Preservation and reflection of size properties of balleans

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Dikran Dikranjan and Nicolò Zava

Department of Mathematics, Computer Science and Physics, University of Udine
Via Delle Scienze 206, 33100 Udine, Italy
dikran.dikranjan@uniud.it, nicolo.zava@gmail.com

Abstract

We study the combinatorial size of subsets of a ballean, as defined in [19, 23] (largeness, smallness, extralargeness, etc.), paying particular attention to the preservation of these properties under taking images and inverse images along various classes of maps (bornologous, effectively proper, (weakly) soft, coarse embeddings, canonical projections of products, canonical inclusions of co-products, etc.). We show by appropriate examples that many of the properties describing the size are not preserved under coarse equivalences (even injective or surjective ones), whereas largeness and smallness are preserved under arbitrary coarse equivalences.

Introduction

Combinatorial size of subsets of a group or semigroup has long been studied in combinatorial group theory and harmonic analysis (see Example 1.2 (c),(d)). The fundamental paper [3] of Bella and Malykhin introduced and studied largeness, smallness and extralargeness in groups as well as their relation in a systematic way. This paper, as well as the consequent ones [4, 1, 6, 10, 13], proposed various challenging problems, many of them arising in the framework of topological groups.

The next fundamental step was the introduction of balleans in the monograph [19] of Protasov and Banakh largely inspired by metric spaces (see Definition 1.1). This construction is equivalent to coarse spaces ([23]), introduced almost simultaneously by Roe ([24]). Protasov and Banakh showed that the balleans provide a nice and unifying way to describe size in a sufficiently general setting. The leading example of these concepts of size is largeness, generalizing the well-known notion of a net in a metric space: if $\mathfrak{B} = (X, P, B)$ is a ballean, a subset $L \subseteq X$ is *large* in $\mathfrak{B}$ if there exists a radius $\alpha \in P$ such that $B(L, \alpha) = \bigcup_{l\in L} B(l, \alpha) = X$. These authors extended to this general level also smallness and extralargeness, introducing also piecewise largeness along the way. The monograph [23] contains another concept of size, namely thickness. Since then the size and various cardinal invariants of balleans related to size have been intensively studied by the Ukrainian school [11, 12, 14, 16, 17, 20, 21, 22]. The survey [18] is very helpful to get a better idea on the topic. A relevant connection between size and categorical properties of the coarse category was pointed out recently in [7].

As already noticed in [19, 23], the above concepts of size are related to each other: for example a subset A of a ballean X is large in X if and only if $X \setminus A$ is not thick in X; A is small if and only if A is not piecewise large if and only if $X \setminus A$ is extralarge (see Theorem 2.5 for further relations). On the other hand, it was noticed in [19], that these sizes are preserved under asymorphisms, yet the question of whether they are preserved also by coarse equivalences was never discussed. The aim of this paper is to address these two issues as follows.

First of all, following the idea suggested by the above mentioned relations, we define new notions of size, introducing *slim* sets, *meshy* sets and sets *with slim interior* (Definition 2.8). It turns out that these are exactly the negations of largeness, thickness and extralargness, respectively (although these three properties are well related to the first five also via complements, Proposition 2.9). All these eight properties can be divided in two groups: "largeness" properties (largeness, thickness, extralargeness and piecewise largeness) and "smallness" properties (smallness, meshiness, slimness and having slim interior): the first ones are stable under taking supersets, while the second ones are stable under taking subsets.

Secondly, we study the stability of these properties under taking images and inverse images along maps between balleans. It is easy to see that the new notions of size are invariant for asymorphisms as well. A natural question is whether a class of maps preserves or reflects a property W of subsets of balleans in the

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following sense: a map $f\colon X \to Y$ between balleans is said to be *W-preserving* (*W-reflecting*) if for every $A \subseteq X$ the image $f(A)$ has the property W in Y whenever A has W in X (if for every $B \subseteq Y$ the preimage $f^{-1}(B)$ has the property W in X whenever B has W in Y, respectively). For example, largeness is preserved by surjective bornologous maps and reflected by surjective effectively proper ones ([19, Lemma 11.3]). Here we separately study the case of surjective maps and injective maps (in the latter case the main role is obviously played by the size of the image). Then we combine these results and obtain similar properties of general maps. We pay special attention to several classes of maps for which we obtain a complete description of which preservation properties are available. These are: coarse equivalences (including the case of injective or surjective ones), bornologous maps, effectively proper maps, weakly soft and soft maps, coarse embeddings, the canonical projections (inclusions, respectively) of a (co-)product ballean. Somewhat surprisingly, coarse equivalences do not preserve and reflect all types of size even when they are injective or surjective (although the bijective coarse equivalences preserve all size properties). Nevertheless, coarse equivalences preserve largeness and smallness, as well as their negations: slimness and piecewise largeness (Theorem 4.27).

This paper is organized as follows. In §1.1, we recall the basic notions of balleans and their maps, §1.2 contains some categorical constructions: product, coproduct and quotients, with a particular emphasis on some specific quotients maps, the weakly soft and soft maps recently introduced in [7]. In Section 2 we recall the five known notions of sizes (largeness, thickness, extralargeness, piecewise largeness and smallness) as well as the relations between them which naturally lead to the definition of the three new properties (§2.1). In Section 3, we define the concept of W-preserving and W-reflecting maps, where W is a property of some subsets of balleans, and we recall the notion of W-copreserving map, due to Dydak and Virk [9]. We describe how these preservation/copreservation/reflection properties vary when the property W varies in appropriate way. This allows for a relevant reduction to a smaller collection of properties (namely, the W-preserving maps) that are sufficient to describe the remaining ones (Corollary 3.8 and Remark 3.9). Section 4 starts with a large subsection §4.1 examining the preservation properties of surjective maps. Similarly, we describe completely, in §4.2, the preservation properties of inclusion maps of a subballean (Theorems 4.13 and 4.12) and we pay a special attention to the case of canonical inclusions in a coproduct (Theorem 4.15). This allows for a reduction of the study of sizes to the case of connected balleans. Finally, in §4.3, we use the results from §§4.1,4.2 about surjective and injective maps to deduce properties for general maps; in particular we determine which size-preserving and size-reflecting properties coarse equivalences may have. This section contains also many examples and counterexamples.

The main relations between the sizes can be found in Table 1, whereas Table 2 collects all informations concerning the preservation properties each one of the five main classes of surjective maps may have. Figure 1 summarizes the connections between the various preservation or reflection properties.

Unfortunately, it was not possible to include many relevant facts and properties related to this paper. In Remark 2.15 we give a more detailed comment on this issue.

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1 Background on balleans

1.1 Balleans and maps between balleans

Definition 1.1. ([23]) A *ball structure* is a triple $\mathfrak{B} = (X, P, B)$ where X and P are sets (simultaneously empty or non-empty), and $B : X \times P \to \mathcal{P}(X)$ is a map, such that $x \in B(x, \alpha)$ for every $x \in X$ and every $\alpha \in P$. The set X is called *support of the ball structure*, P – *set of radii*, and $B(x, \alpha)$ – *ball of center x and radius α*. In case $X = P = \emptyset$, the map B is the empty map.

For a ball structure (X, P, B), $x \in X$, $\alpha \in P$ and a subset A of X, one puts

$$B^*(x, \alpha) = \{y \in X \mid x \in B(y, \alpha)\} \qquad B(A, \alpha) = \bigcup_{x \in A} B(x, \alpha).$$

A ball structure (X, P, B) is said to be
(a) *upper symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha') \text{ and } B^*(x, \beta) \subseteq B(x, \beta');$$

(b) *symmetric* if, for any $\alpha \in P$ and any point $x \in X$, we have $B^*(x, \alpha) = B(x, \alpha)$;
(c) *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists a $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Finally a *ballean* is an upper symmetric and upper multiplicative ball structure.

Let $\mathfrak{B} = (X, P, B)$ be a ballean and Y a subset of X. Then we define the *subballean structure* on Y to be $\mathfrak{B}|_Y = (Y, P, B|_Y)$, where $B|_Y(y, \alpha) = B(y, \alpha) \cap Y$, for every $y \in Y$ and $\alpha \in P$.

A ballean $\mathfrak{B} = (X, P, B)$ is said to be *connected* if for every pair of points $x, y \in X$ there exists a radius $\alpha \in P$ such that $y \in B(x, \alpha)$. If $\mathfrak{B} = (X, P, B)$ is a ballean and $x \in X$ is a point, we define the *connected component of* x to be the set

$$\mathcal{C}_{x,X} = \{y \in X \mid \exists \alpha \in P : y \in B(x, \alpha)\} = \bigcup_{\alpha \in P} B(x, \alpha).$$

When no confusion is possible, we simply write $\mathcal{C}_x$, in place of $\mathcal{C}_{x,X}$. Every ballean $\mathfrak{B} = (X, P, B)$ can be partitioned in its connected components $\mathcal{C}_x$, $x \in X$. If Y is a subspace of a ballean X, then $\mathcal{C}_{y,Y} = Y \cap \mathcal{C}_{y,X}$ for all $y \in Y$. In particular, Y is connected, whenever X is connected.

Example 1.2. (a) Let X be a non-empty set. Then there always exist two ballean structures on it, namely the *trivial ballean* $\mathfrak{B}_{\mathcal{T}}$, whose balls are singletons, and the *bounded ballean* $\mathfrak{B}_{\mathcal{M}}$, where the whole space is a ball centered at any point.

(b) A natural source of examples of balleans are (pseudo-)metric spaces (X, d). Then the so-called *metric ballean* is the triple $\mathfrak{B}_d = (X, \mathbb{R}_{>0}, B_d)$, where $B_d(x, R) = \{y \in X \mid d(x, y) \leq R\}$ for every $x \in X$ and $R \in \mathbb{R}_{>0}$. Every metric ballean is connected.

(c) Let G be a group. We denote its identity by e. A family $\mathcal{I}$ of subsets of G is a *group ideal* ([18]) if
   (c_1) there exists a non-empty element in $\mathcal{I}$;
   (c_2) $\mathcal{I}$ is closed under finite unions and under taking subsets (i.e., $\mathcal{I}$ is an ideal of subsets of G);
   (c_3) for every $I_1, I_2 \in \mathcal{I}$, the subset $I_1 I_2 = \{gh \mid g \in I_1,\, h \in I_2\}$ belongs to $\mathcal{I}$; and
   (c_4) for each $I \in \mathcal{I}$, the subset $I^{-1} = \{g^{-1} \mid g \in G\}$ belongs to $\mathcal{I}$.

   If G is a group and $\mathcal{I}$ a group ideal over it, one has a ballean $\mathfrak{B}_{\mathcal{I}} = (G, \mathcal{I}, B_{\mathcal{I}})$, where $B_{\mathcal{I}}(g, I) = g(I \cup \{e\})$ for every $g \in G$ and $I \in \mathcal{I}$. The ballean $\mathfrak{B}_{\mathcal{I}}$ is connected if and only if $\mathcal{I}$ contains the ideal $\mathcal{I}_{fin}$ of all finite sets if G. The *finitary* ballean $\mathfrak{B}_{\mathcal{I}_{fin}}$ has been largely studied in the literature [3, 4, 5, 10].

   A more general version can be obtained by taking an infinite cardinal κ and the ideal $\mathcal{I}_\kappa = [G]^{<\kappa}$ of subsets of cardinality $< \kappa$ of G. Obviously, $\mathcal{I}_\kappa$ is a group ideal, so gives rise to a group ballean $\mathfrak{B}_{\mathcal{I}_\kappa}$. This ballean, and especially the case $\kappa = |G|$ has been intensively studied [11, 12, 14, 16, 17, 20, 21, 22].

(d) It is possible to nicely unify items (b) and (c) in the case of a countably infinite group G. It was proved by Smith [25] that every such group G admits a left invariant proper metric d (i.e., such that $d(gx, gy) = d(x, y)$, for every $g, x, y \in G$, and whose closed balls are compact) and every pair of such metrics are coarsely equivalent (actually asymorphic). Since all balls of a proper left invariant metric on a countable group are finite, the metric ballean of (G, d) coincides with the finitary one.

