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Dirichlet sets vs characterized subgroups

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ABSTRACT

A subset A of the circle group $\mathbb{T}$ is a Dirichlet set if there exists an increasing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}_0}$ in $\mathbb{N}$ such that $\|u_n x\| \to 0$ uniformly on A. In particular, A is contained in the subgroup $t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : \|u_n x\| \to 0\}$, which is the subgroup of $\mathbb{T}$ characterized by $\mathbf{u}$.

Using strictly increasing sequences $\mathbf{u}$ in $\mathbb{N}$ such that u_n divides u_{n+1} for every $n \in \mathbb{N}$, we find in $\mathbb{T}$ a family of closed perfect D-sets that are also Cantor-like sets. Moreover, we write $\mathbb{T}$ as the sum of two closed perfect D-sets. As a consequence, we solve an open problem by showing that $\mathbb{T}$ can be written as the sum of two of its proper characterized subgroups, i.e., $\mathbb{T}$ is factorizable. Finally, we describe all countable subgroups of $\mathbb{T}$ that are factorizable and we find a class of uncountable characterized subgroups of $\mathbb{T}$ that are factorizable.

1. Introduction

Let $\mathbb{N}$ denote the set of all strictly positive natural numbers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The behavior of the sequence of multiples $(u_n\alpha)_{n\in\mathbb{N}_0}$, where $(u_n)_{n\in\mathbb{N}_0}$ is a sequence of integers and $\alpha \in [0, 1]$, considered modulo 1 has deep roots in Topology (precompact topologies on $\mathbb{Z}$ with or without converging sequences [4,15]), Harmonic Analysis (sets of convergence of trigonometric series, thin sets) and Topological Algebra (topo-

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logically torsion elements and characterized subgroups), as well as in Number Theory (uniform distribution of sequences) and Dynamical Systems.

On the other hand, we are interested in the following two kinds of thin sets; the first one was introduced by Arbault in [1] in studying the sets of absolute convergence of trigonometric series, the second one analogously by Kahane in [28] (see also [29]). For a general treatment of thin sets in Harmonic Analysis we refer to the survey paper [10].

Definition 1.1. A set $A \subseteq [0, 1]$ is:

(a) an *Arbault set* (briefly, *A-set*) if there exists an increasing sequence $\mathbf{u}$ in $\mathbb{N}$ such that $(\sin(\pi u_n x))_{n\in\mathbb{N}_0}$ converges to 0 for all $x \in A$.
(b) a *Dirichlet set* (briefly, *D-set*) if there exists an increasing sequence $\mathbf{u}$ in $\mathbb{N}$ such that $(\sin(\pi u_n x))_{n\in\mathbb{N}_0}$ converges uniformly to 0 on A.

Clearly, a D-set is necessarily an A-set. Moreover, every D-set is included in a closed *perfect* (i.e., with no isolated points) D-set and has Lebesgue measure zero (see [22]). The classical Dirichlet Theorem on Diophantine approximation implies that each finite set is a D-set (see [28], for a proof see [9, 8.133]).

A general aim of this paper is to study D-sets and A-sets in the light of the theory of characterized subgroups of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and vice versa. So, we start by "moving" D-sets and A-sets to $\mathbb{T}$ as follows. Let $\varpi : \mathbb{R} \to \mathbb{T}$ be the canonical projection, then $\varphi := \varpi \restriction_{[0,1)}: [0, 1) \to \mathbb{T}$ is a bijection.

Definition 1.2. A subset A of $\mathbb{T}$ is a *D-set* (respectively, *A-set*) if $\varphi^{-1}(A) \subseteq [0, 1]$ is a D-set (respectively, A-set).

We give several basic properties of A-sets and D-sets in Section 2.1. In particular, subsets of A-sets are clearly A-sets and subsets of D-sets are clearly D-sets. We see that also the closure of a D-set is still a D-set (see Proposition 2.5). Moreover, A-sets and D-sets have Haar measure zero (see Lemma 2.6).

It will be immediately clear from the definitions that a subset A of $\mathbb{T}$ is an A-set precisely when A is contained in some characterized subgroup of $\mathbb{T}$. Indeed, following [20] (the terminology was given in [7]), a subgroup H of $\mathbb{T}$ is *characterized* if there exists a sequence $\mathbf{u}$ in $\mathbb{Z}$ such that H coincides with

$$t_{\mathbf{u}}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0\}.$$

Note that $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{T}$ precisely when $\mathbf{u}$ is eventually 0. If $\mathbf{u}$ is not eventually 0, we suppose without loss of generality that all members of $\mathbf{u}$ are non-zero, so we can also assume that $\mathbf{u}$ is in $\mathbb{N}$, since $t_{\mathbf{u}}(\mathbb{T}) = t_{|\mathbf{u}|}(\mathbb{T})$, where $|\mathbf{u}| = (|u_n|)_{n\in\mathbb{N}_0}$. It is worth to recall also that every countable subgroup of $\mathbb{T}$ is characterized (see [7], and see [6] and [19] for more general results).

The characterized subgroups of $\mathbb{T}$ are studied in Topological Algebra (e.g., see [7,14,18,30,33] and the papers [24–27] by Gabriyelyan), but also in relation to Diophantine approximation and Ergodic Theory (see [7,31,38]). We refer to the survey paper [13] for a comprehensive discussion on the characterized subgroups of $\mathbb{T}$.

In this paper, we are interested in the following special kind of sequences in $\mathbb{N}$ and characterized subgroups of $\mathbb{T}$.

Definition 1.3. A sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}_0}$ in $\mathbb{N}$ is *arithmetic* (briefly, an *a-sequence*) if $\mathbf{u}$ is strictly increasing, $u_0 = 1$ and u_n divides u_{n+1} for all $n \in \mathbb{N}$. A subgroup H of $\mathbb{T}$ is *a-characterized* if there exists an a-sequence $\mathbf{u}$ in $\mathbb{N}$ such that $H = t_{\mathbf{u}}(\mathbb{T})$.

Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. Since u_n divides u_{n+1} for all $n \in \mathbb{N}_0$, it is possible to define the sequence of positive integers $(q_n^{\mathbf{u}})_{n\in\mathbb{N}}$ by letting, for every $n \in \mathbb{N}$,

$$q_n^{\mathbf{u}} := \frac{u_n}{u_{n-1}}.$$

Using this sequence, we can rewrite the sequence $\mathbf{u}$ as

$$1 = u_0 < u_1 = q_1^{\mathbf{u}} < u_2 = q_1^{\mathbf{u}} q_2^{\mathbf{u}} < u_3 = q_1^{\mathbf{u}} q_2^{\mathbf{u}} q_3^{\mathbf{u}} < \dots .$$

We recall also that an a-sequence $\mathbf{u}$ in $\mathbb{N}$ is called *q-bounded* if $\{q_n^{\mathbf{u}} : n \in \mathbb{N}\}$ is bounded.

By using the a-sequence $\mathbf{u}$ and the sequence $(q_n^{\mathbf{u}})_{n\in\mathbb{N}}$ one can find the following representation for every element x of $\mathbb{T}$. Indeed, by identifying $\mathbb{T}$ with $[0, 1)$ by means of the bijection $\varphi : [0, 1) \to \mathbb{T}$, it is known (see [35]) that for every $x \in \mathbb{T}$ there exists a unique sequence $(c_n(x))_{n\in\mathbb{N}}$ in $\mathbb{N}_0$ such that

$$x = \sum_{n=1}^{\infty} \frac{c_n(x)}{u_n}, \quad c_n(x) < q_n^{\mathbf{u}} \text{ for every } n \in \mathbb{N}, \quad \text{and } c_n(x) < q_n^{\mathbf{u}} - 1 \text{ for infinitely many } n \in \mathbb{N}. \tag{1.1}$$

For $x \in \mathbb{T}$, let

$$\mathrm{supp}_{\mathbf{u}}(x) := \{n \in \mathbb{N} : c_n(x) \neq 0\}.$$

For more details on the canonical representation (1.1), see Section 2.2.

The a-characterized subgroups $t_{\mathbf{u}}(\mathbb{T})$ of $\mathbb{T}$ are precisely the subgroups of topologically $\mathbf{u}$-torsion elements of $\mathbb{T}$; this concept generalizes that of topologically torsion elements (for $u_n = n!$) and that of topologically p-torsion elements (for $u_n = p^n$), which were introduced to study the structure of locally compact abelian groups (see [2,8,12,21,37] and the survey [11]). The subgroup of the topologically p-torsion elements of $\mathbb{T}$ was shown to be $\mathbb{Z}(p^\infty)$ in [2]. Generalizing this fact, it was proved in [21, Chap. 4], that for a q-bounded a-sequence $\mathbf{u}$ an element $x \in \mathbb{T}$ belongs to $t_{\mathbf{u}}(\mathbb{T})$ precisely when all but finitely many $c_n(x)$ are 0. On the other hand, it was proved in [21, Chap. 4] that when $q_n^{\mathbf{u}} \to +\infty$ for an a-sequence $\mathbf{u}$, then $x \in t_{\mathbf{u}}(\mathbb{T})$ whenever $\frac{c_n(x)}{q_n^{\mathbf{u}}} \to 0$ in $\mathbb{R}$. The following more precise theorem can be obtained from [12, Corollary 2.3] by taking $I = \mathbb{N}$ in that theorem:

Theorem 1.4 *(See [16, Corollary 3.7]). Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ with $q_n^{\mathbf{u}} \to +\infty$, and let $x \in \mathbb{T}$. Then*

$$x \in t_{\mathbf{u}}(\mathbb{T}) \quad \textit{if and only if} \quad \frac{c_n(x)}{q_n^{\mathbf{u}}} \to 0 \ \textit{in} \ \mathbb{T}.$$

A complete description of the a-characterized subgroups of $\mathbb{T}$ for an arbitrary a-sequence $\mathbf{u}$ can be found in [16, Theorem 2.3].

Following [34] (see also [10]), where only the sequence $(2^n)_{n\in\mathbb{N}_0}$ was considered, we introduce the following subsets of $\mathbb{T}$.

Definition 1.5. Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. For $L \subseteq \mathbb{N}$, let

$$K_L^{\mathbf{u}} := \{x \in \mathbb{T} : \mathrm{supp}_{\mathbf{u}}(x) \subseteq L\}.$$

Clearly, $0 \in K_L^{\mathbf{u}}$ since $\mathrm{supp}_{\mathbf{u}}(0) = \emptyset$.

We are mainly interested in understanding when $K_L^{\mathbf{u}}$ is a D-set. Since finite $K_L^{\mathbf{u}}$ are always D-sets and $K_L^{\mathbf{u}}$ D-set implies L non-cofinite (see Claim 3.2), we consider subsets L of $\mathbb{N}$ that are infinite and non-cofinite.

In Section 3, we see that, under this assumption on L, the set $K_L^{\mathbf{u}}$ is closed (see Proposition 3.5) and it is a Cantor-like set in the sense of the following definition, so in particular $K_L^{\mathbf{u}}$ is a perfect subset of $\mathbb{T}$ (see Theorem 3.8).

Definition 1.6. For every $n \in \mathbb{N}$, let $\mathcal{K}_n$ be a non-empty finite set of pairwise disjoint closed intervals in $[0, 1)$. Assume that:

(a) for every $n \in \mathbb{N}$ and for every $K \in \mathcal{K}_n$ there exist $L', L'' \in \mathcal{K}_{n+1}$ such that $L' \neq L''$ and $L' \cup L'' \subseteq K$;
(b) for every $n \in \mathbb{N}$ and for every $L \in \mathcal{K}_{n+1}$ there exists $K \in \mathcal{K}_n$ such that $L \subseteq K$;
(c) $\max\{\lambda(K) : K \in \mathcal{K}_n\} \to 0$, where λ denotes the Lebesgue measure on $\mathbb{R}$.

The set $C = \bigcap_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} K$ is a *Cantor-like set*.

The aim of Section 4 is to describe the sets $K_L^{\mathbf{u}}$ that are D-sets in terms of properties of the set L and the sequence of ratios $(q_n^{\mathbf{u}})_{n\in\mathbb{N}}$ of $\mathbf{u}$. Since L is an infinite non-cofinite subset of $\mathbb{N}$, one has a partition $L = \bigcup_{n\in\mathbb{N}} L_n$, where each

$$L_n = \{m_n^L, \dots, M_n^L\}$$

is a finite set of consecutive naturals and the consecutive intervals L_n and L_{n+1} are not adjacent (i.e., $M_n^L < m_{n+1}^L - 1$). The non-empty set

$$G_n = \{M_n^L + 1, \dots, m_{n+1}^L - 1\}$$

"between" L_n and L_{n+1} is called *gap*. For an a-sequence $\mathbf{u}$ in $\mathbb{N}$, for every $n \in \mathbb{N}$ let

$$\widetilde{q}_n^L := \prod_{i\in G_n} q_i^{\mathbf{u}} = \frac{u_{m_{n+1}^L - 1}}{u_{M_n^L}}.$$

As we shall show in the sequel (see Proposition 3.3), the case $L = 2\mathbb{N}$ will become of extraordinary importance, this is why we shall rewrite the above data in this case:

$$m_n^L = M_n^L = 2n \quad \text{and} \quad \widetilde{q}_n^L = q_{2n+1}^{\mathbf{u}}, \quad \text{for every } n \in \mathbb{N}.$$

One of the main results of this paper is the following description of the sets $K_L^{\mathbf{u}}$ that are D-sets.

Theorem 1.7 *(see Theorem 4.8). If $\mathbf{u}$ is an a-sequence and L is an infinite non-cofinite subset of $\mathbb{N}$, then $K_L^{\mathbf{u}}$ is a D-set precisely when $\sup_{n\in\mathbb{N}} \widetilde{q}_n^L = +\infty$.*

We recall the following notion, to better explain the meaning of Theorem 1.7 through some consequences.

Definition 1.8. An infinite non-cofinite subset L of $\mathbb{N}$ is *large* if the sequence $(|G_n^L|)_{n\in\mathbb{N}}$ is bounded (equivalently, there exists a finite $F \subseteq \mathbb{Z}$ such that $\mathbb{N} \subseteq F + L$).

