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Ph.D. Thesis

Markov topologies on groups

Candidate:
Daniele Toller

Supervisor:
Dikran Dikranjan

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Author's e-mail: tollerdaniele@gmail.com

Author's address:

Dipartimento di Matematica e Informatica
Università degli Studi di Udine
Via delle Scienze, 206
33100 Udine
Italia

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Preface

In 1944, Markov introduced four special families of subsets of a group G:

Definition P.1. *([41]) A subset X of a group G is called:*

(a) elementary algebraic *if there exist an integer $n > 0$, elements $g_1, \ldots, g_n \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$, such that*

$$X = \{x \in G : g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} = e_G\};$$

(b) additively algebraic *if X is a finite union of elementary algebraic subsets of G;*

(c) algebraic *if X is an intersection of additively algebraic subsets of G;*

(d) unconditionally closed *if X is closed in* every *Hausdorff group topology on G.*

Definition P.2. If G is a group, take x as a symbol for a variable, and denote $G[x] = G * \langle x \rangle$ the free product of G and the infinite cyclic group $\langle x \rangle$ generated by x. We call $G[x]$ *the group of words with coefficients in G*, or *the group of words in G*, and its elements $w(x)$, or simply w, *words in G*.

An elementary algebraic subset X of G as in Definition P.1 (a) will be denoted by E_w^G (or simply, E_w), where w is an abbreviation for the defining word

$$w(x) = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \tag{1}$$

considered as an element of the free product $G[x] = G * \langle x \rangle$. We often consider w as a function from G to G, and in this sense one can consider the elementary algebraic subset E_w as the solution-set of the equation $w(x) = e_G$ in G.

Let $\mathbb{E}_G$ denote the family of elementary algebraic subsets of a group G. Obviously, every singleton is an elementary algebraic subset, so every finite subset is additively algebraic (for other examples see below, and Example 2.43). Then the family of algebraic subsets is closed under finite unions and arbitrary intersections, and contains G and all finite subsets of G. So it can be taken as the family of closed sets of a unique T_1 topology $\mathfrak{Z}_G$ on G, called the *Zariski topology* ([19, 20, 21, 24, 25, 4, 26]).

Markov did not explicitly introduce this topology, although it was implicitly present in [41, 42, 43] (through the *algebraic closure* of a subset X, i.e., the smallest algebraic subset of the group G that contains X). It was explicitly introduced only in 1977 by Bryant [11] under the name *verbal topology*. Here we keep the name

Zariski topology and the notation $\mathfrak{Z}_G$ for this topology following this use since 2003 when the first drafts of [21, 22] were ready (their main results were reported later in [18, 19, 20]).

Here comes the first supply of less trivial elementary algebraic subsets. Let G be a group and $g \in G$. We shall see in Example 2.43, item 1, that the one-element centralizer $C_G(g)$ is an elementary algebraic subset, and so the centralizer $C_G(S) = \bigcap_{g\in S} C_G(g)$ of any subset S of G is an algebraic subset. In particular, the center $Z(G)$ is an algebraic subset.

In some cases these are essentially all the elementary algebraic subsets (see §7 about free groups).

A case when the Zariski topology has a very transparent description is that of abelian groups. If G is an abelian group, then $\mathbb{E}_G = \{g + G[n] \mid g \in G, n \in \mathbb{N}\}$, so the family of algebraic subsets of G is $\mathbb{E}_G^{\cup}$. In other words, $\mathbb{E}_G^{\cup}$ is the family of all $\mathfrak{Z}_G$-closed subsets of an abelian group G.

More recently, in a series of papers starting in 1999 with [8], Baumslag, Myasnikov and Remeslennikov developed the study of algebraic geometry over an abstract group G: in analogy to the well-known Zariski topology from algebraic geometry, the authors consider finite powers G^n of a group G, and introduce the *Zariski topology on G^n* using solution sets of n-variable equations. In the case $n = 1$, this topology is $\mathfrak{Z}_G$.

In *linear* groups the term *Zariski topology* is used for a different standardly defined topology. Namely, for a field K consider the topology $\mathcal{A}_{K^n}$ of the affine space K^n having as a subbase of the closed sets the family of zero-sets of n-variable polynomials over K. The full linear group $\mathrm{GL}_n(K)$ (as well as its subgroups) carry the topology induced by $\mathcal{A}_{K^{n^2}}$ (via the embedding in K^{n^2}). Usually *this* topology of the linear groups is called Zariski topology. In order to avoid confusion, we use the term *affine topology*, when we refer to this topology for a linear group G and denote it by $\mathcal{A}_G$. In general, $\mathfrak{Z}_G \subseteq \mathcal{A}_G$ for a linear group G (Example 10.2 (a)), and they need not coincide (Corollaries 8.23 and 9.25).

The cardinality

$$\mathrm{bd}(G) = \min\{|S| : S \subseteq \mathbb{E}_G, \ \bigcup S = G \setminus \{e_G\}\}$$

is called the *bound* of G in[12]; G is called *κ-bound* for a (possibly finite) cardinal κ if $\mathrm{bd}(G) \leq \kappa$ (see [34, Definition 1]). Clearly, $\mathrm{bd}(G) \leq |G|$; while $\mathrm{bd}(G) < \infty$ if and only if $\mathfrak{Z}_G$ is discrete. This definition was inspired by Podewski [52], where κ-bound groups are called *κ-gebunden* (see also [33]), and groups G that are not κ-bound for every $\kappa < |G|$ are called *ungebunden*. In other words, ungebunden groups are those G such that $\mathrm{bd}(G) = |G|$.

To connect the bound to the Zariski topology we need to recall some notions from topology. If $(X, \mathcal{T})$ is a topological space and κ is an infinite cardinal, a union of $\leq \kappa$-many closed subsets is called an *F_κ-set.* The topology $\mathcal{T}$ is called a *P_κ-topology,* if every F_κ-set is closed. Every topology $\mathcal{T}$ on X admits a coarsest P_κ-topology $P_\kappa\mathcal{T}$

containing $\mathcal{T}$, called the P_κ*-modification* of $\mathcal{T}$ (namely the topology having as a base of its closed sets all F_κ-sets of $(X, \mathcal{T})$).

Let G be a group and let δ_G denote the discrete topology of G. For an infinite cardinal κ, the P_κ-modification $P_\kappa\mathfrak{Z}_G$ of the Zariski topology of G will be called κ*-Zariski topology* of G. Clearly, $P_\kappa\mathfrak{Z}_G$ is discrete for all $\kappa \geq |G|$, but $P_\kappa\mathfrak{Z}_G$ may be non-discrete for some infinite $\kappa < |G|$. We call the chain of topologies

$$\mathfrak{Z}_G \leq P_\omega\mathfrak{Z}_G \leq \ldots \leq P_\kappa\mathfrak{Z}_G \leq \ldots \leq P_{|G|}\mathfrak{Z}_G = \delta_G$$

the *Zariski rod* of G. If $\mathfrak{Z}_G$ is not discrete, then $\mathrm{bd}(G)$ is infinite and the least cardinal κ with $P_\kappa\mathfrak{Z}_G = \delta_G$ coincides with $\mathrm{bd}(G)$. This allows us to consider $\mathrm{bd}(G)$ also as a *measure of the failure of* $\mathfrak{Z}_G$ *to be discrete.*

P.1 The Markov topologies on a group and Markov's problems

As noticed in [20], the family of unconditionally closed subsets of G coincides with the family of closed subsets of a T_1 topology $\mathfrak{M}_G$ on G, called *the Markov topology of* G. It coincides with the infimum (taken in the lattice of all topologies on G) of all Hausdorff group topologies on G. As every elementary algebraic subset is closed in every Hausdorff group topology, one has that $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$.

A Hausdorff topological group is said to be *precompact*, if G is topologically isomorphic to a subgroup of a compact Hausdorff topological group. In analogy with $\mathfrak{M}_G$, let $\mathfrak{P}_G$ be the infimum of all precompact Hausdorff group topologies on G. If G admits no such topologies, then $\mathfrak{P}_G = \delta_G$ is discrete. Call $\mathfrak{P}_G$ *the precompact Markov topology of* G [21]. Clearly, $\mathfrak{M}_G \subseteq \mathfrak{P}_G$, so that in general

$$\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G.$$

We refer to these three topologies on a group as *the Markov topologies.*

If G is abelian, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ is a Noetherian topology (see Theorem 4.10), although $\mathfrak{Z}_G = \mathfrak{M}_G \neq \mathfrak{P}_G$ may occur in some nilpotent groups of class 2 (see Proposition 11.39). In this work we also provide a large series of examples of solvable groups G with discrete $\mathfrak{P}_G$.

Note that $(G, \mathfrak{Z}_G)$, $(G, \mathfrak{M}_G)$ and $(G, \mathfrak{P}_G)$ are quasi-topological groups in the sense of [2], i.e., the inversion and shifts are continuous (see also Corollary 4.4). Nevertheless, these groups are almost never topological (for example, if a group is abelian, this holds only if it is also finite). One of the aims of the paper is to provide a series of example of infinite groups G such that $(G, \mathfrak{Z}_G)$, $(G, \mathfrak{M}_G)$ and $(G, \mathfrak{P}_G)$ are (necessarily Hausdorff) topological groups.

Markov posed the following problem (without the explicit use of topologies):

Markov's First Problem: Does $\mathfrak{Z}_G = \mathfrak{M}_G$ hold true for an arbitrary group G?

He proved that these two topologies coincide in the countable case [41], and attributed the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ for abelian groups to Perel'man. However, a proof of this fact never appeared in print until the independently obtained [20], [60], where the authors prove that $\mathfrak{Z}_G = \mathfrak{M}_G$ for groups $G = A \times \bigoplus_{i \in I} H_i$, where A is an abelian group, and each H_i is a countable group (see Theorem 4.11).

If the group G is infinite, the topology $\mathfrak{M}_G$ is discrete if and only if G is *non-topologizable*, i.e., it does not admit a non-discrete Hausdorff group topology (see Definition 11.13). In 1945, Markov [42, Problem 4] asked:

Markov's Second Problem: Does there exist an infinite non-topologizable group?

This problem remained unsolved for many years, until Shelah [57] constructed under CH a non-topologizable group of size ω_1. Actually, Hesse [33] eliminated CH from Shelah's construction, thus presenting the first example of a non-topologizable group in ZFC of size ω_1. Finally, in 1980, Ol′shanskij [49] built up a countable non-topologizable group in ZFC (for more details see §11.2).

An important connection between this problem of Markov and the bound was found by Podewski [52]. He proved that a group G with $\mathrm{bd}(G) = |G|$ is always topologizable. (More precisely, it admits the maximum number $2^{2^{|G|}}$ of Hausdorff group topologies, see Theorem 11.27.) Hesse [33] showed that this condition is not necessary: for any uncountable cardinal λ he found a topologizable group G of size λ with $\mathrm{bd}(G) = \omega$ (so $\mathrm{bd}(G) < |G|$).

Markov was interested also to describe the groups admitting a *connected* Hausdorff group topology. If τ is such a topology on a group G, then every proper τ-closed subgroup H of G has index at least $\mathfrak{c}$. (Indeed, if τ is a connected Hausdorff group topology on G, then the non-trivial completely regular quotient space G/H is connected, hence $|G/H| = [G : H] \geq \mathfrak{c}$.) In particular, all proper $\mathfrak{M}_G$-closed subgroups of G must have index at least $\mathfrak{c}$. This is why Markov asked:

Markov's Third Problem: If every proper $\mathfrak{M}_G$-closed subgroup of a group G has index $\geq \mathfrak{c}$, does G admit a connected Hausdorff group topology?

Pestov [51] answered negatively the third Markov's problem by a rather complex counter-example. Later, Remus [55] noticed that a quite easy example can be obtained using the symmetric groups $G = S(X)$. Indeed, $\mathfrak{M}_G = \tau_p(G)$ is the point-wise convergence topology on G by Theorem 12.8, so every Hausdorff group topology on G is totally disconnected (as $\mathfrak{M}_G$ is totally disconnected and this property is preserved by taking finer topologies). Hence one must only ensure to have all proper $\mathfrak{M}_G$-closed subgroups of index at least $\mathfrak{c}$. This is possible choosing X with $|X| \geq \mathfrak{c}$, so obtaining a counter-example to Markov's third problem (see Theorem 12.9 and Corollary 12.10 for details).

Introduction

If $\mathcal{P}$ is a property of a topological space, we define a group G to be $\mathfrak{Z}$-$\mathcal{P}$ if the space $(G, \mathfrak{Z}_G)$ satisfies $\mathcal{P}$. Similarly, we introduce the notion of a $\mathfrak{M}$-$\mathcal{P}$ (respectively, $\mathfrak{P}$-$\mathcal{P}$) group, provided that $(G, \mathfrak{M}_G)$ (respectively, $(G, \mathfrak{P}_G)$) satisfies $\mathcal{P}$. In particular, we will consider the following properties as property $\mathcal{P}$: being cofinite (topological space), Noetherian, compact, Hausdorff, discrete, irreducible, connected.

One of the aims of this work is to deduce topological properties of the space $(G, \mathfrak{Z}_G)$ from the algebraic properties of G, or vice versa. Another purpose is to study the behaviour of the Markov topologies under the standard passages to subgroups, quotients, products (direct or semi-direct). Also, we consider some easier to deal with topologies (as the *Taĭmanov topology* recalled in Definition 1.13, the *monomial topology*, or the other *partial Zariski topologies* introduced in §§4.2-4.3) that nicely approssimate the Zariski topology and often coincide with it.

The last main issue of this thesis is to compare the properties of the Zariski topology in the non-abelian case to those in the abelian one. We do this mainly with nilpotent groups.

In §I.1, we fix the basic notation, while §1 is devoted to the the necessary preliminaries. In particular, in §1.1 we recall some algebraic definitions from group theory, in §1.2 we cover the set-theoretical topological background, and in §1.3 we give some general results on quasi-topological groups (see Definition 1.5) that are well-known to hold for topological groups. In the final §1.3.1 we recall the definition of the *Taĭmanov topology* $\mathcal{T}_G$ on a group G, we give a few of its properties, and we introduce its T_1 refinement $\mathcal{T}'_G = \mathcal{T}_G \vee cof_G$, as in general $\mathcal{T}_G$ is not T_1.

In §2 we begin to study one of the main tools of this work: the group $G[x]$ of words over a group G. First, §2.1 is dedicated to $G[x]$, defined through a universal property in §2.1.1. Then, in §2.1.2 we focus on its elements, the words w over G, and we introduce various notions related to a word. For example, if w is as in (1), we define the *content* $\epsilon(w) \in \mathbb{Z}$ of w, as $\epsilon(w) = \sum_{j=1}^{n} \varepsilon_j$ (Definition 2.6), and we say that w is a *singular word* if $\epsilon(w) = 0$ (Definition 2.11).

In §2.2 we introduce the notion of *verbal function of* G, namely the evaluation function $f_w\colon G \to G$, determined by a word $w \in G[x]$, mapping $g \mapsto w(g)$ (Definition 2.14). We dedicate the introductory §2.2.1 to definitions and to show that many natural functions $G \to G$ are verbal (Example 2.15).

Then, in §2.2.2 we define the *universal words of* G, namely words $w \in G[x]$ such that $f_w \equiv e_G$ (Definition 2.17). The component-wise operation defines a group structure on the set $\mathscr{F}(G)$ of verbal functions, so that $\mathscr{F}(G)$ is a quotient of $G[x]$ by the normal subgroup $\mathcal{U}_G$ of $G[x]$ consisting of the universal words of G, and defined in (2.5). We define also the subgroup $\mathcal{U}_G^{sing}$ of *singular* universal words, and through this we introduce the invariant $\mathrm{u}(G)$ of G in Definition 2.19, called the *universal exponent of* G. Then $\mathrm{u}(G) \in \mathbb{N}$ is the minimum natural such that every universal word of G has content multiple of $\mathrm{u}(G)$. For example, it is immediate from the definition that if $\exp(G) > 0$, then $\mathrm{u}(G) \mid \exp(G)$ (Lemma 2.20).

Using $\mathrm{u}(G)$ we define for every $n \in \mathbb{N}$ the class $\mathcal{W}_n$ of groups G such that $n \mid \mathrm{u}(G)$ (Definition 2.21), and we study the first properties of the invariant $\mathrm{u}(\cdot)$ and of the classes $\mathcal{W}_n$ (see Lemmata 2.23 and 2.24).

In §2.2.3 we talk about *monomials over* G, i.e. words of the form $w = gx^n \in G[x]$, and we show in (2.8) how to associate a monomial w_{ab} to an arbitrary word w. In §2.2.4 we prove that $\mathrm{u}(G) = \exp(G)$ for an abelian group G (Lemma 2.32) and we see that $\mathscr{F}(G)$ is represented by f_w for monomials w, when G is abelian (Proposition 2.33). In §2.3 we study further the stability properties of the classes $\mathcal{W}_n$, and we prove $\mathrm{u}(S_3) = 2$ (Example 2.37), and $\mathrm{u}(G) = 2$ for a class of semidirect products (see Example 2.38).

Finally, in §2.4 we present the elementary algebraic set E_w (already introduced in Definition P.1 (a)) as the preimage $f_w^{-1}(\{e_G\})$ of the verbal function f_w associated to w (see Definition 2.39). In §2.4.1 we give some basic properties of the family $\mathbb{E}_G$ for an abelian group G. Equation (2.13) classifies the elementary algebraic subsets of an abelian group G, so that the following (2.14) completely describes $\mathbb{E}_G$, using the already mentioned description of $\mathscr{F}(G)$ given in §2.2.4. Consequently, the non-empty elementary algebraic subsets of an abelian group G are the cosets of the n-socle subgroups $G[n]$. Then, in §2.4.2 we provide further natural examples of (elementary) algebraic subsets (Examples 2.43 and 2.45), and we prove that the family $\mathbb{E}_G$ is stable under taking inverse image under verbal functions (Lemma 2.44).

One of the aims of this work is to study the Zariski topology $\mathfrak{Z}_G$, having as closed sets the algebraic subsets of G. To this end, we have to first study the family $\mathbb{E}_G$ of elementary algebraic subsets $E_w = f_w^{-1}(\{e_G\})$. Then, it is sufficient to consider appropriate subsets $W \subseteq G[x]$ such that $\mathbb{E}_G = \{E_w \mid w \in W\}$. The final §2.4.3 treats this argument.

In §3 we deepen the study of elementary algebraic subsets. We give some technical results which lead us to Theorem 3.4 and Theorem 3.6, describing some cases when an elementary algebraic subset is a coset of a subgroup.

The class $\mathscr{N}_2$ of nilpotent groups of nilpotency class 2 is studied in §3.1. Here we give some conditions under which an elementary algebraic subset is a coset (or a union of cosets) of a subgroup, and in Theorem 3.14 we describe $\mathscr{F}(G)$ for $G \in \mathscr{N}_2$. As an application, Corollaries 3.15 and 3.16 describe more specifically the words $w \in G[x]$ which determine all elementary algebraic subsets of a group $G \in \mathscr{N}_2$.

We use these results to completely describe $\mathbb{E}_G$ when $G \in \mathscr{N}_2$ is either such that $G/Z(G)$ is torsion-free (Lemma 3.17), or G has prime exponent $p > 2$ (Lemma 3.18).

Then, in §3.2 we consider groups $G \in \mathscr{N}_2$ with $\exp(G) = 2$, giving some general properties in §3.2.1. In particular we obtain Corollary 3.19 which we use in §§3.2.2-3.2.3 to describe $\mathbb{E}_{Q_8}$ and $\mathbb{E}_{D_8}$, and to compute $\mathrm{u}(Q_8) = 4$ in Lemma 3.21, and $\mathrm{u}(D_8) = 4$ in Lemma 3.22.

In §3.3 we introduce and describe the classes $\mathcal{W}_n^* \subseteq \mathcal{W}_n$, for an integer $n \in \mathbb{N}$ (see Definition 3.23), and we study when some groups are contained in some $\mathcal{W}_n^*$. Using the classes $\mathcal{W}_n^*$, we define the invariants $\mathrm{u}^\circ(G) \in \mathbb{N}$ in (3.14), and $\mathrm{u}^*(G) \in \mathbb{N}$ in (3.15) of a group G. We call u^*-*exponent of* G the natural $\mathrm{u}^*(G)$. Then $\mathrm{u}^\circ(G) \mid \mathrm{u}^*(G)$ and $\mathrm{u}^*(G) \mid \mathrm{u}(G)$, and for example $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$ for every finite group G (Lemma 3.26), while $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = \exp^*(G)$ when G is abelian (Corollary 3.28).

In §3.4 we introduce the notion of δ-*word* in $G[x]$ (Definition 3.29). Lemma 3.30 shows that only center-free groups may admit a δ-word, and in Proposition 3.31 we construct a δ-word for every free non-abelian group. The close connection of δ-words to products becomes clear in §6.1.1.

In §4 we begin studying quasi-topological group topologies, and in particular the topologies $\mathfrak{Z}_G$, $\mathfrak{M}_G$, $\mathfrak{P}_G$. In Lemma 4.1 we classify quasi-topological group topologies in term of continuity of an appropriate family of verbal functions. Proposition 4.3 gives some properties of $\mathscr{F}(G)$, and $\mathfrak{Z}_G$. In Corollary 4.4 we provide a new argument using verbal functions to prove the already discussed inclusions $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$. In Proposition 4.6 we prove that, given a normal subgroup N of a group G, it is $\mathfrak{Z}_G$-closed if and only if the canonical map $\pi\colon (G, \mathfrak{Z}_G) \to (G/N, \mathfrak{Z}_{G/N})$ is continuous. As a corollary, we get that $Z_n(G)$ is $\mathfrak{Z}_G$-closed for every positive integer n.

In §4.1 we recall results from [21] about the properties of $\mathfrak{Z}_G$ for an abelian group G. Then, we see in Theorem 4.10 that $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ is a Noetherian topology, whose closed sets are the elements of $\mathbb{E}_G^\cup$. Fact 4.12 (b) and Corollary 4.13 determine the connected component of the identity in $(G, \mathfrak{Z}_G)$. Then, Fact 4.14 classifies abelian $\mathfrak{Z}$-irreducible groups, while Proposition 4.15 describes the subclass of abelian $\mathfrak{Z}$-cofinite groups.

In §4.2 we introduce some *partial Zariski topologies* on G, namely topologies having *some* elementary algebraic sets as a subbase for the closed sets. For example, we introduce the *monomial topology* $\mathfrak{T}_{mon}$ on a group G in Definition 4.22, we note that $\mathfrak{T}_{mon} = \mathfrak{Z}_G$ when G is abelian (Example 4.23), and we prove that $\mathfrak{T}_{mon}$ is the cofinite topology when G is nilpotent, torsion free (Corollary 4.25). Then we dedicate §4.3 to the *centralizer topologies* $\mathfrak{C}_G$ and $\mathfrak{C}'_G$, introduced in Definition 4.20 and Definition 4.26 (see Lemma 4.28 for the first few properties they satisfy). Then Lemma 4.32 proves that $\mathfrak{C}_G = \mathcal{T}_G$ for an FC-group G, while $\mathfrak{Z}_G = \mathfrak{T}_{mon} \vee \mathfrak{C}_G$ if $G \in \mathscr{N}_2$ and $G/Z(G)$ is torsion-free (Corollary 4.33). We conclude this chapter proving that if $G \in \mathscr{N}_2$ is torsion-free, then $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I}$ for every non-empty set I in Theorem 4.34. The same conclusion holds for groups $G \in \mathscr{N}_2$ such that G has prime exponent $p > 2$ (Theorem 4.35).

We begin §5 noting in (5.1) that $\mathfrak{M}_G\restriction_H \supseteq \mathfrak{Z}_G\restriction_H \supseteq \mathfrak{Z}_H \subseteq \mathfrak{M}_H$ hold for every subgroup H of a group G. Then we recall the definitions of some particular subgroup embeddings in a group (Definition 5.2), and we show how they are related (see diagram (5.2) for a quick reference). These definitions were introduced in [20] to guarantee that also either $\mathfrak{Z}_G\restriction_H \subseteq \mathfrak{Z}_H$ or $\mathfrak{M}_G\restriction_H \subseteq \mathfrak{M}_H$ hold.

The whole §6 is devoted to the study of the direct products and sums of groups. We begin giving some general results that lead us to prove that if $G = \prod_{i \in I} G_i$ is a direct product, and $w \in G[x]$, then $E_w^G = \prod_{i\in I} E_{w_i}^{G_i}$, for an appropriate family of words $w_i \in G_i[x]$, for $i \in I$, uniquely determined by w (see Theorem 6.4). This yields that the Zariski topology $\mathfrak{Z}_G$ of the direct product is contained in the product topology $\prod_{i\in I} \mathfrak{Z}_{G_i}$.

Then we dedicate §6.1 to finite direct products. Here we mainly discuss when a pair of groups G_1, G_2 satisfies the equality $\mathfrak{Z}_{G_1\times G_2} = \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$ (such a pair is called $\mathfrak{Z}$*-productive* in Definition 6.19). We also define the weaker notion of *semi* $\mathfrak{Z}$*-productive pair* G_1*,* G_2, provided that both $G_1 \times \{e_{G_2}\}$ and $\{e_{G_1}\} \times G_2$ are $\mathfrak{Z}_{G_1\times G_2}$-closed subsets of $G_1 \times G_2$. Obviously, this is a necessary condition to have $\mathfrak{Z}$-productivity, and we ask in Question 6 if it is also sufficient. In §6.1.1 we characterize the groups that admit a δ-word as those G_2 such that $G_1 \times \{e_{G_2}\} \in \mathbb{E}_{G_1\times G_2}$ for every group G_1 (see Corollary 6.24). In §6.1.2 we focus on semi $\mathfrak{Z}$-productive pairs. Corollary 6.30 proves that $\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_{G_1\times G_2}} \subseteq Z(G_1)[u(G_2)] \times G_2$ for every pair G_1, G_2, so that a pair G_1, G_2 is semi $\mathfrak{Z}$-productive whenever

$$Z(G_1)[\mathrm{u}(G_2)] = \{e_{G_1}\} \text{ and } Z(G_2)[\mathrm{u}(G_1)] = \{e_{G_2}\}$$

(see Corollary 6.31). In particular, if both G_1, G_2 are center-free, then the pair G_1, G_2 is semi $\mathfrak{Z}$-productive.

On the other hand, we prove in Theorem 6.32 that $Z(G_1)[\mathrm{u}^*(G_2)] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_{G_1\times G_2}}$, so that if a pair G_1, G_2 is semi $\mathfrak{Z}$-productive, then

$$Z(G_1)[\mathrm{u}^*(G_2)] = \{e_{G_1}\} \text{ and } Z(G_2)[\mathrm{u}^*(G_1)] = \{e_{G_2}\}$$

by Corollary 6.33.

Then, Theorem 6.36 gives a characterization of center-free groups in these terms, proving that G is center-free if and only if $G_1 \times \{e_G\}$ is a Zariski closed subset of $G_1 \times G$ for every group G_1.

In §6.1.3 we consider only abelian pairs of groups, and we prove in Theorem 6.38 that for these pairs the semi $\mathfrak{Z}$-productivity is equivalent to the $\mathfrak{Z}$-productivity, positively answering Question 6. This theorem also describes the structure of such pairs of groups.

The following §6.2 is focused on direct sums. In Proposition 6.40 we describe the elementary algebraic subsets of such groups (see equation (6.6)), and then we prove in Corollary 6.41 that the Zariski topology $\mathfrak{Z}_S$ of a direct sum $S = \bigoplus_{i\in I} G_i$ is coarser than the sum topology $(\prod_{i\in I} \mathfrak{Z}_{G_i}) \restriction_S$ induced by the product topology.

We study the centralizer topologies $\mathfrak{C}_G$ and $\mathfrak{C}'_G$ on products in §6.3, where we prove the equality $\mathfrak{Z}_G = \mathfrak{C}'_G$ for a class of groups $G \in \mathscr{N}_2$ in Theorem 6.48 (see also Theorem 6.49 for a more topological description of $\mathfrak{Z}_G$).

In §6.4 we consider direct products of *finite* groups, we give some general results (see for example Remark 6.54), and then we determine the Zariski topology of direct products and sums of *finite, center-free* groups in Theorem 6.55, proving for example

$$\mathfrak{C}_G = \mathfrak{C}'_G = \mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i\in I} \mathfrak{Z}_{F_i}$$

for such a direct product $G = \prod_{i \in I} F_i$.

In §6.5 we prove that an infinite direct product of groups in $\mathcal{W}_n$ belongs to $\mathcal{W}_n^*$ (Theorem 6.59). As a consequence, Corollary 6.61 shows that

$$\mathrm{u}^{\circ}(G^I) = \mathrm{u}^{*}(G^I) = \mathrm{u}(G^I) = \mathrm{u}(G)$$

for every group G and infinite set I. Then we conclude this part giving some results on the Zariski closure of direct summands of groups (see Corollaries 6.63 and 6.65 for general results, and Corollaries 6.66 and 6.67 for more concrete cases). The final §6.6 gives a complete description of the Zariski topology of the group $\mathbb{Z}_2 \times S_3^I$.

§7 provides a description of the Zariski topology of a free non-abelian group F, and the main Theorem 7.5 proves that $\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F$. The following Corollary 7.7 generalizes this result to prove that $\mathfrak{C}_G = \mathfrak{Z}_G = \prod_{i \in I} \mathfrak{Z}_{G_i}$ hold whenever $\{G_i \mid i \in I\}$ is a family of free non-abelian groups, and $G = \prod_{i \in I} G_i$.

§8 is dedicated to *Heisenberg groups*. We first study groups $H = H(n, K)$ of some particular $n \times n$ upper uni-triangular matrixes over a field K. The group H is nilpotent of class two, and depending on the characteristic of the field K, H is torsion-free (when $\operatorname{char} K = 0$), or has exponent 4 (when $\operatorname{char} K = 2$), or has exponent a prime number $p > 2$ (when $\operatorname{char} K = p$), according to Lemma 8.2. The group H has two normal, not super-normal, subgroups defined in (8.2), L and M, both isomorphic to the abelian group $(K^{n+1}, +)$, that are $\mathfrak{Z}_H$-closed sets by Lemma 8.3.

In Lemma 8.7 we give a first description of the elementary algebraic subsets of H, which is complete if $\operatorname{char} K \neq 2$, allowing us to conclude $\mathfrak{Z}_H = \mathfrak{C}'_H$ in this case, by Corollary 8.9. When $\operatorname{char} K = 2$ we also need Lemma 8.11 to describe $\mathfrak{Z}_H$ in Corollary 8.12.

In §8.3 we turn our attention to the case $n = 1$, considering from now on the group $H = H(1, K)$. In Lemma 8.13 (resp., Theorem 8.18) we show that the family $\mathcal{C}_K \subseteq \mathcal{P}(H)$ defined in (8.8) is a subbase of the $\mathfrak{Z}_H$-closed sets if $\operatorname{char} K \neq 2$ (resp., if $\operatorname{char} K = 2$). This allows us to completely describe the topological space $(H, \mathfrak{Z}_H)$, and in particular to prove that $\dim(H, \mathfrak{Z}_H) = 3$ in Corollary 8.22. All three subgroups L, M, $Z(H)$ are $\mathfrak{Z}_H$-closed irreducible subsets of H, as well as H itself, by Proposition 8.21.

Moreover, $\mathfrak{M}_L = \mathfrak{Z}_L \subsetneq \mathfrak{Z}_H \restriction_L \subsetneq \mathcal{A}_L = \mathcal{A}_H \restriction_L$ (and the same holds for M), so that also $\mathfrak{Z}_H \subsetneq \mathcal{A}_H$ by Lemma 8.3 and Corollary 8.23. It is also proved in Lemma 8.3 that the subgroups L and M are neither Zariski, nor Markov embedded in H (yet $L \cap M = Z(H)$, being super-normal, is Zariski, Markov and Hausdorff embedded in H).

The final §8.5 is dedicated to *generalized Heisenberg groups*. We first study groups H_R of 3×3 upper uni-triangular matrixes over a Unique Factorization Domain R, for which we prove that $\mathfrak{Z}_{H_R} = \mathfrak{C}'_{H_R}$ is Noetherian and $\dim(H_R, \mathfrak{Z}_{H_R}) = 3$ (Theorem 8.27). Then, for a K-vector space V, we use the canonical scalar product in V to define the group H_V of 3×3 upper uni-triangular formal matrixes (see the definition in §8.5.1). We give some general properties and definitions, and then we pay attention to the case when K is finite, and $\dim_K V$ is infinite, that we assume from now on. If $\mathrm{char}(K) = p > 2$, then $\mathfrak{Z}_{H_V} = \mathfrak{C}'_{H_V} = \mathcal{T}'_{H_V}$, while if $\mathrm{char}(K) = 2$, then $\mathfrak{C}'_{H_V} = \mathcal{T}'_{H_V} \lneq \mathfrak{Z}_{H_V}$ (Theorem 8.29).

Finally, the only $\mathfrak{Z}$-irreducible sets of H_V are the singletons, so $\dim(H_V, \mathfrak{Z}_{H_V}) = 0$ and H_V has no $\mathfrak{Z}_G$-atoms (Corollary 8.30).

In §9 the linear group $G = G_K$ over a field K is studied. If V is a K-vector space, then $G = K^* \ltimes V$ is the semidirect product of K^* with V, where K^* acts on V via scalar multiplications. If $\dim_K V = n$ is finite, then G is a linear group (a subgroup of upper triangular matrixes in $GL_{n+1}(K)$, see (9.1)). The group G is not nilpotent, being center-free, but it is solvable of class two, as $V = G'$ is the commutator subgroup of G by Lemma 9.1 (d). In particular, V is a normal subgroup of G, it is a semidirect factor of G, but V is not super-normal in G by Lemma 9.5. Moreover, V is a centralizer in G, hence an elementary algebraic subset of G, by Lemma 9.1 (a). Furthermore, V is Zariski embedded in G, and $\mathfrak{Z}_G \restriction_V = \mathfrak{Z}_V = \mathfrak{M}_V \subseteq \mathfrak{M}_G \restriction_V$ by Lemma 9.5 and Corollary 9.18.

The question when the subgroup V is Hausdorff embedded in G is more subtle. If $\dim_K V$ is infinite, then V is Hausdorff embedded in G exactly when $K = \mathbb{F}_p$ for some prime $p \in \mathbb{P}$, by Corollary 9.10. So for all fields $K \neq \mathbb{F}_p$ for every $p \in \mathbb{P}$, V is not Hausdorff embedded in G. If in addition G is also countable (i.e. both K and $\dim_K V$ are countable), then V is also Markov embedded. For finite-dimensional V we impose on the field K a condition (†) (i.e. either $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K = p > 0$ and the extension $K/\mathbb{F}_p$ is not algebraic), ensuring that the subgroup V is not Hausdorff embedded in G (Corollary 9.7). We do not know whether this condition is necessary (see Question 15).

§9.1 is focused on $\mathfrak{Z}_G$. In particular, Theorem 9.15 describes explicitly a subbase $\mathcal{B}$ of $\mathfrak{Z}_G$. Using this explicit description of $\mathfrak{Z}_G$ we prove that G is $\mathfrak{Z}$-Noetherian (Proposition 9.21) and that the dimension of the space $(G, \mathfrak{Z}_G)$ is either 1 or 2, depending on whether K is finite or infinite (Corollary 9.23). If $\dim_K V$ is finite, then also the affine topology $\mathcal{A}_G$ is defined on G, and $\mathfrak{Z}_G \subsetneq \mathcal{A}_G$ by Corollary 9.25.

In §10 we study $\mathfrak{Z}$-Noetherian, $\mathfrak{M}$-Noetherian and $\mathfrak{P}$-Noetherian groups (see Definition 10.1). We first recall known examples of classes of $\mathfrak{Z}$-Noetherian groups, as for example abelian groups and linear groups (Example 10.2), and free groups by Theorem 7.5. In §10.1 we prove the important criterion Theorem 10.6 for a group to be $\mathfrak{Z}$-Noetherian. In §10.2 we give Theorem 10.12 that classifies directs products or sums that are $\mathfrak{Z}$-Noetherian, thus extending Bryant's results reported in Fact 10.3. In §10.3 we study $\mathfrak{Z}$-compact and $\mathfrak{M}$-compact groups, and in the final §10.4 we resume the permanence properties of the classes $\mathfrak{Z}$-Noetherian and $\mathfrak{Z}$-compact groups.

Question 1. *[21, Question 12.2] Let G be a group. If $\mathfrak{Z}_G$ is compact, must $\mathfrak{Z}_G$ be necessarily Noetherian?*

We answer negatively this question (see Example 11.7).

We dedicate §11 to $\mathfrak{Z}$-Hausdorff, $\mathfrak{M}$-Hausdorff and $\mathfrak{P}$-Hausdorff groups, introduced in Definition 11.1.

According to Fact 4.12 (a), the Zariski topology $\mathfrak{Z}_G$ of an infinite abelian group is never Hausdorff (while it is Noetherian by a theorem of Bryant, see Example 10.2 (c)). This motivated the following question from [21].

Question 2. *[21, Question 12.3] Does there exist an infinite group G such that its Zariski topology $\mathfrak{Z}_G$ is compact Hausdorff?*

We answer positively this question by means of series of examples (see again Example 11.7). This also provides a negative answer to Question 1.

We study finite-center direct products in §11.1. Then in §11.2 we pay attention to $\mathfrak{Z}$-discrete and $\mathfrak{M}$-discrete groups, and we recall classic and recent results on these classes of groups. §11.3 is dedicated to $\mathfrak{P}$-Hausdorff groups, and in particular in §11.3.1 we consider $\mathfrak{P}$-discrete groups, i.e. groups that do not admit a precompact Hausdorff group topology. It is proved in Theorem 11.37 that a solvable divisible non-abelian group G is $\mathfrak{P}$-discrete. In particular, $\mathfrak{Z}_G = \mathfrak{M}_G \neq \mathfrak{P}_G$ may occur for some linear groups G that are nilpotent of class 2 (see Proposition 11.39). This partially answers the following question from [21] about the coincidence of the three topologies $\mathfrak{Z}_G, \mathfrak{M}_G, \mathfrak{P}_G$ on a nilpotent group G.

Question 3. *[21, Question 12.1] Which of the equalities $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ is true for nilpotent non-abelian groups?*

In §12 we consider minimal group topologies (see Definition 12.1), we recall classic examples, and then in §12.1 we specialize our study to the class of algebraically minimal groups, introduced in Definition 12.6 as groups G such that $\mathfrak{M}_G$ is a compact Hausdorff group topology. For example, some permutation groups are algebraically minimal, see §12.1.1. In the final §12.1.2 we determine when $\mathfrak{M}_G = \mathfrak{P}_G$ for algebraically minimal groups.

We give three diagrams recalling implications and counter-examples to non-implications among properties considered in this work in §13, while the final §14 collects some open questions. For better understanding the problems, some other questions are also spread in the text, next to the results that motivated them.

I.1 Notation

The set of natural numbers, integers, rationals and reals are denoted respectively by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ as usual. We denote by $\mathbb{N}_+$ the set of positive naturals, and by $\mathbb{P}$ the set of prime numbers.

If $n, m \in \mathbb{Z}$, we say that n *divides* m, and write $n \mid m$, if $m\mathbb{Z} \leq n\mathbb{Z}$. For example, $n \mid 0$ for every $n \in \mathbb{Z}$, and $0 \mid m$ if and only if $m = 0$.

If X is an (infinite) set, we denote by $S(X)$ the *symmetric group of* X, consisting of the permutations of X, and with $S_\omega(X)$ its subgroup consisting of the permutations having finite support. Then the *alternating subgroup* $A(X)$ is the subgroup of $S_\omega(X)$ consisting of even permutations; $A(X)$ has index two in $S_\omega(X)$, and in particular is normal. If $F \subseteq X$, then we will denote the *point-wise stabilizer* of F by $S_F(X) = \{\phi \in S(X) \mid \phi(f) = f \text{ for every } f \in F\}$.

When X is a finite set with n elements, we will also write S_n for $S(X) = S_\omega(X)$, and A_n for $A(X)$.

The finite cyclic group with n elements is denoted by $\mathbb{Z}_n$, while D_{2n} will stand for the dihedral group of order $2n$.

If X is a set, and α is a cardinal, we will let $[X]^{<\alpha} = \{Y \subseteq X \mid |Y| < \alpha\}$. In particular, we will often consider $[X]^{<\omega}$, the family of finite subsets of X.

For a set X, and a family $\mathcal{B} \subseteq \mathcal{P}(X)$ of subsets of X, let $\mathcal{B}^\cup$ denote the family of finite unions of elements of $\mathcal{B}$, and $\mathcal{B}^\cap$ the family of finite intersections of elements of $\mathcal{B}$.

1

Preliminaries

1.1 Algebraic facts

A group G is called *divisible* if for every $n \in \mathbb{N}$ and $g \in G$ there exists $x \in G$ such that $x^n = g$.

Given two elements $g, h \in G$, their commutator element is $[g, h] = ghg^{-1}h^{-1} \in G$. Note that $[g, h] = e_G$ if and only if $gh = hg$.

If $A, B \subseteq G$ are subsets of G, we denote by $[A, B]$ the subgroup of G generated by the elements $[a, b]$, for $a \in A$, $b \in B$. Then the *commutator subgroup* $G' = [G, G]$, or *derived subgroup*, is the subgroup of G generated by all the commutators $[g, h]$ of elements $g, h \in G$. Then one can iterate this procedure defining $G^{(2)} = (G')'$, and in general $G^{(n+1)} = (G^{(n)})'$, obtaining a descending chain of characteristic subgroups $G^{(n)}$. If $G^{(n)} = \{e_G\}$ for some $n \in \mathbb{N}_+$, then G is called *solvable*, and its *solvability class*, or *derived length*, is the least $n \in \mathbb{N}_+$ such that $G^{(n)} = G$. For example, a group is solvable of solvability class 1 exactly when $G' = \{e_G\}$, i.e. it is abelian. Note that G is solvable of solvability class 2 exactly when the commutator $G^{(2)}$ is trivial, i.e. G' is abelian; such groups are also called *meta-abelian*.

The commutator G' also has the following universal property: for every normal subgroup H of G, the quotient group G/H is abelian if and only if $G' \leq H$.

The *center* of a group G is the normal subgroup $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$. Let $Z_1(G) = Z(G)$. Consider the quotient group $G/Z(G)$, its center $Z\big(G/Z(G)\big)$, and its preimage $Z_2(G) \leq G$ under the canonical projection $\pi\colon G \to G/Z(G)$. Proceed by induction to define an ascending chain of characteristic subgroups $Z_n(G)$. A group G is called *nilpotent* if $Z_n(G) = G$ for some $n \in \mathbb{N}_+$, and its *nilpotency class* is the least $n \in \mathbb{N}_+$ such that $Z_n(G) = G$. For an $n \in \mathbb{N}_+$, we denote by $\mathscr{N}_n$ the class of nilpotent groups of nilpotency class n. Note that $G \in \mathscr{N}_1$ exactly when $Z(G) = G$, i.e. it is abelian. On the other hand, $G \in \mathscr{N}_2$ exactly when the quotient group $G/Z(G)$ is abelian and non-trivial, i.e. $G' \leq Z(G) \lneq G$. In this work, we will pay particular attention to groups belonging to $\mathscr{N}_2$.

It can be shown that a nilpotent group (of nilpotency class n) is also solvable (of solvability class at most n). So one can think of both the solvability class and (especially) the nilpotency class as a measure of the failure of G to be abelian.

Consider the characteristic subgroup $H_1(G)$ of G consisting of elements which have only finitely many conjugates in G, i.e. $H_1(G) = \{x \in G \mid [G : C_G(x)] < \omega\}$. If $G = H_1(G)$, then G is called *FC-group* (*F*inite-*C*onjugates classes). It can be easily seen that G is an FC-group if and only if $[G : C_G(F)] < \omega$ for every $F \in [G]^{<\omega}$. If $G \neq H_1(G)$, consider $H_1\big(G/H_1(G)\big)$, and its preimage $H_2(G) \leq G$ under the canonical projection $\pi\colon G \to G/H_1(G)$. Proceed by induction, to define an ascending chain of characteristic subgroups $H_n(G)$. Then G is called *FC-nilpotent* if $H_n(G) = G$ for some $n \in \mathbb{N}_+$.

As $Z_n(G) \leq H_n(G)$ for every $n \in \mathbb{N}_+$, it immediately follows that nilpotent groups are FC-nilpotent.

If $N \trianglelefteq G$, a subset $T \subseteq G$ is called a *transversal for N in G* if T intersects every coset of N at exactly one element.

A *torsion group*, or *periodic group*, is a group in which each element has finite order. All finite groups are torsion.

For an integer n, the *n-socle of* G is the subset $G[n] = \{g \in G \mid g^n = e_G\}$. If G is abelian, then $G[n]$ is a subgroup. For example, $G[1] = \{e_G\}$ and $G[0] = G$. Then G is said to be *almost torsion-free* , if $G[n]$ is finite for every $n > 0$.

The *exponent* $\exp(G)$ of a torsion group G is the least common multiple, if it exists, of the orders of the elements of G. In this case, the group is called *bounded*, and $\exp(G) > 0$. Otherwise, or if G is not even torsion, it will be called *unbounded*, and we conventionally define $\exp(G) = 0$. Any finite group has positive exponent: it is a divisor of $|G|$. Let G be an abelian group and $m > 1$ be an integer. Following Givens and Kunen [30], we say that G has essential exponent m (denoted by $\exp^*(G) = m$), if mG is finite, but dG is infinite for every proper divisor of m.

If $p \in \mathbb{P}$ is a prime number, an abelian group G is said to be an *elementary abelian p-group* if $\exp(G) = p$. Note that, if G is an abelian group such that $G[p]$ is not trivial, then $G[p]$ is an elementary abelian p-group. There exists a (unique) cardinal number, denoted by $r_p(G)$ and called *p-rank of* G, such that $G[p] \cong \oplus_{r_p(G)}\mathbb{Z}_p$.

For an abelian group G, recall that $\pi(G) = \{p \in \mathbb{P} \mid r_p(G) > 0\}$, and, if $p \in \mathbb{P}$, then it is defined the subgroup $G_p = \{g \in G \mid \exists n \in \mathbb{N}\ p^n g = 0\} = \bigcup_{n\in\mathbb{N}} G[p^n]$.

Let $\mathcal{G} = \{G_i \mid i \in I\}$ be a family of groups. Adopting terminology and notation from abelian group theory, we denote by $G = \prod_{i\in I} G_i$ the group having the cartesian product of $\mathcal{G}$ as underlying set, with componentwise defined operation, and we call G *direct product* of $\mathcal{G}$. For an element $g = (g_i)_{i\in I} \in G$, we denote by $\mathrm{supp}(g) = \{i \in I \mid g_i \neq e_{G_i}\} \subseteq I$ the set of indexes such that the correspondent coordinates of g are non-trivial.

The subgroup S of G consisting of the elements g such that $\mathrm{supp}(g)$ is finite will be called *direct sum* of $\mathcal{G}$, and denoted by $S = \bigoplus_{i\in I} G_i$.

Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ denote the *quaternion group with* 8 *elements*, given by the group presentation

$$Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1\rangle$$

where 1 is the identity element and -1 commutes with the other elements of the group. Then

$$Z(Q_8) = Q_8' = \{1, -1\} = \{g^2 \mid g \in Q_8\} = Q_8[2],$$

and in particular $Q_8 \in \mathscr{N}_2$.

The *dihedral group* D_8 is defined by the following presentation, with e denoting the identity element:

$$D_8 = \langle \rho, \sigma \mid \rho^4 = \sigma^2 = e,\ \sigma\rho\sigma^{-1} = \rho^{-1} \rangle.$$

Then $D_8 = \{e, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$, $D_8[2] = D_8 \setminus \{\rho, \rho^3\}$ and

$$Z(D_8) = D_8' = \{e, \rho^2\} = \{g^2 \mid g \in D_8\}.$$

From this, it follows that $D_8 \in \mathscr{N}_2$.

If A and B are subsets of a group G, we denote by $A \cdot B = AB = \{ab \mid a \in A, b \in B\} \subseteq G$. If $n \in \mathbb{Z}$, we sometimes denote by $A^n = \{a^n \mid a \in A\} \subseteq G$. This notation may be confused with the cartesian product $A^n = \prod_{i=1}^n A$, but will always be clear by the context. For example, we have noted above that $Q_8^2 = Z(Q_8)$ and $D_8^2 = Z(D_8)$.

1.2 Topological preliminaries

If X is a set, we denote by:

- $\iota_X = \{X, \emptyset\}$ the *indiscrete* topology on X;
- $\delta_X = \mathcal{P}(X)$ the *discrete* topology on X;
- $cof_X = \{X \setminus F \mid F \in [X]^{<\omega}\} \cup \{\emptyset\}$ the *cofinite* topology on X.

If (X, τ) is a topological space, and $Y \subseteq X$, we denote by $\overline{Y}^\tau$ the τ-closure of Y, i.e. the smallest τ-closed subset of X that contains Y.

Let us denote by τ^c the family of τ-closed subsets. A subfamily $\mathcal{B} \subseteq \tau^c$ is said to be:

- a base for τ^c, or a *base for τ-closed sets*, if every element of τ^c is an intersection of elements of $\mathcal{B}$;
- a subbase for τ^c, or a *subbase for τ-closed sets*, if $\mathcal{B}^\cup$ is a base for τ^c, i.e. if every element of τ^c is an intersection of a finite union of elements of $\mathcal{B}$.

In both cases, note that τ is the smallest topology such that $\mathcal{B} \subseteq \tau^c$. Obviously, a base for τ-closed sets is a subbase for τ-closed sets.

For example, $\mathcal{B} = \{\{x\} \mid x \in X\}$ is a subbase for cof_X-closed sets, while $\mathbb{E}_G$ is a subbase for $\mathfrak{Z}_G$-closed sets.

On the other hand, if X is a set, and $\mathcal{B} \subseteq \mathcal{P}(X)$, then $\mathcal{B}$ can be considered as the subbase of the closed sets a topology $\tau_{\mathcal{B}}$, such that $\mathcal{B}^{\cup}$ is a base for $\tau_{\mathcal{B}}$. That is, the family of intersections of elements of $\mathcal{B}^{\cup}$ is the family of the closed sets of the topology $\tau_{\mathcal{B}}$.

Let $\{(X_i, \tau_i) \mid i \in I\}$ be a family of topological spaces, and $\mathcal{B}_i \subseteq \mathcal{P}(X_i)$ be a subbase of the closed sets in X_i. Then $X = \prod_{i \in I} X_i$ can be a equipped with the *product topology* τ, denoted by $\prod_{i \in I} \tau_i$, having the family $\{B_i \times \prod_{i \neq j \in I} X_j \mid i \in I,\ B_i \in \mathcal{B}_i\}$ as a subbase for closed sets. Equivalently, τ is the initial topology of the family of canonical projections $\pi_i \colon X \to (X_i, \tau_i)$, for $i \in I$.

The topology cof_X can be generalized as follows: let λ be an infinite cardinal number. As the family $\mathcal{B} = [X]^{<\lambda} \cup \{X\}$ is stable under taking finite unions and arbitrary intersections, it is the family of closed sets of a topology on X, denoted by co-λ_X. Then $[X]^{<\lambda}$ is a subbase for co-λ_X-closed sets. For example, taking $\lambda = \omega$, we obtain the topology co-$\omega_X = cof_X$.

A function $f \colon X \to Y$ will be called τ, σ-continuous if $f \colon (X, \tau) \to (Y, \sigma)$ is continuous. If $(Y, \sigma) = (X, \tau)$, then f will just be called τ-continuous.

Definition 1.1. *A topological space X is:*

(a) Noetherian*, if X satisfies the descending chain condition on closed sets (or equivalently, if it satisfies the ascending chain condition on open sets);*

(b) irreducible*, if $X = F_1 \cup F_2$ for closed subsets F_1, F_2 of X always implies $X = F_1$ or $X = F_2$;*

(c) connected*, if $X = F_1 \cup F_2$ for closed* disjoint *subsets F_1, F_2 of X always implies $X = F_1$ or $X = F_2$.*

Obviously, an irreducible space is connected.

If $Y \subseteq X$, then Y is an irreducible (resp., connected) subset of X if the subspace $(Y, \tau \restriction_Y)$ is irreducible (resp., connected).

The continuous image of an irreducible (resp., connected) space is still irreducible (resp., connected).

If $x \in (X, \tau)$, let $\{C_i \mid i \in I\} \subseteq \mathcal{P}(X)$ be the family of connected subsets of X containing x. Then the *connected component of x in (X, τ)* is defined as $c(x, X, \tau) = \bigcup_{i \in I} C_i$. One can prove that $c(x, X, \tau)$ is a closed subset of X and that the family $\{c(x, X, \tau) \mid x \in X\}$ is a partition of X.

A topological space X is *totally disconnected* if $c(x, X, \tau) = \{x\}$ for every $x \in X$.

Remark 1.2. (a) Obviously, a Noetherian space is compact, and a subspace of a Noetherian space is Noetherian itself. Actually, a space is Noetherian if and only if all its subspaces are compact. Hence, an infinite Noetherian space is never Hausdorff.

(b) Given a topological space X and a natural number n, we write $\dim X \geq n$ if there exists a strictly increasing chain

$$F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_n \tag{1.1}$$

of non-empty irreducible closed subsets of X. The *combinatorial dimension* $\dim X$ of a space X is the smallest number $n \in \mathbb{N}$ satisfying $\dim X \leq n$, if such a number exists, or ∞ otherwise. Clearly, every Hausdorff space (as well as every indiscrete space) has combinatorial dimension 0.

The following useful technique for building Noetherian spaces was proposed by Bryant [11]:

Proposition 1.3 ([11])**.** *Let X be a set, and $X \in \mathcal{B} \subseteq \mathcal{P}(X)$ be such that:*

- *$\mathcal{B}$ is stable under taking finite intersections;*
- *$\mathcal{B}$ satisfies the descending chain condition.*

Let $\tau_{\mathcal{B}}$ be the topology having $\mathcal{B}$ as a subbase for $\tau_{\mathcal{B}}$-closed sets. Then $\tau_{\mathcal{B}}$ is Noetherian, and $\mathcal{B}^{\cup}$ is the family of closed sets of $\tau_{\mathcal{B}}$.

If A is a ring, then $\operatorname{char} A$ denotes its characteristic.

Let K be a field, and d be a positive integer. Consider the vector space K^d, and the ring $K[x_1, \ldots, x_d]$ of polynomials in d variables with coefficients in the field K. Recall that the family $\mathcal{B}$ of zero-sets of those polynomials satisfies $\mathcal{B} = \mathcal{B}^{\cup}$ and is a subbase of the closed sets of a Noetherian T_1 topology $\mathcal{A}_{K^d}$ on K^d, which we will call the *affine topology.* If X is a subset of K^d, the affine topology of X is defined as $\mathcal{A}_X = \mathcal{A}_{K^d} \restriction_X$. In particular, the linear group $\mathrm{GL}_n(K)$ and all its subgroups carry the topology induced by $\mathcal{A}_{K^{n^2}}$ (via the embedding in K^{n^2}). The following result is folklore.

Fact 1.4. *For every field K and for every positive integer d, the affine topology $\mathcal{A}_{K^d}$ is Noetherian. In particular, for every linear group G, the affine topology $\mathcal{A}_G$ is Noetherian.*

1.3 Quasi-topological groups

Definition 1.5. *Let G be a group, and τ a topology on G. The pair (G,τ) is called* quasi-topological group *provided that the functions*

$$\begin{array}{ccc}(G,\tau) & \to & (G,\tau) \\ x & \mapsto & x\cdot g\end{array}, \quad \begin{array}{ccc}(G,\tau) & \to & (G,\tau) \\ x & \mapsto & g\cdot x\end{array} \quad \text{and} \quad \begin{array}{ccc}(G,\tau) & \to & (G,\tau) \\ x & \mapsto & x^{-1}\end{array}$$

are continuous.

For example, for every infinite cardinal number λ, the space $(G, co\text{-}\lambda_G)$ is a T_1 quasi-topological group. In particular, (G, cof_G) is a T_1 Noetherian quasi-topological group. So if G is infinite, (G, cof_G) is not Hausdorff, hence not a topological group. On the other hand, $(G, co\text{-}\lambda_G)$ is not Noetherian when G is uncountable and $\omega < \lambda \leq |G|$.

In what follows we give some general results for quasi-topological groups. For a reference on this topic, see for example [2].

Theorem 1.6. *Let (G,τ) be a quasi-topological group.*

(a) If $S \subseteq G$, then the τ-closure of S is

$$\overline{S} = \bigcap_{U\in\mathcal{V}_\tau(e_G)} U\cdot S = \bigcap_{V\in\mathcal{V}_\tau(e_G)} S\cdot V.$$

(b) If H is a subgroup with non-empty interior, then H is open.

(c) A finite-index closed subgroup of G is open.

(d) the τ-closure of a (normal) subgroup is again a (normal) subgroup.

Proof. To prove (a), (b) and (c) one only needs the inversion and shifts to be continuous, so proceed as in the case of topological groups.

(d) Let H be a subgroup of G, and $\overline{H}$ be its τ-closure. We have to show that $\overline{H}$ is a subgroup, that is: $\overline{H}^{\,-1} \subseteq \overline{H}$ and $\overline{H}\cdot\overline{H} \subseteq \overline{H}$.

The hypothesis that the inversion function is τ-continuous guarantees that $\overline{H}^{\,-1} \subseteq \overline{H^{-1}} = \overline{H}$.

In the same way, for every $h \in H$, the (left) traslation in G by h is τ-continuous, so $h\cdot\overline{H} \subseteq \overline{h\cdot H} = \overline{H}$; as this holds for every $h\in H$, we get $H\cdot\overline{H}\subseteq\overline{H}$.

Now consider the (right) traslation in G by an element $c \in \overline{H}$: it's again continuous, so

$$\overline{H}\cdot c \subseteq \overline{H\cdot c} \subseteq \overline{H\cdot\overline{H}} \subseteq \overline{\overline{H}} = \overline{H}.$$

Considering the union over all $c\in\overline{H}$, we finally obtain $\overline{H}\cdot\overline{H}\subseteq\overline{H}$.

Composing translations we obtain that also conjugations are τ-continuous; so if H is a normal subgroup and $g\in G$, then

$$g\cdot\overline{H}\cdot g^{-1} \subseteq \overline{g\cdot H\cdot g^{-1}} = \overline{H}.$$

□

Let (G,τ) be a quasi-topological group, and N be a normal subgroup of G. Consider the quotient group $\overline{G} = G/N$ and the canonical map $\pi\colon (G,\tau) \to \overline{G}$. The quotient topology $\overline{\tau}$ of τ on $\overline{G}$ is final topology of π, namely $\overline{\tau} = \{A \subseteq \overline{G} \mid \pi^{-1}(A) \in \tau\}$. In the notation above, the following results hold.

Proposition 1.7. *If (G,τ) is a quasi-topological group, then $(\overline{G},\overline{\tau})$ is a quasi-topological group, and the map $\pi\colon (G,\tau) \to (\overline{G},\overline{\tau})$ is continuous and open. In particular, $\overline{\tau} = \{\pi(X) \subseteq \overline{G} \mid X \in \tau\}$.*

Proof. Proceed as in the case of topological groups to verify that $(\overline{G},\overline{\tau})$ is a quasi-topological group.

We prove that π is open. Let $A \subseteq G$, and note that $\pi(A) \in \overline{\tau}$ if and only if $\pi^{-1}\pi(A) \in \tau$. As

$$\pi^{-1}\pi(A) = A \cdot N = \bigcup_{n\in N} An,$$

and $(\overline{G},\overline{\tau})$ is a quasi-topological group, we are done. □

Proposition 1.8. *If (G,τ) is a quasi-topological group, then the following are equivalent.*

(1) N is τ-closed;

(2) $\{e_{\overline{G}}\}$ is $\overline{\tau}$-closed.

(3) $\overline{\tau}$ is a T_1 topology;

Proof. (1) implies (2). Let us prove that $A = \overline{G} \setminus \{e_{\overline{G}}\}$ is $\overline{\tau}$-open, and note that this holds if and only if $\pi^{-1}(A)$ is τ-open. Being N is τ-closed, we have that $\pi^{-1}(A) = G \setminus N$ is τ-open.

(2) implies (3) holds as $(\overline{G},\overline{\tau})$ is a quasi-topological group by Proposition 1.7.

(3) implies (1). If $\overline{\tau}$ is a T_1 topology, in particular $\{e_{\overline{G}}\}$ is $\overline{\tau}$-closed, so that $N = \pi^{-1}(\{e_{\overline{G}}\})$ is τ-closed, being π continuous. □

For a quasi-topological group (G,τ), we denote by $c(G) = c(e_G, G, \tau)$ the connected component of the identity element. The next result proves that it is a closed normal subgroup of G.

Proposition 1.9. *If (G,τ) is a quasi-topological group, then the connected component of the identity element $c(G) = c(e_G, G, \tau)$ is a closed normal subgroup of G.*

Proof. The connected component of a point is always a closed subset.

