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The rationality of the moduli space of one-pointed ineffective spin hyperelliptic curves via an almost del Pezzo threefold

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Abstract. Using the geometry of an almost del Pezzo threefold, we show that the moduli space $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ of genus g one-pointed ineffective spin hyperelliptic curves is rational for every $g \geq 2$.

0. Introduction

Throughout this paper, we work over $\mathbb{C}$, the complex number field. The purpose of this paper is to show the following result:

Theorem 0.0.1 (=Theorem 5.0.2)**.** *The moduli space $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ of one-pointed genus g hyperelliptic ineffective spin curves is an irreducible rational variety.*

We have the following immediate corollary:

Corollary 0.0.2. *The moduli space $\mathcal{S}_{g}^{0,\mathrm{hyp}}$ of genus g hyperelliptic ineffective spin curves is an irreducible unirational variety.*

Now we give necessary definitions and notions to understand the statement of the above results. We recall that a couple (C, θ) is called a *genus g spin curve* if C is a genus g curve and θ is a theta characteristic on C, namely, a half canonical divisor of C. If the linear system $|\theta|$ is empty, then θ is called an *ineffective* theta characteristic, and we also say that such a *spin curve is ineffective.* A *hyperelliptic spin curve* (C, θ) means that C is hyperelliptic. A pair of a spin curve (C, θ) and a point $p \in C$ is called a *one-pointed spin curve.* One-pointed spin curves (C, θ, p) and (C', θ', p') are said to be *isomorphic* to each other if there exists an isomorphism $\xi\colon C \to C'$ such that $\xi^*\theta' \simeq \theta$ and $\xi^*p' = p$. Finally, we denote by $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ (resp. $\mathcal{S}_{g}^{0,\mathrm{hyp}}$) the coarse moduli space of isomorphism classes of one-pointed genus g hyperelliptic ineffective spin curves (resp. genus g hyperelliptic ineffective spin curves).

Main motivations of our study are the rationalities of the moduli spaces of hyperelliptic curves [2] and of pointed hyperelliptic curves [3].

One feature of the paper is that the above rationality is proved via the geometry of a certain smooth projective threefold. We developed such a method in our previous works [10, 11, 12]. In these works, we established the interplay between

- even spin trigonal curves, where even spin curve means that the considered theta characteristics have even-dimensional spaces of global sections, and

- the quintic del Pezzo threefold B, which is known to be unique up to isomorphisms and is isomorphic to a codimension three linear section of $\mathrm{G}(2,5)$.

The relationship between curves and 3-folds are a kind of mystery but many such relationships have been known to nowadays. A common philosophy of such works is that a family of certain objects in a certain threefold is an algebraic curve with some extra data. In [10, Cor. 4.1.1], we showed that a genus $d-2$ trigonal curve appears as the family of lines on B which intersect a fixed another rational curve of degree $d \geq 2$, and, in [11, Prop. 3.1.2], we constructed a theta characteristic on the trigonal curve from the incidence correspondence of intersecting lines on B. The mathematician who met first such an interplay is S. Mukai, who discovered that lines on a genus twelve prime Fano threefold V is parameterized by a genus three curve, and constructed a theta characteristic on the the genus three curve from the incidence correspondence of intersecting lines on V [8, 9]. In our previous works [10, 11, 12], we interpreted Mukai's work from the view point of the quintic del Pezzo threefold B and generalized it.

The study of this paper is directly related to our paper [12], in which we showed that the moduli of even spin genus four curves is rational by using the above mentioned interplay.

We are going to show our main result also by using such an interplay, but we replace the quintic del Pezzo threefold by a certain degeneration of it. This is a new feature of this paper. The degeneration is a quintic del Pezzo threefold with one node, which is also known to be unique up to isomorphisms and is isomorphic to a codimension three linear section of $\mathrm{G}(2,5)$ by [4]. Moreover, it is not factorial at the node, and hence it admits two small resolutions, which we call B_a and B_b in this paper. Actually, we do not work on this singular threefold directly but work on small resolutions, mainly on B_a. Along the above mentioned philosophy, we consider a family of 'lowest degree' rational curves on B_a, which we call B_a-lines, intersecting a fixed another 'higher degree' rational curve R. Then we show such B_a-lines are parameterized by a hyperelliptic curve C_R, and we construct an ineffective theta characteristic θ_R on it from the incidence correspondence of intersecting B_a-lines. Then we may reduce the rationality problem of the moduli to that of a certain quotient of family of rational curves on B_a by the group acting on B_a, and solve the latter by computing invariants.

Finally, we sketch the structure of the paper. In the section 1, we define a projective threefold B_a, which is the key variety for our investigation of one-pointed ineffective spin hyperelliptic curves. In this section, we also review several properties of B_a. In the section 2, we construct the above mentioned families of rational curves R on B_a, and the family of B_a-lines. Then, in the section 3, we construct hyperelliptic curves C_R as the parameter space of B_a-lines intersecting each fixed R. In the section 4, we construct an ineffective theta characteristic θ_R on C_R from the incidence correspondence of intersecting B_a-lines parameterized by C_R. We also remark that C_R comes with a marked point from its construction. Finally in this section, we interpret the moduli $\mathcal{S}^{0,\mathrm{hyp}}_{g,1}$ by a certain group quotient of the family of R. Then, in the section 5, we show the rationality of the latter by computing invariants.

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1. The key projective threefold B_a

1.1. **Definition of B_a.** The key variety to show the rationality of $\mathcal{S}^{0,\mathrm{hyp}}_{g,1}$ is the threefold, which we denote by B_a in this paper, with the following properties:

(1) B_a is a smooth *almost del Pezzo threefold*, which is, by definition, a smooth projective threefold with nef and big but non-ample anticanonical divisor divisible by 2 in the Picard group.
(2) If we write $-K_{B_a} = 2M_{B_a}$, then $M^3_{B_a} = 5$.
(3) $\rho(B_a) = 2$.
(4) B_a has two elementary contractions, one of which is the anticanonical model $B_a \to B$ and it is a small contraction, and another is a $\mathbb{P}^1$-bundle $\pi_a \colon B_a \to \mathbb{P}^2$.

1.2. **Descriptions of B_a.** Many people met the threefold B_a in several contexts. The first one is probably T. Fujita. In his classification of singular del Pezzo threefolds [4], B_a appears as a small resolution of the quintic del Pezzo threefold B. Here we do not review Fujita's construction of B_a in detail except that we sum up his results as follows:

Proposition 1.2.1. *B_a is unique up to isomorphism, and the anti-canonical model $B_a \to B$ contracts a single smooth rational curve, say, γ_a to a node of B. In particular the normal bundle of γ_a is $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.*

Fujita treats B_a less directly, so descriptions of B_a by [7], [6], [13] and [5], which we review below, are more convenient for our purpose.

By [7, §3] and [6, Thm. 3.6], we may write $B_a \simeq \mathbb{P}(\mathcal{E})$ with a stable rank two bundle $\mathcal{E}$ on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 2$ fitting in the following exact sequence:

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}(-2) \to \mathcal{E} \to 0. \tag{1.1}$$

Let $H_{\mathcal{E}}$ be the tautological divisor for $\mathcal{E}$ and L the π_a-pull back of a line in $\mathbb{P}^2$. By the canonical bundle formula for projective bundle, we may write $-K_{B_a} = 2H_{\mathcal{E}} + 4L$. Therefore, by the definition of M_{B_a}, we see that M_{B_a} is the tautological line bundle associated to $\mathcal{E}(2)$.

Generally, let $\mathcal{F}$ be a stable bundle on $\mathbb{P}^2$ with $c_1(\mathcal{F}) = -1$. In [5], Hulek studies jumping lines for such an $\mathcal{F}$, where a line j on $\mathbb{P}^2$ is called a *jumping line* for $\mathcal{F}$ if $\mathcal{F}_{|\mathsf{j}} \not\simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. We also recall that a line l on $\mathbb{P}^2$ is called a *jumping line of the second kind* for $\mathcal{F}$ if $h^0(\mathcal{F}_{|2\mathsf{l}}) \neq 0$. In [ibid. Thm. 3.2.2], it is shown that the locus $C(\mathcal{F})$ in the dual projective plane $(\mathbb{P}^2)^*$ parameterizing jumping lines of the second kind is a curve of degree $2(c_2(\mathcal{F}) - 1)$. Therefore, in our case, $C(\mathcal{E})$ is a conic. Moreover the following properties of $\mathcal{E}$ hold by [ibid.]:

Proposition 1.2.2. (1) *$\mathcal{E}$ is unique up to an automorphism of $\mathbb{P}^2$,*
(2) *$C(\mathcal{E}) \subset (\mathbb{P}^2)^*$ is a line pair, which we denote by $\ell_1 \cup \ell_2$,*
(3) *$\mathcal{E}$ has a unique jumping line $\subset \mathbb{P}^2$, which we denote by j, and the point $[\mathsf{j}]$ in the dual projective plane $(\mathbb{P}^2)^*$ is equal to $\ell_1 \cap \ell_2$, and*
(4) *$\mathcal{E}_{|\mathsf{j}} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.*

Proof. (1)–(3) follow from [ibid. Prop. 8.2], and (4) follows from [ibid. Prop. 9.1]. □

Notation 1.2.3. For a line $\mathsf{m} \subset \mathbb{P}^2$, we set $L_{\mathsf{m}} := \pi_a^{-1}(\mathsf{m}) \subset B_a$. We denote by $C_0(\mathsf{m})$ the negative section of L_{m}.

Here we can interpret the jumping line of $\mathcal{E}$ by the birational geometry of B_a as follows:

Corollary 1.2.4. *The π_a-image on $\mathbb{P}^2$ of the exceptional curve γ_a of $B_a \to B$ is the jumping line j.*

Proof. By the uniqueness of γ_a, we have only to show that the negative section $C_0(\mathsf{j})$ of L_{j} is numerically trivial for $-K_{B_a}$. By Proposition 1.2.2 (4), we have $H_{\mathcal{E}} \cdot C_0(\mathsf{j}) = -2$. Therefore, since $-K_{B_a} = 2H_{\mathcal{E}} + 4L_{\mathsf{j}}$, we have $-K_{B_a} \cdot C_0(\mathsf{j}) = 2 \times (-2) + 4 = 0$. □

1.3. **Two-ray link.** By [6, Thm. 3.5 and 3.6] and [13, Thm. 2.3], a part of the birational geometry of B_a is described by the following two-ray link:

(1.2)
$$\begin{tikzcd} & B_a \arrow[dl, "\pi_a"'] \arrow[dr] \arrow[rr, dashed] & & B_b \arrow[dl] \arrow[dr, "\pi_b"] & \\ \mathbb{P}^2 & & B & & \mathbb{P}^1, \end{tikzcd}$$

where

(i) $B_a \dashrightarrow B_b$ is the flop of a single smooth rational curve γ_a.
(ii) π_b is a quadric bundle.
(iii) Let L be the pull-back of a line by π_a, and H a fiber of π_b. Then

$$-K = 2(H + L), \tag{1.3}$$

where we consider this equality both on B_a and B_b, and $-K$ denotes both of the anti-canonical divisors.