As a consequence of Theorem 1.7, we find that: when $\{q_n^{\mathbf{u}} : n \in \mathbb{N} \setminus L\}$ is not bounded, $K_L^{\mathbf{u}}$ is always a D-set; on the other hand, when $\mathbf{u}$ is q-bounded, $K_L^{\mathbf{u}}$ is a D-set if and only if L is non-large (see Corollary 4.10).

Moreover, $K_L^{\mathbf{u}}$ is a D-set when L is non-large. So, taking an infinite non-cofinite subset L of $\mathbb{N}$ which is non-large and with $G := \mathbb{N} \setminus L$ non-large, since $\mathbb{T} = K_L^{\mathbf{u}} + K_G^{\mathbf{u}}$ (see Theorem 5.1), we have the following result.

Theorem 1.9. *The group $\mathbb{T}$ can be written as the sum of two closed perfect D-sets (which have necessarily Haar measure zero).*

This is related to the result, proved by Erdös, Kunen and Mauldin in 1981, stating that, if P is a non-empty perfect subset of $\mathbb{R}$, then there exists a perfect subset M of $\mathbb{R}$ of Lebesgue measure zero such that $\mathbb{R} = P + M$ (see [23]). Thirty years later, Eliaš proved that M can be found to be a D-set; in other words, for every perfect subset P of $\mathbb{T}$ there exists a perfect D-set D such that $P + D = \mathbb{T}$ (see [22, Theorem 3.2]).

We recall Problem 1.11 below, which was proposed in [3]. To better express it, we introduce the following notion.

Definition 1.10. Call a subgroup H of $\mathbb{T}$ *factorizable* (respectively, *a-factorizable*), if $H = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ with proper characterized (respectively, a-characterized) subgroups $t_{\mathbf{v}}(\mathbb{T})$ and $t_{\mathbf{w}}(\mathbb{T})$.

Clearly, a-factorizable implies factorizable. In the sequel we discuss the connection between the properties of being (a-)characterized and (a-)factorizable.

Problem 1.11. [3, Question 5.1] *For sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{Z}$, describe $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T})$ in terms of $\mathbf{u}$ and $\mathbf{v}$.*

(a) When is a given factorizable subgroup of $\mathbb{T}$ characterized?
(b) When is a given characterized subgroup of $\mathbb{T}$ factorizable? In particular, is $\mathbb{T}$ factorizable?

Clearly, it is worth considering the counterpart of Problem 1.11 for a-characterized and a-factorizable subgroups of $\mathbb{T}$ as well.

In Section 5, as a consequence of Theorem 1.7 and other results in the previous sections, we give answers to items (b) of these problems. First, we deduce from Theorem 1.9 the following

Theorem 1.12 *(see Theorem 5.1). The circle group $\mathbb{T}$ is a-factorizable.*

Then, we consider the countable subgroups H of $\mathbb{T}$ (as recalled above, these are all characterized subgroups of $\mathbb{T}$) and we find equivalent conditions for H to be factorizable (respectively, a-factorizable): H is factorizable if and only if H is non-cocyclic (see Theorem 5.7), and H is a-factorizable precisely when H is a-characterized and non-cocyclic (see Theorem 5.9).

Since these results solve completely the countable case, we are left with the uncountable characterized subgroups of $\mathbb{T}$ and the following general problem.

Problem 1.13. *Describe the uncountable (a-)factorizable (a)-characterized subgroups.*

In this case we find a family of uncountable a-characterized subgroups of $\mathbb{T}$ that are a-factorizable:

Theorem 1.14 *(see Theorem 5.16). Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ such that $q_n^{\mathbf{u}} \to +\infty$. Then the a-characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

It remains unclear whether one can weaken the condition $q_n^{\mathbf{u}} \to +\infty$ to $\sup_{n\in\mathbb{N}} q_n^{\mathbf{u}} = +\infty$ in the above theorem:

Question 1.15. Suppose that the a-sequence $\mathbf{u}$ in $\mathbb{N}$ is not q-bounded. Is it true that the subgroup $t_{\mathbf{u}}(\mathbb{T})$ is factorizable? What about a-factorizable?

The following question is left open as well.

Question 1.16. Suppose that an (a-)characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ is factorizable. Is it true that it is also a-factorizable?

In the countable case, as a consequence of our results, we see that a countable factorizable subgroup H of $\mathbb{T}$ is a-factorizable precisely when H is a-characterized.

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2. Preliminaries

2.1. Dirichlet sets in $\mathbb{T}$

For every $x \in \mathbb{R}$, let $\phi(x)$ be the unique element in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ such that $\phi(x) \equiv_{\mathbb{Z}} x$, that is,

$$\phi(x) = \begin{cases} x - \lfloor x \rfloor & \text{if } x - \lfloor x \rfloor \in \left[0, \frac{1}{2}\right), \\ x - \lfloor x \rfloor - 1 & \text{if } x - \lfloor x \rfloor \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Moreover, denote by $\|x\| = d(x, \mathbb{Z})$ the distance of x from integers, in other words, $\|x\| = |\phi(x)|$.

Claim 2.1. *For $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$,*

(a) $\sin(\pi\|x\|) = |\sin(\pi x)|$;
(b) $2\|nx\| \leq |\sin(\pi n x)| \leq \pi\|nx\|$.

Proof. (a) For every $x, y \in \mathbb{R}$, the condition $x \equiv_{\mathbb{Z}} y$ implies $|\sin(\pi x)| = |\sin(\pi y)|$. Since $x \equiv_{\mathbb{Z}} \phi(x)$, we have

$$|\sin(\pi x)| = |\sin(\pi\phi(x))| = |\sin(\pi\|x\|)| = \sin(\pi\|x\|).$$

(b) Let $y = nx$. For $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq \frac{\pi}{2}$, we get $\frac{2}{\pi}\alpha \leq \sin\alpha \leq \alpha$. So, for $\alpha := \pi\|y\|$, and applying item (a), we have $2\|y\| \leq |\sin(\pi y)| \leq \pi\|y\|$. □

We give now equivalent conditions for a set $A \subseteq [0, 1]$ to be an A-set.

Lemma 2.2. *Let $A \subseteq [0, 1]$. Then the following conditions are equivalent:*

(a) A is an A-set;
(b) there exists an increasing sequence $\mathbf{u}$ in $\mathbb{N}$ such that $(\|u_n x\|)_{n\in\mathbb{N}_0}$ converges to 0 for every $x \in A$;
(c) $\varpi(A) \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some increasing sequence $\mathbf{u}$ in $\mathbb{N}$.

Proof. (a)⇔(b) follows from Claim 2.1 and (b)⇔(c) is clear since $\mathbf{u}$ is in $\mathbb{N}$. □

The following is the counterpart of Lemma 2.2 for D-sets.

Lemma 2.3. *Let $A \subseteq [0, 1]$. The following conditions are equivalent:*

(a) A is a D-set;

(b) there exists an increasing sequence $\mathbf{u}$ *in* $\mathbb{N}$ *such that* $(\|u_n x\|)_{n\in\mathbb{N}_0}$ *converges uniformly to* 0 *in* A*;*
(c) for every $\varepsilon > 0$ *there exists* $v \in \mathbb{N}$ *such that* $\sup_{x\in A} \|vx\| \leq \varepsilon$*.*

Proof. (a)⇔(b) follows from Claim 2.1.

(b)⇒(c) Clearly, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sup_{x\in A} \|u_{n_0} x\| \leq \varepsilon$.

(c)⇒(b) By hypothesis, for every $n \in \mathbb{N}$ there exists $v_n \in \mathbb{N}$ such that $\sup_{x\in A} \|v_n x\| \leq \frac{1}{n}$. If the subset $\{v_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is infinite, then there exists an increasing subsequence $\mathbf{u}$ of $\mathbf{v}$ with the required property. If $\{v_n : n \in \mathbb{N}\}$ is finite, then there exists $w \in \mathbb{N}$ such that $v_n = w$ for infinitely many $n \in \mathbb{N}$, hence $\sup_{x\in A} \|wx\| = 0$. Thus, A is finite and letting $u_n := nw$ for every $n \in \mathbb{N}$, we get $\|u_n x\| = 0$ for every $x \in A$. □

Clearly, subsets of D-sets are D-sets. Moreover, we see in Proposition 2.5 that the closure of a D-set is still a D-set in view of the next lemma.

Lemma 2.4. *If* X *is a topological space and* $(f_n)_{n\in\mathbb{N}}$ *is a sequence of continuous real-valued functions on* X *that converges uniformly to* 0 *on a dense subset* D *of* X*, then* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* 0 *on* X*.*

Proof. Pick an $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|f_n(x)| \leq \varepsilon$ for all $n \geq n_0$ and every $x \in D$. For $n \geq n_0$ the set $f_n^{-1}([-\varepsilon, +\varepsilon])$ is closed and contains the dense subset D of X, so coincides with X; in other words, $|f_n(x)| \leq \varepsilon$ for all $n \geq n_0$ and $x \in X$. Therefore, f_n converges uniformly to 0 on X. □

Proposition 2.5. *The closure of a* D*-set in* $\mathbb{T}$ *is a* D*-set.*

Proof. Assume that H is a D-set. Then there exists an increasing sequence $\mathbf{u}$, such that the sequence $(u_n x)_{n\in\mathbb{N}_0}$ converges uniformly to 0 on H. By Lemma 2.4, $(u_n x)_{n\in\mathbb{N}_0}$ converges uniformly to 0 on $\overline{H}$. This implies that $\overline{H}$ is a D-set. □

We denote by μ the normalized Haar measure on $\mathbb{T}$. For a sequence $\mathbf{u}$ in $\mathbb{N}$, since $t_{\mathbf{u}}(\mathbb{T})$ is a Borel set, so it is measurable.

Lemma 2.6. *A proper characterized subgroup* H *of* $\mathbb{T}$ *has Haar measure* 0*. Consequently, a set of positive Haar measure in* $\mathbb{T}$ *is not an* A*-set, and so not a* D*-set.*

Proof. Let $a := \mu(H) \geq 0$. Since $\mathbb{T}/H$ is a non-trivial divisible abelian group, $\mathbb{T}/H$ is infinite. Then, for every $n \in \mathbb{N}$, there exist $x_1 + H, \ldots, x_n + H$ distinct cosets of H in $\mathbb{T}$; clearly, $na = \sum_{i=1}^{n} \mu(x_i + H) \leq 1$. Therefore, $a \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, that is, $a = 0$.

Assume that $A \subseteq \mathbb{T}$ has $\mu(A) > 0$ and assume for a contradiction that A is an A-set. So $A \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some increasing sequence $\mathbf{u}$ in $\mathbb{N}$. By the first statement, $\mu(t_{\mathbf{u}}(\mathbb{T})) = 0$, a contradiction. □

The first statement of the following corollary follows from Proposition 2.5 and Lemma 2.6, the second one follows from the first one since every infinite subgroup of $\mathbb{T}$ is dense.

Corollary 2.7. *No dense subset of* $\mathbb{T}$ *is a* D*-set. If* H *is an infinite subgroup of* $\mathbb{T}$*, then* H *is not a* D*-set.*

In particular, infinite characterized subgroups of $\mathbb{T}$ cannot be D-sets:

Corollary 2.8. *If* $\mathbf{u}$ *is a strictly increasing sequence in* $\mathbb{N}$*, then* $t_{\mathbf{u}}(\mathbb{T})$ *is a* D*-set if and only if it is finite.*

Finally, we discuss below when the union or the sum of two D-sets is again a D-set. An example showing that this need not occur in general will be given after Example 5.2.

Lemma 2.9. *Suppose that D_1 and D_2 are D-sets containing* 0*. Then $D_1 + D_2$ is a D-set if and only if $D_1 \cup D_2$ is a D-set.*

Proof. Let us prove first the case when $D_1 = D_2$. Notice now that the sum $D_1 + D_1$ is obviously a D-set whenever D_1 is a D-set.

In the general case, $D_1 \cup D_2 \subseteq D_1 + D_2$. So it suffices to notice that if $D_1 \cup D_2$ is a D-set, then also the sum $(D_1 \cup D_2) + (D_1 \cup D_2)$ is a D-set. As this sum contains $D_1 + D_2$, the latter sum is a D-set as well. □

2.2. The canonical representation related to an a-sequence

Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. For $x \in \mathbb{T}$, recall that we identify $\mathbb{T}$ with $[0, 1)$ and we use the canonical representation (1.1).

For a subset $S \subseteq \mathbb{N}$, the S-*truncation of* x is

$$x_S = \sum_{n \in S} \frac{c_n(x)}{u_n}, \quad \text{so} \quad c_n(x_S) = \begin{cases} c_n(x) \text{ for } n \in S, \\ 0 \text{ for } n \in \mathbb{N} \setminus S. \end{cases}$$

Remark 2.10. Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. For $x \in (0, 1)$, set $c^*_n(x) = q^{\mathbf{u}}_n - 1 - c_n(x)$ for all $n \in \mathbb{N}$. It is straightforward to verify that

$$1 - x = \begin{cases} \sum_{n=1}^{\infty} \frac{c^*_n(x)}{u_n}, \text{ if } \operatorname{supp}_{\mathbf{u}}(x) \text{ is infinite}, \\ \frac{c^*_1(x)}{u_1} + \frac{c^*_2(x)}{u_2} + \ldots + \frac{c^*_{k-1}(x)}{u_{k-1}} + \frac{c^*_k(x)+1}{u_k}, \text{ if } \operatorname{supp}_{\mathbf{u}}(x) \subseteq \{1, 2, \ldots, k\} \text{ and } c_k(x) \neq 0. \end{cases} \tag{2.1}$$

Using this equality, one can obtain the nice description of the open ball

$$B_{\frac{1}{u_n}}(0) := \left\{ x \in \mathbb{T} : \|x\| < \frac{1}{u_n} \right\}$$

in Proposition 2.13. We need first the next claim, which will be frequently used in the sequel.