To prove that $c(G)\cdot c(G) \subseteq c(G)$, we show that $g\cdot c(G) \subseteq c(G)$ for every $g \in c(G)$. So let $g \in c(G)$. As both $c(G)$ and $g \cdot c(G)$ are connected subsets of G, containing g, also their union is connected, and obviously still contains e_G. By maximality of $c(G)$, we get $g \cdot c(G) \subseteq c(G)$.

Now we prove that $c(G)^{-1} \subseteq c(G)$. As $e_G \in c(G)^{-1} \cap c(G)$, the same argument used above gives that $c(G)^{-1} \cup c(G)$ is a connected subset, containing $c(G)$, hence coincide with $c(G)$.

So $c(G)$ is a subgroup of G. To prove that is normal, let $x \in G$. Again, $S = xc(G)x^{-1}$ is a connected subset of G, containing e_G, hence $S \subseteq c(G)$. □

The two following results are proved as in the case of topological groups.

Proposition 1.10. *Let (G, τ) be a quasi-topological group, and N be a normal subgroup of G. If both N with the induced topology, and G/N with the quotient topology are connected, then also G is connected.*

As a consequence, we obtain the following.

Corollary 1.11. *If (G, τ) is a quasi-topological group, then the quotient space $G/c(G)$ is totally disconnected.*

1.3.1 Topological groups

Definition 1.12. *A topological group (G, τ) is called* totally bounded *if for every non-empty open subset A of G there exists $F \in [G]^{<\omega}$ such that $G = F \cdot A$.*

Note that the above definition is only apparently asymmetric, as one can easily see that (G, τ) is totally bounded if and only if for every non-empty open subset A of G there exists $F \in [G]^{<\omega}$ such that $G = A \cdot F$.

We conclude this part giving the definition and a few properties of the *Taĭmanov topology* of a group.

Definition 1.13. *The* Taĭmanov topology $\mathcal{T}_G$ *on a group G is the topology having the family of the centralizers of the elements of G as a subbase of the filter of the neighborhoods of e_G.*

Then $\mathcal{T}_G$ is a group topology, and for every element $g \in G$ the subgroup $C_G(g)$ is a $\mathcal{T}_G$-open (hence, closed) subset of G. In particular, $\overline{\{e_G\}}^{\mathcal{T}_G} = Z(G)$.

Lemma 1.14. *If G is a group, then the following hold for $\mathcal{T}_G$.*

1. *$\mathcal{T}_G$ is Hausdorff if and only if G is center-free.*

2. *$\mathcal{T}_G$ is totally bounded if and only if G is an FC-group.*

Proof. 1. It immediately follows from the above osservation that $\overline{\{e_G\}}^{\mathcal{T}_G} = Z(G)$.

2. As the family $C_G(F)$, for $F \in [G]^{<\omega}$, is a local subbase of the filter of the neighborhoods of e_G, note that $\mathcal{T}_G$ is totally bounded if and only if each of them has finite index in G. And this is the definition of an FC-group.

□

In particular, from Lemma 1.14, item 1, it follows that $\mathcal{T}_G$ is not a T_1 topology in general. For this reason, we introduce here its T_1-refinement.

Definition 1.15. *The* T_1 Taĭmanov topology *is the supremum (in the lattice of all topologies on* G*)* $\mathcal{T}'_G = \mathcal{T}_G \vee cof_G$.

2

The group of words, verbal functions and elementary algebraic subsets

2.1 The group of words $G[x]$

2.1.1 The categorical aspect of $G[x]$

In the following fact we briefly recall the categorical definition of the free product of two groups.

Fact 2.1. *Let G, H be groups. Then there exist a unique (up to isomorphism) group $G * H$, together with two injective group homomorphism $i_G \colon G \to G * H$, $i_H \colon H \to G * H$ satisfying the following universal property:*
for every group Γ, for every group homomorphisms $f_G \colon G \to \Gamma$, $f_H \colon H \to \Gamma$, there exists a unique group homomorphism $f \colon G*H \to \Gamma$ such that $f \circ i_G = f_G$, $f \circ i_H = f_H$.

$$\begin{tikzcd}
G \arrow[dr, hook, "i_G"] \arrow[ddr, bend right, "f_G"'] & & H \arrow[dl, hook', "i_H"'] \arrow[ddl, bend left, "f_H"] \\
& G * H \arrow[d, dashed, "f"'] & \\
& \Gamma &
\end{tikzcd}$$

In the notation of Fact 2.1, taking the infinite cyclic group $H = \langle x \rangle$, given a group G we obtain a categorical description of $G[x] = G * \langle x \rangle$ (see Definition P.2) in the following corollary. The injective homomorphism $i_G \colon G \to G[x]$, is the map $G \ni g \mapsto g \in G[x]$. Then, $G[x]$ is determined by the universal property stated below.

Corollary 2.2. *Let G be a group. Then $G[x]$ is the unique (up to isomorphism) group satisfying the following universal property:*
for every group Γ, for every group homomorphism $\phi \colon G \to \Gamma$, and for every $\gamma \in \Gamma$, there exists a unique group homomorphism $\widetilde{\phi} \colon G[x] \to \Gamma$ such that $\widetilde{\phi} \restriction_G = \phi$, $\widetilde{\phi}(x) =$

γ.

$$\begin{tikzcd} & G[x] \arrow[d, "\tilde{\phi}"] \ni x \arrow[d, mapsto] \\ G \arrow[ur, hook, "i_G"] \arrow[r, "\phi"] & \Gamma \ni \gamma \end{tikzcd}$$

In the following example we illustrate a few particular cases when Corollary 2.2 can be applied.

Example 2.3. 1. Consider the identity map $\mathrm{id}_G\colon G \to G$. By Corollary 2.2, for every $g \in G$ there exists a unique map $\mathrm{ev}_g\colon G[x] \to G$, with $\mathrm{ev}_g \restriction_G = \mathrm{id}_G$ and $\mathrm{ev}_g(x) = g$, that we call *evaluation map*. Then we define $w(g) = \mathrm{ev}_g(w(x))$.

$$\begin{tikzcd} & G[x] \arrow[d, "\mathrm{ev}_g"] \ni x \arrow[d, mapsto] \\ G \arrow[ur] \arrow[r, "\mathrm{id}_G"] & G \ni g \end{tikzcd}$$

2. A *G-endomorphism of $G[x]$* is a group homomorphism $f\colon G[x] \to G[x]$ such that $f \restriction_G = \mathrm{id}_G$, i.e. $f \circ i_G = i_G$, and the following diagram commutes:

$$\begin{tikzcd} & G[x] \arrow[d, "f"] \\ G \arrow[ur, "i_G"] \arrow[r, "i_G"] & G[x] \end{tikzcd}$$

Then f is uniquely determined by the element $w = f(x) \in G[x]$, and now we show that every choice of $w \in G[x]$ can be made, thus classifying the G-endomorphisms of $G[x]$. To this end, consider the map $i_G\colon G \to G[x]$. By Corollary 2.2, for every $w \in G[x]$ there exists a unique G-endomorphism $\xi_w\colon G[x] \to G[x]$, with $x \mapsto w$.

$$\begin{tikzcd} & G[x] \arrow[d, "\xi_w"] \ni x \arrow[d, mapsto] \\ G \arrow[ur] \arrow[r, "i_G"] & G[x] \ni w \end{tikzcd}$$

Proposition 2.4. *Let $f\colon G_1 \to G_2$ be a group homomorphism. Then there exists a unique group homomorphism $F\colon G_1[x] \to G_2[x]$ such that $F \restriction_{G_1} = f$, $F(x) = x$. In particular, if f is surjective (resp., injective), then F is surjective (resp., injective).*

Moreover, the following hold:

1. *if $H \leq G$ is a subgroup of G, then $H[x] \leq G[x]$;*

2. *if $H \trianglelefteq G$ is a normal subgroup of G, and $\overline{G} = G/H$, then $\overline{G}[x]$ is a quotient of $G[x]$.*

Proof. Composing $f\colon G_1 \to G_2$ and the map $i_{G_2}\colon G_2 \to G_2[x]$, we obtain $\widetilde{f} = i_{G_2} \circ f\colon G_1 \to G_2[x]$. Then apply Corollary 2.2 and use the universal property of $G_1[x]$ to get $F\colon G_1[x] \to G_2[x]$ such that $F(x) = x$ and $F\restriction_{G_1} = \widetilde{f}$.

If f is surjective, then F is surjective too, as $F(G_1[x])$ contains both x and $f(G_1)$.

In Remark 2.16, item 1, we will explicitly describe the map F, so that by (2.3) it will immediately follows that F is injective when f is injective.

1. In this case, the injection $f\colon H \hookrightarrow G$ gives the injection $F\colon H[x] \to G[x]$.

2. The canonical projection $f\colon G \to \overline{G}$ gives $F\colon G[x] \to \overline{G}[x]$, and F is surjective.

□

The following corollary immediately follows by Proposition 2.4.

Corollary 2.5. *The assignment $G \mapsto G[x]$, and the canonical embedding $G \xrightarrow{i_G} G[x]$, define a pointed endofunctor $\varpi\colon \mathbf{Gr} \to \mathbf{Gr}$ in the category of groups and group homomorphism. In other words, for every group homomorphism $\phi\colon G \to H$, the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "i_G"] \arrow[d, "\phi"'] & G[x] \arrow[d, "\varpi(\phi)"] \\ H \arrow[r, "i_H"] & H[x]. \end{tikzcd}$$

2.1.2 The concrete form of $G[x]$

Here we recall the concrete definition of $G[x]$ in terms of products of the form (2.1) below that will be called *words*. In particular, if $g \in G$, then $w = g \in G[x]$ will be called *constant word*, and we define its *lenght* to be $\mathrm{l}(w) = 0 \in \mathbb{N}$. In the general case, for $w \in G[x]$ there exist $n \in \mathbb{N}_+$, elements $g_1, \dots, g_n, g_{n+1} \in G$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$, such that

$$w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_{n+1}. \tag{2.1}$$

If, $g_i \neq e_G$ whenever $\varepsilon_{i-1} = -\varepsilon_i$ for $i = 2, \dots, n$, we say that w is a *reduced word* in the free product $G[x] = G * \langle x \rangle$ and we define lenght of w by $\mathrm{l}(w) = n$, where $n \in \mathbb{N}_+$ is the least natural such that w is as in (2.1).

Definition 2.6. If $w \in G[x]$ is as in (2.1), we define the following notions.

- The *constant term of* w is $\mathrm{ct}(w) = w(e_G) = g_1 g_2 \cdots g_n g_{n+1} \in G$;

- The *content* of w is $\epsilon(w) = \sum_{j=1}^{n} \varepsilon_j \in \mathbb{Z}$, which will also be denoted simply by ϵ when no confusion is possible.

If $w = g$, then we define $\epsilon(w) = 0$, $\mathrm{ct}(w) = w(e_G) = g$. We call *singular* a word w such that $\epsilon(w) = 0$. Note that by definition, all constant words are singular.

Below, we state a few easy equalities that will be used in the sequel (see also §3.1).

Remark 2.7. 1. If $\varepsilon_2 = -\varepsilon_1$, then $g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} = g_1(g_2 g_2^{-1}) x^{\varepsilon_1} g_2 x^{-\varepsilon_1} = g_1 g_2 [g_2^{-1}, x^{\varepsilon_1}]$.

2. If $\varepsilon_2 = \varepsilon_1$, then $g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} = g_1(g_2^{-1} g_2) x^{\varepsilon_1} g_2 x^{\varepsilon_1} = g_1 g_2^{-1} (g_2 x^{\varepsilon_1})^2$.

3. Note that $g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} = g_1 g_2 x^{\varepsilon_1} [x^{-\varepsilon_1}, g_2^{-1}] x^{\varepsilon_2}$.

Both the functions $\mathrm{ct}\colon G[x] \to G$, mapping $w \mapsto \mathrm{ct}(w)$, and $\epsilon\colon G[x] \to \mathbb{Z}$, mapping $w \mapsto \epsilon(w)$, are surjective group homomorphisms. In particular, $\mathrm{ct}(G[x]') \leq G'$ and $\epsilon(G[x]') \leq \mathbb{Z}' = \{0\}$, so that $G[x]' \leq \mathrm{ct}^{-1}(G') \cap \ker(\epsilon)$. In the following theorem, we prove the reverse inclusion.

Theorem 2.8. *For every group G, $G[x]' = \mathrm{ct}^{-1}(G') \cap \ker(\epsilon)$.*

Proof. Let $U = \mathrm{ct}^{-1}(G') \cap \ker(\epsilon) = \{w \in G[x] \mid \mathrm{ct}(w) \in G', \epsilon(w) = 0\}$. We have already noted above that $G[x]' \subseteq U$, and we prove the other inclusion by induction on $\mathrm{l}(w)$ for a word $w \in U$.

Let $w \in G[x]$ and assume $w \in U$. We first consider the case when $\mathrm{l}(w) = 0$, i.e. $w = \mathrm{ct}(w)$ is a constant word, so that $w \in G' \leq G[x]'$ as desired. So now let $w \in U$ be as in (2.1), and note that $n = \mathrm{l}(w) > 0$ is even, so that for the base case we have to consider $n = 2$, for which $w = g_1 x^{\varepsilon} g_2 x^{-\varepsilon} (g_1 g_2)^{-1} c$, with $c = \mathrm{ct}(w) \in G'$. Let $g = g_1 g_2$, and $w_0 = [g_2^{-1}, x^{\varepsilon}] \in G[x]'$, so that by Remark 2.7, item 1, we have $w = g w_0 g^{-1} c = [g, w_0] w_0 c \in G[x]'$.

Now assume $n > 2$. As $\epsilon(w) = 0$, we have $\varepsilon_{i+1} = -\varepsilon_i$ for some $1 \leq i \leq n-1$. Then $w = w_1 w_2 w_3$ for the words

$$\begin{aligned} w_1 &= g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_{i-1} x^{\varepsilon_{i-1}}, \\ w_2 &= g_i x^{\varepsilon_i} g_{i+1} x^{\varepsilon_{i+1}} (g_i g_{i+1})^{-1}, \\ w_3 &= (g_i g_{i+1}) g_{i+2} x^{\varepsilon_{i+2}} \cdots g_n x^{\varepsilon_n} g_{n+1}. \end{aligned}$$

Note that $w = [w_1, w_2] w_2 w_1 w_3$, and that $w_2 \in G[x]'$ for the base case, so that we only have to show that $w_1 w_3 \in G[x]'$. As

$$\mathrm{ct}(w) = \mathrm{ct}(w_1)\,\mathrm{ct}(w_2)\,\mathrm{ct}(w_3) = \mathrm{ct}(w_1) e_G\,\mathrm{ct}(w_3) = \mathrm{ct}(w_1 w_3),$$

we have $\mathrm{ct}(w_1 w_3) \in G'$, and similarly $\epsilon(w_1 w_3) = 0$. Then $w_1 w_3 \in G[x]'$ by the inductive hypothesis. □

Definition 2.9. Let $w \in G[x]$ be as in (2.1). Then w is called *positively homogeneous* (resp., *negatively homogeneous*) if $\varepsilon_i = 1$ (resp., $\varepsilon_i = -1$) for every $i = 1, \dots, n$. We call *homogeneous* a word that is either positively or negatively homogeneous.

Note that $w \in G[x]$ is positively homogeneous if and only if $w^{-1} \in G[x]$ is negatively homogeneous. For example, if $n \in \mathbb{N}_+$, then the words x^n and x^{-n} are homogeneous. If w is positively (resp., negatively) homogeneous, then $\epsilon(w) = \mathrm{l}(w)$ (resp., $\epsilon(w) = -\mathrm{l}(w)$).

If $n \in \mathbb{N}_+$, let

$$W_{hom,n} = \{g_1 x g_2 x \cdots g_n x g_{n+1} \mid g_1, g_2, \dots, g_n, g_{n+1} \in G\} \subseteq G[x] \tag{2.2}$$

be the family of homogeneous words $w \in G[x]$ with $\epsilon(w) = n$.

Below, we state a few easy general equalities that will be used in the sequel (see also §3.1).

Remark 2.10. Let $w = b_1 x b_2 x \cdots b_s x \in G[x]$ be a positively homogeneous word. Then $w = \overline{w}$ for a word $\overline{w} \in G[x]$ such that:

1. if $s = \epsilon(w)$ is even, then $\overline{w} = b_1'(b_2 x)^2 b_3'(b_4 x)^2 \cdots b_{s-1}'(b_s x)^2$;
2. if $s = \epsilon(w)$ is odd, then $\overline{w} = b_1'(b_2 x)^2 b_3'(b_4 x)^2 \cdots b_{s-2}'(b_{s-1} x)^2 b_s x$.

In fact, just define $b_{2i+1}' = b_{2i+1} b_{2i+2}^{-1}$ for every integer $0 \leq i \leq s/2$.

Definition 2.11. Let $w \in G[x]$. If w is as in (2.1), then we define the set of *coefficients of* w as $\mathrm{coeff}(w) = \{g_1, g_2, \dots, g_n, g_{n+1}\} \subseteq G$. If $w = g$ is constant, we let $\mathrm{coeff}(w) = \{g\}$.

Then we define the following notions.

- $C_w = C_G(\mathrm{coeff}(w)) \leq G$.
- $N_w = [\mathrm{coeff}(w), G] \trianglelefteq G$.

The next lemma is straightforward.

Lemma 2.12. *For $w \in G[x]$, one has $N_w \leq G'$ and the following are equivalent:*

- *$C_w = G$;*
- *$\mathrm{coeff}(w) \subseteq Z(G)$;*
- *$N_w = \{e_G\}$.*

See Corollary 2.27 for further properties on the case when the conditions in the above lemma hold.

Finally, we give the following definition.

Definition 2.13. If $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_{n+1} \in G[x]$ is a word in G, we define $\epsilon_i(w) = \sum_{j=1}^{i} \varepsilon_j$ for integers $i = 1, \dots, n$.

If w is a word as in Definition 2.13, then obviously $\epsilon_n(w) = \epsilon(w)$, as defined in Definition 2.6.

2.2 Verbal functions

2.2.1 Definition and examples

Definition 2.14. A word $w \in G[x]$ determines the associated evaluation function $f_w^G \colon G \to G$. We will often write f_w for f_w^G. We call *verbal function* of G a function $G \to G$ of the form f_w, and we will denote by $\mathscr{F}(G)$ the set of verbal functions on G.

If $w \in G[x]$ and $g \in G$, sometimes we will also write $w(g)$ for the element $f_w(g)$. So *a priori*, if f is a verbal function, then $f = f_w$ for a word $w \in G[x]$ as in (2.1).

Moreover, $f \in \mathscr{F}(G)$ is called *homogeneous* (resp., *positively*, *negatively*) *verbal function* if $f = f_w$ for a homogeneous (resp., positively, negatively) word $w \in G[x]$.

Note that $f_w \colon G \to G$ is the only map such that $f_w \circ \mathrm{ev}_g = \mathrm{ev}_g \circ \xi_w$ for every $g \in G$, i.e. making the following diagram commute:

$$\begin{tikzcd} G[x] \arrow[r, "\xi_w"] \arrow[d, "\mathrm{ev}_g"] & G[x] \arrow[d, "\mathrm{ev}_g"] \\ G \arrow[r, "f_w"] & G. \end{tikzcd}$$

Obviously, $f = f_w \in \mathscr{F}(G)$ is positively homogeneous if and only if $f^{-1} = f_{w^{-1}} \in \mathscr{F}(G)$ is negatively homogeneous.

Example 2.15. Some very natural functions $G \to G$ are verbal. Those one presented in items 2–5 are also homogeneous:

1. The constant functions.
2. The identity map of G is the function $f_x \colon g \mapsto g$.
3. The inversion function of G is $f_{x^{-1}} \colon g \mapsto g^{-1}$.
4. More generally, for every integer $n \in \mathbb{Z}$, the word $x^n \in G[x]$ determines the verbal function $f_{x^n} \colon g \mapsto g^n$.
5. The left translation in G by an element $a \in G$ is the function $f_{ax} \colon g \mapsto ag$, and the right translation is the function $f_{xa} \colon g \mapsto ga$.
6. For an element $a \in G$, the word $w = axa^{-1}$ determines the conjugation by a, as $f_w \colon g \mapsto aga^{-1}$.
7. If $\varepsilon \in \{\pm 1\}$, and $a \in G$, the word $w = [a, x^\varepsilon] = ax^\varepsilon a^{-1} x^{-\varepsilon} \in G[x]$ determines the verbal function $f_w \colon g \mapsto [a, g^\varepsilon]$. We will call *commutator verbal function* a function of this form.

Remark 2.16. 1. Let $\phi\colon G_1 \to G_2$ be a group homomorphism, and

$$F = \varpi(\phi)\colon G_1[x] \to G_2[x]$$

be as in Proposition 2.4. If $w \in G[x]$ is as in (2.1), then

$$F\colon w \mapsto F(w) = \phi(g_1)x^{\varepsilon_1}\phi(g_2)x^{\varepsilon_2}\cdots\phi(g_n)x^{\varepsilon_n}\phi(g_{n+1}). \tag{2.3}$$

By (2.3), it immediately follows that F is injective when f is injective.

Moreover, one can easily see that $\phi \circ f_w = f_{F(w)} \circ \phi$, i.e. the following diagram commutes:

$$\begin{tikzcd} G_1 \arrow[r, "f_w"] \arrow[d, "\phi"'] & G_1 \arrow[d, "\phi"] \\ G_2 \arrow[r, "f_{F(w)}"] & G_2 \end{tikzcd} \tag{2.4}$$

2. In particular, we will often consider the case when ϕ is the canonical projection $\pi\colon G \to G/N$, if N is a normal subgroup of G. In this case, let $\overline{G} = G/N$ be the quotient group, and for an element $g \in G$, denote $\overline{g} = \pi(g) \in \overline{G}$. If $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n} \in G[x]$, denote also $\overline{w} = \overline{g}_1x^{\varepsilon_1}\overline{g}_2x^{\varepsilon_2}\cdots\overline{g}_nx^{\varepsilon_n} \in \overline{G}[x]$. Then (2.4) (with $\phi = \pi$) gives $\pi \circ f_w = f_{\overline{w}} \circ \pi$.

2.2.2 Universal words

The group operation on $G[x]$ induces a group operation on $\mathscr{F}(G)$ as follows. If $w_1, w_2 \in G[x]$, let $w = w_1w_2 \in G[x]$ be their product, and consider the verbal functions $f_{w_1}, f_{w_2}, f_w \in \mathscr{F}(G)$. Obviously, f_w is the pointwise product $f_{w_1} \cdot f_{w_2}$ of f_{w_1} and f_{w_2}, namely the map $f_w\colon g \mapsto f_{w_1}(g)f_{w_2}(g) = f_w(g)$. With this operation, $(\mathscr{F}(G), \cdot)$ is a group, with identity element the constant function $e_{\mathscr{F}(G)}\colon g \mapsto e_G$ for every $g \in G$. If w^{-1} is the inverse of $w \in G[x]$, then the inverse of $f_w \in \mathscr{F}(G)$ is $f_{w^{-1}}$, and will be denoted $(f_w)^{-1}$.

For $S \subseteq G$,

$$(f_w)^{-1}(S) = \{(f_w)^{-1}(s) \mid s \in S\} = \{f_{w^{-1}}(s) \mid s \in S\}$$

will denote the *image* of S under $(f_w)^{-1} = f_{w^{-1}}$, while $f_w^{-1}(S) = \{g \in G \mid f_w(g) \in S\}$ will denote the preimage of S under f_w.

Consider the surjective group homomorphism $\Phi_G\colon G[x] \to \mathscr{F}(G)$, $w \mapsto f_w$. Then $\mathscr{F}(G) \cong G[x]/\mathcal{U}_G$, where $\mathcal{U}_G$ is the kernel

$$\mathcal{U}_G = \ker(\Phi_G) = \{w \in G[x] \mid \forall g \in G \quad f_w(g) = e_G\}. \tag{2.5}$$

Definition 2.17. *If G is a group, and $w \in G[x]$, we will say that w is a* universal word for G *if $w \in \mathcal{U}_G$.*

Note that a word $w \in G[x]$ is universal exactly when $E_w = G$.

Example 2.18. 1. If $w \in \mathcal{U}_G$, then obviously $\mathrm{ct}(w) = f_w(e_G) = e_G$.

2. If G has $k = \exp(G) > 0$, then $w = x^k \in G[x]$ is a non-singular universal word for G, i.e. $f_w \equiv e_G$ is the constant function.

The singular universal words will play a prominent role, so we set

$$\mathcal{U}_G^{sing} = \{w \in \mathcal{U}_G : \epsilon(w) = 0\}.$$

Obviously, $\mathcal{U}_G^{sing} = \mathcal{U}_G \cap \ker \epsilon$, so $\mathcal{U}_G^{sing}$ is a normal subgroup of $G[x]$ and $\mathcal{U}_G/\mathcal{U}_G^{sing}$, being isomorphic to a subgroup of the cyclic group $G[x]/\ker \epsilon \cong \mathbb{Z}$ is cyclic as well. Let $\mathrm{u}(G) \in \mathbb{N}$ be the generator of the cyclic subgroup of $\mathbb{Z}$, corresponding to $\mathcal{U}_G/\mathcal{U}_G^{sing}$ under this isomorphism. In other words,

$$\epsilon(\mathcal{U}_G) = \{\epsilon(w) : w \in \mathcal{U}_G\} = \mathrm{u}(G)\mathbb{Z}.$$

Definition 2.19. *Given a group G, the natural* $\mathrm{u}(G)$ *is called the* universal exponent *of G (*u*-exponent, for short).*

As $\mathcal{U}_G/\mathcal{U}_G^{sing} \cong \mathcal{U}_G \ker \epsilon / \ker \epsilon$, one can easily deduce that

$$G[x]/\mathcal{U}_G \ker \epsilon \cong \epsilon(G[x])/\epsilon(\mathcal{U}_G) \cong \mathbb{Z}/\, \mathrm{u}(G)\mathbb{Z}.$$

Hence, either $\mathrm{u}(G) = 0$, i.e. $\mathcal{U}_G = \mathcal{U}_G^{sing}$, or $\mathrm{u}(G) > 0$ and in this case $\mathrm{u}(G) = [G[x] : \mathcal{U}_G \ker \epsilon]$.

Lemma 2.20. *If G is a bounded group, then* $\mathrm{u}(G) > 0$ *and* $\mathrm{u}(G) \mid \exp(G)$.

Proof. Let $\exp(G) = n$. Then $n > 0$ and obviously $x^n \in \mathcal{U}_G$, so that $n \in \mathrm{u}(G)\mathbb{Z}$. □

In particolar, note that if G is the trivial group, then $\exp(G) = 1$, so that $\mathrm{u}(G) = 1$, as in fact $x \in \mathcal{U}_G$.

We will see in Lemma 2.32 that actually $\mathrm{u}(G) = \exp(G)$ for an *abelian* group G. This equality does not hold in general even for finite groups, see Example 2.37, item 1, where we will consider the case of $G = S_3$, for which $\mathrm{u}(S_3) = 2$, while $\exp(S_3) = 6$.

Definition 2.21. Define $\mathcal{W}_0$ as the class of all groups G having $\mathrm{u}(G) = 0$. For $n > 0$ let

$$\mathcal{W}_n = \{G : n \mid \mathrm{u}(G)\} = \{G : n \mid \epsilon(w) \text{ for every } w \in \mathcal{U}_G\}.$$

Obviously, every group is in $\mathcal{W}_1$. Note that $G \in \mathcal{W}_0$ if and only if $\epsilon(\mathcal{U}_G) = \{0\}$, i.e. every universal word is singular; equivalently, G has no non-singular universal words.

It immediately follows by the definitions that $\mathcal{W}_0 \subseteq \mathcal{W}_n$ for every $n \in \mathbb{N}$, and that $\mathcal{W}_m \subseteq \mathcal{W}_n$ whenever $n \mid m$. In particular, if $n \in \mathbb{N}_+$, then

$$\mathcal{W}_n = \bigcap\{\mathcal{W}_d \mid 1 \leq d \leq n,\ d \mid n\}.$$

Note that for example also the following holds:

$$\mathcal{W}_1 \supseteq \mathcal{W}_2 \supseteq \ldots \supseteq \mathcal{W}_{n!} \supseteq \ldots \supseteq \mathcal{W}_0.$$

In the first item of the following lemma we prove that $\mathcal{W}_0$ is the intersection of any infinite family of classes $\mathcal{W}_n$.

Lemma 2.22. *(a) Let $N \subseteq \mathbb{N}$ be an infinite subset of $\mathbb{N}$. Then*

$$\mathcal{W}_0 = \bigcap_{n \in N} \mathcal{W}_n.$$

(b) If $n_1, \ldots, n_k \in \mathbb{N}_+$, and n is the least common multiple of these numbers, then

$$\mathcal{W}_n = \bigcap\{\mathcal{W}_{n_i} \mid 1 \leq i \leq k\}. \tag{2.6}$$

Proof. (a). Let $G \in \bigcap_{n \in N} \mathcal{W}_n$, and let $w \in \mathcal{U}_G$. Then, $\epsilon(w) \in n\mathbb{Z}$ for every $n \in N$, and being N infinite, we get $\epsilon(w) = 0$.

(b). For every $i = 1, \ldots, k$ we have $n_i \mid n$, so that $\mathcal{W}_n \subseteq \mathcal{W}_{n_i}$ and the inclusion $\mathcal{W}_n \subseteq \bigcap\{\mathcal{W}_{n_i} \mid 1 \leq i \leq k\}$ is obvious. For the converse, let $G \in \bigcap_{i=1}^k \mathcal{W}_{n_i}$, and let $w \in \mathcal{U}_G$. Then, $n_i \mid \epsilon(w)$ for every $i = 1, \ldots, k$, so that $n \mid \epsilon(w)$. □

Lemma 2.23. *Let $\mathrm{u}(G) = m$. Then the following hold.*

1. *There exists $w_0 \in \mathcal{U}_G$ with $\epsilon(w_0) = m$.*
2. *$G \in \mathcal{W}_m$.*
3. *On the other hand, if $m \in \mathbb{N}$ satisfies condition 1 and 2, then $m = \mathrm{u}(G)$.*
4. *If $k \in \mathbb{N}$, and $G \in \mathcal{W}_k$, then either $k = m = 0$ or $0 \neq k \mid m$ (so that $\mathcal{W}_m \subseteq \mathcal{W}_k$). In particular,*

$$\mathcal{W}_m = \bigcap\{\mathcal{W}_k \mid k \in \mathbb{N},\ G \in \mathcal{W}_k\}. \tag{2.7}$$

Proof.

1. Just note that $m \in m\mathbb{Z} = \epsilon(\mathcal{U}_G)$.
2. Immediately follows by the definition of $\mathcal{W}_m$.
3. If $m = 0$, then $G \in \mathcal{W}_0$ implies $\mathrm{u}(G) = 0$.

   So let $m \in \mathbb{N}_+$ be such that $G \in \mathcal{W}_m$ and there exists $\widetilde{w} \in \mathcal{U}_G$ with $\epsilon(\widetilde{w}) = m$. In particular, $m = \epsilon(\widetilde{w}) \in \epsilon(\mathcal{U}_G) = \mathrm{u}(G)\mathbb{Z}$, so that $\mathrm{u}(G) \neq 0$ and $\mathrm{u}(G) \mid m$. As $G \in \mathcal{W}_m$, we have $m \mid \mathrm{u}(G)$. Being $m, \mathrm{u}(G) \in \mathbb{N}_+$, they coincide.

4. If $G \in \mathcal{W}_k$, then $\epsilon(\mathcal{U}_G) = m\mathbb{Z} \le k\mathbb{Z}$. From this, it follows the inclusions $\mathcal{W}_m \subseteq \mathcal{W}_k$ and $\mathcal{W}_m \subseteq \bigcap_{\substack{k\in\mathbb{N}\\ G\in\mathcal{W}_k}} \mathcal{W}_k$. The reverse inclusion of (2.7) follows by item 2.

□

By equation (2.7), it follows that $\mathcal{W}_{\mathrm{u}(G)}$ is the smallest among the classes $\mathcal{W}_n$ containing G. Equivalently, either $\mathrm{u}(G) = 0$, and $G \in \mathcal{W}_n$ for every $n \in \mathbb{N}$, or $\mathrm{u}(G) \neq 0$ is the greatest $n \in \mathbb{N}$ such that $G \in \mathcal{W}_n$. In particular, by Lemma 2.22 (a) and Lemma 2.23, item 4, it follows that

$$\mathrm{u}(G) = \begin{cases} 0 & \text{if } G \in \mathcal{W}_0, \\ \text{the least common multiple of } \{k \in \mathbb{N} \mid G \in \mathcal{W}_k\} & \text{otherwise.} \end{cases}$$

In the following lemma we will use the notation introduced in Remark 2.16, item 2.

Lemma 2.24. *Let $\phi\colon G \to H$ be a surjective group homomorphism, and $F = \varpi(\phi)\colon G[x] \to H[x]$ be as in Proposition 2.4. If $w \in \mathcal{U}_G$, then $F(w) \in \mathcal{U}_H$.*

In particular, if $n \in \mathbb{N}$ and $H \in \mathcal{W}_n$, then $G \in \mathcal{W}_n$.

As a consequence, $\mathrm{u}(H) \mid \mathrm{u}(G)$.

Proof. If $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_{n+1}$, then $F(w) = \phi(g_1)x^{\varepsilon_1}\phi(g_2)x^{\varepsilon_2}\cdots\phi(g_n)x^{\varepsilon_n}\phi(g_{n+1})$ by equation (2.3). Now, if $h = \phi(g) \in H$, we have $F(w)(h) = \phi(w(g)) = \phi(e_G) = e_H$, so that $F(w) \in \mathcal{U}_H$.

If $H \in \mathcal{W}_n$, and $w \in \mathcal{U}_G$, then $\epsilon(w) = \epsilon(F(w)) \in n\mathbb{Z}$. This proves that $G \in \mathcal{W}_n$.

Now, from $H \in \mathcal{W}_{\mathrm{u}(H)}$, we deduce that $G \in \mathcal{W}_{\mathrm{u}(H)}$ as well. This proves that $\mathrm{u}(H) \mid \mathrm{u}(G)$, according to Lemma 2.23, item 4. □

Remark 2.25. Let $[G, x] = \langle [g, x] \mid g \in G\rangle$ be the subgroup of $G[x]$ generated by all commutators $[g, x]$, $g \in G$. It is easy to see that $[G, x]$ is a normal subgroup of $G[x]$, being the kernel of the natural surjective homomorphism $G[x] \to G \times \langle x\rangle$ mapping $w \mapsto (\mathrm{ct}(w), x^{\epsilon(w)})$. In particular,

$$[G, x] = \ker(\mathrm{ct}) \cap \ker(\epsilon).$$

Then, we have the following map of relevant subgroups of $G[x]$ considered so far.

$$\begin{tikzcd}[column sep=small]
 & & G[x] \arrow[dll, dash] \arrow[drr, dash, "G[x]/\ker(\epsilon)\cong\mathbb{Z}"] & & \\
\mathrm{ct}^{-1}(G') \arrow[d, dash] \arrow[drr, dash] & & & & \ker(\epsilon) \arrow[dll, dash] \arrow[dd, dash] \\
\ker(\mathrm{ct}) \arrow[d, dash] \arrow[drrrr, dash] & & \mathrm{ct}^{-1}(G') \cap \ker(\epsilon) = G[x]' \arrow[dd, dash] & & \\
\mathcal{U}_G \arrow[drr, dash, "\mathcal{U}_G/\mathcal{U}_G^{sing}\cong\mathrm{u}(G)\mathbb{Z}"'] & & & & \ker(\mathrm{ct}) \cap \ker(\epsilon) = [G,x] \arrow[dll, dash] \\
 & & \mathcal{U}_G^{sing} = \mathcal{U}_G \cap \ker(\epsilon) = \mathcal{U}_G \cap G[x]' & &
\end{tikzcd}$$

Using the normal subgroup $\mathcal{U}_G$ of $G[x]$, we can define a congruence relation $\approx$ on $G[x]$ as follows: for a pair of words $w_1, w_2 \in G[x]$, we define $w_1 \approx w_2$ if $w_1\mathcal{U}_G = w_2\mathcal{U}_G$. Then

$$w_1 \approx w_2 \ \text{ if and only if } \ \Phi_G(w_1) = \Phi_G(w_2), \text{ i.e., } f_{w_1} = f_{w_2}.$$

In particular, a word w is universal when $w \approx e_{G[x]}$, i.e. f_w is the constant function e_G on G. Note that the quotient group is $G[x]/_{\approx} = G[x]/\mathcal{U}_G \cong \mathscr{F}(G)$.

A second monoid operation in $\mathscr{F}(G)$ can be introduced as follows. If w is as in (2.1), and $w_1 \in G[x]$, one can consider the word $\xi_w(w_1) = g_1 w_1^{\varepsilon_1} g_2 w_1^{\varepsilon_2} \cdots g_n w_1^{\varepsilon_n} g_{n+1}$ obtained substituting w_1 to x in w and taking products in $G[x]$. We shall also denote by $w \circ w_1$ the word $\xi_w(w_1)$. On the other hand, one can consider the usual composition of the associated verbal functions $f_w, f_{w_1} \in \mathscr{F}(G)$. Then the composition of words is compatible with the composition of functions, in the sense that

$$f_w \circ f_{w_1} = f_{w \circ w_1} \in \mathscr{F}(G).$$

With this operation, $(\mathscr{F}(G), \circ)$ is a monoid, with identity element the identity function $\mathrm{id}_G = f_x$ of G, mapping $\mathrm{id}_G \colon g \mapsto g$ for every $g \in G$. Obviously, $(\mathscr{F}(G), \circ)$ is a submonoid of the monoid $(G^G, \circ)$ of all self-maps $G \to G$.

2.2.3 Monomials

Even if a group G has a quite simple structure (for example, is abelian), the group of words $G[x]$ may be more difficult to study (for example, $G[x]$ is never abelian, unless G is trivial). As we are more interested in its quotient group of verbal function $\mathscr{F}(G)$, it will be useful to consider some subset $W \subseteq G[x]$ such that $G[x] = W \cdot \mathcal{U}_G$, i.e. $\Phi_G(W) = \{f_w \mid w \in W\} = \mathscr{F}(G)$, i.e. $\mathscr{F}(G) = W/_{\approx}$. In the following §2.2.4 we will present such an appropriate subset $W \subseteq G[x]$ in the case when G is abelian, while in §3.1 we will consider the case of groups $G \in \mathscr{N}_2$.

A homogeneous word of the form $w = gx^m$, $g \in G, m \in \mathbb{Z}$, will be called a *monomial.* One can associate a monomial to an arbitrary word $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n}g_{n+1} \in G[x]$ as follows, letting

$$w_{ab} = \mathrm{ct}(w)x^{\epsilon(w)} = g_1g_2\cdots g_ng_{n+1}x^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_n} \in G[x]. \tag{2.8}$$

The monomials in $G[x]$ do not form a subgroup unless G is trivial. Nevertheless, one can "force" them to form a group, by taking an appropriate *quotient* of $G[x]$. Indeed, recall the surjective homomorphism $G[x] \to G \times \langle x\rangle$ mapping $w \mapsto (\mathrm{ct}(w), x^{\epsilon(w)})$ considered in Remark 2.25. Then the group $G \times \langle x\rangle$ "parametrizes" in the obvious way all monomials of $G[x]$ (although the group operation is not the one from $G[x]$).

Note that if $w \in \mathcal{U}_G$, then $w_{ab} = x^{\epsilon(w)}$ by Example 2.18, item 1. In particular, $x^{\epsilon(w)} \in \mathcal{U}_{Z(G)}$. Now we will slightly generalize this result. While $w \neq w_{ab}$ as elements of $G[x]$ (except in trivial cases), the next theorem (see item 2) proves that $f_w \restriction_{C_w} = f_{w_{ab}} \restriction_{C_w}$. In item 3, we will use the notation introduced in Remark 2.16, item 2.

Theorem 2.26. *Let G be a group, and $w \in G[x]$.*

1. *If $g \in G$ and $z \in C_G(g) \cap C_w$, then $w(zg) = w(gz) = w(g)\cdot z^{\epsilon(w)}$.*
2. *If $z \in C_w$, then $w(z) = \mathrm{ct}(w)\cdot z^{\epsilon(w)}$. In particular, $f_w \restriction_{C_w} = f_{w_{ab}} \restriction_{C_w}$.*
3. *Consider the canonical map $\pi\colon G \to G/N_w = \overline{G}$. Then the words $\overline{w}, (\overline{w})_{ab} \in \overline{G}[x]$ satisfy $\overline{w} \approx (\overline{w})_{ab}$. In particular, $\pi\circ f_w = \pi\circ f_{w_{ab}}$, so that for every $g \in G$*

$$w(g)N_w = w_{ab}(g)N_w. \tag{2.9}$$

Proof. 1. If $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n}g_{n+1} \in G[x]$, then

$$\begin{aligned} w(zg) = g_1(zg)^{\varepsilon_1}g_2(zg)^{\varepsilon_2}\cdots g_n(zg)^{\varepsilon_n}g_{n+1} = \\ = g_1g^{\varepsilon_1}g_2g^{\varepsilon_2}\cdots g_ng^{\varepsilon_n}g_{n+1}\cdot z^{\epsilon(w)} = w(g)\cdot z^{\epsilon(w)}. \end{aligned}$$

2. Follows taking $g = e_G$ in item 1.
3. Note that $\mathrm{coeff}(\overline{w}) \le Z(\overline{G})$, so $C_{\overline{w}} = \overline{G}$, and Corollary 2.27 implies that $\overline{w} \approx (\overline{w})_{ab}$ in $\overline{G}[x]$. Obviously, $(\overline{w})_{ab} = \overline{w_{ab}}$.

   By Remark 2.16, item 2, we have $\pi\circ f_w = f_{\overline{w}}\circ\pi$ and $\pi\circ f_{w_{ab}} = f_{\overline{w_{ab}}}\circ\pi$, so that $\pi\circ f_w = \pi\circ f_{w_{ab}}$.

□

The following corollary is a straightforward application of Theorem 2.26, item 2.

Corollary 2.27. *If $w \in G[x]$ satisfies $C_w = G$, then $w \approx w_{ab}$.*

The next result is a corollary of Theorem 2.26, item 3.

Corollary 2.28. *If $w \in G[x]$ is a non-singular universal word for G, then the quotient group G/N_w is bounded and its exponent divides $\epsilon(w)$.*

In particular, G/G' is bounded and its exponent divides $\epsilon(w)$.

Proof. As $E_w = G$, it follows from (2.9) that $g^{\epsilon(w)} \in N_w$ for every $g \in G$, which gives the conclusion.

The last assertion follows from the fact that G' contains N_w. □

Remark 2.29. Let $m \in \mathbb{Z}$, G be a group, $N \trianglelefteq G$, and finally let $\overline{G} = G/N$. Then the following conditions on N are equivalent, and will be denoted as condition (E_m).

- Either $m = 1$ or the quotient group G/N is bounded, and its exponent divides $m - 1$;
- $(G/N)[m-1] = G/N$;
- $g^{m-1} \in N$ for every $g \in G$;
- $g^m N = gN$ for every $g \in G$;
- $x^m \approx x$ as words in $G/N[x]$.

In this case, if $w \in G[x]$ has $\epsilon(w) = m$, then the words $\overline{w}_{ab} = \mathrm{ct}(\overline{w})x^m$ and $\mathrm{ct}(\overline{w})x$ in $(G/N)[x]$ satisfy

$$\overline{w}_{ab} \approx \mathrm{ct}(\overline{w})x.$$

Corollary 2.30. *Let $w \in G[x]$ and consider a subgroup N satisfying $N_w \leq N \trianglelefteq G$ and condition $(E_{\epsilon(w)})$. Then for every $g \in G$ we have*

$$w(g)N = \mathrm{ct}(w)gN. \tag{2.10}$$

Proof. Follows from (2.9) and the fact that $g^{\epsilon(w)}N = gN$ for every $g \in G$. □

Let us say immediately that we do not know if a non-trivial group G with $\mathrm{u}(G) = 1$ exist (see Question 4 below). The following theorem gives some necessary conditions on a group G to to have $\mathrm{u}(G) = 1$, i.e. to admit a word $w \in \mathcal{U}_G$ with $\epsilon(w) = 1$.

Theorem 2.31. *Let G be a group, and $w \in \mathcal{U}_G$ with $\epsilon(w) = 1$ (so that $\mathrm{u}(G) = 1$). Then the following hold.*

1. *$N_w = G$. In particular, $G' = G$.*

2. $C_w = \{e_G\}$, *so that* $\mathcal{T}_G = \delta_G$. *In particular,* G *is center-free.*

3. *If* $N \trianglelefteq G$, *then* $\mathrm{u}(G/N) = 1$.

Proof. Note that $w_{ab} = \mathrm{ct}(w)x^{\epsilon(w)} = x \in G[x]$.

1. Consider the canonical map $\pi \colon G \to G/N_w = \overline{G}$, and let $\overline{w} = \vartheta(\pi)(w) \in \overline{G}[x]$. Note that $(\overline{w})_{ab} = \overline{\mathrm{ct}(w)}x^{\epsilon(w)} = x \in \overline{G}[x]$, so that $\overline{w} \approx x$ by Theorem 2.26, item 3.

   By Lemma 2.24, $\overline{w} \in \mathcal{U}_{\overline{G}}$. Then also $x \in \mathcal{U}_{\overline{G}}$, so that $\overline{G}$ is trivial, and $N_w = G$.

2. If $g \in C_w$, then $w(g) = w_{ab}(g)$ by Theorem 2.26, item 2, so that $e_G = g$.

3. Immediately follows by Lemma 2.24, as $\mathrm{u}(G/N) \mid 1$.

□

Question 4. *Does there exist a non-trivial group* G *with* $\mathrm{u}(G) = 1$*?*

In Lemma 2.50 we will prove that only the trivial group admit a $w \in \mathcal{U}_G$ with $\epsilon(w) = 1$ and $\mathrm{l}(w) \leq 3$.

2.2.4 A leading example: the abelian case

A case when $\mathscr{F}(G)$ has a very transparent description is that of abelian groups. Let $(G, +, 0_G)$ be an abelian group. While $G[x]$ is not abelian in any case, its quotient $\mathscr{F}(G)$ becomes indeed abelian, and so we will keep additive notation also to denote words $w \in G[x]$. Remind that we really are interested only in the evaluation function $f_w \in \mathscr{F}(G)$ associated to w, and to its preimage $E_w^G = f_w^{-1}(\{0_G\}) = \{g \in G \mid f_w(g) = 0_G\}$ (see Definition 2.39).

Then, Corollary 2.27 applies to every word $w \in G[x]$, giving $w \approx w_{ab} = ct(w) + \epsilon(w)x$, and in particular, letting

$$W = \{w_{ab} \mid w \in G[x]\} = \{g + nx \mid g \in G, n \in \mathbb{Z}\} \subseteq G[x],$$

we have $W/_{\approx} = G[x]/_{\approx}$, so that

$$\mathscr{F}(G) = \{f_{g+nx} \mid g \in G, n \in \mathbb{Z}\}.$$

For these reasons, when G is abelian, we will only consider such words $w \in W$. These observations are heavily used in computing $\mathscr{F}(G)$ for an abelian group G (hence also $\mathbb{E}_G$, see Example 2.4.1).

Now we prove that $\mathrm{u}(G) = \exp(G)$ for an abelian group G. Compare this result with Lemma 2.20.

Lemma 2.32. *Let* G *be an abelian group, with* $\exp(G) = n$. *Then* $\mathrm{u}(G) = n$. *In particular,* $G \in \mathcal{W}_n$.

Proof. As $nx \in G[x]$ obviously is a universal word for G, we have $n \in \epsilon(\mathcal{U}_G)$, so that $\mathrm{u}(G) \mid n$.

Now we prove that $\epsilon(\mathcal{U}_G) = \mathrm{u}(G)\mathbb{Z} \leq n\mathbb{Z}$. Let $w \in \mathcal{U}_G$. Then also $w_{ab} \in \mathcal{U}_G$, and $\epsilon(w) = \epsilon(w_{ab})$, so we can assume $w = g + kx$. Then $g = 0_G$ by Example 2.18, item 1, so that $w = kx$. If $\exp(G) = 0$, then $k = 0$. If $\exp(G) = n > 0$, then either $k = 0$, or $k \neq 0$ and $n \mid k$. In any case, $k \in n\mathbb{Z}$. □

Note that the surjective homomorphism $\Psi_G \colon G[x] \to G \times \mathbb{Z}$, mapping $w \mapsto (\mathrm{ct}(w), \epsilon(w))$, has kernel $\ker(\Psi_G) = \ker(\mathrm{ct}) \cap \ker(\epsilon) = G[x]'$ by Theorem 2.8, so that $G[x]/G[x]' \cong G \times \mathbb{Z}$. So, if one considers the quotient $G[x]/G[x]'$, the canonical projection $G[x] \to G[x]/G[x]'$ is exactly $w \mapsto w_{ab} = \mathrm{ct}(w) + x^{\epsilon(w)}$. Moreover, being $\mathscr{F}(G) \cong G[x]/\mathcal{U}_G$ abelian, we have that $\mathcal{U}_G \geq G[x]'$, and that $\mathscr{F}(G) \cong \frac{G[x]/G[x]'}{G[x]'/\mathcal{U}_G}$ is a quotient of $G \times \mathbb{Z}$.

Here we give an explicit description of the group $\mathscr{F}(G)$.

Proposition 2.33. *If G is an abelian group, then:*

$$\mathscr{F}(G) \cong \begin{cases} G \times \mathbb{Z} & \text{if } \exp(G) = 0, \\ G \times \mathbb{Z}_n & \text{if } \exp(G) = n > 0. \end{cases}$$

Proof. Let $n = \exp(G) \in \mathbb{N}$. Note that $\Psi'_G \colon G \times \mathbb{Z} \to \mathscr{F}(G)$, mapping $(g, k) \to f_{g+kx}$, is a surjective group homomorphism, and that $(g, k) \in \ker(\Psi'_G)$ if and only if $g + kx \in \mathcal{U}_G$. Then $\ker(\Psi'_G) = \{0_G\} \times n\mathbb{Z}$ by Lemma 2.32. □

In §3.1 we use similar reductions for groups $G \in \mathscr{N}_2$.

2.3 Further properties of the universal exponent

In the following lemma we give an easy generalization of the last part of Lemma 2.32, about the relation between the content of universal words and the exponent of the center of the group.

Lemma 2.34. *Let G be a group, and $n \in \mathbb{N}$. If $Z(G) \in \mathcal{W}_n$, then $G \in \mathcal{W}_n$. In particular, if $\exp(Z(G)) = n$, then $G \in \mathcal{W}_n$.*

Proof. Let $w \in \mathcal{U}_G$. Then $w_{ab} = x^{\epsilon(w)} \in Z(G)[x]$ by Example 2.18, item 1, and so $w_{ab} \in \mathcal{U}_{Z(G)}$. As $Z(G) \in \mathcal{W}_n$, we conclude $\epsilon(w) = \epsilon(w_{ab}) \in n\mathbb{Z}$ as desired.

The last part follows by Lemma 2.32, as $Z(G) \in \mathcal{W}_{\exp(Z(G))}$. □

As a consequence of the above lemma one obtains the following dichotomy. If $\mathrm{u}(Z(G)) = 0$ (i.e., $Z(G)$ is unbounded), then $\mathrm{u}(G) = 0$ as well. Otherwise, $\mathrm{u}(Z(G)) > 0$ and $\mathrm{u}(Z(G)) \mid \mathrm{u}(G)$.

Corollary 2.35. *Let $n \in \mathbb{N}$, and N be a normal subgroup of a group G such that $G/N \in \mathcal{W}_n$. Then $G \in \mathcal{W}_n$.*

In particular, if $[G : N] = p \in \mathbb{P}$, then $G \in \mathcal{W}_p$.

Proof. The first part immediately follows by Lemma 2.24. If $[G : N] = p$, then $G/N \cong \mathbb{Z}_p$ is abelian and has exponent p, so that $G/N \in \mathcal{W}_p$ by Lemma 2.32. $\square$

Let $\{G_i \mid i \in I\}$ be a family of groups. If $i_0 \in I$, then obviously $\prod_{i_0 \neq i \in I} G_i$ is a normal subgroup of $\prod_{i \in I} G_i$, and $G_{i_0} \cong \big(\prod_{i \in I} G_i\big)/\big(\prod_{i_0 \neq i \in I} G_i\big)$. Similarly, $\bigoplus_{i_0 \neq i \in I} G_i$ is a normal subgroup of $\bigoplus_{i \in I} G_i$, and $G_{i_0} \cong \big(\bigoplus_{i \in I} G_i\big)/\big(\bigoplus_{i_0 \neq i \in I} G_i\big)$.

So we can apply Corollary 2.35 to obtain the following result.

Corollary 2.36. *Let $n \in \mathbb{N}$, and $\{G_i \mid i \in I\}$ be a family of groups. Assume that $G_{i_0} \in \mathcal{W}_n$ for some $i_0 \in I$. Then the following hold.*

1. $\prod_{i \in I} G_i \in \mathcal{W}_n$.

2. $\bigoplus_{i \in I} G_i \in \mathcal{W}_n$.

In Lemma 6.13 we will generalize Corollary 2.36, item 1, using a different, direct proof.

The next example uses Corollary 2.35 to show that every non-trivial permutation group $G = S_\omega(X)$ is in $\mathcal{W}_2$. Then we also prove that $\mathrm{u}(S_3) = 2$.

Example 2.37.
1. Let X be a set with $|X| > 2$, and $G = S_\omega(X)$. As $G' = A(X)$ has index 2 in G, Corollary 2.35 applies, and $G \in \mathcal{W}_2$. In particular, $S_n \in \mathcal{W}_2$ for every $n > 2$.

2. Here we build a universal word $\widetilde{w}$ for S_3 with $\epsilon(\widetilde{w}) = 2$. Note that $S_3 \in \mathcal{W}_2$ by item 1, thus 2 will be the minimum among positive contents of universal words for S_3, so $\mathrm{u}(S_3) = 2$ by Lemma 2.23, item 3. Moreover, $\exp(S_3) = 6$, so that $\mathrm{u}(S_3) \neq \exp(S_3)$.

   Let
$$\widetilde{w} = (12)x(12)x(12)x(12)x(12)x^{-1}(12)x^{-1} = \big((12)x\big)^4\big((12)x^{-1}\big)^2.$$

   To prove that $E_{\widetilde{w}} = S_3$, it suffices to check that $\big((12)g\big)^2 = \big((12)g^{-1}\big)^2$ for every $g \in S_3$.

   - If $g \in A_3$, then also $g^{-1} \in A_3$, so both $(12)g$ and $(12)g^{-1}$ are elements of $S_3 \setminus A_3$, i.e. 2-cycles (transpositions), so $\big((12)g\big)^2 = \big((12)g^{-1}\big)^2 = \mathrm{id}$.
   - If $g \in S_3 \setminus A_3$, then $g = g^{-1}$ is a 2-cycle, so $(12)g = (12)g^{-1}$.

In the following example, we produce a family of non-abelian bounded groups K with $\mathrm{u}(K) = 2$ or $\mathrm{u}(K) = 4$.

Example 2.38. Let B be an abelian bounded group, with $\exp(B) = n > 2$. Then the inversion $-\operatorname{id}_B$ is a non-trivial automorphism of B of order two.

(a) Consider the semidirect product $K = B \rtimes \langle -\operatorname{id}_B \rangle$. (For example, if $B = \mathbb{Z}_n$ is the cyclic group of order n, then K is the dihedral group D_{2n} of order $2n$.)

It can be easily verified that $Z(K) = B[2] \times \langle \operatorname{id}_B \rangle$, so that K is center-free whenever B has no non-trivial 2-torsion elements. Note also that

$$\exp(K) = \begin{cases} n & \text{if } n \text{ is even,} \\ 2n & \text{otherwise.} \end{cases}$$

Observe that for every $g \in K \setminus B$ one has $g^2 = e_K$ and $gB = K \setminus B$, as B has index two in K. Then, $K \in \mathcal{W}_2$ by Corollary 2.35, so that in particular every universal word for K has even content.

(b) Suppose that $\exp(B) = n$ is not divisible by 4, i.e. n is either odd or $n = 2k$ for an odd integer k. Here we explicitly build a word $\widetilde{w} \in \mathcal{U}_K$ with $\epsilon(\widetilde{w}) = 2$. Thus 2 will be the minimum among positive contents of universal words for K, and so $\mathrm{u}(K) = 2$ by Lemma 2.23, item 3.

Let

$$n' = \begin{cases} n, & \text{if } n \text{ is odd;} \\ n/2, & \text{otherwise,} \end{cases}$$

and observe that n' is an odd integer. Now fix an element $\sigma \in K \setminus B$. Finally, let

$$\widetilde{w} = \big(\sigma x\big)^{n'+1} \big(\sigma x^{-1}\big)^{n'-1}. \tag{2.11}$$

To prove that $\widetilde{w}$ is universal, consider an element $g \in K$:

- if $g \in B$, then also $g^{-1} \in B$, and both σg and σg^{-1} are elements of $K \setminus B$. So $(\sigma g)^{n'+1} = e_K = (\sigma g^{-1})^{n'-1}$ and $\widetilde{w}(g) = e_K$;
- if $g \in K \setminus B$, then $g = g^{-1}$ and $\sigma g = \sigma g^{-1} \in B$. So $\widetilde{w}(g) = (\sigma g)^{2n'} = e_K$, as n divides $2n'$.

(c) If $4 \mid n = \exp(B)$, we don't know if K has any universal word of content 2. Anyway, in the general case (i.e. for every $n > 2$) the word

$$\widetilde{v} = \big(\sigma x\big)^{n+2} \big(\sigma x^{-1}\big)^{n-2} \in K[x] \tag{2.12}$$

has $\epsilon(\widetilde{v}) = 4$, and arguments as those above show that $\widetilde{v} \in \mathcal{U}_K$. In particular, if $K \in \mathcal{W}_m$, then $m \neq 0$ and $m \mid 4$, i.e. $m \in \{1, 2, 4\}$. We have already established in item (a) that $K \in \mathcal{W}_2$, and $K \in \mathcal{W}_1$ is obvious, so what we really do not know is whether $K \in \mathcal{W}_4$. So if $4 \mid n$, then either $\mathrm{u}(K) = 2$ and $K \notin \mathcal{W}_4$, or $\mathrm{u}(K) = 4$ and $K \in \mathcal{W}_4$.

2.4 Elementary algebraic subsets

This subsection is focused on the family $\mathbb{E}_G \subseteq \mathcal{P}(G)$, consisting of preimages $f_w^{-1}(\{e_G\})$, rather than on the group $G[x]$, consisting of words w, or its quotient $\mathscr{F}(G)$, consisting of verbal functions f_w.

Definition 2.39. *If $w \in G[x]$, we define an* elementary algebraic subset of G *to be the preimage*

$$E_w^G = f_w^{-1}(\{e_G\}) = \{g \in G \mid f_w(g) = e_G\} \subseteq G.$$

The above definition is of course equivalent to Definition P.1 (a), and we keep the definition of (additively) algebraic subsets accordingly. Then the algebraic subsets form the family of $\mathfrak{Z}_G$-closed sets, and $\mathbb{E}_G$ is a subbase for $\mathfrak{Z}_G$-closed sets; while the additively algebraic subsets are exactly the members of $\mathbb{E}_G^{\cup}$, and are a base for $\mathfrak{Z}_G$-closed sets.

Example 2.40. Note that if $w = g$ is a constant word, then either $E_w = G$ or $E_w = \emptyset$ (depending on whether $g = e_G$ or $g \neq e_G$).

2.4.1 A leading example: the abelian case II

Let G be an abelian group (see §2.2.4). Then the elementary algebraic subset of G determined by f_{g+nx} is

$$E_{g+nx} = \begin{cases} \emptyset & \text{if } g + nx = 0_G \text{ has no solution in } G, \\ G[n] + x_0 & \text{if } x_0 \text{ is a solution of } g + nx = 0_G. \end{cases} \tag{2.13}$$

On the other hand, if $n \in \mathbb{Z}$, and $g \in G$, then $G[n] + g = E_{nx-ng}$. So the non-empty elementary algebraic subsets of G are exactly the cosets of the torsion subgroups of G:

$$\mathbb{E}_G \setminus \{\emptyset\} = \{G[n] + g \mid n \in \mathbb{N}, g \in G\}. \tag{2.14}$$

One can verify that $\mathbb{E}_G$ is stable under taking finite intersections, and satisfies the descending chain condition. Then Proposition 1.3 implies that $\mathbb{E}_G^{\cup}$ is the family of all the $\mathfrak{Z}_G$-closed subsets of an abelian group G. In other words, *every algebraic subset of G is additively algebraic.* See §4.1 for consequences and more results on the Zariski topology of an abelian group.

Remark 2.41. It follows from Example 2.40 and (2.13) that if G is abelian, and $w \in G[x]$ is singular, then either $E_w = G$ or $E_w = \emptyset$.

Now we prove an easy result that we will use later in Corollary 3.28 and Corollary 4.13.