We anticipate the main property of subsets of balleans regarding "size" because of its importance, the remaining ones will be given in Section §2. A subset L of X is *large* in a ballean $\mathfrak{B} = (X, P, B)$ if there exists a radius $\alpha \in P$ such that $B(L, \alpha) = X$.

Let $f, g \colon S \to X$ be two maps from a set S to a ballean $\mathfrak{B} = (X, P, B)$. We say that these two maps are *close* (and we write $f \sim g$) if there exists $\alpha \in P$ such that $f(x) \in B(g(x), \alpha)$ for every $x \in S$.

Definition 1.3 ([19, 24])**.** Let $\mathfrak{B}_X = (X, P_X, B_X)$ and $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ be two balleans. Then a map $f \colon X \to Y$ is called

(a) *bornologous* (or $\prec$*-mapping*) if for every radius $\alpha \in P_X$ there exists another radius $\beta \in P_Y$ such that $f(B_X(x, \alpha)) \subseteq B_Y(f(x), \beta)$ for every point $x \in X$;
(b) a $\succ$*-mapping* if for every $\alpha \in P_Y$ there exists a radius $\beta \in P_X$ such that $B_Y(f(x), \alpha) \subseteq f(B_X(x, \beta))$ for every $x \in X$;
(c) *effectively proper* if for every $\alpha \in P_Y$ there exists a radius $\beta \in P_X$ such that $f^{-1}(B_Y(f(x), \alpha)) \subseteq B_X(x, \beta)$ for every $x \in X$;
(d) a *coarse embedding* (or *quasi-asymorphic embedding*) if it is both bornologous and effectively proper;
(e) an *asymorphism* if it is bijective and both f and f^{-1} are bornologous;
(f) a *coarse equivalence* (or *quasi-asymorphism*) if it is a coarse embedding such that $f(X)$ is large in $\mathfrak{B}_Y$ or, equivalently, if it is bornologous and there exists another bornologous map $g \colon Y \to X$, called *coarse inverse*, such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$.

Our choice to use terms from coarse geometry (as bornologous, effectively proper map, coarse embedding and coarse equivalence [24]) is justified by the fact these notions (usually used in the framework of entourages, etc.) are equivalent to those given above (see [23, 7]).

Remark 1.4. One may be tempted to consider items (b) and (c) as equivalent. Let us see the precise relation between effectively proper maps and $\succ$-mappings. Firstly, effectively proper maps between balleans have uniformly bounded fibers. Moreover, every *surjective* effectively proper map is a $\succ$-mapping. Actually, this can be inverted: a *surjective* map between balleans with uniformly bounded fibers is effectively proper if and only if it is a $\succ$-mapping. For non surjective maps the notion of a $\succ$-mapping may become too restrictive, as noted in [26], where an appropriately weakened version of this notion is investigated.

Fix a non-empty set X and two ballean structures $\mathfrak{B} = (X, P, B)$ and $\mathfrak{B}' = (X, P', B')$ on X. We write $\mathfrak{B} \prec \mathfrak{B}'$ if the identity map $id\colon \mathfrak{B} \to \mathfrak{B}'$ is a $\prec$-mapping and we say $\mathfrak{B}$ *is finer than* $\mathfrak{B}'$. We write $\mathfrak{B} = \mathfrak{B}'$, in case $\mathfrak{B}' \prec \mathfrak{B}$ holds as well, i.e., when for every $\alpha \in P$ and $\alpha' \in P'$ there exist $\beta \in P'$ and $\beta' \in P$ such that

$$B(x, \alpha) \subseteq B'(x, \beta) \quad \text{and} \quad B'(x, \alpha') \subseteq B(x, \beta') \quad \text{for every} \quad x \in X. \tag{1}$$

In other words, we identify two ballean structures $\mathfrak{B}$ and $\mathfrak{B}'$ on X satisfying (1).

Remark 1.5. Every ballean $\mathfrak{B} = (X, P, B)$ admits an "equivalent", in the sense of (1), symmetric upper multiplicative ball structure $\mathfrak{B}_{sim} = (X, P, B_{sim})$, where $B_{sim}(x, \alpha) = B(x, \alpha) \cap B^*(x, \alpha)$ for every $x \in X$ and $\alpha \in P$ ([23]). Indeed, for every $\alpha \in P$ there exists a $\beta \in P$ such that $B_{sim}(x, \alpha) \subseteq B(x, \alpha) \subseteq B_{sim}(x, \beta)$.

1.2 Product, coproduct and quotients of balleans

Let $\{\mathfrak{B}_i = (X_i, P_i, B_i)\}_{i \in I}$ be a family of balleans. Let $X = \Pi_i X_i$ and $p_i\colon X \to X_i$, for every $i \in I$, be the projection maps. For the sake of simplicity, we denote the subset $\bigcap_{i\in I} p_i^{-1}(A_i)$ of X by $\Pi_{i\in I} A_i$, where $A_i \subseteq X_i$, for every index $i \in I$. The *product ballean structure* on X can be described as follows: this is the ball structure $\Pi_{i\in I}\mathfrak{B}_i = (X, \Pi_{i\in I} P_i, B_X)$, where, for each $(x_i)_{i\in I} \in X$ and each $(\alpha_i)_i \in \Pi_i P_i$, we put

$$B_X((x_i)_i, (\alpha_i)_i) = \Pi_{i\in I} B_i(x_i, \alpha_i)$$

If $\mathfrak{B}_Z = (Z, P_Z, B_Z)$ is a ballean such that for each $i \in I$ there exists a bornologous map $f_i\colon \mathfrak{B}_Z \to \mathfrak{B}_i$, then the unique map $f\colon \mathfrak{B}_Z \to \Pi_{i\in I}\mathfrak{B}_i$ such that $p_i \circ f = f_i$ for every $i \in I$ is bornologous.

Let $\{\mathfrak{B}_i = (X_i, P_i, B_i)\}_{i \in I}$ be a family of balleans. The *coproduct ballean structure* on $X = \bigsqcup_{i\in I} X_i$ is

$$\coprod_{i\in I} \mathfrak{B}_i = (X, \Pi_{i\in I} P_i, B_X),$$

where $B_X(i_\nu(x), (\alpha_i)_i) = i_\nu(B_\nu(x, \alpha_\nu))$, for every $i_\nu(x) \in X$ and $(\alpha_i)_i \in \Pi_i P_i$.

Every ballean $\mathfrak{B} = (X, P, B)$ allows for a partition of X in connected components $\mathcal{C}_x$, for $x \in X$. Let Λ be an index set for the family of all the connected components of X. This gives rise to the coproduct $\mathfrak{B} = \coprod_{\lambda\in\Lambda} \mathfrak{B}|_{\mathcal{C}_\lambda}$.

Remark 1.6. If the product and coproduct constructions are applied to a family of symmetric balleans, one obtains symmetric balleans.

Let $q\colon X \to Y$ be a surjective map from a ballean $\mathfrak{B}_X = (X, P, B)$ to a non-empty set Y. In the sequel, for $A \subseteq X$ we denote by $R_q[A]$ the set $q^{-1}(q(A))$; in case $A = \{a\}$ is a singleton, we simply write $R_q[a]$. The quotient ballean $\widetilde{\mathfrak{B}}^q$ is the ballean structure on Y which satisfies these equivalent conditions:

(a) $\widetilde{\mathfrak{B}}^q$ the finest ballean structure on Y such that $q : \mathfrak{B}_X \to \widetilde{\mathfrak{B}}^q$ is bornologous;

(b) for every ballean Z and every map $f : Y \to Z$ such that $f \circ q : X \to Z$ is bornologous, also $f : Y \to Z$ is bornologous.

The quotient ballean structure always exists, since the category **Coarse** is topological (see [7]), but its explicit description could be somewhat complicated. However there are some special cases for which the quotient structure can be quite easily defined.

Definition 1.7. Let $q\colon X \to Y$ be a surjective map and $\mathfrak{B}_X = (X, P, B)$ be a ballean. We call q:

(a) *soft* if for every $\alpha \in P$ there exists $\beta \in P$ such that for every $x \in X$ we have

$$R_q[B(R_q[x], \alpha)] \subseteq B(R_q[x], \beta);$$

(b) *weakly soft* if for every $\alpha \in P$ there exists $\beta \in P$ such that for every $x \in X$ we have

$$B(R_q[B(x, \alpha)], \alpha) \subseteq R_q[B(R_q[x], \beta)].$$

In [7] an equivalent description of these properties of maps is given in terms of entourages.

Let $\mathfrak{B} = (X, P, B)$ be a ballean, Y be a non-empty set and $q\colon X \to Y$ be a surjective map. Define the *quotient ball structure* to be the ball structure $\overline{\mathfrak{B}}^q = (Y, P, \overline{B}^q)$, where $\overline{B}^q(y, \alpha) = q(B(q^{-1}(y), \alpha))$, for every $\alpha \in P$ and $y \in Y$. Actually this is not a ballean in general, since $\overline{\mathfrak{B}}^q$ may fail to be upper multiplicative, although it is always upper symmetric (see [7]). Since $\overline{\mathfrak{B}}^q \prec \widetilde{\mathfrak{B}}^q$, the quotient ball structure is a ballean if and only if $\overline{\mathfrak{B}}^q = \widetilde{\mathfrak{B}}^q$. If we start with a symmetric ballean $\mathfrak{B}_X = (X, P, B)$ and a surjective map $q\colon X \to Y$, also the ball structure $\overline{\mathfrak{B}}^q$ is symmetric.

Theorem 1.8. [7, Remark 4.8(b), Lemma 4.9, Theorem 4.12] *Let $\mathfrak{B} = (X, P, B)$ be a ballean, Y be a non-empty set and $q\colon X \to Y$ be a surjective map. Then:*
(a) if q is soft, then it is weakly soft;
(b) q is weakly soft if and only if $\overline{\mathfrak{B}}^q$ is a ballean;
(c) if q is soft, then $\overline{\mathfrak{B}}^q$ is a ballean; in such a case q is a $\succ$-mapping provided Y is equipped with $\overline{\mathfrak{B}}^q$.

In particular, it is proved in [7] that the projection maps from the product ballean onto its components are soft. Moreover, if $q\colon G \to H$ is a surjective group homomorphism and $\mathfrak{B}_{\mathcal{I}} = (G, \mathcal{I}, B_{\mathcal{I}})$ is a ballean (see Example 1.2(c)), then q is soft ([7]).

2 Sizes in balleans

Following [19], for a ballean $\mathfrak{B} = (X, P, B)$, $\alpha \in P$ and a subset A of X define the *α-interior* of A (or simply *interior*, when α is either clear or irrelevant), by $\mathrm{Int}(A, \alpha) = \{a \in A \mid B^*(a, \alpha) \subseteq A\}$ (if $\mathfrak{B}$ is also symmetric, then $\mathrm{Int}(A, \alpha) = \{a \in A \mid B(a, \alpha) \subseteq A\}$). The interior $\mathrm{Int}(A, \alpha)$ is very helpful in describing the complement of balls, as $\mathrm{Int}(A, \alpha) = X \setminus B(X \setminus A, \alpha)$.

Definition 2.1. For a ballean (X, P, B) and a subset A of X we say that
(a) A is *large* (denoted briefly by L) in X, if there exists $\alpha \in P$ such that $B(A, \alpha) = \bigcup_{x \in A} B(x, \alpha) = X$;
(b) A is *thick* (briefly, T) in X, if $\mathrm{Int}(A, \alpha) \neq \emptyset$ for every $\alpha \in P$;
(c) A is *piecewise large* (briefly, PL) in X, if there exists $\alpha \in P$ such that $B(A, \alpha)$ is thick;
(d) A is *small* (briefly, S) in X, if for each $\alpha \in P$ the set $X \setminus B(A, \alpha)$ is large;
(e) A is *extralarge* (briefly, XL) in X, if $X \setminus A$ is small.

These five properties are invariant under asymorphisms (see [19], for largeness, extralargeness and smallness). Therefore, one can actually make recourse only to symmetric balleans, according to Remark 1.5.

We start relating "large" and "thick". In general, there is no implication between large and thick as Example 2.3 shows. Nevertheless, there is a nice connection between "large" and "thick" realized via passage to complements described in the next lemma, where the equivalences (a) $\leftrightarrow$ (b) and (b) $\leftrightarrow$ (c) follow from the definitions.