Claim 2.11. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$, $x \in \mathbb{T} \setminus \{0\}$ and $m, M \in \mathbb{N}$ with $m < M$.*

(a) If $\operatorname{supp}_{\mathbf{u}}(x) \subseteq \mathbb{N} \setminus \{1, \ldots, m\}$, then $x < \frac{1}{u_m}$.
(b) If $\operatorname{supp}_{\mathbf{u}}(x) \subseteq \{m + 1, \ldots, M\}$, then $\frac{1}{u_M} \leq x \leq \frac{1}{u_m} - \frac{1}{u_M} = \sum_{n=m+1}^{M} \frac{q^{\mathbf{u}}_n - 1}{u_n}$.
(c) If $x \in B_{\frac{1}{u_m}}(0)$, then $\operatorname{supp}_{\mathbf{u}}(x) \nsubseteq \{1, \ldots, m\}$.

Proof. (a) Write $x = \sum_{n=m+1}^{\infty} \frac{c_n(x)}{u_n}$. Since $0 \leq c_n(x) \leq q^{\mathbf{u}}_n - 1$ for every $n \in \mathbb{N}$ and $c_n(x) < q^{\mathbf{u}}_n - 1$ for infinitely many $n \in \mathbb{N}$, it follows that

$$x < \sum_{n=m+1}^{\infty} \frac{q^{\mathbf{u}}_n - 1}{u_n} = \frac{1}{u_m} - \frac{1}{u_{m+1}} + \frac{1}{u_{m+1}} - \frac{1}{u_{m+2}} + \ldots = \frac{1}{u_m}.$$

(b) Write $x = \sum_{n=m+1}^{M} \frac{c_n(x)}{u_n}$. Clearly, $x \geq \frac{1}{u_M}$. Moreover,

$$x \leq \sum_{n=m+1}^{M} \frac{q^{\mathbf{u}}_n - 1}{u_n} = \frac{1}{u_m} - \frac{1}{u_{m+1}} + \frac{1}{u_{m+1}} - \frac{1}{u_{m+2}} + \ldots + \frac{1}{u_{M-1}} - \frac{1}{u_M} = \frac{1}{u_m} - \frac{1}{u_M}.$$

(c) Assume that $\mathrm{supp}_{\mathbf{u}}(x) \subseteq \{1, \ldots, m\}$ and let $k \in \{1, \ldots, m\}$ be the maximum with $c_k(x) \neq 0$. Then, by item (b), $x \geq \frac{1}{u_k} \geq \frac{1}{u_m}$ and, by (2.1),

$$1 - x = \sum_{n=1}^{k-1} \frac{c_n^*(x)}{u_n} + \frac{c_k^*(x) + 1}{u_k} \geq \frac{1}{u_k} \geq \frac{1}{u_m}.$$

Hence, $x \notin B_{\frac{1}{u_m}}(0)$. □

The following lemma will be useful in the description of the sets $K_L^{\mathbf{u}}$.

Lemma 2.12. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* $m, M \in \mathbb{N}$ *with* $m \leq M$*. Then*

$$S := \left\{ \sum_{n=m}^{M} \frac{c_n}{u_n} :\ c_n \in \mathbb{N}_0,\ c_n < q_n^{\mathbf{u}} \right\} = \left\{ 0, \frac{1}{u_M}, \frac{2}{u_M}, \frac{3}{u_M}, \ldots, \frac{i}{u_M} \right\},$$

where $i = \frac{u_M}{u_{m-1}} - 1$.

Proof. The inclusion $S \subseteq \left\{0, \frac{1}{u_M}, \frac{2}{u_M}, \frac{3}{u_M}, \ldots, \frac{i}{u_M}\right\}$ follows from the fact that all members of S are non-negative multiples of $\frac{1}{u_M}$ and the greatest member of S is

$$\sum_{n=m}^{M} \frac{q_n^{\mathbf{u}} - 1}{u_n} = \frac{1}{u_{m-1}} - \frac{1}{u_M} = \frac{i}{u_M}$$

by Claim 2.11(b).

To prove the converse inclusion, it suffices to check that if $x = \sum_{n=m}^{M} \frac{c_n}{u_n} \in S$ and $x < \frac{i}{u_M}$, then $x + \frac{1}{u_M} \in S$. This is clear whenever $c_M < q_M^{\mathbf{u}} - 1$. Now suppose that $c_M = q_M^{\mathbf{u}} - 1$. As $x < \frac{i}{u_M}$, there exists $m \leq k < M$ such that $c_k < q_k - 1$. Let k be the maximum with this property, so $c_j = q_j^{\mathbf{u}} - 1$ for all $k < j \leq M$. Then

$$x + \frac{1}{u_M} = \sum_{n=m}^{k-1} \frac{c_n}{u_n} + \frac{c_k + 1}{u_k}$$

belongs to S. □

Now we describe the open balls of 0 of radius $\frac{1}{u_m}$, where u_m is some term of the a-sequence $\mathbf{u}$ in $\mathbb{N}$.

Proposition 2.13. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$*. For* $m \in \mathbb{N}$ *and* $x \in \mathbb{T} \setminus \{0\}$*, the following conditions are equivalent:*

(a) $x \in B_{\frac{1}{u_m}}(0)$*;*
(b) $\mathrm{supp}_{\mathbf{u}}(x) \nsubseteq \{1, \ldots, m\}$ *and either* $c_i(x) = 0$ *for every* $i \in \{1, \ldots, m\}$ *or* $c_i(x) = q_i^{\mathbf{u}} - 1$ *for every* $i \in \{1, \ldots, m\}$*.*

Proof. (a)⇒(b) Let $x \in B_{\frac{1}{u_m}}(0)$, equivalently, either $x \in \left[0, \frac{1}{u_m}\right)$ or $x \in \left(1 - \frac{1}{u_m}, 1\right)$. By Claim 2.11(c), $\mathrm{supp}_{\mathbf{u}}(x) \nsubseteq \{1, \ldots, m\}$.

If $x \in \left[0, \frac{1}{u_m}\right)$, then $c_i(x) = 0$ for every $i \in \{1, \ldots, m\}$, otherwise $x \geq \frac{1}{u_m}$ (indeed, if $c_k(x) \neq 0$ for some $k \in \{1, \ldots, m\}$ then $x \geq \frac{1}{u_k} \geq \frac{1}{u_m}$).

Assume now that $x \in \left(1 - \frac{1}{u_m}, 1\right)$. Equivalently, $1 - x \in \left(0, \frac{1}{u_m}\right)$. If $\mathrm{supp}_{\mathbf{u}}(x)$ is infinite, by (2.1) this means that $c_i^*(x) = 0$ for $i \in \{1, \ldots, m\}$, otherwise $1 - x \geq \frac{1}{u_m}$. If $\mathrm{supp}_{\mathbf{u}}(x)$ is finite, since $\mathrm{supp}_{\mathbf{u}}(x) \not\subseteq \{1, \ldots, m\}$, by (2.1) we have that $c_i^*(x) = 0$ for $i = 1, \ldots, m$, otherwise $1 - x \geq \frac{1}{u_m}$.

(b)⇒(a) Assume that $\mathrm{supp}_{\mathbf{u}}(x) \not\subseteq \{1, \ldots, m\}$.

If $c_i(x) = 0$ for every $i \in \{1, \ldots, m\}$, then $x = \sum_{n=m+1}^{\infty} \frac{c_n(x)}{u_n} < \frac{1}{u_m}$ by Claim 2.11(a), hence $x \in B_{\frac{1}{u_m}}(0)$.

Assume now that $c_i(x) = q_i^{\mathbf{u}} - 1$, that is, $c_i^*(x) = 0$, for every $i \in \{1, \ldots, m\}$. If $\mathrm{supp}_{\mathbf{u}}(x)$ is infinite, by (2.1) and Claim 2.11(a) we have

$$1 - x = \sum_{n=m+1}^{\infty} \frac{c_n^*(x)}{u_n} < \frac{1}{u_m}.$$

If $\mathrm{supp}_{\mathbf{u}}(x)$ is finite, let $k \in \mathbb{N}$ be the maximum with $c_k(x) \neq 0$ ($k > m$, since $\mathrm{supp}_{\mathbf{u}}(x) \not\subseteq \{1, \ldots, m\}$); then (2.1) and Claim 2.11(b) yield

$$1 - x = \sum_{n=m+1}^{k-1} \frac{c_n^*(x)}{u_n} + \frac{c_k^*(x) + 1}{u_k} < \frac{1}{u_m}.$$

In both cases, $x \in B_{\frac{1}{u_m}}(0)$. □

Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. Identify the finite cyclic group $\mathbb{Z}(q_n^{\mathbf{u}})$ with $\{0, 1, \ldots, q_n^{\mathbf{u}} - 1\}$. Let

$$C_{\mathbf{u}}^* = \prod_{n \in \mathbb{N}} \mathbb{Z}(q_n^{\mathbf{u}}), \quad q_{\mathbf{u}}^* := (q_n^{\mathbf{u}} - 1) \in C_{\mathbf{u}}^* \quad \text{and} \quad C_{\mathbf{u}} := C_{\mathbf{u}}^* \setminus \left(q_{\mathbf{u}}^* + \bigoplus_{n \in \mathbb{N}} \mathbb{Z}(q_n^{\mathbf{u}}) \right).$$

In these terms, the canonical representation (1.1) induces a bijection

$$c_{\mathbf{u}} : [0, 1) \to C_{\mathbf{u}}, \quad x \mapsto c_{\mathbf{u}}(x) = (c_n(x)). \tag{2.2}$$

Consider on $C_{\mathbf{u}}$ the lexicographical order $\prec$. The bijection in (2.2) preserves the order:

Lemma 2.14. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$. If $x < y$ in $[0, 1)$, then $c_{\mathbf{u}}(x) \prec c_{\mathbf{u}}(y)$.*

Proof. Assume that $c_1(y) < c_1(x)$. Since $y_{\mathbb{N}+1} = \sum_{n=2}^{\infty} \frac{c_n(y)}{u_n} < \frac{1}{u_1}$ by Claim 2.11(a), it follows that $y < \frac{c_1(y)+1}{u_1} \leq x$, against the hypothesis. Therefore, $c_1(x) \leq c_1(y)$. If $c_1(x) < c_1(y)$, then $c(x) \prec c(y)$. If $c_1(x) = c_1(y)$, consider the truncations $x_{\mathbb{N}+1}$ and $y_{\mathbb{N}+1}$; reasoning as above, we conclude that $c_2(x) \leq c_2(y)$. If $c_2(x) < c_2(y)$, then $c_{\mathbf{u}}(x) \prec c_{\mathbf{u}}(y)$. If $c_2(x) = c_2(y)$, consider the truncations $x_{\mathbb{N}+2}$ and $y_{\mathbb{N}+2}$. Proceeding this way by induction, since $c_{\mathbf{u}}(x) \neq c_{\mathbf{u}}(y)$ as $c_{\mathbf{u}}$ is injective, one finds a minimum $n_0 \in \mathbb{N}$ such that $c_{n_0}(x) < c_{n_0}(y)$. Hence, $c_{\mathbf{u}}(x) \prec c_{\mathbf{u}}(y)$. □

3. The set $K_L^{\mathbf{u}}$

In this section we consider only a-sequences $\mathbf{u}$ in $\mathbb{N}$ and we examine the properties of $K_L^{\mathbf{u}}$ for a subset L of $\mathbb{N}$. We will assume L to be infinite and not cofinite in view of Claim 3.2.

3.1. Basic properties of $K_L^{\mathbf{u}}$

We start with some basic properties:

Lemma 3.1. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* $L_1, L_2 \subseteq \mathbb{N}$*.*

(a) If $L_1 \subseteq L_2$*, then* $K^{\mathbf{u}}_{L_1} \subseteq K^{\mathbf{u}}_{L_2}$*.*
(b) If $L = L_1 \cup L_2$*, then* $K^{\mathbf{u}}_L \subseteq K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2}$*. Moreover, if* $L_1 \cap L_2 = \emptyset$*, then* $K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2} = K^{\mathbf{u}}_L$*.*
(c) If $L_1 \cup L_2 = \mathbb{N}$*, then* $K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2} = \mathbb{T}$*.*

Proof. (a) is clear and (c) follows from (b).

(b) Assume that $L = L_1 \cup L_2$. If $x \in K^{\mathbf{u}}_L$, then $x = \sum_{n\in L} \frac{c_n(x)}{u_n}$. Clearly, we can write

$$x = \sum_{n\in L'_1} \frac{c_n(x)}{u_n} + \sum_{n\in L'_2} \frac{c_n(x)}{u_n},$$

where $L'_i \subseteq L_i$ and so $\sum_{n\in L'_i} \frac{c_n(x)}{u_n} \in K^{\mathbf{u}}_{L'_i}$, for $i = 1, 2$. Hence, $x \in K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2}$. This proves that $K^{\mathbf{u}}_L \subseteq K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2}$.

We verify the converse inclusion under the assumption that $L_1 \cap L_2 = \emptyset$. Let $x \in K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2}$, then $x = x_1 + x_2$, where $x_i = \sum_{n\in L_i} c_n(x_i) \in K^{\mathbf{u}}_{L_i}$, for $i = 1, 2$. Since $L_1 \cap L_2 = \emptyset$, we have that

$$c_n(x) = \begin{cases} c_n(x_1) & \text{if } n \in L_1; \\ c_n(x_2) & \text{if } n \in L_2; \\ 0 & \text{if } n \in \mathbb{N} \setminus L. \end{cases}$$

In particular, $\mathrm{supp}(x) \subseteq L$, that is, $x \in K^{\mathbf{u}}_L$. This proves the inclusion $K^{\mathbf{u}}_{L_1} + K^{\mathbf{u}}_{L_2} \subseteq K^{\mathbf{u}}_L$. $\square$

The next claim justifies the assumption on L to be an infinite non-cofinite subset of $\mathbb{N}$.

Claim 3.2. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* L *a subset of* $\mathbb{N}$*.*

(a) If L *is finite, then* $K^{\mathbf{u}}_L$ *is finite as well and so* $K^{\mathbf{u}}_L$ *is a closed D-set.*
(b) If L *is a cofinite subset of* $\mathbb{N}$*, then:*
  (i) when L *is proper,* $K^{\mathbf{u}}_L$ *is not closed;*
  (ii) $\mu(K^{\mathbf{u}}_L) > 0$ *and so* $K^{\mathbf{u}}_L$ *is not a D-set.*
  In particular, L *is not cofinite whenever* $K^{\mathbf{u}}_L$ *is a D-set.*

Proof. (a) is clear.