Lemma 2.42. *Let G be an abelian group, and assume that G is a finite union of elementary algebraic subsets determined by non-singular words. Then G is bounded.*

Proof. Let $G = \bigcup_{i=1}^{k} G[n_i] + g_i$ for elements $g_i \in G$ and integers $n_i \in \mathbb{N}_+$, as $1 \leq i \leq k$. If $m = \prod_{i=1}^{k} n_i$, then $G[n_i] \subseteq G[m]$, so that $G = \bigcup_{i=1}^{k} G[m] + g_i$. Then $[G : G[m]]$ is finite, and so $mG \cong G/G[m]$ is finite. As $m \neq 0$, we deduce that G is bounded. □

2.4.2 Further examples

Here we provide examples in the non-abelian case.

Example 2.43.

1. If $g \in G$, then the centralizer $C_G(g)$ coincides with E_w, where $w = gxg^{-1}x^{-1} \in G[x]$ (see also Example 2.15, item 7). Hence the centralizer $C_G(g)$ is an elementary algebraic subset of G. Therefore, the centralizer $C_G(S) = \bigcap_{g \in S} C_G(g)$ of any subset S of G is an algebraic subset. In particular, the center $Z(G)$ is an algebraic subset.

2. By Example 2.15, item 4, for every $n \in \mathbb{N}$ the word $x^n \in G[x]$ determines the verbal function $f_{x^n} \colon g \mapsto g^n$. Hence, the elementary algebraic subset $E_w = G[n]$ is the n-torsion subset of G. If G is abelian, every $G[n]$ is a subgroup of G, and these (together with their cosets, of course) are all the non-empty elementary algebraic subsets of G (see (2.14)).

3. Let $n \in \mathbb{N}$. Here we shall provide some easy examples of cases when the elementary algebraic subset $E^G_{x^n} = G[n]$ is not a coset of a subgroup, as indeed $e_G \in G[n]$ and its generated subgroup may be the whole group. To this end, it suffices to consider a simple group G such that $\{e_G\} \neq G[n] \neq G$, as then, being the subset $G[n]$ invariant under conjugations, the subgroup N it generates is normal in G, and we conclude $N = G$.

   Let G be a non-abelian finite simple group. Then $|G|$ is even by Feit-Thompson theorem, so that $\{e_G\} \neq G[2] \neq G$.

   As another example, let G be a compact, connected, simple Lie group (for example, the group $G = \mathrm{SO}_3(\mathbb{R})$ will do). Then G is covered by copies of $\mathbb{T}$ (see for example [1]), so that $\{e_G\} \neq G[n] \neq G$ for every $n > 1$.

4. By item 2, we have that $G[2] = E_{x^2}$. Here we slightly generalize this example studying E_w for a homogeneous word $w = g_1xg_2x$ (note that $w = x^2$ when $g_1 = g_2 = e_G$).

   Then $w = a^{-1}(g_2x)^2$, for $a = g_2g_1^{-1}$, by Remark 2.7, item 2, so that

   $$E_w = \{g \in G \mid (g_2g)^2 = a\} = \{g_2^{-1}h \in G \mid h^2 = a\} = g_2^{-1}\{g \in G \mid g^2 = a\}$$

   is a coset of the 'square roots' of the element $a \in G$.

   Note that if $E_w \neq \emptyset$, i.e. if $a = b^2$ for some $b \in G$, then $g_2^{-1}\big(C_G(b)[2]\big)b \subseteq E_w$.

Lemma 2.44. *For every group G, the family $\mathbb{E}_G$ is stable under taking inverse image under verbal functions.*

Proof. For every pair $w, w' \in G[x]$, consider the verbal function f_w and the elementary algebraic subset $E_{w'}$. Then

$$f_w^{-1}E_{w'} = f_w^{-1}f_{w'}^{-1}(\{e_G\}) = (f_{w'} \circ f_w)^{-1}(\{e_G\}) = f_{w' \circ w}^{-1}(\{e_G\}) = E_{w' \circ w}, \qquad (2.15)$$

so that $f_w^{-1}E_{w'} \in \mathbb{E}_G$. □

As a first application of Lemma 2.44, we see that the translate of an elementary algebraic subset is still an elementary algebraic subset.

Example 2.45. 1. By Example 2.15 (5), the left translation in G by an element $g \in G$ is the verbal function f_{gx}, and so by (2.15) we have

$$gE_w = f_{g^{-1}x}^{-1}E_w = E_{w \circ g^{-1}x} = E_{w(g^{-1}x)}. \qquad (2.16)$$

Similarly, $E_w g = E_{w \circ xg^{-1}}$. Note that $\epsilon(w \circ g^{-1}x) = \epsilon(w) = \epsilon(w \circ xg^{-1})$.

2. If $a \in G$, then $C_G(a) = E_w$, for the word $w = axa^{-1}x^{-1} \in G[x]$ by Example 2.43, item 1. By (2.16), its left coset determined by an element $g \in G$ is $gC_G(a) = E_{w_1}$ for

$$w_1 = w \circ (g^{-1}x) = a(g^{-1}x)a^{-1}(g^{-1}x)^{-1} = ag^{-1}xa^{-1}x^{-1}g.$$

Note also that, for $w_2 = gw_1g^{-1} = (gag^{-1})xa^{-1}x^{-1}$, we have $E_{w_2} = E_{w_1} = gC_G(a)$.

On the other hand, $C_G(a)g = gg^{-1}C_G(a)g = gC_G(g^{-1}ag)$, so that $\{gC_G(a) \mid a, g \in G\}$ is the family of all cosets of one-element centralizers in G. It coincides with $\{E_w \mid \exists a, g \in G\ w = (gag^{-1})xa^{-1}x^{-1}\} \subseteq \mathbb{E}_G$.

Remark 2.46. Let us fix a group G. We will now consider the iterated images of G under ϖ^n, for $n \in \mathbb{N}_+$, and to this end, we need to introduce a countable set of variables $\{x_n \mid n \in \mathbb{N}_+\}$. Then applying ϖ we obtain the following diagram:

$$G \xrightarrow{\varpi} G[x_1] \xrightarrow{\varpi} (G[x_1])[x_2] \xrightarrow{\varpi} \big((G[x_1])[x_2]\big)[x_3] \xrightarrow{\varpi} \dots \qquad (2.17)$$

If $n \in \mathbb{N}_+$, we let $G_n = G[x_1, \dots, x_n] = \varpi^n(G)$, and it can be proved that if $\sigma \in S_n$, then

$$G_n \cong G[x_{\sigma(1)}, \dots, x_{\sigma(n)}] \cong G * \langle x_1 \rangle * \langle x_2 \rangle * \dots * \langle x_n \rangle.$$

Every $w = w(x_1, \dots, x_n) \in G_n$ determines the associated evaluation function of n variables over G, that we denote $f_w \colon G^n \to G$, in analogy with Definition 2.14.

Finally, one can define $E_w \subseteq G^n$ as the preimage $E_w = f_w^{-1}(\{e_G\})$, and consider the family $\{E_w \mid w \in G_n\}$ as a subbase of the closed sets of a topology on G^n.

These observations are the basis of a theory of algebraic geometry over groups, recently started with [8] and developed in a series of subsequent papers. In this work, we will concentrate on the case when $n = 1$, considering only verbal functions $f_w \colon G \to G$ of one variable, and elementary algebraic subsets $E_w \subseteq G$.

2.4.3 Further reductions

As already noted above, to study $\mathscr{F}(G)$ it is sufficient to consider a subset $W \subseteq G[x]$ such that $\mathscr{F}(G) = \Phi_G(W) = W/_{\approx}$. Since our effort is really devoted to the study of the Zariski topology $\mathfrak{Z}_G$ on a group G, hence to the family $\mathbb{E}_G$, a further reduction is also possible as follows.

As an example to introduce this reduction, consider the abelian group $G = \mathbb{Z}\times\mathbb{Z}_2$, and the verbal functions $f_w, f_{w'} \in \mathscr{F}(G)$, associated to $w = 2x, w' = 4x \in G[x]$. Then $f_w \neq f_{w'}$, and yet $E_w = f_w^{-1}(\{0_G\}) = \{0_\mathbb{Z}\} \times \mathbb{Z}_2 = f_{w'}^{-1}(\{0_G\}) = E_{w'}$.

Another example of a more general property could be the following: consider a word $w \in G[x]$, and its inverse $w^{-1} \in G[x]$. Obviously $f_w \neq f_{w^{-1}}$ in general, but for an element $g \in G$ we have $f_{w^{-1}}(g) = e_G$ if and only if $f_w(g) = e_G$. In particular, the preimage under f_w and $f_{w^{-1}}$ of $\{e_G\}$ coincide:

$$E_{w^{-1}} = {f_{w^{-1}}}^{-1}(\{e_G\}) = \{g \in G \mid f_{w^{-1}}(g) = e_G\} = \\ = \{g \in G \mid f_w(g) = e_G\} = f_w^{-1}(\{e_G\}) = E_w.$$

So $E_w = E_{w^{-1}}$ in every group G, and in Remark 2.47 below we slightly generalize this result.

So we will consider another equivalence relation $\sim$ on $G[x]$ defined as follows: for a pair of words $w_1, w_2 \in G[x]$, we define $w_1 \sim w_2$ if $E_{w_1} = E_{w_2}$. Obviously, $w \approx w'$ implies $w \sim w'$.

For example, as noted above $w \sim w^{-1}$ for every $w \in G[x]$.

Remark 2.47. Let $w \in G[x]$, and $s \in \mathbb{Z}$. Consider the element $w^s \in G[x]$, and note that $\epsilon(w^s) = s\epsilon(w)$ and $w^s(g) = (w(g))^s$ for every $g \in G$. Hence, $E_{w^s} = \{g \in G \mid (w(g))^s = e_G\} = f_w^{-1}(G[s])$ is the preimage of $G[s]$ under f_w.

In particular, if $G[s] = \{e_G\}$, then $E_{w^s} = E_w$, i.e. $w \sim w^s$.

Then, in describing $\mathbb{E}_G$, we can restrict ourselves to a subset $W \subseteq G[x]$ of representants with respect to the equivalence $\sim$, that is such that the quotient set $W/_\sim = G[x]/_\sim$. For example, if $W \subseteq G[x]$ satisfies $\mathscr{F}(G) = \{f_w \mid w \in W\}$, that is $G[x]/_\approx = W/_\approx$, then $G[x]/_\sim = W/_\sim$. As we have seen in §2.2.4, in the *abelian* case the set $W = \{gx^n \in G[x] \mid n \in \mathbb{N},\ g \in G\}$ satisfies $G[x]/_\approx = W/_\approx$, so that this W will do.

We shall see in §3.1 the case of groups $G \in \mathscr{N}_2$.

Remark 2.48. Note that from Theorem 2.26, item 2, it follows that $E_w^G \cap C_w = E_{w_{ab}}^G \cap C_w$ for every word $w \in G[x]$. In particular, if $C_w = G$, then $E_w^G = E_{w_{ab}}^G$.

Finally, note that $f_w(g) = e_G$ if and only if $f_{awa^{-1}}(g) = e_G$ holds for every $a \in G$, so that $w \sim awa^{-1}$. Then, in describing $\mathbb{E}_G$, there is no harm in assuming that a word $w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n}g_{n+1} \in G[x]$ has $g_{n+1} = e_G$ (or $g_1 = e_G$); indeed, *from now on, we will often consider exclusively words w of the form*

$$w = g_1x^{\varepsilon_1}g_2x^{\varepsilon_2}\cdots g_nx^{\varepsilon_n} \in G[x]. \tag{2.18}$$

Lemma 2.49. *Let $v \in G[x]$. Then $v \sim w$ for a word $w \in G[x]$ as in (2.18), with $\epsilon(w) = |\epsilon(v)| \geq 0$.*

Proof. By Remark 2.47, we have that $v \sim v^{-1}$, and $\epsilon(v^{-1}) = -\epsilon(v)$, so that we can assume $\epsilon(v) \geq 0$.

Then, by the above discussion, $v \sim w$ for a word w as in (2.18), and with $\epsilon(w) = \epsilon(v)$. □

Recall (Question 4) that we do not know if a non-trivial group G can have $\mathrm{u}(G) = 1$, i.e. can admit a $w \in \mathcal{U}_G$ with $\epsilon(w) = 1$. Now we give some necessary conditions such a word must satisfy.

Let $w \in \mathcal{U}_G$, and assume w to be as in (2.18). Note that if w has $\epsilon(w) = 1$, then $\mathrm{l}(w)$ is odd. If $\mathrm{l}(w) = 1$, then $w = gx$, and $g = \mathrm{ct}(w) = w(e_G) = e_G$, so that $w = x$ and G is trivial. The following lemma proves that $\mathrm{l}(w) \neq 3$.

Lemma 2.50. *Let G be a group, and let $w \in \mathcal{U}_G$ with $\epsilon(w) = 1$ and $\mathrm{l}(w) \leq 3$. Then $\mathrm{l}(w) = 1$ and G is trivial.*

Proof. Let $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} g_3 x^{\varepsilon_3} \in \mathcal{U}_G$ have $\epsilon(w) = 1$. Then exactly one among $\varepsilon_1, \varepsilon_2, \varepsilon_3$ equals -1, while the others equal 1.

Moreover, for every $g \in G$ we have $g_1 g^{\varepsilon_1} g_2 g^{\varepsilon_2} = \left(g_3 g^{\varepsilon_3}\right)^{-1}$, so that also the word $g_3 x^{\varepsilon_3} g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \in \mathcal{U}_G$. Similarly, $g_2 x^{\varepsilon_2} g_3 x^{\varepsilon_3} g_1 x^{\varepsilon_1} \in \mathcal{U}_G$ too. Then we can assume $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = -1$, so that $w = g_1 x g_2 x^{-1} g_3 x$. Now $w(e_G) = e_G$ gives $g_1 g_2 g_3 = e_G$, while $w(g_2) = e_G$ gives $g_1 g_2 g_3 g_2 = e_G$, from which we deduce $g_2 = e_G$. Then $w = g_1 g_3 x$ has $\mathrm{l}(w) = 1$ and G is trivial. □

3

General properties of words and elementary algebraic subsets

We begin this section with an immediate corollary of Theorem 2.26, item 1.

Lemma 3.1. *Let G be a group, $w \in G[x]$, and let S be a subset of G such that $e_G \in S \subseteq C_G(E_w) \cap C_w \cap G[\epsilon(w)]$. Then $S \cdot E_w = E_w$.*

Proof. By Theorem 2.26, item 1, if $z \in S$ and $g \in E_w$, then $w(zg) = w(g)z^{\epsilon(w)} = e_G$, so that also $zg \in E_w$. This proves that $S \cdot E_w \subseteq E_w$, while $E_w \subseteq S \cdot E_w$ trivially holds as $e_G \in S$. $\square$

A relevant corollary of this lemma tells us that some elementary algebraic subsets are union of cosets of $Z(G)$.

Corollary 3.2. *Let G be a group, and let $w \in G[x]$. If $Z(G) \subseteq G[\epsilon(w)]$, then E_w is a union of cosets of $Z(G)$.*

Proof. Apply Lemma 3.1 to $S = Z(G) \subseteq C_G(E_w) \cap C_w$. $\square$

Lemma 3.3. *Let $w \in G[x]$ be such that $e_G \in E_w$ (i.e. $\mathrm{ct}(w) = e_G$), and let $\epsilon = \epsilon(w)$. Then the following hold.*

(a) If $S \subseteq C_w$, then $S \cap E_w = S[\epsilon]$. In particular $C_w \cap E_w = C_w[\epsilon]$ and $Z(G) \cap E_w = Z(G)[\epsilon]$.

(b) If w is a singular word, then $Z(G) \subseteq C_w \subseteq E_w$. In particular, E_w is a union of cosets of $Z(G)$.

(c) If w is a non-singular universal word for G, then $Z(G)$ is bounded and its exponent divides ϵ.

Proof. Note that if $g \in C_w$, then $w(g) = g^\epsilon$ by Theorem 2.26, item 2.

(a). In particular, for an element $g \in S \subseteq C_w$ we have that $g \in E_w$ if and only if $g \in G[\epsilon]$, which yields $S \cap E_w = S \cap G[\epsilon] = S[\epsilon]$.

(b). Immediately follows from (a) and Corollary 3.2.

(c). In this case, $E_w = G$ in (a) gives $C_w = C_w[\epsilon]$ and $Z(G) = Z(G)[\epsilon]$. $\square$

Note that if G is an abelian group, and w is a non-trivial universal word for G, then w is also non-singular. Then by Lemma 3.3 (c) it follows that if G is an abelian group and w is a non-trivial universal word for G, then G is bounded and $\exp(G) \mid \epsilon(w)$.

Theorem 3.4. *Let $w \in G[x]$, and consider a subgroup $N_w \leq N \trianglelefteq G$. Then:*

(a)

$$(E_w)^{\epsilon(w)} \subseteq (f_w^{-1}(N))^{\epsilon(w)} \subseteq \mathrm{ct}(w)^{-1}N. \tag{3.1}$$

(b) If N satisfies condition $(E_{\epsilon(w)})$, then

$$E_w \subseteq f_w^{-1}(N) \subseteq \mathrm{ct}(w)^{-1}N. \tag{3.2}$$

Proof. (a). The first inclusion is obvious, as $E_w = f_w^{-1}(\{e_G\})$. To prove the second one, we are going to apply Theorem 2.26, item 3.

If $f_w^{-1}(N) = \emptyset$, there is nothing to prove, so assume this is not the case, and let $g \in f_w^{-1}(N)$, i.e. $w(g) \in N$. Then $\mathrm{ct}(w)g^{\epsilon(w)} \in N$ by equation (2.9), so that $g^{\epsilon(w)} \in \mathrm{ct}(w)^{-1}N$.

(b). As N satisfies condition $(E_{\epsilon(w)})$, we have $gN = g^{\epsilon(w)}N$ for every $g \in G$. Now (3.1) applies. □

The following is a straightforward application of Theorem 3.4 (b).

Example 3.5. Let X be a set, consider the group $G = S_\omega(X)$, and recall that its subgroup $N = G' = A(X)$ has index 2 in G. Hence, N satisfies condition (E_m) for every odd m.

Then, for every word $w \in G[x]$ such that $\epsilon(w)$ is odd, we have that $E_w \subseteq \mathrm{ct}(w)^{-1}N$ by equation (3.2), as N satisfies condition $(E_{\epsilon(w)})$. In particular, if $w \in \mathcal{U}_G$, then $\epsilon(w)$ must be even, i.e. $G \in \mathcal{W}_2$. This was also already established in Example 2.37, item 1.

In the following result, we show that under certain conditions an elementary algebraic subset is a coset of a subgroup. This result should be compared with the abelian case, where *every* elementary algebraic subset has this form (see (2.14)). In §3.1 later, we shall give other results in this sense, for groups $G \in \mathscr{N}_2$.

Theorem 3.6. *Let $w \in G[x]$ be such that $E_w \neq \emptyset$, and consider a subgroup $N_w \leq N \trianglelefteq G$ satisfying condition $(E_{\epsilon(w)})$. If moreover N is abelian, and $N \leq C_w$, then E_w is a coset of $N[\epsilon(w)]$.*

Proof. Let $\epsilon = \epsilon(w)$, and $x_0 = \mathrm{ct}(w)^{-1} = (g_1 \cdots g_n)^{-1}$. As $\emptyset \neq E_w \subseteq x_0N$ by equation (3.2), there exists $z_0 \in N$ be such that $x_0z_0 \in E_w$. We aim to prove that $E_w = x_0z_0 \cdot N[\epsilon]$.

In order to establish a connection between x_0 and z_0, we note that $C_G(x_0) \geq C_w \geq N$, so that $w(x_0z_0) = w(x_0)z_0^\epsilon$ by Theorem 2.26, item 1. As $w(x_0z_0) = e_G$, we get $w(x_0) = z_0^{-\epsilon}$.

As $E_w \subseteq x_0N$, so $x_0^{-1}E_w \subseteq N$, to prove the required equality $E_w = x_0z_0 \cdot N[\epsilon]$ we prove that $x_0^{-1}E_w = z_0 \cdot N[\epsilon]$ by checking when $z \in N$ belongs to $x_0^{-1}E_w$, equivalent to when belongs to $z_0 \cdot N[\epsilon]$.

Let $z \in N$. Then, another application of Theorem 2.26, item 1, gives $w(x_0z) = w(x_0)z^\epsilon = z_0^{-\epsilon}z^\epsilon = (z_0^{-1}z)^\epsilon$, where the last equality holds as N is abelian. In particular, $x_0z \in E_w$ if and only if $z_0^{-1}z \in N[\epsilon]$, so $z \in x_0^{-1}E_w$ if and only if $z \in z_0 \cdot N[\epsilon]$. This gives $x_0^{-1}E_w = z_0 \cdot N[\epsilon]$. □

In Lemma 3.11, we shall consider a more particular case of Theorem 3.6, for groups $G \in \mathscr{N}_2$.

Example 3.7. To better understand the proof of Theorem 3.6, we shall see what happens considering the group $G = Q_8$, its subgroup $N = Q_8' = Z(Q_8) = \{1, -1\} \cong \mathbb{Z}_2$, and a word $w \in Q_8[x]$ such that $\epsilon(w)$ is odd and $E_w \neq \emptyset$. As the quotient $G/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ has exponent 2, N satisfies condition (E_m) for every odd m. In particular, N satisfies condition $(E_{\epsilon(w)})$ and all the hypotheses of Theorem 3.6 are satisfied, so one can conclude that E_w is a coset of $N[\epsilon(w)]$, hence a singleton.

Let us prove it applying Theorem 3.4 (b). Letting $x_0 = \mathrm{ct}(w)^{-1}$, by equation (3.2) we obtain that $E_w \subseteq x_0 \cdot N = \{x_0, -x_0\}$. Then, by Theorem 2.26, item 1, $w(-x_0) = (-1)^{\epsilon(w)}w(x_0) = -1 \cdot w(x_0)$, so that E_w is properly contained in $\{x_0, -x_0\}$.

On the other hand, if $w \in Q_8[x]$ is such that $\epsilon(w)$ is even, then E_w is a union of cosets of $Z(Q_8)$ by Corollary 3.2. By the way, these results for the group $G = Q_8$ will be completely covered by Corollary 3.13.

See also §3.2.2, where we completely describe $\mathbb{E}_{Q_8}$.

3.1 Canonical words for groups in $\mathscr{N}_2$

We begin giving a few general properties on the class $\mathscr{N}_2$.

Fact 3.8. *If $G \in \mathscr{N}_2$, then $[ab, c] = [a, c][b, c]$ and $[a, bc] = [a, b][a, c]$ for every $a, b, c \in G$. In particular, $[a, c^{-1}] = [a, c]^{-1} = [a^{-1}, c]$ for every $a, c \in G$.*

As a consequence, the commutator verbal functions are group homorphisms of G, with range contained in $G' \leq Z(G)$.

Then, we obtain the following result using the identities above.

Proposition 3.9. *If $G \in \mathscr{N}_2$, and $\overline{G} = G/Z(G)$, then there is an injective group homomorphism $\overline{G} \to Z(G)^G$. As a consequence:*

- $\exp(\overline{G}) \mid \exp(Z(G))$, *so* $\exp(G) \mid \exp\big(Z(G)\big)^2$. *In the particular case when* $\exp(Z(G)) = p \in \mathbb{P}$, *then also* $\exp(\overline{G}) = p$.

- *G is torsion-free if and only if $Z(G)$ is torsion-free. In this case, also $\overline{G}$ is torsion-free.*

Proof. As $G \in \mathscr{N}_2$, the map $G \times G \to Z(G)$, $(a,b) \mapsto [a,b]$ is well-defined, and it is bilinear by Fact 3.8. As $(G \times Z(G)) \cup (Z(G) \times G)$ is mapped to e_G, we obtain a well defined, still bilinear map $\psi\colon \overline{G} \times \overline{G} \to Z(G)$, mapping $(g_1 Z(G), g_2 Z(G)) \mapsto [g_1, g_2]$.

In particular, for every $\overline{g} = gZ(G) \in \overline{G}$ we have a linear map $\psi_{\overline{g}}\colon \overline{G} \to Z(G)$, $xZ(G) \mapsto [g,x]$. Note that if $g \notin Z(G)$, then there exists $x \in G$ such that $[g,x] \neq e_G$ (in particular, $x \notin Z(G)$), so that $\psi_{\overline{g}}$ is not the trivial homomorphism. Then the correspondence $\overline{g} \mapsto \psi_{\overline{g}}$ defines an injective homomorphism $\overline{G} \to Z(G)^{\overline{G}}$.

If $Z(G)$ is torsion-free, then $\overline{G}$ is torsion-free by the first part of the proof, so that G is torsion-free too. Obviously, if G is torsion-free, then also $Z(G)$ is torsion-free. □

Note that when $G \in \mathscr{N}_2$ and $Z(G)$ is finite, then G has positive exponent (more precisely, $\exp(G) \mid \exp\big(Z(G)\big)^2$ by Proposition 3.9).

Corollary 3.10. *If $G \in \mathscr{N}_2$ and $Z(G)$ is finite, then G is an FC-group.*

Proof. To verify that G is an FC-group, let $g \in G$, and we have to check that $[G : C_G(g)]$ is finite.

Let $w = [x,g] \in G[x]$. By Fact 3.8, it follows that the verbal function f_w is a group homomorphism of G, with range contained in $G' \leq Z(G)$, hence finite. As $\ker(f_w) = C_G(g)$, we conclude that $[G : C_G(g)]$ is finite as well. □

Note that the converse implication in the above corollary does not hold in general, i.e. there exist FC-groups in $\mathscr{N}_2$ that have infinite center. For example, consider the group $G = \mathbb{Z} \times D_8$. Obviously $G \in \mathscr{N}_2$, has infinite center, and the centralizer of a generic element $(n,g) \in G$ is $\mathbb{Z} \times C_{D_8}(g)$, that has finite index in G, so that G is FC.

The following result immediately follows by Theorem 3.6, and will be applied to a class of nilpotent groups $G \in \mathscr{N}_2$, thus satisfying $G' \leq Z(G)$.

Lemma 3.11. *Let $G \in \mathscr{N}_2$ and let $w \in G[x]$ with $E_w \neq \emptyset$. Let $G' \leq N \leq Z(G)$ be a subgroup of G satisfying condition $(E_{\epsilon(w)})$. Then E_w is a coset of $N[\epsilon(w)]$.*

Proof. In this case, $N_w \leq G' \leq N \leq Z(G) \leq C_w$, so that Theorem 3.6 applies. □

Corollary 3.12. *Let $G \in \mathscr{N}_2$, and let $w \in G[x]$ be such that $E_w \neq \emptyset$. Assume the following hypotheses:*

- *the quotient $G/Z(G)$ is bounded, and let $\exp(G/Z(G)) = r > 0$;*
- *$(\epsilon(w), r) = 1$, and $G[s] = \{e_G\}$, where $s \in \mathbb{N}$ is such that $s\epsilon(w) \equiv_r 1$.*

Then E_w is a coset of $Z(G)[\epsilon(w)]$.

Proof. Consider $w^s \in G[x]$, and note that $E_w = E_{w^s}$ by Remark 2.47. Then $\epsilon(w^s) = s\epsilon(w) \equiv_r 1$, so $Z(G)$ satisfies condition $(E_{s\epsilon(w)})$. By Lemma 3.11, applied to the word w^s and to $N = Z(G)$, we get that $E_w = E_{w^s}$ is a coset of $Z(G)[s\epsilon(w)] = Z(G)[\epsilon(w)]$, where the last equality holds as $G[s] = \{e_G\}$. □

The following result describes $\mathbb{E}_G$ for a particular class of groups $G \in \mathscr{N}_2$. It turns out that every non-empty $E_w \in \mathbb{E}_G$ is a union of cosets of subgroups of G, as in the abelian case (see (2.14)).

Corollary 3.13. *Let $G \in \mathscr{N}_2$. Assume that $Z(G)$ is bounded, with $\exp Z(G) = p \in \mathbb{P}$, and let $w \in G[x]$ with $E_w \neq \emptyset$.*

(i) If $(\epsilon(w), p) = 1$, then E_w is a singleton.

(ii) Otherwise, E_w is a union of cosets of $Z(G)$.

Proof. First, note that also $\exp(\overline{G}) = p$ by Proposition 3.9, so G itself is bounded, of exponent either p or p^2. In particular, $G[s] = \{e_G\}$ for every integer s coprime with p.

(i) In this case, note that $Z(G)[\epsilon(w)]$ is the trivial subgroup, and apply Corollary 3.12 with $r = p$ to obtain that E_w is a singleton.

(ii) If $(\epsilon(w), p) \neq 1$, then $Z(G) \subseteq G[p] \subseteq G[\epsilon(w)]$, and the conclusion directly follows by Corollary 3.2. $\square$

It is possible for an E_w as in Corollary 3.13 (ii) to be an infinite union of cosets of $Z(G)$. For example, let A be an infinite set, consider the group $G = Q_8^A$, and note that $Z(G) = \{\pm 1\}^A$. Fix an index $\alpha_0 \in A$ and let $g = (g_\alpha)_{\alpha \in A}$, where $g_{\alpha_0} = i \in Q_8$, and $g_\alpha = 1 \in Q_8$ for $\alpha_0 \neq \alpha \in A$. Then $C_G(g) = \langle i \rangle \times Q_8^{A \setminus \{\alpha_0\}} = E_w$ for $w = [g, x] \in G[x]$. As the index $[C_G(g) : Z(G)]$ is infinite, E_w cannot be expressed as a finite union of cosets of $Z(G)$.

Theorem 6.48 and Theorem 6.49 will describe the Zariski topology of direct products of groups as in Corollary 3.13.

The following results will be used to compute the elementary algebraic subsets for groups $G \in \mathscr{N}_2$.

Theorem 3.14. *Let $G \in \mathscr{N}_2$, $T \subseteq G$ be a transversal of $Z(G)$ in G, and $w \in G[x]$. Then $w \approx \widetilde{w}$, for the word*

$$\widetilde{w} = w_{ab}[x, a] \in G[x], \tag{3.3}$$

where a can be chosen in T.

As a consequence, the group $\mathscr{F}(G)$ is generated by the monomials and the commutator verbal functions.

Proof. We prove the case when $\mathrm{l}(w) = 3$, i.e. $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} g_3 x^{\varepsilon_3}$, as the proof in the general case is similar.

First, let $w_0 = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2}$, so that $w_0 = g_1 g_2 x^{\varepsilon_1} [x^{-\varepsilon_1}, g_2^{-1}] x^{\varepsilon_2}$ by Remark 2.7, item 3. As $G \in \mathscr{N}_2$, for every $g \in G$ we have $[g^{-\varepsilon_1}, g_2^{-1}] = [g, g_2^{\varepsilon_1}]$ by Fact 3.8, and this is a central element. In particular, $w_0 \approx g_1 g_2 x^{\varepsilon_1 + \varepsilon_2} [x, g_2^{\varepsilon_1}]$, and

$$w = w_0 g_3 x^{\varepsilon_3} \approx w_1 [x, g_2^{\varepsilon_1}], \tag{3.4}$$

where $w_1 = g_1 g_2 x^{\varepsilon_1+\varepsilon_2} g_3 x^{\varepsilon_3}$. Applying the idea above to w_1 we obtain $w_1 \approx g_1 g_2 g_3 x^{\varepsilon_1+\varepsilon_2+\varepsilon_3}[x, g_3^{\varepsilon_1+\varepsilon_2}]$, so that (3.4) implies that

$$w \approx g_1 g_2 g_3 x^{\varepsilon_1+\varepsilon_2+\varepsilon_3}[x, g_2^{\varepsilon_1}][x, g_3^{\varepsilon_1+\varepsilon_2}] \approx g_1 g_2 g_3 x^{\varepsilon_1+\varepsilon_2+\varepsilon_3}[x, g_2^{\varepsilon_1} g_3^{\varepsilon_1+\varepsilon_2}],$$

where the last equivalence follows from Fact 3.8. This proves that $w \approx \widetilde{w}$, for a word $\widetilde{w}$ as in (3.3).

Finally, let $a = tz$, for elements $t \in T$, $z \in Z(G)$. Then, for every $g \in G$ we have $[g, a] = [g, t]$, so that $[x, a] \approx [x, t]$, and $w \approx w_{ab}[x, t]$. □

Corollary 3.15. *Let $G \in \mathscr{N}_2$, $T \subseteq G$ be a transversal of $Z(G)$ in G, and $w \in G[x]$. Then $w \sim \widetilde{w}$ for a word*

$$\widetilde{w} = g_1 x^m [x, g_2] \in G[x], \tag{3.5}$$

where $m = \epsilon(\widetilde{w}) = |\epsilon(w)|$, and $g_2 \in T$.

Proof. As $w \sim w^{-1}$ by Lemma 2.49, we can assume $\epsilon(w) = |\epsilon(w)| \geq 0$. Apply Theorem 3.14 to get $\widetilde{w} \in G[x]$ as in (3.3), such that $w \sim \widetilde{w}$. Obviously, w_{ab} has the form $g_1 x^m$, and $m = \epsilon(w) \geq 0$. □

Sometimes, especially when the quotient group $G/Z(G)$ is finitely generated and the generators are images of some especially simple to deal with elements $t_1, \dots, t_r$ of G, then we can replace the single commutator $[x, a]$ in (3.3) by a product of commutators of the form $[x, t_i]$, arising by the replacement of the generic element a by a product of the generators t_i. We formalize this observation in the following corollary of Theorem 3.14.

Corollary 3.16. *Let $G \in \mathscr{N}_2$, and $T \subseteq G$ be such that the quotient group $G/Z(G)$ is generated by its image $\{tZ(G) \mid t \in T\}$. If $w \in G[x]$, then $w \approx \widetilde{w}$ for a word*

$$\widetilde{w} = w_{ab}[x, t_1]^{m_1}[x, t_2]^{m_2} \cdots [x, t_r]^{m_r} \in G[x], \tag{3.6}$$

where $r \in \mathbb{N}$, $t_i \in T$ and $m_i \in \mathbb{Z}$ for $i = 1, \dots, r$.

In particular,

$$w \sim g x^m [x, t_1]^{m_1}[x, t_2]^{m_2} \cdots [x, t_r]^{m_r} \in G[x], \tag{3.7}$$

for an element $g \in G$, and $m = |\epsilon(w)|$.

Proof. By Theorem 3.14, we have $w \approx w_{ab}[x, a]$ for an $a \in G$, and let $a = t_1^{m_1} t_2^{m_2} \cdots t_r^{m_r} z$, for $z \in Z(G)$, $r \in \mathbb{N}$, and $t_i \in T$, $m_i \in \mathbb{Z}$ for $i = 1, \dots, r$.

By Fact 3.8, $[g, a] = [g, t_1]^{m_1}[g, t_2]^{m_2} \cdots [g, t_r]^{m_r}$ for every $g \in G$, so that $[x, a] \approx [x, t_1]^{m_1}[x, t_2]^{m_2} \cdots [x, t_r]^{m_r}$, which gives (3.6).

For the last part, given $w \in G[x]$, apply Corollary 3.15 to get a word $\widetilde{w}$ as in (3.5) such that $w \sim \widetilde{w}$. Applying the first part of the proof, we can assume $\widetilde{w}$ to be as in (3.7). □

Note that if the quotient $G/Z(G)$ is torsion-free, then $G[m] \subseteq Z(G)$ for every $m \in \mathbb{N}_+$. In particular,

$$Z(G)[m] = G[m] \cap Z(G) = G[m].$$

Let $G \in \mathscr{N}_2$ be such that $G/Z(G)$ is torsion-free. We shall see now that if $\emptyset \neq E_w \in \mathbb{E}_G$, then E_w is the translate of either the m-socle $Z(G)[m]$ of $Z(G)$ (if $m = |\epsilon(w)| \neq 0$), or of some centralizer $C_G(a)$ (if $\epsilon(w) = 0$).

Lemma 3.17. *If $G \in \mathscr{N}_2$ and $G/Z(G)$ is torsion-free, then every non-empty elementary algebraic subset of G is a coset of a subgroup of G that is either central or contains $Z(G)$. More precisely, if $w \in G[x]$, and $e_G \in E_w$, then either*

(a) $m = \epsilon(w) \neq 0$ and $E_w = Z(G)[m] = G[m]$, or

(b) $\epsilon(w) = 0$ and E_w is centralizer of a single element, so contains $Z(G)$.

Proof. Let $w \in G[x]$ with $E_w \neq \emptyset$. If $g \in E_w$, then $e_G \in g^{-1}E_w = E_{w'}$, with $w' = w \circ gx$ by Example 2.45, item 1. Then $\epsilon(w') = \epsilon(w)$, and $\mathrm{ct}(w') = e_G$, so that $w' \sim x^m[x, a]$, with $m \geq 0$, and $a \in G$ by Corollary 3.15.

So we assume $e_G \in E_w$ and $w(x) = x^m[x, a]$, with $m \geq 0$, and $a \in G$.

If $m \neq 0$ and $g \in E_w$, then obviously $g^m \in Z(G)$. Hence our hypothesis implies $g \in Z(G)$, so that $e_G = w(g) = g^m$. This proves that $E_w \subseteq Z(G)[m]$. The other inclusion immediately follows noting that $w(g) = g^m$ for $g \in Z(G)$.

If $m = 0$, then $w = [x, a]$ is a commutator, hence f_w is an endomorphism of G by Fact 3.8. Since obviously $Z(G) \leq C_G(a) = \ker f_w = E_w$, we are done. □

In the following lemma we consider groups $G \in \mathscr{N}_2$ of prime exponent $p > 2$ (note that the groups in $\mathscr{N}_2$ are non-abelian, hence cannot have exponent 2). For such groups, we prove that a dichotomy holds for $E \in \mathbb{E}_G$, similar to that proved in Lemma 3.17 for groups G such that $G/Z(G)$ is torsion-free.

Lemma 3.18. *Let $G \in \mathscr{N}_2$ of prime exponent $p > 2$, and $w \in G[x]$ with $e_G \in E_w$. Then the following hold.*

(a) Either $E_w = \{e_G\}$, or E_w is a normal subgroup of G, the centralizer of a single element.

(b) If $Z(G)$ is finite, and $E_w \neq \{e_G\}$, then E_w is a normal subgroup of G of finite index.

Proof. (a). Recall that according to Theorem 3.14 $w \sim \widetilde{w}$, where $\widetilde{w} = x^m[x, g]$, with $m = |\epsilon(w)|$. Since $\exp(G) = p$, we can assume also that $0 \leq m < p$. If $m > 0$, then $(m, p) = 1$, so $E_w = \{e_G\}$ by Corollary 3.13 (i).

Now assume that $m = 0$, so $w = [x, g]$. Then the verbal function $f = f_w : G \to Z(G) \subseteq G$ is a homomorphism by Fact 3.8. Clearly, $E_w = \ker f = C_G(g)$ is a normal subgroup of G.

(b). Immediately follows from item (a) and Corollary 3.10. □

3.2 Some examples on finite groups in $\mathscr{N}_2$

In this section we will classify the elementary algebraic subsets of the groups Q_8 and D_8, the two non-abelian groups of order eight.

Let us consider first the family $\mathbb{E}_G$ for abelian groups G with eight elements, using (2.14).

If $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $G[2] = G$, so that the only elements of $\mathbb{E}_G$ are the singletons, G, and $\emptyset$, and $|\mathbb{E}_G| = 10$.

If $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, then $G[4] = G$, and $G[2] = \mathbb{Z}_2 \times 2\mathbb{Z}_4$ has index 2 in G, so that it has two cosets. Then the elements of $\mathbb{E}_G$ are the singletons, G, $\emptyset$, and the two cosets of $G[2]$, hence $|\mathbb{E}_G| = 12$.

Finally, if $G = \mathbb{Z}_8$, then $G[8] = G$, $G[4] = 2G$ has index 2, and $G[2] = 4G$ has index 4. Then the elements of $\mathbb{E}_G$ are the singletons, G, $\emptyset$, the two cosets of $G[4]$, and the four cosets of $G[2]$, so that $|\mathbb{E}_G| = 16$.

Now recall Corollary 3.13, in the special case when $p = 2$. Let $G \in \mathscr{N}_2$ be a group with $\exp(Z(G)) = 2$, and $w \in G[x]$ be such that $E_w \neq \emptyset$.

If $\epsilon(w)$ is odd, then E_w is a singleton.

If $\epsilon(w)$ is even, then E_w is a union of cosets of $Z(G)$.

Obviously, every singleton belongs to $\mathbb{E}_G$, so what really remains to study is whether all possible unions of cosets of $Z(G)$ are elements of $\mathbb{E}_G$. We will prove that this actually happens when $G = Q_8$ or $G = D_8$, while we will show in Remark 6.17 that this does not happen in general. In particular, $|\mathbb{E}_G| = 24$ for those two groups G. In fact, there are 16 subsets of G that are union of cosets of $Z(G)$ (as $[G : Z(G)] = 4$), and the 8 singletons of G.

3.2.1 Some properties of groups $G \in \mathscr{N}_2$ with $\exp(Z(G)) = 2$

Here we collect a few general properties of groups $G \in \mathscr{N}_2$ with $\exp(Z(G)) = 2$. First we study the consequences of Corollary 3.16 for such groups. Recall that in this case also $\exp(G/Z(G)) = 2$ by Proposition 3.9, so that $g^2 \in Z(G)$ for every $g \in G$. In particular, both $Z(G)$ and $G/Z(G)$ are 2-elementary abelian groups.

If moreover $G/Z(G)$ is finite, then $G/Z(G) \cong \mathbb{Z}_2^h$ for some $h \in \mathbb{N}_+$, so that $G/Z(G)$ has size 2^h and is generated by a set of h elements.

Corollary 3.19. *Let $G \in \mathscr{N}_2$ be such that $\exp(Z(G)) = 2$, and $|G/Z(G)|$ is finite. Let $T = \{t_1, \ldots, t_k\} \subseteq G$ be such that the quotient group $G/Z(G)$ is generated by its image $\{tZ(G) \mid t \in T\}$.*

Finally, let $w \in G[x]$ be such that $\epsilon(w)$ is even, and $E_w \neq \emptyset$. Then $w \sim \widetilde{w}$, for the word

$$\widetilde{w} = zx^{2m_0}[x, t_1]^{m_1}[x, t_2]^{m_2} \cdots [x, t_k]^{m_k} \in G[x], \tag{3.8}$$

satisfying $m_0, m_i \in \{0, 1\}$ for every $i = 1, \ldots, k$, and $z \in Z(G)$.

Proof. Let $|\epsilon(w)| = 2m_0$ for an $m_0 \in \mathbb{N}$. By Corollary 3.16, we can assume $\widetilde{w}$ to be as in (3.6), i.e. as follows:

$$\widetilde{w} = gx^{2m_0}[x, t_{i_1}]^{m_1}[x, t_{i_2}]^{m_2} \cdots [x, t_{i_r}]^{m_r}.$$

Then, recall that $[g, t]$ is a central element for every $g \in G$ and $t \in T$, so one can arrange the factors $[x, t]$ with the same coefficient $t \in T$. Moreover, also $g^2 \in Z(G)$ for every $g \in G$, so that we can assume $m_0, m_i \in \{0, 1\}$ for every $i = 1, \ldots, k$, as $\exp(Z(G)) = 2$.

Finally, if $x_0 \in E_w$, then $e_G = w(x_0) = gz_0$ for a central element z_0 depending on x_0 (namely, $z_0 = x_0^{2m_0}[x_0, t_{i_1}]^{m_1}[x_0, t_{i_2}]^{m_2} \cdots [x_0, t_{i_r}]^{m_r}$). So also $g = z_0^{-1} \in Z(G)$. □

Lemma 3.20. *Let G be a group.*

(a) Let $S = \{e_G, s\} \subseteq G$ be a doubleton, and $w \in G[x]$ be such that $w(g) \in S$ for every $g \in G$. If $w' = s^{-1}w \in G[x]$, then $G \setminus E_w = E_{w'}$.

(b) In particular, if $G \in \mathscr{N}_2$, and $|Z(G)| = 2$, let $e_G \neq z \in Z(G)$. Then $G \setminus E_w = E_{zw}$ for every $w \in G[x]$ as in (3.8).

Proof. (a). Just note that $g \in G \setminus E_w$ if and only if $w(g) \neq e_G$, i.e. $w(g) = s$. In other words, $w'(g) = e_G$.

(b). If $w \in G[x]$ is as in (3.8), then $w(g) \in Z(G)$ for every $g \in G$, so that item (a) applies. □

Now we will apply Corollary 3.19 to groups $G \in \mathscr{N}_2$ such that $|Z(G)| = 2$ and $[G : Z(G)] = 4$. Obviously, only Q_8 and D_8 satisfy these restraints, and for both groups G we have $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, $G/Z(G)$ is generated by two elements $t_1Z(G)$ and $t_2Z(G)$, so that if we let

$$w_1 = x^{2m_0},$$
$$w_2 = [t_1, x]^{m_1}[t_2, x]^{m_2},$$

then a word $w \in G[x]$ as in (3.8) can also be written as $w = zw_1w_2$.

3.2.2 Description of $\mathbb{E}_{Q_8}$ and $\mathcal{U}_{Q_8}$

First we consider $G = Q_8$. Here $Z(G) = \{1, -1\}$, and note that the subset $T = \{i, j\} \subseteq G$ generates G, so the images of its elements in $G/Z(G)$ generate $G/Z(G)$. Then,

$$w_1 = x^{2m_0},$$
$$w_2 = [i, x]^{m_1}[j, x]^{m_2}.$$

Recall that $G^2 \subseteq Z(G) = G[2]$, so that for every $g \in G$

$$w_1(g) = \begin{cases} 1 & \text{if } g \in Z(G), \\ -1^{m_0} & \text{if } g \in G \setminus Z(G) = iZ(G) \cup jZ(G) \cup kZ(G). \end{cases}$$

As $[a,b] \in G' = Z(G) = \{1,-1\}$ for every $a,b \in G$, we have $[a,b] = -1$ if and only if a and b do not commute. In particular,

$$w_2(g) = \begin{cases} 1 & \text{if } g \in Z(G), \\ -1^{m_2} & \text{if } g \in \langle i \rangle \setminus Z(G) = \{i,-i\} = iZ(G), \\ -1^{m_1} & \text{if } g \in \langle j \rangle \setminus Z(G) = \{j,-j\} = jZ(G), \\ -1^{m_1+m_2} & \text{if } g \in \langle k \rangle \setminus Z(G) = \{k,-k\} = kZ(G). \end{cases}$$

Note that $z = w(e_G) = \mathrm{ct}(w)$. Then, we can compute $w(g) = zw_1(g)w_2(g)$, obtaining

$$w(g) = \begin{cases} z = \mathrm{ct}(w) & \text{if } g \in Z(G), \\ -1^{m_0+m_2}z & \text{if } g \in \langle i \rangle \setminus Z(G) = \{i,-i\} = iZ(G), \\ -1^{m_0+m_1}z & \text{if } g \in \langle j \rangle \setminus Z(G) = \{j,-j\} = jZ(G), \\ -1^{m_0+m_1+m_2}z & \text{if } g \in \langle k \rangle \setminus Z(G) = \{k,-k\} = kZ(G). \end{cases} \tag{3.9}$$

Recall that $Z(G) = G[2] = E_{x^2}$, so that $Z(G) \in \mathbb{E}_G$. *As a consequence, every coset of $Z(G)$ is an elementary algebraic subset of G.*

Let us deduce that $Z(G) \in \mathbb{E}_G$ using (3.9). Then, we will find all $w \in G[x]$ as in (3.8) such that $Z(G) = E_w$. By (3.9), $z = 1$ for such words w. We have to determine $m_0, m_1, m_2 \in \{0,1\}$ such that

$$\begin{cases} -1^{m_0+m_2} \neq 1 \\ -1^{m_0+m_1} \neq 1 \\ -1^{m_0+m_1+m_2} \neq 1, \end{cases} \quad \text{i.e.} \quad \begin{cases} m_0 + m_2 \in 1 + 2\mathbb{Z} \\ m_0 + m_1 \in 1 + 2\mathbb{Z} \\ m_0 + m_1 + m_2 \in 1 + 2\mathbb{Z}. \end{cases}$$

The (unique) solution of the above system is $m_0 = 1$, $m_1 = m_2 = 0$. As $z = 1$, we obtain $w = x^2 \in G[x]$, according to the fact that $Z(G) = G[2] = E_{x^2}$.

By Lemma 3.20, as $Z(G) = G[2] = E_{x^2}$, we get $G \setminus Z(G) = E_{-x^2}$. A direct argument is the following: as $G^2 = \{1,-1\}$, and $Z(G) = G[2]$, we obtain that $G \setminus Z(G) = \{g \in G \mid g^2 = -1\} = E_{-x^2}$.

Then, $G \setminus Z(G) \in \mathbb{E}_G$, and as a consequence every translate of $G \setminus Z(G)$ is an elementary algebraic subset. In other words, *every union of three cosets of $Z(G)$ is an elementary algebraic subset of G.*

We consider now the union of *two* cosets of $Z(G)$. Let us first consider

$$I = Z(G) \cup iZ(G) = \langle i \rangle.$$

As $C_G(i) = C_G(I) = I$, we have $I = E_w$ for $w = [i,x] \in G[x]$, so $I \in \mathbb{E}_G$. Taking translations in G, one obtains $jI = kI = jZ(G) \cup kZ(G)$, so that we also have

$jZ(G) \cup kZ(G) \in \mathbb{E}_G$. It is possible to conclude $jI \in \mathbb{E}_G$ also noting that jI is the complement of I in G, and then applying Lemma 3.20.

Now we classify also all words $w \in G[x]$ as in (3.8) such that $I = E_w$, i.e. such that $w(g) = 1$ if and only if $g \in Z(G) \cup iZ(G)$. By (3.9), $z = 1$ and we have to find $m_0, m_1, m_2 \in \{0, 1\}$ such that

$$\begin{cases} -1^{m_0+m_2} = 1 \\ -1^{m_0+m_1} \neq 1 \\ -1^{m_0+m_1+m_2} \neq 1, \end{cases} \quad \text{i.e.} \quad \begin{cases} m_0 + m_2 \in 2\mathbb{Z} \\ m_0 + m_1 \in 1 + 2\mathbb{Z} \\ m_0 + m_1 + m_2 \in 1 + 2\mathbb{Z}. \end{cases}$$

The unique solution of this system is $m_1 = 1$, and $m_0 = m_2 = 0$. The word w obtained this way is $w = [i, x] \in G[x]$, according to the fact that $[i, g] = 1$ if and only if $g \in \langle i \rangle$.

Similarly, letting $J = Z(G) \cup jZ(G) = \langle j \rangle$, we obtain that $J = E_{[j,x]}$, so that $J \in \mathbb{E}_G$. Then also its translate $iJ = iZ(G) \cup kZ(G) \in \mathbb{E}_G$. Finally, again the same argument proves that both $Z(G) \cup kZ(G)$ and its translate $iZ(G) \cup jZ(G)$ are elements of $\mathbb{E}_G$.

As $\emptyset$ and the whole G are elementary algebraic subsets, we can conclude that *every union of cosets of $Z(G)$ is an elementary algebraic subset of G.*

Now we are interested in studying $\mathcal{U}_{Q_8}$ and $\mathrm{u}(Q_8)$.

Lemma 3.21. $\mathrm{u}(Q_8) = 4$.

Proof. As $\exp(Q_8) = 4$, we have that $x^4 \in \mathcal{U}_{Q_8}$, so that $\mathrm{u}(Q_8) \mid 4$, and $\mathrm{u}(Q_8) \in \{1, 2, 4\}$. By Corollary 3.13, $\epsilon(w)$ is even for every $w \in \mathcal{U}_{Q_8}$, so that $\mathrm{u}(Q_8) \neq 1$. At this point, we will only have to exclude $\mathrm{u}(Q_8) = 2$.

So assume by contradiction $\epsilon(w) = 2$ for a $w \in \mathcal{U}_{Q_8}$. By Corollary 3.19, we can assume

$$w = zx^2[i, x]^{m_1}[j, x]^{m_2},$$

for $z \in Z(Q_8)$, and $m_1, m_2 \in \{0, 1\}$. Moreover, note that w has $m_0 = 1$, and $z = w(1) = 1$. Then equation (3.9) gives

$$\begin{cases} -1^{1+m_2} = 1 \\ -1^{1+m_1} = 1 \\ -1^{1+m_1+m_2} = 1, \end{cases} \quad \text{i.e.} \quad \begin{cases} 1 + m_2 \in 2\mathbb{Z} \\ 1 + m_1 \in 2\mathbb{Z} \\ 1 + m_1 + m_2 \in 2\mathbb{Z}. \end{cases} \tag{3.10}$$

Now system (3.10) implies $3 + 2(m_1 + m_2) \in 2\mathbb{Z}$, a contradiction. □

3.2.3 Description of $\mathbb{E}_{D_8}$ and $\mathcal{U}_{D_8}$

Now we consider $G = D_8$. Now $Z(G) = \{e, \rho^2\}$, and G is generated by $T = \{\rho, \sigma\} \subseteq G$, so that the image of T generates the quotient $G/Z(G)$. Letting

$$w_1 = x^{2m_0},$$
$$w_2 = [\rho, x]^{m_1}[\sigma, x]^{m_2}$$

(where $m_0, m_1, m_2 \in \{0, 1\}$), a word $w \in G[x]$ as in (3.8) can be written as $w = zw_1w_2$.

Now recall that $G^2 \subseteq Z(G)$ and that $G[2] = G \setminus \rho Z(G)$, so that $g^2 = \rho^2$ if and only if $g^2 \neq e$, i.e. $g \in \rho Z(G)$. In particular,

$$w_1(g) = \begin{cases} \rho^{2m_0} & \text{if } g \in \rho Z(G), \\ e & \text{if } g \in G \setminus \rho Z(G) = Z(G) \cup \sigma Z(G) \cup \rho\sigma Z(G). \end{cases}$$

As $[a, b] \in G' = Z(G) = \{e, \rho^2\}$ for every $a, b \in G$, we have $[a, b] = \rho^2$ if and only if a and b do not commute. In particular,

$$w_2(g) = \begin{cases} e & \text{if } g \in Z(G), \\ \rho^{2m_2} & \text{if } g \in \rho Z(G), \\ \rho^{2m_1} & \text{if } g \in \sigma Z(G), \\ \rho^{2(m_1+m_2)} & \text{if } g \in \rho\sigma Z(G). \end{cases}$$

Then, we can compute $w(g) = zw_1(g)w_2(g)$, obtaining

$$w(g) = \begin{cases} z & \text{if } g \in Z(G), \\ z\rho^{2(m_0+m_2)} & \text{if } g \in \rho Z(G), \\ z\rho^{2m_1} & \text{if } g \in \sigma Z(G), \\ z\rho^{2(m_1+m_2)} & \text{if } g \in \rho\sigma Z(G). \end{cases} \tag{3.11}$$

First of all, we show that $Z(G) \in \mathbb{E}_G$ using (3.11). We are looking for a word $w \in G[x]$ such that $Z(G) = E_w$, i.e. $w(g) = 1$ if and only if $g \in Z(G)$. By (3.11), $z = 1$ and we have to find $m_0, m_1, m_2 \in \{0, 1\}$ such that

$$\begin{cases} \rho^{2(m_0+m_2)} \neq e \\ \rho^{2m_1} \neq e \\ \rho^{2(m_1+m_2)} \neq e, \end{cases} \quad \text{i.e.} \quad \begin{cases} m_0 + m_2 \in 1 + 2\mathbb{Z} \\ m_1 \in 1 + 2\mathbb{Z} \\ m_1 + m_2 \in 1 + 2\mathbb{Z}. \end{cases} \tag{3.12}$$

The solution of system (3.12) is $m_0 = m_1 = 1$, and $m_2 = 0$. In this case, we obtain $w = x^2[\rho, x] \in G[x]$, and $Z(G) = E_w$. *As a consequence, every coset of $Z(G)$ is an elementary algebraic subset of G.*

As $Z(G) \in \mathbb{E}_G$, by Lemma 3.20 also $G \setminus Z(G) \in \mathbb{E}_G$, and so all translates of $G \setminus Z(G)$ are elements of $\mathbb{E}_G$ too. Then *every union of three cosets of* $Z(G)$ *is an elementary algebraic subset of* G.

Another argument to show that the cosets of $Z(G)$ and the unions of three cosets of $Z(G)$ are elementary algebraic subsets of G is the following. Recall that $G[2] = G \setminus \{\rho, \rho^3\} = G \setminus \rho Z(G)$, so that $G \setminus \rho Z(G) = E_{x^2} \in \mathbb{E}_G$. Then, $\rho Z(G) \in \mathbb{E}_G$ by Lemma 3.20, and we can conclude taking translations in G.

Now we will prove that every union of two cosets of $Z(G)$ is an elementary algebraic subset of G. We begin with $R = Z(G) \cup \rho Z(G) = \langle \rho \rangle$. As $\rho \notin Z(G)$, we have $R = C_G(\rho)$, so that we can immediately conclude $R = E_{[\rho,x]} \in \mathbb{E}_G$, and $\sigma R = \sigma Z(G) \cup \sigma\rho Z(G) \in \mathbb{E}_G$.

Similarly, if $S = Z(G) \cup \sigma Z(G) = \langle \rho^2, \sigma \rangle$, then $S = C_G(\sigma)$, so that $S = E_{[\sigma,x]} \in \mathbb{E}_G$, and $\rho S = \rho Z(G) \cup \rho\sigma Z(G) \in \mathbb{E}_G$.

The same argument applies to $P = Z(G) \cup \rho\sigma Z(G) = \langle \rho^2, \rho\sigma \rangle = C_G(\rho\sigma)$, so that $P \in \mathbb{E}_G$, and $\rho P = \rho Z(G) \cup \sigma Z(G) \in \mathbb{E}_G$. Finally, we conclude that *every union of cosets of* $Z(G)$ *is an elementary algebraic subset of* G.

Now we compute $\mathrm{u}(D_8)$.

Lemma 3.22. $\mathrm{u}(D_8) = 4$.

Proof. Note that $x^4 \in \mathcal{U}_{D_8}$, so that $\mathrm{u}(D_8) \mid 4$. Moreover, $\mathrm{u}(D_8) \neq 1$ by Corollary 3.13, so that now it suffices to prove that D_8 has no $w \in \mathcal{U}_{D_8}$ with $\epsilon(w) = 2$.

By Corollary 3.19, if $w \in \mathcal{U}_{D_8}$ and $\epsilon(w) = 2$, then $\mathrm{ct}(w) = e$ and we can assume

$$w = zx^2[\rho, x]^{m_1}[\sigma, x]^{m_2},$$

for $m_1, m_2 \in \{0, 1\}$. Moreover, note that such a w has $m_0 = 1$.

As $w(g) = 1$ for every $g \in D_8$, equation (3.11) implies

$$\begin{cases} \rho^{2(1+m_2)} = e \\ \rho^{2m_1} = e \\ \rho^{2(m_1+m_2)} = e \end{cases} \quad \text{i.e.} \quad \begin{cases} 1 + m_2 \in 2\mathbb{Z} \\ m_1 \in 2\mathbb{Z} \\ m_1 + m_2 \in 2\mathbb{Z}. \end{cases} \tag{3.13}$$

As system (3.13) implies $1 + 2(m_1 + m_2) \in 2\mathbb{Z}$, we obtain a contradiction. □

3.3 The universal exponent of a group

Recall the classes $\mathcal{W}_n$ introduced in Definition 2.21. We have already noted that a group $G \in \mathcal{W}_n$ if and only if $\epsilon(w) \in n\mathbb{Z}$ for every $w \in \mathcal{U}_G$ (i.e. $E_w = G$). Now, for every $n \in \mathbb{N}$ we introduce a subclass $\mathcal{W}_n^* \subseteq \mathcal{W}_n$ consisting of groups satisfying a stronger property.

Definition 3.23. If $n \in \mathbb{N}$ is a non-negative integer, we denote by $\mathcal{W}_n^*$ the class of groups G satisfying the following property: if $G = \bigcup_{i=1}^k E_{w_i}$, then $\epsilon(w_i) \in n\mathbb{Z}$ for some $i = 1, 2, \ldots, k$.

Clearly, $\mathcal{W}_n^* \subseteq \mathcal{W}_n$, and if $G \in \mathcal{W}_n$ is $\mathfrak{Z}$-irreducible, then $G \in \mathcal{W}_n^*$.

Example 3.24. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2^{\mathbb{N}}$. Then $\mathrm{u}(G) = \exp(G) = 4$ by Lemma 2.32, so that $G \in \mathcal{W}_4$. Let us see that $G \notin \mathcal{W}_4^*$. Let $c \in \mathbb{Z}_4$ be a generator of $\mathbb{Z}_4 = \{0, c, 2c, 3c\}$. Note that $G[2] = \{0, 2c\} \times \mathbb{Z}_2^{\mathbb{N}}$, so that $G = G[2] \cup (c + G[2])$. By (2.13), neither $G[2]$ nor $c + G[2]$ are of the form E_w for $w \in G[x]$ with $\epsilon(w) \in 4\mathbb{Z}$.

Note that every group is in $\mathcal{W}_1^* = \mathcal{W}_1$. Moreover, it immediately follows from the definitions that $\mathcal{W}_0^* \subseteq \bigcap_{n \in \mathbb{N}_+} \mathcal{W}_n^*$. In the following lemma (a counterpart of Lemma 2.22 (a) about the class $\mathcal{W}_0$), we see that also the reverse inclusion holds.