Notation 1.3.1. (1) We denote by γ_a and γ_b the flopping curves on B_a and B_b, respectively.

(2) It is important to notice that there exist exactly two singular π_b-fibers, which are isomorphic to the quadric cone (this follows from the calculation of the topological Euler number of B_a and invariance of Euler number under flop). We denote them by F_1 and F_2.

Though we mainly work on B_a, the threefold B_b is also useful to understand the properties of B_a related to the jumping lines of the second kind since the definition of such jumping lines is less geometric (see the subsection 2.3).

1.4. **Group action on B_a.** In this subsection, we show that B_a has a natural action by the subgroup of $\mathrm{Aut}\,(\mathbb{P}^2)^*$ fixing $\ell_1 \cup \ell_2$. This fact should be known for experts but we do not know appropriate literatures.

Our way to see this is based on the elementary transformation of the $\mathbb{P}^2$-bundle $\pi_a \colon B_a \to \mathbb{P}^2$ centered at the flopping curve γ_a. This make it possible to describe the group action quite explicitly.

Proposition 1.4.1. *Let $\mu\colon \widetilde{B}_a \to B_a$ be the blow-up along the flopping curve γ_a. Let $\nu\colon \widetilde{B}_a \to B_c$ be the blow down over $\mathbb{P}^2$ contracting the strict transform of $L_{\mathsf{j}} = \pi_a^{-1}(\mathsf{j})$ to a smooth rational curve γ_c (the existence of the blow down follows from Mori theory in a standard way). Then $B_c \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Moreover, γ_c is a divisor of type $(1,2)$ in $\mathbb{P}^1 \times \mathsf{j}$.*

(1.4)
$$\begin{tikzcd} & \widetilde{B}_a \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ B_a \arrow[dr, "\pi_a"'] & & B_c \arrow[dl] \\ & \mathbb{P}^2 & \end{tikzcd}$$

Proof. This follows from [4, p.166, (si111o) Case (a)]. □

Let $(x_1 : x_2)$ be a coordinate of $\mathbb{P}^1$ and $(y_1 : y_2 : y_3)$ be a coordinate of $\mathbb{P}^2$. By a coordinate change, we may assume that $\mathsf{j} = \{y_3 = 0\} \subset \mathbb{P}^2$ and the two ramification points of $\gamma_c \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{p_1} \mathbb{P}^1$ are $(0:1) \times (1:0:0)$ and $(1:0) \times (0:1:0)$. Then $\gamma_c = \{\alpha x_1 y_1^2 + \beta x_2 y_2^2 = y_3 = 0\}$ with $\alpha\beta \neq 0$. By a further coordinate change, we may assume that

$$\gamma_c = \{x_1 y_1^2 + x_2 y_2^2 = y_3 = 0\}. \tag{1.5}$$

Let us denote by G the automorphism group of B_a. Now we can easily obtain the following description of G from Proposition 1.4.1. For this, we denote by $G_m \simeq \mathbb{C}^*$ the multiplicative group and by $G_a \simeq \mathbb{C}$ the additive group.

Corollary 1.4.2. *The automorphism group G of B_a is isomorphic to the subgroup of the automorphism group of B_c which preserves γ_c. Explicitly, let an element $(A,B) \in \mathrm{PGL}_2 \times \mathrm{PGL}_3$ acts on $B_c \simeq \mathbb{P}^1 \times \mathbb{P}^2$ as $(\mathbf{x},\mathbf{y}) \mapsto (A\mathbf{x}, B\mathbf{y})$ by matrix multiplication. If (A,B) preserve γ_c with the equation (1.5) as above, then (A,B) is of the form*

(i) $A = \begin{pmatrix} a_1^2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & a_1 & b_2 \\ 0 & 0 & a_2 \end{pmatrix}$, *or*

(ii) $A = \begin{pmatrix} 0 & a_1^2 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & b_1 \\ a_1 & 0 & b_2 \\ 0 & 0 & a_2 \end{pmatrix}$,

where $a_1, a_2 \in G_m$ and $b_1, b_2 \in G_a$ in both cases.

In particular, the G-orbit of $(1:1) \times (0:0:1)$ in $\mathbb{P}^1 \times \mathbb{P}^2$ is open. Therefore, the action of G on B_c is, and hence the one on B_a is quasi-homogeneous.

It is also easy and is convenient to write down the G-action on the base $\mathbb{P}^2$.

Corollary 1.4.3. (1) *The projective plane $\mathbb{P}^2$ consists of the following three orbits of G :*

$$\mathbb{P}^2 = G \cdot (0:0:1) \sqcup G \cdot (1:1:0) \sqcup \{(1:0:0) \sqcup (0:1:0)\},$$

where $G \cdot (0:0:1)$ is the open orbit, $G \cdot (1:1:0)$ is an open subset of the jumping line $\mathsf{j} := \{y_3 = 0\}$, and the two points $(1:0:0), (0:1:0) \in \mathsf{j}$ form one orbit and correspond to the lines ℓ_1 and ℓ_2 by projective duality.

(2) *The dual projective plane $(\mathbb{P}^2)^*$ has the following three orbits of G by the contragredient action of G :*

$$(\mathbb{P}^2)^* = G \cdot (1:1:0) \sqcup \{G \cdot (1:0:0) \sqcup G \cdot (0:1:0)\} \sqcup (0:0:1),$$

where $G \cdot (1:1:0)$ is the open orbit, the closures of $G \cdot (1:0:0)$ and $G \cdot (0:1:0)$ are the two lines ℓ_1 and ℓ_2.

Proof. We only show that the two points $(1:0:0), (0:1:0) \in \mathsf{j}$ correspond to the lines ℓ_1 and ℓ_2 by projective duality. This follows from the orbit decomposition of $\mathbb{P}^2$ by the identity component G_0 of G since the two points $\in \mathbb{P}^2$ corresponding to the lines ℓ_1 and ℓ_2 are fixed by G_0, and G_0 has only two fixed points. □

In the section 5, a central role is played by the following explicit description of the action of G on B_a preserving L_{m} for a general m. By quasi-homogenousity of the action on B_a, we may assume that $\mathsf{m} = \{y_1 = y_2\}$.

Lemma 1.4.4. *An element $(A,B) \in \mathrm{PGL}_2 \times \mathrm{PGL}_3$ of G preserves L_{m}, equivalently, preserves m if and only if (A,B) is of the form*

$$(a)\ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_1 \\ 0 & 0 & a_2 \end{pmatrix},$$

or

$$(b)\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 1 & b_1 \\ 1 & 0 & b_1 \\ 0 & 0 & a_2 \end{pmatrix},$$

where $a_2 \in G_m$ and $b_1 \in G_a$ in both cases.

In particular, such elements form a subgroup $\Gamma \simeq (\mathbb{Z}_2 \times G_a) \rtimes G_m$ and Γ is generated by the following three type elements:

- $G_m : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$ *with $a \in G_m$,*
- $G_a : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$ *with $b \in G_a$, and*
- $\mathbb{Z}_2 : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2. Families of rational curves on B_a

In this section, we construct families of rational curves on B_a, which will ties the geometries of B_a and one-pointed ineffective spin hyperelliptic curves. We start by some preliminary discussions.

Lemma 2.0.5. *If a line m is not equal to the jumping line j, then $(H-L)_{|L_\mathsf{m}}$ is linearly equivalent to the negative section $C_0(\mathsf{m})$ of $L_\mathsf{m} \simeq \mathbb{F}_1$. If $\mathsf{m} = \mathsf{j}$, then $(H-L)_{|L_\mathsf{m}}$ is linearly equivalent to the negative section $C_0(\mathsf{j})$ plus a ruling of $L_\mathsf{j} \simeq \mathbb{F}_3$.*

Proof. As we mention in the subsection 1.2, M_{B_a} is the tautological line bundle on B_a associated to the bundle $\mathcal{E}(2)$. Therefore, by (1.3), $H - L = M_{B_a} - 2L$ is the tautological line bundle associated to the bundle $\mathcal{E}$. If m is not equal to the jumping line j, then $\mathcal{E}_{|\mathsf{m}} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and hence $(H-L)_{|L_\mathsf{m}}$ is linearly equivalent to the negative section $C_0(\mathsf{m})$ of $L_\mathsf{m} \simeq \mathbb{F}_1$. If $\mathsf{m} = \mathsf{j}$, then $\mathcal{E}_{|\mathsf{m}} \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and hence $(H-L)_{|L_\mathsf{m}}$ is linearly equivalent to the negative section $C_0(\mathsf{j})$ plus a ruling of $L_\mathsf{j} \simeq \mathbb{F}_3$. □

By this lemma, it is easy to show the following proposition:

Proposition 2.0.6. *Let $\mathsf{m} \subset \mathbb{P}^2$ be a line and $g \geq -1$ an integer. If $\mathsf{m} \neq \mathsf{j}$ (resp. $\mathsf{m} = \mathsf{j}$ and $g \geq 1$), then a general element R of the linear system $|(H+gL)_{|L_\mathsf{m}}|$ is a smooth rational curve with $H \cdot R = g+1$ and $L \cdot R = 1$. Moreover, if $\mathsf{m} \neq \mathsf{j}$ and $g \geq 0$ (resp. $\mathsf{m} = \mathsf{j}$ and $g \geq 1$), then $|(H+gL)_{|L_\mathsf{m}}|$ has no base point.*

Definition 2.0.7. We define $\mathcal{L}$ to be the following subvariety of $B_a \times (\mathbb{P}^2)^*$:

$$\mathcal{L} := \{(x, [\mathsf{m}]) \mid x \in L_\mathsf{m} = \pi_a^{-1}(\mathsf{m})\}.$$

Let $p_1 \colon \mathcal{L} \to B_a$ and $p_2 \colon \mathcal{L} \to (\mathbb{P}^2)^*$ be the first and the second projections, respectively. Note that the p_2-fiber over a point $[\mathsf{m}]$ is nothing but L_m.