(b) Since L is cofinite, L contains $\{n \in \mathbb{N} : n \geq k\}$ for some $k \in \mathbb{N}$, and we choose $k \in \mathbb{N}$ to be the minimum with this property.

(i) Since L is proper, $k > 1$ and so $x := \frac{1}{u_{k-1}} \notin K^{\mathbf{u}}_L$. For every $n \in \mathbb{N}_0$, let $x_n = \sum_{i=0}^{n} \frac{q^{\mathbf{u}}_{k+i}-1}{u_{k+i}} \in K^{\mathbf{u}}_L$; then the sequence $(x_n)_{n\in\mathbb{N}_0}$ in $K^{\mathbf{u}}_L$ converges to x.

(ii) Since L is cofinite, L contains a subset $L' = \{n \in \mathbb{N} : n > m\}$ for some $m \in \mathbb{N}$. Then $K^{\mathbf{u}}_L$ contains $K^{\mathbf{u}}_{L'} = \left[0, \frac{1}{u_m}\right)$. Hence, $\mu(K^{\mathbf{u}}_L) > 0$. By Lemma 2.6, we conclude that $K^{\mathbf{u}}_L$ cannot be a D-set. $\square$

3.2. Reduction to the cases $L = 2\mathbb{N}$ *or* $L = 2\mathbb{N} - 1$

Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and L an infinite non-cofinite subset of $\mathbb{N}$. For every $k \in \mathbb{N}$, let

$$\mathbb{A}_k := \{x \in \mathbb{T} : \mathrm{supp}_{\mathbf{u}}(x) \subseteq L_k\} = \{x \in \mathbb{T} : c_n(x) = 0 \text{ for } n \in \mathbb{N} \setminus L_k\}.$$

Clearly,

$$K_L^{\mathbf{u}} = \sum_{k\in\mathbb{N}} \mathbb{A}_k. \tag{3.1}$$

The choice of the letter "A" is motivated by the fact that $\mathbb{A}_k$ is an arithmetic progression; indeed, according to Lemma 2.12,

$$\mathbb{A}_k = \left\{0, \frac{1}{u_{M_k}}, \frac{2}{u_{M_k}}, \dots, \frac{i_k}{u_{M_k}}\right\}, \text{ where } i_k = \frac{u_{M_k}}{u_{m_k-1}} - 1; \tag{3.2}$$

namely, $\frac{i_k}{u_{M_k}} = \frac{1}{u_{m_k-1}} - \frac{1}{u_{M_k}}$.

We see now that the study of the sets $K_L^{\mathbf{u}}$ can be reduced to the cases when either $L = 2\mathbb{N}$ or $L = 2\mathbb{N}-1$.

Proposition 3.3. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* L *an infinite non-cofinite subset of* $\mathbb{N}$.

(a) If $1 \notin L$*, then* $K_L^{\mathbf{u}} = K_{2\mathbb{N}}^{\mathbf{v}}$ *and* $\widetilde{q}_n^L = q_{2n+1}^{\mathbf{v}}$ *for every* $n \in \mathbb{N}$*, where* $v_{2k} := u_{M_k}$ *and* $v_{2k-1} := u_{m_k-1}$ *for every* $k \in \mathbb{N}$.
(b) If $1 \in L$*, then* $K_L^{\mathbf{u}} = K_{2\mathbb{N}-1}^{\mathbf{w}}$ *and* $\widetilde{q}_n^L = q_{2n}^{\mathbf{w}}$ *for every* $n \in \mathbb{N}$*, where* $w_{2k-1} := u_{M_k}$ *and* $w_{2k} := u_{m_{k+1}-1}$ *for every* $k \in \mathbb{N}$.

Proof. Let $x \in K_L^{\mathbf{u}}$. In view of (3.1) and (3.2), $x = \sum_{k\in\mathbb{N}} \frac{j_k}{u_{M_k}}$ for some $j_k \in \mathbb{N}_0$ with $0 \le j_k \le i_k$.

(a) By the choice of $\mathbf{v}$, we have $i_k = \frac{u_{M_k}}{u_{m_k-1}} - 1 = q_{2k}^{\mathbf{v}} - 1$. Therefore, $x = \sum_{k\in\mathbb{N}} \frac{j_k}{v_{2k}} \in K_{2\mathbb{N}}^{\mathbf{v}}$. Moreover, $\widetilde{q}_n^L = \frac{u_{m_{n+1}-1}}{u_{M_n}} = q_{2n+1}^{\mathbf{v}}$ for every $n \in \mathbb{N}$.

(b) By the choice of $\mathbf{w}$, we have $i_k = \frac{u_{M_k}}{u_{m_k-1}} - 1 = q_{2k-1}^{\mathbf{w}} - 1$. Therefore, $x = \sum_{k\in\mathbb{N}} \frac{j_k}{w_{2k-1}} \in K_{2\mathbb{N}-1}^{\mathbf{w}}$. Moreover, $\widetilde{q}_n^L = \frac{u_{m_{n+1}-1}}{u_{M_n}} = q_{2n}^{\mathbf{w}}$ for every $n \in \mathbb{N}$. □

By Proposition 3.3 and the next remark, for our aims we can always reduce to the case $K_{2\mathbb{N}}^{\mathbf{u}}$.

Remark 3.4. Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and L an infinite non-cofinite subset of $\mathbb{N}$ such that $1 \in L$.

(a) Then

$$K_L^{\mathbf{u}} = K_{\{1\}}^{\mathbf{u}} + K_{L^*}^{\mathbf{u}},$$

where $L^* = L \setminus \{1\}$. Note that $1 \notin L^*$ and that $K_{\{1\}}^{\mathbf{u}} = \left\{0, \frac{1}{u_1}, \dots, \frac{q_1^{\mathbf{u}}-1}{u_1}\right\}$; moreover, $K_{L^*}^{\mathbf{u}} \subseteq \left[0, \frac{1}{u_1}\right)$ by Claim 2.11(a). Moreover, we can write $K_L^{\mathbf{u}}$ as the disjoint union

$$K_L^{\mathbf{u}} = \dot{\bigcup}_{a\in K_{\{1\}}^{\mathbf{u}}} a + K_{L^*}^{\mathbf{u}}. \tag{3.3}$$

(b) If $2 \in L$, then $\widetilde{q}_n^{L^*} = \widetilde{q}_n^L$ for every $n \in \mathbb{N}$; otherwise, we are in the case $L_1 = \{1\}$ and $\widetilde{q}_n^{L^*} = \widetilde{q}_{n+1}^L$ for every $n \in \mathbb{N}$.
(c) We will use that

$$\sup_{n\in\mathbb{N}} \widetilde{q}_n^{L^*} = +\infty \quad \Leftrightarrow \quad \sup_{n\in\mathbb{N}} \widetilde{q}_n^L = +\infty. \tag{3.4}$$

We see now that $K_L^{\mathbf{u}}$ is closed in $\mathbb{T}$ when L is an infinite non-cofinite subset of $\mathbb{N}$. We apply Proposition 3.3 and Remark 3.4 to reduce to the case when $L = 2\mathbb{N}$.

Proposition 3.5. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* L *an infinite non-cofinite subset of* $\mathbb{N}$*. Then* $K_L^{\mathbf{u}}$ *is closed.*

Proof. We prove first that

$$K_{2\mathbb{N}}^{\mathbf{u}} \text{ is closed.} \tag{3.5}$$

To this end, we verify that the complement of $K_{2\mathbb{N}}^{\mathbf{u}}$ is open. Let $x \in \mathbb{T} \setminus K_{2\mathbb{N}}^{\mathbf{u}}$ (so, $x \neq 0$) and let $\bar{n} \in \mathbb{N} \setminus 2\mathbb{N}$ be the minimum such that $c_{\bar{n}}(x) \neq 0$. Moreover, let $t \in \{n \in \mathbb{N} : n \geq \bar{n} + 2\}$ be the minimum such that $c_t(x) < q_t - 1$. Note that, in view of (2.1),

$$1 - x \geq \frac{1}{u_t}. \tag{3.6}$$

We verify now that

$$B_{\frac{1}{u_t}}(x) \subseteq \mathbb{T} \setminus K_{2\mathbb{N}}^{\mathbf{u}} \tag{3.7}$$

and this will give that $\mathbb{T} \setminus K_{2\mathbb{N}}^{\mathbf{u}}$ is open, hence (3.5) holds true. To verify (3.7), we take $y \in K_{2\mathbb{N}}^{\mathbf{u}}$ and prove that

$$\|x - y\| \geq \frac{1}{u_t}. \tag{3.8}$$

Let $n_0 \in \mathbb{N}$ be the minimum such that $c_{n_0}(x) \neq c_{n_0}(y)$ and let

$$z := \begin{cases} 0 & \text{if } n_0 = 1 \\ \sum_{n=1}^{n_0-1} \frac{c_n(x)}{u_n} & \text{if } n_0 > 1. \end{cases}$$

In particular, $n_0 \leq \bar{n}$; moreover,

$$n_0 < \bar{n} \implies n_0 \in 2\mathbb{N}, \tag{3.9}$$

as $c_n(x) = c_n(y) = 0$ for every $n \in \mathbb{N} \setminus 2\mathbb{N}$ with $n < n_0$.

We consider first the case when $y > x$, that is, $y - x > 0$. Then $c_{n_0}(y) > c_{n_0}(x)$ in view of Lemma 2.14; hence, $n_0 < \bar{n}$ and $n_0 \in 2\mathbb{N}$ by (3.9). Since $y < \frac{1}{u_1}$, we have

$$1 - (y - x) \geq 1 - y \geq 1 - \frac{1}{u_1} \geq \frac{1}{u_t}. \tag{3.10}$$

Moreover,

$$\begin{aligned} y \geq y' &:= z + \frac{c_{n_0}(y)}{u_{n_0}} \\ x \leq x' &:= z + \frac{c_{n_0}(y) - 1}{u_{n_0}} + \sum_{n=n_0+1}^{t} \frac{q_n - 1}{u_n}. \end{aligned}$$

Then

$$y - x \geq y' - x' = \frac{1}{u_{n_0}} - \sum_{n=n_0+1}^{t} \frac{q_n - 1}{u_n} = \frac{1}{u_t}. \tag{3.11}$$

Thus, in this case (3.10) and (3.11) yield $\|x-y\| \geq \frac{1}{u_t}$, that is, (3.8).

We consider now the case when $y < x$, that is, $x - y > 0$. Then $c_{n_0}(y) < c_{n_0}(x)$ in view of Lemma 2.14. By (3.6), we have that

$$1-(x-y) \geq 1-x \geq \frac{1}{u_t}. \tag{3.12}$$

We have two cases.

First assume that $n_0 < \bar{n}$, so $n_0 \in 2\mathbb{N}$ by (3.9). Then

$$x \geq x'' := z + \frac{c_{n_0}(x)}{u_{n_0}} + \frac{1}{u_{\bar{n}}} \quad \text{and} \quad y \leq y'' := z + \frac{c_{n_0}(x)}{u_{n_0}} = x'' - \frac{1}{u_{\bar{n}}}.$$

Hence, $x - y \geq x'' - y'' = \frac{1}{u_{\bar{n}}} \geq \frac{1}{u_t}$. This, together with (3.12), gives $\|x-y\| \geq \frac{1}{u_t}$, that is, (3.8).

Assume now that $n_0 = \bar{n}$. We have that

$$\begin{aligned} x \geq x''' &:= z + \frac{1}{u_{\bar{n}}} \\ y \leq y''' &:= z + \frac{0}{u_{\bar{n}}} + \frac{q_{\bar{n}+1}-1}{u_{\bar{n}+1}} + \frac{1}{u_{\bar{n}+2}}. \end{aligned}$$

Therefore,

$$x - y \geq x''' - y''' = \frac{1}{u_{\bar{n}}} - \frac{q_{\bar{n}+1}-1}{u_{\bar{n}+1}} - \frac{1}{u_{\bar{n}+2}} = \frac{1}{u_{\bar{n}+1}} - \frac{1}{u_{\bar{n}+2}} = \frac{q_{\bar{n}+1}-1}{u_{\bar{n}+2}} \geq \frac{1}{u_t}.$$

This, together with (3.12), gives $\|x-y\| \geq \frac{1}{u_t}$, that is, (3.8). Hence, (3.5) holds true.

We consider now the general case. If $1 \notin L$, then $K_L^{\mathbf{u}} = K_{2\mathbb{N}}^{\mathbf{v}}$ for a suitable subsequence $\mathbf{v}$ of $\mathbf{u}$ by Proposition 3.3, so $K_L^{\mathbf{u}}$ is closed by (3.5). If $1 \in L$, letting $L^* = L \setminus \{1\}$, we have that $K_L^{\mathbf{u}} = \dot{\bigcup}_{a \in K_{\{1\}}^{\mathbf{u}}} a + K_{L^*}^{\mathbf{u}}$ by (3.3). Since $1 \notin L^*$, we have that $K_{L^*}^{\mathbf{u}}$ is closed by the previous case. Moreover, $a + K_{L^*}^{\mathbf{u}}$ is closed for every $a \in K_{\{1\}}^{\mathbf{u}}$. Since $K_{\{1\}}^{\mathbf{u}}$ is finite, we can conclude that $K_L^{\mathbf{u}}$ is closed. □

3.3. $K_L^{\mathbf{u}}$ is a Cantor-like set

Fix an a-sequence $\mathbf{u}$ in $\mathbb{N}$. In view of Proposition 3.3 and Remark 3.4, first we prove that $K_{2\mathbb{N}}^{\mathbf{u}}$ is a Cantor-like set and then we deduce the general case in Theorem 3.8. When $L = 2\mathbb{N}$, for every $k \in \mathbb{N}$, we have $L_k = \{2k\}$ and so

$$\mathbb{A}_k = \left\{0, \frac{1}{u_{2k}}, \frac{2}{u_{2k}}, \ldots, \frac{q_{2k}-1}{u_{2k}}\right\}, \quad \text{where } \frac{q_{2k}-1}{u_{2k}} = \frac{1}{u_{2k-1}} - \frac{1}{u_{2k}}. \tag{3.13}$$

As observed in (3.1),

$$K_{2\mathbb{N}}^{\mathbf{u}} = \sum_{k=1}^{\infty} \mathbb{A}_k. \tag{3.14}$$

Let

$$\mathbb{I}_k = \left[0, \frac{1}{u_{2k-1}}\right].$$

Clearly, $\mathbb{A}_k \subseteq \mathbb{I}_k$, $\mathbb{I}_{k+1} \cap \mathbb{A}_k = \{0\}$ and $\mathbb{A}_k + \mathbb{I}_{k+1} \subseteq \mathbb{I}_k$.