Lemma 3.25. *Let $N \subseteq \mathbb{N}$ be an infinite subset of $\mathbb{N}$. Then*

$$\mathcal{W}_0^* = \bigcap_{n \in N} \mathcal{W}_n^*.$$

Proof. Indeed, let $G \in \bigcap_{n \in N} \mathcal{W}_n^*$, and assume that $G = \bigcup_{i=1}^k E_{w_i}$. Then, for every $n \in N$ there is an $i_n = 1, \ldots, k$ such that $\epsilon(w_{i_n}) \in n\mathbb{Z}$. Being N infinite, there is an $i = 1, \ldots, k$ and an infinite $M \subseteq N$ such that $\epsilon(w_i) \in n\mathbb{Z}$ for every $n \in M$. Then $\epsilon(w_i) = 0$. $\square$

As $\mathcal{W}_m^* \subseteq \mathcal{W}_n^*$ whenever $0 \neq n \mid m$, if $n \in \mathbb{N}_+$, and $d \mid n$, then

$$\mathcal{W}_n^* = \bigcap \{\mathcal{W}_d^* \mid 1 \leq d \leq n,\ d \mid n\}.$$

In particular,

$$\mathcal{W}_1^* \supseteq \mathcal{W}_2^* \supseteq \ldots \supseteq \mathcal{W}_{n!}^* \supseteq \ldots \supseteq \mathcal{W}_0^*.$$

By Lemma 3.25, a group $G \in \mathcal{W}_0^*$ if and only if $G \in \mathcal{W}_n^*$ fon infinitely many $n \in \mathbb{N}$. Then, for any group G we define $\mathrm{u}^\circ(G) \in \mathbb{N}$ by

$$\mathrm{u}^\circ(G) = \begin{cases} 0 & \text{if } G \in \mathcal{W}_0^*, \\ \max\{k \in \mathbb{N} \mid G \in \mathcal{W}_k^*\} & \text{otherwise.} \end{cases} \tag{3.14}$$

By definition, when $\mathrm{u}^\circ(G) \neq 0$, it is the greatest $n \in \mathbb{N}$ such that $G \in \mathcal{W}_n^*$.

For every group G we introduce also $\mathrm{u}^*(G) \in \mathbb{N}$ by

$$\mathrm{u}^*(G) = \begin{cases} 0 & \text{if } G \in \mathcal{W}_0^*, \\ \text{the least common multiple of } \{k \in \mathbb{N} \mid G \in \mathcal{W}_k^*\} & \text{otherwise.} \end{cases} \tag{3.15}$$

The natural $\mathrm{u}^*(G)$ is called the u^*-*exponent of* G.

Then, in analogy with (2.7), we have the inclusion

$$\mathcal{W}^*_{\mathrm{u}^*(G)} \subseteq \bigcap\{\mathcal{W}^*_k \mid k \in \mathbb{N},\ G \in \mathcal{W}_k\}. \tag{3.16}$$

Obviously, $\mathrm{u}^\circ(G) = 0$ if and only if $\mathrm{u}^*(G) = 0$ if and only if $G \in \mathcal{W}^*_k$ for infinitely many $k \in \mathbb{N}$ if and only if $G \in \mathcal{W}^*_0$ by Lemma 3.25. In this case also $\mathrm{u}(G) = 0$.

Clearly, $\mathrm{u}^*(G) \mid \mathrm{u}(G)$ when $\mathrm{u}(G) = 0$. Let us see that this remains true also when $\mathrm{u}(G) \neq 0$. In fact, for every $k \in \mathbb{N}_+$, if $G \in \mathcal{W}^*_k$ then also $G \in \mathcal{W}_k$, as $\mathcal{W}^*_k \subseteq \mathcal{W}_k$, so that $k \mid \mathrm{u}(G)$ by Lemma 2.23, item 4. This gives $\mathrm{u}^*(G) \mid \mathrm{u}(G)$.

In particular, if $\mathrm{u}(G) \neq 0$, letting

$$S = \{k \in \mathbb{N} \mid G \in \mathcal{W}_k\},$$
$$S^* = \{k \in \mathbb{N} \mid G \in \mathcal{W}^*_k\},$$

we have $0 \notin S \supseteq S^*$ and both S and S^* are finite. Moreover, $\mathrm{u}(G) \in S$ is both the maximum and the least common multiple of S, while $\mathrm{u}^\circ(G)$ is the maximum of S^*, and $\mathrm{u}^*(G)$ is the least common multiple of S^*.

Moreover, $\mathrm{u}^\circ(G) \mid \mathrm{u}^*(G)$ obviously by the definitions, so that

$$\mathrm{u}^\circ(G) \mid \mathrm{u}^*(G), \text{ and } \mathrm{u}^*(G) \mid \mathrm{u}(G).$$

Let us say now that we do not know if the equality $\mathrm{u}^\circ(G) = \mathrm{u}^*(G)$ hold for every group G. Of course, this is equivalent to ask whether $G \in \mathcal{W}^*_{\mathrm{u}^*(G)}$ is true for every G, i.e. if the equality holds in (3.16).

Question 5. *Does there exist a group G such that* $\mathrm{u}^\circ(G) \neq \mathrm{u}^*(G)$*?*

Now we prove that if G is a finite group, then $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$.

Lemma 3.26. *Let G be a group, and $w \in G[x]$ be such that $N \subseteq E_w$ for a subgroup $N \leq G$ having finite index. Then* $\mathrm{u}^*(G) \mid \epsilon(w)$*, so that* $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$ *whenever* $\epsilon(w) = 1$.

In particular, $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$ *for every finite group G.*

Proof. Let $[G : N] = k$, and let $g_1, \dots, g_k \in G$ be so that $G = \bigcup_{i=1}^k g_i N$. Then also $G = \bigcup_{i=1}^k g_i E_w$. By Example 2.45, item 1, for every $i = 1, \dots, k$ we have $g_i E_w = E_{w_i}$ for $w_i = w \circ g_i^{-1} x$, and $\epsilon(w_i) = \epsilon(w)$. If $G \in \mathcal{W}^*_n$, then $n \mid \epsilon(w)$, so that $\mathrm{u}^*(G) \mid \epsilon(w)$. When $\epsilon(w) = 1$ one immediately obtains $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$ by definition.

When G is finite, it suffice to note that $E_x \supseteq \{e_G\}$. □

Now we give a combinatorial lemma about finite coverings of a group with cosets of subgroups. This result will be used in the following Corollary 3.28.

Lemma 3.27 ([46, Lemma 4.1]). *Let G be a group covered by finitely many cosets of subgroups, say $G = \bigcup_{i=1}^{r} g_i H_i$. Then $r \geq min\{[G : H_i] \mid 1 \leq i \leq r\}$. In particular, at least one of those subgroups has finite index.*

Compare the following result with Lemma 2.32 about the equality $\exp(G) = \mathrm{u}(G)$ for an abelian group G.

Corollary 3.28. *Let G be an abelian group. Then $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = \exp^*(G)$.*

Proof. We shall first consider the case when $\exp^*(G) = 0$. Then G is unbounded, so that $G \in \mathcal{W}_0^*$ by Lemma 2.42, and $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 0$.

When $\exp^*(G) = 1$, i.e. when G is finite, then $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = 1$ too by Lemma 3.26.

So now assume $n = \exp^*(G) > 1$. As $G/G[n] \cong nG$ is finite, we have $G = \bigcup_{i=1}^{r} g_i + G[n]$ for some elements $g_1, \dots, g_r \in G$. So if $G \in \mathcal{W}_k^*$ then $k \mid n$.

To conclude the proof, we see that also $G \in \mathcal{W}_n^*$, so that $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = n$. So let $G = \bigcup_{i=1}^{r} g_i + G[n_i]$. By Lemma 3.27, $[G : G[n_i]]$ is finite for some $i = 1, \dots, r$, i.e. $n_i G$ is finite. Then $n \mid n_i$ by definition. $\square$

3.4 δ-words

Definition 3.29. If G is a group, and $w \in G[x]$ is singular, we will say that w is a *δ-word for G* if $E_w^G = \{e_G\}$.

Let us immediately see that a non-trivial abelian group G never has any δ-word. Indeed, a singular word $w \in G[x]$ is a constant word $w = g$ for an element $g \in G$, so E_w^G is either empty, or the whole group G (see also Remark 2.41).

In the following lemma, we give a much more precise result.

Lemma 3.30. *If a group G has a δ-word, then its Taĭmanov topology $\mathcal{T}_G$ is discrete. In particular, G has trivial center.*

Proof. Assume $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x]$ to be a δ-word for G. Then in particular $e_G \in E_w$ and $\epsilon(w) = 0$, so that Lemma 3.3 (b) applies, giving $C_w = \{e_G\}$. As C_w is a $\mathcal{T}$-neighborhood of e_G, we have $\mathcal{T}_G = \delta_G$. $\square$

It directly follows by Lemma 3.30 that a group $G \in \mathscr{N}_2$ has no δ-words. Let us provide a different argument to prove this fact using Theorem 3.14.

If w is a δ-word for such a group G, then w_{ab} is trivial, being $\epsilon(w) = 0$ and $\mathrm{ct}(w) = e_G$, so we can assume $w = [x, a]$ by Theorem 3.14. Then, every central element of G lies in E_w, which is a contradiction.

In the following proposition we show a δ-word for every free non-abelian group.

Proposition 3.31. *Let F be a free non-abelian group, generated by the elements $\{a_i \mid i \in I\}$, and let a, b be two of them. Then*

$$w = [a,x][b,x] = axa^{-1}x^{-1}bxb^{-1}x^{-1} \in F[x]$$

is a δ-word for F.

Proof. Obviously w is singular, $w(e_F) = e_F$, and we have to prove that $f_w(g) \neq e_G$ for every $g \in F$, $g \neq e_F$. To this end, let $f_1 = f_{w_1}$ and $f_2 = f_{w_2}$, where

$$\begin{aligned} w_1 &= [a,x]^{-1} = [x,a] = xax^{-1}a^{-1}, \\ w_2 &= [b,x] = bxb^{-1}x^{-1}. \end{aligned}$$

As $w = w_1^{-1}w_2$, we have that $f_w = (f_1)^{-1}f_2$, and so $f_w(g) = e_G$ if and only if $f_1(g) = f_2(g)$, for every $g \in F$. So it suffices to prove that $f_1(g) \neq f_2(g)$ for every $g \in F$, $g \neq e_F$.

So let $e_F \neq g \in F$, and we are going to show that $f_1(g) \neq f_2(g)$. Note that we can assume $g \notin \bigcup_{i\in I}\langle a_i \rangle$, so let $g = a_i^n h a_j^m$ be the reduced form of g, for $h \in F$, $0 \neq n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. In particular, if $h = e_F$, then $g = a_i^n a_j^m$, with $i \neq j$. Then

$$\begin{aligned} f_1(g) &= a_i^n h a_j^m \cdot a \cdot (a_i^n h a_j^m)^{-1} \cdot a^{-1} = a_i^n h \underline{a_j^m \cdot a \cdot a_j^{-m}} h^{-1} \underline{a_i^{-n} \cdot a^{-1}}, \\ f_2(g) &= b \cdot a_i^n h a_j^m \cdot b^{-1} \cdot (a_i^n h a_j^m)^{-1} = \underline{b \cdot a_i^n} h \underline{a_j^m \cdot b^{-1} \cdot a_j^{-m}} h^{-1} a_i^{-n}. \end{aligned}$$

As the only possible cancellations are between underlined elements, we can immediately say that $f_1(g)$ begins with $a_i^n h \ldots$; on the other hand, $f_2(g)$ either begins with $a_i^{n+1} h \ldots$ (if $a_i = b$), or it begins with $b \cdot a_i^n h \ldots$ (if $a_i \neq b$). In either case, $f_1(g) \neq f_2(g)$. □

We shall see in §6.1.1 some important applications of these facts.

Problem 1. *Determine which groups admit a δ-word.*

Recall that a group G with a δ-word has a discrete Taĭmanov topology.

By Proposition 6.12, every product of free non-abelian groups has a δ-word.

4

Quasi-topological group topologies

We will now characterize which topologies on a group make it a quasi-topological group, in term of continuity of an appropriate family of verbal functions.

Lemma 4.1. *Let G be a group, and τ a topology on G. Then (G, τ) is a quasi-topological group if and only if f_w is τ-continuous for every word $w = gx^\varepsilon$, with $g \in G$ and $\varepsilon = \pm 1$.*

In particular, if a topology σ on a group G makes continuous every verbal function, then (G, σ) is a quasi-topological group. If σ is also T_1, then $\mathfrak{Z}_G \subseteq \sigma$.

Proof. Let ι denote the inversion function of G. If (G, τ) is a quasi-topological group, then every function f_{gx}, being a left translation, is τ-continuous. Also every $f_{gx^{-1}} = f_{gx} \circ \iota$ is τ-continuous.

For the converse, let τ be a topology on G such that $f_w \colon (G, \tau) \to (G, \tau)$ is continuous for every word $w = gx^\varepsilon$, with $g \in G$ and $\varepsilon = \pm 1$. Then points 3 and 5 in Example 2.15 show that the inversion and the left translations are τ-continuous. Finally, the right translation by an element g is $f_{xg} = f_{x^{-1}} \circ f_{g^{-1}x^{-1}}$.

For the last part, just note that if $\{e_G\}$ is σ-closed and every f_w is σ-continuous, then also every $E_w = f_w^{-1}(\{e_G\})$ is σ-closed. As $\mathbb{E}_G$ is a subbase for $\mathfrak{Z}_G$-closed sets, we conclude $\mathfrak{Z}_G \subseteq \sigma$. $\square$

Example 4.2. Let (G, τ) be a T_1 quasi-topological group. By Lemma 4.1, every verbal function in $\{f_{gx^\varepsilon} \mid g \in G, \varepsilon = \pm 1\}$ is τ-continuous. We shall see that not every verbal function needs to be τ-continuous. To this end, recall that the space $(G, co\text{-}\lambda_G)$ is a T_1 quasi-topological group for every infinite cardinal number λ.

So let $\omega \leq \lambda < \kappa = |G|$, and consider $\tau = co\text{-}\lambda_G \neq \delta_G$.

1. Let G be a group having a non-central element a such that $|C_G(a)| \geq \lambda$ (for example, the group $G = \oplus_\kappa S_3$ will do). Then let $w = [a, x] \in G[x]$, and consider the commutator verbal function $f_w \in \mathscr{F}(G)$. As $f_w^{-1}(\{e_G\}) = C_G(a)$, we have that f_w is not τ-continuous.

2. Let G be a group such that $|G[2]| \geq \lambda$ (also in this case the group $G = \oplus_\kappa S_3$ considered above will do). Then let $w = x^2 \in G[x]$, and consider the homogeneous verbal function $f_w \in \mathscr{F}(G)$. As $f_w^{-1}(\{e_G\}) = G[2]$, we have that f_w is not τ-continuous.

Proposition 4.3. *For every group G, the following hold.*

1. *Every verbal function is $\mathfrak{Z}_G$-continuous.*

2. *The pair $(G, \mathfrak{Z}_G)$ is a quasi-topological group.*

3. *$\mathfrak{Z}_G$ is the initial topology of the family of all verbal functions $\{f\colon G \to (G, \mathfrak{Z}_G) \mid f \in \mathscr{F}(G)\}$.*

Proof.

1. Follows from the fact that $\mathbb{E}_G$ is a subbase for the $\mathfrak{Z}_G$-closed subsets of G, and from Lemma 2.44.

2. Immediately follows by Lemma 4.1 and item 1.

3. Also follows by item 1.

□

Corollary 4.4. *Every group topology on a group G makes continuous every verbal function of G. In particular $\mathfrak{M}_G$ and $\mathfrak{P}_G$ make continuous every verbal function of G, so $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$, and all the three are quasi-topological group topologies.*

Proof. As a verbal function is a composition of products and inversions, it is continuous with respect to every group topology. The same is true for $\mathfrak{M}_G$ and $\mathfrak{P}_G$, which are intersections of group topologies, then Lemma 4.1 applies. □

Proposition 4.5. *Let N be a normal subgroup of a group G, and let $\overline{G} = G/N$. Then the quotient topology $\overline{\mathfrak{Z}}_G$ makes continuous every verbal function of $\overline{G}$.*

Proof. Let $\overline{w} = \overline{g}_1 x^{\varepsilon_1} \overline{g}_2 x^{\varepsilon_2} \cdots \overline{g}_n x^{\varepsilon_n} \in \overline{G}[x]$, and we have to prove that

$$f_{\overline{w}}\colon (\overline{G}, \overline{\mathfrak{Z}}_G) \to (\overline{G}, \overline{\mathfrak{Z}}_G)$$

is continuous. Let $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G[x]$. In the notation of Remark 2.16, item 2, recall that we have seen there that the following diagram commutes.

$$\begin{tikzcd} (G, \mathfrak{Z}_G) \arrow[r, "f_w"] \arrow[d, "\pi"'] & (G, \mathfrak{Z}_G) \arrow[d, "\pi"] \\ (\overline{G}, \overline{\mathfrak{Z}}_G) \arrow[r, "f_{\overline{w}}"] & (\overline{G}, \overline{\mathfrak{Z}}_G). \end{tikzcd} \tag{4.1}$$

As a consequence, also $f_{\overline{w}}$ is continuous, as f_w is continuous and $\overline{\mathfrak{Z}}_G$ is the final topology of the canonical projection $\pi\colon (G, \mathfrak{Z}_G) \to \overline{G}$. □

Proposition 4.6. *Let N be a normal subgroup of a group G, and let $\overline{G} = G/N$. Then the following conditions are equivalent:*

(1) N is $\mathfrak{Z}_G$-closed;

(2) $\overline{\mathfrak{Z}}_G$ *is a* T_1 *topology;*

(3) $\mathfrak{Z}_{\overline{G}} \subseteq \overline{\mathfrak{Z}}_G$*;*

(4) the canonical map $\pi\colon (G, \mathfrak{Z}_G) \to (\overline{G}, \mathfrak{Z}_{\overline{G}})$ *is continuous.*

Proof. (1) $\leftrightarrow$ (2) follows by Proposition 1.8.

(2) $\to$ (3) follows by Proposition 4.5 and Lemma 4.1.

(3) $\to$ (4). In this case, the map $\mathrm{id}\colon (\overline{G}, \overline{\mathfrak{Z}}_G) \to (\overline{G}, \mathfrak{Z}_{\overline{G}})$ is continuous, and so also the composition

$$(G, \mathfrak{Z}_G) \xrightarrow{\pi} (\overline{G}, \overline{\mathfrak{Z}}_G) \xrightarrow{\mathrm{id}} (\overline{G}, \mathfrak{Z}_{\overline{G}})$$

as desired.

(4) $\to$ (1) holds as $N = \pi^{-1}(e_{\overline{G}})$ and $\{e_{\overline{G}}\}$ is $\mathfrak{Z}_{\overline{G}}$-closed. □

We shall see examples of groups G having $\mathfrak{Z}_G$-closed subgroup N such that the inclusion $\mathfrak{Z}_{\overline{G}} \subseteq \overline{\mathfrak{Z}}_G$ in Proposition 4.6, item (3), manifestally fails to be an equality. See for example Remark 6.50 where we have an infinite quotient $\overline{G}$ with $\mathfrak{Z}_{\overline{G}} = cof_{\overline{G}}$, while $\overline{\mathfrak{Z}}_G$ is a compact Hausdorff totally disconnected group topology.

Corollary 4.7. *For every group* G*, and every positive integer* n*, the subgroup* $Z_n(G)$ *is* $\mathfrak{Z}_G$*-closed.*

Proof. We have already seen in Example 2.43, item 1, that $Z_1(G) = Z(G)$ is $\mathfrak{Z}_G$-closed in any group G. Let $\overline{G} = G/Z(G)$. As $Z(\overline{G})$ is $\mathfrak{Z}_{\overline{G}}$-closed, the projection $\pi\colon (G, \mathfrak{Z}_G) \to (\overline{G}, \mathfrak{Z}_{\overline{G}})$ is continuous by Proposition 4.6, so $Z_2(G) = \pi^{-1}(Z(\overline{G}))$ is $\mathfrak{Z}_G$-closed.

Proceed by induction to get the thesis. □

Remark 4.8. Corollary 4.7 can be proved observing that it is possible to define by induction $Z_1(G) = Z(G)$ and, for an integer $i \geq 1$, $x \in Z_{i+1}(G)$ if and only if $[x, g] = xgx^{-1}g^{-1} \in Z_i(G)$ for every $g \in G$. Equivalently:

$$Z_{i+1}(G) = \bigcap_{g\in G} \{x \in G \mid [x, g] \in Z_i(G)\} = \bigcap_{g\in G} [\,\cdot\,, g]^{-1}(Z_i(G)).$$

For every $g \in G$, the commutator verbal function $[\,\cdot\,, g]\colon x \mapsto [x, g]$ is $\mathfrak{Z}_G$-continuous, and $Z_i(G)$ is $\mathfrak{Z}_G$-closed by inductive hypothesis, so $[\,\cdot\,, g]^{-1}(Z_i(G))$ is $\mathfrak{Z}_G$-closed, and $Z_{i+1}(G)$ is an intersection of $\mathfrak{Z}_G$-closed subsets.

Definition 4.9. *A countably infinite subset* A *of a group* G *is called* $\mathfrak{Z}_G$*-atom if* $\mathfrak{Z}_G\restriction_A = cof_A$*.*

We shall see in Fact 4.12 that the $\mathfrak{Z}_G$-atoms determine the Zariski topology on an abelian group G. This fails to be true in general, as a group $G \in \mathscr{N}_2$ may have no $\mathfrak{Z}_G$-atoms at all (see Corollary 8.30).

4.1 The Markov topologies of the abelian groups

Theorem 4.10 ([21]). *If G is an abelian group, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ is a Noetherian topology, whose family of closed sets is $\mathbb{E}_G^{\cup}$.*

A different proof of the fact that $\mathfrak{Z}_G = \mathfrak{M}_G$ for an abelian group G can be found in [20] or [60], where the authors independently proved the following more general result.

Theorem 4.11 ([20, 60]). *Let A be an abelian group, $\{H_i \mid i \in I\}$ be a family of countable groups, and $G = A \times \bigoplus_{i\in I} H_i$. Then $\mathfrak{Z}_G = \mathfrak{M}_G$.*

The following properties of the Zariski topology of an abelian group G were established in [21]:

Fact 4.12 ([21]). *Let G be an abelian group.*

(a) [21, Theorem 3.5, Corollary 3.6] The space $(G, \mathfrak{Z}_G)$ is Noetherian, so it is Hausdorff if and only if G is finite.

(b) [21, Theorem 4.6 (ii)] For a bounded abelian group G, the connected component $c(G, \mathfrak{Z}_G) = G[\exp^*(G)]$ has finite index in G, so $(G, \mathfrak{Z}_G)$ has only finitely many connected components. In particular, if G is an infinite abelian group, $(G, \mathfrak{Z}_G)$ is not totally disconnected.

(c) [21] For every infinite set $X \subseteq G$ and each point $x \in \overline{X}^{\mathfrak{Z}_G} \setminus X$ one can find a faithfully indexed subset $Y = \{y_n \mid n \in \mathbb{N}\} \subseteq X$ having the cofinite topology such that the sequence y_n converges to x in the Zariski topology $\mathfrak{Z}_G$.

Then note that the subset Y as in Fact 4.12 (c) is a $\mathfrak{Z}_G$-atom. An equivalent reformulation of Fact 4.12 (c) is the following: a subset $X \subseteq G$ is $\mathfrak{Z}_G$-closed if and only if either $A \subseteq X$ or $A \cap X$ is finite, for every $\mathfrak{Z}_G$-atom A.

From Lemma 2.42, it immediately follows the following.

Corollary 4.13 ([21, Theorem 4.6 (ii)]). *If G is an unbounded abelian group, then $(G, \mathfrak{Z}_G)$ is irreducible, and consequently connected.*

As Proposition 4.15 shows, the implication from the above corollary cannot be inverted. It provides a complete description of the abelian groups that are $\mathfrak{Z}$-irreducible. The $\mathfrak{Z}$-irreducible subsets of an abelian group are classified in [21].

Fact 4.14 ([21, Corollary 4.7]). *An abelian group G is $\mathfrak{Z}$-irreducible if and only if G is either unbounded or bounded with* $\exp(G) = \exp^*(G)$.

The following result from [61] classifies the smaller class of abelian groups that have a cofinite Zariski topology.

Proposition 4.15 ([61, Theorem 5.1]). *An abelian group G is $\mathfrak{Z}$-cofinite if and only if either G is almost torsion-free, or* $\exp(G) = p$ *for some $p \in \mathbb{P}$.*

The following result is a corollary of Proposition 4.15.

Proposition 4.16. *Let K be a field, λ a cardinal, and $m > 0$ an integer. Then:*

(a) the group $(K, +)^\lambda$ is $\mathfrak{Z}$-cofinite;

(b) the group $(K^*, \cdot)^m$ is $\mathfrak{Z}$-cofinite.

As a consequence, all their subgroups are $\mathfrak{Z}$-cofinite; in particular, the group $(\bigoplus_\lambda K, +)$ is $\mathfrak{Z}$-cofinite.

Proof. In both cases, we are going to apply Proposition 4.15.

(a) Note that K^λ is either torsion-free, if $\operatorname{char} K = 0$, or $\exp(K^\lambda) = p$, if $\operatorname{char} K = p > 0$.

(b) It follows from the elementary properties of fields, that $(K^*, \cdot)$ is almost torsion-free. This entails that $(K^*, \cdot)^m$ is almost torsion-free as well, so Proposition 4.15 applies.

Finally, the last assertion follows by equation (5.1), where $\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$ is noted for every subgroup H of an arbitrary group G. □

Problem 2. *Describe the class of $\mathfrak{Z}$-cofinite groups. Does there exist an infinite, non-abelian, $\mathfrak{Z}$-cofinite group?*

4.2 Partial Zariski topologies

Given a subset $W \subseteq G[x]$, we consider the family $\mathcal{E}(W) = \{E_w^G \mid w \in W\} \subseteq \mathbb{E}_G$ of elementary algebraic subsets of G determined by the words $w \in W$. Then, following [5] and [6], we consider the topology $\mathfrak{T}_W$ having $\mathcal{E}(W)$ as a subbase for its closed sets.

Example 4.17. 1. Note that $\mathcal{E}(G[x]) = \mathbb{E}_G$, so $\mathfrak{T}_{G[x]} = \mathfrak{Z}_G$.

2. Taking $W = \{gx \mid g \in G\}$, one obtains that $\mathcal{E}(W) = \{\{g\} \mid g \in G\}$, so that $\mathfrak{T}_W = cof_G$.

Lemma 4.18. *Assume that $gw \in W$, for every $w \in W$ and every $g \in G$. Then $\mathfrak{T}_W$ is the initial topology of the family of verbal functions $\{f_w \colon G \to (G, cof_G) \mid w \in W\}$.*

Proof. If τ is such initial topology, then the subsets $f_w^{-1}(g) = f_{g^{-1}w}^{-1}(e_G)$, for $w \in W$ and $g \in G$, form a subbase for τ-closed sets. On the other hand, the elementary algebraic subsets $E_w = f_w^{-1}(e_G)$, for $w \in W$, form a subbase for the family of $\mathfrak{T}_W$-closed subsets. By assumption, those families of subsets coincide. □

In particular, $\mathfrak{Z}_G$ can be equivalently defined as the initial topology of the family of all verbal functions $\{f \colon G \to (G, cof_G) \mid f \in \mathscr{F}(G)\}$.

Example 4.19. Let $a, b \in G$, and $w = axbx^{-1} = ab[b^{-1}, x] \in G[x]$. Note that $E_w \neq \emptyset$ if and only if there exists an element $g \in G$ such that $a = gb^{-1}g^{-1}$, i.e. a and b^{-1} are conjugated elements in G. In this case, $w = (gb^{-1}g^{-1})xbx^{-1}$.

In particular, letting $V = \{axbx^{-1} \mid a, b \in G\} \subseteq G[x]$ and

$$W_{\mathfrak{C}} = \{[g, a][a, x] = (gag^{-1})xa^{-1}x^{-1} \mid a, g \in G\} \subseteq V, \tag{4.2}$$

we obtain that $\mathcal{E}(V)\setminus\emptyset = \mathcal{E}(W_{\mathfrak{C}}) \subseteq \mathcal{E}(V)$, so that $\mathfrak{T}_V = \mathfrak{T}_{W_{\mathfrak{C}}}$. Moreover, by Example 2.45, item 2, we have

$$\mathcal{E}(W_{\mathfrak{C}}) = \{gC_G(a) \mid a, g \in G\}.$$

Definition 4.20. *Given a group G, the* centralizer topology $\mathfrak{C}_G$ *is the topology $\mathfrak{T}_{W_{\mathfrak{C}}}$, for $W_{\mathfrak{C}} \subseteq G[x]$ as in (4.2) in Example 4.19.*

By definition, the family $\{gC_G(a) \mid a, g \in G\}$ is a subbase for the $\mathfrak{C}_G$-closed subsets of G. So note that $\mathfrak{C}_G \subseteq \mathcal{T}_G$ in general (more on this in §4.3).

Remark 4.21. If $S \subseteq G$, let

$$C(S) = \{[g, a][a, x] = (gag^{-1})xa^{-1}x^{-1} \mid g \in G, a \in S\} \subseteq G[x],$$
$$D(S) = \{[xcx^{-1}, b] \mid b, c \in S\} \subseteq G[x].$$

For example, $C(G) = W_{\mathfrak{C}}$ as in (4.2), so that $\mathfrak{T}_{C(G)} = \mathfrak{C}_G$.

In [4], the authors introduced two *restricted Zariski topologies* $\mathfrak{Z}'_G$, $\mathfrak{Z}''_G$ on a group G, that in our notation are respectively $\mathfrak{Z}'_G = \mathfrak{T}_{C(G[2]) \cup D(G[2])}$, and $\mathfrak{Z}''_G = \mathfrak{T}_{C(G[2])}$. Obviously, $\mathfrak{Z}''_G \subseteq \mathfrak{Z}'_G \subseteq \mathfrak{Z}_G$ and $\mathfrak{Z}''_G \subseteq \mathfrak{C}_G$ hold for every group G.

In the following definition we introduce the partial Zariski topology $\mathfrak{T}_{mon}$ determined by the monomials. Note that by Lemma 2.49 there is no harm in considering only the monomials with non-negative content. Moreover, by Example 2.40 we can indeed consider only positive-content monomials.

Definition 4.22. *If $M = \{gx^n \mid g \in G,\ n \in \mathbb{N}_+\} \subseteq G[x]$ is the family of the monomials with positive content, then we denote $\mathfrak{T}_{mon}$ the topology having $\mathcal{E}(M)$ as a subbase of its closed sets, and we call it the* monomial topology.

Note that $gx \in M$ for every $g \in G$, so that $\mathfrak{T}_{mon}$ is T_1 topology.

Example 4.23. We have seen in §2.2.4 that $w \approx w_{ab}$ for every $w \in G[x]$, when G is abelian. As in studying E_w we can assume $\epsilon(w) \geq 0$ by Lemma 2.49, this shows that $\mathfrak{T}_{mon} = \mathfrak{Z}_G$.

Now we recall a classical result due to Chernikov, that we will use in the subsequent corollary.

Fact 4.24. *If G is a nilpotent, torsion-free group, then G satisfies the 'cancellation law', i.e. for every $n \in \mathbb{N}_+$ and $x, y \in G$, if $x^n = y^n$ then $x = y$.*

Corollary 4.25. *If G is a nilpotent, torsion-free group, then $\mathfrak{T}_{mon} = cof_G$.*

Proof. If $w = gx^m \in G[x]$ is a monomial with $m > 0$, it will suffice to prove that E_w has at most one element. Assume $a \in E_w$, so that $a^m = g^{-1}$. Then $E_w = \{p \in G \mid p^m = a^m\}$, so that $E_w = \{a\}$ by Fact 4.24. □

4.3 Centralizer topologies

In this section, we study two partial Zariski topologies. The first one is the topology $\mathfrak{C}_G$ introduced in Definition 4.20. As we shall see in Lemma 4.28 (3), the topology $\mathfrak{C}_G$ is not T_1 in general. For this reason, in analogy with Definition 1.15, we also introduce the following topology.

Definition 4.26. *The* T_1 centralizer topology $\mathfrak{C}'_G$ *on a group* G *is the supremum (in the lattice of all topologies on* G*)* $\mathfrak{C}'_G = \mathfrak{C}_G \vee cof_G$*.*

Remark 4.27. Note that $\mathfrak{C}'_G$ is T_1, and that $\mathfrak{C}_G \subseteq \mathfrak{C}'_G \subseteq \mathfrak{Z}_G$, so that $\mathfrak{C}_G = \mathfrak{C}'_G$ if and only if $\mathfrak{C}_G$ is T_1.

Let $W = W_{\mathfrak{C}} \cup \{gx \mid g \in G\}$. Then

$$\mathcal{E}(W) = \mathcal{E}(W_{\mathfrak{C}}) \cup \{\{g\} \mid g \in G\} = \{gC_G(a) \mid a, g \in G\} \cup \{\{g\} \mid g \in G\}$$

by Example 4.19, and $\mathfrak{C}'_G = \mathfrak{T}_W$. Obviously, $\mathfrak{C}'_G = \mathfrak{T}_{W'}$ also for $W = \{axbx^{-1} \mid a, b \in G\} \cup \{xg \mid g \in G\}$.

Here follows some easy to establish properties of $\mathfrak{C}_G$ and $\mathfrak{C}'_G$.

Lemma 4.28. *Let* G *be a group. Then:*

1. *both the pair* $(G, \mathfrak{C}_G)$ *and* $(G, \mathfrak{C}'_G)$ *are quasi-topological groups;*

2. *the closure* $\overline{\{e_G\}}^{\mathfrak{C}_G} = Z(G)$*;*

3. $\mathfrak{C}_G$ *is* T_1 *(so* $\mathfrak{C}_G = \mathfrak{C}'_G$*) if and only if* $Z(G) = \{e_G\}$*, while* $\mathfrak{C}_G = \iota_G$ *is indiscrete if and only if* $G = Z(G)$ *is abelian;*

4. *if* $H \leq G$*, then* $\mathfrak{C}_H \subseteq \mathfrak{C}_G \restriction_H$ *and* $\mathfrak{C}'_H \subseteq \mathfrak{C}'_G \restriction_H$*.*

Proof. (2). As $Z(G) = \bigcap_{g \in G} C_G(g)$ is $\mathfrak{C}_G$-closed, one only has to verify that every $\mathfrak{C}_G$-closed subset containing e_G must also contain $Z(G)$.

(3). Immediately follows from (2) and (1).

(4). To prove that $\mathfrak{C}_H \subseteq \mathfrak{C}_G \restriction_H$, it suffices to note that for every element $h \in H$ we have that $C_H(h) = C_G(h) \cap H$ is a $\mathfrak{C}_G \restriction_H$-closed subset of H.

Then,

$$\mathfrak{C}'_H = \mathfrak{C}_H \vee cof_H \subseteq (\mathfrak{C}_G \restriction_H \vee cof_G \restriction_H) \subseteq (\mathfrak{C}_G \vee cof_G) \restriction_H = \mathfrak{C}'_G \restriction_H .$$

□

Example 4.29. Let F be a free non-abelian group. As F is center-free, then $\mathfrak{C}_F = \mathfrak{C}'_F$ by Lemma 4.28, item 3. Indeed, we will prove in Theorem 7.5 that $\mathfrak{Z}_F = \mathfrak{C}'_F$, so that $\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F$.

Example 4.30. Let us show that the inclusion $\mathfrak{C}_H \subseteq \mathfrak{C}_G \restriction_H$ in Lemma 4.28, item 4, may be proper. To this end, it will suffice to consider a group G having an abelian, non-central subgroup H, so that

$$\iota_H = \mathfrak{C}_H \subsetneq \mathfrak{C}_G \restriction_H .$$

Indeed, $\iota_H = \mathfrak{C}_H$ holds by Lemma 4.28, item 3, as H is abelian, while $\emptyset \neq Z(G) \cap H \subsetneq H$ is a $\mathfrak{C}_G \restriction_H$-closed subset of H.

Lemma 4.31. *Let G be a group, $\overline{G} = G/Z(G)$, and τ be the initial topology on G of the map*

$$\pi\colon G \to (\overline{G}, cof_{\overline{G}}). \tag{4.3}$$

Then $\tau \subseteq \mathfrak{C}_G$.

Moreover, $\mathfrak{C}_G = \tau$ if and only if for every $g \in G \setminus Z(G)$ the index $[C_G(g) : Z(G)]$ is finite.

Proof. As the family of singletons of $\overline{G}$ is a subbase for $cof_{\overline{G}}$-closed sets, and $\pi^{-1}(\{gZ(G)\}) = gZ(G)$ is $\mathfrak{C}_G$-closed for every $g \in G$ by Lemma 4.28, item 1 and 2, we immediately obtain $\tau \subseteq \mathfrak{C}_G$.

For the reverse inclusion, we have that $\mathfrak{C}_G \subseteq \tau$ if and only if $C_G(g)$ is τ-closed for every $g \in G$, if and only if $C_G(g)$ is τ-closed for every $g \in G \setminus Z(G)$, as $C_G(g) = G$ is certanly τ-closed for a central element g.

Finally note that, if $g \in G \setminus Z(G)$, then $G \gneq C_G(g)$ is τ-closed exactly when $\pi(C_G(g)) = C_G(g)/Z(G)$ is finite. □

Now we prove that $\mathfrak{C}_G$ and $\mathcal{T}_G$ coincide on an FC-group G.

Lemma 4.32. *If G is an FC-group, then $\mathfrak{C}_G = \mathcal{T}_G$.*

In particular, if $G \in \mathscr{N}_2$ and $Z(G)$ is finite, then G is an FC-group, so $\mathfrak{C}_G = \mathcal{T}_G$.

Proof. The inclusion $\mathfrak{C}_G \subseteq \mathcal{T}_G$ holds for every group, so we prove the reverse one. To this end, it suffices to prove that $C_G(F)$ is a $\mathfrak{C}_G$-neighborhood of e_G, for every $F \in [G]^{<\omega}$. So let $F \in [G]^{<\omega}$, and note that $C_G(F)$ is a finite-index subgroup as G is an FC-group. As $(G, \mathfrak{C}_G)$ is a quasi-topological group by Lemma 4.28, item 1, we can apply Theorem 1.6 (c) to conclude that $C_G(F)$ is $\mathfrak{C}_G$-open.

If $G \in \mathscr{N}_2$ and $Z(G)$ is finite, then G is an FC-group by Corollary 3.10. □

We conclude this chapter with a couple of results on the centralizer topologies on groups in $\mathscr{N}_2$.

Here follows an immediate corollary of Lemma 3.17.

Corollary 4.33. *If $G \in \mathscr{N}_2$ and $G/Z(G)$ is torsion-free, then $\mathfrak{Z}_G = \mathfrak{T}_{mon} \vee \mathfrak{C}_G$.*

Proof. It suffices to prove $\mathfrak{Z}_G \leq \mathfrak{T}_{mon} \vee \mathfrak{C}_G$. To this end, we see that every $E \in \mathbb{E}_G$ is either $\mathfrak{T}_{mon}$-closed, or $\mathfrak{C}_G$-closed. By Lemma 3.17, if $e_G \in E \in \mathbb{E}_G$ then either $E = G[m] = E_{x^m}$ is a $\mathfrak{T}_{mon}$-closed set, or E is the centralizer of a single element, hence a $\mathfrak{C}_G$-closed set. □

The following theorem is another application of Lemma 3.17.

Theorem 4.34. *If $G \in \mathscr{N}_2$ is torsion-free, then $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I}$ for every non-empty set I.*

Proof. Recall that also $G/Z(G)$ is torsion-free by Proposition 3.9.

Note that the power $G^I \in \mathscr{N}_2$ has the same properties as G, so we can simply replace G^I by G. Now Lemma 3.17 applies, giving that the translate of every $E \in \mathbb{E}_G$ is either a singleton, or the centralizer of a single element of G, so that $\mathfrak{Z}_G \subseteq \mathfrak{C}'_G$. Hence, the two topologies coincide. □

Recall the definition of the T_1 Taĭmanov topology $\mathcal{T}'_G = cof_G \vee \mathcal{T}_G$ given in Definition 1.15. In the following result, we use Lemma 3.18 to compare $\mathcal{T}'_G$ with the Zariski topology for a class of groups in $\mathscr{N}_2$.

Theorem 4.35. *Let $G \in \mathscr{N}_2$ of prime exponent $p > 2$. Then $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I} \leq \mathcal{T}'_{G_I}$ for every set I.*

If moreover $Z(G)$ is finite, and I is finite, then $\mathfrak{C}_{G^I} = \mathcal{T}_{G^I}$, so that $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I} = \mathcal{T}'_{G^I}$.

Proof. Note that $G^I \in \mathscr{N}_2$ also has exponent p, se we can simply replace G^I by G in the first part of the assertion. The inclusion $\mathfrak{C}'_G \leq \mathfrak{Z}_G$ is trivial, while the converse immediately follows from Lemma 3.18, as every elementary algebraic subset is a singleton or the coset of a centralizer. This proves $\mathfrak{Z}_G = \mathfrak{C}'_G$. As the inclusion $\mathfrak{C}_G \leq \mathcal{T}_G$ always holds, we also have $\mathfrak{C}'_G \leq \mathcal{T}'_G$.

When both $Z(G)$ and I are finite, then $Z(G^I)$ is also finite, so $\mathfrak{C}_{G^I} = \mathcal{T}_{G^I}$ by Lemma 4.32. □

5

Embeddings

Remark 5.1. If H is a subgroup of a group G, then H carries its own Zariski and Markov topologies $\mathfrak{Z}_H$ and $\mathfrak{M}_H$, as well as the induced topologies $\mathfrak{Z}_G\restriction_H$ and $\mathfrak{M}_G\restriction_H$. If $w \in H[x]$, then one can consider w also in $G[x]$, so that both E_w^H and E_w^G make sense, and $E_w^H = E_w^G \cap H$. From this, one can deduce the inclusion $\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$. This gives the following inclusions between the four mentioned topologies on H:

$$\mathfrak{M}_G\restriction_H \supseteq \mathfrak{Z}_G\restriction_H \supseteq \mathfrak{Z}_H \subseteq \mathfrak{M}_H. \tag{5.1}$$

To describe better the cases when some of the inclusions can be equalities, the following definition was given in [20].

Definition 5.2 ([20, Definitions 2.1, 3.1])**.** *A subgroup H of a group G is called:*

(a) super-normal in G *if for every $g \in G$ there exist $h \in H$ such that $gxg^{-1} = hxh^{-1}$ for every $x \in H$;*

(b) Zariski embedded in G, or $\mathfrak{Z}$-embedded *if the injection $(H, \mathfrak{Z}_H) \hookrightarrow (G, \mathfrak{Z}_G)$ is continuous;*

(c) Markov embedded in G, or $\mathfrak{M}$-embedded *if the injection $(H, \mathfrak{M}_H) \hookrightarrow (G, \mathfrak{M}_G)$ is continuous.*

(d) Hausdorff embedded in G *if every Hausdorff group topology on H can be extended to a Hausdorff group topology on G.*

Remark 5.3. Note that H is Markov embedded in G if and only if $\mathfrak{M}_G \restriction_H \subseteq \mathfrak{M}_H$.

Similarly, H is Zariski embedded in G if and only if $\mathfrak{Z}_G \restriction_H \subseteq \mathfrak{Z}_H$, but in this case the topologies coincide by (5.1). It is also equivalent to ask $E_w^G \cap H$ to be an algebraic subset of H for every word $w \in G[x]$.

Note also that all four properties in Definition 5.2 are transitive with respect to composition of injections. There are easy examples where those properties fail (see for Example Remark 12.15).

Below we will discuss how the properties in Definition 5.2 are related. Proposition 5.10 (a) and (b) give respectively the implications (1) and (3) in the following diagram, while (2) follows by Lemma 5.9.

$$\begin{tikzcd}
 & \text{Hausdorff embedded} \arrow[r, "(3)"'] \arrow[dr, "/" marking, "(5)"'] & \text{Markov embedded} \\
\text{super-normal} \arrow[ur, "(1)"] \arrow[dr, "(2)"'] & & \\
 & \text{Zariski embedded \& Markov embedded} \arrow[uu, "/" marking, "(4)"'] \arrow[r] & \text{Zariski embedded} \arrow[uu, "?" marking, "(6)"']
\end{tikzcd} \tag{5.2}$$

We will provide evidence for the failure of the implications (4) (see Remark 9.20) and (5) (see Example 5.4 below). This shows, among others, that none of the reverse implications of (1), (2), (3) and (6) holds true). The missing implication (6) is left as an open question (see Question 11).

Example 5.4. Let us verify now the non-implication (5) in the above diagram. This will imply that the reverse implication of (1) and (6) fail as well.

To this end we provide an argument that is essentially [20, Corollary 6.17]. Take a $\mathfrak{M}$-discrete, not $\mathfrak{Z}$-discrete group H (for example, one of the groups constructed by Hesse in Theorem 11.25). By Example 11.17, item 4, embed H in a $\mathfrak{Z}$-discrete group G. Then $\mathfrak{Z}_G\restriction_H = \mathfrak{M}_G\restriction_H = \mathfrak{M}_H$ is discrete, while $\mathfrak{Z}_H$ is not discrete. Note that H is trivially Hausdorff embedded, but not Zariski embedded in G.

In the rest of this subsection, we will mainly discuss some results about the definitions given in Definition 5.2.

Proposition 5.5 ([20, Lemma 3.2])**.** *A subgroup H of a group G is super-normal if and only if $G = HC_G(H)$. In particular, if H is abelian, then H is super-normal in G if and only if it is central.*

Note that every central subgroup is super-normal by Proposition 5.5.

Proposition 5.6. *Let G be a group, S be a countable subset of G, and $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ be a countable family in $\mathbb{E}_G^\cup$. Then there exists a countable subgroup H of G containing S and such that $F_n \cap H \subseteq H \in \mathbb{E}_H^\cup$ for every $n \in \mathbb{N}$.*

Proof. For every $n \in \mathbb{N}$, let $F_n = \bigcup_{i=1}^{k_n} E^G_{w_i^{(n)}}$ for words $w_1^{(n)}, \dots, w_{k_n}^{(n)} \in G[x]$.

Let H be the (countable) subgroup of G generated by S and $\bigcup_{n\in\mathbb{N}} \bigcup_{i=1}^{k_n} \operatorname{coeff}(w_i^{(n)})$. Then $E^G_{w_i^{(n)}} \cap H = E^H_{w_i^{(n)}}$ is an elementary algebraic subset of H for every n and i. So $F_n \cap H = \bigcup_{i=1}^{k_n} E^H_{w_i^{(n)}}$ is an additively algebraic subset of H. $\square$

Remark 5.7. In the notation of Proposition 5.6, for every subgroup K of G containing H, $F_n \cap K \in \mathbb{E}_K^\cup$ for every $n \in \mathbb{N}$.

Lemma 5.8. *Every word $w \in G[x]$ can be written in the form*

$$w = (\alpha_1 x^{\varepsilon_1} \alpha_1^{-1})(\alpha_2 x^{\varepsilon_2} \alpha_2^{-1}) \cdots (\alpha_{n-1} x^{\varepsilon_{n-1}} \alpha_{n-1}^{-1}) \alpha_n x^{\varepsilon_n}, \tag{5.3}$$

for $\alpha_1, \ldots, \alpha_n \in G$.

Proof. If $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n}$, for $i = 1, \ldots, n$ let $\alpha_i = g_1 g_2 \cdots g_i$. $\square$

Lemma 5.9. *Let H be a super-normal subgroup of a group G. If $E \in \mathbb{E}_G$, then $E \cap H \in \mathbb{E}_H$. In particular, H is Zariski embedded in G.*

Proof. Let $E = E_w$ for some $w \in G[x]$. By Lemma 5.8, it is not restrictive to consider only words $w \in G[x]$ as in (5.3), for whom

$$E_w^G \cap H = \{x \in H \mid (\alpha_1 x^{\varepsilon_1} \alpha_1^{-1})(\alpha_2 x^{\varepsilon_2} \alpha_2^{-1}) \cdots (\alpha_n x^{\varepsilon_n} \alpha_n^{-1}) = \alpha_n^{-1}\}.$$

As H is super-normal in G, for every $i = 1, \ldots, n$ there exists $\beta_i \in H$ such that, for every $x \in H$, $\alpha_i x^{\varepsilon_i} \alpha_i^{-1} = \beta_i x^{\varepsilon_i} \beta_i^{-1}$. So

$$E_w^G \cap H = \{x \in H \mid (\beta_1 x^{\varepsilon_1} \beta_1^{-1})(\beta_2 x^{\varepsilon_2} \beta_2^{-1}) \cdots (\beta_n x^{\varepsilon_n} \beta_n^{-1}) = \alpha_n^{-1}\}. \tag{5.4}$$

If $\alpha_n \in G \setminus H$, then $E_w^G \cap H = \emptyset$ and there is nothing to prove. If $\alpha_n \in H$, then every coefficient in the equation (5.4) defining $E_w^G \cap H$ is an element of H, so $E_w^G \cap H$ is an elementary algebraic subset of H. $\square$

By Proposition 5.5 and Lemma 5.9, every central subgroup is Zariski embedded, unlike G' (see Example 11.7 (c)).

The following proposition shows how some of the properties in Definition 5.2 are related.

Proposition 5.10. *(a)* ([20, Corollary 2.6]) *A super-normal subgroup is Hausdorff embedded.*

(b) ([20, Lemma 2.7]) *A Hausdorff embedded subgroup is also Markov embedded.*

The next theorem characterizes normal Hausdorff embedded subgroups of a group.

Theorem 5.11 ([20, Theorem 3.4])**.** *Let N be a normal subgroup of a group G. Then N is Hausdorff embedded in G if and only if all the automorphisms of N induced by conjugation by elements of G are continuous for every Hausdorff group topology on N.*

The following fact from [21] shows that distinguishing between Zariski embedded and Markov embedded subgroups requires groups or subgroups on which the Zariski and Markov topologies differ.

Fact 5.12 ([21, Lemma 2.2])**.** *Let H be a subgroup of a group G.*

(a) *If* $\mathfrak{Z}_H = \mathfrak{M}_H$ *and* H *is Markov embedded, then* H *is also Zariski embedded.*

(b) *If* $\mathfrak{Z}_G = \mathfrak{M}_G$ *and* H *is Zariski embedded, then* H *is also Markov embedded.*

In analogy to Definition 5.2, we introduce here the following notion.

Definition 5.13. *A subgroup* H *of a group* G *is called* $\mathfrak{P}$-embedded in G *if the injection* $(H, \mathfrak{P}_H) \hookrightarrow (G, \mathfrak{P}_G)$ *is continuous (i.e., if* $\mathfrak{P}_G\restriction_H \subseteq \mathfrak{P}_H$*).*

6

Direct products and direct sums

If $I \neq \emptyset$ is a set, and $\{G_i \mid i \in I\}$ is a family of groups, throughout this section we will consider the direct product $G = \prod_{i\in I} G_i$, and denote $\prod_{i\in I} \mathfrak{Z}_{G_i}$ the product topology on G of the Zariski topologies $\mathfrak{Z}_{G_i}$ on each factor G_i.

We will consider also the direct sum $S = \bigoplus_{i\in I} G_i$, and denote $\sigma = \prod_{i\in I} \mathfrak{Z}_{G_i} \restriction_S = \bigoplus_{i\in I} \mathfrak{Z}_{G_i}$ the topology on S induced by $\prod_{i\in I} \mathfrak{Z}_{G_i}$. If all the groups G_i coincide, we will denote G and S respectively G_i^I and $G_i^{(I)}$.

If $J \subseteq I$, we will denote $\prod_{i\in J} G_i$ the subgroup $\prod_{i\in J} G_i \times \prod_{i\in I\setminus J}\{e_{G_i}\}^{\{i\}} \leq G$, in order to omit the trivial factors when no confusion is possible.

Lemma 6.1. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. Then there exists a canonical map $\vartheta\colon G[x] \to \prod_{i\in I}\big(G_i[x]\big)$.*

Proof. For every $i \in I$, let $p_i\colon G \to G_i$ be the i-th canonical projection. Apply Proposition 2.4 to obtain the homomorphism $\pi_i\colon G[x] \to G_i[x]$, such that $\pi_i \restriction_G = p_i$, and $\pi_i(x) = x$. Finally, consider the diagonal map ϑ of the family $\{\pi_i \mid i \in I\}$, so that $\vartheta\colon G[x] \to \prod_{i\in I}\big(G_i[x]\big)$.

The map $\vartheta\colon G[x] \to \prod_{i\in I}\big(G_i[x]\big)$ has the following explicit form. Let

$$w = g^{(1)}x^{\varepsilon_1}g^{(2)}x^{\varepsilon_2}\cdots g^{(n)}x^{\varepsilon_n} \in G[x],$$

where $g^{(j)} = (g_i^{(j)})_{i\in I} \in G$ for elements $g_i^{(j)} = p_i(g^{(j)}) \in G_i$, for $i \in I$ and $j = 1, \dots, n$. Denote by $w_i = g_i^{(1)}x^{\varepsilon_1}g_i^{(2)}x^{\varepsilon_2}\cdots g_i^{(n)}x^{\varepsilon_n} \in G_i[x]$ the word in G_i obtained by taking the i-th coordinate of the coefficients of w. Then $w_i = \pi_i(w)$, and $\vartheta(w) = (w_i)_{i\in I} \in \prod_{i\in I}\big(G_i[x]\big)$. □

Definition 6.2. *In the notation of Lemma 6.1, we will call $\vartheta(w) = (w_i)_{i\in I}$ the* coordinates *of w in $\prod_{i\in I}\big(G_i[x]\big)$. Note that $\epsilon(w) = \epsilon(w_i)$ for every $i \in I$.*

The map ϑ in Lemma 6.1 is not injective, if $|I| > 1$ and if the groups under consideration are not trivial (we will discuss $\ker(\vartheta)$ in Example 6.5 below). Nonetheless, Lemma 6.1 suffices to obtain the following corollary which describes the verbal functions of a direct product.

Corollary 6.3. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. If $w \in G[x]$ has coordinates $\vartheta(w) = (w_i)_{i\in I} \in \prod_{i\in I}\big(G_i[x]\big)$, then the verbal function $f_w\colon G \to G$ is the mapping $(g_i)_{i\in I} \mapsto (f_{w_i}(g_i))_{i\in I}$.*

In the following theorem we show that the elementary algebraic subset E_w of a direct product is the direct product of the elementary algebraic subsets E_{w_i}, where $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}\bigl(G_i[x]\bigr)$.

Theorem 6.4. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. If $w \in G[x]$, and $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}\bigl(G_i[x]\bigr)$, then E_w^G has the form*

$$E_w^G = \prod_{i\in I} E_{w_i}^{G_i}. \tag{6.1}$$

In particular, $w \in \mathcal{U}_G$ if and only if $w_i \in \mathcal{U}_{G_i}$ for every $i \in I$.

As a consequence, the Zariski topology $\mathfrak{Z}_G$ of the direct product is coarser than the product topology $\prod_{i\in I}\mathfrak{Z}_{G_i}$.

Proof. It suffices to note that $g = (g_i)_{i\in I} \in G$ satisfies $w(g) = e_G$ if and only if $g_i \in G_i$ satisfies $w_i(g_i) = e_{G_i}$ for every $i \in I$, by Corollary 6.3. Thus E_w^G is as in (6.1), and $E_w^G = G$ if and only if $E_{w_i}^{G_i} = G_i$ for every $i \in I$.

By (6.1), it follows that E_w^G is closed in the product topology $\prod_{i\in I}\mathfrak{Z}_{G_i}$. Being $\mathbb{E}_G$ a subbase for $\mathfrak{Z}_G$-closed sets, we conclude $\mathfrak{Z}_G \subseteq \prod_{i\in I}\mathfrak{Z}_{G_i}$. □

Example 6.5. Let G_1, G_2 be non-trivial groups, $g_i \in G_i \setminus \{e_{G_i}\}$, and $G = G_1 \times G_2$. Consider the word

$$w = (g_1^{-1}, e_{G_2})x(e_{G_1}, g_2)x^{-1}(g_1, e_{G_2})x(e_{G_1}, g_2^{-1})x^{-1} \in G[x].$$

Then w is non-trivial, as in fact $\mathrm{l}(w) = 4$. As

$$\begin{aligned} w_1 = \pi_1(w) = g_1^{-1}xe_{G_1}x^{-1}g_1xe_{G_1}x^{-1} = e_{G_1[x]},\\ w_2 = \pi_2(w) = e_{G_2}xg_2x^{-1}e_{G_2}xg_2^{-1}x^{-1} = e_{G_2[x]}, \end{aligned}$$

we have $w \in \ker(\vartheta)$, in the notation of Lemma 6.1.

Note that, if $w \in \ker(\vartheta)$, then $w_i = e_{G_i[x]}$ is the trivial word for every $i \in I$, so that in particular $w_i \in \mathcal{U}_{G_i}$. Then also $w \in \mathcal{U}_G$ by Theorem 6.4.

Corollary 6.6. *Let G_1, G_2 be non-trivial groups, and $G = G_1 \times G_2$. Then G has a singular, non-trivial universal word.*

Proof. Consider the singular, non-trivial word $w \in G[x]$ defined in Example 6.5. Its coordinates in $G_1[x] \times G_2[x]$ are $(w_1, w_2) = (e_{G_1[x]}, e_{G_2[x]})$, so that equation (6.1) gives $E_w^G = E_{e_{G_1[x]}}^{G_1} \times E_{e_{G_2[x]}}^{G_2} = G_1 \times G_2$. □

These two topologies $\mathfrak{Z}_G$ and $\prod_{i\in I}\mathfrak{Z}_{G_i}$ on a product group $G = \prod_{i\in I} G_i$ need not coincide even in very simple cases. For example the Zariski topology of $G = \mathbb{Z}\times\mathbb{Z}$ is the cofinite topology by Proposition 4.15, so neither $\mathbb{Z}\times\{0\}$ nor $\{0\}\times\mathbb{Z}$ are Zariski closed in G, whereas they are certainly closed in the product topology (see §6.1 for more details).

Remark 6.7. Let $\mathcal{F} = \{G_i \mid i \in I\}$ be a family of non-trivial groups, and $G = \prod_{i\in I} G_i$. For every $i \in I$, let $e_{G_i} \neq g_i \in G_i$, and $D_i = \{e_{G_i}, g_i\} \subseteq G_i$. Obviously, $\mathfrak{Z}_{G_i} \restriction_{D_i} = \delta_{D_i}$ being $\mathfrak{Z}_{G_i}$ a T_1 topology, so that $D = \prod_{i\in I} D_i \subseteq G$ is a (compact) Hausdorff topological space when equipped with the topology $\sigma = \prod_{i\in I} \delta_{D_i}$. In particular, being D infinite, the topological space (D, σ) is not Noetherian, by Remark 1.2 (a). As $\sigma = (\prod_{i\in I} \mathfrak{Z}_{G_i}) \restriction_D$, also $(G, \prod_{i\in I} \mathfrak{Z}_{G_i})$ is not Noetherian again by Remark 1.2 (a).

These easy observations produce plenty of examples showing that $\mathfrak{Z}_G$ needs not coincide with $\prod_{i\in I} \mathfrak{Z}_{G_i}$: for example, it suffices to consider a family $\mathcal{F}$ such that $\mathfrak{Z}_G$ *is* Noetherian (such families will be classified in Theorem 10.12).

The next definition will be used in the following Lemma 6.9 to give a sufficient condition on an I-ple $(w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$ to belong to $\vartheta\big((\prod_{i\in I} G_i)[x]\big)$.

Definition 6.8. *Let $w \in G[x]$. If $\mathrm{l}(w) = n \in \mathbb{N}_+$ and $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} g_{n+1}$ is as in (2.1), we define $\vec{\epsilon}(w) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{1, -1\}^n$.*

Lemma 6.9. *Let $n \in \mathbb{N}_+$, $\vec{\epsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) \in \{1, -1\}^n$, and $\{G_i \mid i \in I\}$ be a family of groups. For every $i \in I$, let $w_i \in G_i[x]$ be such that $\mathrm{l}(w_i) = n$ and $\vec{\epsilon}(w_i) = \vec{\epsilon}$.*

If $G = \prod_{i\in I} G_i$, then there exists $w \in G[x]$ such that $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}\big(G_i[x]\big)$.

If in addition every $w_i \in G_i[x]$ is a δ-word (resp., a universal word) for G_i, then also $w \in G[x]$ is a δ-word (resp., a universal word) for G.