Lemma 2.2. *For a ballean (X, P, B) and a subset A of X the following are equivalent:*
(a) A is large;
(b) there exists $\alpha \in P$ such that $A \cap B(x, \alpha) \neq \emptyset$ for every $x \in X$;
(c) $X \setminus A$ is not thick.

The obvious topological interpretation of item (b) is "(uniform) density" of the large sets. This suggests also the following equivalent way to state the above lemma: *a subset A is large if and only if it non-trivially meets every thick subset* (or, equivalently, *a subset A is thick if and only if it non-trivially meets every large subset*) ([23, Proposition 9.1.2]).

Example 2.3. Consider $\mathbb{Z}$, endowed with the euclidean metric ballean structure $\mathfrak{B}_{\mathbb{Z}}^d$.
(a) A subset A of $\mathbb{Z}$ is large if and only if A is unbouded from above and from below and if $A = \{a_n \mid n \in \mathbb{Z}\}$ is an increasing enumeration of A, then the sequence $d_n = |a_n - a_{n+1}|$ is bounded ([5]).
(b) Both $2\mathbb{Z} = \{2x \mid x \in \mathbb{Z}\}$ and $\mathbb{Z} \setminus 2\mathbb{Z}$ are large in $\mathbb{Z}$, by (a). By the above lemma, $2\mathbb{Z}$ is not thick as $\mathbb{Z} \setminus 2\mathbb{Z}$ is large. The same lemma implies that $\mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\}$ is thick in $\mathbb{Z}$, but it is not large in $\mathbb{Z}$. Other instances of large subsets that are not thick can be found in Example 2.12.

Extralarge sets can be described as follows:

Theorem 2.4. [19, Theorem 11.1] *For a ballean (X, P, B) and a subset A of X the following conditions are equivalent:*
(a) $\mathrm{Int}(A, \alpha)$ is large for every $\alpha \in P$;
(b) for each large set L of X the set $L \cap A$ remains large in X;
(c) A is extralarge (i.e., $X \setminus A$ is small).

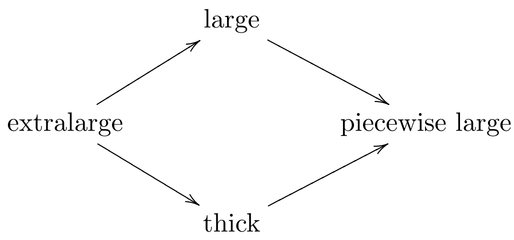
We collect in the next theorem the equivalent smallness conditions. The equivalence of (a) and (b) follows from the previous theorem and the others are proved in [23, Proposition 9.1.1].

Theorem 2.5. *For a ballean (X, P, B) and a subset A of X the following conditions are equivalent:*
(a) A is small;
(b) the set $L \setminus A$ is large in X for each large set L of X;
(c) $X \setminus A$ is extralarge;
(d) A is not piecewise large;
(e) $\mathrm{Int}(X \setminus A, \alpha)$ is large for all $\alpha \in P$.

Following [19], we denote by $\mathcal{S}(X)$ the family of all small subsets of the ballean X and we let $\mathcal{E}(X)$ denote the family of all extralarge subsets of X.

Corollary 2.6. *[19, Theorem 11.2] For a ballean (X, P, B), the family $\mathcal{S}(X)$ is an ideal, while the family $\mathcal{E}(X)$ is a filter on X. Consequently, if a piecewise large subset A of X is finitely partitionated, $A = A_1 \cup \cdots \cup A_n$, then at least one of these A_i, $i \in \{1, \ldots, n\}$, is piecewise large.*

If A is extralarge, then $X \setminus A \in \mathcal{S}(X)$, so $A = X \setminus (X \setminus A)$ is large by Theorem 2.5. On the other hand, A is also thick by Theorem 2.4. This gives the following connections between the four concepts of largeness



(2)

In §2.1, we give a diagram of the four concepts of smallness obtained by taking the negations in (2).

Example 2.7. (a) A thick and large subset Y of $X = \mathbb{Z}$, that is not XL. Take $Y = \mathbb{Z}_+ \cup 2\mathbb{Z}$. This works as the subset $L = 1 + 2\mathbb{Z}$ is large in $\mathbb{Z}$, but $L \cap Y$ is not large in X.

(b) A thick subset Y of $X = \mathbb{Z}$, such that for some XL set A in X the intersection $A \cap Y$ is not large even in Y (so not large in X either). Let $Y = \{n^2 \mid n \in \mathbb{Z}_+\} \cup (\mathbb{Z} \setminus \mathbb{Z}_+)$ and $A = \mathbb{Z} \setminus \{n^2 \mid n \in \mathbb{Z}_+\}$. This works.

2.1 Other sizes

A closer look at Lemma 2.2 and Theorem 2.5 shows a striking similarity between the pair of properties large-thick and the pair extralarge-piecewise large (see (i) and (ii) below)). The following definitions allow us to handle better this relation. In the sequel, throughout the whole paper, W will denote a generic property of subsets of balleans.

Let X be a ballean and A a subset of X. We say that

- *A has the property W^c in X* if $X \setminus A$ has the property W in X;
- *A has the property $\neg W$ in X* if A does not have the property W in X;
- *A has the property $\neg W^c$ in X* if $X \setminus A$ does not have the property W in X.

In the sequel we denote the property $\neg W^c$ also by W^* and call it *dual* to W. Let us note that if W is stable under taking subsets (supersets), then W^* has the same property, while $\neg W$ and W^c are stable under taking supersets (resp., subsets).

In these terms one can immediately describe the connection between the already known properties L, T, PL, XL, S, established in Lemma 2.2 and Theorem 2.5, as follows:

(i) $L = \neg T^c$, while $T = \neg L^c$; similarly,

(ii) $XL = \neg PL^c$; while $PL = \neg XL^c$; and

(iii) $S = XL^c$ and $S = \neg PL$.

Comparing (a), (b) and (c) one can realize that the property S plays a somewhat singular role, with respect to the remaining properties, i.e., something seems to be "missing". In order to complete the picture and make it totally symmetric we introduce the missing three properties as follows:

Definition 2.8. Let (X, P, B) be a ballean and $A \subseteq X$. Then we say that A is:

(a) *slim* (briefly, SL) if for every $\alpha \in P$ there exists $x \in X$ such that $B(x, \alpha) \cap A = \emptyset$;

(b) *with slim interior* (briefly, SI) if there exists $\alpha \in P$ such that $\mathrm{Int}(A, \alpha)$ is slim;

(c) *meshy* (briefly, M) if there exists $\alpha \in P$ such that $B(x, \alpha) \setminus A \neq \emptyset$ for every $x \in X$.

These properties are obviously invariant under asymorphisms.

Now we see that actually these three properties are closely related to the old ones by the operations of negation and passage to the complement. In other words these are precisely the "missing" ones.

Proposition 2.9. *Let $\mathfrak{B} = (X, P, B)$ be a ballean. Then*

(a) $SL = \neg L = T^c = \neg M^c$;

(b) $M = \neg T = L^c = \neg SL^c$;

(c) $SI = \neg XL = PL^c = \neg S^c$.

W	$\neg W$	W^c	$W^* = \neg W^c$
L	SL	M	T
T	M	SL	L
SL	L	T	M
M	T	L	SL
S	PL	XL	SI
SI	XL	PL	S
PL	S	SI	XL
XL	SI	S	PL

Table 1: In this table we summarize all the eight properties of subsets of balleans and the relationships between them. We devote a line to each property.

Proof. (a) The equalities $SL = \neg L$ and $SL = T^c$ are obvious (taking into account $L = \neg T^c$ from (i)). As obviously $M = L^c$, this yields $SL = \neg M^c$. (b) is a trivial consequence of (a).

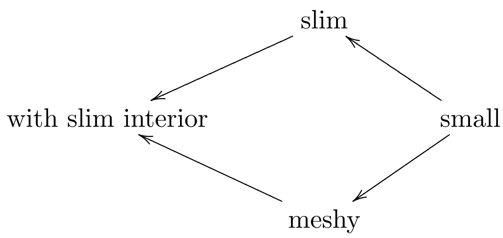
(c) The equalities $\neg XL = PL^c = \neg S^c$ follow from Theorem 2.5. The equality $SI = \neg XL$ follows from Theorem 2.4, as there exists a radius $\alpha \in P$ with $\mathrm{Int}(A, \alpha)$ slim (i.e., non-large) precisely when A is not extralarge. $\square$

In Table 1, we summarize the relations established in Proposition 2.9, as well as in Lemma 2.2 and Theorem 2.5.

In the sequel W will be usually one of the eight properties $\{L, T, XL, PL, SL, M, SI, S\}$ describing the size of subsets in balleans. We shall briefly denote by $\mathfrak{S}$ this set of eight properties. Note that the properties of $\mathfrak{S}^{\downarrow} = \{SL, M, SI, S\}$ are stable under taking subsets, while $\mathfrak{S}^{\uparrow} = \{L, T, PL, XL\}$ consists of all properties in $\mathfrak{S}$ stable under taking supersets. Motivated by this and also by the obvious intuitive idea behind the definitions, we refer to the properties of $\mathfrak{S}^{\uparrow}$ (resp., $\mathfrak{S}^{\downarrow}$) as *largeness type* (*smallness type*) *properties.*

Remark 2.10. In every non-empty ballean X, the set X has the four largeness type properties, while $\emptyset$ has the four smallness type ones. Moreover $\emptyset$ never has a property $W \in \mathfrak{S}^{\uparrow}$ and X never has $W \in \mathfrak{S}^{\downarrow}$.

We can now rewrite the counterpart of diagram (2), by using the specific names of the smallness type properties that are negations of the largeness type properties from (2):



Using these new concepts we can give a different characterization of smallness.

Proposition 2.11. *Let X be a ballean and A be a subset of X. Then A is small if and only if for every meshy subset C of X, $A \cup C$ is still meshy.*

Proof. A is small if and only if for every large subset L of X, $L \setminus A$ is still large in X (Theorem 2.5). Thanks to Proposition 2.9, it is equivalent to say that $X \setminus (L \setminus A) = (X \setminus L) \cup A$ is meshy in X. We conclude by observing that L is large in X precisely when $X \setminus L$ is meshy in X. $\square$

Example 2.12. Let X be a ballean and let A be a subset of X.
(a) Suppose that X is bounded (see Example 1.2(a)). Then
 (a1) A is large if and only if $A \neq \emptyset$ if and only if A is piecewise large;
 (a2) A is thick if and only if $A = X$ if and only if A is extralarge (since $T = SL^c$ and $XL = S^c$);
 (a3) A is small if and only if $A = \emptyset$ if and only if A is slim (since $S = \neg PL$ and $SL = \neg L$);
 (a4) A is meshy if and only if $A \neq X$ if and only if A is with slim interior (since $M = \neg T$ and $SI = \neg XL$).
(b) Suppose that X is the trivial ballean (see Example 1.2(a)). Then
 (b1) A is large if and only if $A = X$ if and only if A is extralarge;
 (b2) A is thick if and only if $A \neq \emptyset$ if and only if A is piecewise large (since $T = L^*$ and $PL = XL^*$);

(b3) A is small if and only if $A = \emptyset$ if and only if A is meshy (since $S = \neg PL$ and $M = \neg T$);
(b4) A is slim if and only if $A \neq X$ if and only if A is with slim interior (since $SL = \neg L$ and $SI = \neg XL$).

We have just noticed that the only small subset of a ballean is the empty set, provided the ballean is either bounded or trivial. Here is an example of space which has non-empty small subset.

Example 2.13. All singletons of $X = \mathbb{Z}$ endowed with the euclidean metric ballean structure, are small. Hence every finite set is small in X.

Now we generalize this example as follows:

Theorem 2.14. *Let X be a connected ballean. Then the following are equivalent:*
(a) X is unbounded;
(b) every finite subset of X is small;
(c) there is a small singleton of X.

Proof. To prove the implication (a)→(b) pick $x \in X$ and a radius $\alpha \in P$, our aim is to prove that $Y = X \setminus B(x, \alpha)$ is large in X. Since X is unbounded, there exists $y \in Y$. As X is connected, there exists $\beta \in P$ such that $x \in B(y, \beta)$. Pick $\gamma \in P$ such that $B(B(z, \beta), \alpha) \subseteq B(z, \gamma)$ for all $z \in X$. Then $B(x, \alpha) \subseteq B(y, \gamma)$. Obviously, $X = B(Y, \gamma)$, so Y is large. This proves that all singletons of X are small. We are done as finite unions of small sets are small (Corollary 2.6).