Remark 2.0.8. To follow the sequel easily, it is useful to notice that $\mathcal{L}$ is the pull-back by the composite $B_a \times (\mathbb{P}^2)^* \overset{\pi_a \times \mathrm{id}}{\longrightarrow} \mathbb{P}^2 \times (\mathbb{P}^2)^*$ of the point-line incidence variety $\{(x, [\mathsf{m}]) \mid x \in \mathsf{m}\} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$. Therefore, we also see that $\mathcal{L}$ is G-invariant, where the G-action is induced on $B_a \times (\mathbb{P}^2)^*$ by the G-action on B_a defined as above and the contragredient G-action on $(\mathbb{P}^2)^*$.

2.1. **Higher degree case.**

Definition 2.1.1. (1) For an integer $g \geq 0$, we set

$$\mathcal{R}_g := p_{2*}p_1^*\mathcal{O}_{B_a}(H + gL).$$

We see that $\dim H^0(\mathcal{O}_{L_\mathsf{m}}(H+gL))$ is constant since $H^1(\mathcal{O}_{L_\mathsf{m}}(H+gL)) = \{0\}$ for any m and $g \geq 0$. Therefore, by Grauert's theorem, $\mathcal{R}_g$ is a locally free sheaf on $(\mathbb{P}^2)^*$. Set

$$\Sigma_g := \mathbb{P}(\mathcal{R}_g^*),$$

which is nothing but the projective bundle over $(\mathbb{P}^2)^*$ whose fiber over a point $[\mathsf{m}]$ is the projective space $\mathbb{P}(H^0(\mathcal{O}_{L_\mathsf{m}}(H + gL)))$.

(2) We denote by $\mathcal{H}_g \subset \Sigma_g$ the sublocus parameterizing smooth rational curves. Note that $\mathcal{H}_g$ is a non-empty open subset of Σ_g by Proposition 2.0.6.

2.2. B_a**-Lines.** Now we construct a family of curves parameterizing the negative section of L_m for an $\mathsf{m} \neq \mathsf{j}$, and the negative section plus a ruling of L_j. Intuitively, it is easy to imagine such a family but a rigorous construction needs some works.

Lemma 2.2.1. *The following hold*:

(1) $H^0(\mathcal{O}_{B_a}(H - L)) = \{0\}$ *and* $H^1(\mathcal{O}_{B_a}(H - L)) = \mathbb{C}$.

(2) $H^0(\mathcal{O}_{B_a}(H - 2L)) = \{0\}$, $H^1(\mathcal{O}_{B_a}(H - 2L)) = \mathbb{C}^2$, *and* $H^2(\mathcal{O}_{B_a}(H - 2L)) = \{0\}$.

Proof. The results follow easily from the exact sequence (1.1). Here we only show that $H^2(\mathcal{O}_{B_a}(H - 2L)) \simeq H^2(\mathbb{P}^2, \mathcal{E}(-1)) = \{0\}$, By the Serre duality, we have $H^2(\mathbb{P}^2, \mathcal{E}(-1)) \simeq H^0(\mathbb{P}^2, \mathcal{E}^*(-2))^* \simeq H^0(\mathbb{P}^2, \mathcal{E}(-1))^*$, which is zero by (1.1). □

Notation 2.2.2. Let $b\colon \widetilde{(\mathbb{P}^2)^*} \to (\mathbb{P}^2)^*$ be the blow-up at the point $[\mathsf{j}]$. Let E_0 be the b-exceptional curve, and r be a ruling of $\widetilde{(\mathbb{P}^2)^*} \simeq \mathbb{F}_1$. The surface $\widetilde{(\mathbb{P}^2)^*}$ will be the parameter space of the family of rational curves which we are going to construct.

For a point $[\mathsf{m}] \in (\mathbb{P}^2)^* \setminus [\mathsf{j}]$, we use the same character $[\mathsf{m}]$ for the corresponding point on $\widetilde{(\mathbb{P}^2)^*}$.

Let $b_{\mathcal{L}}\colon \widetilde{\mathcal{L}} \to \mathcal{L}$ be the blow-up along the fiber of $p_2\colon \mathcal{L} \to (\mathbb{P}^2)^*$ over $[\mathsf{j}]$. By universality of blow-up, the variety $\widetilde{\mathcal{L}}$ is contained in $B_a \times \widetilde{(\mathbb{P}^2)^*}$ and a unique map $\tilde{p}_2\colon \widetilde{\mathcal{L}} \to \widetilde{(\mathbb{P}^2)^*}$ is induced. We denote by $\tilde{p}_1\colon \widetilde{\mathcal{L}} \to B_a$ the map obtained by composing $\widetilde{\mathcal{L}} \to \mathcal{L}$ with $p_1\colon \mathcal{L} \to B_a$.

$$\begin{tikzcd} \widetilde{\mathcal{L}} \arrow[r, "b_{\mathcal{L}}"] \arrow[rr, bend left, "\tilde{p}_1"] \arrow[d, "\tilde{p}_2"'] & \mathcal{L} \arrow[r, "p_1"] \arrow[d, "p_2"'] & B_a \\ \widetilde{(\mathbb{P}^2)^*} \arrow[r, "b"] & (\mathbb{P}^2)^* & \end{tikzcd} \tag{2.1}$$

Lemma 2.2.3. *It holds that* $H^0(\tilde{p}_1^*\mathcal{O}_{B_a}(H-L) \otimes \tilde{p}_2^*\mathcal{O}(E_0+2r)) \simeq \mathbb{C}$.

Proof. Let $\tilde{\rho}_1\colon B_a \times \widetilde{(\mathbb{P}^2)^*} \to B_a$ and $\rho_1\colon B_a \times (\mathbb{P}^2)^* \to B_a$ be the first projections, and $\tilde{\rho}_2\colon B_a \times \widetilde{(\mathbb{P}^2)^*} \to \widetilde{(\mathbb{P}^2)^*}$ and $\rho_2\colon B_a \times (\mathbb{P}^2)^* \to (\mathbb{P}^2)^*$ the second projections. By Remark 2.0.8, as a divisor on $B_a \times (\mathbb{P}^2)^*$, $\mathcal{L}$ is linearly equivalent to $\rho_1^*L + \rho_2^*\mathcal{O}_{(\mathbb{P}^2)^*}(1)$. Since $\mathcal{L}$ does not contain the fiber of $B_a \times (\mathbb{P}^2)^* \to (\mathbb{P}^2)^*$ over $[\mathsf{j}]$, the variety $\widetilde{\mathcal{L}}$ is the total pull-back of $\mathcal{L}$ by $B_a \times \widetilde{(\mathbb{P}^2)^*} \to B_a \times (\mathbb{P}^2)^*$. Hence $\widetilde{\mathcal{L}}$ is linearly equivalent to $\tilde{\rho}_1^*L + \tilde{\rho}_2^*(E_0+r)$ since $\mathcal{O}(E_0+r) = b^*\mathcal{O}_{(\mathbb{P}^2)^*}(1)$.

Now let us consider the following exact sequence:

$$\begin{aligned} 0 \to \widetilde{\rho}_1^*\mathcal{O}_{B_a}(H-2L) \otimes \widetilde{\rho}_2^*\mathcal{O}(-E_0-r) \to \\ \widetilde{\rho}_1^*\mathcal{O}_{B_a}(H-L) \to \widetilde{p}_1^*\mathcal{O}_{B_a}(H-L) \to 0, \end{aligned}$$

which is obtained from the natural exact sequence

$$0 \to \mathcal{O}_{B_a \times \widetilde{(\mathbb{P}^2)^*}}(-\widetilde{\mathcal{L}}) \to \mathcal{O}_{B_a \times \widetilde{(\mathbb{P}^2)^*}} \to \mathcal{O}_{\widetilde{\mathcal{L}}} \to 0.$$

by tensoring $\widetilde{\rho}_1^*\mathcal{O}_{B_a}(H-L)$. By Lemma 2.2.1, the pushforward of the exact sequence by $\widetilde{\rho}_2$ is

$$0 \to \tilde{p}_{2*}\tilde{p}_1^*\mathcal{O}_{B_a}(H-L) \to \mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O} \to R^1\tilde{p}_{2*}\tilde{p}_1^*\mathcal{O}_{B_a}(H-L) \to 0.$$

Note that, for a point $[\mathsf{m}] \neq [\mathsf{j}]$, it holds that $H^0(\mathcal{O}_{L_\mathsf{m}}(H-L)) \simeq \mathbb{C}$ and $H^1(\mathcal{O}_{L_\mathsf{m}}(H-L)) = \{0\}$ by Lemma 2.0.5. Therefore, by Grauert's theorem, $\tilde{p}_{2*}\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)$ is an invertible sheaf possibly outside E_0, and the support of $R^1 := R^1\tilde{p}_{2*}\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)$ is contained in E_0.

We show that the support of R^1 is equal to E_0. Indeed, let $\mathcal{I}$ be the image of the map $\mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O}$ in the above exact sequence, which is an ideal sheaf. Then the closed subscheme Δ defined by $\mathcal{I}$ is the intersection of one or two members of $|E_0+r|$. In particular, Δ is non-empty. Noting $\mathcal{O}_\Delta = R^1$ and the support of R^1 is contained in E_0, the subscheme Δ must be equal to E_0.

Therefore, the map $\mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O}$ is decomposed as $\mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O}(-E_0) \hookrightarrow \mathcal{O}$ and $\mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O}(-E_0)$ is surjective. Hence the kernel $\tilde{p}_{2*}\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)$ of the map $\mathcal{O}(-E_0-r)^{\oplus 2} \to \mathcal{O}(-E_0)$ is isomorphic to

$\mathcal{O}(-E_0 - 2r)$. Now we can compute

$$\begin{aligned} H^0(\widetilde{p}_1^*\mathcal{O}_{B_a}(H-L)\otimes \widetilde{p}_2^*\mathcal{O}(E_0+2r)) &\simeq \\ H^0(\widetilde{p}_{2*}\widetilde{p}_1^*\mathcal{O}_{B_a}(H-L)\otimes \mathcal{O}(E_0+2r)) &\simeq \\ H^0(\mathcal{O}(-E_0-2r)\otimes \mathcal{O}(E_0+2r)) &\simeq \mathbb{C}. \end{aligned}$$

□

In the next proposition, we obtain the desired family of curves.

Proposition 2.2.4. *Let $\mathcal{U}_1$ be the unique member of $|\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)\otimes \tilde{p}_2^*\mathcal{O}(E_0+2r)|$. Then the natural map $\mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ is flat. Moreover, the fibers are described as follows:*

(1) *the fiber over a point $[\mathsf{m}] \neq [\mathsf{j}]$ is the negative section of L_{m}, and*
(2) *the fiber over a point x of E_0 is the negative section plus a ruling of L_{j}.*

Proof. Note that $\mathcal{U}_1$ is Cohen-Macaulay since it is a divisor on a smooth variety. Therefore the flatness follows from the smoothness of $\widetilde{(\mathbb{P}^2)^*}$ and the descriptions of fibers, which we are going to give below.