Proof. By assumption, for every $i \in I$, the word w_i has the form

$$w_i = g_i^{(1)} x^{\varepsilon_1} g_i^{(2)} x^{\varepsilon_2} \cdots g_i^{(n)} x^{\varepsilon_n} \in G_i[x].$$

Defining $g^{(j)} = (g_i^{(j)})_{i\in I} \in G$ for $j = 1, \dots, n$, the word $w = g^{(1)} x^{\varepsilon_1} g^{(2)} x^{\varepsilon_2} \cdots g^{(n)} x^{\varepsilon_n} \in G[x]$ is such that $\vartheta(w) = (w_i)_{i\in I}$, where $\vartheta\colon G[x] \to \prod_{i\in I}\big(G_i[x]\big)$ is the map defined in Lemma 6.1, i.e. $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}\big(G_i[x]\big)$. By (6.1), w is a δ-word (resp., a universal word) for G, if every $w_i \in G_i[x]$ is a δ-word (resp., a universal word). □

Remark 6.10. Let $\{B_i \mid i \in I\}$ be a family of bounded abelian groups, with $\exp(B_i) = n_i > 2$, and assume the set $F = \{n_i \mid i \in I\}$ to be finite. For every $i \in I$, let K_i be the group constructed from B_i as in Example 2.38 (a), and fix an element $\sigma_i \in K_i \setminus B_i$. Finally, let $H = \prod_{i\in I} K_i$, and $\Sigma = (\sigma_i)_{i\in I} \in H$.

(a) For every $i \in I$, let $v_i = (\sigma_i x)^{n_i+2}(\sigma_i x^{-1})^{n_i-2} \in \mathcal{U}_{K_i}$ be as in equation (2.12). If $N = \prod_{n\in F} n$, then also $\overline{v}_i = (\sigma_i x)^{N+2}(\sigma_i x^{-1})^{N-2} \in \mathcal{U}_{K_i}$, for every $i \in I$.

By Lemma 6.9, there is a universal word $\overline{v} \in \mathcal{U}_H$ such that $(\overline{v}_i)_{i\in I} \in \prod_{i\in I}(K_i[x])$ are the coordinates of $\overline{v}$, and $\epsilon(\overline{v}) = \epsilon(\overline{v}_i) = 4$.

Now we explicitly produce such a $\overline{v} \in \mathcal{U}_H$, letting

$$\overline{v} = (\Sigma x)^{N+2}(\Sigma x^{-1})^{N-2}. \tag{6.2}$$

By construction, the coordinates of $\overline{v} \in H[x]$ are $(\overline{v}_i)_{i\in I} \in \prod_{i\in I}(K_i[x])$, so

$$E^H_{\overline{v}} = \prod_{i\in I} E^{K_i}_{\overline{v}_i} = \prod_{i\in I} K_i = H.$$

(b) If in addition the groups K_i are as in Example 2.38 (b), then $\mathrm{u}(K_i) = 2$ for every $i \in I$, and let $w_i = (\sigma_i x)^{n'_i+1}(\sigma_i x^{-1})^{n'_i-1} \in \mathcal{U}_{K_i}$ be as in equation (2.11). If $N = \prod_{n\in F} n'$, then also $\overline{w}_i = (\sigma_i x)^{N+1}(\sigma_i x^{-1})^{N-1} \in \mathcal{U}_{K_i}$, and

$$\overline{w} = (\Sigma x)^{N+1}(\Sigma x^{-1})^{N-1} \in \mathcal{U}_H$$

is a universal word for H, with $\epsilon(\overline{w}) = 2$. As $H \in \mathcal{W}_2$ by Corollary 2.36, item 1, in particular $\mathrm{u}(H) = 2$ by Lemma 2.23, item 3.

Lemma 6.11. *Let G be a group, and I be a set. Then G has a δ-word if and only if G^I does.*

Proof. Let $w \in G[x]$ be a δ-word. Then Lemma 6.9 trivially applies to give that there exists a word $v \in G^I[x]$ such that $(w)_{i\in I} \in G[x]^I$ are the coordinates of v, and v is a δ-word.

On the other hand, $w \in G^I[x]$ with coordinates $(w_i)_{i\in I}$ is a δ-word if and only if $w_i \in G[x]$ is a δ-word for every $i \in I$, again by (6.1). □

As a consequence of Proposition 3.31 and Lemma 6.11, we get that every power of a free non-abelian group has a δ-word. In the following result, we show that every *product* of free non-abelian groups has a δ-word.

Proposition 6.12. *Let $\{G_i \mid i \in I\}$ be a family of free non-abelian groups. Then $G = \prod_{i\in I} G_i$ has a δ-word.*

Proof. For every $i \in I$, let $a_i, b_i \in G_i$ be two of the generators of G_i, and $w_i = [a_i, x][b_i, x] = a_i x a_i^{-1} x^{-1} b_i x b_i^{-1} x^{-1} \in G_i[x]$ be the δ-word for G_i constructed in Proposition 3.31. As $\mathrm{l}(w_i) = 4$, and $\vec{\epsilon}(w_i) = (1, -1, 1, -1)$ for every $i \in I$, Lemma 6.9 applies and there exists a δ-word $w \in G[x]$ such that $(w_i)_{i\in I}$ are the coordinates of w in $\prod_{i\in I}\big(G_i[x]\big)$. □

In the following lemma, we use Theorem 6.4 to study when a direct product of groups lies in some $\mathcal{W}_n$.

Lemma 6.13. *Let $N = \{n_i \mid i \in I\} \subseteq \mathbb{N}$ be a set of naturals, and $\{G_i \mid i \in I\}$ be a family of groups. For every $i \in I$, assume that $G_i \in \mathcal{W}_{n_i}$, and let $G = \prod_{i\in I} G_i$.*

- *If either N is unbounded, or $0 \in N$, then $G \in \mathcal{W}_0$.*

- *Otherwise, let n be the least common multiple of N. Then $G \in \mathcal{W}_n$.*

Proof. Let $w \in \mathcal{U}_G$ have coordinates $(w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$. Then $w_i \in \mathcal{U}_{G_i}$ for every $i \in I$ by Theorem 6.4, hence $\epsilon(w_i) \in n_i\mathbb{Z}$.

As $\epsilon(w) = \epsilon(w_i)$ for every $i \in I$, we get $\epsilon(w) \in \bigcap_{i\in I} n_i\mathbb{Z}$. Finally, note that if N is either unbounded, or $0 \in N$, then $\bigcap_{i\in I} n_i\mathbb{Z} = \{0\}$. Otherwise, $0 \notin N$ is bounded, and $\bigcap_{i\in I} n_i\mathbb{Z} = n\mathbb{Z}$. □

Recall that every group lies in the class $\mathcal{W}_1$. Then note that Corollary 2.36, item 1 is a particular case of Lemma 6.13, when one assumes that, for some index $i_0 \in I$, the group $G_{i_0} \in \mathcal{W}_n$, and $G_i \in \mathcal{W}_1$ for every $i_0 \neq i \in I$.

Corollary 6.14. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. Then $\mathrm{u}(G_i) \mid \mathrm{u}(G)$ for every $i \in I$.*

Moreover, if either $N = \{\mathrm{u}(G_i) \mid i \in I\} \subseteq \mathbb{N}$ is unbounded, or $0 \in N$, then $\mathrm{u}(G) = 0$.

Proof. If either N is unbounded, or $0 \in N$, then $G \in \mathcal{W}_0$ by Lemma 6.13, so $\mathrm{u}(G) = 0$ by Lemma 2.23, item 3.

Otherwise, if n is the least common multiple of N, then $G \in \mathcal{W}_n$ by Lemma 6.13, so that $n \mid \mathrm{u}(G)$. □

Corollary 6.15. *Let G be a group, and I be a set. Then $\mathrm{u}(G^I) = \mathrm{u}(G)$.*

Proof. By Lemma 2.23, item 1, there exists $w_0 \in \mathcal{U}_G$ with $\epsilon(w_0) = \mathrm{u}(G)$. Then G^I has a universal word w with $\epsilon(w) = \mathrm{u}(G)$ by Lemma 6.9, so that $\mathrm{u}(G^I) \mid \mathrm{u}(G)$. On the other hand, $\mathrm{u}(G) \mid \mathrm{u}(G^I)$ by Corollary 6.14. □

We conclude this part with a few results on the Zariski topology of a direct product.

Lemma 6.16. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $X_i \subseteq G_i$ be a subset for every $i \in I$. If $G = \prod_{i\in I} G_i$, then $\prod_{i\in I} C_{G_i}(X_i)$ is a $\mathfrak{Z}_G$-closed subgroup of G.*

In particular, if G_{i_0} is center-free for some $i_0 \in I$, then $\prod_{i_0\neq i\in I} G_i$ is $\mathfrak{Z}_G$-closed.

Proof. Follows from the fact that $\prod_{i\in I} C_{G_i}(X_i) = C_G(\prod_{i\in I} X_i)$, then Example 2.43, item 1, applies.

In the special case when G_{i_0} is center-free, then

$$\{e_{G_{i_0}}\} \times \prod_{i_0\neq i\in I} G_i = C_G(G_{i_0} \times \prod_{i_0\neq i\in I} \{e_{G_i}\}).$$

□

Remark 6.17. Recall that if $G \in \mathscr{N}_2$, and $\exp Z(G) = p \in \mathbb{P}$, then the elementary algebraic subsets with more than one element are union of cosets of $Z(G)$, by Corollary 3.13. Then, we have showed in §3.2 that for $G = Q_8$ and $G = D_8$ *all* possible unions of cosets of $Z(G)$ are actually elementary algebraic subsets.

Now, let us note that this does not happen in general, for example for a group $G = G_1 \times G_2$. In fact, let $w \in G[x]$ have coordinates $(w_1, w_2) \in G_1[x] \times G_2[x]$, so that $E_w^G = E_{w_1}^{G_1} \times E_{w_2}^{G_2}$ by equation (6.1). Now assume $E_w = T \cdot Z(G)$ for some subset $T \subseteq G$. Then we have $E_{w_1}^{G_1} \times E_{w_2}^{G_2} = T \cdot \big(Z(G_1) \times Z(G_2)\big)$. If $\pi_i \colon G \to G_i$ for $i = 1, 2$ are respectively the projections on the first and second coordinate, then we obtain $E_{w_1}^{G_1} = \pi_1(T) \cdot Z(G_1)$ and $E_{w_2}^{G_2} = \pi_2(T) \cdot Z(G_2)$, so that finally

$$E_w^G = \big(\pi_1(T) \times \pi_2(T)\big) \cdot Z(G).$$

In particular, only the unions of cosets taken over rectangular subsets of G are possible.

6.1 Finite products

We begin giving a sufficient condition for a direct product to belong to $\mathcal{W}_n^*$.

Lemma 6.18. *Let G_1, G_2 be groups, with $G_1 \in \mathcal{W}_n^*$. Then $G_1 \times G_2 \in \mathcal{W}_n^*$.*

Proof. Let $G = G_1 \times G_2$, and $G = \bigcup_{i=1}^k E_{w_i}^G$. If $(w_i', w_i'') \in G_1[x] \times G_2[x]$ are the coordinates of w_i, then $G = \bigcup_{i=1}^k E_{w_i'}^{G_1} \times E_{w_i''}^{G_2}$. In particular, $G_1 = \bigcup_{i=1}^k E_{w_i'}^{G_1}$, so that $\epsilon(w_i') \in n\mathbb{Z}$ for some $i = 1, \dots, k$ as $G_1 \in \mathcal{W}_n^*$. We conclude recalling $\epsilon(w_i) = \epsilon(w_i')$. $\square$

Definition 6.19. Let G_1, G_2 be groups, and $G = G_1 \times G_2$. Then the pair G_1, G_2 will be called:

- $\mathfrak{Z}$-*productive*, if $\mathfrak{Z}_G = \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$.
- *semi* $\mathfrak{Z}$-*productive*, if both $G_1 \times \{e_{G_2}\}$ and $\{e_{G_1}\} \times G_2$ are $\mathfrak{Z}_G$-closed subsets of G;
- *strongly semi* $\mathfrak{Z}$-*productive*, if both $G_1 \times \{e_{G_2}\}$ and $\{e_{G_1}\} \times G_2$ are additively algebraic subsets of G;

From the definitions, it immediately follows the implications below, for every pair G_1, G_2:

$$\mathfrak{Z}\text{-productive} \Rightarrow \text{semi } \mathfrak{Z}\text{-productive} \Leftarrow \text{strongly semi } \mathfrak{Z}\text{-productive}.$$

Remark 6.20. 1. As noted above, a strongly semi $\mathfrak{Z}$-productive pair is semi $\mathfrak{Z}$-productive. If every $\mathfrak{Z}_{G_1 \times G_2}$-closed subset is additively algebraic, then these two conditions are equivalent. According to Theorem 4.10, this happens if G_1, G_2 are abelian.

2. Note that the pair G_1, G_2 is $\mathfrak{Z}$-productive exactly when $\mathfrak{Z}_{G_1\times G_2} \supseteq \mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}$, by Theorem 6.4.

As already noted, if G_1, G_2 is $\mathfrak{Z}$-productive, then it is semi $\mathfrak{Z}$-productive. We are interested in studying when the converse implication holds true, so we explicitly state the following question.

Question 6. *Let G_1, G_2 be a semi $\mathfrak{Z}$-productive pair. Is G_1, G_2 then $\mathfrak{Z}$-productive?*

According to Corollary 6.31, to answer negatively this question it suffices to find a pair of center-free groups G_1, G_2 that is not $\mathfrak{Z}$-productive.

Theorem 6.38 will answer the above question when G_1, G_2 are abelian, thus classifying the abelian $\mathfrak{Z}$-productive pairs.

6.1.1 Groups with δ-words

Lemma 6.21. *Let G_1 be a group, G_2 be a group having a δ-word, and $G = G_1 \times G_2$. Then $G_1 \times \{e_{G_2}\} = E^G_w$, for a singular word $w \in G[x]$.*

Proof. Let $w_0 = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n} \in G_2[x]$ be a δ-word for G_2. Define the elements $\widetilde{g}_i = (e_{G_1}, g_i) \in G$, and the word $w = \widetilde{g}_1 x^{\varepsilon_1} \widetilde{g}_2 x^{\varepsilon_2} \cdots \widetilde{g}_n x^{\varepsilon_n} \in G[x]$. Then $\epsilon(w) = \epsilon(w_0) = 0$, and $(w_1, w_0) \in G_1[x] \times G_2[x]$ are the coordinates of w in $G_1[x] \times G_2[x]$. Note that $w_1 = e_{G_1} x^{\varepsilon_1} e_{G_1} x^{\varepsilon_2} \cdots e_{G_1} x^{\varepsilon_n} = x^{\epsilon(w_0)} = x^0$ is the neutral element of $G_1[x]$, so that $E^G_w = E^{G_1}_{w_1} \times E^{G_2}_{w_0} = G_1 \times \{e_{G_2}\}$. $\square$

Example 6.22. Let G_1 be a group, F a free non-abelian group, and $G = G_1 \times F$. By Proposition 3.31, F has δ-words, so that $G_1 \times \{e_F\}$ is an elementary algebraic subset of G by Lemma 6.21, hence a $\mathfrak{Z}_G$-closed subset of G.

Theorem 6.23. *Let $G = G_1 \times G_2$, for groups $G_1 \in \mathcal{W}_0^*$ and G_2. Then the following conditions are equivalent.*

(a) *G_2 has a δ-word;*

(b) *$G_1 \times \{e_{G_2}\} = E^G_w$, for a singular word $w \in G[x]$;*

(c) *$G_1 \times \{e_{G_2}\} \in \mathbb{E}_G$;*

(d) *$G_1 \times \{e_{G_2}\} \in \mathbb{E}^{\cup}_G$.*

Proof. (a) implies (b) follows by Lemma 6.21.

(b) implies (c), and (c) implies (d) are trivial.

(d) implies (a). Assume $G_1 \times \{e_{G_2}\} = \bigcup_{i=1}^k E^G_{w_i}$ for a positive integer k, and words $w_i \in G[x]$ for $i = 1, \ldots, k$ such that $E^G_{w_i} \neq \emptyset$.

By equation (6.1), every elementary algebraic subset E^G_w of G has the form $E^G_w = E^{G_1}_{w'} \times E^{G_2}_{w''}$ for words $w' \in G_1[x]$ and $w'' \in G_2[x]$. So $G_1 \times \{e_{G_2}\} = \bigcup_{i=1}^k E^{G_1}_{w'_i} \times E^{G_2}_{w''_i}$, from which we deduce

$$G_1 = \bigcup_{i=1}^{k} E^{G_1}_{w'_i}, \tag{6.3}$$

$$\text{and } \{e_{G_2}\} = \bigcup_{i=1}^{k} E^{G_2}_{w''_i}, \text{ i.e. } E^{G_2}_{w''_i} = \{e_{G_2}\} \text{ for every } i = 1, \dots, n. \tag{6.4}$$

As $G_1 \in \mathcal{W}_0$, (6.3) implies that w'_i is singular for some $i = 1, \dots, k$. This implies that also w''_i is singular. By (6.4), w''_i is a δ-word for G_2. □

From Lemma 6.21 and Theorem 6.23 it immediately follows the corollary below. In particular, the equivalence between conditions (b) and (c) in the next corollary provides a converse to Lemma 6.21.

Corollary 6.24. *Let G_2 be a group. Then, the following conditions are equivalent.*

(a) G_2 has a δ-word;

(b) $G_1 \times \{e_{G_2}\} \in \mathbb{E}_{G_1 \times G_2}$ for every group G_1;

(c) $G_1 \times \{e_{G_2}\} \in \mathbb{E}_{G_1 \times G_2}$ for every $G_1 \in \mathcal{W}_0^*$;

(d) $G_1 \times \{e_{G_2}\} \in \mathbb{E}_{G_1 \times G_2}$ for some $G_1 \in \mathcal{W}_0^*$.

Corollary 6.25. *Let G_1, G_2 be abelian groups, with G_1 unbounded and $G_2 \neq \{0\}$. Then $G_1 \times \{0_{G_2}\}$ is not a Zariski closed subset of $G = G_1 \times G_2$.*

Proof. If G_1 is an unbounded abelian group, then trivially $G_1 \in \mathcal{W}_0$ by Lemma 2.32, and G_1 is $\mathfrak{Z}$-irreducible by Corollary 4.13, so that $G_1 \in \mathcal{W}_0^*$.

By Lemma 3.30, the abelian group G_2 has no δ-words, so that $G_1 \times \{0_{G_2}\} \notin \mathbb{E}_G$ by Theorem 6.23. Then we conclude by Theorem 4.10. □

Remark 6.26. We point out here that the implication in Corollary 6.25 needs not hold if one of the groups G_1, G_2 is non-abelian. Indeed, consider an arbitrary group G_1, a free non-abelian group F, and let $G = G_1 \times F$. By Example 6.22, we have that $G_1 \times \{e_F\}$ is $\mathfrak{Z}_G$-closed, independently on G_1.

6.1.2 Semi $\mathfrak{Z}$-productive pairs

Lemma 6.27. *Let G_1, G_2 be groups, $H_i \leq G_i$, for $i = 1, 2$ be subgroups, $G = G_1 \times G_2$ and $H = H_1 \times H_2$. Then the following hold.*

1. *If the pair G_1, G_2 is semi $\mathfrak{Z}$-productive, and H is Zariski embedded in G, then also the pair H_1, H_2 is semi $\mathfrak{Z}$-productive.*

2. *If the pair G_1, G_2 is $\mathfrak{Z}$-productive, and $H_i \le G_i$, for $i = 1,2$ are Zariski embedded, then also the pair H_1, H_2 is $\mathfrak{Z}$-productive.*

Proof. 1. By assumption, $G_1 \times \{e_{G_2}\}$ is a $\mathfrak{Z}_G$-closed subset of G, so $H_1 \times \{e_{H_2}\}$ is a $\mathfrak{Z}_G \restriction_H$-closed subsets of H. As $\mathfrak{Z}_G \restriction_H = \mathfrak{Z}_H$, this proves that $H_1 \times \{e_{H_2}\}$ is a $\mathfrak{Z}_H$-closed subset of H. The same argument holds for $\{e_{H_1}\} \times H_2$.

2. As H_i is Zariski embedded in G_i for $i = 1, 2$, we have that $\mathfrak{Z}_{G_1\restriction_{H_1}} \times \mathfrak{Z}_{G_2\restriction_{H_2}} = \mathfrak{Z}_{H_1} \times \mathfrak{Z}_{H_2}$. Then

$$\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H = (\mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2}) \restriction_H = \mathfrak{Z}_{G_1\restriction_{H_1}} \times \mathfrak{Z}_{G_2\restriction_{H_2}} = \mathfrak{Z}_{H_1} \times \mathfrak{Z}_{H_2},$$

where the first equality holds as G_1, G_2 is $\mathfrak{Z}$-productive.

□

Corollary 6.28. *If G_1, G_2 is a (semi) $\mathfrak{Z}$-productive pair, and $H_i \le Z(G_i)$, for $i = 1, 2$ are subgroups, then also H_1, H_2 is (semi) $\mathfrak{Z}$-productive.*

In particular, if G_1, G_2 is an abelian (semi) $\mathfrak{Z}$-productive pair, and $H_i \le G_i$, for $i = 1, 2$ are subgroups, then also H_1, H_2 is (semi) $\mathfrak{Z}$-productive.

Proof. As central subgroups are super-normal, hence Zariski embedded, we have that $H_i \le G_i$, for $i = 1, 2$ are Zariski embedded subgroups. The same argument applies to $H = H_1 \times H_2 \le Z(G_1) \times Z(G_2) = Z(G_1 \times G_2)$, giving that $H \le G_1 \times G_2$ is Zariski embedded.

Finally, Lemma 6.27 applies. □

Lemma 6.29. *Let G_1, G_2 be groups, and let $w \in \mathcal{U}_{G_2}$ with $\epsilon(w) = m$. If $G = G_1 \times G_2$, then $Z(G_1)[m] \times G_2$ is $\mathfrak{Z}_G$-closed. In particular, $\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G} \subseteq Z(G_1)[m] \times G_2$.*

Proof. As

$$Z(G_1)[m] \times G_2 = \big(G_1[m] \times G_2\big) \cap \big(Z(G_1) \times G_2\big),$$

and $Z(G_1) \times G_2$ is $\mathfrak{Z}_G$-closed by Lemma 6.16, it only remains to prove that $G_1[m] \times G_2$ is $\mathfrak{Z}_G$-closed. To this end, we will build a word $\widetilde{w} \in G[x]$ such that $G_1[m] \times G_2 = E^G_{\widetilde{w}}$.

Let $w = g_1 x^{\varepsilon_1} g_2 x^{\varepsilon_2} \cdots g_n x^{\varepsilon_n}$. Define the elements $\widetilde{g_i} = (e_{G_1}, g_i) \in G$, and the word $\widetilde{w} = \widetilde{g_1} x^{\varepsilon_1} \widetilde{g_2} x^{\varepsilon_2} \cdots \widetilde{g_n} x^{\varepsilon_n} \in G[x]$. Then the coordinates of $\widetilde{w}$ in $G_1[x] \times G_2[x]$ are $(w_1, w) \in G_1[x] \times G_2[x]$, where $w_1 = e_{G_1} x^{\varepsilon_1} e_{G_1} x^{\varepsilon_2} \cdots e_{G_1} x^{\varepsilon_n} = x^{\epsilon(w)} = x^m \in G_1[x]$, so that $E^G_{\widetilde{w}} = E^{G_1}_{w_1} \times E^{G_2}_{w} = G_1[m] \times G_2$. □

As every group G_2 has a universal word with content $\mathrm{u}(G_2)$, from Lemma 6.29 it immediately follows the corollary below taking $m = \mathrm{u}(G_2)$. Moreover, being $\epsilon(\mathcal{U}_{G_2}) = \mathrm{u}(G_2)\mathbb{Z}$, note that $Z(G_1)[u(G_2)] \times G_2$ is the smallest subgroup of G containing $\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}$ of the form $Z(G_1)[m] \times G_2$, for $m \in \epsilon(\mathcal{U}_{G_2})$.

Corollary 6.30. *Let G_1, G_2 be groups and $G = G_1 \times G_2$. Then $Z(G_1)[u(G_2)] \times G_2$ is $\mathfrak{Z}_G$-closed, so*

$$\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G} \subseteq Z(G_1)[u(G_2)] \times G_2.$$

This corollary gives a nice sufficient condition for semi $\mathfrak{Z}$-productivity of a pair of groups:

Corollary 6.31. *If G_1, G_2 is a pair of groups with*

$$Z(G_1)[\mathrm{u}(G_2)] = \{e_{G_1}\} \quad \text{and} \quad Z(G_2)[\mathrm{u}(G_1)] = \{e_{G_2}\},$$

then the pair G_1, G_2 is semi $\mathfrak{Z}$-productive. In particular, every pair of center-free groups is semi $\mathfrak{Z}$-productive.

Note that the last assertion of Corollary 6.31 follows also by Lemma 6.16.

Theorem 6.32. *Let G_1 be a group, $n \in \mathbb{N}$ and $G_2 \in \mathcal{W}_n^*$. Consider the group $G = G_1 \times G_2$. Then $Z(G_1)[n] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}$.*

In particular, for every group G_2

$$Z(G_1)[\mathrm{u}^*(G_2)] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}.$$

Proof. We will prove that every $\mathfrak{Z}_G$-closed subset C of G containing $\{e_{G_1}\} \times G_2$ must also contain $Z(G_1)[n] \times G_2$.

Let $\{e_{G_1}\} \times G_2 \subseteq C \neq G$ be a basic $\mathfrak{Z}_G$-closed subset, i.e. C is an additively algebraic subset of G. So let $w_1, \dots, w_k \in G[x]$ be such that $C = \bigcup_{\nu=1}^k E_{w_\nu}^G \supseteq \{e_{G_1}\} \times G_2$.

For $\nu = 1, \dots, k$ let $(w'_\nu, w''_\nu) \in G_1[x] \times G_2[x]$ be the coordinates of w_ν. So $E_{w_\nu}^G = E_{w'_\nu}^{G_1} \times E_{w''_\nu}^{G_2}$, and from $\{e_{G_1}\} \times G_2 \subseteq C$ we obtain $G_2 = \bigcup_{\nu=1}^k E_{w''_\nu}^{G_2}$. As $G_2 \in \mathcal{W}_n^*$, $\epsilon(w''_\nu) \in n\mathbb{Z}$ for some $\nu = 1, \dots, k$. As $\epsilon(w''_\nu) = \epsilon(w_\nu) = \epsilon(w'_\nu)$, we conclude $\epsilon(w'_\nu) \in n\mathbb{Z}$, so apply Lemma 3.3 (a) to get $Z(G_1)[\epsilon(w'_\nu)] = Z(G_1) \cap E_{w'_\nu}^{G_1}$. In particular,

$$Z(G_1)[n] \subseteq Z(G_1)[\epsilon(w'_\nu)] \subseteq E_{w'_\nu}^{G_1},$$

so that

$$Z(G_1)[n] \times G_2 \subseteq E_{w'_\nu}^{G_1} \times E_{w''_\nu}^{G_2} = E_{w_\nu}^G \subseteq C.$$

As $Z(G_1)[n] \times G_2 \subseteq C$ holds for any $\mathfrak{Z}_G$-basic closed set C containing $\{e_{G_1}\} \times G_2$, we conclude that $Z(G_1)[n] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}$.

If $\mathrm{u}^*(G_2) = 0$, then $G_2 \in W_0^*$, and taking $n = 0 = \mathrm{u}^*(G_2)$ we obtain

$$Z(G_1) \times G_2 = Z(G_1)[0] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}.$$

If $\mathrm{u}^*(G_2) \neq 0$, then $\mathrm{u}^*(G_2)$ is the least common multiple of the set $S = \{n \in \mathbb{N} \mid G_2 \in \mathcal{W}_n^*\}$, so that

$$\langle Z(G_1)[n] \times G_2 \mid n \in S \rangle = Z(G_1)[\mathrm{u}^*(G_2)] \times G_2.$$

Being $\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}$ a subgroup of G by Theorem 1.6 (d), we conclude

$$Z(G_1)[\mathrm{u}^*(G_2)] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G}.$$

□

By Theorem 6.32, the direct summand $G_2 = \{e_{G_1}\}\times G_2$ of the group $G = G_1\times G_2$ is not $\mathfrak{Z}_G$-closed when $Z(G_1)[\mathrm{u}^*(G_2)] \neq \{e_{G_1}\}$. Note that this happens exactly when $Z(G_1)[n] \neq \{e_{G_1}\}$ and $G_2 \in \mathcal{W}_n^*$, for some $n \in \mathbb{N}$.

We do not know whether G_2 need to be $\mathfrak{M}_G$-closed, so we ask the following question.

Question 7. Do there exist two groups G_1, G_2 such that $Z(G_1)[\mathrm{u}^*(G_2)] \neq \{e_{G_1}\}$, and G_2 is $\mathfrak{M}_{G_1\times G_2}$-closed?

A positive answer to Question 7 would mean that G_2 is $\mathfrak{M}_G$-closed. Since G_2 is not $\mathfrak{Z}_G$-closed by Theorem 6.32, this will provide a large class of examples of groups G satisfying $\mathfrak{Z}_G \neq \mathfrak{M}_G$.

From Theorem 6.32, one can deduce a necessary condition for semi $\mathfrak{Z}$-productivity of a pair of groups:

Corollary 6.33. *If a pair G_1, G_2 is semi $\mathfrak{Z}$-productive, then*

$$Z(G_1)[\mathrm{u}^*(G_2)] = \{e_{G_1}\} \quad \textit{and} \quad Z(G_2)[\mathrm{u}^*(G_1)] = \{e_{G_2}\}.$$

Note that if $G_2 \in \mathcal{W}^*_{\mathrm{u}(G_2)}$, then $\mathrm{u}(G_2) \mid \mathrm{u}^*(G_2)$, so that $\mathrm{u}(G_2) = \mathrm{u}^*(G_2)$. As a consequence, $Z(G_1)[\mathrm{u}^*(G_2)] = Z(G_1)[\mathrm{u}(G_2)]$ for every group G_1. Then, by Corollary 6.30 and Theorem 6.32, it follows:

Corollary 6.34. *Let G_1, G_2 be groups, and $G = G_1 \times G_2$. Then*

$$Z(G_1)[\mathrm{u}^*(G_2)] \times G_2 \subseteq \overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G} \subseteq Z(G_1)[\mathrm{u}(G_2)] \times G_2.$$

In particular, if $Z(G_1)[\mathrm{u}^*(G_2)] = Z(G_1)[\mathrm{u}(G_2)]$, then

$$\overline{\{e_{G_1}\} \times G_2}^{\mathfrak{Z}_G} = Z(G_1)[\mathrm{u}(G_2)] \times G_2.$$

Corollary 6.35. *Let H be an abelian unbounded group. For every group G, $\overline{\{e_G\} \times H}^{\mathfrak{Z}_{G\times H}} = Z(G) \times H$.*

Proof. As $\mathrm{u}(H) = 0$ by Lemma 2.32, and H is $\mathfrak{Z}$-irreducible by Corollary 4.13, we have $H \in \mathcal{W}_0^*$, so that also $\mathrm{u}^*(H) = 0$. So we can apply Corollary 6.34 to get

$$\overline{\{e_G\} \times H}^{\mathfrak{Z}_{G\times H}} = Z(G)[0] \times H = Z(G) \times H.$$

□

By Corollary 6.24, a group G_2 has a δ-word if and only if $G_1 \times \{e_{G_2}\} \in \mathbb{E}_{G_1\times G_2}$ for every group G_1. In particular, $G_1 \times \{e_{G_2}\}$ is a Zariski closed subset of $G_1 \times G_2$ for every group G_1. The next theorem characterizes the groups G_2 with the latter (weaker) property.

Theorem 6.36. *For a group G the following are equivalent:*
(a) G is center-free;
(b) $G_1 \times \{e_G\}$ is a Zariski closed subset of $G_1 \times G$ for every group G_1.

Proof. (b) $\to$ (a). Corollary 6.35, applied with $H = G_1 = \mathbb{Z}$, implies $Z(G) = \{e_G\}$.
(a) $\to$ (b). Since G is a center-free group, Lemma 6.16 applies to conclude that G satisfies (b). □

6.1.3 Abelian $\mathfrak{Z}$-productive pairs

Lemma 6.37. *Let G_1, G_2 be bounded abelian groups having coprime exponents. Then G_1, G_2 is $\mathfrak{Z}$-productive.*

Proof. Let $G = G_1 \times G_2$, and $\exp(G_i) = m_i$ for $i = 1, 2$. By (2.14), the $\mathfrak{Z}_{G_1}$- (resp., $\mathfrak{Z}_{G_2}$)-closed subsets are generated by the cosets of the n-torsion subgroups $G_1[n]$ (resp., $G_2[n]$), for $n \in \mathbb{N}$. So it will suffice to show that, for every $n \in \mathbb{N}$, the subgroups $G_1[n] \times G_2$ and $G_1 \times G_2[n]$ are $\mathfrak{Z}_G$-closed subsets. Indeed $G_1[n] \times G_2$ is an elementary algebraic subset of G, as

$$G_1[n] \times G_2 = G_1[n] \times G_2[nm_2] = G_1[nm_2] \times G_2[nm_2] = G[nm_2],$$

where the first equality holds as $m_2 = \exp(G_2)$, and the second one as $(\exp(G_1), m_2) = 1$.
Similarly, $G_1 \times G_2[n] = G_1[nm_1] \times G_2[nm_1] = G[nm_1]$. □

In the following theorem, we answer positively Question 6 for abelian $\mathfrak{Z}$-productive pairs, and we describe the structure of abelian groups G_1, G_2 such that the pair G_1, G_2 is $\mathfrak{Z}$-productive. Moreover, the implication (b) $\to$ (c) is a 'symmetric' form of Corollary 6.25, giving a much more precise conclusion.

Theorem 6.38. *Let G_1, G_2 be abelian groups, and $G = G_1 \times G_2$. Then the following conditions are equivalent:*

(a) the pair G_1, G_2 is $\mathfrak{Z}$-productive;

(b) the pair G_1, G_2 is semi $\mathfrak{Z}$-productive;

(c) G_1 and G_2 are bounded, $G_1 = F_1 \oplus G_1^*$, and $G_2 = F_2 \oplus G_2^*$, for finite subgroups $F_i \leq G_i$ for $i = 1, 2$, and subgroups $G_i^* \leq G_i$ for $i = 1, 2$ such that $(\exp(G_1^* \oplus G_2^*), |F_1|) = 1$, $(\exp(G_1^* \oplus G_2^*), |F_2|) = 1$, $(\exp(G_1^*), \exp(G_2^*)) = 1$.

Proof. (a) $\to$ (b) follows by the definitions.
(b) $\to$ (c). As both $G_1 \times \{0_{G_2}\}$ and $\{0_{G_1}\} \times G_2$ are $\mathfrak{Z}_G$-closed subsets of G, then both G_1 and G_2 are bounded by Corollary 6.25. Let $G_i = \bigoplus_{p \in \pi(G_i)} G_{i,p}$, where $\pi(G_i)$ is finite, for $i = 1, 2$.

Let $\pi = \pi(G_1) \cap \pi(G_2)$. If $\pi = \emptyset$, let F_1 and F_2 be the trivial subgroups of G_1 and G_2 respectively. Otherwise, let

$$F_1 = \bigoplus_{p \in \pi} G_{1,p} \quad \text{and} \quad F_2 = \bigoplus_{p \in \pi} G_{2,p}.$$

Let also be

$$G_1^* = \bigoplus_{p \in \pi(G_1) \setminus \pi(G_2)} G_{1,p} \quad \text{and} \quad G_2^* = \bigoplus_{p \in \pi(G_2) \setminus \pi(G_1)} G_{2,p},$$

so that

$$G_1 = F_1 \oplus G_1^* \quad \text{and} \quad G_2 = F_2 \oplus G_2^*.$$

It only remains to prove that both F_1, F_2 are finite groups, that is: if $p \in \pi$, then both $G_{1,p}$ and $G_{2,p}$ are finite. So let $p \in \pi$ and by contradiction assume $G_{1,p}$ to be infinite. Then also $r_p(G_1)$ is infinite, and let $H_1 = G_1[p]$. Fix an element $x \in G_2$ of order p, and let $H_2 = \langle x \rangle \leq G_2$. Finally, let $H = H_1 \times H_2$, and note that $\exp(H) = p$, so that $\mathfrak{Z}_H = cof_H$ by Proposition 4.15. Being $H_0 = H_1 \times \{0_{G_2}\}$ an infinite proper subgroup of H, it is not $\mathfrak{Z}_H$-closed. This contradicts Corollary 6.28.

(c) $\to$ (a). Assume $G_1 = F_1 \oplus G_1^*$ and $G_2 = F_2 \oplus G_2^*$, with F_1, F_2 finite, G_1^*, G_2^* bounded, with coprime exponents as in the statement of (c). Then $\mathfrak{Z}_{G_i} = \mathfrak{Z}_{F_i} \times \mathfrak{Z}_{G_i^*}$ for $i = 1, 2$ by Lemma 6.37, so that

$$\mathfrak{Z}_{G_1} \times \mathfrak{Z}_{G_2} = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{F_2} \times \mathfrak{Z}_{G_2^*}.$$

Finally, let $F = F_1 \times F_2$ and note that $\mathfrak{Z}_F = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{F_2}$ is the discrete topology on the finite group F. So

$$\mathfrak{Z}_{G_1 \times G_2} = \mathfrak{Z}_{F_1 \oplus G_1^* \times F_2 \oplus G_2^*} = \mathfrak{Z}_{F \times G_1^* \times G_2^*} \stackrel{(*)}{=} \mathfrak{Z}_F \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{G_2^*} = \mathfrak{Z}_{F_1} \times \mathfrak{Z}_{F_2} \times \mathfrak{Z}_{G_1^*} \times \mathfrak{Z}_{G_2^*},$$

where the starred equality follows again from Lemma 6.37, as the three groups F, G_1^* and G_2^* are all bounded with mutually coprime exponents. This concludes the proof. $\square$

The following immediate consequence of Corollary 6.28 and Corollary 6.35 could also be used to give a different proof of the implication (b) $\to$ (c) of Theorem 6.38.

Corollary 6.39. *Let G_1, G_2 be an abelian semi $\mathfrak{Z}$-productive pair. Then neither G_1, nor G_2, can contain as a subgroup any of the following groups: the group of integers $\mathbb{Z}$; the p-Prüfer group $\mathbb{Z}_{p^\infty}$; $\bigoplus_{n=1}^\infty \mathbb{Z}_{p^n}$ for a prime number $p \in \mathbb{P}$; $\bigoplus_{n=1}^\infty \mathbb{Z}_{p_n}$ for infinitely many different prime numbers $p_n \in \mathbb{P}$, as $n \in \mathbb{N}$.*

Proof. By contradiction, let H be one of those groups, and assume $H \leq G_2$.

By Corollary 6.28, the pair G_1, H is semi $\mathfrak{Z}$-productive, so $\{e_{G_1}\} \times H$ is $\mathfrak{Z}_{G_1 \times H}$-closed. On the other hand, H is abelian unbounded, so $\overline{\{e_{G_1}\} \times H}^{\mathfrak{Z}_{G_1 \times H}} = Z(G_1) \times H = G_1 \times H$ by Corollary 6.35. $\square$

It follows from Theorem 6.38 that for every non trivial abelian group G there exists a bounded abelian group H such that G, H is not a $\mathfrak{Z}$-productive pair.

6.2 Direct Sums

Let $\{G_i \mid i \in I\}$ be a family of groups, $G = \prod_{i\in I} G_i$, and $S = \bigoplus_{i\in I} G_i$. We begin this subsection with a description of the elementary algebraic subsets of S. First, given a family of subsets $X_i \subseteq G_i$ for every $i \in I$, we define

$$\bigoplus_{i\in I} X_i = \big(\prod_{i\in I} X_i\big) \cap S.$$

Obviously, if $I = I_1 \cup I_2$ is a partition of I, then $G = G_{I_1} \times G_{I_2}$, where $G_{I_k} = \prod_{i\in I_k} G_i$ for $k = 1, 2$. As $S[x] \leq G[x]$ by Proposition 2.4, item 1, if $w \in S[x]$ we can consider in particular the coordinates of $w \in G[x]$ in $G_{I_1}[x] \times G_{I_2}[x]$ and denote them $(w_{I_1}, w_{I_2}) \in G_{I_1}[x] \times G_{I_2}[x]$.

Proposition 6.40. *Let $w \in S[x]$. Then there exists $F \in [I]^{<\omega}$ such that, letting $G_F = \prod_{i\in F} G_i$, the coordinates of w in $\prod_{i\in I}(G_i[x])$ split as follows:*

$$\big(w_F, (x^{\epsilon(w)})_{i\in I\setminus F}\big) \in G_F[x] \times \prod_{i\in I\setminus F} \big(G_i[x]\big). \tag{6.5}$$

In particular, E^S_w has the following form:

$$E^S_w = E^{G_F}_{w_F} \times \bigoplus_{i\in I\setminus F} G_i[\epsilon(w)] = \bigoplus_{i\in F} E^{G_i}_{w_i} \times \bigoplus_{i\in I\setminus F} G_i[\epsilon(w)]. \tag{6.6}$$

Proof. We define $F = \bigcup_{g\in \mathrm{coeff}(w)} \mathrm{supp}(g) \subseteq I$, and we note that F is finite. Letting $G_{I\setminus F} = \prod_{i\in I\setminus F} G_i$, then $G = G_F \times G_{I\setminus F}$.

We first consider the projections $\pi_F \colon G[x] \to G_F[x]$ and $\pi_{I\setminus F} \colon G[x] \to G_{I\setminus F}[x]$, so that $(w_F, w_{I\setminus F})$ are the coordinates of w in $G_F[x] \times G_{I\setminus F}[x]$.

Then, let $(w_i)_{i\in I\setminus F} \in \prod_{i\in I\setminus F}\big(G_i[x]\big)$ be the coordinates of $w_{I\setminus F} \in G_{I\setminus F}[x]$, and note that $w_i = x^{\epsilon(w)} \in G_i[x]$ for $i \in I \setminus F$. Then the coordinates of w split as in (6.5) and E^S_w is as in (6.6). $\square$

Now we give a corollary of Theorem 6.4. Note that this corollary could also be proved using Proposition 6.40.

Corollary 6.41. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $S = \bigoplus_{i\in I} G_i$. Then the Zariski topology $\mathfrak{Z}_S$ of the direct sum is coarser than σ.*

Proof. Easily follows from Theorem 6.4, as $\mathfrak{Z}_S \subseteq \mathfrak{Z}_G \restriction_S \subseteq \sigma = (\prod_{i\in I} \mathfrak{Z}_{G_i}) \restriction_S$. $\square$

In the following lemma, we give a direct proof of Corollary 2.36, item 2, using Proposition 6.40. Compare this result also with Lemma 6.13.

Lemma 6.42. *Let $n \in \mathbb{N}$, and $\{G_i \mid i \in I\}$ be a family of groups. If $G_{i_0} \in \mathcal{W}_n$ for some $i_0 \in I$, then $S = \bigoplus_{i\in I} G_i \in \mathcal{W}_n$.*

Proof. Let $w \in \mathcal{U}_S$, and consider its coordinates $(w_i)_{i\in I} \in \prod_{i\in I}(G_i[x])$. Then $w_{i_0} \in \mathcal{U}_{G_{i_0}}$ by Proposition 6.40, hence $\epsilon(w_{i_0}) \in n\mathbb{Z}$. As $\epsilon(w) = \epsilon(w_{i_0})$, we conclude $S \in \mathcal{W}_n$. $\square$

Proposition 6.43. *A direct sum $\bigoplus_{i\in I} G_i$ is a super-normal subgroup of a direct product $\prod_{i\in I} G_i$ if and only if all but finitely many of the groups G_i are abelian.*

Proof. Let $S = \bigoplus_{i\in I} G_i$ and $G = \prod_{i\in I} G_i$. Then, $C_G(S) = Z(G) = \prod_{i\in I} Z(G_i)$ and so

$$S \cdot C_G(S) = \bigoplus_{i\in I} G_i \cdot \prod_{i\in I} Z(G_i).$$

Then, $S \cdot C_G(S) = G$ if and only if $G_i = Z(G_i)$ for all but finitely many i, i.e. if and only if all but finitely many G_i's are abelian. Now Proposition 5.5 applies. $\square$

Compare the following lemma about direct sum of centralizers with Lemma 6.16 about direct *products* of centralizers.

Lemma 6.44. *Let $\{G_i \mid i \in I\}$ be a family of groups, and let $X_i \subseteq G_i$ be a subset for every $i \in I$. If $S = \bigoplus_{i\in I} G_i$, then $\bigoplus_{i\in I} C_{G_i}(X_i)$ is a $\mathfrak{Z}_S$-closed subgroup of S.*

Proof. Follows from the fact that $\bigoplus_{i\in I} C_{G_i}(X_i) = C_S(\bigoplus_{i\in I} X_i)$, then Example 2.43, item 1, applies. $\square$

6.3 Centralizer topologies on products

Lemma 6.45. *Let $\{G_i \mid i \in I\}$ be a family of groups, $G = \prod_{i\in I} G_i$ and $S = \bigoplus_{i\in I} G_i$. Then:*

1. $\mathfrak{C}_G = \prod_{i\in I} \mathfrak{C}_{G_i}$.
2. $\mathfrak{C}_S = \mathfrak{C}_G \restriction_S = \bigoplus_{i\in I} \mathfrak{C}_{G_i}$.
3. $\mathfrak{C}'_G \subseteq \prod_{i\in I} \mathfrak{C}'_{G_i}$.

Proof. As all the topologies listed above are stable under taking translations, when compairing them (actually, their closed subsets) we will only consider subsets containing the identity element.

1. Let $\rho = \prod_{i\in I} \mathfrak{C}_{G_i}$ denote the product topology, and $b = (b_i)_{i\in I} \in G$. As $C_G(b) = \prod_{i\in I} C_{G_i}(b_i)$ is ρ-closed, we conclude that $\mathfrak{C}_G \subseteq \rho$.

On the other hand, the translates of subsets X of the form $X = C_{G_i}(b_i) \times \prod_{i\neq j\in I} G_j$, for $i \in I$, and $b_i \in G_i$ are a subbase for the ρ-closed subsets. Letting $b_j = e_{G_j}$ for $i \neq j \in I$, and denoting $b = (b_i)_{i\in I} \in G$, we get that $X = C_G(b)$ is a $\mathfrak{C}_G$-closed subset of G, so that $\rho \subseteq \mathfrak{C}_G$.

2. Let $\sigma = \bigoplus_{i\in I} \mathfrak{C}_{G_i} = \prod_{i\in I} \mathfrak{C}_{G_i} \restriction_S$. By Lemma 4.28, item (4), $\mathfrak{C}_S \subseteq \mathfrak{C}_G \restriction_S$, so that $\mathfrak{C}_S \subseteq \sigma$ by item 1.

Now we shall prove the reverse inclusion $\sigma \subseteq \mathfrak{C}_S$. To this end, it will suffice to show that for every $g = (g_i)_{i\in I} \in G$, the subset $C_G(g) \cap S$ is $\mathfrak{C}_S$-closed. In fact, we will prove that $C_G(g) \cap S = \bigcap_{i\in I} C_S(s^{(i)})$, for suitable elements $s^{(i)} \in S$ for every $i \in I$. As $C_G(g) = \prod_{i\in I} C_{G_i}(g_i)$, we have that $C_G(g) \cap S = \bigoplus_{i\in I} C_{G_i}(g_i)$. So, for every $i \in I$, we define the element $s^{(i)} = (s^{(i)}_j)_{j\in I} \in S$ as follows: $s^{(i)}_i = g_i$, and $s^{(i)}_j = e_{G_j}$ whenever $i \neq j \in I$. Then $\bigcap_{i\in I} C_S(s^{(i)}) = C_G(g) \cap S$ as desired.

3. Let $\tau = \prod_{i\in I} \mathfrak{C}'_{G_i}$ be the product topology. Then τ is T_1, that is $cof_G \subseteq \tau$. Moreover, obviously $\prod_{i\in I} \mathfrak{C}_{G_i} \subseteq \tau$, so that $\mathfrak{C}_G \subseteq \tau$ by item 1. Then $\mathfrak{C}'_G = \mathfrak{C}_G \vee cof_G \subseteq \tau$. $\square$

Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. Then

$$\prod_{i\in I} \mathfrak{C}_{G_i} = \mathfrak{C}_G \subseteq \mathfrak{C}'_G \subseteq \mathfrak{Z}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{G_i} \tag{6.7}$$

$$\prod_{i\in I} \mathfrak{C}_{G_i} = \mathfrak{C}_G \subseteq \mathfrak{C}'_G \subseteq \prod_{i\in I} \mathfrak{C}'_{G_i} \subseteq \prod_{i\in I} \mathfrak{Z}_{G_i}. \tag{6.8}$$

From (6.7), it immediately follows the following result.

Lemma 6.46. *Let $\{G_i \mid i \in I\}$ be a family of groups, and $G = \prod_{i\in I} G_i$. If $\mathfrak{C}_{G_i} = \mathfrak{Z}_{G_i}$ for every $i \in I$, then $\mathfrak{C}_G = \mathfrak{Z}_G = \prod_{i\in I} \mathfrak{Z}_{G_i}$.*

Note that, if G_i is a center-free group, then the hypothesis $\mathfrak{Z}_{G_i} = \mathfrak{C}_{G_i}$, as in Lemma 6.46, is equivalent to $\mathfrak{Z}_{G_i} = \mathfrak{C}'_{G_i}$.

We shall see in the following example that every inclusion in (6.7) can be proper, as well as the inclusion in Lemma 6.45 (3), even if $\mathfrak{Z}_{G_i} = \mathfrak{C}'_{G_i}$ for every $i \in I$. In particular, the conclusion of Lemma 6.46 needs not hold relaxing its hypotheses to $\mathfrak{Z}_{G_i} = \mathfrak{C}'_{G_i}$ for every $i \in I$.

Example 6.47. Consider an infinite family $\{G_i \mid i \in I\}$ of *finite* groups. Then $\mathfrak{Z}_{G_i} = \mathfrak{C}'_{G_i}$ is the discrete topology on G_i for every $i \in I$, so that $\prod_{i\in I} \mathfrak{Z}_{G_i} = \prod_{i\in I} \mathfrak{C}'_{G_i}$ is a compact Hausdorff topology.

If we take every group G_i also *abelian*, then G is abelian too, so that $\mathfrak{C}_G = \iota_G$ is the indiscrete topology, and $\mathfrak{C}'_G$ is the cofinite topology. Moreover, according to Fact 4.12 (a), $\mathfrak{Z}_G$ is Noetherian on G infinite abelian, hence not Hausdorff, so $\mathfrak{Z}_G \subsetneq \prod_{i\in I} \mathfrak{Z}_{G_i}$.

Finally, if we want $\mathfrak{Z}_G$ to differ also from $\mathfrak{C}'_G$, we should manage to have $\mathfrak{Z}_G$ not cofinite, and this can be achieved taking for example $G_i = \mathbb{Z}_4$ for every $i \in I$ (see Proposition 4.15). Then, for an infinite set I and the group $G = \mathbb{Z}_4^I$, the following hold

$$\mathfrak{C}_G = \iota_G \subsetneq \mathfrak{C}'_G = cof_G \subsetneq \mathfrak{Z}_G \subsetneq \prod_{i\in I} \mathfrak{Z}_{G_i} = \prod_{i\in I} \mathfrak{C}'_{G_i}.$$

We shall see in §6.6 a more curious example. There, we will consider the group $G = \mathbb{Z}_2 \times S_3^I$, and prove that

$$\mathfrak{C}_G \subsetneq \mathfrak{Z}_G \subsetneq \mathfrak{Z}_{\mathbb{Z}_2} \times \prod_{i \in I} \mathfrak{Z}_{G_i}.$$

This example will show that the conclusion of Lemma 6.46 may fail even if just only one of the groups in the family under consideration (in this case, $\mathbb{Z}_2$) does not satisfy the condition $\mathfrak{Z}_{G_i} = \mathfrak{C}'_{G_i}$.

In the following results, we describe the Zariski topology on a class of products in $\mathscr{N}_2$. Recall that $\exp(G/Z(G)) \mid \exp(Z(G))$ for a group $G \in \mathscr{N}_2$ by Proposition 3.9.

Theorem 6.48. *Let p be a prime number, and $\{G_i \mid i \in I\} \subseteq \mathscr{N}_2$ be a family of groups such that $\exp(Z(G_i)) = p$, and $G_i/Z(G_i)$ is finite, for every $i \in I$.*

If $G = \prod_{i \in I} G_i$, then $\mathfrak{Z}_G = \mathfrak{C}'_G$.

Proof. We have to prove that $\mathfrak{Z}_G \subseteq \mathfrak{C}'_G = cof_G \vee \mathfrak{C}_G$. Let $w \in G[x]$, and assume that $E_w^G \neq \emptyset$. It will suffice to show that either E_w^G is a singleton, or E_w^G is $\mathfrak{C}_G$-closed.

If $(w_i)_{i \in I} \in \prod_{i \in I}(G_i[x])$ are the coordinates of w in $\prod_{i \in I}(G_i[x])$, recall that $E_w^G = \prod_{i \in I} E_{w_i}^{G_i}$ by (6.1) in Theorem 6.4, and that $\epsilon(w) = \epsilon(w_i)$ for every $i \in I$.

If $(\epsilon(w), p) = 1$, then for every $i \in I$ also $(\epsilon(w_i), p) = 1$, so that each $E_{w_i}^{G_i}$ is a singleton by Corollary 3.13 (i), and E_w^G is a singleton.

Otherwise, for every $i \in I$, $E_{w_i}^{G_i}$ is a finite union of cosets of $Z(G_i)$ by Corollary 3.13 (ii). In particular, $E_{w_i}^G$ is $\mathfrak{C}_{G_i}$-closed, so that $E_w^G = \prod_{i \in I} E_{w_i}^{G_i}$ is $\prod_{i \in I} \mathfrak{C}_{G_i}$-closed. As $\prod_{i \in I} \mathfrak{C}_{G_i} = \mathfrak{C}_G$ by Lemma 6.45 (1), we have that E_w^G is $\mathfrak{C}_G$-closed as desired. □

In the notation of Theorem 6.48, for every $i \in I$ consider the quotient group $\overline{G_i} = G_i/Z(G_i)$. Let $\overline{G} = G/Z(G) \cong \prod_{i \in I} \overline{G_i}$, and equip it with the product topology $\tau = \prod_{i \in I} \delta_{\overline{G_i}}$ of the discrete topologies on each (finite) factor group $\overline{G_i}$. Note that τ is a compact Hausdorff totally disconnected group topology on $\overline{G}$. The following theorem provides a more topological description of $\mathfrak{Z}_G$.

Theorem 6.49. *Let G be a group as in Theorem 6.48, and let $\pi^{-1}\tau$ denote the initial topology of the canonical projection*

$$\pi\colon G \to (G/Z(G), \tau). \tag{6.9}$$

Then $\mathfrak{C}_G = \pi^{-1}\tau$, so that $\mathfrak{Z}_G = cof_G \vee \pi^{-1}\tau$. In particular, the connected component $c(G, \mathfrak{Z}_G) = Z(G)$ is $\mathfrak{Z}$-irreducible.

Proof. Note that, for every $i \in I$, the group homomorphisms in the following diagram make it commutative:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi_i"'] & (\prod_{i \in I} \overline{G_i}, \tau) \cong \overline{G} \arrow[d, "p_i"] \\ (G_i, \mathfrak{C}_{G_i}) \arrow[r, "\pi_{G_i}"] & (\overline{G}_i, \delta_{\overline{G_i}}). \end{tikzcd} \tag{6.10}$$

Then note that:

1. Let $i \in I$. By Lemma 4.31, $\mathfrak{C}_{G_i}$ is the initial topology on G_i of the map $\pi_{G_i}\colon G_i \to (\overline{G_i}, \delta_{\overline{G_i}})$, so that the subsets $\pi_{G_i}^{-1}(\{x\})$, for $x \in \overline{G_i}$, form a subbase for $\mathfrak{C}_{G_i}$-closed sets.

2. By Lemma 6.45 (1), $\mathfrak{C}_G = \prod_{i\in I} \mathfrak{C}_{G_i}$, i.e. $\mathfrak{C}_G$ is the initial topology on G of the maps $\{\pi_i\colon G \to (G_i, \mathfrak{C}_{G_i}) \mid i \in I\}$. Then, by the previous point, the subsets $\pi_i^{-1}\pi_{G_i}^{-1}(\{x\})$, for $i \in I$ and $x \in \overline{G_i}$, form a subbase for $\mathfrak{C}_G$-closed sets.

3. As $\tau = \prod_{i\in I} \delta_{\overline{G_i}}$ is the initial topology on $\overline{G}$ of the maps $\{p_i\colon \overline{G} \to (\overline{G_i}, \delta_{\overline{G_i}}) \mid i \in I\}$, the subsets $p_i^{-1}(\{x\})$, for $x \in \overline{G_i}$, form a subbase for τ-closed sets.

4. By the previous point, and the definition of $\pi^{-1}\tau$, the subsets $\pi^{-1}p_i^{-1}(\{x\})$, for $x \in \overline{G_i}$, form a subbase for $\pi^{-1}\tau$-closed sets.

As (6.10) is commutative, item 2 and 4 imply that the families of $\mathfrak{C}_G$-closed sets and $\pi^{-1}\tau$-closed sets have the same subbase, so that $\mathfrak{C}_G = \pi^{-1}\tau$. Then apply Theorem 6.48 to conclude $\mathfrak{Z}_G = cof_G \vee \pi^{-1}\tau$.

To prove the last assertion, note that $Z(G) = \prod_{i\in I} Z(G_i)$ has exponent p, so that is $\mathfrak{Z}$-cofinite by Proposition 4.15, hence $\mathfrak{Z}$-irreducible. As $\mathfrak{Z}_{Z(G)} = \mathfrak{Z}_G \restriction_{Z(G)}$, we have in particular that $(Z(G), \mathfrak{Z}_G \restriction_{Z(G)})$ is connected, so that $Z(G) \subseteq c(G, \mathfrak{Z}_G)$.

From the equality $\mathfrak{Z}_G = cof_G \vee \pi^{-1}\tau$ already proved, it follows that $\pi\colon (G, \mathfrak{Z}_G) \to (G/Z(G), \tau)$ is continuous, so that also $X = \pi(c(G, \mathfrak{Z}_G)) = c(G, \mathfrak{Z}_G)/Z(G)$ is τ-connected. But τ is totally disconnected, hence $|X| = 1$ and $c(G, \mathfrak{Z}_G) = Z(G)$. $\square$

Remark 6.50. Let G be as in Theorem 6.48. By Theorem 6.49, $c(G, \mathfrak{Z}_G) = Z(G)$, so that the quotient space $(\overline{G}, \overline{\mathfrak{Z}_G})$ is totally disconnected by Corollary 1.11. Indeed, it can be easily verified that $\overline{\mathfrak{Z}_G} = \tau$, so that $\overline{\mathfrak{Z}_G}$ is a compact Hausdorff totally disconnected group topology.

On the other hand, $\overline{G} \cong \prod_{i\in I} \overline{G_i}$, and $\exp(Z(G_i)) = p$ for every $i \in I$, so that also $\exp(\overline{G}) = p$ by Proposition 3.9. In particular, $\overline{G}$ is $\mathfrak{Z}$-cofinite by Proposition 4.15, so that $(\overline{G}, \mathfrak{Z}_{\overline{G}})$ is irreducible, hence connected.

Then obviously $\mathfrak{Z}_{\overline{G}} \subsetneq \overline{\mathfrak{Z}_G}$, and this example shows that the quotient group $\overline{G} = G/c(G, \mathfrak{Z}_G)$, with its own Zariski topology, need not be totally disconnected.

Remark 6.51. Let G be a group. If $C = c(G, \mathfrak{Z}_G)$, then $(C, \mathfrak{Z}_G \restriction_C)$ is connected by definition, and $\mathfrak{Z}_C \subseteq \mathfrak{Z}_G \restriction_C$ implies that also $(C, \mathfrak{Z}_C)$ is connected.

This easy observation shows that only $\mathfrak{Z}$-connected groups can be realized as $c(G, \mathfrak{Z}_G)$ for some group G.

6.4 The Zariski topology of direct products of finite groups

From now on in this section, $\{F_i \mid i \in I\}$ will be a non-empty family of finite groups.

Let $G = \prod_{i\in I} F_i$. For every $i \in I$, the Zariski topology $\mathfrak{Z}_{F_i}$ is the discrete one, and trivially a compact Hausdorff group topology on F_i. We shall be interested in the cases when G is $\mathfrak{M}$-Hausdorff, or even $\mathfrak{Z}$-Hausdorff, so by Proposition 6.52 we must necessarily have $Z(G) = \prod_{i\in I} Z(F_i)$ finite, i.e. all but a finite number of the groups F_i must be center-free. So it is not restrictive to consider the case when all but one of the groups F_i are center-free. This is why we impose in Theorem 6.55 *all* groups F_i to be center-free. The general case will be discussed in §11.1.