Let now

$$\mathbb{F}_k = \mathbb{A}_1 + \mathbb{A}_2 + \ldots + \mathbb{A}_k.$$

Then $\mathbb{F}_{k+1} = \mathbb{F}_k + \mathbb{A}_{k+1}$. Clearly, $\bigcup_{k\in\mathbb{N}} \mathbb{F}_k \subseteq K^{\mathbf{u}}_{2\mathbb{N}}$.

Unlike the sets $\mathbb{A}_k$, the sets $\mathbb{F}_k$ need not be progressions, yet some nice properties of the sets $\mathbb{F}_k$ are available:

Lemma 3.6. *Let $k \in \mathbb{N}$.*

(a) All elements of $\mathbb{F}_k$ are multiples of $\frac{1}{u_{2k}}$.
(b) If $x, y \in \mathbb{F}_k$ are distinct, then $|x - y| \geq \frac{1}{u_{2k}}$ and so $(x + \mathbb{I}_{k+1}) \cap (y + \mathbb{I}_{k+1}) = \emptyset$.
(c) For every $x \in \mathbb{F}_k$, $x + \mathbb{A}_{k+1} + \mathbb{I}_{k+2} \subseteq x + \mathbb{I}_{k+1}$.

Proof. (a) is obvious and implies (b), while (c) follows from the fact that $\mathbb{A}_{k+1} + \mathbb{I}_{k+2} \subseteq \mathbb{I}_{k+1}$. □

For every $k \in \mathbb{N}$, let

$$\mathcal{J}_k := \{f + \mathbb{I}_{k+1} : f \in \mathbb{F}_k\}.$$

Lemma 3.7. *Let $k \in \mathbb{N}$. Then:*

(a) $\mathcal{J}_k$ is a family of pairwise disjoint intervals;
(b) each interval of $\mathcal{J}_{k+1}$ is contained in some interval of $\mathcal{J}_k$ (i.e., $\mathbb{F}_{k+1} + \mathbb{I}_{k+2} \subseteq \mathbb{F}_k + \mathbb{I}_{k+1}$);
(c) each interval of $\mathcal{J}_k$ contains at least two intervals of $\mathcal{J}_{k+1}$;
(d) $K^{\mathbf{u}}_{2\mathbb{N}} \subseteq \bigcup \mathcal{J}_k = \mathbb{F}_k + \mathbb{I}_{k+1}$.

Proof. (a) By Lemma 3.6(b), $\mathcal{J}_k$ is a family of pairwise disjoint intervals.

(b) Let $g + \mathbb{I}_{k+2} \in \mathcal{J}_{k+1}$. Then $g = f + a$, where $f \in \mathbb{F}_k$ and $a \in \mathbb{A}_{k+1}$, so $g + \mathbb{I}_{k+2} \subseteq f + \mathbb{I}_{k+1} \in \mathcal{J}_k$ by Lemma 3.6(c).

(c) Take $f + \mathbb{I}_{k+1} \in \mathcal{J}_k$. By Lemma 3.6(c), $f + \mathbb{I}_{k+1} \supseteq f + a + \mathbb{I}_{k+2} \in \mathcal{J}_{k+2}$ for every $a \in \mathbb{A}_{k+1}$.

(d) Since $K^{\mathbf{u}}_{2\mathbb{N}} = \sum_{n\in\mathbb{N}} \mathbb{A}_n$ by (3.14), where $\sum_{n=1}^{k} \mathbb{A}_n = \mathbb{F}_k$ by definition and $\sum_{n=k+1}^{\infty} \mathbb{A}_n \subseteq \mathbb{I}_{k+1}$ in view of Claim 2.11(a), we conclude that $K^{\mathbf{u}}_{2\mathbb{N}} \subseteq \bigcup \mathcal{J}_k$. □

We are now in position to prove the property announced in the subsection title.

Theorem 3.8. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and L an infinite non-cofinite subset of $\mathbb{N}$. Then $K^{\mathbf{u}}_L$ is a Cantor-like set.*

Proof. We prove first that $K^{\mathbf{u}}_{2\mathbb{N}}$ is a Cantor-like set. By Lemma 3.7, $\bigcup_{k\in\mathbb{N}} \mathbb{F}_k \subseteq K^{\mathbf{u}}_{2\mathbb{N}} \subseteq \bigcap_{k\in\mathbb{N}} \bigcup \mathcal{J}_k$. Since $\bigcup_{k\in\mathbb{N}} \mathbb{F}_k$ is dense in $\bigcap_{k\in\mathbb{N}} \bigcup \mathcal{J}_k$ and $K^{\mathbf{u}}_{2\mathbb{N}}$ is closed by Lemma 3.5, we conclude that $K^{\mathbf{u}}_{2\mathbb{N}} = \bigcap_{k\in\mathbb{N}} \bigcup \mathcal{J}_k$.

We consider now that general case. If $1 \notin L$, then $K^{\mathbf{u}}_L = K^{\mathbf{v}}_{2\mathbb{N}}$ for a suitable subsequence $\mathbf{v}$ of $\mathbf{u}$ in view of Proposition 3.3, so $K^{\mathbf{u}}_L$ is a Cantor-like set by the previous case. If $1 \in L$ and $L^* = L \setminus \{1\}$, then $K^{\mathbf{u}}_L = \dot{\bigcup}_{a\in K^{\mathbf{u}}_{\{1\}}} a + K^{\mathbf{u}}_{L^*}$ is a disjoint union by (3.3). Since $1 \notin L^*$, we have that $K^{\mathbf{u}}_{L^*}$ is a Cantor-like set by the previous case. Consequently, $a + K^{\mathbf{u}}_{L^*}$ is a Cantor-like set for every $a \in K^{\mathbf{u}}_{\{1\}}$. Since $K^{\mathbf{u}}_{\{1\}}$ is finite and the union $K^{\mathbf{u}}_L = \dot{\bigcup}_{a\in K^{\mathbf{u}}_{\{1\}}} a + K^{\mathbf{u}}_{L^*}$ is disjoint, we conclude that $K^{\mathbf{u}}_L$ is a Cantor-like set, as well. □

4. When $K^{\mathbf{u}}_L$ is a D-set

The aim of this section is to describe the sets $K^{\mathbf{u}}_L$ that are D-sets. In the first subsection we consider the special case when $L = 2\mathbb{N}$, that is, the sets $K^{\mathbf{u}}_{2\mathbb{N}}$; in the second subsection we deduce the general case.

4.1. When $K^{\mathbf{u}}_{2\mathbb{N}}$ is a D-set

Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and let $\mathbf{b} = (b_n)_{n\in\mathbb{N}}$ be a sequence of integers with $0 \le b_n < q^{\mathbf{u}}_n$ for every $n \in \mathbb{N}$. Let

$$A(\mathbf{u},\mathbf{b}) := \left\{ \frac{c_n}{u_n} : n \in \mathbb{N},\ c_n \in \mathbb{N},\ c_n \le b_n \right\} \quad \text{and} \quad B(\mathbf{u},\mathbf{b}) := \left\{ \sum_{n=1}^{\infty} \frac{c_n}{u_n} : c_n \in \mathbb{N}_0,\ c_n \le b_n \right\}.$$

Clearly, $A(\mathbf{u},\mathbf{b}) \subseteq B(\mathbf{u},\mathbf{b})$.

These sets will turn out to be relevant due to the property that they “capture” the set $K^{\mathbf{u}}_{2\mathbb{N}}$:

$$A(\mathbf{w},\mathbf{b}) \subseteq K^{\mathbf{u}}_{2\mathbb{N}} \subseteq B(\mathbf{w},\mathbf{b}), \quad \text{for } b_n := q^{\mathbf{u}}_{2n} - 1 \text{ for every } n \in \mathbb{N} \text{ and } w_n := u_{2n} \text{ for every } n \in \mathbb{N}_0. \tag{4.1}$$

Indeed, clearly $\frac{c}{u_{2n}} \in K^{\mathbf{u}}_{2\mathbb{N}}$ for every $c \in \mathbb{N}$ with $c \le q^{\mathbf{u}}_{2n} - 1$, and every $x = \sum_{n\in\mathbb{N}} \frac{c_{2n}(x)}{u_{2n}} \in K^{\mathbf{u}}_{2\mathbb{N}}$ belongs to $B(\mathbf{w},\mathbf{b})$ since $c_{2n}(x) \le q^{\mathbf{u}}_{2n} - 1$ for every $n \in \mathbb{N}$ by (1.1).

The inclusions in (4.1) and Proposition 4.3 will ensure that $K^{\mathbf{u}}_{2\mathbb{N}}$ is a D-set if and only if $A(\mathbf{w},\mathbf{b})$ and $B(\mathbf{w},\mathbf{b})$ are D-sets.

Lemma 4.1. *Let $0 < b_n < q^{\mathbf{u}}_n$ for every $n \in \mathbb{N}$. If $\frac{b_{n_k+1}}{q^{\mathbf{u}}_{n_k+1}} \to 0$, then $(u_{n_k})_{k\in\mathbb{N}}$ witnesses that $B(\mathbf{u},\mathbf{b})$ is a D-set.*

Proof. Let $x = \sum_{n\in\mathbb{N}} \frac{c_n(x)}{u_n} \in B(\mathbf{u},\mathbf{b})$. Then $u_n x = u_n \sum_{i\in\mathbb{N}} \frac{c_i(x)}{u_i} \equiv_{\mathbb{Z}} \sum_{i>n} \frac{c_i(x) u_n}{u_i}$. Therefore, also by Claim 2.11(a),

$$\|u_n x\| = \left\| \sum_{i>n} \frac{c_i(x) u_n}{u_i} \right\| \le u_n \sum_{i>n} \frac{c_i(x)}{u_i} \le \frac{b_{n+1}}{q^{\mathbf{u}}_{n+1}} + u_n \sum_{i>n+1} \frac{c_i(x)}{u_i} \le \frac{b_{n+1}}{q^{\mathbf{u}}_{n+1}} + \frac{1}{q^{\mathbf{u}}_{n+1}} \le 2\frac{b_{n+1}}{q^{\mathbf{u}}_{n+1}}.$$

This entails $\sup_{x\in B(\mathbf{u},\mathbf{b})} \|u_n x\| \le 2 \cdot \frac{b_{n+1}}{q^{\mathbf{u}}_{n+1}}$ for every $n \in \mathbb{N}$, so the thesis follows. □

The following lemma is folklore (e.g., can be found in [36]).

Lemma 4.2. *Let $\varepsilon \in \left(0, \frac{1}{3}\right)$ and $x \in [0, \varepsilon]$ with $\|ix\| \le \varepsilon$ for $i = 1, \dots, N$. Then $Nx \le \varepsilon$.*

The next result is fundamental for the complete description of the sets $K^{\mathbf{u}}_L$ that are D-sets. In its proof we use the fact that, given an a-sequence $\mathbf{u}$ in $\mathbb{N}$, one can write every natural number $v \in \mathbb{N}$ in a unique way as a finite sum

$$v = \sum_{i\ge k} a_i u_i \tag{4.2}$$

with $k \in \mathbb{N}_0$, $a_i \in \mathbb{N}_0$, $a_i < q^{\mathbf{u}}_{i+1}$, $a_k \ne 0$. Note that $u_k | v$, while $u_i \nmid v$ for all $i > k$. Note that for every natural v divisible by u_1 there exists a uniquely determined index $k \in \mathbb{N}_0$ with this property. We denote it by $o_{\mathbf{u}}(v) := k$.

Proposition 4.3. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* $0 < b_n < q_n^{\mathbf{u}}$ *for every* $n \in \mathbb{N}$. *Then the following conditions are equivalent:*

(a) $A(\mathbf{u}, \mathbf{b})$ *is a D-set;*
(b) $B(\mathbf{u}, \mathbf{b})$ *is a D-set;*
(c) $\inf_{n\in\mathbb{N}} \frac{b_n}{q_n^{\mathbf{u}}} = 0$.

Proof. (b)⇒(a) is clear.

(a)⇒(c) We have to prove that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}_0$ such that $\frac{b_{k+1}}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon$. To this end we pick $\varepsilon \in \left(0, \min\left\{\frac{1}{u_1}, \frac{1}{3}\right\}\right)$. By Lemma 2.3(c) there exists $v \in \mathbb{N}$ such that $\sup_{x\in A(\mathbf{u},\mathbf{b})} \|vx\| \leq \varepsilon$. Since $\varepsilon < \frac{1}{u_1}$ and $\frac{1}{u_1} \in A(\mathbf{u}, \mathbf{b})$, we deduce that $u_1|v$. Let $k \in \mathbb{N}_0$ such that $u_k|v$ and $u_{k+1} \not| v$, i.e., $k = o_{\mathbf{u}}(v)$. We show that

$$\frac{b_{k+1}}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon. \tag{4.3}$$

Write v as in (4.2) and let $k := o_{\mathbf{u}}(v)$. For $c \in \mathbb{N}$ with $c \leq b_{k+1}$, since $\|v \cdot \frac{c}{u_{k+1}}\| \leq \varepsilon$ and $v \cdot \frac{c}{u_{k+1}} \equiv_{\mathbb{Z}} c \cdot \frac{a_k}{q_{k+1}^{\mathbf{u}}}$, we get

$$\left\| c \cdot \frac{a_k}{q_{k+1}^{\mathbf{u}}} \right\| \leq \varepsilon \quad \text{for every } c = 1, 2, \ldots, b_{k+1}. \tag{4.4}$$

Moreover,

$$\frac{1}{q_{k+1}^{\mathbf{u}}} \leq \frac{a_k}{q_{k+1}^{\mathbf{u}}} \leq (q_{k+1}^{\mathbf{u}} - 1)\frac{1}{q_{k+1}^{\mathbf{u}}} = 1 - \frac{1}{q_{k+1}^{\mathbf{u}}}. \tag{4.5}$$

We now distinguish two cases.