Note that, by the uniqueness of $\mathcal{U}_1$, the group G acts on $\mathcal{U}_1$, where G acts on $\mathcal{L}$ and hence on $\widetilde{\mathcal{L}}$ by Remark 2.0.8. Let x be a point of $\widetilde{(\mathbb{P}^2)^*}$. Set $[\mathsf{m}] := b(x) \in (\mathbb{P}^2)^*$. Note that the fiber of $\widetilde{\mathcal{L}} \to \widetilde{(\mathbb{P}^2)^*}$ over x is L_{m}.

If $x \notin E_0$, then $L_{\mathsf{m}} \subset \mathcal{U}_1$ or $\mathcal{U}_{1|L_{\mathsf{m}}}$ is the negative section of $L_{\mathsf{m}} \simeq \mathbb{F}_1$ since $\mathcal{U}_1 \in |\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)\otimes \tilde{p}_2^*\mathcal{O}(E_0+2r)|$. We show that the latter occurs for any $x \notin E_0$, which implies the assertion (1). If $L_{\mathsf{m}} \subset \mathcal{U}_1$ for an $x \notin E_0$ such that $[\mathsf{m}] \notin \ell_1 \cup \ell_2$, then, by the description of the group action of G (Corollary 1.4.3), $L_{\mathsf{m}} \subset \mathcal{U}_1$ hold for all such x's, which implies that $\mathcal{U}_1 = \widetilde{\mathcal{L}}$, a contradiction. If $L_{\mathsf{m}} \subset \mathcal{U}_1$ for an $x \notin E_0$ such that $[\mathsf{m}] \in \ell_i$ for $i = 1, 2$, then, again by the group action of G, $L_{\mathsf{m}} \subset \mathcal{U}_1$ hold for all such x's, which implies that $\mathcal{U}_1$ contains the pull-back of the strict transform $\ell'_i \subset \widetilde{(\mathbb{P}^2)^*}$ of ℓ_i. Since ℓ'_i is a ruling of $\widetilde{(\mathbb{P}^2)^*}$, this implies that $H^0(\tilde{p}_1^*\mathcal{O}_{B_a}(H-L)\otimes \tilde{p}_2^*\mathcal{O}(E_0+r)) \neq 0$, which is impossible by the proof of Lemma 2.2.3.

Now assume that $x \in E_0$. By a similar argument to the above one using the group action, we see that $\mathcal{U}_{1|L_{\mathsf{j}}\times\{x\}}$ is the negative section plus a ruling if x is not contained in the strict transforms ℓ'_i of ℓ_i $(i = 1, 2)$. Therefore $\mathcal{U}_{1|L_{\mathsf{j}}\times E_0}$ is a member of the linear system $|\mathcal{O}_{L_{\mathsf{j}}}(H-L)\boxtimes \mathcal{O}_{E_0}(1)|$ on $L_{\mathsf{j}} \times E_0$. Suppose by contradiction that $\mathcal{U}_{1|L_{\mathsf{j}}\times\{x\}} = L_{\mathsf{j}} \times \{x\}$ for $x = \ell'_1 \cap E_0$ or $\ell'_2 \cap E_0$. Then, since the group action interchanges $\ell'_1 \cap E_0$ and $\ell'_2 \cap E_0$, $\mathcal{U}_{1|L_{\mathsf{j}}\times\{x\}} = L_{\mathsf{j}} \times \{x\}$ for both $x = \ell'_1 \cap E_0$ and $\ell'_2 \cap E_0$. This would imply that $|\mathcal{O}_{L_{\mathsf{j}}}(H-L)\boxtimes \mathcal{O}_{E_0}(-1)|$ is nonempty, which is absurd.

Therefore the assertion (2) follows. □

Definition 2.2.5. We call a fiber of $\mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ *a B_a-line.* Explicitly, by Proposition 2.2.4, a B_a-line is the negative section $C_0(\mathsf{m})$ of L_{m} for $[\mathsf{m}] \neq [\mathsf{j}]$, or the negative section $C_0(\mathsf{j})$ plus a ruling of L_{j}.

The name comes from the fact that the image of a B_a-line on the anti-canonical model B is a line in the usual sense when B is embedded by $|M_B|$, where M_B is the ample generator of $\operatorname{Pic} B$.

2.3. **B_a-Lines interpreted on B_b.** In the section 3, we will construct hyperelliptic curves using the map $\tilde{p}_{1|\mathcal{U}_1}\colon \mathcal{U}_1 \to B_a$. To understand the map $\tilde{p}_{1|\mathcal{U}_1}$, it is convenient to interpret B_a-lines by the geometry of B_b.

Notation 2.3.1. (1) We denote by F_1 and F_2 the two singular π_b-fibers and by v_i the vertex of F_i $(i = 1, 2)$.
(2) We denote by F_i' the strict transform on B_a of F_i $(i = 1, 2)$.
(3) By Corollary 1.2.4, we have $L \cdot \gamma_a = 1$, and, by a standard property of flop, we have $L \cdot \gamma_b = -1$. This and the equality (1.3) imply that γ_b is a π_b-section. Therefore γ_b does not pass through v_1 nor v_2 and so $B_b \dashrightarrow B_a$ is isomorphic near v_1 and v_2. We denote by v_i' the point on B_a corresponding to v_i $(i = 1, 2)$.

Proposition 2.3.2. (1) *The π_a-images of v_1' and v_2' in $\mathbb{P}^2$ correspond to the lines ℓ_1 and ℓ_2 in $(\mathbb{P}^2)^*$ by projective duality.*
(2) *For a line $\mathsf{m} \neq \mathsf{j}$ on $\mathbb{P}^2$, the negative section $C_0(\mathsf{m})$ of L_{m} is disjoint from γ_a.*
(3) *For a line $\mathsf{m} \neq \mathsf{j}$ on $\mathbb{P}^2$, the curve $C_0(\mathsf{m})$ is the strict transform of a ruling of a π_b-fiber disjoint from γ_b, and vice-versa. Moreover, under this condition, $C_0(\mathsf{m})$ is the strict transform of a ruling of F_1 or F_2 if and only if $[\mathsf{m}] \in \ell_1 \cup \ell_2$.*
(4) *A ruling f of L_{j} is the strict transform of a ruling of a π_b-fiber intersecting γ_b, and vice-versa (note that $f \cap \gamma_a \neq \emptyset$, and $\gamma_a \cup f$ is a B_a-line). Moreover, under this condition, f is the strict transform of a ruling of F_1 or F_2 if and only if the point $\pi_a(f) \in \mathbb{P}^2$ corresponds to the line ℓ_1 or ℓ_2 in $(\mathbb{P}^2)^*$ by projective duality.*

Proof. We show the assertion (1). We use the group actions of G on B_a and B_b. The action of G on B_b fixes or interchanges F_1 and F_2, and hence v_1 and v_2. Since $B_b \dashrightarrow B_a$ is isomorphic near v_1 and v_2 as we noted in Notation 2.3.1, the group action on B_a fixes or interchanges v_1' and v_2'. By Corollary 1.4.3, this implies that the images of v_1' and v_2' correspond to the lines ℓ_1 and ℓ_2 by projective duality.

We show the assertion (2). Let $C_0'(\mathsf{m})$ be the strict transform of $C_0(\mathsf{m})$ on B_b. Note that $H \cdot C_0(\mathsf{m}) = 0$ by Lemma 2.0.5. If $C_0(\mathsf{m}) \cap \gamma_a \neq \emptyset$, then $H \cdot C_0'(\mathsf{m}) < H \cdot C_0(\mathsf{m}) = 0$ by a standard property of flop, which is a contradiction since H is nef on B_b.

We show the first assertions of (3) and (4). Since the proofs are similar, we only show (4), which is more difficult. We also only prove the only if part since the if part follows by reversing the argument. Recall that $\gamma_a + f \sim (H - L)_{|L_{\mathsf{j}}}$. Thus $H \cdot f = 1$. Since f intersects γ_a transversely at one point, and $H \cdot \gamma_a = -1$, we have $H \cdot f' = 0$, where f' is the strict transform of f on B_b. Hence f' is contained in a π_b-fiber F. By the equality

(1.3), we have $-K_F = -K_{B_b}|_F = 2L|_F$. Therefore f' is a ruling of F since $L \cdot f' = L \cdot f + 1 = 1$.

The latter assertions of (3) and (4) follows from (1). $\square$

Corollary 2.3.3. *Let x be a point of $B_a \setminus \gamma_a$. If x is not in the strict transform of F_1 nor F_2, then x is contained in exactly two B_a-lines. If x is in the strict transform of F_1 or F_2 and is not equal to v_1' nor v_2', then x is contained in exactly one B_a-line.*

In particular, outside $\gamma_a \cup v_1' \cup v_2'$, the map $\tilde{p}_{1|\mathcal{U}_1} \colon \mathcal{U}_1 \to B_a$ is finite of degree two and is branched along F_1' and F_2'.

Proof. The assertions follow from Proposition 2.3.2 (3) and (4), and the description of rulings on quadric surfaces. $\square$

3. Hyperelliptic curves parameterizing B_a-lines

Definition 3.0.4. Let $\mathsf{m} \subset \mathbb{P}^2$ be a line and $R \subset L_{\mathsf{m}}$ a (not necessarily irreducible) member of the linear system $|(H + gL)_{|L_{\mathsf{m}}}|$ (cf. the subsection 2.1).

(1) We define
$$C_R := \tilde{p}_1^{-1}(R) \cap \mathcal{U}_1 \subset \widetilde{\mathcal{L}}$$
with the notation as in the diagram (2.1). Here we take the intersection scheme-theoretically as follows: first we consider $\tilde{p}_1^{-1}(L_m) \cap \mathcal{U}_1$, which is a divisor in $\mathcal{U}_1$. Second, we consider $C_R = \tilde{p}_1^{-1}(R) \cap \mathcal{U}_1$ as a divisor in $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1$.

(2) We define $\widetilde{M}_R$ to be the image of C_R on $\widetilde{(\mathbb{P}^2)^*}$, and M_R to be the image of C_R on $(\mathbb{P}^2)^*$. Note that $\widetilde{M}_R$ parameterizes B_a-lines intersecting R.