The implication (b)→(c) is trivial. Finally (c)→(a) follows from Example 2.12, since, if X is bounded, then the empty set is the only small subset. □

Remark 2.15. Here we briefly list some properties that are not discussed in this paper for the lack of space. We are aware that group balleans were not paid sufficient attention, they will be discussed in the forthcoming manuscript [8].

(a) We are not going to consider combinations of the above properties, as the conjunction $L\&T$, for example (by Example 2.7 (a), this property is strictly weaker that XL, although it is strictly stronger than both L and T). Similarly, the property $SL\&PL$ will not be paid attention here. These sets are precisely those that are neither large nor small. In the special case of a finitary group balleans (see Example 1.2(c)) these sets were introduced and studied in [3], under the name of *medium* sets.

(b) Another notion of "smallness" can be found in the literature. For a ballean (X, P, B) a subset A of X is *thin*, if for every $\alpha \in P$ there exists a bounded set $V \subseteq X$ such that $B(a, \alpha) \cap B(a', \alpha) = \emptyset$ whenever $a, a' \in A \setminus V$ are distinct [14].

The following brief comment motivates our choice to discard this property in the paper.

Although this property gives the intuitive idea of smallness, there are no clear connections between smallness and thinness. Indeed, the ballean $B = \{n^2 \mid n \in \mathbb{N}\}$, with the metric ballean structure, is thin and even extralarge in iteself (so definitely non-small). On the other hand, thin subsets of a ballean do not form an ideal, providing easy examples of non-thin small subsets (take the euclidean metric ballean $\mathbb{Z}$, its subsets B and $B + 1 = \{b + 1 \mid b \in B\}$ are thin and small, but $A = B \cup (B + 1)$ is not thin, even if it is small).

(c) Finally, we are not going to discuss notions of size typical for group balleans, as sparseness ([11, 12]), or P-smallness [5] (see also Prodanov's original paper [15], where the notion "P-small" was introduced under the name "small").

(d) A remarkable connection between small sets in the metric ballean $\mathbb{R}^n$ and asymptotic dimension asdim was pointed out by Banakh, Chervak and Lyaskovska [2]. They proved that a set $A \subseteq \mathbb{R}^n$ is small if and only if asdim $A < n$, and extended this property also to all groups of the form $G = \mathbb{R}^n \times K \times \mathbb{Z}^m$, where K is a compact abelian group and $m \in \mathbb{N}$, providing G with group ball structure generated by the ideal of relatively compact sets of G.

3 Size preserving and size reflecting maps

3.1 Preservation and reflection of properties along maps

First of all we need to fix the terminology for the maps that preserve or reflect properties of subsets of the domain or the codomain, respectively.

Definition 3.1. Let $f : X \to Y$ be a map between balleans, we say that
(a) f is *W-preserving* if for every subset A of X which has property W in X, the subset $f(A)$ has W in Y;

(b) f is W-*reflecting* if for every subset A of Y which has property W in Y, the subset $f^{-1}(A)$ has W in X;
(c) f is W-*copreserving* if every subset A of X such that $f(A)$ has the property W in Y, has W in X ([9]).

In the sequel, for a subset $\mathfrak{A}$ of $\mathfrak{S}$ we shall briefly write $\mathfrak{A}$-preserving in place of "W-preserving for every $W \in \mathfrak{A}$".

Remark 3.2. It is easy to check that, if $f: X \to Y$ and $g: Y \to Z$ are maps between balleans, then $g \circ f$ is W-preserving (or W reflecting) if both f and g have this property.

The next goal is to give some results connecting the properties of a map to preserve, to copreserve or to reflect some of the properties W, $\neg W$, W^c or W^*.

Claim 3.3. *Let $f: X \to Y$ be a map between balleans.*
(a) f is W-reflecting if and only if it is W^c-reflecting.
(b) f is W-copreserving if and only if it is $\neg W$-preserving.

Proof. (a) Assume that f is W-reflecting and let A be a subset of Y which has property W^c. Then $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ has property W in X, hence $f^{-1}(A)$ has W^c. This proves that f is W^c-reflecting whenever it is W-reflective. Exchanging W and W^c we prove the opposite implication.

(b) Assume that f is $(\neg W)$-preserving. If for $A \subseteq X$ the set $f(A)$ satisfies W, then A satisfies W, since otherwise f cannot be $(\neg W)$-preserving. Now assume that f is W-copreserving. If $A \subseteq X$ fails to satisfy W, then $f(A)$ fails to satisfy W either, as f is W-copreserving. So f is $(\neg W)$-preserving. □

Claim 3.4. *Let $f: X \to Y$ be a map between balleans. If f is surjective or W is stable under taking supersets, then the following implications hold:*
(a) if f is W-preserving, then it is $\neg W$-reflecting;
(b) if f is W^*-preserving, then it is W-reflecting.

Proof. (a) Let A be a subset of Y such that $f^{-1}(A)$ has property W in X. Then, if f is surjective, $A = f(f^{-1}(A))$ has the same property in Y; otherwise A has W in Y, since $A \cap f(X) = f(f^{-1}(A))$ has property W in X and $A \cap f(X) \subseteq A$.

(b) Let A be a subset of Y and suppose that $f^{-1}(A)$ has property $\neg W$ in X. Since f is $\neg W^c$-preserving and $X \setminus f^{-1}(A)$ has property $\neg W^c$ in X, $f(X \setminus f^{-1}(A)) = f(X) \setminus A$ has property $\neg W^c$ in Y. If f is surjective, then A has property $\neg W$ in Y. Otherwise, since $f(X) \setminus A \subseteq Y \setminus A$ and W^* is stable under taking supersets, we conclude that also $Y \setminus A$ has property $\neg W^c$. □

Claim 3.5. *Let $f: X \to Y$ be a map between balleans. Suppose that f is injective or W is stable under taking supersets. If f is W^*-reflecting, then it is W-preserving.*

Proof. Let A be a subset of X and suppose that $f(A)$ has the property $\neg W$ in Y. By applying the hypotesis, we obtain that $f^{-1}(Y \setminus f(A)) = X \setminus f^{-1}(f(A))$ has property $\neg W^c$ in X and so $f^{-1}(f(A))$ has property $\neg W$ in X. If f is injective, then there is nothing left to be proved. Otherwise, since $A \subseteq f^{-1}(f(A))$ and the property W is stable under taking superset, then A itself has property $\neg W$ in X. □

By taking $\neg W$ in place of W in Claim 3.5 and using Claim 3.3 (a), we can deduce that for injective f or V stable under taking subsets V-reflecting implies $\neg V$-preserving.

Claim 3.6. *Let $f: X \to Y$ be a map between balleans. Suppose one of the following conditions holds:*
(a) W is stable under taking subsets and f is surjective;
(b) W is stable under taking supersets and f is injective.
If f is W-preserving, then it is W^c-preserving.

Proof. Let A be a subset of X which has the property W^c in X. Then $f(X \setminus A)$ has the property W in Y. If f is surjective and W is stable under taking subsets, $Y \setminus f(A)$ has the property W too, since $Y \setminus f(A) \subseteq f(X \setminus A)$. If f is injective, then $f(X \setminus A) = f(X) \setminus f(A)$ and so $Y \setminus f(A)$ has the property W, since $Y \setminus f(A) \supseteq f(X) \setminus f(A)$ and W is stable under taking supersets. □

3.2 The four blocks of equivalent properties

In the next theorem we concentrate on all basic relations between the preservation/reflection properties in the case when W is stable under taking supersets.

Theorem 3.7. *Let $f: X \to Y$ be a map between balleans and let W be a property of subsets of balleans stable under taking supersets. Then for f the following equivalences hold*

$$\neg W\text{-copreserving} \Leftrightarrow W\text{-preserving} \Leftrightarrow W^*\text{-reflecting} \Leftrightarrow \neg W\text{-reflecting}. \tag{3}$$

If f is in addition also surjective or injective, then one can add more implications as follows

$$\begin{aligned} &W^c\text{-preserving} \xRightarrow{f\ surj.} \neg W\text{-copreserving} \Leftrightarrow W\text{-preserving} \Leftrightarrow W^*\text{-reflecting} \Leftrightarrow \neg W\text{-reflecting} \xRightarrow{f\ inj.} \\ &\qquad \xRightarrow{f\ inj.} W^c\text{-preserving} \Leftrightarrow W^*\text{-copreserving}. \end{aligned} \tag{4}$$

Proof. By Claim 3.3(b), $\neg W$-copreserving $\Leftrightarrow$ W-preserving and, by Claim 3.4(b), W^*-preserving $\Rightarrow$ W-reflecting. Since W^* is still stable under taking supersets and $W^{**} = W$, one can deduce from that claim also the implication W-preserving $\Rightarrow$ W^*-reflecting. On the other hand, Claim 3.5 ensures that W^*-reflecting implies W-preserving. To check the third equivalence in (3) apply Claim 3.3(a) to the property W^* (noting that $(W^*)^c = \neg W$) to conclude that $\neg W$-reflecting is equivalent to W^*-reflecting. This proves (3).

If, in addition f is also injective, we apply Claim 3.6(b) to obtain W-preserving $\Rightarrow$ W^c-preserving. In conjunction with (3) and Claim 3.3(b), this provides the last implication in (4).

Assume now that in addition f is also surjective. By Claim 3.6(a) applied to W^c, which is stable under under taking subsets, we get W^c-preserving $\Rightarrow$ W-preserving. Therefore, (4) holds true. □

Applying Theorem 3.7 to the size properties defined in Definitions 2.1 and 2.8 we obtain:

Corollary 3.8. *If $f\colon X \to Y$ is a map between balleans, then the following implications and equivalences hold.*

$$\begin{aligned} &SL\text{-preserving} \xRightarrow{f\ surj.} M\text{-copreserving} \Leftrightarrow T\text{-preserving} \Leftrightarrow L\text{-reflecting} \Leftrightarrow M\text{-reflecting} \xRightarrow{f\ inj.} \\ &\qquad \xRightarrow{f\ inj.} SL\text{-preserving} \Leftrightarrow L\text{-copreserving}. \end{aligned}$$

$$\begin{aligned} &M\text{-preserving} \xRightarrow{f\ surj.} SL\text{-copreserving} \Leftrightarrow L\text{-preserving} \Leftrightarrow T\text{-reflecting} \Leftrightarrow SL\text{-reflecting} \xRightarrow{f\ inj.} \\ &\qquad \xRightarrow{f\ inj.} M\text{-preserving} \Leftrightarrow T\text{-copreserving}. \end{aligned}$$

$$\begin{aligned} &SI\text{-preserving} \xRightarrow{f\ surj.} S\text{-copreserving} \Leftrightarrow PL\text{-preserving} \Leftrightarrow XL\text{-reflecting} \Leftrightarrow S\text{-reflecting} \xRightarrow{f\ inj.} \\ &\qquad \xRightarrow{f\ inj.} SI\text{-preserving} \Leftrightarrow XL\text{-copreserving}. \end{aligned}$$

$$\begin{aligned} &S\text{-preserving} \xRightarrow{f\ surj.} SI\text{-copreserving} \Leftrightarrow XL\text{-preserving} \Leftrightarrow PL\text{-reflecting} \Leftrightarrow SI\text{-reflecting} \xRightarrow{f\ inj.} \\ &\qquad \xRightarrow{f\ inj.} S\text{-preserving} \Leftrightarrow PL\text{-copreserving}. \end{aligned}$$

Proof. By choosing $W \in \mathfrak{S}^\uparrow$ in (4) and taking into account that $L = T^*$, $T = L^*$, $XL = PL^*$, $PL = XL^*$, $M = \neg T$, $SL = \neg L$, $SI = \neg XL$, $S = \neg PL$ and $SL = T^c$, $M = L^c$, $SI = PL^c$, $S = XL^c$, one obtains the desired four chains of implications and equivalences. □

In particular, the previous corollary shows that for every $W \in \mathfrak{S}^\downarrow$ a map is W-reflecting if and only if it is W-copreserving (this follows also from Theorem 3.7 for an arbitrary W stable under taking subsets).

