sigma(a_O)) \circ G_{\widehat{I}(\mathcal{A})}(\sigma(a)_O) = G_{I(\mathcal{A})}(\sigma(a_O)) \circ \Xi(\sigma(a)_O, \sigma(a_O))$ follows: Then, it is enough to mimic the proof for Proposition 3.7. □

The proof goes along the same lines of the one for Proposition 3.7: Additionally, now the "normalization" along $\widehat{I}$ picks up a single representative for each orbit, which is mirrored by the introduction of the family Ξ_{a_O}. Using the previously defined functor, it is easy to realize that named sets are just a different presentation for finite kernel permutation algebras.

Theorem 3.14 $\mathbf{NSet} \cong \mathit{FSAlg}^{\mathit{fk}}_{\pi}$.

Proof. Let N be a named set: It is easy to prove that it is isomorphic to $\widehat{I}(F(N))$. Thanks to Lemmata 3.8 and 3.11, the set of states of the latter is $\bigcup_{q \in Q_N}(([q]_{\equiv_N})_O)$, its enumerating function is $|\mathit{supp}_{F(N)}(([q]_{\equiv_N})_O)| = \|q\|_N$, and its set of permutations $G_{\widehat{I}(F(\mathcal{N}))}(([q]_{\equiv_N})_O) \subseteq \mathrm{Aut}(\|q\|_N)$ satisfies

$$G_{\widehat{I}(F(\mathcal{N}))}(([q]_{\equiv_N})_O) \circ \Xi(([q]_{\equiv_N})_O, [q]_{\equiv_N})) = \Xi(([q]_{\equiv_N})_O, [q]_{\equiv_N})) \circ G_{I(F(\mathcal{N}))}([q]_{\equiv_N}).$$

Now, since $G_{I(F(\mathcal{N}))}([q]_{\equiv_N}) = G_N(q)$, the corresponding isomorphism is given by $\langle([-]_{\equiv_N})_O, G_{\widehat{I}(F(\mathcal{N}))}(([q]_{\equiv_N})_O) \circ \Xi(([q]_{\equiv_N})_O, [q]_{\equiv_N}))\rangle$, which is also natural.

Analogous considerations hold for the endomorphism $F(\widehat{I}(\mathcal{A}))$ on algebras. The element $a_O \in \widehat{I}(\mathcal{A})$ generates the whole orbit of $[a_O]_{\equiv_{F(\widehat{I}(\mathcal{A}))}}$, and the algebra isomorphism σ maps the latter is to $\widehat{\pi}_A(a_O)$, for π any permutation extending $\mathit{norm}_A(a_O)$. ☐

Remark 3.15 As a final note, we remark that the canonical representative a_O of each orbit can be constructively defined. In fact, $\mathrm{Aut}(\omega)$ can be naturally equipped with a total order, which is then lifted to sets of permutations. Hence, for each orbit an element a_c can be chosen, such that $|\mathit{supp}_A(a_c)| = \mathit{supp}_A(a_c)$, and which has the minimal permutation group associated to it. The definition is well-given, since it is easy to prove that $\mathit{fix}_A(a) = \mathit{fix}_A(a')$ implies $a = a'$ for all finitely supported $a \in A$ and $a' \in \mathit{Orb}_A(a)$.

4 Conclusions

We investigated the connections between three different approaches to the treatment of nominal calculi, such as calculi for name passing or location generation, comparing meta-models based on (pre)sheaf categories, on algebras over permutation signatures, and on sets enriched with names and permutation groups. We proved that the category of named sets is equivalent to the category of permutation algebras with finite support on the signature with finite kernel permutations; which in turn, when *all* permutations are considered, is equivalent to the category of sheaves over $\mathbb{I}$, i.e., the Schanuel topos.

Our characterization results are summarized in Figure 2. They confirm that permutation algebras are well suited for modeling the semantics of nominal calculi. Moreover, we can import from the (pre)sheaf approach the initial algebra/final coalgebra machinery. Our next step will be to compare the mod-

$$\begin{tikzcd}
Alg_\pi \arrow[rrr, hook] \arrow[dr, "\cong"] & & & Alg_\pi^{fk} \arrow[dr, "\cong"] & \\
& \mathbf{B}\mathrm{Aut}(\omega)^\delta \arrow[rrr, hook] \arrow[dd, shift left] & & & \mathbf{B}\mathrm{Aut}^{fk}(\omega)^\delta \arrow[dd, shift left] \\
FSAlg_\pi \arrow[uu, hook] \arrow[rrr, hook] \arrow[dr, "\cong"] & \arrow[d, phantom, "\dashv"] & & FSAlg_\pi^{fk} \cong \mathbf{NSet} \arrow[uu, hook] \arrow[dr, "\cong"] & \arrow[d, phantom, "\dashv"] \\
& \mathbf{B}\mathrm{Aut}(\omega) \cong \mathrm{Sh}(\mathbb{I}^{op}) \arrow[uu, hook, shift left] \arrow[rrr, hook] & & & \mathbf{B}\mathrm{Aut}^{fk}(\omega) \arrow[uu, hook, shift left]
\end{tikzcd}$$

Fig. 2. The Permutation Algebra Cube, strengthened version.

els obtained by bialgebras on named sets and permutation algebras (see [14, Section 4] and [3]), with the coalgebraic models over presheaves categories [5].

Beside this, it seems natural to develop further our research in terms of categorical logic. We would aim to define a suitable internal language for the three meta-models we analyzed so far. The connection with the Schanuel topos, and its correspondence with Fraenkel-Mostowski set theory, would lead us to consider some variant (e.g., higher-order) of Pitts' Nominal Logic [15].

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