A note on truncations in fractional Sobolev spaces

Published
DOI:10.1142/S1664360719500012

A note on truncations in fractional Sobolev spaces

Roberta Musina* and Alexander I. Nazarov†

Abstract

We study the Nemytskii operators $u \mapsto |u|$ and $u \mapsto u^{\pm}$ in fractional Sobolev spaces $H^s(\mathbb{R}^n)$, $s > 1$.

Keywords: Fractional Laplacian - Sobolev spaces - Truncation operators

2010 Mathematics Subject Classification: 46E35, 47H30.

1 Introduction. Main result

In this paper we discuss the relation between the map $u \mapsto |u|$ and the *Dirichlet Laplacian*. Recall that the Dirichlet Laplacian $(-\Delta_{\mathbb{R}^n})^s u$ of order $s > 0$ of a function $u \in L^2(\mathbb{R}^n)$, $n \geq 1$, is the distribution

$$\langle (-\Delta_{\mathbb{R}^n})^s u, \varphi \rangle \equiv \int\limits_{\mathbb{R}^n} u \ (-\Delta_{\mathbb{R}^n})^s \varphi \, dx := \int\limits_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[\varphi] \, \overline{\mathcal{F}[u]} \, d\xi \ , \quad \varphi \in \mathcal{C}^\infty_0(\mathbb{R}^n),$$

where

$$\mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx$$

*Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy. Email: roberta.musina@uniud.it. Partially supported by Miur-PRIN 2015233N54.

†St.Petersburg Department of Steklov Institute, Fontanka, 27, St.Petersburg, 191023, Russia and St.Petersburg State University, Universitetskii pr. 28, St.Petersburg, 198504, Russia. E-mail: al.il.nazarov@gmail.com.

is the Fourier transform in $\mathbb{R}^n$. The Sobolev–Slobodetskii space

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid (-\Delta_{\mathbb{R}^n})^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \}$$

naturally inherits an Hilbertian structure from the scalar product

$$(u, v) = \langle (-\Delta_{\mathbb{R}^n})^s u, v \rangle + \int\limits_{\mathbb{R}^n} uv \, dx \, .$$

The standard reference for the operator $(-\Delta_{\mathbb{R}^n})^s$ and functions in $H^s(\mathbb{R}^n)$ is the monograph [8] by Triebel.

For any positive order $s \notin \mathbb{N}$ we introduce the constant

$$C_{n,s} = \frac{2^{2s} s}{\pi^{\frac{n}{2}}} \frac{\Gamma\big(\frac{n}{2} + s\big)}{\Gamma\big(1 - s\big)} \, . \tag{1}$$

Notice that

$$C_{n,s} > 0 \quad \text{if} \quad \lfloor s \rfloor \quad \text{is even;} \qquad C_{n,s} < 0 \quad \text{if} \quad \lfloor s \rfloor \quad \text{is odd,} \tag{2}$$

where $\lfloor s \rfloor$ stands for the integer part of s. It is well known that for $s \in (0, 1)$ and $u, v \in H^s(\mathbb{R}^n)$ one has

$$\langle (-\Delta_{\mathbb{R}^n})^s u, v \rangle = \frac{C_{n,s}}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy \, . \tag{3}$$

Let us recall some known facts about the Nemytskii operator $| \cdot | : u \mapsto |u|$.

1. $| \cdot |$ is a Lipschitz transform of $H^0(\mathbb{R}^n) \equiv L^2(\mathbb{R}^n)$ into itself.

2. Let $0 < s \leq 1$. Then $| \cdot |$ is a continuous transform of $H^s(\mathbb{R}^n)$ into itself, by general results about Nemytskii operators in Sobolev/Besov spaces, see [7, Theorem 5.5.2/3]. Also it is obvious that for $u \in H^1(\mathbb{R}^n)$

$$\langle -\Delta |u|, |u| \rangle = \langle -\Delta u, u \rangle = \int\limits_{\mathbb{R}^n} |\nabla u|^2 \, dx \; , \qquad \langle -\Delta u^+, u^- \rangle = \int\limits_{\mathbb{R}^n} \nabla u^+ \cdot \nabla u^- \, dx = 0 \, .$$