Remark 3.9. As Corollary 3.8 shows, every size reflecting or size copreserving property is equivalent to a preservation property with respect to another appropriate size type. Hence, for the sake of brevity, in the sequel we can use the preservation properties as representatives of their equivalence classes of properties. In particular, for every $W \in \mathfrak{S}^\uparrow$ there are three properties equivalent to W-preservation, while for every $V \in \mathfrak{S}^\downarrow$ there is one property equivalent to V-preservation. For example we will write that a map is T-preserving, instead of T-preserving, $\{L, M\}$-reflecting and M-copreserving (or SL-preserving, instead of SL-preserving and L-copreserving). Similarly, we shall only say $\mathfrak{S}^\uparrow$-preserving instead of $\mathfrak{S}^\uparrow$-preserving and $\mathfrak{S}$-reflecting, etc.

For reader's convenience, all implications from Corollary 3.8, as well as from the next three lemmas, are conveniently visualized in Figure 1, providing a complete summary of all these results.

Finally, we observe that for bijective maps all lines in Corollary 3.8 become equivalences, so in the case of bijective maps one is left with only four representatives of these properties, namely W-preserving, for $W \in \mathfrak{S}^\uparrow$.

The next lemmas involve also specific large scale properties of the maps.

Lemma 3.10. *If $f\colon (X, P_X, B_X) \to (Y, P_Y, B_Y)$ is a T-preserving, bornologous map between two balleans, then it is PL-preserving. Moreover, if f is also injective, then it is SI-preserving.*

Proof. Fix a piecewise large subset A of X, a radius $\alpha \in P_X$ such that $B_X(A, \alpha)$ is thick and a radius $\beta \in P_Y$ such that $f(B_X(x, \alpha)) \subseteq B_Y(f(x), \beta)$ for every $x \in X$. Then $f(B_X(A, \alpha))$ is thick in Y and so is $B_Y(f(A), \beta)$ since it contains that subset. The last assertion follows from Corollary 3.8. □

Lemma 3.11. *If $f\colon (X,P_X,B_X)\to(Y,P_Y,B_Y)$ is a L-preserving, effectively proper map between two balleans, then it is S-preserving. Moreover, if it is surjective, then it is XL-preserving.*

Proof. Let $A\subseteq X$ be a small subset of X. To prove that $f(A)$ is small in Y fix a radius $\alpha\in P_Y$. One has to check that $Y\setminus B_Y(f(S),\alpha)$ is large in Y. Let $\beta\in P_X$ be a radius such that $f^{-1}(B_Y(f(x),\alpha))\subseteq B_X(x,\beta)$ for each point $x\in X$. Then, since $X\setminus B_X(S,\beta)$ is large in X and f is L-preserving, we have that $f(X\setminus B_X(S,\beta))$ is large in Y and so it is $Y\setminus B_Y(f(S),\alpha)$, since

$$f(X\setminus B_X(S,\beta))\subseteq f(X\setminus f^{-1}(B_Y(f(S),\alpha)))=f(X)\setminus B_Y(f(S),\alpha)\subseteq Y\setminus B_Y(f(S),\alpha).$$

This proves that f is S-preserving. To deduce that f is also XL-preserving, one can use surjectivity of f to apply Corollary 3.8. □

Lemma 3.12. *Let $f\colon X\to Y$ be a map between balleans.*
(a) If f is both L-preserving and T-preserving, then it is $\{L,T,XL\}$-preserving. Moreover,
  (a1) if f is also bornologous, then it is $\mathfrak{S}^{\uparrow}$-preserving;
  (a2) if f is injective, then it is $\mathfrak{S}$-preserving.
(b) If f is both M-preserving and T-preserving, then it is $\{T,M,PL\}$-preserving. Moreover, if f is also surjective, then it is $\mathfrak{S}\setminus\{SL,SI\}$-preserving.

Proof. (a) To show that f is XL-preserving pick an extralarge set A of X. According to Theorem 2.4 it is enough to check that for an arbitrary large subset L of Y the intersection $f(A)\cap L$ is still large. Since A is extralarge and $f^{-1}(L)$ is large in X (Corollary 3.8), then $A\cap f^{-1}(L)$ is large in X and so is $f(A\cap f^{-1}(L))=f(A)\cap L$ in Y.

(a1) If f is bornologous, then by Lemma 3.10 it is PL-preserving. So f is $\mathfrak{S}^{\uparrow}$-preserving.

(a2) Suppose now that f is injective. Since f is $\{L,T,XL\}$-preserving, we aim to prove that f is also PL-preserving, so $\mathfrak{S}^{\uparrow}$-preserving. Then injectivity if f and Corollary 3.8 will imply that f is $\mathfrak{S}$-preserving. To check that f is XL-reflecting fix an extralarge subset A of Y and a large subset B of X. Since $f(B)$ is large in Y and $f(B)\cap A$ has the same property,

$$B\cap f^{-1}(A)=f^{-1}(f(B))\cap f^{-1}(A)=f^{-1}(f(B)\cap A)$$

is large in X, where the first equality follows from the injectivity of f. This shows again that f is XL-reflecting. In particular, f is $\mathfrak{S}^{\uparrow}$-preserving.

(b) First we show that f is S-reflecting. Let A be a small subset of Y and fix a meshy subset C of X. In order to apply Proposition 2.11 we need to prove that $f^{-1}(A)\cup C$ is still meshy in X. By our assumption that f is M-preserving it follows that $f(C)$ and so $f(C)\cup A$ are meshy in Y and then, since f is also M-reflecting, $f^{-1}(f(C)\cup A)=f^{-1}(f(C))\cup f^{-1}(A)$ has the same property in X. We are done since the inclusion $C\cup f^{-1}(A)\subseteq f^{-1}(f(C))\cup f^{-1}(A)$ holds and M is stable under taking subsets.

By Corollary 3.8, f is also PL-preserving.

Suppose that f is surjective. We show first that f is S-preserving. Fix a small subset A of X and a meshy subset B of Y. By our assumption that f is M-reflecting it follows that $f^{-1}(B)$ is meshy in X and so it is $f^{-1}(B)\cup A$. Then also $f(A\cup f^{-1}(B))=f(A)\cup(B\cap f(X))$ is meshy in Y and thus $f(A)$ is small, since f is surjective.

Again surjectvity of f, along with Corollary 3.8, ensures that f is $\{L,XL\}$-preserving. Therefore, f is $\mathfrak{S}\setminus\{SL,SI\}$-preserving. □

We shall see in the sequel that an XL-preserving bornologous surjective map need not to be W-preserving for any $W\in\mathfrak{S}^{\downarrow}$ (see Examples 4.10 and 4.17).

Example 3.13. Here we provide two examples which show that a map which is $\{S,SI,PL,XL\}$-preserving needs not be L-preserving or T-preserving.

Let X be a set with at least two points. Denote by $X^{\mathcal{T}}$ and X^b the set X endowed with the trivial and the bounded ballean structure respectively. By applying Example 2.12, both the map $id_1\colon X^b\to X^{\mathcal{T}}$ and $id_2\colon X^{\mathcal{T}}\to X^b$ are $\{S,SI,PL,XL\}$-preserving, while the first one is not L-preserving and the second one is not T-preserving.

4 Size preservation along large scale maps

4.1 When surjective maps are size preserving

Our aim here is to apply the tools collected so far to determine when a surjective map with some additional property (bornologous, effectively proper, weakly soft or soft, etc.) has the size preserving or size reflecting

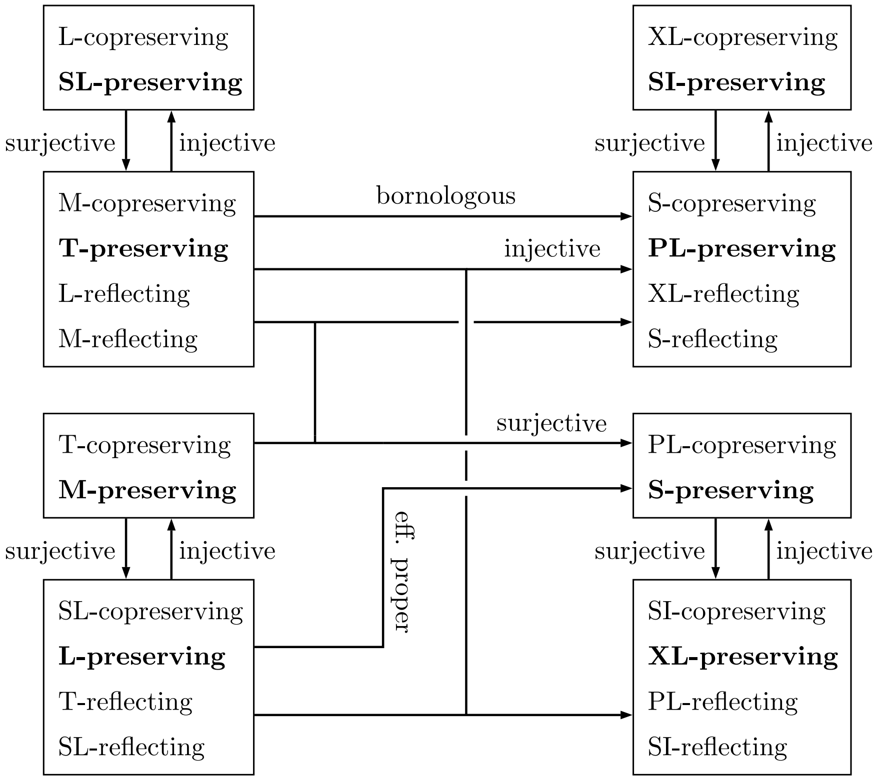


Figure 1: In this diagram we summarize all the implications which are proved in Corollary 3.8 and Lemmas 3.10, 3.11 and 3.12. The properties included in the same rectangular are equivalent.

properties introduced in the previous section. The positive results are collected in Theorems 4.1, 4.2, 4.4, 4.9 and Corollary 4.3, we provide counterexamples for all missing properties. The first results about preservation and reflection of largeness appeared already in [19]:

Theorem 4.1. *[19, Lemma 11.3] For balleans $\mathfrak{B}_X = (X, P_X, B_X)$, $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ and a surjective map $f\colon X \to Y$ the following holds:*
(a) if f is bornologous, then f is L-preserving;
(b) if f is a $\succ$-mapping, then f is T-preserving.

Theorem 4.2. *The following properties hold for balleans $\mathfrak{B}_X = (X, P_X, B_X)$, $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ and a surjective map $f\colon X \to Y$:*
(a) if f is bornologous, then f is L-preserving;
(b) if f is effectively proper, then f is $\{T, SL\}$-preserving;

Proof. Item (a) and T-preservation of item (b) are proved in Theorem 4.1 (as surjective effectively proper maps are $\succ$-mappings). The only thing that remains to be checked is that f is SL-preserving whenever it is effectively proper. According to Corollary 3.8, it suffices to check that f is L-copreserving. Let A be a subset of X such that $f(A)$ if large in Y. Then $Y = B_Y(f(A), \alpha)$ for some radius $\alpha \in P_Y$. Fix $\beta \in P_X$ such that $f^{-1}(B_Y(f(x), \alpha)) \subseteq B_X(x, \beta)$ for every $x \in X$. Then $X = f^{-1}(Y) = f^{-1}(B_Y(f(A), \alpha)) \subseteq B_X(A, \beta)$, i.e., A is large in X. □

As one can expect, surjective coarse embeddings have almost all the properties of preservation and hence of reflection. Surprisingly they do not have all of them, as we see later on (Example 4.5).

Corollary 4.3. *If $f\colon X \to Y$ is a surjective coarse embedding between two balleans, then it is $\mathfrak{S} \setminus \{M, SI\}$-preserving.*

Proof. Applying Theorems 4.1 and 4.2 we conclude that f is $\{L, T, SL\}$-preserving. Moreover, by Lemma 3.10, f is PL-preserving, while Lemma 3.11 yields that f is also $\{S, XL\}$-preserving. Therefore, f is $\mathfrak{S} \setminus \{M, SI\}$-preserving. □

Now we focus on particular bornologous maps: weakly soft and soft maps, which have been introduced in Definition 1.7. These maps become automatically bornologous, when the target space carries the quotient ball structure. Since we always consider them in this context, they become automatically L-preserving by

Theorem 4.1(a). In case they are soft, then they become also $\succ$-mappings, and so T-preserving, by Theorem 4.1(b).