If $\frac{a_k}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon$, from Lemma 4.2 and by (4.4) one can deduce $b_{k+1} \cdot \frac{a_k}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon$, and so $\frac{b_{k+1}}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon$.

If $\frac{a_k}{q_{k+1}^{\mathbf{u}}} > \varepsilon$, from (4.4) and (4.5) it follows that

$$1 - \varepsilon \leq \frac{a_k}{q_{k+1}^{\mathbf{u}}} \leq 1 - \frac{1}{q_{k+1}^{\mathbf{u}}},$$

whence

$$\frac{1}{q_{k+1}^{\mathbf{u}}} \leq 1 - \frac{a_k}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon. \tag{4.6}$$

This yields $\left\|\left(1 - \frac{a_k}{q_{k+1}^{\mathbf{u}}}\right) \cdot c\right\| \leq \varepsilon$ for $c = 1, \ldots, b_{k+1}$. Again by Lemma 4.2, $\left(1 - \frac{a_k}{q_{k+1}^{\mathbf{u}}}\right) \cdot b_{k+1} \leq \varepsilon$ and finally, thanks to (4.6), one concludes that $\frac{b_{k+1}}{q_{k+1}^{\mathbf{u}}} \leq \varepsilon$. Hence (4.3) is proved.

(c)⇒(b) is contained in Lemma 4.1. □

The following is an interesting consequence of Proposition 4.3.

Corollary 4.4. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$. *Then* $\left\{\frac{1}{u_n} : n \in \mathbb{N}\right\}$ *is a D-set precisely when* $(q_n^{\mathbf{u}})_{n\in\mathbb{N}}$ *is not bounded.*

Proof. Apply Proposition 4.3 (a)⇔(c) with $b_n = 1$ for all $n \in \mathbb{N}$. □

Remark 4.5. In the notations of Proposition 4.3, $\inf_{n\in\mathbb{N}} \frac{b_n}{q_n^{\mathbf{u}}} = 0$ if and only if $\inf_{n\in\mathbb{N}} \frac{b_n+1}{q_n^{\mathbf{u}}} = 0$. This follows from the clear inequalities $\frac{b_n}{q_n^{\mathbf{u}}} \le \frac{b_n+1}{q_n^{\mathbf{u}}} \le 2\frac{b_n}{q_n^{\mathbf{u}}}$.

By applying Proposition 4.3, we can now give an equivalent condition for $K_{2\mathbb{N}}^{\mathbf{u}}$ to be a D-set.

Corollary 4.6. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$. *Then* $K_{2\mathbb{N}}^{\mathbf{u}}$ *is a D-set if and only if* $\sup_{n\in\mathbb{N}} q_{2n+1}^{\mathbf{u}} = +\infty$. *More precisely, if* $q_{2n_k+1}^{\mathbf{u}} \to +\infty$, *then the sequence* $(u_{2n_k})_{k\in\mathbb{N}}$ *witnesses that* $K_{2\mathbb{N}}^{\mathbf{u}}$ *is a D-set.*

Proof. Put $b_n := q_{2n}^{\mathbf{u}} - 1$ and $w_n := u_{2n}$ for every $n \in \mathbb{N}$; then $A(\mathbf{w},\mathbf{b}) \subseteq K_{2\mathbb{N}}^{\mathbf{u}} \subseteq B(\mathbf{w},\mathbf{b})$ as observed in (4.1). Consequently, by Proposition 4.3 and Remark 4.5, and in view of the fact that a subset of a D-set is a D-set as well, we obtain that $K_{2\mathbb{N}}^{\mathbf{u}}$ is a D-set if and only if $\inf_{n\in\mathbb{N}} \frac{b_n+1}{q_n^{\mathbf{w}}} = 0$. This condition is equivalent to $\sup_{n\in\mathbb{N}} q_{2n+1}^{\mathbf{u}} = +\infty$, since

$$\frac{b_n+1}{q_n^{\mathbf{w}}} = \frac{q_{2n}^{\mathbf{u}}}{q_n^{\mathbf{w}}} = \frac{u_{2n}}{u_{2n-1}} \cdot \frac{u_{2n-2}}{u_{2n}} = \frac{u_{2n-2}}{u_{2n-1}} = \frac{1}{q_{2n-1}^{\mathbf{u}}}. \tag{4.7}$$

To prove that last assertion, assume that $q_{2n_k+1}^{\mathbf{u}} \to +\infty$. Then, by (4.7), $\frac{b_{n_k+1}}{q_{n_k+1}^{\mathbf{w}}} \to 0$. Moreover, $b_n < q_n^{\mathbf{w}}$ again by (4.7). Thus, Lemma 4.1 applies. □

4.2. When $K_L^{\mathbf{u}}$ *is a D-set*

Now we go back to the general set $K_L^{\mathbf{u}}$ and see when it is a D-set. Since we apply Corollary 4.6 about $K_{2\mathbb{N}}^{\mathbf{u}}$, we need the following lemma.

Lemma 4.7. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$, L *an infinite non-cofinite subset of* $\mathbb{N}$ *such that* $1 \in L$ *and* $L^* = L \setminus \{1\}$. *If* $\mathbf{v}$ *is a subsequence of* $\mathbf{u}$ *witnessing that* $K_{L^*}^{\mathbf{u}}$ *is a D-set, then* $\mathbf{v}$ *witnesses that* $K_L^{\mathbf{u}}$ *is a D-set.*

Proof. By (3.3) we have $K_L^{\mathbf{u}} = \dot\bigcup_{a\in K_{\{1\}}^{\mathbf{u}}} a + K_{L^*}^{\mathbf{u}}$, where $K_{\{1\}}^{\mathbf{u}} = \left\{0, \frac{1}{u_1}, \dots, \frac{q_1^{\mathbf{u}}-1}{u_1}\right\}$. Being $\mathbf{v}$ a subsequence of $\mathbf{u}$, u_1 divides v_n for every $n \in \mathbb{N}$; hence, for $a \in K_{\{1\}}^{\mathbf{u}}$ and $x \in K_{L^*}^{\mathbf{u}}$, we have $\|v_n(a+x)\| = \|v_n x\|$. This implies that $\mathbf{v}$ witnesses that $a + K_{L^*}^{\mathbf{u}}$ is a D-set, hence $\mathbf{v}$ witnesses that $K_L^{\mathbf{u}}$ is a D-set. □

Now we are in position to prove our main theorem on the description of all sets of the form $K_L^{\mathbf{u}}$ that are D-sets.

Theorem 4.8. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$, L *an infinite non-cofinite subset of* $\mathbb{N}$, *and let* $\mathbf{m}^L = (u_{M_n^L})_{n\in\mathbb{N}}$. *Then the following conditions are equivalent:*

(a) $\sup_{n\in\mathbb{N}} \widetilde{q}_n^L = +\infty$;
(b) $K_L^{\mathbf{u}}$ *is a D-set;*
(c) some subsequence $\mathbf{w}$ *of* $\mathbf{m}^L$ *witnesses that* $K_L^{\mathbf{u}}$ *is a D-set;*
(d) some subsequence $\mathbf{w}$ *of* $\mathbf{m}^L$ *witnesses that* $K_L^{\mathbf{u}}$ *is an A-set (i.e.,* $K_L^{\mathbf{u}} \subseteq t_{\mathbf{w}}(\mathbb{T})$*).*

Proof. We start by proving the equivalence of (a), (b) and (c).

Assume first that $1 \notin L$. By Proposition 3.3, $K_L^{\mathbf{u}} = K_{2\mathbb{N}}^{\mathbf{v}}$ and $\widetilde{q}_n^L = q_{2n+1}^{\mathbf{v}}$ for every $n \in \mathbb{N}$, where $v_{2k} := u_{M_k}$ and $v_{2k-1} := u_{m_k-1}$ for every $k \in \mathbb{N}$.

(a)⇔(b) By Corollary 4.6 we get that $K_{2\mathbb{N}}^{\mathbf{v}}$ is a D-set if and only if $\sup_{n\in\mathbb{N}} q_{2n+1}^{\mathbf{v}} = +\infty$, that is, $\sup_{n\in\mathbb{N}} \widetilde{q}_n^L = +\infty$.

(c)⇒(b) is obvious and (a)⇒(c) follows from the last statement of Corollary 4.6.

Assume now that $1 \in L$ and let $L^* = L \setminus \{1\}$. By the previous case, the thesis holds true for $K^{\mathbf{u}}_{L^*}$, so we call (a*), (b*) and (c*) the conditions of the statement for $K^{\mathbf{u}}_{L^*}$. (a) is equivalent to (a*) by (3.4) in Remark 3.4 and (c) is equivalent to (c*) by Lemma 4.7 since $K^{\mathbf{u}}_{L^*} \subseteq K^{\mathbf{u}}_L$. Hence, we have (a)⇔(c). Moreover, (c)⇒(b) is obvious. To conclude, note that (b) implies (b*) since $K^{\mathbf{u}}_{L^*} \subseteq K^{\mathbf{u}}_L$, now (b*) is equivalent to (c*), which is equivalent to (c).

Thus, we have that (a), (b) and (c) are equivalent.

(c)⇒(d) is clear.

(d)⇒(a) Let $\mathbf{w} = (u_{M_{n_k}})_{k\in\mathbb{N}}$ be a subsequence of $\mathbf{m}^L$. Assume for a contradiction that the sequence $\widetilde{q}^L_n$ is bounded. Hence, $\widetilde{q}^L_n \leq C$ for some $C > 0$ and all $n \in \mathbb{N}$. To prove that $K^{\mathbf{u}}_L \not\subseteq t_{\mathbf{w}}(\mathbb{T})$, consider

$$x = \sum_{j=1}^{\infty} \frac{c_j}{u_{m_{n_j}}} \in K^{\mathbf{u}}_L, \text{ where } c_j = \left\lceil \frac{q^{\mathbf{u}}_{m_{n_j}} - 1}{2} \right\rceil .$$

Then, $\|u_{M_{n_k}} x\| = \|u_{M_{n_k}} x'\|$, with $x' = \sum_{j=k+1}^{\infty} \frac{c_j}{u_{m_{n_j}}}$. Since

$$x' \leq \sum_{j=k+1}^{\infty} \frac{q^{\mathbf{u}}_{m_{n_j}} - 1}{u_{m_{n_j}}} \leq \frac{1}{u_{m_{n_{k+1}}-1}},$$

we have

$$u_{M_{n_k}} x' \leq \frac{u_{M_{n_k}}}{u_{m_{n_{k+1}}-1}} \leq \frac{1}{2}.$$

Therefore, $\|u_{M_{n_k}} x'\| = u_{M_{n_k}} x'$. Moreover,

$$u_{M_{n_k}} x' \geq \frac{u_{M_{n_k}}}{2} \frac{q^{\mathbf{u}}_{m_{n_{k+1}}} - 1}{u_{m_{n_{k+1}}}} = \frac{u_{M_{n_k}}}{2} \left(\frac{1}{u_{m_{n_{k+1}}-1}} - \frac{1}{u_{m_{n_{k+1}}}} \right) =$$
$$= \frac{u_{M_{n_k}}}{2u_{m_{n_{k+1}}-1}} \left(1 - \frac{1}{q^{\mathbf{u}}_{m_{n_{k+1}}}} \right) = \frac{1}{2\widetilde{q}^L_{n_k}} \left(1 - \frac{1}{q^{\mathbf{u}}_{m_{n_{k+1}}}} \right) \geq \frac{1}{4C}.$$

This shows that $u_{M_{n_k}} x' \not\to 0$. □

Needless to say, the equivalence of (c) and (d) in the above theorem brings more light on the strong connection between the notions of D-set and A-set. Note that $\mathbf{w}$ in (c) and (d) of the above theorem is a subsequence of $\mathbf{m}^L$ such that $\widetilde{q}^L_{n_k} \uparrow +\infty$.

The next lemma will be used in its counter-positive form; its proof is straightforward.

Lemma 4.9. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and* L *an infinite non-cofinite subset of* $\mathbb{N}$. *Then* $\sup_{n\in\mathbb{N}} \widetilde{q}^L_n < +\infty$ *if and only if* $\{q^{\mathbf{u}}_n : n \in \mathbb{N} \setminus L\}$ *is bounded and* L *is large.*

By Lemma 4.9, it is immediate to get the following consequence of Theorem 4.8.

Corollary 4.10. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *and let* L *be an infinite non-cofinite subset of* $\mathbb{N}$.

(a) If L *is non-large, then* $K^{\mathbf{u}}_L$ *is a D-set (and a subsequence of* $\mathbf{u}$ *witnesses that* $K^{\mathbf{u}}_L$ *is a D-set).*
(b) If $\mathbf{u}$ *is q-bounded, then* $K^{\mathbf{u}}_L$ *is a D-set if and only if* L *is non-large.*
(c) If $\{q^{\mathbf{u}}_n : n \in \mathbb{N} \setminus L\}$ *is not bounded, then* $K^{\mathbf{u}}_L$ *is a D-set.*

Here we stress the fact that the right condition in order to ensure that $K_L^{\mathbf{u}}$ is a D-set involves not only largeness of L, but the more subtle condition $\sup_{n\in\mathbb{N}} \tilde{q}_n^L = +\infty$ of Theorem 4.8(a). The relationship between the largeness of L and this property is described in Lemma 4.9. Now we give an example of a D-set $K_L^{\mathbf{u}}$, with a large set L.

Example 4.11. Let $L = 2\mathbb{N}$ and $\mathbf{u} = ((n+1)!)_{n\in\mathbb{N}_0}$. Then $K_L^{\mathbf{u}}$ is a D-set by Theorem 4.8, since $\tilde{q}_n^L = q_{2n+1}^{\mathbf{u}} = 2n+2 \to +\infty$. Obviously, $L = 2\mathbb{N}$ is large.