(3) For a π_a-fiber f, we also define C_f, $\widetilde{M}_f$, and M_f in a similar fashion to (1) and (2).

In Proposition 3.0.8 below, we are going to show that C_R is a hyperelliptic curve of genus g under the following generality conditions for m and R as in Definition 3.0.4:

Generality Condition 3.0.5. Let $\mathsf{m} \subset \mathbb{P}^2$ be a line and $R \subset L_{\mathsf{m}}$ a member of the linear system $|(H + gL)_{|L_{\mathsf{m}}}|$. We consider the following conditions for m and R:

(a) $[\mathsf{m}] \notin \ell_1 \cup \ell_2$. In particular, $v_1', v_2' \notin R$ by Proposition 2.3.2 (1).
(b) R is smooth.
(c) $R \cap \gamma_a = \emptyset$.
(d) R intersects F_1' and F_2' transversely at $g + 1$ points, respectively (note that, by $R \sim (H + gL)_{|L_{\mathsf{m}}}$, we have $F_i' \cdot R = H \cdot R = g + 1$).

Note that the condition (c) implies that $R \cap F_1' \cap F_2' = R \cap \gamma_a = \emptyset$.

It is easy to see that, if $g \geq 0$, then general m and R satisfy these conditions by Proposition 2.0.6.

Lemma 3.0.6. *If* $[\mathsf{m}] \notin \ell_1 \cup \ell_2$*, then* $F'_{i|L_{\mathsf{m}}}$ *is linearly equivalent to* $C_0(\mathsf{m}) + L_{|L_{\mathsf{m}}}$*, and is irreducible* ($i = 1, 2$)*. In particular,* $C_0(\mathsf{m})$ *is disjoint from* F'_i*.*

Proof. We see that $F'_{i|L_{\mathsf{m}}}$ is linearly equivalent to $C_0(\mathsf{m}) + L_{|L_{\mathsf{m}}}$ by (1.3) since $F'_i \sim H$. Assume by contradiction that $F'_{i|L_{\mathsf{m}}}$ is reducible. Then $C_0(\mathsf{m}) \subset F'_{i|L_{\mathsf{m}}}$. Therefore, by Proposition 2.3.2 (3), $C_0(\mathsf{m})$ passes through v'_i, which contradicts the assumption that $[\mathsf{m}] \notin \ell_1 \cup \ell_2$. □

Lemma 3.0.7. *Assume that a* π_a*-fiber* f *is disjoint from* γ_a*. Then the following hold:*

(1) $v'_1, v'_2 \notin f$.

(2) *f intersects F'_1 and F'_2 at one point, respectively, and* $f \cap F'_1 \cap F'_2 = \emptyset$*.*

Proof. The assumption $f \cap \gamma_a = \emptyset$ is equivalent to that $\pi_a(f)$ belongs to the open orbit of G. Therefore the assertion (1) follows from Corollary 1.4.3 and Proposition 2.3.2 (1).

We show that f intersects F'_i ($i = 1, 2$) at one point. By (1.3), we have $F'_i \cdot f = H \cdot f = 1$ since $-K_{B_a} \cdot f = 2$ and $L \cdot f = 0$. Therefore, we have only to show that f is not contained in F'_i. If $f \subset F'_i$, then the strict transform f' of f is contained in F_i. Then, however, $-K_{F_i} \cdot f' = 2L_{|F_i} \cdot f' = 0$, a contradiction.

The assumption implies that $f \cap F'_1 \cap F'_2 = f \cap \gamma_a = \emptyset$. Therefore we have the assertion (2).

□

Proposition 3.0.8. *Assume that* $g \geq 2$*, and* m *and* R *satisfy Generality Condition* 3.0.5 (a)–(d)*. Then the following hold:*

(1) *C_R is a smooth hyperelliptic curve of genus g. The hyperelliptic structure is given by the map* $\tilde{p}_{1|C_R} \colon C_R \to R$ *and the map is branched at* $R \cap (F'_1 \cup F'_2)$*.*

(2) *Assume that a π_a-fiber f is disjoint from γ_a. Then C_f is a smooth rational curve and $C_f \to f$ is a double cover branched at the two points* $f \cap (F'_1 \cup F'_2)$*. Moreover, M_f is the line of* $(\mathbb{P}^2)^*$ *corresponding to the point* $\pi_a(f) \in \mathbb{P}^2$ *by projective duality.*

(3) $\widetilde{M}_R \subset \widetilde{(\mathbb{P}^2)^*}$ *is a curve which is smooth outside the point* $[\mathsf{m}]$ *and has a g-ple point at* $[\mathsf{m}]$*.*

(4) $\deg M_R = g + 2$ *and* $M_R \simeq \widetilde{M}_R$*. Moreover, the unique* g^1_2 *on* C_R *is given by the pull-back of the pencil of lines through* $[\mathsf{m}]$*.*

Proof. We use the notation in Section 2 freely.

(1). By Corollary 2.3.3 and Generality Condition 3.0.5 (a)–(d), we see that C_R with reduced structure satisfies all of the claimed properties. Therefore, we have only to show C_R is reduced. It suffices to show this for a general R since C_R form a flat family for R's with Generality Condition 3.0.5 (a)–(d). By the Bertini theorem on $\mathcal{L}$, the divisor $\tilde{p}_1^{-1}(L_m) \cap \mathcal{U}_1$ is a reduced surface since $|L|$ has no base point. Now, again by the Bertini theorem in

$\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1$, the intersection $C_R = \tilde{p}_1^{-1}(R) \cap \mathcal{U}_1$ is also reduced, and we are done.

(2). The assertions for C_f can be proved similarly to (1) by Lemma 3.0.7. As for M_f, note that f intersects the negative sections of L_{m}'s such that $\mathsf{m} \ni \pi_a(f)$. Therefore the assertion follows since M_f is the image of the smooth curve C_f.

To show the remaining assertions, we investigate fibers of $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$. For a point $s \in \widetilde{(\mathbb{P}^2)^*}$, the fiber over s is the intersection between L_{m} and the B_a-line corresponding to s. Therefore the fiber over $[\mathsf{m}]$ can be identified with the negative section $C_0(\mathsf{m})$ of L_{m}. Recall that E_0 is as in Notation 2.2.2. Let t be the point of E_0 over which the fiber of $\mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ is the union of γ_a and the ruling of L_{j} over $\mathsf{j} \cap \mathsf{m}$. Then the fiber of $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ over t is the ruling of L_{j} over $\mathsf{j} \cap \mathsf{m}$. Besides, over $\widetilde{(\mathbb{P}^2)^*} \setminus ([\mathsf{m}] \cup t)$, the map $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ is one to one, hence is an isomorphism by the Zariski main theorem.

Note that the map $\mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ is smooth over $\widetilde{(\mathbb{P}^2)^*} \setminus E_0$. Therefore, $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1$ is smooth possibly outside the fiber over t. Therefore $\tilde{p}_1^{-1}(L_{\mathsf{m}}) \cap \mathcal{U}_1 \to \widetilde{(\mathbb{P}^2)^*}$ is the blow-up at $[\mathsf{m}]$ near $[\mathsf{m}]$. We denote by E_{m} and E_t the exceptional curves over $[\mathsf{m}]$ and t, respectively.

(3). By Lemma 3.0.6, $\tilde{p}_1^{-1}(C_0(\mathsf{m})) \cap \mathcal{U}_1 \to C_0(\mathsf{m})$ is an étale double cover, and hence $\tilde{p}_1^{-1}(C_0(\mathsf{m})) \cap \mathcal{U}_1$ consists of two disjoint smooth rational curves. Since $E_{\mathsf{m}} \subset \tilde{p}_1^{-1}(C_0(\mathsf{m})) \cap \mathcal{U}_1$, we see that E_{m} is one of its components by the above description of E_{m}. Thus, since R intersects $C_0(\mathsf{m})$ transversely at g points, the curve C_R intersects E_{m} transversely at g points. Therefore we have the assertion (3) by blowing down E_{m}, except that we postpone proving $\widetilde{M}_R$ is smooth outside $[\mathsf{m}]$ in (4).

(4). To compute $\deg M_R$, we take a general fiber f of $L_{\mathsf{m}} \to \mathsf{m}$ such that $f \cap \gamma_a = \emptyset$ and $R \cap f \not\in F_1' \cup F_2' \cup C_0(\mathsf{m})$. Then, since $R \cap f \not\in F_1' \cup F_2'$, C_R and C_f intersect transversely at two points, which is the inverse image of one point $R \cap f$. Since $R \cap f \not\in C_0(\mathsf{m})$, C_R and C_f does not intersect on E_{m}. Therefore, the intersection multiplicity of M_R and M_f at $[\mathsf{m}]$ is g. Thus we conclude that $\deg M_R = M_R \cdot M_f = g + 2$ since M_f is a line by generality of f and the assertion (2).

Now the facts that $\widetilde{M}_R$ is smooth outside $[\mathsf{m}]$ and $\widetilde{M}_R \simeq M_R$ follow since $g(C_R) = g$, $\deg M_R = g + 2$ and M_R has a g-ple point at $[\mathsf{m}]$.

□

Remark 3.0.9 (Hyperelliptic structure of C_R via the geometry of B_b)**.** Using the interpretation of B_a-lines on B_b, we may describe the hyperelliptic structure of C_R on B_b. We think that it is helpful for the readers to bear this in mind, so we give a sketch of it.

By Proposition 2.3.2 (3) and (4), B_a-lines correspond to rulings of π_b-fibers in a one to one way. Thus we may identify C_R with the relative Hilbert scheme H_R of rulings of π_b-fibers. The natural map $H_R \to \mathbb{P}^1$, where $\mathbb{P}^1$ is the target of π_b, is a double cover branched at the images of two singular π_b-fibers since a smooth quadric has two families of rulings while a singular quadric has one such a family.

Notation 3.0.10 (The marked point $[\mathsf{j}]_R$ on C_R). Assume that R satisfies Generality Condition 3.0.5 (a)–(d). Then there is a unique B_a-line intersecting R of the form γ_a plus a ruling, which is $\gamma_a \cup (L_{\mathsf{m}} \cap L_{\mathsf{j}})$. We denote by $[\mathsf{j}]_R$ the point of the hyperelliptic curve C_R corresponding to this B_a-line since this point is mapped to $[\mathsf{j}] \in M_R \subset (\mathbb{P}^2)^*$.