Under the stronger condition of softness one can get more properties of preservation listed in the following theorem:

Theorem 4.4. *Let $\mathfrak{B}_X = (X, P, B_X)$ be a ballean and $q: X \to Y$ a surjective map. If Y carries the quotient ball structure $\overline{\mathfrak{B}}^q$ and if q is soft (equivalently, if q is a $\succ$-mapping), then f is $\mathfrak{S}^\uparrow$-preserving.*

Proof. By Theorem 4.1, q is T-preserving. Now we exploit this property of q to get all the other ones.

Since q is surjective and bornologous, it is L-preserving by Theorem 4.1, hence also PL-preserving, by Lemma 3.10. Moreover q is both L-preserving and T-preserving. Therefore, q is $\mathfrak{S}^\uparrow$-preserving, by Lemma 3.12(a). □

Let G be a group, equipped with ballean structure $\mathfrak{B}_{\mathcal{I}} = (G, \mathcal{I}, B_{\mathcal{I}})$ as in Example 1.2(c). Since every quotient homomorphism $q: G \to H$ is soft (see [7]), such a map q is $\mathfrak{S}^\uparrow$-preserving. Gusso proved in [10, Proposition 2.2, Proposition 2.3] proved that a quotient map is L-preserving, L-reflecting and S-copreserving once the groups are endowed with the finitary ballean structure (note that, if $q: G \to H$ is a quotient homomorphism and G is endowed with the finitary ballean structure, then the quotient ballean structure over H is the finitary one.

Now we give counterexamples to show that the above results are sharp, so they cannot be extended further. In particular: Example 4.5 shows that a surjective coarse embedding need not be M-preserving or SI-preserving; Examples 4.5, 4.10, 4.17 and 4.17 witness that a soft map may fail to be W-preserving for any property $W \in \mathfrak{S}^\downarrow$; Examples 4.6, 4.7 and 4.18 show that a weakly soft map (hence a bornologous surjective map as well) need not be W-preserving, for any $W \in \mathfrak{S} \setminus \{L\}$, and W'-reflecting, for any $W' \in \mathfrak{S} \setminus \{T, SL\}$. Finally Example 4.8 shows that an effectively proper surjective map need not to be W-preserving, for any $W \in \mathfrak{S} \setminus \{T, SL\}$, and W'-reflecting, for any $W' \in \mathfrak{S} \setminus \{L, M\}$.

Example 4.5. Here we show that a surjective coarse embedding need not to be M-preserving or SI-preserving.

Let $\mathbb{R}$ be endowed with the usual metric ballean structure. Then the floor map $\lfloor\cdot\rfloor: \mathbb{R} \to \mathbb{Z}$ (such that $\lfloor x\rfloor$ is the greatest integer below $x \in \mathbb{R}$) is a surjective coarse embedding. However, while the subset $\mathbb{Z} \subseteq \mathbb{R}$ is obviously non-thick (hence neither extralarge) in $\mathbb{R}$, its image is the whole space which is both extralarge and thick. Hence, this map is neither XL-copreserving, nor T-copreserving, hence it is neither M-preserving nor SI-preserving, by Corollary 3.8.

A different example of surjective coarse embedding which is not M-preserving is given by a bounded ballean, whose support has at least two points, which is sent to a one point-space. Then any proper subset of the first ballean is meshy while its image is not.

Finally, note that the map $\lfloor\cdot\rfloor: \mathbb{R} \to \mathbb{Z}$ is also soft.

Example 4.6. We construct here a weakly soft quotient map between two balleans which is neither XL-preserving nor T-preserving (consequently, it is non-SL-preserving and non-S-preserving by Corollary 3.8).

To this end we need a quotient map $q: X \to Y$, where $\mathfrak{B}_X = (X, P, B_X)$ is a ballean, such that $\overline{\mathfrak{B}}^q$ is bounded (hence $\overline{\mathfrak{B}}^q$ is a ballean and q is weakly soft by Theorem 1.8). Then q is S-reflecting and SL-reflecting, as Y has no non-empty small sets and no non-empty slim sets, by Example 2.12. Hence, it is also PL-preserving and L-preserving (the last assertion follows also from Theorem 4.2, as q is also bornologous).

Now assume that there exists $y \in Y$ such that the fiber $F = q^{-1}(y)$ is small in X, i.e., $E = X \setminus F$ is extralarge. Then F witnesses the fact that q is not S-preserving, while E witnesses the fact that q is not XL-preserving. Moreover, the (large) singelton $\{y\}$ is witnessing that q is not L-reflecting, as $F = q^{-1}(y)$ is non-large (actually, small) in X. By Corollary 3.8, q is neither T-preserving, nor SL-preserving.

Now we build an example of a quotient map $q: X \to Y$ satisfying the conditions from Example 4.6.

Example 4.7. Let $\mathfrak{B}_X = (\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}, B_d)$ be the euclidean metric ballean and $Y = \mathbb{T}$. Our aim is to to define a suitable map $q: X \to Y$ satisfying the conditions of the above example. This map will parametrize the winding of $\mathbb{R}_{\geq 0}$ around $\mathbb{T}$, but not at "constant speed". Define $q(0) = 0$ and let q map $[0, 1]$ onto $\mathbb{T}$ in the canonical way. As for the second "lap", q maps $[1, 3] \subseteq \mathbb{R}_{\geq 0}$ bijectively onto $\mathbb{T}$, then $[3, 7]$ onto $\mathbb{T}$ and so on, by descreasing speed by a factor 2 after each lap completion.

First we should prove that $\overline{\mathfrak{B}}^q$ is bounded. For every point $y \in Y$, there exists a (unique) element $x \in [0, 1) \cap q^{-1}(y)$; then

$$\mathbb{T} = q([0, 1]) \subseteq q(B_d(x, 1)) \subseteq \overline{B}^q(y, 1).$$

It is not hard to check that $q^{-1}(0) = \{0, 1, 3, 7, \dots\} = \{2^n - 1 \mid n \in \mathbb{N}\}$ is small in $\mathbb{R}_{\geq 0}$.

In the next example we focus on bijective maps that are bornologous and effectively proper. A bijective bornologous map is also $\{L, M\}$-preserving, by Theorem 4.1 and Corollary 3.8. Item (a) provides an example of a bijective bornologous map which is not W-preserving for $W \in \{T, SL, S, XL\}$, while the map in (b) is bijective and bornologous, but it is not W-preserving for $W \in \{T, SL, SI, PL\}$. In (c) we define a bijective effectively proper map which is not PL-preserving (so it is not SI-preserving by Corollary 3.8), while in (d) the map is bijective and effectively proper, but it is not W-preserving for $W \in \{S, XL, L, M\}$.

Example 4.8. Let X be a ballean such that there exists a small singleton $\{x\}$. For example X can satisfy the hypothesis of Theorem 2.14. Denote by $X^{\mathcal{T}}$ and X^b, respectively, the trivial ballean and the bounded ballean over the same support X.

(a) The identity map $id\colon X \to X^b$ is bornologous. Since $\{x\}$ is small in X, $X \setminus \{x\}$ is extralarge and hence thick in X. However, it is neither thick nor extralarge in X^b (Example 2.12) and then id is neither T-preserving, nor XL-preserving (hence, by Corollary 3.8, nor SL-preserving, nor S-preserving).

(b) Since $\{x\}$ is small in X, $\{x\}$ is neither piecewise large, nor thick, while this singleton has this two properties in $X^{\mathcal{T}}$ (Example 2.12), and then the bornologous map $id\colon X^{\mathcal{T}} \to X$ is neither PL-preserving nor T-preserving. By applying Corollary 3.8, the map is neither SL-preserving, nor SI-preserving.

(c) The identity map $id\colon X^b \to X$ is an effectively proper map. The singleton $\{x\}$ is small in X, but it is not small (actually it is large) in X^b, according to Example 2.12. Hence, id is not S-reflecting. By Corollary 3.8, it is not PL-preserving and neither SI-preserving, being bijective.

(d) Consider the identity map $id\colon X \to X^{\mathcal{T}}$, which is effectively proper. The singleton $\{x\}$ is small in X, but it is not small in $X^{\mathcal{T}}$ (see Example 2.12), so id is not S-preserving, so neither XL-preserving, by Corollary 3.8. On the other hand, by Lemma 3.12 applied to id we conclude that id is not L-preserving, so by Corollary 3.8, it is not M-preserving.

Now we focus on the product ballean and, in particular, on sizes of images and preimages through projection maps. Fix a family of balleans $(\mathfrak{B}_i = (X_i, P_i, B_i))_{i\in I}$ and consider the product ballean $\Pi_i\mathfrak{B}_i = (X, P, B_X)$, where $X = \Pi_i X_i$ and $P = \Pi_i P_i$. We are interested in *rectangular subset* A of the product, namely such that $A = \prod_{i\in I} A_i$. One can compute the interior of a rectangular subset $A = \prod_{i\in I} A_i$ as follows

$$\begin{aligned}\operatorname{Int}(A, (\alpha_i)_i) &= \{(x_i)_i \mid B_X^*((x_i)_i, (\alpha_i)_i) \subseteq A\} = \{(x_i)_i \mid \Pi_{i\in I} B_i^*(x_i, \alpha_i) \subseteq A\} = \\ &= \{(x_i)_i \mid B_i^*(x_i, \alpha_i) \subseteq A_i \,\forall i \in I\} = \Pi_{i\in I}\operatorname{Int}(A_i, \alpha_i).\end{aligned} \tag{5}$$

Theorem 4.9. *Let $(\mathfrak{B}_i = (X_i, P_i, B_i))_{i\in I}$ be a family of balleans and $\Pi_i\mathfrak{B}_i$ their product, with support $X = \Pi_i X_i$ and projection maps $p_j\colon X \to X_j$, $j \in I$.*

(a) For every $i \in I$, p_i is $\mathfrak{S}^{\uparrow}$-preserving.

(b) Fix for every $i \in I$ a subset $A_i \subseteq X_i$. If $W \in \mathfrak{S}$ and A_i has property W in $\mathfrak{B}_i$ for every $i \in I$, then $\prod_{i\in I} A_i$ has the same property in $\Pi_i\mathfrak{B}_i$.

Proof. Item (a) is a consequence of Theorem 4.4, since the projections p_i are soft. Hence, only item (b) remains to be proved.

Case $W = L$. Let $\alpha_i \in P_i$ be a radius which witnesses that A_i is large in $\mathfrak{B}_i$ for every $i \in I$. Then, by the definition of product balls, it is easy to check that $(\alpha_i)_i \in \Pi_i P_i$ witnesses that $\Pi_i A_i$ is large in $\Pi_i\mathfrak{B}_i$.

Cases $W = XL$ *and* $W = T$. The case $W = XL$ follows from the previous case, (5) and Theorem 2.4. The case $W = T$ is an easy consequence of (5).

Case $W = PL$. Let $\alpha_i \in P_i$ be a radius such that $B_i(A_i, \alpha_i)$ is thick for every $i \in I$. Then by using the previous option we conclude that $\Pi_{i\in I} B_i(A_i, \alpha_i) = B_X(\Pi_{i\in I} A_i, (\alpha_i)_i)$ is thick and this finishes our proof.

Case $W \in \mathfrak{S}^{\downarrow}$. Obviously, (a) implies that p_i is $\mathfrak{S}$-reflecting for every $i \in I$. From this and from the fact that W is stable under taking subsets we can conclude. □

Example 4.10. This example provides a soft quotient map, which is actually a projection map of a product of balleans, which is not W-preserving for any $W \in \mathfrak{S}^{\downarrow}$.

Let X be a ballean such that there exists a small singleton $\{x\}$ (see Theorem 2.14) and consider the product ballean $X \times X$. Consider the subset $Y = X \times \{x\}$, which is small (since the projection map $p_2 : X \times X \to X$ is S-reflecting, by Theorem 4.9) and then meshy, slim and with slim interior in the ballean $X \times X$. However, $p_1(Y) = X$ is surely not meshy, small, slim, nor with slim interior in X.