Here and elsewhere $u^\pm = \max\{\pm u, 0\} = \frac{1}{2}(|u| \pm u)$, so that $u = u^+ - u^-$, $|u| = u^+ + u^-$. On the other hand, for $s \in (0, 1)$ and $u \in H^s(\mathbb{R}^n)$ formula (3) gives

$$\langle (-\Delta_{\mathbb{R}^n})^s u^+, u^- \rangle = - \, C_{n,s} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{u^+(x) u^-(y)}{|x - y|^{n+2s}} \, dxdy. \tag{4}$$

From (4) we infer by the polarization identity

$$4\langle(-\Delta_{\mathbb{R}^n})^s u^+, u^-\rangle = \langle(-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle - \langle(-\Delta_{\mathbb{R}^n})^s u, u\rangle$$

that if u changes sign then

$$\langle(-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle < \langle(-\Delta_{\mathbb{R}^n})^s u, u\rangle, \qquad s \in (0,1). \tag{5}$$

We mention also [4, Theorem 6] for a different proof and explanation of (5), that includes the case when $(-\Delta_{\mathbb{R}^n})^s$ is replaced by the *Navier* (or *spectral Dirichlet*) *Laplacian* on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$.

3. Let $1 < s < \frac{3}{2}$. The results in [2] and [6] (see also Section 4 of the exhaustive survey [3]) imply that $|\cdot|$ is a bounded transform of $H^s(\mathbb{R}^n)$ into itself. That is, there exists a constant $c(n,s)$ such that

$$\langle(-\Delta_{\mathbb{R}^n})^s |u|, |u|\rangle \le c(n,s)\langle(-\Delta_{\mathbb{R}^n})^s u, u\rangle, \qquad u \in H^s(\mathbb{R}^n)\,.$$

In particular, $|\cdot|$ is continuous at $0 \in H^s(\mathbb{R}^n)$.

It is easy to show that the assumption $s < \frac{3}{2}$ can not be improved, see Example 1 below and [2, Proposition p. 357], where a more general setting involving Besov spaces $B^{s,q}_p(\mathbb{R}^n)$, $s \ge 1 + \frac{1}{p}$, is considered.

At our knowledge, the continuity of $|\cdot| : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, $s \in (1, \frac{3}{2})$, is an open problem. We can only point out the next simple result.

Proposition 1 *Let $0 < \tau < s < \frac{3}{2}$. Then $|\cdot| : H^s(\mathbb{R}^n) \to H^\tau(\mathbb{R}^n)$ is continuous.*

Proof. Recall that $H^s(\mathbb{R}^n) \hookrightarrow H^\tau(\mathbb{R}^n)$ for $0 < \tau < s$. Actually, the Hölder inequality readily gives the well known interpolation inequality

$$\langle(-\Delta_{\mathbb{R}^n})^\tau v, v\rangle = \int\limits_{\mathbb{R}^n} |\xi|^{2\tau}|\mathcal{F}[v]|^2 d\xi \le \Big(\langle(-\Delta_{\mathbb{R}^n})^s v, v\rangle\Big)^{\frac{\tau}{s}} \Big(\int\limits_{\mathbb{R}^n} |v|^2\, dx\Big)^{\frac{s-\tau}{s}}, \quad v \in H^s(\mathbb{R}^n).$$

Since $|\cdot|$ is continuous $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and bounded $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, the statement follows immediately. □

Now we formulate our main result. It provides the complete proof of [5, Theorem 1] for s below the threshold $\frac{3}{2}$ and gives a positive answer to a question raised in [1, Remark 4.2] by Nicola Abatangelo, Sven Jahros and Albero Saldaña.

Theorem 1 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n)$. Then formula (4) holds. In particular, if u changes sign then*

$$\langle (-\Delta_{\mathbb{R}^n})^s |u|, |u| \rangle > \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle .$$

Our proof is deeply based on the continuity result in Proposition 1. The knowledge of continuity of $|\cdot| : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ could considerably simplify it.

We denote by c any positive constant whose value is not important for our purposeses. Its value may change line to line. The dependance of c on certain parameters is shown in parentheses.

2 Preliminary results and proof of Theorem 1

We begin with a simple but crucial identity that has been independently pointed out in [5, Lemma 1] and [1, Lemma 3.11] (without exact value of the constant). Notice that it holds for general fractional orders $s > 0$.