4. Theta characteristics on the hyperelliptic curves.

4.1. **Constructing theta characteristics.** By the above understanding of the hyperelliptic double cover $C_R \to R$, we may construct an ineffective theta characteristic on C_R as follows:

Proposition 4.1.1. *For a curve R satisfying Generality Condition* 3.0.5 (a)–(d) *and $g \geq 2$, we denote by h_R the unique g^1_2 on the hyperelliptic curve C_R (cf. Proposition* 3.0.8 (1) *and* (4)*). Let $\nu\colon C_R \to M_R$ be the morphism constructed in Proposition* 3.0.8*, which is the normalisation. Then*

$$\mathcal{O}_{C_R}(\theta_R) := \nu^*\mathcal{O}_{M_R}(1) \otimes_{\mathcal{O}_{C_R}} \mathcal{O}_{C_R}(-h_R - [\mathsf{j}]_R)$$

is an ineffective theta characteristic on C_R.

Proof. Let F be one of the two singular π_b-fibers and F' its strict transform on B_a. By Generality Condition 3.0.5 (d), R intersects F' transversely at $g+1$ points, which we denote by $s_1, \ldots, s_{g+1}$. By Proposition 3.0.8 (1), these points are contained in the branched locus of the hyperelliptic double cover $C_R \to R$. We denote by $t_1, \ldots, t_{g+1}$ the inverse images on C_R of $s_1, \ldots, s_{g+1}$, and by $u_1, \ldots, u_{g+1}$ the images on M_R of $t_1, \ldots, t_{g+1}$. Then, by Proposition 2.3.2 (3) and (4), $u_1, \ldots, u_{g+1}$ are contained in $\ell := \ell_1$ or ℓ_2. We show that the points $u_1, \ldots, u_{g+1}$ are different from $[\mathsf{j}]$. Note that the unique B_a-line through a point u_i is the strict transform l_i of a ruling of F, or the union of l_i and γ_a by Proposition 2.3.2 (3) and (4). Assume by contradiction that $u_i = [\mathsf{j}]$ for some i. Then, by Proposition 2.2.4, the latter occurs, namely, $l_i \cap \gamma_a \neq \emptyset$ and l_i is a B_a-fiber. Moreover, by Proposition 2.3.2 (4), the point $\pi_a(l_i) \in \mathbb{P}^2$ corresponds to $\ell \subset (\mathbb{P}^2)^*$ by projective duality. This implies that $[\mathsf{m}] \in \ell$, a contradiction to Generality Condition 3.0.5 (a).

Therefore, since ℓ and M_R contain $[\mathsf{j}]$, and $\deg M_R = g + 2$, we have $\ell_{|M_R} = u_1 + \cdots + u_{g+1} + [\mathsf{j}]$. Then, by the definition of θ_R, we have $\theta_R = t_1 + \cdots + t_{g+1} - h_R$. Now the assertion follows from [1, p.288, Exercise 32]. □

Remark 4.1.2. (1) In the proof of Proposition 4.1.1, we obtain the presentation $\theta_R = t_1 + \cdots + t_{g+1} - h_R$. So there are two such presentations according to choosing ℓ_1 or ℓ_2. This is compatible with [1, p.288, Exercise 32 (ii)].

(2) In the introduction, we say that we construct the theta characteristic from the incidence correspondence of intersecting B_a-lines. We add explanations about this since this is not obvious from the above construction.

The flow of the consideration below is quite similar to the proof of Proposition 4.1.1. Instead of a singular π_b-fiber, we consider a smooth π_b-fiber $H \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let r_1 and r_2 be the two rulings of H intersecting γ_b, and r_1' and r_2' the strict transforms on B_a of r_1 and r_2, respectively. By Proposition 2.3.2 (4), r_1' and r_2' are two π_a-fibers such that $\pi_a(r_1'), \pi_a(r_2') \in \mathsf{j}$. Let δ_1 and δ_2 are the families of rulings of F containing r_1 and r_2 respectively. By Proposition 2.3.2 (3) and (4), there exists a family δ_i' of B_a-lines corresponding to δ_i $(i = 1, 2)$. Note that this is nothing but $\widetilde{M}_{r_{3-i}}$ defined as in Definition 3.0.4 (3). By the same proof as that of Proposition 3.0.8 (2), we see that $M_{r_{3-i}}$ is a line in $(\mathbb{P}^2)^*$. Let $l_1, \dots, l_{d-1} \in \delta_1'$ be the B_a-lines intersecting R. Note that the B_a-line $r_1' \cup \gamma_a$ intersects r_2' and corresponds to the point $[\mathsf{j}]$. Therefore $[\mathsf{j}], [l_1], \dots, [l_{d-1}] \in M_R \cap M_{r_2}$. In a similar way to the proof of Proposition 4.1.1, we can show that $[\mathsf{j}]$ is different from $[l_1], \dots, [l_{d-1}]$. Hence we have $M_{r_2} \cap M_R = [l_1] + \ldots + [l_{d-1}] + [l]$. Let $m_1, \dots, m_{d-1} \in \delta_2'$ be the B_a-lines intersecting R. By relabelling if necessary, we have $h_R \sim [l_i] + [m_i]$ by Remark 3.0.9. Choose one of m_i's, say, m_1. Then, by the definition of θ_R, we have $\theta_R + [m_1] = [l_2] + \cdots + [l_{d-1}]$. The B_a-lines $l_2, \dots, l_{d-2}$ are nothing but those intersecting m_1 and R (l_1 is excluded since it will be disjoint from m_1 after the blow-up along R. See [10, §4] and [11, §3.1] for this consideration).

4.2. **Reconstructing rational curves.** Let $g \geq 2$. By Propositions 3.0.8, and 4.1.1 (see also Notation 3.0.10), we obtain a rational map

$$\pi_{g,1} \colon \mathcal{H}_{g+2} \dashrightarrow \mathcal{S}_{g,1}^{0,\mathrm{hyp}},\ [R] \mapsto [C_R, [\mathsf{j}]_R, \theta_R], \tag{4.1}$$

which is fundamental for our purpose.

The next theorem shows how to construct the rational curve R such that $\pi_{g,1}([R]) = [(C, p, \theta)]$ for a general element $[(C, p, \theta)]$ in $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$.

This is one of our key result to show the rationality of $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$.

Theorem 4.2.1. (Reconstruction theorem) *The map $\pi_{g,1}$ is dominant. More precisely, let $[(C, p, \theta)] \in \mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ be any element such that p is not a Weierstrass point, then there exists a point $[R] \in \mathcal{H}_{g+2}$ such that R satisfies Generality Condition* 3.0.5 (a)–(d) *and $\pi_{g,1}([R]) = [(C, p, \theta)]$.*

For our proof of the theorem, we need the following general results for an element of $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$. The proof given below is slightly long but it is elementary and only uses standard techniques from algebraic curve theory.

Lemma 4.2.2. *Let $[(C,p,\theta)]$ be any element of $\mathcal{S}^{0,\mathrm{hyp}}_{g,1}$. Let $\{p_1,\dots,p_{g+1}\}\cup\{p'_1,\dots,p'_{g+1}\}$ be the partition of the set of the Weierstrass points of C such that θ has the following two presentations*:

$$\theta\sim p_1+\cdots+p_{g+1}-g^1_2\sim p'_1+\cdots+p'_{g+1}-g^1_2 \tag{4.2}$$

(*cf.* [1, p.288, Exercise 32]). *The following assertions hold*:

(1) *The linear system $|\theta+g^1_2+p|$ defines a birational morphism from C to a plane curve of degree $g+2$.*

(2) *$|\theta+p|$ has a unique member D and it is mapped to a single point t by the map $\varphi_{|\theta+g^1_2+p|}$.*

For the assertions (3) *and* (4), *we set $S:=\{p,p_1,\dots,p_{g+1},p'_1,\dots,p'_{g+1}\}$.*

(3) *The support of D contains no point of S.*

(4) *The point t as in* (2) *is different from the $\varphi_{|\theta+g^1_2+p|}$-images of points of S. Besides, by the map $\varphi_{|\theta+g^1_2+p|}$, no two points of S are mapped to the same point.*

Proof. (1). We show that the linear system $|\theta+g^1_2+p|$ has no base points. By (4.2), we see that $\mathrm{Bs}\,|\theta+g^1_2+p|\subset\{p\}$. By the Serre duality, we have

$$H^1(\theta+g^1_2+p)\simeq H^0(K_C-\theta-g^1_2-p)^*=H^0(\theta-g^1_2-p)=0$$

since θ is ineffective. Similarly, we have $H^1(\theta+g^1_2)=\{0\}$. Therefore, by the Riemann-Roch theorem,

$$h^0(\theta+g^1_2+p)-h^0(\theta+g^1_2)=\chi(\theta+g^1_2+p)-\chi(\theta+g^1_2)=1,$$

which implies that $p\notin\mathrm{Bs}\,|\theta+g^1_2+p|$.

By the above argument, we see that $h^0(\theta+g^1_2+p)=\deg(\theta+g^1_2+p)+1-g=3$. Therefore, $|\theta+g^1_2+p|$ gives a morphism $\varphi_{|\theta+g^1_2+p|}\colon C\to\mathbb{P}(V)\simeq\mathbb{P}^2$ with $V=H^0(C,\mathcal{O}_C(\theta+g^1_2+p))^*$. Let $M:=\varphi_{|\theta+g^1_2+p|}(C)$ be the image of C. We show that $C\to M$ is birational. Note that by the Riemann-Roch theorem and $h^1(\theta+p)=h^0(K-\theta-p)=0$, we have $h^0(\theta+p)=1$. Therefore the hyperelliptic double cover $\varphi_{|g^1_2|}\colon C\to\mathbb{P}^1$ factors through the map $\varphi_{|g^1_2+\theta+p|}$. So we have only to show that $|\theta+g^1_2+p|$ separates the two points in a member of $|g^1_2|$. This is equivalent to $h^0(\theta+g^1_2+p-g^1_2)=h^0(\theta+g^1_2+p)-2$, which follows from the above computations. Since $C\to M$ is birational, the degree of M is $g+2$.

(2). Since $h^0(\theta+p)=1$ as in the proof of (1), the linear system $|\theta+p|$ has a unique member D. We see that D is mapped to a point since $\theta+g^1_2+p$ is the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)|_M$ and $h^0(\theta+g^1_2+p-(\theta+p))=h^0(g^1_2)=2$.

(3). The point p is not contained in the support of D since $h^0(\theta+p-p)=h^0(\theta)=0$. Let's us consider points of $S\setminus\{p\}$. Without loss of generality, we have only to show that $h^0(\theta+p-p_1)=0$. By the Riemann-Roch theorem, the assertion is equivalent to $h^1(\theta+p-p_1)=0$. By (4.2), $\theta+p-p_1=p_2+\cdots+p_{g+1}+p-g^1_2$. Therefore, by the Serre duality, we have

$$h^1(\theta+p-p_1)=h^0(g\times g^1_2-(p_2+\cdots+p_{g+1}+p))$$

since $K_C = (g-1)g^1_2$. Now it is easy to verify this is zero by using the hyperelliptic morphism $C \to \mathbb{P}^1$.