Remark 4.11. Let us point out two aspects of Theorem 4.9.

(a) Arbitrary finite intersections of extralarge subsets is extralarge and arbitrary finite unions of small subsets is small (Corollary 2.6). In item (b) of Theorem 4.9 one has an extralarge intersection of extralarge subsets in $\Pi_i X_i$ (namely, inverse images of extralarge subsets through projections). This intersection will be infinite in the case the product is infinite and will give rise also to a small set that is a union of infinitely many small subsets.

(b) Example 4.10 shows that projections are not necessarely W-preserving for any $W \in \mathfrak{S}^{\downarrow}$.

property	bornologous	effectively proper	coarse embedding	weakly soft	soft
L-copreserving	×	✓	✓	×	×
SL-preserving	×	✓	✓	×	×
M-copreserving	×	✓	✓	×	✓
T-preserving	×	✓	✓	×	✓
L-reflecting	×	✓	✓	×	✓
M-reflecting	×	✓	✓	×	✓
T-copreserving	×	×	×	×	×
M-preserving	×	×	×	×	×
SL-copreserving	✓	×	✓	✓	✓
L-preserving	✓	×	✓	✓	✓
T-reflecting	✓	×	✓	✓	✓
SL-reflecting	✓	×	✓	✓	✓
XL-copreserving	×	×	×	×	×
SI-preserving	×	×	×	×	×
S-copreserving	×	×	✓	×	✓
PL-preserving	×	×	✓	×	✓
XL-reflecting	×	×	✓	×	✓
S-reflecting	×	×	✓	×	✓
PL-copreserving	×	×	✓	×	×
S-preserving	×	×	✓	×	×
SI-copreserving	×	×	✓	×	✓
XL-preserving	×	×	✓	×	✓
PL-reflecting	×	×	✓	×	✓
SI-reflecting	×	×	✓	×	✓

Table 2: This table resumes the results of this subsection: the symbol ✓ stands for the fact that the class of surjective maps in a given column has the relevant property of the corresponding crossing line, otherwise we use the symbol ×. Theorems 4.1, 4.2, 4.4 and Corollary 4.3 cover the results proving the checkmarks, while Examples 4.5–4.8, 4.10, 4.17–4.18 contain counterexamples witnessing the crosses.

4.2 When inclusions are size preserving

Here we focus on injective maps.

Let X be a ballean, Y be a subset of X, endowed with the subballean structure and let $i\colon Y\to X$ be the inclusion map. It is easy to see that:

(a) i is W-preserving if and only if every subset $A\subseteq Y$ having the property W in Y, has the same property in X;

(b) i is W-reflecting if and only if $A\cap Y$ has property W in Y for every subset $A\subseteq X$ having the property W in X. m

Since bijective coarse equivalences (i.e., asymorphisms) are $\mathfrak{S}$-preserving, it is clear that as far as properties concerning preservation of size are involved, one can replace arbitrary injective coarse equivalences by inclusions.

The next two theorems completely describe all preservation properties of the inclusion map of a subballean. Observe that this map is always a coarse embedding.

Theorem 4.12. *Let $\mathfrak{B}=(X,P,B_X)$ be a ballean and $\mathfrak{B}|_Y=(Y,P,B_X|_Y)$ be a subballean of X. Let $i\colon Y\to X$ be the inclusion map.*
(a) i is $\mathfrak{S}^{\downarrow}$-preserving.
(b) The following are equivalent:

(b1) Y is large in X;
(b2) i is $\mathfrak{S} \setminus \{T, XL\}$-preserving;
(b3) i is L-preserving.

Proof. (a) We start with the case S. To prove that i is S-preserving it is enough to check that i is PL-copreserving. Assume that a subset A of Y is a piecewise large set in X. Then there exists a radius $\gamma \in P$ such that $B_X(A, \gamma)$ is thick. Fix now an arbitrary radius $\alpha \in P$ and let $\beta \in P$ such that $B_X(B_X(x, \gamma), \alpha) \subseteq B_X(x, \beta)$. Then there exists a point $x_\beta \in B_X(A, \gamma)$ such that $B_X(x_\beta, \beta) \subseteq B_X(A, \gamma)$. Finally, if $x_\alpha \in A$ is a point such that $x_\alpha \in B_X(x_\beta, \gamma)$, then

$$B_X(x_\alpha, \alpha) \subseteq B_X(B_X(x_\beta, \gamma), \alpha) \subseteq B_X(x_\beta, \beta) \subseteq B_X(A, \gamma),$$

and this conclude the proof, since $x_\alpha \in A \subseteq Y$ and so A is piecewise large in Y.

For the remaining cases we use Corollary 3.8 and check that i is $\{XL, T, L\}$-copreserving. The case L is a trivial applications of the definitions. To check that i is T-copreserving pick a subset A of Y that is thick in X. Let $\alpha \in P$ be a radius. Then $Int_X(A, \alpha) \neq \emptyset$. As $Int_Y(A, \alpha) = Int_X(A, \alpha) \cap Y \subseteq A$, we deduce that $Int_Y(A, \alpha) \neq \emptyset$ as well. Thus A is thick in Y.

It remains to prove that i is XL-copreserving. Let A be a subset of Y such that A is XL in X. Then Y is XL in X and for every radius $\alpha \in P$ the set $Int_X(A, \alpha)$ is large in X. Hence $\mathrm{Int}_Y(A, \alpha)$ is large in X, since $\mathrm{Int}_Y(A, \alpha) = \mathrm{Int}_X(A, \alpha) \cap Y$ and Y is extralarge in X. By the option L proved above, one can claim that , $\mathrm{Int}_Y(A, \alpha)$ is large in Y as well. Hence, A is XL in Y.

(b) Obviously, (b2) $\to$ (b3) $\to$ (b1).

To prove the implication (b1) $\to$ (b2) assume that Y is large in X. Then f is L-preserving because of the upper multiplicativity of $\mathfrak{B}$.

To check that i is XL-reflecting pick an extralarge subset A of X. Fix a large subset L of Y. Then L, being large in Y, is large in X as well. Finally $A \cap L = (A \cap Y) \cap L$ is large in X and so it is in Y. By Corollary 3.8, i is also PL-preserving. □

The counterpart of the above theorem fails for L traded for T (namely if Y is thick, then i need not be T-preserving). Indeed, take a ballean Z having a point z such that the singleton $\{z\}$ is small (then $\{z\}$ is neither thick nor piecewise large). Let $X = Z \sqcup Z$, $Y = i_1(Z) \sqcup \{i_2(z)\}$ and $i : Y \to X$ the inclusion map. Then Y is thick in X but i is not T-preserving. Indeed, $A = \{i_2(z)\}$ is thick in Y, yet $A = i(A)$ is not thick in X.

The next theorem resolves the case of the properties $\{T, XL\}$ missing in the above result.

Theorem 4.13. *Let $\mathfrak{B} = (X, P, B_X)$ be a ballean and $\mathfrak{B}|_Y = (Y, P, B_X|_Y)$ be a subballean of X. Then for the inclusion map $i\colon Y \to X$ the following are equivalent:*
(a) Y is extralarge in X;
(b) i is L-preserving and T-preserving;
(c) i is $\mathfrak{S}$-preserving.

Proof. (a) $\to$ (b) Let A be a large subset of X. Since Y is extralarge in X, the intersection $A \cap Y$ is still large in X and so, in particular, in Y, since $A \cap Y \subseteq Y$. Therefore, i is L-reflecting and so T-preserving by Corollary 3.8. By Theorem 4.12, i is L-preserving, as Y is large.

(b) $\to$ (c) By applying Lemma 3.12, we deduce that i is $\mathfrak{S}$-preserving.

(c) $\to$ (a) Since i is XL-preserving and Y itself is extralarge in Y, we deduce that Y is extralarge in X as well. □

This theorem tells, among others, that the apparently strong hypothesis "extralarge" in item (a) is necessary (so cannot be relaxed to "thick and large", the relevant example of a thick and large set that is not extralarge can be found in Example 2.7).

The next example is a counterpart of Example 4.5 showing that surjective coarse embeddings need not to be M-preserving or SI-preserving.

Example 4.14. Here we show that an injective coarse embedding need not to be T-preserving or XL-preserving. Indeed, according to Theorems 4.12 and 4.13, it suffices to consider the inclusion map $i : Y \to X$, where X is a ballean and Y a large subset of X that is not extralarge. Then i is not XL-preserving, as Y is extralarge in Y, but not in X. Since i is L-preserving and Y is not extralarge, it cannot be T-preserving, by Theorem 4.13.

Now we study the size preservation and the size reflection properties of the canonical maps of a coproduct.

Theorem 4.15. *Let $\{\mathfrak{B}_\nu = (X_\nu, P_\nu, B_\nu) \mid \nu \in I\}$ be a family of balleans, let $X = \bigsqcup_{\nu \in I} X_\nu$, endowed with the coproduct ballean structure $\coprod_\nu \mathfrak{B}_\nu$ and let $i_\nu\colon X_\nu \to X$ be the inclusion maps.*

(a) If $A = \bigcup_\nu i_\nu(A_\nu)$ is a subset of X, then:

(a1) A is large (respectively meshy, extralarge, small) if and only if A_ν is large (respectively meshy, extralarge, small) for every $\nu \in I$;

(a2) A is thick (respectively slim, piecewise large, with slim interior) if and only if A_ν is thick (respectively piecewise large, slim, with slim interior) for some $\nu \in I$.

(b) For every $\nu \in I$, i_ν is $\mathfrak{S} \setminus \{L, XL\}$-preserving.

Proof. (a1) The case L is trivial, thanks to the definition of balls of the coproduct. This settles also the case meshy, since $M = L^c$.

In order to handle the case XL, we need the following chain of equalities valid for every subset $D = \bigcup_\nu i_\nu(D_\nu)$ of X:

$$A \cap D = \bigcup_{\nu \in I} i_\nu(A_\nu) \cap \bigcup_{\nu \in I} i_\nu(D_\nu) = \bigcup_{\nu \in I} i_\nu(A_\nu \cap D_\nu). \tag{6}$$

Now assume that A is extralarge and fix $\nu \in I$. To check that A_ν is extralarge in X_ν pick a large set D_ν in X_ν and for every $\nu' \in I \setminus \{\nu\}$ define $D_{\nu'} = X_{\nu'}$. Then $D = \bigcup_{\mu \in I} D_\mu$ is large, by the case considered above. Hence, in view of (6), $A \cap D = \bigcup_{\nu \in I} i_\mu(A_\mu \cap D_\mu)$ is large, so $A_\nu \cap D_\nu$ is large. This proves that A_ν is extralarge in X_ν. Conversely, if A_ν is extralarge in X_ν for every $\nu \in I$, then we check that $A \cap D$ is large in X for every large set D of X, using the fact that $D = \bigcup_\nu i_\nu(D_\nu)$ and each D_ν is large, so $A_\nu \cap D_\nu$ is large in X_ν.

The case $S = XL^c$ easily follows now from the definitions.

(a2) Since $T = L^*$, A is thick precisely when $X \setminus A = \bigcup_\nu i_\nu(X_\nu \setminus A_\nu)$ is not large. By applying item (a1), the latter fact is equivalent to the existence of $\mu \in I$ such that $X_\mu \setminus A_\mu$ is not large in X_μ or, equivalently, A_μ is thick in X_μ. The argument for the other options is similar, if we use the relations $PL = XL^*$, $SL = M^*$ and $SI = S^*$.

(b) Fix an index $\nu \in I$. By item (a2), i_ν is $\{T, SL, SI, PL\}$-preserving. Moreover, the conjunction of item (a1) and the fact that the empty set has all the $\mathfrak{S}^\downarrow$ properties, entail that i_ν is also $\{M, S\}$-preserving.

Alternatively, it is possible to use Theorem 4.12, in order to prove that i_ν is $\mathfrak{S}^\downarrow$-preserving. □

Item (b) cannot be reinforced to produce $\mathfrak{S}$-preservation, as witnessed by item (a1) in the case of non-trivial coproducts (i.e., $|I| > 1$ and $X_\nu \neq \emptyset$ for all $\nu \in I$).

Here are two applications of the above theorem, using the fact that every ballean is equal to the coproduct of its connected components (see §1.2).