Proposition 6.52. *Let $\{F_i \mid i \in I\}$ be a non-empty family of finite groups, $G = \prod_{i\in I} F_i$, and $S = \bigoplus_{i\in I} F_i$. If either G or S is an $\mathfrak{M}$-Hausdorff group, then all but finitely many of the groups F_i are center-free.*

Proof. Will easily follow from Corollary 11.4, giving that (respectively) either $Z(G) = \prod_{i\in I} Z(F_i)$ or $Z(S) = \bigoplus_{i\in I} Z(F_i)$ is finite. In both cases, obviously all but finitely many of the groups F_i must be center-free. □

Lemma 6.53. *If $\{F_i \mid i \in I\}$ is a non-empty family of finite groups, and $G = \prod_{i\in I} F_i$, then the product topology $\prod_{i\in I} \mathfrak{Z}_{F_i}$ is a compact Hausdorff group topology on G, so*

$$\mathfrak{C}_G \subseteq \mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{F_i}. \tag{6.11}$$

Proof. Being $\prod_{i\in I} \mathfrak{Z}_{F_i}$ a compact (hence, precompact) Hausdorff group topology on G, we have $\mathfrak{P}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{F_i}$. The others inclusions follow from the definitions. □

Observe that Theorem 6.4 applied to a family $\{F_i \mid i \in I\}$ of (not necessarily finite) groups gives only the result $\mathfrak{Z}_G \subseteq \prod_{i\in I} \mathfrak{Z}_{F_i}$, weaker than (6.11).

Remark 6.54. Let $\{F_i \mid i \in I\}$ be a non-empty family of finite groups, $G = \prod_{i\in I} F_i$, and $H \leq G$. As $\mathfrak{C}_H \subseteq \mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$, from equation (6.11) it follows that

$$\mathfrak{C}_H \subseteq \mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H \subseteq \mathfrak{M}_G \restriction_H \subseteq \mathfrak{P}_G \restriction_H \subseteq \Big(\prod_{i\in I} \mathfrak{Z}_{F_i}\Big) \restriction_H . \tag{6.12}$$

Moreover, $\left(\prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_H$ is a precompact Hausdorff group topology on H by Lemma 6.53, so $\mathfrak{P}_H \subseteq \left(\prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_H$ and

$$\mathfrak{C}_H \subseteq \mathfrak{Z}_H \subseteq \mathfrak{M}_H \subseteq \mathfrak{P}_H \subseteq \Big(\prod_{i\in I} \mathfrak{Z}_{F_i}\Big) \restriction_H . \tag{6.13}$$

The following theorem determines the Markov topologies on directs products and sums of finite center-free groups.

Theorem 6.55. *Let $\{F_i \mid i \in I\}$ be a non-empty family of finite center-free groups, $G = \prod_{i\in I} F_i$, and $S = \bigoplus_{i\in I} F_i$. Then:*

(a) $\mathfrak{C}_G = \mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \prod_{i\in I} \mathfrak{Z}_{F_i}$.

(b) $\mathfrak{C}_S = \mathfrak{Z}_S = \mathfrak{M}_S = \mathfrak{P}_S = \left(\prod_{i\in I}\mathfrak{Z}_{F_i}\right)\restriction_S = \mathfrak{Z}_G\restriction_S = \mathfrak{M}_G\restriction_S = \mathfrak{P}_G\restriction_S$.

Proof. (a). By Lemma 4.28, item 3, the topology $\mathfrak{C}_{F_i}$ is T_1 for every $i\in I$, so that $\mathfrak{C}_{F_i} = \mathfrak{Z}_{F_i} = \delta_{F_i}$. Then $\prod_{i\in I}\mathfrak{C}_{F_i} = \prod_{i\in I}\mathfrak{Z}_{F_i}$, so that $\mathfrak{C}_G = \prod_{i\in I}\mathfrak{Z}_{F_i}$ by Lemma 6.45, item 1. The remaining equalities follow by (6.11) in Lemma 6.53.

(b). We have that $\mathfrak{C}_S = \mathfrak{C}_G\restriction_S$ by Lemma 6.45, item 2, and that $\mathfrak{C}_G\restriction_S = \left(\prod_{i\in I}\mathfrak{Z}_{F_i}\right)\restriction_S$ by the previous point, so that $\mathfrak{C}_S = \left(\prod_{i\in I}\mathfrak{Z}_{F_i}\right)\restriction_S$. The remaining equalities follow by (6.13) in Remark 6.54 and item (a). □

In the following theorem, we point out how the Zariski topology behaves completely differently on abelian and meta-abelian groups, even if we restrict ourselves to the class of almost torsion-free groups. Compare the theorem below with Proposition 4.15, where we see that $\mathfrak{Z}_G$ is cofinite when G is abelian and almost torsion free.

Theorem 6.56. *There exists a center-free, meta-abelian, almost torsion free group G such that $\mathfrak{Z}_G$ is a compact Hausdorff group topology.*

Proof. We will consider an infinite (say, countable) family of finite center-free groups $\{F_n \mid n\in\mathbb{N}\}$ in such a way that $G=\prod_{n\in\mathbb{N}}F_n$ is meta-abelian and almost torsion free. Then, $\mathfrak{Z}_G$ is a compact Hausdorff group topology by Theorem 6.55 (a).

To this end, we need a sequence of distinct primes $(p_n)_{n\in\mathbb{N}}$ such that $p_{2n}|p_{2n+1}-1$ for every $n\in\mathbb{N}$. To find such a sequence one can argue by induction as follows. Let $p_0=2$ and $p_1=3$. Assume that $n\geq 1$ and p_{2k-2},p_{2k-1} are already defined satisfying $p_{2k-2}|p_{2k-1}-1$ for all $k\leq n$.

Pick $p_{2n}>p_{2n-1}$. Now observe that by Dirichlet's theorem the arithmetic progression $\{mp_{2n}+1 : m\in\mathbb{N}\}$ contains infinitely many primes. Choose the prime p_{2n+1} from that progression, such that $p_{2n+1}>p_i$ for every $i<2n+1$.

The next step is to define $F_n=\mathbb{Z}_{p_{2n+1}}\rtimes\mathbb{Z}_{p_{2n}}$, the semidirect product of $\mathbb{Z}_{p_{2n}}$ and $\mathbb{Z}_{p_{2n+1}}$ defined by the embedding of $\mathbb{Z}_{p_{2n}}$ in $\mathbb{Z}_{p_{2n+1}-1}\cong Aut(\mathbb{Z}_{p_{2n+1}})$. Then $\mathbb{Z}_{p_{2n}}$ acts on $\mathbb{Z}_{p_{2n+1}}$ without non-trivial fixed points, so F_n is center-free. Obviously, F_n is meta-abelian, so that $G=\prod_{n\in\mathbb{N}}F_n$ is meta-abelian too.

Finally, we verify that G is almost torsion free. Let $k>1$ be an integer, and let $p_N>k$ for an integer N. Then $G[k]\subseteq\prod_{n\leq N}F_n$, and so it is finite. □

6.5 The universal exponent of infinite products

Now we prove a combinatorial lemma about coverings of a direct product, which are made of rectangular sets.

Lemma 6.57. *Let $\{X_i : i\in I\}$ be a non-empty family of non-empty sets and $X=\prod_{i\in I}X_i$. Let $k>0$ be an integer with $k\leq|I|$. For every $\nu=1,\dots,k$ and $i\in I$, let $Y_i^{(\nu)}\subseteq X_i$ and $Y^{(\nu)}=\prod_{i\in I}Y_i^{(\nu)}$ be such that $X=\bigcup_{i=1}^k Y^{(\nu)}$, and the union is not redundant. Then $|\{i\in I \mid Y_i^{(\nu)}\neq X_i\}|<k$ for every $\nu=1,\dots,k$.*

Proof. For example, let $I_1 = \{i \in I \mid Y_i^{(1)} \neq X_i\}$, and by contradiction assume that $|I_1| \geq k$.

As $Y^{(1)} \nsubseteq \bigcup_{\nu=2}^k Y^{(\nu)}$, there exists some $y = (y_i)_{i\in I} \in Y^{(1)} \setminus \bigcup_{\nu=2}^k Y^{(\nu)}$, so that for every $\nu = 2, \dots, k$ there is $i_\nu \in I$ such that $y_{i_\nu} \notin Y_{i_\nu}^{(\nu)}$. Note that the subset $J = \{i_\nu \mid \nu = 2, \dots, k\} \subseteq I$ has at most $k-1$ elements, so that $I_1 \setminus J \neq \emptyset$.

Then we define an element $x = (x_i)_{i\in I} \in X$ as follows: if $i \in I_1 \setminus J$, then $X_i \setminus Y_i^{(1)} \neq \emptyset$, and take x_i as any of its elements; if $i \in J$, choose $x_i = y_i$. Then $x \notin \bigcup_{\nu=2}^k Y^{(\nu)}$.

By construction, x is not in any of $Y^{(1)}, Y^{(2)}, \dots, Y^{(k)}$, a contradiction. □

Corollary 6.58. *Let $\{G_i : i \in I\}$ be a non-empty family of groups and $G = \prod_{i\in I} G_i$. Let $k > 0$ be an integer with $k \leq |I|$. For every $\nu = 1, \dots, k$, let $w^{(\nu)} \in G[x]$ with coordinates $(w_i^{(\nu)})_{i\in I} \in \prod_{i\in I}(G_i[x])$ be such that $G = \bigcup_{\nu=1}^k E_{w^{(\nu)}}^G$, and the union is not redundant.*

Then $|\{i \in I \mid w_i^{(\nu)} \notin \mathcal{U}_{G_i}\}| < k$ for every $\nu = 1, \dots, k$.

Proof. If $\nu \in \{1, \dots, k\}$, then $E_{w^{(\nu)}}^G = \prod_{i\in I} E_{w_i^{(\nu)}}^{G_i}$. If $I_\nu = \{i \in I \mid w_i^{(\nu)} \notin \mathcal{U}_{G_i}\} \subseteq I$, then $|I_\nu| < k$ by Lemma 6.57, so there is $i \in I \setminus I_\nu$. Obviously, $w_i^{(\nu)} \in \mathcal{U}_{G_i}$ for all $i \in I \setminus I_\nu$. □

In the next theorem we prove that an infinite direct product of groups belonging to $\mathcal{W}_n$ is in $\mathcal{W}_n^*$.

Theorem 6.59. *Let I be an infinite set, $n \in \mathbb{N}$ and $\{K_i : i \in I\} \subseteq \mathcal{W}_n$ a family of groups. Let $H = \prod_{i\in I} K_i$, and $w_1, \dots, w_k \in H[x]$ be such that $H = \bigcup_{\nu=1}^k E_{w_\nu}^H$ and this union is not redundant. Then $\epsilon(w_\nu) \in n\mathbb{Z}$ for every $\nu = 1, \dots, k$.*

In particular, $H \in W_n^*$.

Proof. For $\nu = 1, \dots, k$ let $(w_{\nu,i})_{i\in I} \in \prod_{i\in I}(K_i[x])$ be the coordinates of w_ν.

Since I is infinite, we can apply Corollary 6.58, to get for every $\nu = 1, \dots, k$ an index $i_\nu \in I$ such that $w_{\nu,i_\nu} \in \mathcal{U}_{K_{i_\nu}}$, so that $\epsilon(w_{\nu,i_\nu}) \in n\mathbb{Z}$, since $K_{i_\nu} \in \mathcal{W}_n$. As $\epsilon(w_\nu) = \epsilon(w_{\nu,i_\nu}) \in n\mathbb{Z}$, we are done. □

Note that the hypothesis of Theorem 6.59 on the family $\{K_i : i \in I\}$ to be contained in $\mathcal{W}_n$ could be weakened asking that $K_i \in \mathcal{W}_n$ for infinitely many $i \in I$. By the way, we will immediately obtain this stronger result in the following corollary applying Lemma 6.18.

Corollary 6.60. *Let I be an infinite set, $n \in \mathbb{N}$ and $\{K_i : i \in I\}$ be a family of groups such that $K_i \in \mathcal{W}_n$ for infinitely many $i \in I$. Then $\prod_{i\in I} K_i \in W_n^*$.*

Proof. Let $J = \{i \in I \mid K_i \in \mathcal{W}_n\} \subseteq I$. By assumption, J is infinite, so that $\prod_{i\in J} K_i \in W_n^*$ by Theorem 6.59. Then $\prod_{i\in I} K_i \in W_n^*$ by Lemma 6.18. □

As an application of Theorem 6.59, in the following result we compute the u-exponent and the u*-exponent of the infinite power of a group.

Corollary 6.61. *Let G be a group, and I be an infinite set. Then* $\mathrm{u}^\circ(G^I) = \mathrm{u}^*(G^I) = \mathrm{u}(G^I) = \mathrm{u}(G)$.

Proof. Let $\mathrm{u}(G) = n$. As $G \in \mathcal{W}_n$ by Lemma 2.23, item 2, we have that $G^I \in \mathcal{W}_n^*$ by Theorem 6.59, so that $n \mid \mathrm{u}^*(G^I)$. As $\mathrm{u}(G^I) = n$ by Corollary 6.15, we conclude also $\mathrm{u}^*(G^I) = n$.

Let $w \in \mathcal{U}_{G^I}$ with $\epsilon(w) = n$. If $G^I \in \mathcal{W}_k^*$ for some $k \in \mathbb{N}$, then $G^I = E_w$ implies $k \mid n$. So $\mathrm{u}^\circ(G^I) = n$. □

Note that the assumption on I to be infinite in Corollary 6.61 cannot be dropped, as in this case $\mathrm{u}^*(G^I) = \mathrm{u}(G^I)$ need not hold. In fact, let G be a finite group with $\mathrm{u}(G) \neq 1$ (for example, consider $G = S_3$, that has $\mathrm{u}(S_3) = 2$ by Example 2.37, item 2). If I is also finite, then G^I is finite too, so that $\mathrm{u}^\circ(G^I) = \mathrm{u}^*(G^I) = 1$ by Lemma 3.26, while $\mathrm{u}(G^I) = \mathrm{u}(G) \neq 1$ by Corollary 6.15.

Example 6.62. Recall that the group $G = \mathbb{Z}_4 \times \mathbb{Z}_2^{\mathbb{N}}$ considered in Example 3.24 has $\mathrm{u}(G) = \exp(G) = 4$. By Corollary 3.28, we also have $\mathrm{u}^\circ(G) = \mathrm{u}^*(G) = \exp^*(G) = 2$.

Now consider $G^{\mathbb{N}} \cong \mathbb{Z}_4^{\mathbb{N}} \times \mathbb{Z}_2^{\mathbb{N}}$. Then $\exp(G^{\mathbb{N}}) = \exp^*(G^{\mathbb{N}}) = 4$ implies $\mathrm{u}^\circ(G^{\mathbb{N}}) = \mathrm{u}^*(G^{\mathbb{N}}) = \mathrm{u}(G^{\mathbb{N}}) = 4$, according to Corollary 6.61.

Corollary 6.63. *Let G_0 be a group, I an infinite set, $n \in \mathbb{N}$ and $\{K_i : i \in I\} \subseteq \mathcal{W}_n$ be a family of groups. Consider the group $H = \prod_{i \in I} K_i$ and assume $w \in \mathcal{U}_H$ has $\epsilon(w) = m$. If $G = G_0 \times H$, then $Z(G_0)[n] \times H \subseteq \overline{H}^{\mathfrak{Z}_G} \subseteq Z(G_0)[m] \times H$.*

In particular,

$$Z(G_0)[n] \times H \subseteq \overline{H}^{\mathfrak{Z}_G} \subseteq Z(G_0)[\mathrm{u}(H)] \times H,$$

and H is not $\mathfrak{Z}_G$-closed if $Z(G_0)[n] \neq \{e_{G_0}\}$.

Proof. We have $H \in \mathcal{W}_n^*$ by Theorem 6.59, then Theorem 6.32 applies to give $Z(G_0)[n] \times H \subseteq \overline{H}^{\mathfrak{Z}_G}$. The other inclusion $\overline{H}^{\mathfrak{Z}_G} \subseteq Z(G_0)[m] \times H$ follows by Lemma 6.29, while Corollary 6.30 implies $\overline{H}^{\mathfrak{Z}_G} \subseteq Z(G_0)[\mathrm{u}(H)] \times H$. □

See also Corollary 6.65, where we describe the case when H is $\mathfrak{Z}_G$-dense.

Remark 6.64. Observe that for every group G and for every $n \in \mathbb{N}$ one has the following inclusions between (normal) subgroups of G:

$$Z(G)[n] \leq Z(G) \leq G.$$

- If $n \neq 0$, then $Z(G)[n] = G$ if and only if G is a bounded abelian group, and $\exp(G) \mid n$.
- If $n = 0$, then $Z(G)[0] = Z(G) = G$ if and only if G is an abelian group.

Corollary 6.65. *Let $n \in \mathbb{N}$, G_0 be a group such that $Z(G_0)[n] = G_0$, and let $\{K_i : i \in I\}$, G, and H be as in Corollary 6.63. Then H is $\mathfrak{Z}_G$-dense in G.*

In the following corollary we concentrate on the case $n = 2$.

Next we will consider a family of groups $\{K_i \mid i \in I\} \subseteq \mathcal{W}_2$ introduced in Example 2.38, where we built a universal word of content 2 for each of them.

Corollary 6.66. *Let G_0 be a group, I an infinite set, and $\{K_i \mid i \in I\}$ be a family of groups as in Remark 6.10. Let $H = \prod_{i\in I} K_i$ and $G = G_0 \times H$. Then $Z(G_0)[2] \times H \subseteq \overline{H}^{\mathfrak{Z}_G} \subseteq Z(G_0)[4] \times H$.*

If in addition the groups $\{K_i \mid i \in I\}$ are as in Remark 6.10 (b), then $\overline{H}^{\mathfrak{Z}_G} = Z(G_0)[2] \times H$, so that H is $\mathfrak{Z}_G$-dense if and only if $\exp(G_0) = 2$.

Proof. We have that $\{K_i \mid i \in I\} \subseteq \mathcal{W}_2$ by Example 2.38 (a).

Note that H has a universal word with content 4 by Remark 6.10 (a). Moreover, in the hypothesis that the groups $\{K_i \mid i \in I\}$ are as in Remark 6.10 (b), then H has a universal word with content 2 by Remark 6.10 (b).

In both cases, apply Corollary 6.63.

Finally, when $\overline{H}^{\mathfrak{Z}_G} = Z(G_0)[2] \times H$, obviously H is $\mathfrak{Z}_G$-dense if and only if $G_0 = Z(G_0)[2]$, and this happens exactly when G_0 is an abelian group, of exponent 2, by Remark 6.64. □

Corollary 6.67. *Let G_1, G_2 be groups, I an infinite set, $H = G_2^I$ and consider the group $G = G_1 \times H$. Then*

$$\overline{H}^{\mathfrak{Z}_G} = Z(G_1)[\mathrm{u}(G_2)] \times H.$$

(a) If G_2 is either S_3, or a group H_0 as in Remark 6.10 (b), then $\overline{H}^{\mathfrak{Z}_G} = Z(G_1)[2] \times H$.

(b) If G_2 is either Q_8 or D_8, then $\overline{H}^{\mathfrak{Z}_G} = Z(G_1)[4] \times H$.

(c) If G_2 is an abelian group, with $\exp(G_2) = n$, *then $\overline{H}^{\mathfrak{Z}_G} = Z(G_1)[n] \times H$.*

Proof. As $\mathrm{u}^*(H) = \mathrm{u}(H) = \mathrm{u}(G_2)$ by Corollary 6.61, we have $Z(G_1)[\mathrm{u}^*(H)] = Z(G_1)[\mathrm{u}(H)]$, and Corollary 6.34 applies.

(a) Use the fact that $\mathrm{u}(S_3) = 2$ by Example 2.37, item 2, and $\mathrm{u}(H_0) = 2$ by Remark 6.10 (b).

(b) Use $\mathrm{u}(Q_8) = 4$ by Lemma 3.21 and $\mathrm{u}(D_8) = 4$ by Lemma 3.22.

(c) Use the equality $\mathrm{u}(G_2) = n$ proved in Lemma 2.32. □

Example 6.68. Let I be an infinite set, and consider the groups $\mathbb{Z}_2 = \{1, c\}$ in multiplicative notation, $H = S_3^I$ and $G = \mathbb{Z}_2 \times H$. Then Corollary 6.67 (a) and Remark 6.64 imply $\overline{H}^{\mathfrak{Z}_G} = \mathbb{Z}_2 \times H$, i.e. H is $\mathfrak{Z}_G$-dense, and in particular it is not $\mathfrak{Z}_G$-closed. But it is super-normal, so Zariski embedded: $\mathfrak{Z}_G \restriction_H = \mathfrak{Z}_H$, and $\mathfrak{Z}_H = \prod_{i\in I} \mathfrak{Z}_{S_3}$ is the product topology of the discrete topologies $\mathfrak{Z}_{S_3}$ by Theorem 6.55 (a).

We will explicitly compute $\mathfrak{Z}_G$ in the following §6.6.

6.6 On the Zariski topology of the group $\mathbb{Z}_2 \times S_3^I$

In this subsection, we will describe the Zariski topology of the group $G = \mathbb{Z}_2 \times S_3^I$. Let us consider the cyclic group $\mathbb{Z}_2$ in multiplicative notation, with neutral element 1, so that $\mathbb{Z}_2 = (\{1, c\}, \cdot)$.

First, note that $\mathfrak{C}_{\mathbb{Z}_2} = \iota_{\mathbb{Z}_2}$ and $\mathfrak{C}_{S_3} = \delta_{S_3}$ by Lemma 4.28, item 3, so that Lemma 6.45, item 1, implies that $\mathfrak{C}_{S_3^I} = \prod_{i\in I} \mathfrak{C}_{S_3} = \prod_{i\in I} \delta_{S_3}$ and

$$\mathfrak{C}_G = \mathfrak{C}_{\mathbb{Z}_2} \times \mathfrak{C}_{S_3^I} = \iota_{\mathbb{Z}_2} \times \prod_{i\in I} \delta_{S_3}. \tag{6.14}$$

Obviously $Z(G) = \mathbb{Z}_2$, and let $\overline{G} = G/Z(G) \cong S_3^I$. By Theorem 6.55 (a), it follows that $\mathfrak{C}_{\overline{G}} = \mathfrak{Z}_{\overline{G}} = \prod_{i\in I} \delta_{S_3}$. Note that the quotient map $\pi\colon (G, \mathfrak{Z}_G) \to (\overline{G}, \mathfrak{Z}_{\overline{G}})$ is continuous by Proposition 4.6, so $\pi^{-1}\mathfrak{Z}_{\overline{G}} \subseteq \mathfrak{Z}_G$. Moreover, as the family of all cosets of the subgroups $\mathbb{Z}_2 \times S_3^{I\setminus\{i\}}$, for $i \in I$, is a subbase for $\pi^{-1}\mathfrak{Z}_{\overline{G}}$-closed sets, by (6.14) we obtain that $\mathfrak{C}_G = \pi^{-1}\mathfrak{Z}_{\overline{G}}$.

However, $\mathfrak{C}_G$ is not T_1 by Lemma 4.28, item 3, so $\mathfrak{C}_G \subsetneq \mathfrak{Z}_G$. To be more precise, let us see that the commutator subgroup G' of G is $\mathfrak{Z}_G$-closed, but not $\mathfrak{C}_G$-closed. Indeed, $G' = A_3^I = G[3] = E_{x^3}$, hence it is an elementary algebraic subset of G, while it is not $\mathfrak{C}_G$-closed, since $G' \cap \mathbb{Z}_2 = \{e_G\}$ is not $\mathfrak{C}_G$-closed, as $\mathfrak{C}_G$ is not T_1.

We shall see in Theorem 6.69 that the topology $\mathfrak{Z}_G$ is the coarser topology containing $\mathfrak{C}_G$, and having the cosets of G' as closed sets, i.e. the cosets of centralizers and the cosets of G' form a subbase for $\mathfrak{Z}_G$-closed sets.

Let $V = \{(gx)^3 \mid g \in G\} \subseteq W_{hom,3} \subseteq G[x]$, and note that every word $w \in V$ is homogeneous. Then the family of the cosets of G' in G is

$$\{gG' \mid g \in G\} = \{gE_{x^3} \mid g \in G\} = \{E_{(g^{-1}x)^3} \mid g \in G\} = \mathcal{E}(V).$$

If we let also $W = W_{\mathfrak{C}} \cup V \subseteq G[x]$, then Theorem 6.69 below will prove that $\mathcal{E}(W)$ is a subbase for $\mathfrak{Z}_G$-closed sets, i.e. $\mathfrak{T}_W = \mathfrak{Z}_G$. As a consequence, the Zariski topology $\mathfrak{Z}_G$ is determined by the homogeneous words in V, and by the commutator words of $G[x]$.

By Lemma 6.1, if $w \in G[x]$ then its coordinates in $\mathbb{Z}_2[x] \times S_3[x]^I$ are

$$\big(w_0, (w_i)_{i\in I}\big) \in \mathbb{Z}_2[x] \times S_3[x]^I$$

for some words $w_0 \in \mathbb{Z}_2[x]$ and $w_i \in S_3[x]$ for every $i \in I$. Moreover, $\epsilon(w) = \epsilon(w_0) = \epsilon(w_i)$ for every $i \in I$, and by (6.1) we have

$$E_w^G = E_{w_0}^{\mathbb{Z}_2} \times \prod_{i\in I} E_{w_i}^{S_3}.$$

Theorem 6.69. *Every elementary algebraic subset of G is $\mathfrak{T}_W$-closed. In particular, $\mathfrak{Z}_G = \mathfrak{T}_W$.*

Proof. As $\mathfrak{T}_W \subseteq \mathfrak{Z}_G$, the second part of the statement will immediately follows from the first one, as $\mathbb{E}_G$ is a subbase for $\mathfrak{Z}_G$-closed sets.

Let E_w^G be a non-empty elementary algebraic subset of G.

- If $\epsilon(w) = \epsilon(w_0)$ is even, then $E_{w_0}^{\mathbb{Z}_2} = \mathbb{Z}_2$ and E_w^G has the form $E_w^G = \mathbb{Z}_2 \times \prod_{i\in I} E_{w_i}^{S_3}$, so it is $\mathfrak{C}_G$-closed by (6.14).
- If $\epsilon(w) = \epsilon(w_0)$ is odd, then $E_{w_0}^{\mathbb{Z}_2}$ is a singleton and let $E_{w_0}^{\mathbb{Z}_2} = \{p\}$.

  Let $J = \{i \in I \mid \mathrm{ct}(w_i) \in A_3\} \subseteq I$. For every $i \in I$, as $\epsilon(w_i) = \epsilon(w)$ is odd too, Example 3.5 implies that $E_{w_i}^{S_3} \subseteq \mathrm{ct}(w_i)A_3$, i.e. either $E_{w_i}^{S_3} \subseteq A_3$ (if $i \in J$), or $E_{w_i}^{S_3} \subseteq S_3 \setminus A_3$ (if $i \notin J$). Now fix an element $g \in E_w^G$, so that

  $$E_w^G \subseteq \{p\} \times \prod_{i\in J} A_3 \times \prod_{i\in I\setminus J} S_3 \setminus A_3 = gG_3'.$$

  Note that $F = \mathbb{Z}_2 \cdot E_w^G = \mathbb{Z}_2 \times \prod_{i\in I} E_{w_i}^{S_3}$ is $\mathfrak{C}_G$-closed by (6.14), and so $E_w^G = gG_3' \cap F$ is $\mathfrak{T}_W$-closed.

□

By Remark 5.1, the inequality $\mathfrak{Z}_S \subseteq \mathfrak{Z}_G \restriction_S$ holds true for the subgroup $S = \mathbb{Z}_2 \times S_3^{(I)}$. The following lemma proves the reverse one.

Corollary 6.70. *S is a Zariski embedded subgroup of G.*

Proof. As we have to prove that $\mathfrak{Z}_G \restriction_S \subseteq \mathfrak{Z}_S$, by Theorem 6.69 it will suffice to show that $S \cap G'$ and the subsets $S \cap C_G(g)$, as $g \in G$, are $\mathfrak{Z}_S$-closed subsets of S. To this end, first note that $S \cap G' = S \cap G[3] = S[3] = E_{x^3}^S$ is an elementary algebraic subset of S. While, if $g = (g_0, (g_i)_{i\in I})$, then

$$S \cap C_G(g) = S \cap \Big(\mathbb{Z}_2 \times \prod_{i\in I} C_{S_3}(g_i)\Big) = \mathbb{Z}_2 \oplus \bigoplus_{i\in I} C_{S_3}(g_i)$$

is $\mathfrak{Z}_S$-closed by Lemma 6.44. □

7

The Zariski topology of free non-abelian groups

Let F be a free non-abelian group. In this chapter we will see that the algebraic subsets of F are finite unions of singletons or subsets $fC_F(g)$, for elements $f, g \in F$.

The first step consists in recalling some algebraic properties.

Lemma 7.1. *Let F be a free non-abelian group, and $x, y \in F$ be commuting elements. Then $\langle x, y\rangle$ is cyclic, i.e. x, y are powers of some element $z \in F$.*

Proof. By hypothesis, $\langle x, w\rangle \leq F$ is an abelian subgroup of F, and it is free by the Nielsen−Schreier theorem, so it is cyclic. □

Note that if G is a group, and $x \in G$, then $\langle x\rangle \leq C_G(x)$. Moreover, let us see that a one-element centralizer subgroup cannot be properly contained in a cyclic subgroup. Indeed, if $C_G(x) \leq \langle y\rangle$ for some element $y \in G$, then $\langle x\rangle \leq C_G(x) \leq \langle y\rangle \leq C_G(y)$. Moreover, $\langle x\rangle \leq \langle y\rangle$ implies $C_G(y) \leq C_G(x)$, so that $C_G(x) = \langle y\rangle = C_G(y)$.

The following lemma proves that in a free non-abelian group the centralizer of non-trivial elements are the maximal cyclic subgroups.

Lemma 7.2. *(i) If $\langle x\rangle$ is maximal among cyclic subgroups, then $\langle x\rangle = C_F(x)$.*

(ii) $C_F(x)$ is cyclic, and maximal among cyclic subgroups.

In other words, $\langle x\rangle = C_F(x)$ if and only if $\langle x\rangle$ is maximal among cyclic subgroups.

Proof. (i) Assume $\langle x\rangle$ to be maximal among cyclic subgroups, and let $y \in C_F(x)$. Then $\langle x, y\rangle \leq F$ is cyclic by Lemma 7.1, so it coincides with $\langle x\rangle$ and $y \in \langle x\rangle$.

(ii) Consider $\langle x\rangle \leq C_F(x)$. We shall find an element $x_0 \in C_F(x)$ such that $C_F(x) = C_F(x_0) = \langle x_0\rangle$ is maximal.

In F, every cyclic subgroup is contained in a maximal cyclic subgroup, and let $\langle x_0\rangle$ be the maximal cyclic subgroup containing $\langle x\rangle$. Then $\langle x\rangle \leq \langle x_0\rangle \leq C_F(x_0) \leq C_F(x)$. Note that $\langle x_0\rangle = C_F(x_0)$ by the previous point. Now $y \in C_F(x)$ implies $\langle x, y\rangle \leq F$ to be cyclic again by Lemma 7.1, so that it coincides with $\langle x_0\rangle$; this finally proves that $C_F(x) = C_F(x_0) = \langle x_0\rangle$. □

By Lemma 7.2, when considering a centralizer subgroup $C_F(x)$ in a free non-abelian group F, one can assume $C_F(x) = \langle x\rangle$, i.e. $\langle x\rangle$ to be maximal among cyclic subgroups, i.e. x not to be a proper power of any other element in F.

Let $\mathcal{B} = \{\{f\}, fC_F(g) \mid f, g \in F\} \subseteq \mathbb{E}_F$, and note that $\mathcal{B}$ is a subbase for $\mathfrak{C}'_F$-closed subsets.

Corollary 7.3. *The family $\mathcal{B}$ is stable under finite intersections, satisfies the descending chain condition, and $F \in \mathcal{B}$.*

In particular, $\mathfrak{C}'_F$ is a Noetherian topology on F, and $\mathcal{B}^{\cup}$ is the family of its closed subsets.

Proof. The first assertion will immediately follow from the following: if $x, y \in F \setminus \{e_F\}$, and $C_F(x) \cap C_F(y) \neq \{e_F\}$, then $C_F(x) = C_F(y)$. So let us prove this. By Lemma 7.2 (ii), both $C_F(x)$ and $C_F(y)$ are cyclic and maximal, and we can assume $C_F(x) = \langle x\rangle$ and $C_F(y) = \langle y\rangle$. Then $C_F(x) \cap C_F(y) = \langle x\rangle \cap \langle y\rangle = \langle z\rangle$, with $x^n = z = y^m$ for some integers $n, m \in \mathbb{Z}$. Then $y \in C_F(x^n) = \langle x\rangle$, giving $\langle y\rangle \leq \langle x\rangle$. By maximality, $\langle y\rangle = \langle x\rangle$ as desired.

Then Proposition 1.3 applies. □

Proposition 7.4 ([13, Theorem 5.3])**.** *Arbitrary intersections of proper elementary algebraic subsets of F are elements of $\mathcal{B}^{\cup}$.*

In the original statement of Proposition 7.4, the authors used the family

$$\{\{f\}, fC_F(g)h \mid f, g, h \in F\}$$

instead of $\mathcal{B}$. Recall that $fC_F(g)h = fhC_F(h^{-1}gh)$, so that really the two families coincide.

Theorem 7.5. *If F is a free non-abelian group, then*

$$\mathfrak{C}_F = \mathfrak{C}'_F = \mathfrak{Z}_F.$$

In particular, $\mathfrak{Z}_F$ is Noetherian, the family of algebraic subsets of F coincides with $\mathcal{B}^{\cup}$, and every algebraic subset is additively algebraic.

Proof. It trivially follows by Proposition 7.4 that $\mathbb{E}_F \subseteq \mathcal{B}^{\cup}$, so that $\mathfrak{Z}_F = \mathfrak{C}'_F$ is Noetherian by Corollary 7.3. As F is center-free, we have also $\mathfrak{C}_F = \mathfrak{C}'_F$. □

Corollary 7.6. *The family of $\mathfrak{Z}_F$-closed irreducible sets is $\mathcal{B}$. In particular, F is $\mathfrak{Z}$-irreducible, and* $\dim(F, \mathfrak{Z}_F) = 2$.

Proof. According to Theorem 7.5, if $g \neq e_F$, then $C_F(g)$ is $\mathfrak{Z}$-irreducible by Lemma 7.2 (ii). The whole $F = C_F(e_F)$ is $\mathfrak{Z}$-irreducible as the proper centralizers have infinite index in F, and so the first part of the statement immediately follows by Theorem 7.5. Then, for example consider a chain of the form

$$\{e_F\} \subsetneq C_F(g) \subsetneq F,$$

where $g \in F \setminus \{e_F\}$. □

Corollary 7.7. *If* $\{G_i \mid i \in I\}$ *is a family of free non-abelian groups, and* $G = \prod_{i\in I} G_i$, *then* $\mathfrak{C}_G = \mathfrak{Z}_G = \prod_{i\in I} \mathfrak{Z}_{G_i}$.

Proof. Immediately follows from Lemma 6.46 and Theorem 7.5. □

8

The Zariski topology of the Heisenberg group

If n is a positive integer, and K is an infinite field, the *n-th Heisenberg group with coefficients in K* is the following matrix group:

$$H(n,K) = \left\{ \begin{pmatrix} 1 & x_1 & \cdots & x_n & y \\ & 1 & & 0 & z_1 \\ 0 & & \ddots & & \vdots \\ & 0 & & 1 & z_n \\ 0 & & 0 & & 1 \end{pmatrix} \in \mathrm{GL}_{n+2}(K) \mid x_1, \dots, x_n, z_1, \dots, z_n, y \in K \right\}$$

which we will simply denote H when confusion is not possible. Let $\vec{x} = (x_1, \dots, x_n) \in K^n$ and $\vec{z} = (z_1, \dots, z_n) \in K^n$. Introducing the canonical scalar product over K^n, $\vec{x} \cdot \vec{z} = \sum_{i=1}^n x_i z_i$, the group H can be written as the 'formal matrix' group

$$H = H(n,K) = \begin{pmatrix} 1 & K^n & K \\ & I_n & K^n \\ & & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \mid \vec{x}, \vec{z} \in K^n, y \in K \right\}.$$

The product in H is the following:

$$\begin{pmatrix} 1 & \vec{x}_1 & y_1 \\ & I_n & \vec{z}_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \vec{x}_2 & y_2 \\ & I_n & \vec{z}_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{x}_1 + \vec{x}_2 & y_1 + y_2 + \vec{x}_1 \cdot \vec{z}_2 \\ & I_n & \vec{z}_1 + \vec{z}_2 \\ & & 1 \end{pmatrix},$$

while the commutator of two elements in H is given by

$$\left[\begin{pmatrix} 1 & \vec{x}_1 & y_1 \\ & I_n & \vec{z}_1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \vec{x}_2 & y_2 \\ & I_n & \vec{z}_2 \\ & & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & \vec{0} & \vec{x}_1 \cdot \vec{z}_2 - \vec{x}_2 \cdot \vec{z}_1 \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix}. \tag{8.1}$$

From the above formulas, one can easily see that $Z(H) = H' = \begin{pmatrix} 1 & \vec{0} & K \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix} \cong K$, so that $H/Z(H) \cong (K^n, +) \times (K^n, +)$ is abelian, and $H \in \mathscr{N}_2$.

The proof of the following result is straightforward.

Fact 8.1. *If* $s \in \mathbb{N}$, *and* $h = \begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H$, *then* $h^s = \begin{pmatrix} 1 & s\vec{x} & sy + \frac{s(s-1)}{2}\vec{x} \cdot \vec{z} \\ & I_n & s\vec{z} \\ & & 1 \end{pmatrix}$.

In the following lemma, we shall apply Fact 8.1.

Lemma 8.2. *Depending on the characteristic of the field* K, *the group* H *has the following properties:*

- *If* $\operatorname{char} K = 0$, *then* H *is torsion-free.*
- *If* $\operatorname{char} K = 2$, *then* $\exp(H) = 4$.
- *If* $\operatorname{char} K = p > 2$, *then* $\exp(H) = p$.

Proof. It is obvious that H is torsion-free if $\operatorname{char} K = 0$.

If $\operatorname{char} K = p > 2$, then $\frac{p-1}{2} \in \mathbb{N}$, so that $\exp(H) = p$, as

$$\begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix}^p = \begin{pmatrix} 1 & p\vec{x} & py + \frac{p(p-1)}{2}\vec{x} \cdot \vec{z} \\ & I_n & p\vec{z} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{0} & 0 \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix}.$$

If $\operatorname{char} K = 2$, note that $\exp(H) \neq 2$, as H is non-abelian; moreover,

$$H[2] = \left\{ \begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \mid y \in K,\ \vec{x} \cdot \vec{z} = 0_K \right\} \neq H.$$

But in this case

$$\begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2\vec{x} & 2y + \vec{x} \cdot \vec{z} \\ & I_n & 2\vec{z} \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{0} & \vec{x} \cdot \vec{z} \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix} \in Z(H).$$

As $Z(H) \cong K$, $\exp(Z(H)) = 2$, and $\exp(H) = 4$. □

We distinguish the following subgroups of H:

$$L = \begin{pmatrix} 1 & 0 & K \\ & I_n & K^n \\ & & 1 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 1 & K^n & K \\ & I_n & 0 \\ & & 1 \end{pmatrix}, \tag{8.2}$$

and observe that $Z(H) = L \cap M$.

If $\operatorname{char} K = 2$, then $L \cup M \subseteq H[2]$, and if moreover $n = 1$, then $H[2] = L \cup M$.

Finally, note that H is isomorphic to the semidirect products $(K^n, +) \ltimes M$ and $(K^n, +) \ltimes L$, but this will not be used in the sequel.

Lemma 8.3. *L and M are both $\mathfrak{Z}_H$-closed subsets, that are neither Zariski embedded in H, nor Markov embedded.*

In particular, $\mathfrak{M}_L = \mathfrak{Z}_L \subsetneq \mathfrak{Z}_H \restriction_L \subseteq \mathfrak{M}_H \restriction_L$ and $\mathfrak{M}_M = \mathfrak{Z}_M \subsetneq \mathfrak{Z}_H \restriction_M \subseteq \mathfrak{M}_H \restriction_M$.

Proof. It is easy to verify that L is the centralizer in H of its non-central elements, so L is a $\mathfrak{Z}_H$-closed subset of H. As L is isomorphic to the group $(K^{n+1}, +)$, $\mathfrak{Z}_L$ is the cofinite topology by Proposition 4.16 (a). Being $Z(H)$ an infinite $\mathfrak{Z}_H$-closed subset properly contained in L, thus an infinite proper $\mathfrak{Z}_H \restriction_L$-closed subset, $\mathfrak{Z}_H$ induces on L a topology strictly finer than $\mathfrak{Z}_L$, i.e. L is not Zariski embedded in H. Moreover, $\mathfrak{M}_L = \mathfrak{Z}_L$ for the abelian group L by Theorem 4.11, and L is not Markov embedded in H by Fact 5.12 (a). Finally, $\mathfrak{Z}_H \restriction_L \subseteq \mathfrak{M}_H \restriction_L$ by Remark 5.1.

In the same way one proves that M has the same properties as L. $\square$

Remark 8.4. Let us underline another difference between the Heisenberg groups and the abelian groups, beyond those noted in Remark 11.42. We already noticed that $\mathfrak{Z}_H$ is not the cofinite topology, as for example $Z(H)$, L and M are infinite proper $\mathfrak{Z}_H$-closed subsets. Compare Lemma 8.2 and Proposition 4.15: every abelian group which is either almost torsion-free or of prime exponent is $\mathfrak{Z}$-cofinite. On the other hand, when $\operatorname{char} K$ varies in $\mathbb{N} \setminus \{2\}$, H is either torsion-free or of exponent $p > 2$, but H is not $\mathfrak{Z}$-cofinite.

In the following lemma we describe the cosets of one-element centralizer in H.

Lemma 8.5. *Let $h = \begin{pmatrix} 1 & \vec{a} & b \\ & I_n & \vec{c} \\ & & 1 \end{pmatrix} \in H$, and $g = \begin{pmatrix} 1 & \vec{\alpha} & \beta \\ & I_n & \vec{\gamma} \\ & & 1 \end{pmatrix} \in H$. Then the following hold.*

1. $C_H(h) = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H \mid \vec{c} \cdot \vec{x} - \vec{a} \cdot \vec{z} = 0 \right\}$ *is a normal subgroup of H.*

2. $gC_H(h) = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H \mid \vec{c} \cdot \vec{x} - \vec{a} \cdot \vec{z} = \vec{c} \cdot \vec{\alpha} - \vec{a} \cdot \vec{\gamma} \right\}$.

3. *In particular, the cosets of $C_H(h)$ are the subsets*

$$\left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H \mid \vec{c} \cdot \vec{x} - \vec{a} \cdot \vec{z} = \lambda \right\}, \text{ for } \lambda \in K.$$

Proof. The description of $C_H(h)$ and $gC_H(h)$ immediately follows from the formula given in (8.1) of the commutator of two elements in H. Then, $C_H(h)$ is a normal subgroup of H as it contains H'. $\square$

Note that the family $\{gC_H(h) \mid g, h \in H\}$ of cosets of one-element centralizers in H consists of the solution-sets of polynomial equations of degree 1 in the variables $\vec{x}$, $\vec{z}$.

Lemma 8.6. *Let $n = 1$. Then the following holds for $H = H(1, K)$.*

1. *$L = C_H(l)$, for every $l \in L \setminus Z(H)$, and $M = C_H(m)$, for every $m \in M \setminus Z(H)$.*
2. *For every $h \in H \setminus Z(H)$, the subgroup $C_H(h)$ is normal, but not super-normal in H.*
3. *Let $h_1, h_2 \in H \setminus Z(H)$. If $C_H(h_1) \neq C_H(h_2)$, then $C_H(h_1) \cap C_H(h_2) = Z(H)$.*

Proof. 1. Immediately follows by Lemma 8.5, item 1.

2. By Lemma 8.5, item 1, it only remains to note that $C_H(h)$ is an abelian, non central subgroup, so that Proposition 5.5 applies.

3. For $i = 1, 2$, let $h_i = \begin{pmatrix} 1 & a_i & b_i \\ & 1 & c_i \\ & & 1 \end{pmatrix} \in H$, so that

$$C_H(h_i) = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid c_i \cdot x - a_i \cdot z = 0 \right\}$$

by Lemma 8.5, item 1.

Then

$$C_H(h_1) \cap C_H(h_2) = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid c_i \cdot x - a_i \cdot z = 0, i = 1, 2 \right\},$$

i.e. it is the solution-set in H of the system

$$\begin{cases} c_1 \cdot x - a_1 \cdot z = 0 \\ c_2 \cdot x - a_2 \cdot z = 0. \end{cases}$$

By hypotheses, it follows that the two equations are non-trivial and that they are independent over K, so that the system has a unique solution $x = 0 = z$. □

In the following lemma, we begin the description of $\mathbb{E}_H$ and $\mathfrak{Z}_H$.

Lemma 8.7. *Let $w \in H[x]$ be such that $E_w \neq \emptyset$, and let $\epsilon = \epsilon(w)$.*

1. *If $\epsilon \cdot 1_K \neq 0_K$, then E_w is a singleton.*

2. *If* $\epsilon \cdot 1_K = 0_K$ *and* $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K = 0_K$, *then* E_w *is a coset of* $C_H(h)$, *for some* $h \in H$.

3. *If* $\epsilon \cdot 1_K = 0_K$ *and* $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K \neq 0_K$, *then* $\operatorname{char} K = 2$, *and* $2 \mid \epsilon$ *but* $4 \nmid \epsilon$. *Moreover, there exist* $\vec{A}, \vec{C} \in K^n$, *and* $B \in K$, *such that*

$$E_w = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H \mid (\vec{A} + \vec{x}) \cdot \vec{z} = B + \vec{C} \cdot \vec{x} \right\}. \tag{8.3}$$

Proof. According to Corollary 3.15, we can assume $w = \alpha x^{\epsilon}[\beta, x] \in H[x]$, where

$$\alpha = \begin{pmatrix} 1 & \vec{a} & -B \\ & I_n & \vec{c} \\ & & 1 \end{pmatrix} \in H, \text{ and } \beta = \begin{pmatrix} 1 & \vec{A} & B_0 \\ & I_n & \vec{C} \\ & & 1 \end{pmatrix} \in H,$$

for $\vec{a}, \vec{A}, \vec{c}, \vec{C} \in K^n$ and $B, B_0 \in K$.

In the sequel, if $s \in \mathbb{Z}$ and $\lambda \in K$, the multiple $s\lambda$ in the additive group $(K, +)$ coincides with the product $(s1_K)\lambda$ in K.

If $X = \begin{pmatrix} 1 & \vec{x} & y \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H$, and $D = \frac{\epsilon^2-\epsilon}{2}$, then

$$[\beta, X] = \begin{pmatrix} 1 & \vec{0} & \vec{A} \cdot \vec{z} - \vec{C} \cdot \vec{x} \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix}, \text{ and } X^{\epsilon} = \begin{pmatrix} 1 & \epsilon\vec{x} & \epsilon y + D\vec{x} \cdot \vec{z} \\ & I_n & \epsilon\vec{z} \\ & & 1 \end{pmatrix},$$

so that

$$w(X) = \begin{pmatrix} 1 & \vec{a} + \epsilon\vec{x} & \epsilon y + (\vec{A} + D\vec{x}) \cdot \vec{z} - \vec{C} \cdot \vec{x} - B + \epsilon\vec{a} \cdot \vec{z} \\ & I_n & \vec{c} + \epsilon\vec{z} \\ & & 1 \end{pmatrix}.$$

Then E_w is the solution-set in H of the equation $w(X) = e_H$, i.e. of the system

$$\begin{cases} \vec{a} + \epsilon\vec{x} = \vec{0} \\ \vec{c} + \epsilon\vec{z} = \vec{0} \\ \epsilon y + (\vec{A} + D\vec{x}) \cdot \vec{z} - \vec{C} \cdot \vec{x} - B - \epsilon\vec{a} \cdot \vec{z} = 0. \end{cases} \tag{8.4}$$

If $\epsilon \cdot 1_K \neq 0_K$, then system (8.4) has a unique solution, and E_w is a singleton.

So, from now on, we will assume $\epsilon \cdot 1_K = 0_K$, so that also $\vec{a} = \vec{c} = \vec{0}$ (as we are assuming $E_w \neq \emptyset$). In this case, system (8.4) is equivalent to the equation

$$(\vec{A} + D\vec{x}) \cdot \vec{z} = B + \vec{C} \cdot \vec{x}. \tag{8.5}$$

Claim 8.8. $D \cdot 1_K = 0_K$ *whenever* $\epsilon \cdot 1_K = 0_K$ *and* $\operatorname{char} K \neq 2$.

Proof. If $\operatorname{char} K \neq 2$, then $D \cdot 1_K = 0_K$ if and only if $(\epsilon^2 - \epsilon) \cdot 1_K = 0_K$, and the latter holds by our assumption $\epsilon \cdot 1_K = 0_K$. Note that if $\operatorname{char} K = 2$, then $D \cdot 1_K \in \{0_K, 1_K\}$, and both values can be assumed. □

By Claim 8.8, if $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K = 0_K$, then equation (8.5) becomes $\vec{A} \cdot \vec{z} = B + \vec{C} \cdot \vec{x}$, so that

$$E_w = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & I_n & \vec{z} \\ & & 1 \end{pmatrix} \in H \mid \vec{C} \cdot \vec{x} - \vec{A} \cdot \vec{z} = -B \right\}.$$

By Lemma 8.5, item 2, E_w is a coset of $C_H(h)$, for example for the element $h = \begin{pmatrix} 1 & \vec{A} & 0 \\ & I_n & \vec{C} \\ & & 1 \end{pmatrix}$.

Finally, assume $\epsilon \cdot 1_K = 0_K$ and $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K \neq 0_K$. Then $D \cdot 1_K = 1_K$, and $\operatorname{char} K = 2$ by Claim 8.8, so that E_w is as in (8.3). □

8.1 Case $\operatorname{char} K \neq 2$

If $\operatorname{char} K \neq 2$, Lemma 8.7 is sufficient to describe $\mathfrak{Z}_H$ as a partial Zariski topology as follows.

Corollary 8.9. *If* $\operatorname{char} K \neq 2$, *then* $\mathfrak{Z}_H = \mathfrak{C}'_H$.

Proof. Immediately follows by Lemma 8.7, as every non-empty elementary algebraic subset of H is either a singleton, or a coset of some one-element centralizer. □

Compare the following immediate corollary of Theorem 4.34 and Theorem 4.35 with Corollary 8.9.

Corollary 8.10. *If K is an infinite field with* $\operatorname{char}(K) \neq 2$, *and* $n \in \mathbb{N}_+$, *then* $G = H(n, K)$ *satisfies* $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I}$ *for every non-empty set* I.

Proof. If $\operatorname{char}(K) = 0$, then $G \in \mathscr{N}_2$ is torsion-free, so that Theorem 4.34 applies.

If $\operatorname{char}(K) = p > 2$, then $G^I \in \mathscr{N}_2$ and G^I has exponent p by Lemma 8.2. Now Theorem 4.35 applies. □

8.2 Case $\operatorname{char} K = 2$

If $\operatorname{char} K = 2$, the following result will provide a description of $\mathfrak{Z}_H$ as a partial Zariski topology, which will follow in the subsequent Corollary 8.12. Recall the definition of $W_{hom,2} \subseteq H[x]$ given in (2.2).

Lemma 8.11. *If* $\operatorname{char} K = 2$, *let* $w \in H[x]$ *be such that* $E_w \neq \emptyset$, *and let* $\epsilon = \epsilon(w)$. *If* $\epsilon \cdot 1_K = 0_K$ *and* $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K \neq 0_K$, *then* $w \sim w_0$ *for a word* $w_0 \in W_{hom,2}$.

Proof. By Lemma 8.7, item 3, E_w is as in (8.3). Then, defining

$$\alpha = \begin{pmatrix} 1 & \vec{0} & -B \\ & I_n & \vec{0} \\ & & 1 \end{pmatrix} \in H \text{ and } \beta = \begin{pmatrix} 1 & \vec{A} & B \\ & I_n & \vec{C} \\ & & 1 \end{pmatrix} \in H, \tag{8.6}$$

Lemma 8.7, item 3, applied to the word $w_0 = \alpha x^2[\beta, x] \in H[x]$ gives $E_{w_0} = E_w$.
Finally, as $H' \subseteq Z(H)$ we have $w_0 \approx \alpha[\beta, x]x^2 = \alpha\beta x\beta^{-1}x \in W_{hom,2} \subseteq H[x]$. □

Corollary 8.12. *If* $\operatorname{char} K = 2$, *then* $\mathfrak{Z}_H = \mathfrak{C}'_H \vee \mathfrak{T}_{W_{hom,2}}$.

Proof. Let $w \in H[x]$ be such that $E_w \neq \emptyset$. Then one of the following holds:

1. if $\epsilon \cdot 1_K \neq 0_K$, then E_w is a singleton by Lemma 8.7, item 1;
2. if $\epsilon \cdot 1_K = 0_K$ and $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K = 0_K$, then E_w is a coset of $C_H(h)$, for some $h \in H$, by Lemma 8.7, item 2;
3. if $\epsilon \cdot 1_K = 0_K$ and $\frac{\epsilon^2-\epsilon}{2} \cdot 1_K \neq 0_K$, then $E_w \in \mathcal{E}(W_{hom,2})$ by Lemma 8.11.

This proves $\mathfrak{Z}_H \subseteq \mathfrak{C}'_H \vee \mathfrak{T}_{W_{hom,2}}$. Since the other inclusion is obvious, we are done. □

8.3 The group $H(1, K)$

From now on, we will always assume $n = 1$, so $H = H(1, K) = \begin{pmatrix} 1 & K & K \\ & 1 & K \\ & & 1 \end{pmatrix}$.

Let us define the two subfamilies $\mathcal{C}_{K,singl} \subseteq \mathbb{E}_H$, consisting of the singletons of H, and $\mathcal{C}_{K,centr} \subseteq \mathbb{E}_H$, consisting of cosets of one-element centralizers. Note that $\mathcal{C}_{K,singl}$ is a subbase for cof_H-closed sets, $\mathcal{C}_{K,centr}$ is a subbase for $\mathfrak{C}_H$-closed sets, and that $\mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr}$ is a subbase for $\mathfrak{C}'_H$-closed sets.

Consider the field $K(X)$ of rational functions over K, and a *non-constant* $R \in K(X)$ of the form

$$R = \frac{B + CX}{A + X}$$

where $A, B, C \in K$. Note that R is non-constant if and only if $B \neq AC$, and for such a rational function R, consider the evaluation function $\operatorname{dom} R = K \setminus \{-A\} \to K$, $p \mapsto R(p)$, associated to R. We define the following subset $A_R \subseteq H$

$$A_R = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & R(x) \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R \right\}. \tag{8.7}$$

Note that we will not consider constant rational functions because in this case, letting $R = C \in K \subseteq K(X)$, the subset A_R would become

$$A_R = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & C \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R = K \right\} = \begin{pmatrix} 1 & K & K \\ & 1 & C \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & C \\ & & 1 \end{pmatrix} M.$$

By Lemma 8.6, item 1, A_R would be a coset of a one-element centralizer (namely, some $m \in M \setminus Z(H)$), and we will consider separately this kind of subsets.

Let us define the family $\mathcal{C}_{K,graph} \subseteq \mathcal{P}(H)$, consisting of subsets A_R as in (8.7), for a non-constant $R = \frac{B+CX}{A+X} \in K(X)$. Finally, we define the following family $\mathcal{C}_K \subseteq \mathcal{P}(H)$, whose definition depends on whether $\operatorname{char} K = 2$ or $\operatorname{char} K \neq 2$:

$$\mathcal{C}_K = \begin{cases} \mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr} & \text{if } \operatorname{char} K \neq 2, \\ \mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr} \cup \mathcal{C}_{K,graph} & \text{if } \operatorname{char} K = 2. \end{cases} \tag{8.8}$$

8.3.1 Case $\operatorname{char} K \neq 2$

As a consequence of Corollary 8.9, we obtain that when $\operatorname{char} K \neq 2$ the family

$$\mathcal{C}_K = \mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr} \subseteq \mathbb{E}_H$$

is a subbase for $\mathfrak{Z}_H$-closed sets.

In the following result, we describe the intersections of elements of $\mathcal{C}_K$.

Lemma 8.13. *If $\operatorname{char} K \neq 2$ and $n = 1$, then $\mathcal{C}_K^{\cap} = \mathcal{C}_K \cup \{\emptyset\} \cup \{hZ(H) \mid h \in H\}$. Moreover, both $\mathcal{C}_K$ and $\mathcal{C}_K^{\cap}$ are subbases for $\mathfrak{Z}_H$-closed sets and satisfy the descending chain condition.*

Proof. As $n = 1$, the description of $\mathcal{C}_K^{\cap}$ follows by Lemma 8.6, item 3. Then, one immediately sees that both $\mathcal{C}_K$ and $\mathcal{C}_K^{\cap}$ satisfy the descending chain condition. □

8.3.2 Case $\operatorname{char} K = 2$

In this case,

$$\mathcal{C}_K = \mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr} \cup \mathcal{C}_{K,graph}.$$

We will prove below in Theorem 8.18 that $\mathcal{C}_K$ is a subbase of the $\mathfrak{Z}_H$-closed subsets (recall we are assuming $\operatorname{char} K = 2$ and $n = 1$).

Since $n = 1$, the elementary algebraic subsets E_w, for words $w \in W_{hom,2} \subseteq H[x]$, have a very transparent description. In Lemma 8.14 below, we present such a description.

Lemma 8.14. *Let $\operatorname{char} K = 2$ and $n = 1$. If $w \in W_{hom,2} \subseteq H[x]$, then either $E_w = A_R$ is as in (8.7), or E_w the union of a coset of L and a coset of M. In particular, $E_w \in \mathcal{C}_K^{\cup}$, so that $\mathbb{E}_H \subseteq \mathcal{C}_K^{\cup}$.*

Proof. According to Theorem 3.14 (and its proof), if $w = g_1 x g_2 x \in W_{hom,2}$, then $w \approx g_1 g_2 x^2 [g_2^{-1}, x]$.

By Lemma 8.7, item 3,

$$E_w = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid (A + x)z = B + Cx \right\}$$

for some $A, B, C \in K$.

If $B \neq CA$, then $R = \frac{B+CX}{A+X}$ is non-constant, the system $\begin{cases} A + x = 0 \\ B + Cx = 0 \end{cases}$ has no solution, and $E_w = A_R$. Otherwise, $B = CA$, and the condition $(A+x)z = C(A+x)$ defining E_w is equivalent to the disjunction $x = -A \vee z = C$, so that

$$E_w = \begin{pmatrix} 1 & -A & K \\ & 1 & K \\ & & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & K & K \\ & 1 & C \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -A & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} L \cup \begin{pmatrix} 1 & 0 & 0 \\ & 1 & C \\ & & 1 \end{pmatrix} M.$$

This proves that $\mathcal{E}(W_{hom,2}) \subseteq \mathcal{C}_K^{\cup}$. By Lemmata 8.7 and 8.11, $\mathbb{E}_H \setminus \mathcal{E}(W_{hom,2}) \subseteq \mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr}$. Hence, $\mathbb{E}_H \subseteq \mathcal{C}_K^{\cup}$. $\square$

Remark 8.15. In the notation of Lemma 8.14, note that $E_w = A_R$ when $B \neq AC$.

Lemma 8.16. *If* $\operatorname{char} K = 2$ *and* $n = 1$*, then* $\mathcal{C}_{K,graph} \subseteq \mathcal{E}(W_{hom,2})$*. In particular,* $\mathcal{C}_K \subseteq \mathbb{E}_H$*.*

Proof. Let $R = \frac{B+CX}{A+X} \in K(X)$ be not constant. We are going to define a word $v = \alpha\beta x \beta^{-1} x \in W_{hom,2}$ such that $A_R = E_v$. Note that $v \approx w = \alpha x^2 [\beta, x]$, so we prove $A_R = E_w$.

For such a word w, we have $\epsilon = 2$ and $\frac{\epsilon^2 - \epsilon}{2} = 1$, so that defining $\alpha, \beta \in H$ as in (8.6), we have that E_w is as in (8.3) by Lemma 8.7, item 3, so

$$E_w = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid (A + x)z = B + Cx \right\}.$$

Being R not constant, we have $B \neq CA$, so Remark 8.15 implies

$$E_w = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & R(x) \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R \right\} = A_R.$$

From this, it follows that $\mathcal{C}_K \subseteq \mathbb{E}_H$, as the inclusion $\mathcal{C}_{K,singl} \cup \mathcal{C}_{K,centr} \subseteq \mathbb{E}_H$ is obvious. $\square$

Remark 8.17. Note that in non-trivial cases, i.e. when an elementary algebraic subset is not the whole H, nor a singleton, we can always assume $\mathrm{l}(w) = 2$. In other words, the proper elementary algebraic subsets of H, with more than one element, are all given by words of the form $w = h_1 X^{\varepsilon_1} h_2 X^{\varepsilon_2}$, with $\varepsilon_1, \varepsilon_2 = \pm 1$.