Theorem 2 *Let $s > 0$, $s \notin \mathbb{N}$. Assume that $v, w \in H^s(\mathbb{R}^n)$ have compact and disjoint supports. Then*

$$\langle (-\Delta_{\mathbb{R}^n})^s v, w \rangle = - C_{n,s} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v(x) w(y)}{|x-y|^{n+2s}} \, dxdy. \tag{6}$$

Proof. Let ρ_h be a sequence of mollifiers, and put $w_h := w * \rho_h$. Formula (3) gives

$$\begin{aligned} \langle (-\Delta_{\mathbb{R}^n})^s v, w_h \rangle &= \langle (-\Delta_{\mathbb{R}^n})^{s-\lfloor s \rfloor} v, (-\Delta)^{\lfloor s \rfloor} w_h \rangle \\ &= \frac{C_{n,s-\lfloor s \rfloor}}{2} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big(v(x) - v(y)\big)\big((-\Delta)^{\lfloor s \rfloor} w_h(x) - (-\Delta)^{\lfloor s \rfloor} w_h(y)\big)}{|x-y|^{n+2(s-\lfloor s \rfloor)}} \, dxdy. \end{aligned}$$

Since for large h the supports of v and w_h are separated, we have

$$\langle (-\Delta_{\mathbb{R}^n})^s v, w_h \rangle = -\, C_{n,s-\lfloor s \rfloor} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v(x)\ (-\Delta)^{\lfloor s \rfloor} w_h(y)}{|x-y|^{n+2(s-\lfloor s \rfloor)}}\, dydx.$$

Here we can integrate by parts. Using (1) one computes for $a > 0$

$$\Delta \, \frac{C_{n,a}}{|x-y|^{n+2a}} = \frac{C_{n,a}(n+2a)(2a+2)}{|x-y|^{n+2a+2}} = -\, \frac{C_{n,a+1}}{|x-y|^{n+2(a+1)}}$$

and obtains (6) with w_h instead of w.

Since the supports of v and w are separated, it is easy to pass to the limit as $h \to \infty$ and to conclude the proof. □

Remark 1 *Motivated by (6) and (2), A.I. Nazarov conjectured in [5] that*

$$\begin{array}{ll} \langle (-\Delta_{\mathbb{R}^n})^s |u|, |u| \rangle - \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle < 0 & \text{if } \lfloor s \rfloor \text{ is even;} \\ \langle (-\Delta_{\mathbb{R}^n})^s |u|, |u| \rangle - \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle > 0 & \text{if } \lfloor s \rfloor \text{ is odd} \end{array}$$

for any not integer exponent $s > 0$ and for any changing sign function $u \in H^s(\mathbb{R}^n)$ such that $u^{\pm} \in H^s(\mathbb{R}^n)$.

Lemma 1 *Let $s \in (1, \frac{3}{2})$ and $\varepsilon > 0$. If a function $u \in H^s(\mathbb{R}^n)$ has compact support then $(u-\varepsilon)^+ \in H^s(\mathbb{R}^n)$, and*

$$\langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon)^+, (u-\varepsilon)^+ \rangle \le c(n,s) \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle + c(n,s,\mathrm{supp}(u))\varepsilon^2 \,.$$

Proof. Take a nonnegative function $\eta \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $\mathrm{supp}(u)$. Clearly $u - \varepsilon\eta \in H^s(\mathbb{R}^n)$. Hence, by Item 3 in the Introduction we have that $(u-\varepsilon\eta)^+ = (u-\varepsilon)^+ \in H^s(\mathbb{R}^n)$ and

$$\begin{aligned} \langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon)^+, (u-\varepsilon)^+ \rangle &\le c(n,s)\, \langle (-\Delta_{\mathbb{R}^n})^s (u-\varepsilon\eta), u-\varepsilon\eta \rangle \\ &\le c(n,s)\, \big(\langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle + \varepsilon^2 \langle (-\Delta_{\mathbb{R}^n})^s \eta, \eta \rangle \big). \end{aligned}$$

The proof is complete. □

In order to simplify notation, for $u : \mathbb{R}^n \to \mathbb{R}$ and $s > 0$ we put

$$\Phi^s_u(x,y) = \frac{u^+(x)u^-(y)}{|x-y|^{n+2s}} \,.$$

Lemma 2 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n) \cap \mathcal{C}^0_0(\mathbb{R}^n)$. Then (4) holds, and in particular $\Phi^s_u \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. Thanks to Lemma 1 we have that $(u^- - \varepsilon)^+ \in H^s(\mathbb{R}^n) \cap \mathcal{C}^0_0(\mathbb{R}^n)$ for any $\varepsilon > 0$. Next, the supports of the functions u^+ and $(u^- - \varepsilon)^+$ are compact and disjoint. Thus we can apply Theorem 2 to get

$$\langle (-\Delta_{\mathbb{R}^n})^s u^+, (u^- - \varepsilon)^+ \rangle = -\, C_{n,s} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{u^+(x)(u(y)^- - \varepsilon)^+}{|x-y|^{n+2s}} \, dxdy. \tag{7}$$