Theorem 4.16. *Let X be ballean. Then the following are equivalent:*

(a) every connected component of X is bounded;

(b) the only small subset of X is $\emptyset$;

(c) there is no small singleton of X.

Proof. Since (b)$\leftrightarrow$(c) is trivial, since $\mathcal{S}(X)$ is closed under taking subsets, we only need to prove (a)$\leftrightarrow$(c).

Assume that Y is an unbounded connected component of X. Then by Theorem 2.14 there exists small singleton $\{y\}$ in Y. By Theorem 4.15, $\{y\}$ is small also in X. Now assume that every connected component of X is bounded. If $y \in X$ is small in X and belong to some connected component Y of X, then $\{y\}$ is small in Y as well. Then Y is unbounded. □

As another application of Theorem 4.16 one can reduce the study of size to the case of *connected* balleans. Indeed, if Y is a subspace of a ballean X, then the equality $\mathcal{C}_{y,Y} = Y \cap \mathcal{C}_{y,X}$ for all $y \in Y$ holds, where $\mathcal{C}_{y,Y}$ and $\mathcal{C}_{y,X}$ denote the connected components of y in Y and in X, respectively (see §1.1). Since $X = \bigsqcup_{x \in X} \mathcal{C}_{x,X}$ carries the coproduct ballean structure (see §1.2), largeness of $Y \subseteq X$ in X is completely determined "locally" in the connected components $\mathcal{C}_{x,X}$ of X, by Theorem 4.16. More precisely, largeness of Y is equivalent to the simultaneous largeness of each connected component $\mathcal{C}_{y,Y}$ of Y in the respective component $\mathcal{C}_{y,X}$ of X. Similar simple reduction is available for the remaining properties.

It was shown in Example 4.10 that projections of product balleans provide instances of soft quotient map, which are not W-preserving for any $W \in \mathfrak{S}^\downarrow$. Our first example shows that similar phenomenon can be observed by using appropriate quotient maps defined on coproducts.

Example 4.17. Consider a ballean X and the quotient map $q\colon X \sqcup X \to X$ which glues together the two copies of X. The map q is soft, we show below that q need not to be SL-preserving, nor SI-preserving, not M-preserving.

(a) The subset $A = i_1(X)$ is non-large (hence, also non-extralarge) in $X \sqcup X$, while $q(A) = X$, i.e., $q(A)$ is both extralarge and large. Therefore, q is neither L-copreserving, nor XL-copreserving. According to Corollary 3.8, q is neither SL-preserving, nor SI-preserving.

(b) Assume that X can be partitioned in two large (consequently, also meshy) subsets A and B. (For example we can take $X = \mathbb{Z}$ and $A = 2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ and $B = 2\mathbb{Z}+1 = \{2n+1 \mid n \in \mathbb{Z}\}$.) Then the set $S = i_1(A) \cup i_2(B)$ is meshy in $X \sqcup X$, although $q(S) = X$ is obviously not meshy. Hence, q is not M-preserving.

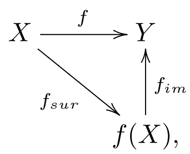
Example 4.18. This is an example of a weakly soft quotient map which is not S-reflecting (so it is neither PL-preserving, nor SI-preserving by Corollary 3.8).

Let X be a ballean with a point $x \in X$ such that $\{x\}$ is small in X. Consider the coproduct ballean $Y = X \sqcup \{x\}$, where $\{x\}$ is endowed with the only possible ballean structure on a singleton. Let $q\colon Y \to X$ be the map that glues together the two copies of x. Then the quotient ball structure of the codomain X coincides with the ballean structure of X, hence q is weakly soft (see Theorem 1.8). Hence, the singleton $\{x\}$ is small in X, but $A = q^{-1}(x)$ is not small in Y (as $i_2^{-1}(A) = \{x\}$ is not small in $\{x\}$, according to item (a) of Theorem 4.15).

4.3 When coarse equivalences are size preserving maps

Until now we focused on surjective or injective maps and we discussed whether particular classes of these maps are W-preserving or W-reflecting. In particular, we proved in Corollary 4.3 and Theorem 4.12 respectively, that coarse equivalences that are either surjective or injective have almost all the properties of preservation and reflection of size. Here we pay attention to *arbitrary* coarse equivalences, in particular, we relax the request of surjectivity or injectivity.

Let now $f\colon X \to Y$ be a map between balleans. Then we can factorize it as in the following diagram:



where $f_{sur}\colon X \to f(X)$ is the co-restriction of f and $f_{im}\colon f(X) \to Y$ is the inclusion map of $f(X)$ into Y. The notation f_{sur}, f_{im} and the factorization $f = f_{im} \circ f_{sur}$ will be frequently used in the section.

As in the rest of the paper, in the sequel W will denote a generic property of subsets of balleans.

According to Remark 3.2, both preservation and reflection are stable under composition. In particular, we have:

Claim 4.19. *Let $f\colon X \to Y$ be a map between balleans. If both f_{im} and f_{sur} are W-preserving (W–reflecting), then f is W-preserving (resp., W-reflecting) as well.*

We omit the easy proof of the next proposition. Item (a) deals with "left cancelation" of injective W-preserving (resp., W-reflecting) maps with respect to the class of W-reflecting (resp., W-preserving) maps. Item (b) deals with "right cancelation" of surjective W-preserving (resp., W-reflecting) maps with respect to the class of W-reflecting (resp., W-preserving) maps.

Proposition 4.20. *Let $g\colon X \to Z$ and $h\colon Z \to Y$ be maps between balleans, $f = h \circ g$.*
(a) Suppose that h is injective. Then:
- *(a1) if h is W-preserving, then g is W-reflecting, whenever f is W-reflecting;*
- *(a2) if h is W-reflecting, then g is W-preserving, whenever f is W-preserving.*

(b) Suppose that g is surjective. Then:
- *(a1) if g is W-preserving, then h is W-reflecting, whenever f is W-reflecting;*
- *(a2) if g is W-reflecting, then h is W-preserving, whenever f is W-preserving.*

We can apply Proposition 4.20 to the factorization $f = f_{im} \circ f_{sur}$ of a map $f\colon X \to Y$ between balleans:

Corollary 4.21. *Let $f\colon X \to Y$ be a map between balleans.*
(a1) If f_{im} is W-preserving, then f_{sur} is W-reflecting, whenever f is W-reflecting.
(a2) If f_{im} is W-reflecting, then f_{sur} is W-preserving, whenever f is W-preserving.
(b1) If f_{sur} is W-preserving, then f_{im} is W-reflecting, whenever f is W-reflecting.
(b2) If f_{sur} is W-reflecting, then f_{im} is W-preserving, whenever f is W-preserving.

By using the results we have collected in Sections 4.1 and 4.2, we can derive properties of some non-injective and non-surjective maps. As a first step in this direction, by combining Corollary 4.21, Theorems 4.12 and 4.13 and Claim 4.19, we obtain the following general fact connecting properties of a map f and its co-restriction component f_{sur}:

Proposition 4.22. *For a map $f\colon X \to Y$ between two balleans the following assertions hold.*

(a) If f_{sur} is W-preserving for some $W \in \mathfrak{S}^{\downarrow}$, then f has the same property.
(b) If f is W-preserving for some $W \in \mathfrak{S}^{\uparrow}$, then f_{sur} has the same property.

Proof. According to Theorem 4.12, f_{im} is $\mathfrak{S}^{\downarrow}$-preserving. Hence, (a) follows from Claim 4.19.

To prove (b) take some $W \in \mathfrak{S}^{\uparrow}$ and note that $\neg W \in \mathfrak{S}^{\downarrow}$. Hence, f_{im} is $\neg W$-preserving, as already mentioned. By Corollary 4.21(a1), we obtain that f_{sur} is $\neg W$-reflecting, whenever f is $\neg W$-reflecting. According to Corollary 3.8, this implication is equivalent to the implication from item (b). □

Corollary 4.23. *Let $f: X \to Y$ be a map between two balleans with $f(X)$ large in Y. Then*
(a) If f_{sur} is W-preserving for some $W \in \mathfrak{S} \setminus \{T, XL\}$, then f has the same property.
(b) If f is W-preserving, for some $W \in \mathfrak{S} \setminus \{M, SI\}$, then f_{sur} has the same property.

Proof. As $f(X)$ is large by assumption, f_{im} is $\mathfrak{S} \setminus \{T, XL\}$-preserving (Theorem 4.12) and, consequently by Corollary 3.8, $\{T, SL, XL, S\}$-reflecting. Item (a) follows from Claim 4.19. Since we can apply Proposition 4.22(b), cases $W \in \{SL, S\}$ of item (b) remain to be proved and this can be done by using Corollary 4.21(a2). □

Let us note that largeness of $f(X)$ in Y is a necessary property in Corollary 4.23 (in the sense that if f is L-preserving, then $f(X)$ in Y is large in Y). Similarly, extralargeness of $f(X)$ in Y is a necessary property in the next corollary (if f is XL-preserving, then $f(X)$ in Y is extralarge in Y).

Corollary 4.24. *Let $f: X \to Y$ be a map between two balleans with $f(X)$ extralarge in Y. Then, for each $W \in \mathfrak{S}$, f_{sur} is W-preserving if and only if f has the same property.*

Proof. If $f(X)$ is extralarge in Y, then f_{im} is $\mathfrak{S}$-preserving (Theorem 4.13) and $\mathfrak{S}$-reflecting (Corollary 3.8). Hence, the "only if" direction follows from Claim 4.19, while the opposite direction from Corollary 4.21(a2). □

Corollary 4.25. *A bornologous map $f: X \to Y$ between balleans is L-preserving if and only if $f(X)$ is large in Y.*

Proof. If f is L-preserving, then $f(X)$ is trivially large in Y. The opposite implication follows from Theorem 4.1 and Corollary 4.23(a). □

Corollary 4.26. *Let $f: X \to Y$ be an effectively proper map between balleans. Then f is SL-preserving. Moreover, if $f(X)$ is extralarge in Y then f is also T-preserving.*

Proof. Since f_{sur} is effectively proper, we deduce from Theorem 4.2(b) that f_{sur} is $\{T, SL\}$-preserving. Hence, f is SL-preserving, by Proposition 4.22(a). If $f(X)$ is extralarge in Y then the last assertion follows from Corollary 4.24. □

The previous corollary cannot be inverted, since, for every ballean X, $i_1: X \to X \sqcup X$ is effectively proper and T-preserving (see Theorem 4.15), although $i_1(X)$ is not extralarge in $X \sqcup X$.

Theorem 4.27. *Let $f: X \to Y$ be a coarse embedding of balleans. Then f is $\{SL, S\}$-preserving. Moreover:*
(a) $f(X)$ is large in Y if and only if f is L-preserving if and only if f is $\{L, SL, S, PL\}$-preserving;
(b) $f(X)$ is extralarge in Y if and only if f is L-preserving and T-preserving if and only if f is $\mathfrak{S} \setminus \{M, SI\}$-preserving.
In particular, every coarse equivalence is $\{L, SL, S, PL\}$-preserving

Proof. Since f is a coarse embedding, this means that f_{sur} is a surjective coarse embedding. Hence, Corollary 4.3 implies that f_{sur} is $\mathfrak{S} \setminus \{M, SI\}$-preserving. In particular, f_{sur} is $\{SL, S\}$-preserving. Now the first assertion follows from Proposition 4.22(a).

The "if" part of both implications in item (a) is trivial. For its counterpart in item (b) use Lemma 3.12 to check that a $\{L, T\}$-preserving map is XL-preserving.

As already observed, f_{sur} is $\mathfrak{S} \setminus \{M, SI\}$-preserving. Therefore, if $f(X)$ is large in Y, Corollary 4.23(a) implies that f is $\{L, SL, S, PL\}$-preserving. Similarly, when $f(X)$ is extralarge in Y, we deduce that f is $\mathfrak{S} \setminus \{M, SI\}$-preserving, applying Corollary 4.24. □

Note that items (a) e (b) concern coarse equivalences. In general, one cannot say more than the conclusion of Theorem 4.27 about coarse equivalences. Indeed, even a surjective coarse embedding (hence a coarse equivalence) may fail to be M-preserving or SI-preserving (Example 4.5), while an injective coarse equivalence, may fail to be T-preserving or XL-preserving (Example 4.14).

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