(4). First we show that t is different from the image of any point x of $C \setminus D$. Indeed, we have

$$h^0(\theta + g^1_2 + p - (\theta + p) - x) = h^0(g^1_2 - x) = 1,$$

which means that $|\theta + g^1_2 + p|$ separates D and x. In particular, we have the former assertion of (4) by (3).

We show that $|\theta + g^1_2 + p|$ separates any two of $p_1, \dots, p_{g+1}$. Without loss of generality, we have only to consider the case of p_1 and p_2. It suffices to show that $h^0(\theta + g^1_2 + p - p_1 - p_2) = h^0(\theta + g^1_2 + p) - 2 = 1$, which is equivalent to $h^1(\theta + g^1_2 + p - p_1 - p_2) = 0$ by the Riemann-Roch theorem. By the presentation (4.2), we have $h^1(\theta + g^1_2 + p - p_1 - p_2) = h^1(p_3 + \cdots + p_{g+1} + p)$. By the Serre duality, we have

$$h^1(p_3 + \cdots + p_{g+1} + p) = h^0((g-1)g^1_2 - p_3 - \cdots - p_{g+1} - p)$$

since $K_C = (g-1)g^1_2$. Now it is easy to verify the r.h.s. is zero by using the hyperelliptic morphism $C \to \mathbb{P}^1$.

The same argument shows that $|\theta + g^1_2 + p|$ separates any two of $p'_1, \dots, p'_{g+1}$. Moreover, if p is distinct from a p_i or p'_j, the same proof works for the separation of p and p_i or p'_j.

It remains to show that $|\theta + g^1_2 + p|$ separates one of $p_1, \dots, p_{g+1}$ and one of $p'_1, \dots, p'_{g+1}$. Without loss of generality, we have only to consider the case of p_1 and p'_1. If $p = p_1$, then $p \neq p'_1$, and hence we have already shown that the images of $p = p_1$ and p'_1 are different. Thus we may assume that $p \neq p_1, p'_1$. By (4.2), $D_1 := p + p_1 + \cdots + p_{g+1}$ and $D_2 := p + p'_1 + \cdots + p'_{g+1}$ are two distinct members of $|\theta + g^1_2 + p|$. If the images of p_1 and p'_1 by the map $\varphi_{|\theta + g^1_2 + p|}$ coincides, then the images of D_1 and D_2 coincides since they are the line through the images of p and p_1, and the line through the images of p and p'_1. This is a contradiction to a property of the map defined by $|\theta + g^1_2 + p|$. $\square$

Proof of Theorem 4.2.1. Let M, $r_1, \dots, r_{g+1}$ and $r'_1, \dots, r'_{g+1} \in M$ be the $\varphi_{|\theta + g^1_2 + p|}$-images of C, the Weierstrass points $p_1, \dots, p_{g+1}$ and $p'_1, \dots, p'_{g+1}$ of C as in (4.2), respectively. Let $r \in M$ be the image of p and $t \in M$ the image of the unique member of $|\theta + p|$. By Lemma 4.2.2 (4), r, t, $r_1, \dots, r_{g+1}$, $r'_1, \dots, r'_{g+1}$ are distinct points (recall that now we are assuming p is not a Weierstrass point). We set $V = H^0(C, \mathcal{O}_C(\theta + g^1_2 + p))^*$. By (4.2), there are two lines $\ell, \ell' \subset \mathbb{P}(V)$ such that $\ell_{|M} = r_1 + \cdots + r_{g+1} + r$ and $\ell'_{|M} = r'_1 + \cdots + r'_{g+1} + r$.

We then identify the polarized space $(\mathbb{P}(V), \ell \cup \ell')$ with $((\mathbb{P}^2)^*, \ell_1 \cup \ell_2)$ (recall the notation as in Proposition 1.2.2). By this identification, the point r corresponds to $[\mathsf{j}]$. Let m be the line of $\mathbb{P}^2$ such that $[\mathsf{m}]$ corresponds to the point t. Since $r \neq t$, the line m is not the jumping line j of the bundle $\mathcal{E}$ such that $B_a \simeq \mathbb{P}(\mathcal{E})$. Moreover, m is not a jumping lines of the second kind of $\mathcal{E}$, equivalently, $[\mathsf{m}] \notin \ell_1 \cup \ell_2$ since t is distinct from r, $r_1, \dots, r_{g+1}$, $r'_1, \dots, r'_{g+1}$. This will show that R constructed below satisfies Generality Condition 3.0.5 (a).

We consider the linear system $|C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}|$ on $L_\mathsf{m} \subset B_a$. We look for a member $R \in |C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}|$ with Generality Condition 3.0.5 (a)–(d) such that $C = C_R$. Note that the condition for an $R \in |C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}|$ to intersect one fixed B_a-line is of codimension 1. Hence there exists at least one $R \in |C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}|$ intersecting the $2g+2$ B_a-lines which correspond to the $2g+2$ points $r_1, \dots, r_{g+1}$ and $r'_1, \dots, r'_{g+1} \in M$, since $\dim H^0(C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}) = 2g+3$. Equivalently, there exists at least one $R \in |C_0(\mathsf{m}) + (g+1)L_{|L_\mathsf{m}}|$ such that $r_1, \dots, r_{g+1}, r'_1, \dots, r'_{g+1} \in M_R$. By Corollary 2.3.3, R intersects F'_1, and F'_2 at $g+1$ points, respectively, corresponding to $r_1, \dots, r_{g+1}$ and $r'_1, \dots, r'_{g+1}$. Therefore, R satisfies Generality Condition 3.0.5 (d). Moreover, R does not pass through $F'_1 \cap F'_2 \cap L_\mathsf{m} = \gamma_a \cap L_\mathsf{m}$ since $r_1, \dots, r_{g+1}$, $r'_1, \dots, r'_{g+1}$ are distinct points. Therefore R satisfies Generality Condition 3.0.5 (c).

We show that R is smooth, namely, R satisfies Generality Condition 3.0.5 (b). Indeed, assume by contradiction that R is reducible. Then R contains a ruling of L_m, say, f. We have $f \cap \gamma_a = \emptyset$ since $R \cap \gamma_a = \emptyset$. Thus M_R contains the curve M_f, which is a line in $(\mathbb{P}^2)^*$ by Proposition 3.0.8 (2), besides M_f contains $t = [\mathsf{m}]$, and one of $r_1, \dots, r_{g+1}$ and one of $r'_1, \dots, r'_{g+1}$ corresponding to $F'_1 \cap f$ and $F'_2 \cap f$, respectively. By reordering the points, we may assume that $r_1, r'_1 \in M_f$. Therefore t, r_1, r'_1 are collinear. This is, however, a contradiction since the line through t and r_1 touches M only at t and r_1 (recall that r_1 is the image of a Weierstrass point).

Finally we show $M = M_R$. We have checked m and R satisfy Generality Condition 3.0.5 (a)–(d). Note that, by the constructions of M and M_R as the images of the map $\varphi_{|\theta + g^1_2 + p|}$ and $\varphi_{|\theta_R + h_R + [\mathsf{j}]_R|}$ respectively, there exists a line through t and touches both M and M_R at r_i with multiplicity two ($i = 1, \dots, g+1$), and the same is true for r'_j ($j = 1, \dots, g+1$). Hence the intersection multiplicities of M_R and M at r_i and r'_j are at least two. Therefore the scheme theoretic intersection $M \cap M_R$ contains r, the $2(g+1)$ points r_i, r'_j, $i, j = 1, ..., g+1$ with multiplicity ≥ 2 and we also have a fat point of multiplicity g^2 at t. This implies that, if $M \neq M_R$, then $M \cdot M_R \geq 1 + 4(g+1) + g^2 = (g+2)^2 + 1$, which is a contradiction since $\deg M = \deg M_R = g+2$. Now we conclude that $M_R = M$. □

Theorem 4.2.1 has a nice corollary, which seems to be unknown.

Corollary 4.2.3. *The moduli space $\mathcal{S}^{0,\mathrm{hyp}}_{g,1}$ and the moduli space $\mathcal{S}^{0,\mathrm{hyp}}_{g}$ of ineffective spin hyperelliptic curves are irreducible.*

Proof. By Definition 2.1.1, $\mathcal{H}_{g+2}$ is an open subset of the projective bundle Σ_{g+2} over the projective plane. Therefore $\mathcal{H}_{g+2}$ is irreducible. By Theorem 4.2.1 we know that the map $\pi_{g,1} \colon \mathcal{H}_{g+2} \dashrightarrow \mathcal{S}^{0,\mathrm{hyp}}_{g,1}$ is dominant to each irreducible component of $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$. The forgetful morphism $\mathcal{S}_{g,1}^{0,\mathrm{hyp}} \to \mathcal{S}_{g}^{0,\mathrm{hyp}}$ is dominant too. Hence the claim follows.

□

4.3. **Birational model of $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$.** Let m be a general line in $\mathbb{P}^2$. By Theorem 4.2.1 and the group action of G on $\mathcal{H}_{g+2}$, the map $\pi_{g,1}\colon \mathcal{H}_{g+2} \dashrightarrow \mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ induces a dominant rational map $\rho_{g,1}\colon |(H+gL)_{|L_{\mathsf{m}}}| \dashrightarrow \mathcal{S}_{g,1}^{0,\mathrm{hyp}}$. Recall the definition of the subgroup Γ of G as in Lemma 1.4.4. By the classical Rosenlicht theorem, we can find an Γ-invariant open set U of $|(H+gL)_{|L_{\mathsf{m}}}|$ such that the quotient U/Γ exists. Since a general Γ-orbit in $|(H+gL)_{|L_{\mathsf{m}}}|$ is mapped to a point by $\rho_{g,1}$, we obtain a dominant map $\overline{\rho}_{g,1}\colon U/\Gamma \to \mathcal{S}_{g,1}^{0,\mathrm{hyp}}$.