Take a decreasing sequence $\varepsilon \searrow 0$. From Lemma 1 we infer that $(u^- - \varepsilon)^+ \to u^-$ weakly in $H^s(\mathbb{R}^n)$, as $(u^- - \varepsilon)^+ \to u^-$ in $L^2(\mathbb{R}^n)$. Hence the duality product in (7) converges to the the duality product in (4). Next, the integrand in the right-hand side of (7) increases to Φ^s_u a.e. on $\mathbb{R}^n \times \mathbb{R}^n$. By the monotone convergence theorem we get the convergence of the integrals, and the conclusion follows immediately. $\square$

Lemma 3 *Let $s \in (1, \frac{3}{2})$ and $u \in H^s(\mathbb{R}^n)$. Then $\Phi^s_u \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. Take a sequence of functions $u_h \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $u_h \to u$ in $H^s(\mathbb{R}^n)$ and almost everywhere. Since $\Phi^s_{u_h} \to \Phi^s_u$ a.e. on $\mathbb{R}^n \times \mathbb{R}^n$, Fatou's Lemma, Lemma 2 for u_h and the boundeness of $v \mapsto v^\pm$ in $H^s(\mathbb{R}^n)$ give

$$\begin{aligned}
\iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \Phi^s_u(x,y)\, dxdy \;\; &\leq \;\; \liminf_{h\to\infty} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \Phi^s_{u_h}(x,y)\, dxdy = c(n,s) \liminf_{h\to\infty} \langle (-\Delta_{\mathbb{R}^n})^s u_h^+, u_h^- \rangle \\
&\leq \;\; c(n,s) \lim_{h\to\infty} \langle (-\Delta_{\mathbb{R}^n})^s u_h, u_h \rangle = c(n,s) \langle (-\Delta_{\mathbb{R}^n})^s u, u \rangle,
\end{aligned}$$

that concludes the proof. $\square$

Proof of Theorem 1. Take a sequence $u_h \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ such that $u_h \to u$ in $H^s(\mathbb{R}^n)$ and almost everywhere. Consider the nonnegative functions

$$v_h := u_h^+ \wedge u^+ = u^+ - (u^+ - u_h^+)^+ \ , \quad w_h := u_h^- \wedge u^- = u^- - (u^- - u_h^-)^+ .$$

Then $v_h, w_h \in H^s(\mathbb{R}^n)$. Next, take any exponent $\tau \in (1, s)$. By Proposition 1 we have that $u^\pm - u_h^\pm \to 0$ in $H^\tau(\mathbb{R}^n)$; hence $(u^\pm - u_h^\pm)^+ \to 0$ in $H^\tau(\mathbb{R}^n)$ by Item 3 in the Introduction. Thus,

$$v_h \to u^+ \ , \quad w_h \to u^- \quad \text{in } H^\tau(\mathbb{R}^n) \text{ and almost everywhere, as } \ h \to \infty. \tag{8}$$

Now we take a small $\varepsilon > 0$. Recall that $(v_h - \varepsilon)^+ \in H^\tau(\mathbb{R}^n)$ by Lemma 1. Moreover, from $0 \le v_h \le u_h^+$, $0 \le w_h \le u_h^-$ it follows that

$$\operatorname{supp}((v_h - \varepsilon)^+) \subseteq \{u_h \ge \varepsilon\}; \qquad \operatorname{supp}(w_h) \subseteq \operatorname{supp}(u_h^-).$$

In particular, the functions $(v_h - \varepsilon)^+, w_h$ have compact and disjoint supports. Thus we can apply Theorem 2 to infer

$$\langle (-\Delta_{\mathbb{R}^n})^\tau (v_h - \varepsilon)^+, w_h \rangle = -\, C_{n,\tau} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v_h(x) - \varepsilon)^+ w_h(y)}{|x-y|^{n+2\tau}}\, dxdy.$$

We first take the limit as $\varepsilon \searrow 0$. The argument in the proof of Lemma 2 gives

$$\langle (-\Delta_{\mathbb{R}^n})^\tau v_h, w_h \rangle = -\, C_{n,\tau} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v_h(x) w_h(y)}{|x-y|^{n+2\tau}}\, dxdy. \tag{9}$$