5. Factorizable subgroups of $\mathbb{T}$

In this section we give a contribution towards the solution of the "factorization" component (b) of Problem 1.11.

5.1. $\mathbb{T}$ is factorizable

The next result is somewhat surprising since D-sets have measure 0. It positively answers the second question in Problem 1.11(b). The same result was proved in [34] for $\mathbf{u} = (2^n)_{n\in\mathbb{N}_0}$.

Theorem 5.1. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and L a non-large subset of $\mathbb{N}$ with non-large complement $G := \mathbb{N} \setminus L$. Then $\mathbb{T} = K_L^{\mathbf{u}} + K_G^{\mathbf{u}}$ and there exist two subsequences $\mathbf{v}$ and $\mathbf{w}$ of $\mathbf{u}$ such that $\mathbf{v}$ and $\mathbf{w}$ witness respectively that $K_L^{\mathbf{u}}$ and $K_G^{\mathbf{u}}$ are D-sets. In particular, $\mathbb{T} = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ is a-factorizable.*

Proof. By Lemma 3.1(c), $\mathbb{T} = K_L^{\mathbf{u}} + K_G^{\mathbf{u}}$. By Corollary 4.10(a), there exist two subsequences $\mathbf{v}$ and $\mathbf{w}$ of $\mathbf{u}$ such that $\mathbf{v}$ and $\mathbf{w}$ witness respectively that $K_L^{\mathbf{u}}$ and $K_G^{\mathbf{u}}$ are D-sets. The fact that $\mathbb{T} = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ follows from the first assertion and from the fact that $K_L^{\mathbf{v}} \subseteq t_{\mathbf{v}}(\mathbb{T})$ and $K_G^{\mathbf{w}} \subseteq t_{\mathbf{w}}(\mathbb{T})$ by Lemma 2.2(c). □

The following example is a particular case of Theorem 5.1 giving a simple way to write $\mathbb{T}$ as the sum of two closed perfect D-sets of the form $K_L^{\mathbf{u}}$, and consequently as the sum of two proper characterized subgroups.

Example 5.2. Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ such that $q_n^{\mathbf{u}} \to +\infty$. By Theorem 4.8, $\mathbf{v} := (u_{2n})_{n\in\mathbb{N}}$ witnesses that $K_{2\mathbb{N}}^{\mathbf{u}}$ is a D-set and $\mathbf{w} := (u_{2n-1})_{n\in\mathbb{N}}$ witnesses that $K_{2\mathbb{N}-1}^{\mathbf{u}}$ is a D-set. Moreover, $\mathbb{T} = K_{2\mathbb{N}}^{\mathbf{u}} + K_{2\mathbb{N}-1}^{\mathbf{u}}$ by Lemma 3.1(c). Consequently, $\mathbb{T} = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ by Lemma 2.2(c).

From the above example and Lemma 2.9 we can easily find two D-sets such that their union is not a D-set. Indeed, take $D_1 = K_{2\mathbb{N}}^{\mathbf{u}}$ and $D_2 = K_{2\mathbb{N}-1}^{\mathbf{u}}$ as in Example 5.2 and observe that D_1 and D_2 are D-sets containing 0. As $D_1 + D_2 = \mathbb{T}$ is not a D-set, we conclude that $D_1 \cup D_2$ is not a D-set either.

5.2. Countable factorizable subgroups of $\mathbb{T}$

Here we describe the countable subgroups of $\mathbb{T}$ that are factorizable and a-factorizable. As recalled in the introduction, it is known from [7] that every countable subgroup of $\mathbb{T}$ is characterized.

The next example shows that for q-bounded a-sequences $\mathbf{u}$ in $\mathbb{N}$, the (necessarily countable) characterized subgroups $t_{\mathbf{u}}(\mathbb{T})$ are not forced to be neither factorizable nor unfactorizable.

Example 5.3.

(a) Let p be a prime and let $\mathbf{u} = (p^n)_{n\in\mathbb{N}_0}$. Then $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$ is an indecomposable group (cannot be written as a sum of two proper subgroups), so $t_{\mathbf{u}}(\mathbb{T})$ is not factorizable.

(b) Let q be a prime distinct from p, and let $\mathbf{u} = (p^n q^n)_{n\in\mathbb{N}_0}$. Then $t_{\mathbf{u}}(\mathbb{T}) = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(q^\infty)$, where with $\mathbf{v} = (p^n)_{n\in\mathbb{N}_0}$ and $\mathbf{w} = (q^n)_{n\in\mathbb{N}_0}$ one has $t_{\mathbf{v}}(\mathbb{T}) = \mathbb{Z}(p^\infty)$ and $t_{\mathbf{w}}(\mathbb{T}) = \mathbb{Z}(q^\infty)$; so, $t_{\mathbf{u}}(\mathbb{T})$ is factorizable.

In order to describe the countable subgroups of $\mathbb{T}$ that are factorizable and a-factorizable, we start from some purely algebraic results. Let $\mathcal{N}$ be the class of all abelian groups G such that, if $G = H + K$ for some subgroups H and K of G, then either $G = H$ or $G = K$.

Lemma 5.4. *If $G \in \mathcal{N}$ and H is a subgroup of G, then $G/H \in \mathcal{N}$.*

Proof. Let $q : G \to G/H$ be the canonical projection. If $G/H \notin \mathcal{N}$, then $G/H = K_1 + K_2$ for some proper subgroups K_1, K_2 of G/H. Then $G = q^{-1}(K_1) + q^{-1}(K_2)$ and $q^{-1}(K_1), q^{-1}(K_2)$ are proper subgroups of G; hence, $G \notin \mathcal{N}$. □

Lemma 5.5. *If $G \in \mathcal{N}$ is divisible or cyclic, then G is cocyclic.*

Proof. In case G is divisible, one can apply the structure theorem of divisible abelian groups. Since $G \in \mathcal{N}$, then either $G = \mathbb{Z}(p^\infty)$ for some prime p and so G is cocyclic, or $G = \mathbb{Q}$. So it is enough to rule out the case $G = \mathbb{Q}$. To this end, note that $\mathbb{Q}/\mathbb{Z}$ is an infinite direct sum of Prüfer groups, so $\mathbb{Q}/\mathbb{Z} \notin \mathcal{N}$ and so $\mathbb{Q} \notin \mathcal{N}$ by Lemma 5.4.

In case G is cyclic we note first that G cannot be infinite, as $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}(2) \times \mathbb{Z}(3)$ and so $\mathbb{Z} \notin \mathcal{N}$ by Lemma 5.4. If G is finite and $m = |G|$ is not a prime power, one can again represent G as a direct sum of two proper subgroups. Therefore, $G \cong \mathbb{Z}(p^n)$ for some prime p and $n \in \mathbb{N}$. □

Proposition 5.6. *Let G be an abelian group. Then $G \in \mathcal{N}$ if and only if G is cocyclic.*

Proof. If G is cocyclic, then obviously $G \in \mathcal{N}$. Assume now that $G \in \mathcal{N}$. When G is divisible, then G is necessarily cocyclic by Lemma 5.5. Suppose now that $G \in \mathcal{N}$ is not divisible; then $G \neq pG$ for some prime p. Then $G/pG \in \mathcal{N}$ by Lemma 5.4 and G/pG is an elementary p-group, so direct sum of cyclic groups of order p, hence G/pG is cyclic. If $x \in G$ projects on the generator of G/pG, then $G = pG + C$, with $C = \langle x\rangle$ cyclic. As $G \neq pG$, we deduce that $G = C$ is cyclic. Now again by Lemma 5.5 we conclude that G is cocyclic. □

Now we can describe all factorizable countable subgroups of $\mathbb{T}$ as follows:

Theorem 5.7. *A countable subgroup H of $\mathbb{T}$ is factorizable if and only if H is not cocyclic.*

Proof. If H is factorizable, in particular $G \notin \mathcal{N}$ and so G is not cocyclic by Proposition 5.6. Assume now that H is not cocyclic; by Proposition 5.6, equivalently $H = H_1 + H_2$ where H_1 and H_2 are proper subgroups of H. Being countable, H_1 and H_2 are characterized (see [7]). □

Next we describe in Theorem 5.9 all a-factorizable countable subgroups of $\mathbb{T}$, we shall see that they are necessarily a-characterized. We recall first known results describing completely the countable a-characterized subgroups of $\mathbb{T}$.

Remark 5.8.

(a) It is known from [12, Corollary 2.8] (see also [16, Corollary 3.8] and [17, Theorem 3.12]) that an a-characterized subgroup $H = t_{\mathbf{u}}(\mathbb{T})$ of $\mathbb{T}$ is countable if and only if $H \subseteq \mathbb{Q}/\mathbb{Z}$. This occurs precisely when $\mathbf{u}$ is q-bounded. Moreover, according to [5], in this case

$$H = \bigoplus_{i=1}^{k} \mathbb{Z}(p_i^{l_i}) \oplus \bigoplus_{i=k+1}^{m} \mathbb{Z}(p_i^{\infty}), \quad \text{for pairwise distinct primes } \{p_1, \dots, p_m\} \text{ and } l_1, \dots, l_k \in \mathbb{N}.$$

(b) On the other hand, every subgroup H of this form is a-characterized; indeed, $H = t_{\mathbf{u}}(\mathbb{T})$ for the a-sequence $\mathbf{u}$ in $\mathbb{N}$ defined by

$$u_1 = q_1^{\mathbf{u}} = \prod_{i=1}^{k} p_i^{l_i} \text{ and } q_n^{\mathbf{u}} = \prod_{i=k+1}^{m} p_i, \quad \text{for every } n \in \mathbb{N},\ n \geq 2.$$

Now we prove that a countable a-characterized subgroup of $\mathbb{T}$ is (a-)factorizable if and only if it is not cocyclic.

Theorem 5.9. *Let H be a countable subgroup of $\mathbb{T}$. Then the following conditions are equivalent:*

(a) H is a-factorizable;
(b) H is a-characterized and non-cocyclic;
(c) $H = \bigoplus_{i=1}^{k} \mathbb{Z}(p_i^{l_i}) \oplus \bigoplus_{i=k+1}^{m} \mathbb{Z}(p_i^{\infty})$, for pairwise distinct primes $\{p_1, \ldots, p_m\}$, with $m \geq 2$;
(d) H is a-characterized and factorizable.

Proof. (b)⇒(c) follows from Remark 5.8(a) and (c)⇒(b) follows from Remark 5.8(b).

(a)⇒(c) Assume that H is a-factorizable. Then $H = H_1 + H_2$, where H_1 and H_2 are proper a-characterized subgroups of H. Then the thesis follows from Remark 5.8(a).

(c)⇒(a) follows from Remark 5.8(b).

The equivalence of (b) and (d) follows from Theorem 5.7. □

5.3. Uncountable characterized subgroups of $\mathbb{T}$ that are factorizable

In this section we consider the a-characterized subgroups $t_{\mathbf{u}}(\mathbb{T})$ of $\mathbb{T}$ such that $q_n^{\mathbf{u}} \to +\infty$, and we prove that they are a-factorizable in Theorem 5.16.

First of all, we offer two technical lemmata:

Lemma 5.10. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ and let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}_0$. Then:*

(a) $\left\| a_n u_n x - a_n \frac{c_{n+1}(x)}{q_{n+1}^{\mathbf{u}}} \right\| \leq \frac{a_n}{q_{n+1}^{\mathbf{u}}}$;
(b) if A is an infinite subset of $\mathbb{N}$ and $\frac{a_n}{q_{n+1}^{\mathbf{u}}} \to 0$, then $\lim_{n\in A, n\to+\infty} \|a_n u_n x\| = 0$ if and only if $\lim_{n\in A, n\to+\infty} \left\| \frac{a_n c_{n+1}(x)}{q_{n+1}^{\mathbf{u}}} \right\| = 0$.

Proof. Observe that

$$u_n x \equiv_{\mathbb{Z}} u_n \frac{c_{n+1}(x)}{u_{n+1}} + u_n \left(\frac{c_{n+2}(x)}{u_{n+2}} + \ldots \right).$$

Hence $\left(a_n u_n x - a_n \frac{c_{n+1}(x)}{q_{n+1}^{\mathbf{u}}} \right) \equiv_{\mathbb{Z}} a_n u_n \left(\frac{c_{n+2}(x)}{u_{n+2}} + \ldots \right)$. So

$$\left\| a_n u_n x - \frac{a_n c_{n+1}(x)}{q_{n+1}^{\mathbf{u}}} \right\| = \left\| a_n u_n \left(\frac{c_{n+2}(x)}{u_{n+2}} + \ldots \right) \right\| \leq \left| a_n u_n \left(\frac{c_{n+2}(x)}{u_{n+2}} + \ldots \right) \right| \leq \frac{a_n}{q_{n+1}^{\mathbf{u}}},$$

where the last inequality holds true thanks to Claim 2.11(a).

(b) The desired equivalence holds in virtue of (a) since $\frac{a_n}{q_{n+1}^{\mathbf{u}}} \to 0$. □

The following second lemma will be used in Proposition 5.13, where both the a-sequence $\mathbf{u}$ and the sequence $(a_n)_{n\in\mathbb{N}_0}$ are subsequences of $(p^n)_{n\in\mathbb{N}_0}$ for some prime p.

Lemma 5.11. *Let* $\mathbf{u}$ *be an a-sequence in* $\mathbb{N}$ *with* $q_n^{\mathbf{u}} \to +\infty$. *Let* $(a_n)_{n\in\mathbb{N}_0}$ *be a sequence in* $\mathbb{N}$ *with:*

(a) $\frac{a_n}{q_{n+1}^{\mathbf{u}}} \to 0$;
(b) $\inf \left\| \frac{a_n^2}{q_{n+1}^{\mathbf{u}}} \right\| = \varepsilon > 0$;
(c) a_n *divides* $q_{n+1}^{\mathbf{u}}$ *for every* $n \in \mathbb{N}$.