Proposition 4.3.1. *The dominant map $\overline{\rho}_{g,1}\colon U/\Gamma \to \mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ is birational.*

Proof. We show that $\overline{\rho}_{g,1}$ is generically injective. We consider two general elements $R, R' \in U$ and the two corresponding Γ-orbits $\Gamma[R], \Gamma[R']$. Note that M_R and $M_{R'}$ both pass through the points $[\mathsf{j}]$ and $[\mathsf{m}]$, and they both have Weierstrass points distributed on the two lines ℓ_1 and ℓ_2. Now assume that $[C_R, p, \theta_R] = [C_{R'}, p', \theta_{R'}] \in \mathcal{S}_{g,1}^{+\mathrm{hyp}}$, equivalently, there exists an isomorphism $\xi\colon C_R \to C_{R'}$ such that $\xi^*\theta_{R'} = \theta_R$ and $\xi(p) = p'$. We consider the following diagram:

$$
\begin{tikzcd}
C_R \arrow[r, "(b\circ\tilde{p}_2)|_{C_R}"] \arrow[d, "\xi"'] & M_R \\
C_{R'} \arrow[r, "(b\circ\tilde{p}_2)|_{C_{R'}}"'] & M_{R'}.
\end{tikzcd}
$$

Note that $(b\circ\tilde{p}_2)|_{C_R}(p) = (b\circ\tilde{p}_2)|_{C_R}(p') = [\mathsf{j}]$ by Notation 3.0.10. Since the g_2^1 is unique on an hyperelliptic curve, we have $\xi^* h_{R'} = h_R$ where h_R and $h_{R'}$ are respectively the g_2^1's of C_R and $C_{R'}$. Therefore there exists a projective isomorphism ξ_M from M_R to $M_{R'}$ such that $(b\circ\tilde{p}_2)|_{C_{R'}} \circ \xi = \xi_M \circ (b\circ\tilde{p}_2)|_{C_R}$ and hence $\xi_M([\mathsf{j}]) = [\mathsf{j}]$ since the morphisms $(b\circ\tilde{p}_2)|_{C_R}\colon C_R \to M_R \subset (\mathbb{P}^2)^*$ and $(b\circ\tilde{p}_2)|_{C_{R'}}\colon C_{R'} \to M_{R'} \subset (\mathbb{P}^2)^*$ are given respectively by $|\theta_R + p + h_R|$ and $|\theta_{R'} + p' + h_{R'}|$. We also have $\xi_M([\mathsf{m}]) = [\mathsf{m}]$ since $[\mathsf{m}]$ is a unique g-ple point of M_R and $M_{R'}$ respectively by Proposition 3.0.8 (3) and (4). Let g be an element of $\mathrm{Aut}\,(\mathbb{P}^2)^*$ inducing the projective isomorphism ξ_M. Since ξ sends the Weierstrass points of C_R to those of $C_{R'}$, the line pair $\ell_1 \cup \ell_2$ must be sent into itself by g. Hence $g \in G$. Moreover, since g fixes $[\mathsf{m}]$ as we noted above, we have $g \in \Gamma$. In summary, we have shown $gM_R = M_{R'}$. It remains to show that $gR = R'$. For this, we have only to show that R is recovered from M_R. Take a general line ℓ through $[\mathsf{m}]$ and set $\ell_{|M_R} = [\mathsf{m}] + [C_1] + [C_2]$ set-theoretically, where C_1 and C_2 are B_a-lines. Note that $C_1 \cap C_2$ is one point. Then R is recovered as the closure of the locus of $C_1 \cap C_2$ when ℓ varies. □

5. Proof of Rationality

Theorem 5.0.2. $\mathcal{S}_{g,1}^{0,\mathrm{hyp}}$ *is a rational variety.*

Proof. As in the subsection 4.3, we fix a general line m in $\mathbb{P}^2$. By Proposition 4.3.1, we have only to show that U/Γ is a rational variety.

Using the elementary transformation as in Proposition 1.4.1, we are going to reduce the problem to that on $\mathbb{P}^1 \times \mathsf{m}$. Let r_v and r_h are rulings of the projections $\mathbb{P}^1 \times \mathsf{m} \to \mathsf{m}$ and $\mathbb{P}^1 \times \mathsf{m} \to \mathbb{P}^1$, respectively. From now on, we identify $\mathbb{P}^1 \times \mathsf{m}$ with $\mathbb{P}^1 \times \mathbb{P}^1$ having the bi-homogeneous coordinate $(x_1' : x_2') \times (y_2 : y_3)$ with $x_1' := (x_1 - x_2)/2$ and $x_2' := (x_1 + x_2)/2$. To clarify the difference of the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$, we keep denoting it by $\mathbb{P}^1 \times \mathsf{m}$. With this coordinate, the action of $\Gamma \simeq (\mathbb{Z}_2 \times G_a) \rtimes G_m$ on $\mathbb{P}^1 \times \mathsf{m}$ is described by multiplications of the following matrices by Lemma 1.4.4:

- $G_m : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ with $a \in G_m$,
- $G_a : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \in G_a$, and
- $\mathbb{Z}_2 : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that members of $|(H+gL)_{|L_{\mathsf{m}}}|$ corresponds to those of the linear system $|r_h + (g+1)r_v|$ through the point $c := \gamma_c \cap (\mathbb{P}^1 \times \mathsf{m}) = (1:0) \times (1:0)$. We denote by Λ the sublinear system consisting of such members. A member of Λ is the zero set of a bi-homogeneous polynomial of bidegree $(1, g+1)$ of the form $x_1' f_{g+1}(y_2, y_3) + x_2' g_{g+1}(y_2, y_3)$, where $f_{g+1}(y_2, y_3)$ and $g_{g+1}(y_2, y_3)$ are binary $(g+1)$-forms

$$f_{g+1}(y_2, y_3) = p_g y_2^g y_3 + \cdots + p_i y_2^i y_3^{g+1-i} + \cdots + p_0 y_3^{g+1},$$
$$g_{g+1}(y_2, y_3) = q_{g+1} y_2^{g+1} + \cdots + q_i y_2^i y_3^{g+1-i} + \cdots + q_0 y_3^{g+1}.$$

Then the linear system Λ can be identified with the projective space $\mathbb{P}^{2g+2}$ with the homogeneous coordinate $(p_0 : \cdots : p_g : q_0 : \cdots : q_{g+1})$. A point $(p_0 : \cdots : p_i : \cdots : p_g : q_0 : \cdots : q_j : \cdots : q_{g+1})$ is mapped by the elements of the subgroups G_m, G_a, and $\mathbb{Z}_2 \subset \Gamma$ as above to the following points:

(a) $G_m : (a^{g+1} p_0 : \cdots : a^{g+1-i} p_i : \cdots : a p_g : a^{g+1} q_0 : \cdots : a^{g+1-j} q_j : \cdots : q_{g+1})$,

(b) G_a : the point $(p_0' : \cdots : p_i' : \cdots : p_g' : q_0' : \cdots : q_j' : \cdots : q_{g+1}')$ with

$$p_i' = \sum_{k=i}^{g} \binom{k}{i} b^{k-i} p_k, \tag{5.1}$$
$$q_j' = \sum_{l=j}^{g+1} \binom{l}{j} b^{l-j} q_l,$$

(c) $\mathbb{Z}_2 : (-p_0 : \cdots : -p_i : \cdots : -p_g : q_0 : \cdots : q_j : \cdots : q_{g+1})$.

Step 1. The quotient $\Lambda_1 := \Lambda/\mathbb{Z}_2$ is rational.

The rationality is well-known by the description of $\mathbb{Z}_2$-action as in (c). In the following steps, it is convenient to show this more explicitly. On the open set $\{q_{g+1} \neq 0\} \subset \Lambda$, which is Γ-invariant, we may consider $q_{g+1} = 1$. Then the action is

$$(p_0, \cdots, p_g, q_0, \cdots, q_g) \mapsto (-p_0, \cdots, -p_g, q_0, \cdots, q_g).$$

Therefore the quotient map can be written on the Γ-invariant open subset $\{p_g \neq 0\}$ as follows:

$$(p_0, \cdots, p_i, \cdots, p_g, q_0, \cdots, q_g) \mapsto (p_0 p_g, \cdots, p_i p_g, \cdots, p_g^2, q_0, \cdots, q_g).$$

We denote by ${}^{\tau}\mathbb{C}^{2g+2}$ the target $\mathbb{C}^{2g+2}$ of this map and by $(\tilde{p}_0, \ldots, \tilde{p}_g, \tilde{q}_0, \ldots, \tilde{q}_g)$ its coordinate. Using this presentation, we compute the quotient by the additive group G_a in the next step.

Step 2. The quotient $\Lambda_2 := \Lambda_1/G_a$ is rational.

Let $(\tilde{p}'_0, \ldots, \tilde{p}'_g, \tilde{q}'_0, \ldots, \tilde{q}'_g)$be the image of the point $(\tilde{p}_0, \ldots, \tilde{p}_g, \tilde{q}_0, \ldots, \tilde{q}_g)$ by the action of an element of G_a as in (b). By the choice of coordinate, it is easy to check $\tilde{p}'_i$ and $\tilde{q}'_j$ can be written by $\tilde{p}_0, \ldots, \tilde{p}_g$ and $\tilde{q}_0, \ldots, \tilde{q}_g$ respectively by the formulas obtained from (5.1) by setting $q_{g+1} = 1$ and replacing p'_i, p_k, q'_j and q_l with $\tilde{p}'_i$, $\tilde{p}_k$, $\tilde{q}'_j$ and $\tilde{q}_l$. Then note that we have $\tilde{q}'_g = \tilde{q}_g + (g+1)b$. Therefore, the stabilizer group of every point is trivial and every G_a-orbit intersects the closed set $\{\tilde{q}_g = 0\}$ at a single point. Hence we may identified birationally the quotient ${}^{\tau}\mathbb{C}^{2g+2}/G_a$ with the closed set $\{\tilde{q}_g = 0\} \subset {}^{\tau}\mathbb{C}^{2g+2}$. In particular, the quotient is rational.

Step 3. The quotient $\Lambda_3 := \Lambda_2/G_m$ is rational.

We may consider the closed set $\{\tilde{q}_g = 0\}$ as the affine space $\mathbb{C}^{2g+1}$ with the coordinate $(\tilde{p}_0, \ldots, \tilde{p}_g, \tilde{q}_0, \ldots, \tilde{q}_{g-1})$. Note that this closed set has the naturally induced G_m-action such that, by the element of G_m as in (a), a point $(\tilde{p}_0, \ldots, \tilde{p}_g, \tilde{q}_0, \ldots, \tilde{q}_{g-1})$ is mapped to $(a^{g+2}\tilde{p}_0, \ldots, a^2\tilde{p}_g, a^{g+1}\tilde{q}_0, \ldots, a^2\tilde{q}_{g-1})$. Therefore the quotient $\mathbb{C}^{2g+1}/G_m$ is a weighted projective space, hence is rational.

□

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