Next we push $h \to \infty$. By (8) we get

$$\lim_{h\to\infty} \langle (-\Delta_{\mathbb{R}^n})^\tau v_h, w_h \rangle = \langle (-\Delta_{\mathbb{R}^n})^\tau u^+, u^- \rangle \, .$$

Further, since the integrand in the right-hand side of (9) does not exceed $\Phi^\tau_u(x,y)$, Lemma 3, (8) and Lebesgue's theorem give

$$\lim_{h\to\infty} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{v_h(x) w_h(y)}{|x-y|^{n+2\tau}}\, dxdy = \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \Phi^\tau_u(x,y)\, dxdy \, .$$

Thus, we proved (4) with s replaced by τ. It remains to pass to the limit as $\tau \nearrow s$. By Lebesgue's theorem, we have

$$\begin{aligned}\lim_{\tau\nearrow s}\langle(-\Delta_{\mathbb{R}^n})^\tau u^+, u^-\rangle &= \lim_{\tau\nearrow s}\int\limits_{\mathbb{R}^n}|\xi|^{2\tau}\mathcal{F}[u^+]\,\overline{\mathcal{F}[u^-]}\,d\xi \\ &= \int\limits_{\mathbb{R}^n}|\xi|^{2s}\mathcal{F}[u^+]\,\overline{\mathcal{F}[u^-]}\,d\xi = \langle(-\Delta_{\mathbb{R}^n})^s u^+, u^-\rangle.\end{aligned}$$

Now we fix $\tau_0 \in (1, s)$ and notice that $0 \le \Phi_u^\tau \le \max\{\Phi_u^{\tau_0}, \Phi_u^s\}$ for any $\tau \in (\tau_0, s)$. Therefore, Lemma 3 and Lebesgue's theorem give

$$\lim_{\tau\nearrow s}\iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\Phi_u^\tau(x,y)\,dxdy = \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\Phi_u^s(x,y)\,dxdy\,.$$

The proof of (4) is complete. The last statement follows immediately from (4), polarization identity and (2). □

Example 1 It is easy to construct a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that $u^+ \in H^s(\mathbb{R}^n)$ if and only if $s < \frac{3}{2}$.

Take $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfying $\varphi(0) = 0, \varphi'(0) > 0$ and $x\varphi(x) \ge 0$ on $\mathbb{R}$. By direct computation one checks that $\varphi^+ = \chi_{(0,\infty)}\varphi \in H^s(\mathbb{R})$ if and only if $s < \frac{3}{2}$. If $n = 1$ we are done. If $n \ge 2$ we take $u(x_1, x_2, \dots, x_n) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)$.

Acknowledgements. The first author wishes to thank Université Libre de Bruxelles for the hospitality in February 2016. She is grateful to Denis Bonheure, Nicola Abatangelo, Sven Jahros and Albero Saldaña for valuable discussion on this subject.

References

[1] N. Abatangelo, S. Jarohs and A. Saldaña, *On the maximum principle for higher-order fractional Laplacians*, preprint arxiv:1607.00929 (2016).

[2] G. Bourdaud and Y. Meyer, *Fonctions qui opèrent sur les espaces de Sobolev*, J. Funct. Anal. **97** (1991), no. 2, 351–360.

[3] G. Bourdaud and W. Sickel, *Composition operators on function spaces with fractional order of smoothness*, Harmonic analysis and nonlinear partial differential equations, 93–132, RIMS Kokyuroku Bessatsu, B26, Res. Inst. Math. Sci. (RIMS), Kyoto (2011).

[4] R. Musina and A. I. Nazarov, *On the Sobolev and Hardy constants for the fractional Navier Laplacian*, Nonlinear Anal. **121** (2015), 123–129. Online version, Url http://www.sciencedirect.com/science/article/pii/S0362546X14003113

[5] A.I. Nazarov, *Remark on fractional Laplacians*, Preprints of St. Petersburg Mathematical Society (March 2016).

[6] P. Oswald, *On the boundedness of the mapping* $f \to |f|$ *in Besov spaces*, Comment. Math. Univ. Carolin. **33** (1992), no. 1, 57–66.

[7] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, **3**, de Gruyter, Berlin (1996).

[8] H. Triebel, *Interpolation theory, function spaces, differential operators*, Deutscher Verlag Wissensch., Berlin (1978).