Take:

(i) an arbitrary partition $\mathbb{N} = A_1 \dot\cup A_2$ *with infinite sets* A_1, A_2, *and*
(ii) $\mathbf{u}^{(i)}$, *for* $i = 1, 2$, *the increasing one-to-one sequence whose terms are* $\{u_n : n \in \mathbb{N}\} \cup \{a_n u_n : n \in A_i\}$.

Then $\mathbf{u}^{(i)}$ *is an a-sequence for* $i = 1, 2$ *and*

$$t_{\mathbf{u}^{(1)}}(\mathbb{T}) + t_{\mathbf{u}^{(2)}}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T}) \quad \textit{and} \quad t_{\mathbf{u}^{(i)}}(\mathbb{T}) \neq t_{\mathbf{u}}(\mathbb{T}),\ i = 1, 2.$$

(In particular $t_{\mathbf{u}}(\mathbb{T})$ *is a-factorizable.)*

Proof. By the condition in item (c) the sequences $\mathbf{u}^{(i)}$ with $i = 1, 2$ are a-sequences, since $u_{n+1} = \frac{q_{n+1}^{\mathbf{u}}}{a_n} \cdot a_n u_n$ for every $n \in \mathbb{N}_0$.

Since $\mathbf{u}$ is a subsequence of $\mathbf{u}^{(i)}$ for $i = 1, 2$, one has $t_{\mathbf{u}^{(1)}}(\mathbb{T}) + t_{\mathbf{u}^{(2)}}(\mathbb{T}) \subseteq t_{\mathbf{u}}(\mathbb{T})$. It remains to check the inclusion

$$t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{u}^{(1)}}(\mathbb{T}) + t_{\mathbf{u}^{(2)}}(\mathbb{T}). \tag{5.1}$$

To this end, pick $x \in t_{\mathbf{u}}(\mathbb{T})$; we have

$$x = z_1 + z_2, \quad \text{where} \quad z_1 := x_{A_2+1} \text{ and } z_2 := \frac{c_1(x)}{u_1} + x_{A_1+1}.$$

We claim that

$$z_1 \in t_{\mathbf{u}^{(1)}}(\mathbb{T}) \quad \text{and} \quad z_2 \in t_{\mathbf{u}^{(2)}}(\mathbb{T}).$$

To verify that $z_2 \in t_{\mathbf{u}^{(2)}}(\mathbb{T})$, it suffices to check that

$$\lim_{n\to+\infty} \|u_n z_2\| = 0 \quad \text{and} \quad \lim_{A_2 \ni n \to +\infty} \|a_n u_n z_2\| = 0. \tag{5.2}$$

The first equality in (5.2) follows from Theorem 1.4, because $x \in t_{\mathbf{u}}(\mathbb{T})$. For the second equality in (5.2), note that $c_{n+1}(z_2) = 0$ if $n \in A_2$, so by (a) and Lemma 5.10 we get

$$\lim_{A_2 \ni n \to +\infty} \|a_n u_n z_2\| = \lim_{A_2 \ni n \to +\infty} \left\| \frac{a_n c_{n+1}(z_2)}{q_{n+1}^{\mathbf{u}}} \right\| = 0.$$

One can check in a similar way that $z_1 \in t_{\mathbf{u}^{(1)}}(\mathbb{T})$. This ends up the proof of (5.1).

Finally, fixed $i = 1, 2$, we check that $t_{\mathbf{u}^{(i)}}(\mathbb{T}) \neq t_{\mathbf{u}}(\mathbb{T})$ by exhibiting an element $y_i \in t_{\mathbf{u}}(\mathbb{T}) \setminus t_{\mathbf{u}^{(i)}}(\mathbb{T})$. To this end, set

$$y_i := \sum_{n \in A_i} \frac{a_n}{u_{n+1}};$$

this is the canonical representation of y_i (i.e., $c_n(y_i) = a_n$ for every $n \in \mathbb{N}$). Then $y_i \in t_{\mathbf{u}}(\mathbb{T})$ by Theorem 1.4, since applying the hypothesis in item (a) we have

$$\left\| \frac{c_{n+1}(y_i)}{q^{\mathbf{u}}_{n+1}} \right\| = \left\| \frac{a_n}{q^{\mathbf{u}}_{n+1}} \right\| \leq \frac{a_n}{q^{\mathbf{u}}_{n+1}} \to 0.$$

We see now that

$$y_i \notin t_{\mathbf{u}^{(i)}}(\mathbb{T}).$$

Observe that, for $k \in A_i$,

$$u_k y_i \equiv_{\mathbb{Z}} u_k \frac{a_k}{u_{k+1}} + u_k \sum_{n \in A_i, n>k} \frac{a_n}{u_{n+1}} = \frac{a_k}{q^{\mathbf{u}}_{k+1}} + u_k \sum_{n \in A_i, n>k} \frac{a_n}{u_{n+1}}.$$

Hence,

$$\left(a_k u_k y_i - \frac{a_k^2}{q^{\mathbf{u}}_{k+1}} \right) \equiv_{\mathbb{Z}} a_k u_k \sum_{n \in A_i, n>k} \frac{a_n}{u_{n+1}}.$$

By Claim 2.11(a),

$$\left\| a_k u_k y_i - \frac{a_k^2}{q^{\mathbf{u}}_{k+1}} \right\| = \left\| a_k u_k \sum_{n \in A_i, n>k} \frac{a_n}{u_{n+1}} \right\| \leq \left| a_k u_k \sum_{n \in A_i, n>k} \frac{a_n}{u_{n+1}} \right| \leq \frac{a_k}{q^{\mathbf{u}}_{k+1}}.$$

Since, by the hypotheses in items (a) and (b), we have

$$\lim_{A_i \ni k \to +\infty} \frac{a_k}{q^{\mathbf{u}}_{k+1}} = 0 \quad \text{and} \quad \inf_{k \in A_i} \left\| \frac{a_n^2}{q^{\mathbf{u}}_{n+1}} \right\| > 0,$$

we conclude that $\|a_k u_k y_i\| \nrightarrow 0$ for $A_i \ni k \to +\infty$, and so that $y_i \notin t_{\mathbf{u}^{(i)}}(\mathbb{T})$. □

Remark 5.12. If one omits the assumption in item (c) of the statement of Lemma 5.11, the sequences $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$ produced in the above proof need not be a-sequences, indeed,

$$u_n^{(i)} \leq a_n u_n \leq u_{n+1} = q^{\mathbf{u}}_{n+1} u_n,$$

so $a_n u_n$ may fail to divide u_{n+1} for example when $q^{\mathbf{u}}_{n+1}$ is a prime number. Nevertheless, the above argument shows that $t_{\mathbf{u}}(\mathbb{T})$ is factorizable anyway.

A sequence $(a_n)_{n \in \mathbb{N}_0}$ with the properties in items (a) and (b) of Lemma 5.11 (with $\varepsilon = \frac{1}{2}$) can be produced by setting $a_n := \left\lceil \sqrt{\frac{q^{\mathbf{u}}_{n+1}}{2}} \right\rceil$ for every $n \in \mathbb{N}_0$.

The next proposition covers the case of Theorem 5.16 of sequences of powers of a fixed prime number.

Proposition 5.13. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ with $q^{\mathbf{u}}_n \to +\infty$ and such that $\mathbf{u}$ is a subsequence of $(p^n)_{n \in \mathbb{N}_0}$ for some prime number p. Then $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

Proof. For every $n \in \mathbb{N}_0$, let $u_n = p^{k_n}$ and $d_n = k_{n+1} - k_n$; so,

$$q^{\mathbf{u}}_{n+1} = p^{d_n}.$$

Since $q_n^{\mathbf{u}} \to +\infty$, also $d_n \to +\infty$. Define now, for every $n \in \mathbb{N}_0$,

$$a_n = p^{\lfloor \frac{d_n}{2} \rfloor - 1}.$$

It is a straightforward verification that $(a_n)_{n\in\mathbb{N}_0}$ fulfills the conditions of Lemma 5.11. □

In [16, Remark 3.9], using an argument from [12, Corollary 2.8] it was proved that $t_{\mathbf{u}}(\mathbb{T})$ is not divisible when $\mathbf{u}$ is an a-sequence with $q_n^{\mathbf{u}} \to +\infty$. The same argument could be used to prove the following lemma. We give a completely different proof here for reader's convenience.

Lemma 5.14. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ such that $q_n^{\mathbf{u}} \to +\infty$. Let p be a prime number such that p divides u_1 and let $v_n := \frac{u_n}{p}$ for every $n \in \mathbb{N}$. Then*

$$pt_{\mathbf{u}}(\mathbb{T}) \le t_{\mathbf{v}}(\mathbb{T}) \lneq t_{\mathbf{u}}(\mathbb{T}).$$

In particular, $t_{\mathbf{u}}(\mathbb{T})$ is not divisible.

Proof. The inclusion $pt_{\mathbf{u}}(\mathbb{T}) \le t_{\mathbf{v}}(\mathbb{T})$ holds true since $x \in t_{\mathbf{u}}(\mathbb{T})$ implies $v_n(px) = u_n x \to 0$ in $\mathbb{T}$. Moreover, $t_{\mathbf{v}}(\mathbb{T}) \le t_{\mathbf{u}}(\mathbb{T})$ as $x \in t_{\mathbf{v}}(\mathbb{T})$ implies $v_n x \to 0$ and so $u_n x = pv_n x \to 0$ in $\mathbb{T}$.

In the rest of the proof we give an element $x \in t_{\mathbf{u}}(\mathbb{T}) \setminus t_{\mathbf{v}}(\mathbb{T})$. Let $c_1 = 1$ and for every $n > 1$, let $c_n \in \{0, 1, \dots, p-1\}$ be such that $q_n^{\mathbf{u}} + c_n \equiv_p 1$. We verify by induction that

$$u_n \left(\sum_{i=1}^{n} \frac{c_i}{u_i} \right) \equiv_p 1. \tag{5.3}$$

For $n = 1$ we have defined $c_1 = 1$. Assume now that $n > 1$; then, by inductive hypothesis,

$$u_{n+1} \sum_{i=1}^{n+1} \frac{c_i}{u_i} \equiv_p q_{n+1}^{\mathbf{u}} + c_{n+1} \equiv_p 1.$$

Set now

$$x := \sum_{n\in\mathbb{N}} \frac{c_n}{u_n}$$

and note that this is the canonical representation of x (i.e., $c_n(x) = c_n$ for every $n \in \mathbb{N}$).

We prove now that

$$v_n x \to \frac{1}{p} \text{ in } \mathbb{T}. \tag{5.4}$$

To this end, write

$$v_n x = \frac{u_n}{p} \left(\sum_{i\le n} \frac{c_i}{u_i} \right) + \frac{u_n}{p} \left(\sum_{i>n} \frac{c_i}{u_i} \right).$$

By (5.3), we have

$$\frac{u_n}{p} \left(\sum_{i\le n} \frac{c_i}{u_i} \right) \equiv_{\mathbb{Z}} \frac{1}{p}.$$

Moreover, applying also Claim 2.11(b),

$$\frac{u_n}{p}\left(\sum_{i>n}\frac{c_i}{u_i}\right)\le\frac{u_n}{p}\left(\sum_{i>n}\frac{p-1}{u_i}\right)\le\frac{u_n}{p}\frac{p-1}{q_n^{\mathbf{u}}-1}\left(\sum_{i>n}\frac{q_n^{\mathbf{u}}-1}{u_i}\right)\le\frac{u_n}{p}\frac{p-1}{q_n^{\mathbf{u}}-1}\frac{1}{u_n}=\frac{p-1}{p(q_n^{\mathbf{u}}-1)}\to 0.$$

Therefore, (5.4) is verified and it implies that $x\notin t_{\mathbf{v}}(\mathbb{T})$ and also that $u_nx=pv_nx\to 0$ in $\mathbb{T}$, that is, $x\in t_{\mathbf{u}}(\mathbb{T})$. $\square$

The following proposition covers the case left open from Proposition 5.13, that is when the a-sequence $\mathbf{u}$ is not a subsequence of $(p^n)_{n\in\mathbb{N}_0}$ for some prime number p.

Proposition 5.15. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ such that $q_n^{\mathbf{u}}\to+\infty$. Let p_1,p_2 be two distinct primes such that p_1p_2 divides u_1. Then $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

Proof. For every $n\in\mathbb{N}$, let

$$\mathbf{u}_n^{(1)}:=\frac{u_n}{p_1}\quad\text{and}\quad\mathbf{u}_n^{(2)}:=\frac{u_n}{p_2}.$$

Then $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are a-sequences.

By Lemma 5.14, for $i=1,2$, we have that $t_{\mathbf{u}^{(i)}}(\mathbb{T})$ is a proper subgroup of $t_{\mathbf{u}}(\mathbb{T})$ containing $p_it_{\mathbf{u}}(\mathbb{T})$. Moreover,

$$t_{\mathbf{u}}(\mathbb{T})\subseteq p_1t_{\mathbf{u}}(\mathbb{T})+p_2t_{\mathbf{u}}(\mathbb{T})\subseteq t_{\mathbf{u}^{(1)}}(\mathbb{T})+t_{\mathbf{u}^{(2)}}(\mathbb{T})\subseteq t_{\mathbf{u}}(\mathbb{T}).$$

Therefore, $t_{\mathbf{u}}(\mathbb{T})=p_1t_{\mathbf{u}}(\mathbb{T})+p_2t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable. $\square$

We are now in position to prove the main theorem of this subsection, by applying Propositions 5.13 and 5.15.

Theorem 5.16. *Let $\mathbf{u}$ be an a-sequence in $\mathbb{N}$ with $q_n^{\mathbf{u}}\to+\infty$. Then $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

Proof. If $\mathbf{u}$ is a subsequence of $(p^n)_{n\in\mathbb{N}_0}$ for some prime number p, then Proposition 5.13 applies. Otherwise, there exist $k\in\mathbb{N}$ and two different prime numbers p_1,p_2 such that the product p_1p_2 divides u_k; we can assume without loss of generality that p_1p_2 divides u_1, and hence Proposition 5.15 applies. $\square$

Note that item (b) of Example 5.3 shows that $q_n^{\mathbf{u}}\to+\infty$ is not a necessary condition for the conclusion of Theorem 5.16.

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