Theorem 8.18. *The family $\mathcal{C}_K$ is a subbase of the $\mathfrak{Z}_H$-closed sets.*

Proof. It follows from Lemmata 8.14 and 8.16 that $\mathcal{C}_K^{\cup} = \mathbb{E}_H^{\cup}$. As $\mathbb{E}_H$ is a subbase of the $\mathfrak{Z}_H$-closed sets by definition, we are done. $\square$

Remark 8.19. In [24], we have defined a family $\mathcal{C}^* \subseteq \mathcal{P}(H)$ in both cases when $\operatorname{char} K \neq 2$ and $\operatorname{char} K = 2$. In the latter case, that we are considering now, $\mathcal{C}^*$ is the union of the following families:

- $\mathcal{C}_{K,singl} \cup \{H\}$, consisting of the singletons of H, and the whole H;
- $\mathcal{C}_{K,cos}$, consisting of cosets of L;
- $\mathcal{C}^*_{K,graph}$, consisting of subsets A_{R^*} as in (8.7), for a rational function $R^* = \frac{\beta+\gamma X}{\alpha+\delta X} \in K(X)$.

Let us prove that $\mathcal{C}_K = \mathcal{C}^*$. We first show that $\mathcal{C}_K \subseteq \mathcal{C}^*$, and to this end it is sufficient to prove that $\mathcal{C}_{K,centr} \setminus \mathcal{C}_{K,cos} \subseteq \mathcal{C}^*_{K,graph}$. So let $h = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \in H \setminus L$, i.e. $a \neq 0$. Then the cosets of $C_H(h)$ are the subsets $\left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid cx - az = \lambda \right\}$, for $\lambda \in K$, by Lemma 8.5, item 3. Letting $R^* = \frac{-\lambda + cx}{a}$, one sees that this subsets have the form A_{R^*}, hence are elements of $\mathcal{C}^*_{K,graph}$.

Now we prove that $\mathcal{C}^* \subseteq \mathcal{C}_K$, showing that $\mathcal{C}^*_{K,graph} \subseteq \mathcal{C}_{K,graph} \cup \mathcal{C}_{K,centr}$. So let $A_{R^*} \in \mathcal{C}^*_{K,graph}$, for $R^* = \frac{\beta+\gamma X}{\alpha+\delta X} \in K(X)$. If $\delta \neq 0$, we can assume $\delta = 1$, so that either R^* is not constant, and $A_{R^*} \in \mathcal{C}_{K,graph}$, or R^* is constant, and A_{R^*} is a coset of M, hence $A_{R^*} \in \mathcal{C}_{K,centr}$ by Lemma 8.6, item 1. Finally, if $\delta = 0$ we can assume $\alpha = 1$, so that $R^* = \beta + \gamma X$ is a polynomial and $A_{R^*} = \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & z \\ & & 1 \end{pmatrix} \in H \mid x \in K, z = \beta + \gamma x \right\}$ is an element of $\mathcal{C}_{K,centr}$ by Lemma 8.5, item 3.

In the following result, we describe the intersections of elements of $\mathcal{C}_K$.

Lemma 8.20. *If $\operatorname{char} K = 2$ and $n = 1$, then $\mathcal{C}_K^{\cap} = \mathcal{C}_K \cup \{F \cdot Z(H) \mid F \in [H]^{<3}\}$. In other words, $\mathcal{C}_K^{\cap} = \mathcal{C}_K \cup \{\emptyset\} \cup \{hZ(H) \mid h \in H\} \cup \{h_1 Z(H) \cup h_2 Z(H) \mid h_1, h_2 \in H\}$.*

Moreover, both $\mathcal{C}_K$ and $\mathcal{C}_K^{\cap}$ satisfy the descending chain condition.

Proof. By Remark 8.19, $\mathcal{C}_K^{\cap} = (\mathcal{C}^*)^{\cap}$, so to classify the elements of $\mathcal{C}_K^{\cap}$ the only non-trivial cases to consider are the intersections between a coset of L and a subset A_{R^*}, or the intersections between different subsets of the form A_{R^*}.

In the first case, the intersection is

$$\begin{pmatrix} 1 & x_0 & K \\ & 1 & K \\ & & 1 \end{pmatrix} \cap \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & R^*(x) \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R^* \right\},$$

non-empty exactly when $x_0 \in \operatorname{dom} R^*$, and in this case is

$$\begin{pmatrix} 1 & x_0 & K \\ & 1 & R^*(x_0) \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 & 0 \\ & 1 & R^*(x_0) \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & K \\ & 1 & 0 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 & 0 \\ & 1 & R^*(x_0) \\ & & 1 \end{pmatrix} Z(H),$$

a coset of $Z(H)$, that is not an element of $\mathcal{C}_K$. In particular, a coset of L cannot contain a subset A_{R^*}, and viceversa.

The second case to consider is the intersection of two different subsets $A_{R_1^*}$, $A_{R_2^*}$:

$$\left\{ \begin{pmatrix} 1 & x & K \\ & 1 & R_1^*(x) \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R_1^* \right\} \cap \left\{ \begin{pmatrix} 1 & x & K \\ & 1 & R_2^*(x) \\ & & 1 \end{pmatrix} \in H \mid x \in \operatorname{dom} R_2^* \right\}.$$

If non-empty, one of its elements is a matrix $\begin{pmatrix} 1 & x_0 & y_0 \\ & 1 & z_0 \\ & & 1 \end{pmatrix}$, with $x_0 \in \operatorname{dom} R_1^* \cap \operatorname{dom} R_2^*$, $R_1^*(x_0) = z_0 = R_2^*(x_0)$, and $y_0 \in K$ arbitrarily chosen. As $R_1^* \neq R_2^*$, x_0 is a solution of the non-trivial rational equation $R_1^*(x) = R_2^*(x)$, which can have either 0, or 1, or 2 solutions, depending on the coefficients of R_1^*, R_2^*. So in this case $A_{R_1^*} \cap A_{R_2^*}$ is the union of at most two cosets of $Z(H)$; as a consequence, a subset $A_{R_1^*}$ cannot contain another subset $A_{R_2^*}$.

Note that $Z(H) \subseteq A_R^*$ if and only if $R^*(0) = 0$, i.e. R^* has the form $R^* = \frac{\gamma X}{\alpha + \delta X}$.

We have just seen that $\mathcal{C}_K$ trivially satisfies the descending chain condition, and one immediately verifies that $\mathcal{C}_K^{\cap}$ satisfies it too. □

8.4 The dimension of $H(1,K)$

In what follows, we consider $H = H(1,K)$, and we compute the combinatorial dimension of the space $(H, \mathfrak{Z}_H)$. First, we recall that the family $\mathcal{C}_K^{\cap}$, whose definition depends on whether $\operatorname{char} K \neq 2$ or $\operatorname{char} K = 2$, has been described in Lemma 8.13 (if $\operatorname{char} K \neq 2$) and Lemma 8.20 (if $\operatorname{char} K = 2$). We resume here this description.

$$\mathcal{C}_K^{\cap} = \begin{cases} \mathcal{C}_K \cup \{F \cdot Z(H) \mid F \in [H]^{<2}\} & \text{if } \operatorname{char} K \neq 2, \\ \mathcal{C}_K \cup \{F \cdot Z(H) \mid F \in [H]^{<3}\} & \text{if } \operatorname{char} K = 2. \end{cases}$$

Moreover, $\mathcal{C}_K^\cap$ contains $\mathcal{C}_K$, so it is a subbase of the $\mathfrak{Z}_H$-closed sets, and $\mathcal{C}_K^\cap$ is also obviously stable under taking finite intersections. So, thanks to Proposition 1.3, we obtain the following result.

Proposition 8.21. *$H = H(1,K)$ is a $\mathfrak{Z}$-Noetherian group and $(\mathcal{C}_K^\cap)^\cup$ is the family of $\mathfrak{Z}_H$-closed sets. Moreover, the family of $\mathfrak{Z}_H$-closed irreducible sets is $\mathcal{C}_K \cup \{hZ(H) \mid h \in H\}$, so that in particular H is $\mathfrak{Z}$-irreducible, hence connected.*

Corollary 8.22. *The topological space $(H, \mathfrak{Z}_H)$ has combinatorial dimension three.*

Proof. It suffices to show a chain of four closed irreducible subsets, for example of the following form:

$$\{e_H\} \subsetneq Z(H) \subsetneq C_H(h) \subsetneq H,$$

where $h \in H$.

If $\operatorname{char} K = 2$, one can also consider a chain of the following form:

$$\{e_H\} \subsetneq Z(H) \subsetneq A_R \subsetneq H,$$

where $R = \frac{CX}{A+X}$. □

Now we can describe the topology $\mathfrak{Z}_H \restriction_L$. Its closed sets are the finite unions of the elements of the family $\{C \cap L \mid C \in \mathcal{C}_K^\cap\}$, which consists of the following subsets of L:

- singletons of L, the whole L;
- cosets of $Z(H)$ contained in L, hence of kind $\begin{pmatrix} 1 & 0 & K \\ & 1 & z_0 \\ & & 1 \end{pmatrix}$.

Identifying L with the group K^2, via the isomorphism

$$\begin{aligned} K^2 &\to L \\ (y,z) &\mapsto \begin{pmatrix} 1 & 0 & y \\ & 1 & z \\ & & 1 \end{pmatrix}, \end{aligned}$$

one can see that the $\mathfrak{Z}_H \restriction_L$-closed are the unions of a finite subset of K^2 and a set $K \times F$, for a finite subset F of K.

Corollary 8.23. *The subgroup $L \leq H = H(1,K)$ satisfies $\mathfrak{Z}_L \subsetneq \mathfrak{Z}_H \restriction_L \subsetneq \mathcal{A}_L$. In particular, the Zariski topology $\mathfrak{Z}_H$ and the affine topology $\mathcal{A}_H$ are different.*

Proof. We have already proved in Lemma 8.3 that $\mathfrak{Z}_L \subsetneq \mathfrak{Z}_H \restriction_L$. To prove the second inclusion to be proper, observe that the subset

$$\{(y,z) \in K^2 \mid z = y\} = \{(y,y) \in K^2 \mid y \in K\}$$

is $\mathcal{A}_{K^2}$-closed, so

$$\left\{ \begin{pmatrix} 1 & 0 & y \\ & 1 & y \\ & & 1 \end{pmatrix} \in H \mid y \in K \right\}$$

is $\mathcal{A}_L$-closed; but it is not $\mathfrak{Z}_H \restriction_L$-closed.

Now recall that $\mathcal{A}_L = \mathcal{A}_H \restriction_L$ by definition; as $\mathfrak{Z}_H \restriction_L \subsetneq \mathcal{A}_H \restriction_L$, we conclude $\mathfrak{Z}_H \subsetneq \mathcal{A}_H$. □

8.5 Generalized Heisenberg groups

In this section, we generalize the definition of Heisenberg group given at the beginning of this chapter, and we give some results for these groups.

If R is a unitary ring, then the Heisenberg group $H_R = H(1,R) = \begin{pmatrix} 1 & R & R \\ & 1 & R \\ & & 1 \end{pmatrix}$ is the group of 3×3 upper unitriangular matrixes with coefficients in R.

It can be easily seen that the commutator of two elements in H_R is

$$\left[\begin{pmatrix} 1 & x_1 & y_1 \\ & 1 & z_1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_2 & y_2 \\ & 1 & z_2 \\ & & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & x_1z_2 - x_2z_1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \tag{8.9}$$

so that

$$C_{H_R}(\begin{pmatrix} 1 & x_1 & y_1 \\ & 1 & z_1 \\ & & 1 \end{pmatrix}) = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \mid x_1z - xz_1 = 0 \right\}. \tag{8.10}$$

In particular, $Z(H_R) = \begin{pmatrix} 1 & 0 & R \\ & 1 & 0 \\ & & 1 \end{pmatrix}$, and the quotient group is $H_R/Z(H_R) \cong (R,+) \times (R,+)$.

Note that by (8.10) it follows that

$$L = \begin{pmatrix} 1 & 0 & R \\ & 1 & R \\ & & 1 \end{pmatrix} = C_{H_V}(\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}), \text{ while } M = \begin{pmatrix} 1 & R & R \\ & 1 & 0 \\ & & 1 \end{pmatrix} = C_{H_V}(\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}).$$

Corollary 8.24. *If R is a unitary ring such that the additive group $(R,+)$ is torsion-free, then the Heisenberg group $G = H_R = H(1,R)$ satisfies $\mathfrak{Z}_{G^I} = \mathfrak{C}'_{G^I}$ for every non-empty set I.*

Proof. It suffices to note that $G \in \mathscr{N}_2$ is torsion-free, then Theorem 4.34 applies. $\square$

From now on, we assume *R to be a Unique Factorization Domain.* We are interested in studying the family of $\mathfrak{Z}_{H_R}$-closed irreducible sets, in order to compute $\dim(H_R, \mathfrak{Z}_{H_R})$. By Corollary 8.24, it is sufficient to study the family of $\mathfrak{C}'_{H_R}$-closed sets. To this end, in the following lemma we describe all one-element centralizers in H_R, when R is a Unique Factorization Domain.

Lemma 8.25. *Let R be a Unique Factorization Domain, and $\gamma = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \in H_R \setminus Z(H_R)$.*

- *If $a = 0$, then $C_{H_R}(\gamma) = L$.*
- *If $c = 0$, then $C_{H_R}(\gamma) = M$.*
- *If $a \neq 0 \neq c$, and d is the greatest common divisor of a and c, let $a = a'd$, $c = c'd$. Then*
$$C_{H_V}(\gamma) = \left\{ \begin{pmatrix} 1 & a't & y \\ & 1 & c't \\ & & 1 \end{pmatrix} \mid y, t \in R \right\}. \tag{8.11}$$

Proof. If $a = 0$, then $c \neq 0$, and $C_{H_R}(\gamma) = L$ by (8.10) and the fact that R is a domain. Similarly, $c = 0$ implies $a \neq 0$, and $C_{H_R}(\gamma) = M$.

Finally, assume $a \neq 0 \neq c$. Then $\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \in C_{H_V}(\gamma)$ if and only if $az - cx = 0$ by (8.10). This equation is equivalent to $a'z = c'x$. Then $a' \mid x$, as a' and c' are coprime, so that $x = a't$ for some $t \in R$. Finally, $a'z = c'a't$ implies $z = c't$. $\square$

Note that a', c' in (8.11) are non-zero coprime elements in R.

The following result determines the intersections of one-element centralizers in H_R.

Lemma 8.26. *Let R be a Unique Factorization Domain. Let $\gamma_1, \gamma_2 \in H_R \setminus Z(H_R)$ be such that $C_{H_R}(\gamma_1) \neq C_{H_R}(\gamma_2)$. Then $C_{H_R}(\gamma_1) \cap C_{H_R}(\gamma_2) = Z(H_R)$.*

Proof. Obviously, it suffices to consider centralizers of element γ as in (8.11), with $a \neq 0 \neq c$. So, for $i = 1, 2$ let $C_i = C_{H_V}(\gamma_i) = \left\{ \begin{pmatrix} 1 & a_i t & y \\ & 1 & c_i t \\ & & 1 \end{pmatrix} \mid y, t \in R \right\}$, with $a_1, a_2, c_1, c_2 \in R \setminus \{0\}$, a_1, c_1 coprime and a_2, c_2 coprime.

So we assume $C_1 \cap C_2 \supsetneq Z(H_V)$, and we are going to prove $C_1 = C_2$. If $\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \in C_1 \cap C_2 \setminus Z(H_V)$, then $x = a_1 t = a_2 s$ and $z = c_1 t = c_2 s$ for some $t, s \in R \setminus \{0\}$.

Then $a_1 c_2 ts = a_2 c_1 ts$, so that $a_1 c_2 = a_2 c_1$. Then a_1, a_2 are associate elements in R, so $a_2 = a_1 u$ for an invertible element $u \in R$. This also yields $c_2 = c_1 u$, so that

$$C_2 = \left\{ \begin{pmatrix} 1 & a_1 ut & y \\ & 1 & c_1 ut \\ & & 1 \end{pmatrix} \mid y, t \in R \right\} = \left\{ \begin{pmatrix} 1 & a_1 t & y \\ & 1 & c_1 t \\ & & 1 \end{pmatrix} \mid y, t \in R \right\} = C_1.$$

□

In the following theorem we use Corollary 8.24 and the above results to prove that $\dim(H_R, \mathfrak{Z}_{H_R}) = 3$.

Theorem 8.27. *Let R be a Unique Factorization Domain, and*

$$\mathcal{B} = \{gC_{H_R}(h) \mid g, h \in H_R\} \cup \{g \mid g \in H_R\}.$$

Then $\mathfrak{Z}_{H_R} = \mathfrak{C}'_{H_R}$ is a Noetherian topology, $\mathcal{B}^\cap$ is its family of closed irreducible sets, and $(\mathcal{B}^\cap)^\cup$ is its family of closed sets.

In particular, $\dim(H_R, \mathfrak{Z}_{H_R}) = 3$.

Proof. Note that $\mathcal{B}^\cap = \mathcal{B} \cup \{gZ(H_R)\}$ by Lemma 8.26, while Corollary 8.24 implies that $\mathfrak{Z}_{H_R} = \mathfrak{C}'_{H_R}$, so that $\mathcal{B}^\cap$ is a subbase for $\mathfrak{Z}_{H_R}$-closed sets.

As $H_R \in \mathcal{B}^\cap$ and $\mathcal{B}^\cap$ obviously satisfies the descending chain condition, Proposition 1.3 implies that $\mathfrak{Z}_{H_R}$ is Noetherian, and that $(\mathcal{B}^\cap)^\cup$ is its family of closed sets.

Obviously, $\mathcal{B}^\cap$ consists of closed irreducible sets, and if $h \in H_R$ is a non-central element, then a chain of the form

$$\{e_{H_R}\} \subsetneq Z(H_R) \subsetneq C_{H_R}(h) \subsetneq H_R$$

shows that $\dim(H_R, \mathfrak{Z}_{H_R}) = 3$. □

8.5.1 The group H_V

If $n > 0$ is a positive cardinal and K is a field, we denote by V the vector space of dimension n over K. Fix a base $\mathcal{B} = \{\vec{e}_i : i < n\}$ of V, so that every $\vec{x} \in V$ can be uniquely identified with a finite set of non-zero coordinates $(x_i)_{i<n}$ (so that $\vec{x} = \sum_i x_i \vec{e}_i$). If $\vec{z} = (z_i)_{i<n} \in V$ is another element of V, the canonical scalar product in V over K is defined by $\vec{x} \cdot \vec{z} = \sum_{i<n} x_i z_i$ (note that this sum is defined

also when K is infinite). This gives a bilinear form $V \times V \to K$ that we use now in order to define the *Heisenberg group* H_V as the following matrix group:

$$H_V = \begin{pmatrix} 1 & V & K \\ & 1 & V \\ & & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & \vec{x} & y \\ & 1 & \vec{z} \\ & & 1 \end{pmatrix} \mid \vec{x}, \vec{z} \in V, y \in K \right\}.$$

The product in H_V is defined as follows:

$$\begin{pmatrix} 1 & \vec{x}_1 & y_1 \\ & 1 & \vec{z}_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \vec{x}_2 & y_2 \\ & 1 & \vec{z}_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \vec{x}_1 + \vec{x}_2 & y_1 + y_2 + \vec{x}_1 \cdot \vec{z}_2 \\ & 1 & \vec{z}_1 + \vec{z}_2 \\ & & 1 \end{pmatrix},$$

while the commutator of two elements in H_V is given by

$$\left[\begin{pmatrix} 1 & \vec{x}_1 & y_1 \\ & 1 & \vec{z}_1 \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & \vec{x}_2 & y_2 \\ & 1 & \vec{z}_2 \\ & & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & \vec{0} & \vec{x}_1 \cdot \vec{z}_2 - \vec{x}_2 \cdot \vec{z}_1 \\ & 1 & \vec{0} \\ & & 1 \end{pmatrix}. \tag{8.12}$$

From the above formulas, one can easily see that $Z(H_V) = H_V' = \begin{pmatrix} 1 & \vec{0} & K \\ & 1 & \vec{0} \\ & & 1 \end{pmatrix} \cong K$, so that $H_V/Z(H_V) \cong (V, +) \times (V, +)$ is abelian, and $H_V \in \mathscr{N}_2$.

The space V carries a linear topology λ defined by means of the base $\mathcal{B}$ as follows. For a finite subset J of $\mathcal{B}$ let V_J be the subspace of V generated by $\mathcal{B} \setminus J$, so that $V = V_J \oplus K^J$, where K^J is the K-linear span of J. Then $\{V_J : J \in [\mathcal{B}]^{<\omega}\}$ forms a base of neighborhoods of 0_V of a Hausdorff group topology λ on $(V, +)$.

As $V \cong K^{(\mathcal{B})} \leq K^{\mathcal{B}}$, note that $\lambda = T\restriction_V$, where $T = \delta_K^{\mathcal{B}}$ is the product topology of the discrete topologies δ_K on each summand K. So λ is non-discrete if and only if $n = |\mathcal{B}|$ is infinite, and it is metrizable if and only if n is countable. Moreover, it is precompact if and only if K is finite.

For $\vec{x} \in V$ there exists a finite $J \subseteq \mathcal{B}$, such that x belongs to the linear span of J. We refer to the minimal J with this property as support of $\vec{x}$ and write $J = \operatorname{supp}(\vec{x})$. For a subset X of V let $X^\perp = \{\vec{y} \in V \mid \vec{x} \cdot \vec{y} = \vec{0} \text{ for every } x \in X\}$. If $X = \{\vec{x}\}$, we write simply $\vec{x}^\perp$. Obviously, $V_J \subseteq \vec{x}^\perp$ where $J = \operatorname{supp}(\vec{x})$.

Consider the product topology $\tau = \lambda \times \iota_K \times \lambda$ on H_V obtained considering the topology λ on both factors V, and the indiscrete topology ι_K on the factor K. Obviously, τ is a (non-Hausdorff) group topology on H_V, and the family $\{O_J \mid J \in [\mathcal{B}]^{<\omega}\}$ is a base of the neighborhood at e_{H_V} for τ, where

$$O_J = \begin{pmatrix} 1 & V_J & K \\ & 1 & V_J \\ & & 1 \end{pmatrix}$$

for a finite $J \subseteq \mathcal{B}$.

Proposition 8.28. *The topologies $\mathcal{T}_{H_V}$ and τ on H_V coincide.*

Proof. Let $\alpha = \begin{pmatrix} 1 & \vec{a} & b \\ & 1 & \vec{c} \\ & & 1 \end{pmatrix}$, where $\vec{a}, \vec{c} \in V$, $b \in K$, and let $J = \mathrm{supp}(\vec{a}) \cup \mathrm{supp}(\vec{c})$. By (8.12),

$$C_{H_V}(\alpha) = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & 1 & \vec{z} \\ & & 1 \end{pmatrix} \mid \vec{a} \cdot \vec{z} - \vec{c} \cdot \vec{x} = 0 \right\} \supseteq O_J,$$

so $\mathcal{T}_{H_V} \subseteq \tau$.

To prove the converse inclusion, it suffices to show that O_J is the centralizer of a finite subset of H_V.

Let

$$O'_J = \begin{pmatrix} 1 & V & K \\ & 1 & V_J \\ & & 1 \end{pmatrix} \quad \text{and} \quad O''_J = \begin{pmatrix} 1 & V_J & K \\ & 1 & V \\ & & 1 \end{pmatrix},$$

so that $O_J = O'_J \cap O''_J$, and we show for example that O'_J is the centralizer of a finite subset of H_V. This follows from the fact that $C_{H_V}(\begin{pmatrix} 1 & \vec{a} & b \\ & 1 & \vec{0} \\ & & 1 \end{pmatrix}) = \begin{pmatrix} 1 & V & K \\ & 1 & \vec{a}^{\perp} \\ & & 1 \end{pmatrix}$ and $O'_J = \bigcap \left\{ \begin{pmatrix} 1 & V & K \\ & 1 & v^{\perp} \\ & & 1 \end{pmatrix} \mid v \in J \right\}$. □

From now on we assume that K is finite. Then the group H_V is infinite precisely when *n is infinite* and we are going to put it as a blanket condition in the sequel.

In this case, H_V is an FC-group and $\mathfrak{C}_{H_V} = \mathcal{T}_{H_V}$ by Lemma 4.32, so that

$$\mathfrak{C}_{H_V} = \mathcal{T}_{H_V} = \tau \tag{8.13}$$

by Proposition 8.28.

If $\mathrm{char}(K) = p > 2$, it can be easily proved that $\exp(H_V) = p$, so that $\mathfrak{Z}_{H_V} = \mathfrak{C}'_{H_V} = \mathcal{T}'_{H_V}$ by Theorem 4.35.

On the other hand, if $\mathrm{char}(K) = 2$, then $\exp(H_V) = 4$ and

$$H_V[2] = \left\{ \begin{pmatrix} 1 & \vec{x} & K \\ & 1 & \vec{z} \\ & & 1 \end{pmatrix} \mid \vec{x} \cdot \vec{z} = 0 \right\} \subsetneq H_V.$$

Note that $H_V[2]$ is not a subgroup of H_V, as it contains the subset

$$\begin{pmatrix} 1 & V & K \\ & 1 & \vec{0} \\ & & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & \vec{0} & K \\ & 1 & V \\ & & 1 \end{pmatrix}$$

that generates the whole H_V.

If $\mathrm{char}(K) = 2$, we still have $\mathfrak{C}'_{H_V} = \mathcal{T}'_{H_V}$, as $\mathfrak{C}_{H_V} = \mathcal{T}_{H_V}$, but now we see that $\mathfrak{Z}_{H_V} \neq \mathfrak{C}'_{H_V}$.

Theorem 8.29. *If* $\mathrm{char}(K) = 2$*, then* $\exp(H_V) = 4$ *and* $\mathfrak{C}_{H_V} = \mathcal{T}_{H_V}$*. Moreover, subset* $H_V[2]$ *is* $\mathfrak{Z}_{H_V}$*-clopen, but not* $\mathfrak{C}'_{H_V}$*-closed, hence the group* H_V *has* $\mathfrak{C}'_{H_V} = \mathcal{T}'_{H_V} \lneq \mathfrak{Z}_{H_V}$.

Proof. We first prove that $H_V[2]$ is a $\mathfrak{Z}_{H_V}$-clopen subset. Obviously, $H_V[2]$ is a $\mathfrak{Z}_{H_V}$-closed subset, as $H_V[2] = E_{x^2} \in \mathbb{E}_{H_V}$. So now we prove that also its complement $C = H_V \setminus H_V[2]$ is $\mathfrak{Z}_{H_V}$-closed. To this end, note that $H_V^2 \subseteq Z(H_V)$, so that

$$C = \bigcup_{z \in Z(H_V) \setminus \{e_{H_V}\}} \{h \in H_V \mid h^2 = z\}.$$

In particular, C is covered by finitely many elementary algebraic subsets E_{zx^2}, for $z \in Z(H_V) \setminus \{e_{H_V}\}$, so that C is $\mathfrak{Z}_{H_V}$-closed (indeed, C is additively algebraic).

Now we prove that $H_V[2]$ is not $\mathfrak{C}'_{H_V}$-closed. For simplicity, fix first $n = \omega$ and let $\{\vec{e}_n \mid n \in \mathbb{N}_+\}$ be the canonical base of V.

For every integer $m \in \mathbb{N}_+$ we define the element

$$\alpha_m = \begin{pmatrix} 1 & \vec{e}_m & 0 \\ & 1 & \vec{e}_m \\ & & 1 \end{pmatrix}.$$

Note that for every finite $J \subseteq \mathcal{B}$ we have $\alpha_m \in O_J$ definitively, i.e. the sequence $(\alpha_m)_{m \in \mathbb{N}_+}$ converges to e_{H_V} in the topology $\tau = \mathfrak{C}_{H_V}$.

Moreover, if $C \subseteq H_V[2]$ is a cofinite subset, then also $\alpha_m \in O_J \cap C$ definitively, i.e. the sequence $(\alpha_m)_{m \in \mathbb{N}_+}$ converges to e_{H_V} also in the topology $\mathfrak{C}'_{H_V} = \mathfrak{C}_{H_V} \vee cof_{H_V}$.

Now let

$$\beta = \begin{pmatrix} 1 & \vec{e}_1 & 0 \\ & 1 & \vec{e}_1 \\ & & 1 \end{pmatrix}, \text{ and } \beta_m = \alpha_m \beta = \begin{pmatrix} 1 & \vec{e}_1 + \vec{e}_m & 0 \\ & 1 & \vec{e}_1 + \vec{e}_m \\ & & 1 \end{pmatrix}$$

for $m \in \mathbb{N}_+$.

By Lemma 4.28, item 1, $(\beta_m)_{m \in \mathbb{N}_+}$ converges to β in the topology $\mathfrak{C}'_{H_V}$.

As $\beta_m \in H_V[2]$ for every m, while $\beta \notin H_V[2]$, we conclude that $H_V[2]$ is not $\mathfrak{C}'_{H_V}$-closed.

If d is uncountable, one simply defines a *net*, in place of a sequence. □

Corollary 8.30. *If* K *is a finite field and* n *is an infinite cardinal, then the only* $\mathfrak{Z}$*-irreducible sets of* H_V *are the singletons. Consequently* $\dim(H_V, \mathfrak{Z}_{H_V}) = 0$ *and* H_V *has no* $\mathfrak{Z}_G$*-atoms.*

Proof. Let $\overline{H_V} = H_V/Z(H_V)$, and consider the canonical projection $\pi\colon H_V \to \overline{H_V}$. Equip the domain with the topologies $\mathfrak{Z}_{H_V}$ and τ, and consider their quotient topologies, respectively $\overline{\mathfrak{Z}_{H_V}}$ and $\overline{\tau}$. As $\overline{H_V} \cong (V,+)\times(V,+)$, we identify them, so that $\overline{\tau} = \lambda \times \lambda$ is a Hausdorff (group) topology.

By (8.13), it follows that $\mathfrak{Z}_{H_V} \supseteq \tau$, so that $\overline{\mathfrak{Z}_{H_V}} \supseteq \overline{\tau}$ and in particular $\overline{\mathfrak{Z}_{H_V}}$ is a Hausdorff topology.

As $\pi\colon (H_V, \mathfrak{Z}_{H_V}) \to (\overline{H_V}, \overline{\mathfrak{Z}_{H_V}})$ is continuous, it maps $\mathfrak{Z}_{H_V}$-irreducible sets in $\overline{\mathfrak{Z}_{H_V}}$-irreducible sets, i.e. singletons. In other words, a $\mathfrak{Z}_{H_V}$-irreducible set is contained in a coset of $Z(H_V)$, hence is a singleton, being $Z(H_V) \cong K$ finite. $\square$

Compare the above result with Fact 4.12 (c). While in the abelian case the $\mathfrak{Z}_G$-atoms essentially determine the Zariski topology via the closure of subsets of G, the group $H_V \in \mathscr{N}_2$ has no $\mathfrak{Z}_G$-atoms at all.

Remark 8.31. In the notation of the above corollary, we have proved that the quotient topology $\overline{\mathfrak{Z}_{H_V}}$ on $\overline{H_V}$ is Hausdorff, so that it is much finer than $\mathfrak{Z}_{\overline{H_V}}$, being $\mathfrak{Z}_{\overline{H_V}} = cof_{\overline{H_V}}$ by Proposition 4.16.

9

The group $K^* \ltimes V$

Let K be a field, V be a K-linear space, and consider the action of the group $(K^*, \cdot)$ on the additive group $(V, +)$ defined by scalar multiplication. Denote by G the semidirect product $G_K = K^* \ltimes V$.

The case when $K = \mathbb{F}_2$ is trivial since then $G \cong V$ is an abelian group of exponent 2. Therefore, in the sequel we assume $K \neq \mathbb{F}_2$. Moreover, if K is finite, we assume V (i.e. $\dim_K V$) to be infinite, otherwise G would be finite itself.

Observe that when $\dim_K V = n$ is finite, G can be realized in a natural way as a subgroup $\overline{G}$ of the linear group $\mathrm{GL}_{n+1}(K)$:

$$\overline{G} = \left\{ \begin{pmatrix} a & b_1 & \cdots & b_n \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(K) \mid a \in K^*, b_1, \dots, b_n \in K \right\} \tag{9.1}$$

that can be written as the matrix group $\begin{pmatrix} K^* & K^n \\ \vec{0}^t & I_n \end{pmatrix}$. The isomorphism is

$$\begin{aligned} K^* \ltimes K^n &\to \begin{pmatrix} K^* & K^n \\ \vec{0}^t & I_n \end{pmatrix} \\ (a, b_1, \dots, b_n) &\mapsto \begin{pmatrix} a & b_1 & \cdots & b_n \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}. \end{aligned}$$

For this reason, for a generic K-vector space V, from now on we will use the notation

$$G = \begin{pmatrix} K^* & V \\ 0 & 1 \end{pmatrix}.$$

Denote by $N_K = \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix}$ the normal subgroup of G corresponding to V, and by $D_K = \begin{pmatrix} K^* & \vec{0} \\ 0 & 1 \end{pmatrix}$ the subgroup of G corresponding to K^*. From now on, we simply denote them by N and D, respectively, when the field K is clear from the context. By the structure of the semidirect product $G = D \ltimes N$, one has $N \cap D = \{e_G\}$, $G = DN$, and the action of D on N is the conjugation in G. The action of K^* on V is fixed-point-free; hence, in terms of the group G, one has $[n, d] \neq e_G$ whenever $n \in N$ and $d \in D$ are non-trivial elements. In particular, G is center-free.

Lemma 9.1. *(a) For every $n \in N$, if $n \neq e_G$ then $N = C_G(n)$. In particular, N is an elementary algebraic subset of G.*

(b) If $g = \begin{pmatrix} a & \vec{b} \\ 0 & 1 \end{pmatrix} \in G \setminus N$ (i.e. $a \neq 1_K$), then $C_G(g) = \left\{ \begin{pmatrix} x & \frac{x-1}{a-1}\vec{b} \\ 0 & 1 \end{pmatrix} \mid x \in K^* \right\}$.

(c) For every $d \in D$, if $d \neq e_G$ then $D = C_G(d)$. In particular, D is an elementary algebraic subset of G.

(d) G is not nilpotent, but it is solvable of class 2.

Proof. (a) – (c) are obvious, while (d) easily follows from the fact that $Z(G) = \{e_G\}$, and that $G' = N$ is abelian. □

In the sequel, for $a \in K^*$ we denote by f_a the group automorphism $f_a\colon (V, +) \to (V, +)$ defined by $f_a(x) = ax$.

Proposition 9.2. *Let K be a field containing a subring A such that:*

(a) A is a Unique Factorization Domain;

(b) there exist two non-associated prime elements r, s in A.

Then there exist a Hausdorff group topology τ on $(K, +)$, and an element $a \in K^*$ such that the automorphism f_a is not τ-continuous.

Proof. For every integer $m \geq 0$, let $U_m = r^m A$ be the principal ideal generated by r^m in A. Then the family $\{U_m \mid m \in \mathbb{N}\}$ is a local base at 0_K of the r-adic group topology τ on $(K, +)$. By (a), $\bigcap_{m\in\mathbb{N}} U_m = \{0_K\}$, so τ is also Hausdorff.

Then let us show that the group automorphism $f_{s^{-1}}$ on $(K, +)$ is not τ-continuous. Indeed, $A = U_0$ is a τ-open neighborhood of 0_K, but $sA = f_{s^{-1}}^{-1}(A)$ is not, as it cannot contain any τ-neighborhood U_m by (b). □

In the next corollary, we give a sufficient condition that implies the hypotheses of Proposition 9.2.

Corollary 9.3. *Let K be a field such that the following condition holds:*

either $\operatorname{char} K = 0$ *or* $\operatorname{char} K = p > 0$ *and the extension $K/\mathbb{F}_p$ is not algebraic.* (†)

Then there exist a Hausdorff group topology τ on $(K, +)$, and an element $a \in K^*$ such that the automorphism f_a is not τ-continuous.

Proof. If $\operatorname{char} K = 0$, then K contains $\mathbb{Z}$, and we can take two different primes in $A = \mathbb{Z}$.

If $\operatorname{char} K = p$ and the extension $K/\mathbb{F}_p$ is not algebraic, fix an element $t \in K$, t transcendent over $\mathbb{F}_p$. Then K contains $A = \mathbb{F}_p[t]$, that is a Unique Factorization Domain, and the elements t and $t-1$ are non-associated primes in A.

In both cases, the hypotheses of Proposition 9.2 are satisfied. $\square$

Remark 9.4. Observe that the ring A as in Proposition 9.2 is not a field. When the field K is an algebraic extension of $\mathbb{F}_p$, then every subring of K is a field so the argument in the proof of Proposition 9.2 cannot be applied. Nevertheless, we are not aware whether the conclusion of this proposition remains true in the general case. See Question 14.

Lemma 9.5. *N is not a super-normal subgroup of G, the topology $\mathfrak{Z}_N = \mathfrak{M}_N$ is cofinite, and is coarser than $\mathfrak{M}_G \restriction_N$.*

Proof. N is a non-central abelian subgroup of G, so it is not super-normal by Proposition 5.5.

As $N \cong V \cong \bigoplus_\lambda K$, for $\lambda = \dim_K V$, we have that $\mathfrak{Z}_N$ is the cofinite topology of N by Proposition 4.16, and it coincides with $\mathfrak{M}_N$ by Theorem 4.11. So $\mathfrak{Z}_N = \mathfrak{M}_N \subseteq \mathfrak{M}_G \restriction_N$. $\square$

Note that Lemma 9.5 gives $\mathfrak{M}_N \subseteq \mathfrak{M}_G \restriction_N$, but we do not know in general if the reverse inclusion holds, i.e. if N is Markov embedded in G (see Question 16).

The following result is a corollary of Theorem 5.11.

Proposition 9.6. *The subgroup $N \cong V$ is Hausdorff embedded in G if and only if for every Hausdorff group topology τ on V, and for every $a \in K^*$, the group automorphism f_a of V is τ-continuous.*

Proof. In view of Theorem 5.11, it will suffice to recall the definition of $G \cong K^* \ltimes V$, as the conjugation of N by elements of $D \cong K^*$ in $G = DN$ is the action of K^* on V of scalar multiplication. $\square$

Corollary 9.7. *If condition (†) holds, then N is not Hausdorff embedded in G.*

Proof. In order to apply Proposition 9.6, it suffices to find an element a of K^* and a Hausdorff group topology τ on V such that the group automorphism f_a of V is not τ-continuous. Consider $a \in K^*$ and τ on K as in Corollary 9.3.

If $V \cong K$ (i.e. $\dim_K V = 1$), we are done. Otherwise, if $\dim_K V = \lambda$, consider the product topology τ^λ on K^λ and the induced topology τ' on $V \cong \bigoplus_\lambda K$. Then the group automorphism f_a of V is not τ'-continuous. $\square$

Note that (†) holds, if the field K is uncountable.

Corollary 9.8. *If K is uncountable, then N is not Hausdorff embedded in G.*

Next we see that when K is finite, we can completely determine when N is Hausdorff embedded in G (see Corollary 9.11). Since this imposes V to be infinite (in order to have the group G infinite) we first pay a special attention to this case:

Lemma 9.9. *Let K be a field with $\operatorname{char} K = p > 0$, and let V be a K-vector space with $\dim_K V$ infinite. Then the following are equivalent:*

(a) $K \neq \mathbb{F}_p$;

(b) there exists a Hausdorff group topology τ on $(V, +)$, and $a \in K^*$ such that the group automorphism f_a of V is not τ-continuous;

(c) N is not Hausdorff embedded in G.

In particular, N is Hausdorff embedded in G if and only if $K = \mathbb{F}_p$.

Proof. (a) $\to$ (b). Consider the topology on the subgroup $U_0 := \bigoplus_\lambda \mathbb{F}_p$ induced by the product topology of $\mathbb{F}_p^\lambda$, and extend it to a group topology τ on $V \cong \bigoplus_\lambda K$ taking U_0 to be τ-open. Pick arbitrarily an element $a \in K \setminus \mathbb{F}_p$. Then continuity of $x \mapsto ax$ would provide a τ-open neighborhood U of 0_V such that $aU \subseteq U_0$. Taking a non-zero element $u \in U$ and a non-zero coordinate $u_i \in \mathbb{F}_p$ of u, we obtain $au_i \in \mathbb{F}_p$. Hence, $a \in \mathbb{F}_p$, a contradiction.

(b) $\to$ (a). It suffices to prove that the element $a \in K^*$ provided by out hypothesis does not belong to $\mathbb{F}_p$. Indeed, the multiplication by any $b \in \mathbb{F}_p$ is continuous with respect to every Hausdorff group topology on $(V, +)$, since bx, for $x \in V$, is a multiple of x in the additive group $(V, +)$.

By Proposition 9.6, (b) $\leftrightarrow$ (c).

The last assertion is simply (a) $\leftrightarrow$ (c) in counter-positive form. $\square$

Corollary 9.10. *Let $\dim_K V$ be infinite. Then N is Hausdorff embedded in G if and only if $K = \mathbb{F}_p$ for some prime p.*

Proof. Assume that N is Hausdorff embedded in G. Then condition (†) fails by Corollary 9.7, so $\operatorname{char} K = p > 0$ (and K is an algebraic extension of $\mathbb{F}_p$). By Lemma 9.9, we obtain $K = \mathbb{F}_p$.

If $K = \mathbb{F}_p$, then N is Hausdorff embedded in G by Lemma 9.9. $\square$

Corollary 9.11. *Let K be finite. Then N is Hausdorff embedded in G if and only if $K = \mathbb{F}_p$, where $p = \operatorname{char} K$.*

So what really remains open here is only the cases when $\dim_K V < \infty$, and $\operatorname{char} K = p > 0$, with (†) failing, i.e. the extension $K/\mathbb{F}_p$ is infinite and algebraic (see Question 15).

9.1 The Zariski topology on $K^* \ltimes V$

In this section we study the Zariski topology $\mathfrak{Z}_G$ and the affine topology $\mathcal{A}_G$ on $G = K^* \ltimes V$.

Let $K(T)$ be the field of rational functions over K. For $r \in K(T)$ with finite set Z of roots in K of its denominator, let $\operatorname{dom} r$ denote the domain $K \setminus Z$ of the rational evaluation function $K \setminus Z \to K$, $x \mapsto r(x)$, associated to r.

Let $\dim_K V = \lambda$. Then the groups $(V, +)$ and $(\bigoplus_\lambda K, +)$ are isomorphic, so we will identify them in the sequel. For a finite subset $I \subseteq \lambda$ we identify $V_I := \bigoplus_I K$ with a subspace of V and the semidirect product $W_I = K^* \ltimes V_I$ with a subgroup of G in the obvious way. For finite subsets $I, J \subseteq \lambda$ one has $W_I W_J = W_{I\cup J}$ and G is the directed limit of these subgroups. So every coset of each W_I is contained in some bigger W_J. In the sequel we describe a subbase of $\mathfrak{Z}_G$. It will contain G and cosets of N_K, all remaining members will be contained in some of the subgroups W_I.

A finite subset $\{R_j : j \in I\}$ of $K(T) \setminus \{0\}$ will be denoted by $\vec{R}$ and considered as an element of the direct sum $\bigoplus_\lambda K(T)$ (considered as a $K(T)$-linear space). Then $\operatorname{dom} \vec{R} = \bigcap_i \operatorname{dom} R_{\lambda_i}$ is a cofinite set in K, and $\vec{R}$ determines the evaluation function $\operatorname{dom} \vec{R} \to V = \bigoplus_\lambda K$, whose range is contained in V_I. For such a function $\vec{R}$ let $supp(\vec{R}) = \{j \in \lambda : R_j \neq 0\}$ and

$$F_{\vec{R}} = \left\{ \begin{pmatrix} x & \vec{R}(x) \\ 0 & 1 \end{pmatrix} \in G \mid x \in K^* \cap \operatorname{dom} \vec{R} = \operatorname{dom} \vec{R} \setminus \{0_K\} \right\}.$$

Clearly, $F_{\vec{R}}$ is contained in the subgroup $W_{supp(\vec{R})}$.

One can identify the subset $F_{\vec{R}}$ of $W_I \subseteq K^* \times \bigoplus_\lambda K$ with the 'graph' of the rational evaluation function $\vec{R}\colon K^* \cap \operatorname{dom} \vec{R} \to V_I$, and in order to keep this intuitive idea about $F_{\vec{R}}$ we often call it just graph in the sequel. A leading example for such a graph is the centralizer $C_G(g)$ of $g = \left(\begin{smallmatrix} a & \vec{b} \\ 0 & 1 \end{smallmatrix}\right) \in G \setminus N$, as $C_G(g) = F_{\vec{R}}$, for $\vec{R} = \frac{x-1}{a-1}\vec{b}$, by Lemma 9.1 (b).

Let $\mathcal{B} = \mathcal{B}_K$ be the family consisting of the following subsets of G:

- singletons of G, and the whole G;
- cosets of the normal subgroup $N = \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix}$ of G;
- graphs $F_{\vec{R}}$, with $\vec{R} \in \bigoplus_\lambda K(T)$.

The next lemma shows that $\mathbb{E}_G \subseteq \mathcal{B}^\cup$.

Lemma 9.12. *The elementary algebraic subsets of G are finite unions of elements of $\mathcal{B}$.*

Proof. Let $E_w \neq \emptyset$ be an elementary algebraic subset of G, where

$$w = (\alpha_1 X^{\varepsilon_1} \alpha_1^{-1})(\alpha_2 X^{\varepsilon_2} \alpha_2^{-1}) \cdots (\alpha_{k-1} X^{\varepsilon_{k-1}} \alpha_{k-1}^{-1}) \alpha_k X^{\varepsilon_k},$$

with $\varepsilon_1, \dots, \varepsilon_k = \pm 1$ (see Lemma 5.8). E_w is the solution set in G of the equation

$$(\alpha_1 X^{\varepsilon_1} \alpha_1^{-1})(\alpha_2 X^{\varepsilon_2} \alpha_2^{-1}) \cdots (\alpha_k X^{\varepsilon_k} \alpha_k^{-1}) = \alpha_k^{-1}. \tag{9.2}$$

For $i = 1, \dots, k$, let $\alpha_i = \begin{pmatrix} a_i & \vec{b}_i \\ 0 & 1 \end{pmatrix} \in G$, where $a_i \in K^*$ and $\vec{b}_i \in V$, and define

$$\delta_i = \begin{cases} 0 & \text{if } \varepsilon_i = 1, \\ -1 & \text{if } \varepsilon_i = -1. \end{cases} \tag{9.3}$$

Write $X = \begin{pmatrix} x & \vec{y} \\ 0 & 1 \end{pmatrix}$ for the variable in G, so that $X^{-1} = \begin{pmatrix} x^{-1} & -x^{-1}\vec{y} \\ 0 & 1 \end{pmatrix}$, and $X^{\varepsilon_i} = \begin{pmatrix} x^{\varepsilon_i} & \varepsilon_i x^{\delta_i} \vec{y} \\ 0 & 1 \end{pmatrix}$ for $\varepsilon_i = \pm 1$.

Recall Definition 2.13 of $\epsilon_i = \epsilon_i(w)$, let $\epsilon_0 = 0$ for convenience, and define

$$\Phi = \sum_{i=1}^{k} \varepsilon_i a_i T^{\epsilon_{i-1} + \delta_i} \qquad \text{and} \qquad \vec{\Psi} = -\vec{b}_1 - a_k^{-1} \vec{b}_k - \sum_{i=1}^{k-1} (\vec{b}_{i+1} - \vec{b}_i) T^{\epsilon_i} + \vec{b}_k T^{\epsilon}. \tag{9.4}$$

Observe that Φ is a rational function (more precisely, $\Phi \in K[T] + K[T^{-1}]$), so can be evaluated at any $x \in K^*$; the same holds for $\vec{\Psi} \in \bigoplus_\lambda K(T)$. (In fact, $\vec{\Psi} \in \bigoplus_\lambda K[T] + \bigoplus_\lambda K[T^{-1}] \subseteq \bigoplus_\lambda K(T)$).

One can prove by induction that the left-hand side in (9.2) is

$$\begin{pmatrix} x^\epsilon & \vec{b}_1 + \sum_{i=1}^{k-1} (\vec{b}_{i+1} - \vec{b}_i) x^{\epsilon_i} - \vec{b}_k x^\epsilon + \sum_{i=1}^{k} \varepsilon_i a_i x^{\epsilon_{i-1} + \delta_i} \vec{y} \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} x^\epsilon & -a_k^{-1} \vec{b}_k - \vec{\Psi}(x) + \Phi(x) \vec{y} \\ 0 & 1 \end{pmatrix}, \tag{9.5}$$

so that equation (9.2) gives

$$x^\epsilon = a_k^{-1} \qquad \text{and} \qquad -a_k^{-1} \vec{b}_k - \vec{\Psi}(x) + \Phi(x) \vec{y} = -a_k^{-1} \vec{b}_k. \tag{9.6}$$

The second equation in (9.6) is $\Phi(x)\vec{y} = \vec{\Psi}(x)$. Denoting by S the solution set in K^* of the first equation of (9.6), the elementary algebraic subset E_w of G is of the following form:

$$E_w = \left\{ \begin{pmatrix} x & \vec{y} \\ 0 & 1 \end{pmatrix} \in G \mid x \in S,\ \Phi(x)\vec{y} = \vec{\Psi}(x) \right\}.$$

Since $S = \{x \in K^* : x^\epsilon = a_k^{-1}\} \neq \emptyset$ (as $E_w \neq \emptyset$), either $\epsilon = 0$ (and necessarily $a_k = 1_K$) and $S = K^*$, or $\epsilon \neq 0$ and $S = \{x_1, \ldots, x_m\}$ is finite.

Case 1. If $\epsilon \neq 0$, then S is finite. So (9.6) is equivalent to the disjunction

$$\Phi(x_1)\vec{y} = \vec{\Psi}(x_1) \vee \Phi(x_2)\vec{y} = \vec{\Psi}(x_2) \vee \ldots \vee \Phi(x_m)\vec{y} = \vec{\Psi}(x_m).$$

Let $A_j = \left\{\begin{pmatrix} x_j & \vec{y} \\ 0 & 1 \end{pmatrix} \in G \mid \Phi(x_j)\vec{y} = \vec{\Psi}(x_j)\right\}$ for $j = 1, 2, \ldots, m$. Then $E_w = \bigcup_{j=1,\ldots,m} A_j$. Since $\Phi(x_j) = 0$ and $\vec{\Psi}(x_j) \neq \vec{0}$ for some j entails $A_j = \emptyset$, we are not interested in those $j = 1, 2, \ldots, m$. So we assume in the sequel that $\vec{\Psi}(x_j) = \vec{0}$ whenever $\Phi(x_j) = 0$. Hence, for every fixed x_j, we distinguish two cases depending on whether $\Phi(x_j) = 0$:

$$A_j = \begin{cases} \left\{\begin{pmatrix} x_j & \frac{\vec{\Psi}(x_j)}{\Phi(x_j)} \\ 0 & 1 \end{pmatrix}\right\} & \text{if } \Phi(x_j) \neq 0, \\ \begin{pmatrix} x_j & V \\ 0 & 1 \end{pmatrix} & \text{if } \Phi(x_j) = 0,\ \vec{\Psi}(x_j) = \vec{0}. \end{cases}$$

Thus, each A_j is either a singleton, or $A_j = \begin{pmatrix} x_j & V \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_j & \vec{0} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix}$ is a coset of N. Therefore, E_w is a finite union of elements of $\mathcal{B}$.

Case 2. If $\epsilon = 0$, i.e. w is singular, then $S = K^*$, so

$$E_w = \left\{\begin{pmatrix} x & \vec{y} \\ 0 & 1 \end{pmatrix} \in G \mid x \in K^*, \Phi(x)\vec{y} = \vec{\Psi}(x)\right\}.$$

Note that $E_w = G \in \mathcal{B}$, if $\Phi = 0 \in K(T)$ and $\vec{\Psi} = \vec{0} \in \bigoplus_\lambda K(T)$, so assume this is not the case in the sequel.

Let $Z = \{x \in K^* \mid \Phi(x) = 0, \vec{\Psi}(x) = \vec{0}\}$ be the finite set of common zeroes of Φ and $\vec{\Psi}$. We distinguish again two cases depending on $\Phi(x) = 0$ by letting

$$E_w^{(1)} = \left\{\begin{pmatrix} x & \vec{y} \\ 0 & 1 \end{pmatrix} \in G \mid x \in K^*, \Phi(x) \neq 0, \Phi(x)\vec{y} = \vec{\Psi}(x)\right\} \quad \text{and}$$

$$E_w^{(2)} = \left\{\begin{pmatrix} x & \vec{y} \\ 0 & 1 \end{pmatrix} \in G \mid \vec{y} \in V, x \in Z\right\} = \bigcup_{x \in Z} \begin{pmatrix} x & \vec{0} \\ 0 & 1 \end{pmatrix} N,$$

so that $E_w = E_w^{(1)} \cup E_w^{(2)}$. As Z is finite, $E_w^{(2)}$ is a finite union of elements of $\mathcal{B}$, so we only pay attention to $E_w^{(1)}$.

Case 2.1 If $\Phi = 0 \in K(T)$ and $\vec{\Psi} \neq \vec{0} \in \bigoplus_\lambda K(T)$, then $E_w^{(1)} = \emptyset$.

Case 2.2 If $\Phi \neq 0 \in K(T)$, let $\vec{R} = \frac{\vec{\Psi}}{\Phi}$ to obtain $E_w^{(1)} = F_{\vec{R}} \in \mathcal{B}$. $\square$

Remark 9.13. Note that $E_w = E_w^{(1)} = F_{\vec{R}} = F_{\frac{\vec{\Psi}}{\Phi}}$ in Case 2.2 of Lemma 9.12, if $\Phi \neq 0 \in K(T)$, and $Z = \emptyset$.

In the following lemma, we show that $\mathcal{B} \subseteq \mathbb{E}_G$, so that $\mathcal{B}^{\cup} \subseteq \mathbb{E}_G^{\cup}$.

Lemma 9.14. *The elements of $\mathcal{B}$ are elementary algebraic subsets of G.*

Proof. Singletons and the whole G are elementary algebraic subsets of G. On the other hand, N is an elementary algebraic subset by Lemma 9.1 (a), and so all its cosets are elementary algebraic subsets too.

We shall see that for every $\vec{R} \in \bigoplus_\lambda K(T)$, the subset $F_{\vec{R}}$ is elementary algebraic, i.e. $F_{\vec{R}} = E_w$ for some word w. Note that $\vec{R} = \vec{0}$ yields $F_{\vec{R}} = D$, which is an elementary algebraic subset by Lemma 9.1 (c). So let $\vec{R} = \frac{\vec{P}}{Q}$ for

$$Q = \mu_0 + \mu_1 T + \cdots + \mu_s T^s, \quad \vec{P} = \vec{\lambda}_0 + \vec{\lambda}_1 T + \cdots + \vec{\lambda}_r T^r. \tag{9.7}$$

We assume that the set of common zeroes of Q and $\vec{P}$ is empty, $\vec{\lambda}_r \neq \vec{0}$, $\mu_s \neq 0$. We can also assume $\mu_0 \neq -1$ (the reason of this request will be clear in the sequel), otherwise multiplying the numerator and the denominator of $\vec{R}$ by an element $a \in K$, $a \neq 0, 1$ (this is possible since $K \neq \mathbb{F}_2$).

As noticed in Remark 9.13, if w is a singular word, $a_k = 1_K$, $\Phi \neq 0$ and $\vec{\Psi}(x), \Phi(x)$ have no common zeroes, then

$$\begin{aligned} E_w = \{X \in G \mid \alpha_1 X^{\varepsilon_1} \alpha_1^{-1} \cdots \alpha_k X^{\varepsilon_k} \alpha_k^{-1} = \alpha_k^{-1}\} = \\ = \left\{ \begin{pmatrix} x & \frac{\vec{\Psi}(x)}{\Phi(x)} \\ 0 & 1 \end{pmatrix} \in G \mid x \in \operatorname{dom} \frac{\vec{\Psi}}{\Phi} \right\}. \end{aligned} \tag{9.8}$$

On the other hand, $F_{\vec{R}} = F_{\frac{\vec{P}}{Q}} = \left\{ \begin{pmatrix} x & \frac{\vec{P}(x)}{Q(x)} \\ 0 & 1 \end{pmatrix} \mid x \in \operatorname{dom} \frac{\vec{P}}{Q} \right\}$.

So we will look for $k \in \mathbb{N}$; $\varepsilon_1, \ldots, \varepsilon_k = \pm 1$ such that $\sum_{i=1}^k \varepsilon_i = 0$; $a_1, \ldots, a_{k-1} \in K^*$, (as already mentioned, we take $a_k = 1_K$); $\vec{b}_1, \ldots, \vec{b}_k \in \bigoplus_\lambda K$ such that the corresponding Φ and $\vec{\Psi}$ defined by relations (9.4) satisfy

$$\Phi = Q \quad \text{and} \quad \vec{\Psi} = \vec{P}.$$

This will suffice, as Φ and $\vec{\Psi}$ will have no common zeroes and $E_w = F_{\frac{\vec{\Psi}}{\Phi}} = F_{\frac{\vec{P}}{Q}} = F_{\vec{R}}$.

Set

$$k = 2(r + s + 1), \quad \varepsilon_1 = \cdots = \varepsilon_{r+s+1} = 1, \quad \text{and} \quad \varepsilon_{r+s+2} = \cdots = \varepsilon_k = -1. \tag{9.9}$$

In this way, $\sum_{i=1}^k \varepsilon_i = 0$, so the word w will be singular. By the definition in (9.3) we obtain

$$\delta_1 = \cdots = \delta_{r+s+1} = 0 \quad \text{and} \quad \delta_{r+s+2} = \cdots = \delta_k = -1 \tag{9.10}$$

and so, for $i = 1, \dots, k$ we have

$$\varepsilon_0 + \cdots + \varepsilon_{i-1} + \delta_i = \begin{cases} i-1 & \text{if } i \le r+s+1, \\ k-i & \text{otherwise.} \end{cases}$$

From (9.4) and the above relations,

$$\begin{aligned}\Phi = \sum_{i=1}^{k} \varepsilon_i a_i T^{\varepsilon_0+\cdots+\varepsilon_{i-1}+\delta_i} = \sum_{i=1}^{r+s+1} a_i T^{i-1} - \sum_{i=r+s+2}^{k} a_i T^{k-i} = \sum_{j=0}^{r+s} (a_{j+1} - a_{k-j}) T^j = \\ = (a_1 - a_k) + (a_2 - a_{k-1})T + \cdots + (a_{s+1} - a_{k-s})T^s + \\ + (a_{s+2} - a_{k-s-1})T^{s+1} + \cdots + (a_{r+s+1} - a_{r+s+2})T^{r+s}.\end{aligned}$$

Recall that $a_k = 1$, so to have $\Phi = Q$ we need $a_1, \dots, a_{k-1} \in K^*$ such that

$$a_1 - 1 = \mu_0,\ a_2 - a_{k-1} = \mu_1, \quad \dots \quad, a_{s+1} - a_{k-s} = \mu_s \tag{9.11}$$

and

$$a_{s+2} - a_{k-s-1} = 0,\ a_{s+3} - a_{k-s-2} = 0, \quad \dots \quad, a_{r+s+1} - a_{r+s+2} = 0. \tag{9.12}$$

Observe that any a_i appears in exactly one of the equations (9.11) and (9.12), so we can first solve system (9.12) taking for example

$$a_{s+2} = \cdots = a_{r+s+1} = a_{r+s+2} = \cdots = a_{k-s-1} = 1.$$

Then $a_1 = \mu_0 + 1 \neq 0$ by the initial choice of $\mu_0 \neq -1$. The remaining equations of (9.11) are of the form $a - a' = \mu$, to be solved in $a, a' \in K^*$, with some specific $\mu \in K$, so they are solvable, as $K \neq \mathbb{F}_2$ has at least three elements. So we have presented Q in the form $Q = \Phi$.

From (9.9) and (9.10) one can also see that for $i = 1, \dots, k$

$$\varepsilon_1 + \cdots + \varepsilon_i = \begin{cases} i & \text{if } i \le r+s+1, \\ k-i & \text{otherwise,} \end{cases}$$

so, by definition (9.4) of $\vec{\Psi}$, with $a_k = 1$ and $\sum_{i=1}^k \varepsilon_i = 0$, we obtain

$$\begin{aligned}\vec{\Psi} = -\vec{b}_1 - a_k^{-1}\vec{b}_k + \sum_{i=1}^{k-1} (\vec{b}_i - \vec{b}_{i+1}) T^{\varepsilon_1+\cdots+\varepsilon_i} + \vec{b}_k T^{\varepsilon_1+\cdots+\varepsilon_k} = \\ = -\vec{b}_1 + \sum_{i=1}^{k-1} (\vec{b}_i - \vec{b}_{i+1}) T^{\varepsilon_1+\cdots+\varepsilon_i} = -\vec{b}_1 + \sum_{i=1}^{r+s+1} (\vec{b}_i - \vec{b}_{i+1}) T^i + \sum_{i=r+s+2}^{k-1} (\vec{b}_i - \vec{b}_{i+1}) T^{k-i} = \\ = \left(-\vec{b}_1 + (\vec{b}_1 - \vec{b}_2)T + (\vec{b}_2 - \vec{b}_3)T^2 + \cdots + (\vec{b}_{r+s+1} - \vec{b}_{r+s+2})T^{r+s+1} \right) + \\ \left(+(\vec{b}_{r+s+2} - \vec{b}_{r+s+3})T^{r+s} + (\vec{b}_{r+s+3} - \vec{b}_{r+s+4})T^{r+s-1} + \cdots + (\vec{b}_{k-1} - \vec{b}_k)T \right).\end{aligned} \tag{9.13}$$

To have $\vec{\Psi} = \vec{P}$, it suffices now to equalize the coefficients of $\vec{\Psi}$ in the last term of (9.13) to the corresponding ones of $\vec{P}$ in (9.7); that is, we need $\vec{b}_i \in V$ such that

$$-\vec{b}_1 = \vec{\lambda}_0, \quad \vec{b}_1 - \vec{b}_2 = \vec{\lambda}_1, \quad \vec{b}_2 - \vec{b}_3 = \vec{\lambda}_2, \quad \dots , \ \vec{b}_r - \vec{b}_{r+1} = \vec{\lambda}_r,$$
$$\vec{b}_{r+1} - \vec{b}_{r+2} = \vec{0}, \ \dots \ , \vec{b}_{k-1} - \vec{b}_k = \vec{0}.$$

A solution of this system can be found easily. □

Theorem 9.15. *The family $\mathcal{B}$ is a subbase of the $\mathfrak{Z}_G$-closed sets.*

Proof. Observe that from Lemmata 9.12 and 9.14 it follows that $\mathcal{B}^{\cup} = \mathbb{E}_G^{\cup}$. As $\mathbb{E}_G$ is a subbase of the $\mathfrak{Z}_G$-closed sets by definition, the theorem holds. □

Remark 9.16. If $\dim_K V = n$ is finite, then $G \leq \mathrm{GL}_{n+1}(K)$ is a linear group, so G is $\mathfrak{Z}$-Noetherian by Example 10.2 (a). On the other hand, if K is finite, then G is an abelian-by-finite group, so again G is $\mathfrak{Z}$-Noetherian, by Example 10.2 (c).

In the sequel (see Proposition 9.21), we will directly prove this result in the general case. Moreover, $\mathcal{B}^{\cup}$ will be proved to be the family of the algebraic subsets of G.

Lemma 9.17. *The intersection of two distinct elements of $\mathcal{B}$ different from G is finite. In particular, if B_1 and B_2 are two distinct elements of $\mathcal{B}$, and $B_1 \subseteq B_2$, then either B_1 is finite or $B_2 = G$.*

Proof. Obviously we only have to consider intersections between graphs, and between a graph and a coset of N.

The intersection of two distinct graphs $F_{\vec{R}_1}$ and $F_{\vec{R}_2}$ is

$$\left\{ \begin{pmatrix} x & \vec{R}_1(x) \\ 0 & 1 \end{pmatrix} \mid x \in \operatorname{dom} \vec{R}_1 \right\} \cap \left\{ \begin{pmatrix} x & \vec{R}_2(x) \\ 0 & 1 \end{pmatrix} \mid x \in \operatorname{dom} \vec{R}_2 \right\}.$$

An element of this set is a matrix $g_0 = \begin{pmatrix} x_0 & \vec{y}_0 \\ 0 & 1 \end{pmatrix}$, where x_0 is a solution of the rational equation $\vec{R}_1(x) = \vec{R}_2(x)$, which has finitely many solutions as desired.

The intersection of a graph $F_{\vec{R}}$ and a coset $g_0 N$ for $g_0 = \begin{pmatrix} x_0 & \vec{y}_0 \\ 0 & 1 \end{pmatrix}$, is

$$\left\{ \begin{pmatrix} x & \vec{R}(x) \\ 0 & 1 \end{pmatrix} \mid x \in \operatorname{dom} \vec{R} \right\} \cap \begin{pmatrix} x_0 & V \\ 0 & 1 \end{pmatrix}.$$

This intersection is non-empty precisely when $x_0 \in \operatorname{dom} \vec{R}$, and namely consists of the single element $g = \begin{pmatrix} x_0 & \vec{R}(x_0) \\ 0 & 1 \end{pmatrix}$. □

Corollary 9.18. *N is a $\mathfrak{Z}_G$-closed, Zariski embedded subgroup of G.*

Proof. We have to prove that $\mathfrak{Z}_G \restriction_N = \mathfrak{Z}_N$. By Theorem 9.15, it follows that $\mathcal{F} = \{B \cap N \mid B \in \mathcal{B}\}$ is a subbase of the $\mathfrak{Z}_G \restriction_N$-closed sets. Now observe that N itself is an element of $\mathcal{B}$, so $\mathcal{F}$ consists of N and finite subsets of N, by Lemma 9.17; consequently, $\mathfrak{Z}_G \restriction_N$ is the cofinite topology on N. On the other hand, we have already proved in Lemma 9.5 that $\mathfrak{Z}_N$ is the cofinite topology of N. $\square$

Remark 9.19. (a) N is a $\mathfrak{Z}_G$-closed, Zariski embedded normal subgroup of G by Corollary 9.18, but N is not super-normal in G by Lemma 9.5.

(b) If K satisfies condition (†), then N is not Hausdorff embedded in G by Corollary 9.7.

(c) If G is countable, i.e. if both K and the dimension $\dim_K V$ are finite or countable, then $\mathfrak{Z}_G = \mathfrak{M}_G$ by Theorem 4.11. In particular, for every subgroup the conditions of being Markov embedded and Zariski embedded coincide, and N is Markov embedded in G.

Remark 9.20. Let $\dim_K V$ be countable, and let K be countable too, and satisfying condition (†) (for example, $K = \mathbb{Q}$ can be considered). Then G_K is countable, so N_K is Zariski and Markov embedded, but not Hausdorff embedded in G_K, by Remark 9.19 (b) and (c). So

$$\text{Zariski embedded \& Markov embedded} \nrightarrow \text{Hausdorff embedded},$$

and in particular, Zariski embedded & Markov embedded do not imply super-normal.

Moreover, N_K is a $\mathfrak{Z}_{G_K}$-closed, $\mathfrak{M}_{G_K}$-closed normal subgroup.

Let $\mathcal{B}_1 = \mathcal{B} \cup [G]^{<\infty}$ be the family obtained adding all finite subsets of G to $\mathcal{B}$, thus consisting of the following subsets of G:

- G and all finite subsets of G;
- cosets of N;
- sets $F_{\vec{R}}$, with $\vec{R} \in \bigoplus_\lambda K(T)$.

Obviously, $\mathcal{B}_1^\cup = \mathcal{B}^\cup$, so $\mathcal{B}_1$ is a subbase of the $\mathfrak{Z}_G$-closed sets too. Lemma 9.17 implies that $\mathcal{B}_1$ is stable under finite intersections and satisfies the descending chain condition; by Proposition 1.3, $\mathcal{B}_1^\cup$ is the family of closed sets of the topology generated by the closed sets in $\mathcal{B}_1$, and this topology is Noetherian. Thus, we have proved the following:

Proposition 9.21. *G is a $\mathfrak{Z}$-Noetherian group, and $\mathcal{B}^\cup$ is the family of the algebraic subsets of G. In particular, every algebraic subset of G is a finite union of elementary algebraic subsets of G.*

Now we focus on $\mathfrak{Z}_G$-closed irreducible subsets of G. To this end, we distinguish the cases when K is infinite or finite. Recall that if K is finite, we assume $\dim_K V$ to be infinite.

Lemma 9.22. *If K is infinite, then $\mathcal{B}$ is the family of closed irreducible subsets of $\mathfrak{Z}_G$. In particular G is $\mathfrak{Z}$-irreducible, hence connected.*

If K is finite and $\dim_K V$ is infinite, the closed irreducible subsets of $\mathfrak{Z}_G$ are the singletons and the cosets of N. In particular, $c(G, \mathfrak{Z}_G) = N$ is $\mathfrak{Z}$-irreducible.

Proof. Singletons are always irreducible.

If K is finite, then the graphs $F_{\vec{R}}$ are finite too, so graphs with more than one element are reducible, while N and its cosets are irreducible as they cannot be expressed as a proper union of elements of $\mathcal{B}$. Moreover, N has finite index in G, so the finite union of cosets of N covers G, and G is reducible, as actually G is not even connected. Then $N \subseteq c(G, \mathfrak{Z}_G) \subsetneq G$, and being $c(G, \mathfrak{Z}_G) \in \mathcal{B}^{\cup}$ we immediately conclude $c(G, \mathfrak{Z}_G) = N$.

If K is infinite, we have to show that G is irreducible. Assume for a contradiction that G is a finite union of proper $\mathfrak{Z}_G$-closed subsets. It is not restrictive to assume that they are members of $\mathcal{B}$:

$$G = g_1 N \cup \ldots \cup g_h N \cup F_{\vec{R}_1} \cup \ldots \cup F_{\vec{R}_k}. \tag{9.14}$$

In this case, N is infinite and has infinite index in G, so there exists a coset $g_0 N$ distinct from the cosets $g_1 N, \ldots, g_h N$. By Lemma 9.17, $g_0 N$ has finite intersection with every $F_{\vec{R}_1}, \ldots, F_{\vec{R}_k}$, thus it should be finite itself, a contradiction.

The same argument shows that the cosets of N and the graphs $F_{\vec{R}}$ are irreducible: they are infinite subsets that have finite intersection with every other element of $\mathcal{B} \setminus \{G\}$. ☐

Corollary 9.23. *If K is finite and $\dim_K V$ is infinite, the topological space $(G, \mathfrak{Z}_G)$ has combinatorial dimension 1. Otherwise, it has dimension 2.*

Proof. If K is finite, a strictly increasing chain of irreducible closed sets, of length two, is

$$\{e_G\} \subsetneq N.$$

If K is infinite, a strictly increasing chain of irreducible closed sets, of length three, can be of one of the following two forms:

$$\{e_G\} \subsetneq N \subsetneq G$$
$$\{g\} \subsetneq F_{\vec{R}} \subsetneq G.$$

☐

Corollary 9.24. *Let $\dim_K V$ be finite, and S be a proper closed subset of the topological space $(G, \mathfrak{Z}_G)$. Then:*

- $\dim S = 0$ *if and only if* S *is finite;*
- $\dim S = 1$ *if and only if* S *is infinite.*

Proof. By Proposition 9.21, S is a finite union of elements of $\mathcal{B}$.

If S is finite, it obviously has dimension zero. Otherwise, S is the union of finitely many cosets of N, finitely many graphs, and a finite subset of G, and so has dimension one. $\square$

We already observed in §9 that if $\dim_K V = n$ is finite, then G is the matrix group $\begin{pmatrix} K^* & K^n \\ \vec{0}^t & I_n \end{pmatrix}$, a subgroup of $\mathrm{GL}_{n+1}(K)$, which can be considered as a subset of $K^{(n+1)^2}$. In particular, we can consider the affine topology $\mathcal{A}_G$ of the group G.

Corollary 9.25. *If* $\dim_K V$ *is finite, then the Zariski topology* $\mathfrak{Z}_G$ *is properly contained in the affine topology* $\mathcal{A}_G$*.*

Proof. By Example 10.2 (a), the inclusion $\mathfrak{Z}_G \subseteq \mathcal{A}_G$ holds for any linear group G.

If $\dim_K V = n > 1$, then the affine topology $\mathcal{A}_N = \mathcal{A}_G \restriction_N$ on the group $N \cong K^n$ is strictly finer than the cofinite topology. On the other hand, $\mathfrak{Z}_G \restriction_N = \mathfrak{Z}_N$ is the cofinite topology on N by Corollary 9.18. Thus $\mathfrak{Z}_N \subsetneq \mathcal{A}_N$, and in particular $\mathfrak{Z}_G \subsetneq \mathcal{A}_G$.

If $\dim_K V = 1$, then $G = \begin{pmatrix} K^* & K \\ 0 & 1 \end{pmatrix}$. We show a subset S of G that is $\mathcal{A}_G$-closed, but not $\mathfrak{Z}_G$-closed. Let

$$S = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in G \mid x = y^2 \right\} = \left\{ \begin{pmatrix} y^2 & y \\ 0 & 1 \end{pmatrix} \mid y \in K^* \right\}.$$

Then S is $\mathcal{A}_G$-closed, as it is the zero-set of the polynomial $P(x,y) = y^2 - x$. Yet S is not $\mathfrak{Z}_G$-closed: otherwise, it would be a finite union of elements of $\mathcal{B} \setminus \{G\}$. Let us see that this is not possible, as in fact S has finite intersection with any member of $\mathcal{B} \setminus \{G\}$. For every $g_0 = \begin{pmatrix} x_0 & y_0 \\ 0 & 1 \end{pmatrix} \in G$,

$$S \cap g_0 N = \left\{ \begin{pmatrix} y^2 & y \\ 0 & 1 \end{pmatrix} \mid y \in K^* \right\} \cap \begin{pmatrix} x_0 & K \\ 0 & 1 \end{pmatrix}$$

has at most two elements: if x_0 is not a square in K, it is empty; if x_0 is a square in K, say $x_0 = x_1^2$, then $S \cap g_0 N = \left\{ \begin{pmatrix} x_0 & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_0 & -x_1 \\ 0 & 1 \end{pmatrix} \right\}$.

If $R \in K(T)$ is a rational function, then

$$S \cap F_R = \left\{ \begin{pmatrix} y^2 & y \\ 0 & 1 \end{pmatrix} \mid y \in K^* \right\} \cap \left\{ \begin{pmatrix} x & R(x) \\ 0 & 1 \end{pmatrix} \in G \mid x \in \operatorname{dom} R \right\}.$$

An element of this intersection is a matrix $\begin{pmatrix} a^2 & a \\ 0 & 1 \end{pmatrix}$ such that $a = R(a^2)$, that is: a is a solution of the non-trivial rational equation $x - R(x^2) = 0$, which admits only finitely many solutions. $\square$

10

$\mathfrak{Z}$-Noetherian and $\mathfrak{M}$-Noetherian groups

Definition 10.1. *A group G is called* $\mathfrak{Z}$-Noetherian *(resp.,* $\mathfrak{Z}$-cofinite*), if $\mathfrak{Z}_G$ is Noetherian (resp., cofinite). Similarly, we will call* $\mathfrak{M}$-Noetherian *(resp.,* $\mathfrak{P}$-Noetherian*) a group G such that $\mathfrak{M}_G$ (resp., $\mathfrak{P}_G$) is Noetherian.*

Obviously, every finite group is $\mathfrak{Z}$-cofinite. The cofinite topology on a set is always Noetherian, and so $\mathfrak{Z}$-cofinite groups are $\mathfrak{Z}$-Noetherian. Bryant [11] studied first the class of $\mathfrak{Z}$-Noetherian groups, under the name '*groups which satisfy* min-*closed*' (i.e. the minimal condition on Zariski closed sets).

Using Proposition 1.3, Bryant proved that the classes of groups in Example 10.2 below are $\mathfrak{Z}$-Noetherian.

Here we recall the definitions of some properties: a group G is abelian-by-finite (or virtually abelian) if it has an abelian subgroup of finite index. Similarly, it is called abelian-by-nilpotent-by-finite group (or virtually abelian-by-nilpotent) if it has an abelian-by-nilpotent subgroup of finite index, while G is abelian-by-nilpotent if it has a normal abelian subgroup such that the quotient group is nilpotent.

Example 10.2. (a) ([11, Theorem 3.5]) If G is a linear group, then $\mathfrak{Z}_G \subseteq \mathcal{A}_G$. In particular, G is $\mathfrak{Z}$-Noetherian by Fact 1.4.

(b) ([11, Corollary 3.7]) Every finitely generated, abelian-by-nilpotent-by-finite group is $\mathfrak{Z}$-Noetherian.

(c) ([11, Theorem 3.8]) Every abelian-by-finite group is $\mathfrak{Z}$-Noetherian.

Recall that also free non-abelian groups are $\mathfrak{Z}$-Noetherian by Theorem 7.5.

We will see in Example 10.13 that a group $G \in \mathscr{N}_2$ need not be $\mathfrak{Z}$-Noetherian (compare this with Example 10.2 (b) and (c)).

Bryant then proved that the class of $\mathfrak{Z}$-Noetherian groups is stable under taking subgroups, and under taking finite products:

Fact 10.3. *(a)* ([11, Lemma 3.3]) *Every subgroup of a $\mathfrak{Z}$-Noetherian group is $\mathfrak{Z}$-Noetherian.*

(b) ([11, Lemma 3.4]) *The finite product of $\mathfrak{Z}$-Noetherian groups is a $\mathfrak{Z}$-Noetherian group.*

Compare the previous result with Lemma 10.11.

Example 10.13 will show that Fact 10.3 (b) cannot be extended to infinite products.

According to Bryant's theorem (see Example 10.2 (c)), $\mathfrak{Z}_G$ is Noetherian for every abelian group G. This fails to be true in general, e.g., that there exist infinite (necessarily non-abelian) groups G with $\mathfrak{Z}_G$ discrete (see §11.2). Nevertheless, there is a huge gap between Noetherian and discrete topologies. In fact, Noetherian is a much stronger property than compactness (see Remark 1.2 (a)). This justified Question 1.

10.1 General properties of $\mathfrak{Z}$-Noetherian groups

In general, the quotient of a $\mathfrak{Z}$-Noetherian group need not be $\mathfrak{Z}$-Noetherian (see Example 10.10). Nevertheless one can prove:

Proposition 10.4. *If N is a $\mathfrak{Z}_G$-closed normal subgroup of a $\mathfrak{Z}$-Noetherian (resp. $\mathfrak{Z}$-compact) group G, then also the quotient group G/N is $\mathfrak{Z}$-Noetherian (resp. $\mathfrak{Z}$-compact).*

Proof. Let us denote by $\overline{G}$ the quotient group G/N, and by $\overline{\mathfrak{Z}}_G$ the quotient topology of $\mathfrak{Z}_G$ on $\overline{G}$. Due to Proposition 4.6, the projection

$$\pi\colon (G, \mathfrak{Z}_G) \to (\overline{G}, \mathfrak{Z}_{\overline{G}})$$

is continuous, so that $(\overline{G}, \mathfrak{Z}_{\overline{G}})$ is Noetherian (resp. compact), as continuous image of a Noetherian (resp. compact) space. □

Now we prove that a group is $\mathfrak{Z}$-Noetherian if all its countable subgroups are $\mathfrak{Z}$-Noetherian.

Proposition 10.5. *If G is not $\mathfrak{Z}$-Noetherian, then it has a countable subgroup that is not $\mathfrak{Z}$-Noetherian.*

Proof. Let

$$C_1 \supsetneq C_2 \supsetneq \ldots \supsetneq C_n \supsetneq \ldots$$

be an infinite descending chain of algebraic subsets of G. Every C_n has the form $C_n = \bigcap_{i_n \in I_n} B_n^{(i_n)}$, where I_n is a set, and each $B_n^{(i_n)}$ is an additively algebraic subset of G. So the chain has the form

$$\bigcap_{i_1 \in I_1} B_1^{(i_1)} \supsetneq \ldots \supsetneq \bigcap_{i_n \in I_n} B_n^{(i_n)} \supsetneq \bigcap_{i_{n+1} \in I_{n+1}} B_{n+1}^{(i_{n+1})} \supsetneq \ldots \tag{10.1}$$

For every $n \geq 1$, fix an element x_n witnessing the n-th strict inclusion in (10.1): $x_n \in B_n^{(i_n)}$ for every $i_n \in I_n$ (so in particular $x_n \in B_k^{(i_k)}$ for every $i_k \in I_k$ and for every $k \leq n$), but there exists $\widetilde{i}_{n+1} \in I_{n+1}$ such that $x_n \notin B_{n+1}^{(\widetilde{i}_{n+1})}$.

Take any $i_1 \in I_1$, and let $B_1 = B_1^{(i_1)}$; then let $B_n = B_n^{(\widetilde{i}_n)}$ for every $n \geq 2$. In this way we obtain a new chain

$$B_1 \supsetneq B_1 \cap B_2 \supsetneq \ldots \supsetneq B_1 \cap B_2 \cap \ldots \cap B_n \supsetneq \ldots \tag{10.2}$$

where elements x_n are again witnessing the strict inclusions:

$$x_n \in \big(B_1 \cap B_2 \cap \ldots \cap B_n\big) \setminus \big(B_1 \cap B_2 \cap \ldots \cap B_n \cap B_{n+1}\big).$$

Apply Proposition 5.6 to the set $S = \{x_n\}_{n=1}^{\infty}$ and the family $\{B_n \mid n \in \mathbb{N}\}$ to find a countable subgroup H containing S and such that $B'_n = B_n \cap H$ is an additively algebraic subset of H.

Now the chain

$$B'_1 \supsetneq B'_1 \cap B'_2 \supsetneq \ldots \supsetneq B'_1 \cap B'_2 \cap \ldots \cap B'_n \supsetneq \ldots \tag{10.3}$$

of $\mathfrak{Z}_H$-closed subsets of H shows that $\mathfrak{Z}_H$ is not a Noetherian topology on H. $\square$

Fact 10.3 (a) and Proposition 10.5 give the following theorem, characterizing the $\mathfrak{Z}$-Noetherian groups.

Theorem 10.6. *A group G is $\mathfrak{Z}$-Noetherian if and only if every countable subgroup of G is $\mathfrak{Z}$-Noetherian.*

As a corollary of Theorem 10.6, one obtains the following result.

Corollary 10.7. *Every free non-abelian group is $\mathfrak{Z}$-Noetherian.*

Proof. Let F be a free non-abelian group, and in view of Theorem 10.6, fix a countable subgroup H of F. Then H is free by Nielson-Schreier Theorem. Now we use a result from [3]: every free non-abelian group H of rank at most $\mathfrak{c}$ is isomorphic to a subgroup of $\mathrm{SO}_3(\mathbb{R})$. Being a linear group, $\mathrm{SO}_3(\mathbb{R})$ is $\mathfrak{Z}$-Noetherian by Example 10.2 (a), so H is $\mathfrak{Z}$-Noetherian too by Fact 10.3 (a). Now Theorem 10.6 applies. $\square$

Obviously, Corollary 10.7 is completely covered by Theorem 7.5, that also describes the Zariski topology of a free non-abelian group. See §7 for more details.

In the following lemma, we consider the group $S(X)$ of the permutations of an infinite set X. For a subset F of X, denote $S_F(X)$ the subgroup of $S(X)$ consisting of the permutations leaving F pointwise fixed.

Lemma 10.8. *If F has at least three elements, then $S_F(X)$ is $\mathfrak{Z}_{S(X)}$-closed.*

Proof. Let $G = S(X)$, and let $S(F)$ be the subgroup of all permutations of F in G. Since the center of $S(F)$ is trivial, the stabilizer $S_F(X)$ is precisely the centralizer $C_G(S(F))$, so it is $\mathfrak{Z}_G$-closed by Example 2.43, item 1. $\square$

Example 10.9. Let X be an infinite set, F_1 be a subset of X with at least three elements, and let $F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \ldots \subsetneq X$ be an infinite chain of subsets of X. Then one has the following stictly descending infinite chain of sugroups of $G = S(X)$:

$$S(X) \supsetneq S_{F_1}(X) \supsetneq S_{F_2}(X) \supsetneq S_{F_3}(X) \supsetneq \ldots$$

By Lemma 10.8, each subgroup $S_{F_i}(X)$ is a $\mathfrak{Z}_G$-closed subset of G, so G is not a $\mathfrak{Z}$-Noetherian group.

In Remark 12.13 we give another argument to show this fact, based on a strong recent result.

The next example shows that the quotient of a $\mathfrak{Z}$-Noetherian group need not be $\mathfrak{Z}$-Noetherian, as in fact *every* group is a quotient of a $\mathfrak{Z}$-Noetherian group.

Example 10.10. It is known that every group G is the quotient of a free non-abelian group F, which is $\mathfrak{Z}$-Noetherian by Corollary 10.7. Taking an infinite set X, a free non-abelian group F and a normal subgroup N of F such that $F/N = S(X)$, one has that the quotient group F/N is not $\mathfrak{Z}$-Noetherian, as seen in Example 10.9.

10.2 When directs products or sums are $\mathfrak{Z}$-Noetherian

We resume here an immediate corollary of Fact 10.3.

Lemma 10.11. *Let $G = G_1 \times \ldots \times G_n$. Then G is $\mathfrak{Z}$-Noetherian if and only if every G_i is $\mathfrak{Z}$-Noetherian.*

Proof. The 'if' part is Fact 10.3 (b), while the 'only if' part is a corollary of Fact 10.3 (a). $\square$

In the following theorem we describe when an arbitrary direct product or sum is a $\mathfrak{Z}$-Noetherian group.

Theorem 10.12. *Let $\{G_i \mid i \in I\}$ be a non-empty family of groups, $G = \prod_{i\in I} G_i$ and $S = \bigoplus_{i\in I} G_i$. Then the following conditions are equivalent.*

(i) every G_i is $\mathfrak{Z}$-Noetherian and all but finitely many of the groups G_i are abelian.

(ii) G is $\mathfrak{Z}$-Noetherian.

(iii) S is $\mathfrak{Z}$-Noetherian.

Proof. If I is finite, G and S coincide and Lemma 10.11 applies, so we will assume I to be infinite.

(i) $\to$ (ii). Assume (i), and let J be the finite subset of I such that G_i is abelian for every $i \in I \setminus J$. Then $G_{I\setminus J} = \prod_{i\in I\setminus J} G_i$ is abelian, so $\mathfrak{Z}$-Noetherian by Example 10.2 (c), and $G = \prod_{i\in J} G_i \times G_{I\setminus J}$ is $\mathfrak{Z}$-Noetherian too by Fact 10.3 (b), being a finite product of $\mathfrak{Z}$-Noetherian groups.

(ii) $\to$ (iii). Immediately follows by Fact 10.3 (a).

(iii) $\to$ (i). Assume S to be $\mathfrak{Z}$-Noetherian, so that for every $i \in I$ the group G_i is $\mathfrak{Z}$-Noetherian too, again by Fact 10.3 (a).

By contradiction, suppose now that $J = \{i \in I \mid G_i \text{ is not abelian}\}$ is infinite. Without loss of generality, we can assume that J contains $\mathbb{N}$, so that for every $n \in \mathbb{N}$ we have a non-abelian group G_n, and let $H = \bigoplus_{n\in\mathbb{N}} G_n \leq S$. We shall see that H is not $\mathfrak{Z}$-Noetherian, contradicting Fact 10.3 (a). In fact, $G_n \supsetneq Z(G_n)$ for every $n \in \mathbb{N}$, so that

$$H \supsetneq Z(G_0) \oplus \bigoplus_{n>0} G_n \supsetneq Z(G_0) \oplus Z(G_1) \oplus \bigoplus_{n>1} G_n \supsetneq$$
$$\supsetneq Z(G_0) \oplus Z(G_1) \oplus Z(G_2) \oplus \bigoplus_{n>2} G_n \supsetneq \dots$$

Now observe that every subset in the above descending chain is the direct sum of centralizers in H, thus it is $\mathfrak{Z}_H$-closed by Lemma 6.44. □

Example 10.13. Consider a (non-abelian) group $G \in \mathscr{N}_2$ (for example $G = Q_8$ will do). Let I be an infinite set, and note that the group $P = G^I \in \mathscr{N}_2$. By Theorem 10.12, P is not $\mathfrak{Z}$-Noetherian.

As $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$ holds for every group G, a $\mathfrak{M}$-Noetherian group is also $\mathfrak{Z}$-Noetherian. The following corollary proves the converse for direct products and direct sums of *countable* groups.

Corollary 10.14. *Let $\{G_i \mid i \in I\}$ be a non-empty family of countable groups, $G = \prod_{i\in I} G_i$ and $S = \bigoplus_{i\in I} G_i$. Then the following conditions are equivalent.*

(i) *every G_i is $\mathfrak{Z}$-Noetherian and all but finitely many of the groups G_i are abelian;*

($i_{\mathfrak{M}}$) *every G_i is $\mathfrak{M}$-Noetherian and all but finitely many of the groups G_i are abelian;*

(ii) *G is $\mathfrak{Z}$-Noetherian;*

($ii_{\mathfrak{M}}$) *G is $\mathfrak{M}$-Noetherian;*

(iii) *S is $\mathfrak{Z}$-Noetherian;*

($iii_{\mathfrak{M}}$) *S is $\mathfrak{M}$-Noetherian.*

In this case, $\mathfrak{Z}_G = \mathfrak{M}_G$.

Proof. By Theorem 4.11, $\mathfrak{Z}_S = \mathfrak{M}_S$ and $\mathfrak{Z}_{G_i} = \mathfrak{M}_{G_i}$ for every $i \in I$, so (i) is equivalent to $(i_{\mathfrak{M}})$ and (iii) is equivalent to $(iii_{\mathfrak{M}})$.

Moreover, (i), (ii) and (iii) are all equivalent by Theorem 10.12.

Since obviously $(ii_{\mathfrak{M}})$ implies (ii), it only remains to prove that $(ii_{\mathfrak{M}})$ follows from (i) and (ii).

Observe that if all but finitely many of the groups G_i are abelian, and all of them are countable, then G is the product of an abelian group and a countable group, so Theorem 4.11 applies again and $\mathfrak{Z}_G = \mathfrak{M}_G$. □

10.3 $\mathfrak{Z}$-compact and $\mathfrak{M}$-compact Groups

This subsection will be devoted to groups G such that $\mathfrak{Z}_G$ is compact or $\mathfrak{M}_G$ is compact, that will be called respectively $\mathfrak{Z}$-compact and $\mathfrak{M}$-compact groups. Obviously, $\mathfrak{M}$-compact groups are $\mathfrak{Z}$-compact.

We begin with a corollary of Theorem 6.4.

Lemma 10.15. *Direct products of $\mathfrak{Z}$-compact groups are $\mathfrak{Z}$-compact.*

Proof. Let $\{G_i \mid i \in I\}$ be a family of $\mathfrak{Z}$-compact groups. Then the product topology $\prod_{i\in I} \mathfrak{Z}_{G_i}$ is compact by Tychonov theorem, so that $\mathfrak{Z}_G$ is compact too by Theorem 6.4. □

Remark 10.16. Theorem 10.12 shows that an infinite product or sum of $\mathfrak{Z}$-Noetherian groups need not be $\mathfrak{Z}$-Noetherian. On the other hand, direct products of $\mathfrak{Z}$-compact groups are $\mathfrak{Z}$-compact groups by Lemma 10.15. We shall see in Example 11.7 (b) that direct sums of $\mathfrak{Z}$-Noetherian groups need not be even $\mathfrak{Z}$-compact.

Lemma 10.17. *Let G be a $\mathfrak{Z}$-compact group, and let H be a $\mathfrak{Z}_G$-closed subgroup of G. Then H is $\mathfrak{Z}$-compact.*

Proof. Consider the topological space $(G, \mathfrak{Z}_G)$. As it is compact, and H is a closed subset, also $(H, \mathfrak{Z}_G \restriction_H)$ is a compact space. Then $(H, \mathfrak{Z}_H)$ is compact too, as $\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$ by (5.1). □

Corollary 10.18. *Let N_1 and N_2 be groups such that $G = N_1 \times N_2$ is $\mathfrak{Z}$-compact and N_1 is center-free. Then N_2 is $\mathfrak{Z}$-compact.*

Proof. By Lemma 6.16, $N_2 \cong \{e_{N_1}\} \times N_2$ is a $\mathfrak{Z}_G$-closed subgroup of G, then Lemma 10.17 applies. □

Remark 10.19. Let $H = S_3^{\mathbb{N}}$, and $G = \mathbb{Z}_2 \times H$. Both the group G and its subgroup H are $\mathfrak{Z}$-compact by Lemma 10.15. On the other hand, H is not $\mathfrak{Z}_G$-closed by Example 6.68.

1. So the condition on H to be $\mathfrak{Z}_G$-closed in Lemma 10.17 is not necessary.

2. This example also shows that the condition on N_1 to be center-free in Corollary 10.18 is not necessary.

Remark 10.19, items 1-2 motivate respectively the first and the second part of the following question. For partial answers, see respectively Lemma 10.17 and Corollary 10.18.

Question 8. If $G = N_1 \times N_2$ is $\mathfrak{Z}$-compact, must N_2 be $\mathfrak{Z}$-compact? What if in addition N_1 is center-free?

10.4 Permanence properties of the classes $\mathfrak{N}$ and $\mathfrak{C}$

Let
$$\mathfrak{N} = \{G : \mathfrak{Z}_G \text{ is Noetherian}\} \subseteq \mathfrak{C} = \{G : \mathfrak{Z}_G \text{ is compact}\}.$$

The class $\mathfrak{N}$ is stable under taking subgroups and finite products by Fact 10.3 (a) and (b), and under taking quotients with respect to Zariski closed normal subgroups (see Proposition 10.4). Moreover, an infinite direct product belongs to $\mathfrak{N}$ if and only if all components belong to $\mathfrak{N}$ and all but finitely many of them are abelian; these two conditions are also equivalent to the fact that an infinite direct sum belongs to $\mathfrak{N}$ (Theorem 10.12).

While $\mathfrak{N}$ contains all abelian groups, it does not contain all nilpotent groups of nilpotency class 2 (see Example 10.13). In particular, it is not stable under taking central extensions.

On the other hand, $\mathfrak{C}$ is stable under taking arbitrary products (Lemma 10.15), Zariski closed subgroups (Lemma 10.17) and quotients with respect to Zariski closed normal subgroups (Proposition 10.4). Finally, the class $\mathfrak{C}$ is not stable under taking central extensions (as $Z(G) \in \mathfrak{N} \subseteq \mathfrak{C}$ for every group G).

11

$\mathfrak{Z}$-Hausdorff, $\mathfrak{M}$-Hausdorff and $\mathfrak{P}$-Hausdorff Groups

The following definition is analogous to Definition 10.1.

Definition 11.1. *A group G such that $\mathfrak{Z}_G$ (resp., $\mathfrak{M}_G$, $\mathfrak{P}_G$) is Hausdorff is called* $\mathfrak{Z}$-Hausdorff *(resp.,* $\mathfrak{M}$-Hausdorff, $\mathfrak{P}$-Hausdorff*).*

As translations are $\mathfrak{Z}_G$-continuous, a group G is $\mathfrak{Z}$-Hausdorff if and only if for every $g \in G$, $g \neq e_G$, there exist $\mathfrak{Z}_G$-closed sets C, D such that $g \notin C$, $e_G \notin D$ and $G = C \cup D$. It is not restrictive to consider only $\mathfrak{Z}_G$-basic closed sets, so G is $\mathfrak{Z}$-Hausdorff if and only if for any $g \in G$, $g \neq e_G$, there exist words $w_1, \dots, w_n$, and an integer $1 \leq k \leq n$ such that

$$G = \bigcup_{i=1}^{n} E_{w_i}, \quad e_G \notin \bigcup_{i=1}^{k} E_{w_i}, \quad g \notin \bigcup_{i=k+1}^{n} E_{w_i}.$$

As $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$ for every group G, it is obvious that $\mathfrak{Z}$-Hausdorff groups are $\mathfrak{M}$-Hausdorff, and $\mathfrak{M}$-Hausdorff groups are $\mathfrak{P}$-Hausdorff. We do not know if also the converse holds true:

Question 9. Is a $\mathfrak{M}$-Hausdorff group necessarily $\mathfrak{Z}$-Hausdorff?

Obviously, $\mathfrak{M}$-Hausdorff implies $\mathfrak{P}$-Hausdorff, but the converse does not hold. We shall see in Remark 11.40 an example showing that $\mathfrak{P}$-Hausdorff $\not\rightarrow$ $\mathfrak{M}$-Hausdorff.

A subgroup H of a $\mathfrak{Z}$-Hausdorff group G need not to be $\mathfrak{Z}$-Hausdorff, as in general only the inclusion $\mathfrak{Z}_H \subseteq \mathfrak{Z}_G \restriction_H$ holds true. Similarly, the subgroups of a $\mathfrak{M}$-Hausdorff group G need not to be $\mathfrak{M}$-Hausdorff. We will present in Example 11.7 (c) a group G such that $\mathfrak{Z}_G = \mathfrak{M}_G$ is a Hausdorff group topology, while the commutator subgroup G' is an infinite $\mathfrak{Z}$-Noetherian group, hence not $\mathfrak{Z}$-Hausdorff. Nevertheless, the following results hold.

Proposition 11.2.
- *Markov embedded subgroups of a $\mathfrak{M}$-Hausdorff group are $\mathfrak{M}$-Hausdorff groups.*
- *Zariski embedded subgroups of a $\mathfrak{Z}$-Hausdorff group are $\mathfrak{Z}$-Hausdorff groups.*

Proof. If H is a Markov embedded subgroup of a $\mathfrak{M}$-Hausdorff group G, then $\mathfrak{M}_G\restriction_H\subseteq\mathfrak{M}_H$. Being $\mathfrak{M}_G\restriction_H$ a Hausdorff topology on H, $\mathfrak{M}_H$ is Hausdorff too. The case of the Zariski topology is similar. □

Corollary 11.3. *Every Zariski embedded abelian subgroup of a $\mathfrak{Z}$-Hausdorff group is finite, and every Markov embedded abelian subgroup of a $\mathfrak{M}$-Hausdorff group is finite.*

Proof. A Zariski embedded abelian subgroup of a $\mathfrak{Z}$-Hausdorff group would be $\mathfrak{Z}$-Hausdorff by Proposition 11.2, and $\mathfrak{Z}$-Noetherian by Example 10.2 (c), thus finite by Remark 1.2 (a).

If G is a $\mathfrak{M}$-Hausdorff group, then a Markov embedded subgroup of G would be $\mathfrak{M}$-Hausdorff itself by Proposition 11.2, and $\mathfrak{Z}$-Hausdorff too by Theorem 4.10. And $\mathfrak{Z}$-Hausdorff abelian groups are finite. □

Corollary 11.4. *If G is a $\mathfrak{M}$-Hausdorff group, then $Z(G)$ is finite.*

Proof. In view of Corollary 11.3, it will suffice to show that $Z(G)$ is Markov embedded in G. This follows from the fact that $Z(G)$ is super-normal in G by Proposition 5.5, so Markov embedded by Proposition 5.10 (a) and (b). □

Lemma 11.5. *Let $\{F_i \mid i\in I\}$ be a non-empty family of finite groups, and $G=\prod_{i\in I}F_i$. Then:*

(a) G is $\mathfrak{Z}$-Hausdorff if and only if $\mathfrak{Z}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$. In this case, $\mathfrak{Z}_G=\mathfrak{M}_G=\mathfrak{P}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$ and all but finitely many F_i are center-free.

(b) G is $\mathfrak{M}$-Hausdorff if and only if $\mathfrak{M}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$. In this case, $\mathfrak{Z}_G\subseteq\mathfrak{M}_G=\mathfrak{P}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$ and all but finitely many F_i are center-free.

(c) G is $\mathfrak{P}$-Hausdorff if and only if $\mathfrak{P}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$. In this case, $\mathfrak{Z}_G\subseteq\mathfrak{M}_G\subseteq\mathfrak{P}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$.

Proof. Recall that $\prod_{i\in I}\mathfrak{Z}_{F_i}$ is a compact Hausdorff group topology on G by Lemma 6.53, and that $\mathfrak{Z}_G\subseteq\mathfrak{M}_G\subseteq\mathfrak{P}_G\subseteq\prod_{i\in I}\mathfrak{Z}_{F_i}$ by (6.11).

(a). If $\mathfrak{Z}_G$ is Hausdorff, then by (6.11) the map $id_G\colon(G,\prod_{i\in I}\mathfrak{Z}_{F_i})\to(G,\mathfrak{Z}_G)$ is continuous from a compact space to a Hausdorff one, hence is open and $\mathfrak{Z}_G=\prod_{i\in I}\mathfrak{Z}_{F_i}$. In particular, G is also $\mathfrak{M}$-Hausdorff, so that $Z(G)=\prod_{i\in I}Z(F_i)$ is finite by Corollary 11.4.

The proof of (b) and (c) is analogous. □

Corollary 11.6. *Let $\{F_i \mid i\in I\}$ be a non-empty family of finite center-free groups, $G=\prod_{i\in I}F_i$, and $S=\bigoplus_{i\in I}F_i$.*

Then, both $\mathfrak{M}_G$ and $\mathfrak{M}_S$ are Hausdorff group topologies, so they are the minimum of the poset of all Hausdorff group topologies on G, and on S respectively. Moreover, they have a local base of clopen subgroups of finite index, so they are zero-dimensional.

Proof. Immediately follows from Theorem 6.55. □

Example 11.7. Let I be an infinite set, and consider the group $G = S_3^I$. By Theorem 6.55 (a), $\mathfrak{Z}_G = \prod_{i\in I} \mathfrak{Z}_{S_3}$ is a compact Hausdorff topology on G, so it is not $\mathfrak{Z}$-Noetherian.

(a) For example, if $I = \mathbb{N}$, one has the following stictly descending infinite chain

$$G = S_3^{\mathbb{N}} \supsetneq \{\mathrm{id}\} \times S_3^{\mathbb{N}\setminus\{0\}} \supsetneq \{\mathrm{id}\} \times \{\mathrm{id}\} \times S_3^{\mathbb{N}\setminus\{0,1\}} \supsetneq$$
$$\supsetneq \{\mathrm{id}\} \times \{\mathrm{id}\} \times \{\mathrm{id}\} \times S_3^{\mathbb{N}\setminus\{0,1,2\}} \supsetneq \dots$$

As the elements of this chain are direct products of centralizers, they are $\mathfrak{Z}_G$-closed sets by Lemma 6.16.

(b) Consider now the subgroup $S = S_3^{(I)}$ of G, and observe that S is not a $\mathfrak{Z}_G$-closed subset of G. Moreover, $\mathfrak{Z}_S = \mathfrak{Z}_G \restriction_S$ by Theorem 6.55 (b), so $\mathfrak{Z}_S$ is a non-compact, precompact Hausdorff topology on S.

In particular, direct sums of finite (hence $\mathfrak{Z}$-Noetherian) groups need not be $\mathfrak{Z}$-compact.

(c) Finally, consider the commutator subgroup of G, $H = S_3^{I\,\prime} = {S_3'}^{I} = A_3^I \cong \mathbb{Z}/3\mathbb{Z}^I$. It is an infinite abelian group, so H is $\mathfrak{Z}$-Noetherian by Example 10.2 (c), hence not $\mathfrak{Z}$-Hausdorff. In particular, $\mathfrak{Z}_H \neq \mathfrak{Z}_G \restriction_H$ and H is not a Zariski embedded subgroup of G. Nor H is Markov embedded: in fact we have $\mathfrak{M}_G = \mathfrak{Z}_G$ by Theorem 6.55, and $\mathfrak{M}_H = \mathfrak{Z}_H$ by Theorem 4.11, so

$$\mathfrak{M}_H = \mathfrak{Z}_H \subsetneq \mathfrak{Z}_G \restriction_H = \mathfrak{M}_G \restriction_H .$$

Note also that $H \in \mathbb{E}_G$, as $H = A_3^{\mathbb{N}} = S_3[3]^{\mathbb{N}} = S_3^{\mathbb{N}}[3] = \{g \in G \mid g^3 = e_G\}$.

In particular, G' need not be Zariski embedded, nor Markov embedded, in G even if it is an elementary algebraic subset of G.

Remark 11.8. In analogy with the definitions given in §10.4, let us introduce also the class $\mathfrak{H} = \{G : \mathfrak{Z}_G \text{ is Hausdorff}\}$.

Note that $\mathfrak{N}\cap\mathfrak{H}$ is the class of finite groups. The group G considered in Example 11.7 satisfies $G \in \mathfrak{C} \cap \mathfrak{H}$ and $G' \in \mathfrak{N} \setminus \mathfrak{H}$.

11.1 Finite-center direct products

Recall that in §6.4 we studied the Zariski topology of groups G of the form $G = \prod_{i\in I} F_i$, where every F_i was finite and center-free.

We consider now a slightly more general case. Let $\{F_i \mid i \in I\}$ be a family of finite groups, and all but finitely many of the groups F_i are center-free. Call G_0 the product of those groups with non-trivial center, so that we can write $G = G_0 \times \prod_{i\in I} F_i$, with G_0 finite, and F_i finite and center-free for every $i \in I$.

Let $H = \prod_i F_i$: by Theorem 6.55 (a) we have $\mathfrak{Z}_H = \prod_{i\in I} \mathfrak{Z}_{F_i}$. Obviously, $H \cong G/G_0$, so we identify these two groups. Being H center-free, note that G_0 is a $\mathfrak{Z}_G$-closed subgroup of G by Lemma 6.16. So the canonical projection

$$\pi\colon (G, \mathfrak{Z}_G) \to (H, \mathfrak{Z}_H) \tag{11.1}$$

is continuous by Proposition 4.6, and as a consequence $\mathfrak{Z}_G \supseteq \iota_{G_0} \times \mathfrak{Z}_H$. In particular, for every $i \in I$, the subgroups $H_i = G_0 \times \{e_{F_i}\} \times \prod_{j\in I\setminus\{i\}} F_j$ are $\mathfrak{Z}_G$-clopen, so $\mathfrak{M}_G$-clopen and $\mathfrak{P}_G$-clopen.

Being $\mathfrak{Z}_G$ a T_1 topology, we have $\mathfrak{Z}_G \supsetneq \iota_{G_0} \times \mathfrak{Z}_H$. Finally, Lemma 6.53 gives the other non-trivial inclusion in the following chain

$$\iota_{G_0} \times \mathfrak{Z}_H \subsetneq \mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G \subseteq \mathfrak{Z}_{G_0} \times \mathfrak{Z}_H. \tag{11.2}$$

Proposition 11.9. *Let $\{F_i \mid i \in I\}$ be a non-empty family of finite center-free groups, and G_0 be a finite group. Then the following are equivalent, for the group $G = G_0 \times \prod_{i\in I} F_i$.*

1. *G is $\mathfrak{Z}$-Hausdorff;*
2. *$\mathfrak{Z}_G = \mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}$;*
3. *the subgroup $H = \prod_{i\in I} F_i$ is $\mathfrak{Z}_G$-closed.*

In this case, $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}$.

Proof. The last part of the statement will immediately follow by (11.2) and item 3.

The equivalence between condition 1 and 2 follows by Lemma 11.5, while the implication $2 \to 3$ is trivial.

So it only remains to prove that 3 implies 2. Consider the subbase for $\big(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\big)$-closed consisting of cosets of H, and of H_i, for $i \in I$. We have already noted that H_i is $\mathfrak{Z}_G$-closed for every $i \in I$, by the inclusion $\iota_{G_0} \times \mathfrak{Z}_H \subseteq \mathfrak{Z}_G$ established in (11.2). So H and its cosets are the only remaining $\big(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\big)$-closed of subbase to check to be $\mathfrak{Z}_G$-closed. ☐

Example 11.10. Let F be either S_3, or a group H_0 as in Remark 6.10 (b) (in the latter case, assume it to be finite and center-free). Let I be an infinite set, and consider the group $H = F^I$. Finally, let G_0 be a finite group such that $Z(G_0)[2] = \{e_{G_0}\}$, and let $G = G_0 \times H$. Then H is $\mathfrak{Z}_G$-closed by Corollary 6.67 (a), so that $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_F$ by Proposition 11.9. For example, one can consider $G_0 = \mathbb{Z}_m$ for every odd integer $m \geq 3$.

Let $S_0 = \bigoplus_{i\in I} F_i$. In the following result, we consider the direct sum $S = G_0 \times S_0 \leq G$.

Proposition 11.11. *Let $\{F_i \mid i \in I\}$ be a non-empty family of finite center-free groups, and G_0 be a finite group. Then the following are equivalent, for the group $S = G_0 \times \bigoplus_{i\in I} F_i$.*

1. $\mathfrak{Z}_S = \left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S$;

2. *the subgroup $S_0 = \{e_{G_0}\} \times \bigoplus_{i\in I} F_i$ is $\mathfrak{Z}_S$-closed.*

In this case, $\mathfrak{Z}_S = \mathfrak{M}_S = \mathfrak{P}_S = \left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S = \mathfrak{Z}_G \restriction_S = \mathfrak{M}_G \restriction_S = \mathfrak{P}_G \restriction_S$.

Proof. Note that Remark 6.54 applies to S, so that (6.12) and (6.13) hold for $H = S$, and the last part of the statement immediately follows by condition 1.

The proof will be very similar to that of Proposition 11.9. For every $i \in I$, consider the subgroups $S_i = G_0 \times \{e_{F_i}\} \times \bigoplus_{j\in I\setminus\{i\}} F_j$.

$1 \to 2$ is trivial.

$2 \to 1$. We only have to prove the inclusion $\left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S \subseteq \mathfrak{Z}_S$. To this end, consider the subbase for $\left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S$-closed consisting of cosets of S_0, and of S_i, for $i \in I$. By Lemma 6.44, every S_i is $\mathfrak{Z}_S$-closed. So S_0 and its cosets are the only remaining $\left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S$-closed of subbase to check to be $\mathfrak{Z}_S$-closed. □

Theorem 11.12. *Let $m > 0$ be an integer, and G_0 be a finite group with $(|G_0|, m) = 1$. If for every $i \in I$, F_i is a finite, center-free, bounded group, with $\exp(F_i) \mid m$, then the following hold.*

1. *$H = \{e_{G_0}\} \times \prod_{i\in I} F_i$ is $\mathfrak{Z}_G$-closed, so $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}$.*

2. *$S_0 = \{e_{G_0}\} \times \bigoplus_{i\in I} F_i$ is $\mathfrak{Z}_S$-closed, so*
$$\mathfrak{Z}_S = \mathfrak{M}_S = \mathfrak{P}_S = \left(\mathfrak{Z}_{G_0} \times \prod_{i\in I} \mathfrak{Z}_{F_i}\right) \restriction_S = \mathfrak{Z}_G \restriction_S = \mathfrak{M}_G \restriction_S = \mathfrak{P}_G \restriction_S .$$

Proof.
1. Follows from the fact that $G[m] = G_0[m] \times \prod_{i\in I} F_i[m] = \{e_{G_0}\} \times \prod_{i\in I} F_i = H$ is an elementary algebraic subset of G, then Proposition 11.9 applies.

2. As in item 1, observe that $S[m] = G[m] \cap S = S_0$ is an elementary algebraic subset of S, then apply Proposition 11.11.

□

11.2 $\mathfrak{Z}$-discrete and $\mathfrak{M}$-discrete groups

Definition 11.13. Recall that a group G is called *topologizable* if it admits a non-discrete Hausdorff group topology (so G is *non-topologizable* if the only Hausdorff group topology of G is the discrete one). As the Markov topology of a non-topologizable group G is the discrete topology, such a group G will also be called $\mathfrak{M}$-*discrete*. Similarly, a group G such that $\mathfrak{Z}_G$ (resp., $\mathfrak{P}_G$) is discrete will be called $\mathfrak{Z}$-*discrete* (resp., $\mathfrak{Z}$-*discrete*).

11.2.1 $\mathfrak{Z}$-discrete groups

We start with a discussion on the $\mathfrak{Z}$-discrete groups, i.e. groups G with finite $\mathrm{bd}(G)$. Obviously, an infinite $\mathfrak{Z}$-*discrete* group is also $\mathfrak{M}$-*discrete*, so provides a solution of Markov's second problem. Since the popularity of this problem has been much wider than that of the first one, and since $\mathfrak{Z}_G = \mathfrak{M}_G$ for a countable group G, this explains the major interest in $\mathfrak{Z}$-discrete groups.

Lemma 11.14. *A group G is $\mathfrak{Z}$-discrete if and only if there exists $F \in [G]^{<\omega}$ such that H is $\mathfrak{Z}$-discrete for every $F \subseteq H \leq G$.*

Proof. One can easily check that G is $\mathfrak{Z}$-discrete if and only if $G \setminus \{e_G\}$ is additively algebraic, i.e. there exist $n \in \mathbb{N}_+$ and $w_1, \ldots, w_n \in G[x]$ such that $E^G_{w_1} \cup \ldots \cup E^G_{w_n} = G \setminus \{e_G\}$. (Recall that the least n with this property is $\mathrm{bd}(G)$.)

Let $F = \bigcup_{i=1}^n \mathrm{coeff}(w_i)$, and $F \subseteq H \leq G$. Then for every $i = 1, \ldots, n$ obviously $w_i \in H[x]$, so that $E^G_{w_i} \cap H = E^H_{w_i}$. From this, it follows that $E^H_{w_1} \cup \ldots \cup E^H_{w_n} = H \setminus \{e_G\}$, i.e. H is $\mathfrak{Z}$-discrete. □

The same idea will actually prove the following more general result.

Proposition 11.15. *Let G be a group, and κ be a cardinal number. Then $P_\kappa \mathfrak{Z}_G = \delta_G$ if and only if there exists $F \in [G]^{\leq\kappa}$ such that $P_\kappa \mathfrak{Z}_H = \delta_H$ for every $F \subseteq H \leq G$.*

The problem to construct a countable $\mathfrak{Z}$-discrete (hence, non-topologizable) group is equivalent to Markov's second problem for countable groups. It was resolved by Ol′shanskij [49] (see also [50, Theorem 31.5]).

Example 11.16 ([49])**.** Ol′shanskij produced a countable $\mathfrak{Z}$-discrete group using an appropriate quotient G of the (countable) Adian group $A(n, m)$. More precisely, he establised that actually $\mathrm{bd}(G) \leq 2m$ holds true.

If a group G is $\mathfrak{Z}$-discrete, then $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ holds because all three topologies become discrete. Such groups are extremely hard to come by, but a variety of examples were constructed since the pioneer work of Ol′shanskij, using his techniques developed in [50] or modifying his example reported above.

Example 11.17.

1. Morris and Obraztsov [45, Theorem L] built, for any sufficiently large $p \in \mathbb{P}$, a continuum of pairwise non-isomorphic infinite $\mathfrak{Z}$-discrete groups of exponent p^2, all of whose proper subgroups are cyclic. Every one of these groups is a central extension of a Tarski monster of exponent p by a central subgroup of order p.

2. Klyachko and Trofimov [37] constructed a finitely generated torsion-free group G such that there exist a word $w \in G[x]$ with $E_w = G \setminus \{e_G\}$, i.e. $\mathrm{bd}(G) = 1$. In particular, G is $\mathfrak{Z}$-discrete.

3. Trofimov [63] constructed an infinite, finitely generated, center-free group G such that every automorphism of G is inner and $\mathrm{bd}(G) = 1$. In particular, $G \cong \mathrm{Aut}\, G$ and G is $\mathfrak{Z}$-discrete.

4. Trofimov [62] proved that every group admits an embedding into a $\mathfrak{Z}$-discrete group.

As already noted above, the infinite groups G built in items 2 and 3 have $\mathrm{bd}(G) = 1$. The groups with finite bound seem to be of interest also in the finite case (see [34] for the description of all finite 2-bound groups, where an unpublished manuscript of G. Cherlin is quoted, describing the finite 1-bound groups).

The next theorem gives an "external" characterization of $\mathfrak{Z}$-discrete groups in terms of Zariski embeddings.

Theorem 11.18 ([20, Theorem 6.14]). *For a group H the following conditions are equivalent:*

(i) H is simultaneously Hausdorff embedded, Markov embedded and Zariski embedded in every group G that contains H as a subgroup,

(ii) H is Zariski embedded in every group G that contains H as a subgroup,

(iii) H is $\mathfrak{Z}$-discrete.

Example 11.19. This theorem gives easy examples of non-Zariski-embedded subgroups. For example, the infinite abelian group $H = \mathbb{Z}$ is not $\mathfrak{Z}$-discrete, hence there exists a group G containing H as a subgroup such that H is not Zariski embedded in G.

11.2.2 $\mathfrak{M}$-discrete groups

Let us recall that a group G with $|G| = \omega_1$ is called a *Kurosh group*, if all proper subgroups of G are countable. Let us introduce the following stronger notion:

Definition 11.20 ([25, Definition 4.3]). *Let $m \in \mathbb{N}$. An uncountable group G is said to be a m-*Kurosh group*, if $A^m = G$ for every subset A of G with $|A| = |G|$ and $A^{-1} = A$.*

Clearly every m-Kurosh group of size ω_1 is a Kurosh group.

For an uncountable group G consider the following condition:

(S) for every subgroup H of G with $|H| < |G|$ there exists $F \in [G]^{<\omega}$ such that $\bigcap_{x\in F} x^{-1}Hx$ is finite.

Clearly, a group G satisfying (S) cannot have countably infinite normal subgroups (actually, (S) implies a stronger property — the core $H_G = \bigcap_{x\in G} x^{-1}Hx$ of any subgroup H of G with $|H| < |G|$ is finite and $H_G = \bigcap_{x\in F} x^{-1}Hx$ for some $F \in [G]^{<\omega}$).

The utility of these notions becomes clear in the following original sufficient condition ensuring $\mathfrak{M}$-discreteness elaborated by Shelah [57].

Proposition 11.21. *If $m \in \mathbb{N}$, then every m-Kurosh group G satisfying (S) is $\mathfrak{M}$-discrete.*

Proof. Let $\mathcal{T}$ be a Hausdorff group topology on G. There exists a $\mathcal{T}$-neighbourhood V of e_G with $V \neq G$. Choose a $\mathcal{T}$-neighbourhood W of e_G with $W^m \subseteq V$. Then $V \neq G$ and our hypothesis on G to be m-Kurosh yield $|W| < |G|$. If $H = \langle W\rangle$, then $|H| = |W| \cdot \omega < |G|$. According to (S), the intersection $O = \bigcap_{x\in F} x^{-1}Hx$ is finite for some $F \in [G]^{<\omega}$. Since each $x^{-1}Hx$ is a $\mathcal{T}$-neighbourhood of e_G, this proves that $e_G \in O \in \mathcal{T}$. As $\mathcal{T}$ is Hausdorff and O is finite, it follows that $\{e_G\}$ is $\mathcal{T}$-open, and therefore $\mathcal{T}$ is discrete. □

Remark 11.22. The above criterion was used by Shelah [57] to produce the first consistent example of a non-topologizable group. He worked under the assumption of CH and produced a 10000-Kurosh group G of size ω_1 satisfying (S) with $|F| = 2$ for every subgroup H of G with $|H| < |G|$ (actually, he managed to have each of these subgroups H *malnormal* in G, i.e. $H \cap x^{-1}Hx = e_G$ for every $x \in G \setminus H$).

Here comes the counterpart of Theorem 11.18 providing an "external" characterization of discreteness of $\mathfrak{M}_H$ in terms of Markov and Hausdorff embeddings.

Theorem 11.23 ([20, Theorem 6.9])**.** *For a group H the following conditions are equivalent:*

(i) H is Hausdorff embedded in every group G that contains H as a subgroup,

(ii) H is Markov embedded in every group G that contains H as a subgroup,

(iii) H is $\mathfrak{M}$-discrete.

Following Lukàcs [39], call a group G *hereditarily $\mathfrak{M}$-discrete*, if for every subgroup H of G all quotients of H are $\mathfrak{M}$-discrete (the term *hereditarily non-topologizable group* is used in [39]). This is the largest class of $\mathfrak{M}$-discrete groups closed with respect to taking subgroups and quotients. The origin of this class comes from the categorically compact groups introduced in [27]. A topological group (G, τ) is called *categorically compact* if, for every topological group H, the projection $G \times H \to H$ sends closed subgroups of $G\times H$ onto closed subgroups of H. By Kuratowski's closed projection theorem, every compact group is categorically compact. This implication is invertible for solvable groups and for connected locally compact groups [27].

It is proved in [27, Corollary 5.4] that if a group G is hereditarily $\mathfrak{M}$-discrete, then (G, δ_G) is categorically compact; the converse holds for countable groups [27, Theorem 5.5].

Very recently, Klyachko, Ol′shanskij and Osin produced in [36] the first examples of infinite hereditarily $\mathfrak{M}$-discrete groups using Ol′shanskij's techniques [50]. More precisely, they gave the following result.

Theorem 11.24 ([36, Theorem 1.2]). *There exist hereditarily $\mathfrak{M}$-discrete groups G, H, I, and J such that:*

(a) G is infinite, finitely generated, and of bounded exponent;

(b) H is finitely generated and of unbounded exponent;

(c) I is countable, but not finitely generated;

(d) J is uncountable.

They managed to produce examples with those properties to answer [27, Question 1.2 and Question 5.2] (later reported also in [19, Question 929]) in the negative.

11.2.3 $\mathfrak{M}$-discrete groups that are not $\mathfrak{Z}$-discrete

Here we discuss the first of Markov's problems, namely when $\mathfrak{Z}_G = \mathfrak{M}_G$ occurs for a group G.

Although Hesse gave a strong and very impressive negative solution of Markov's first problem by 1979, it is fair to say that his solution remained completely unknown to the large audience for long time. Indeed, in 1992, this problem is qualified as "still open" in the survey [15]. Similarly, the quite recent survey [38] does not mention Hesse's solution and quotes the recent consistent counterexample announced in [58], where an appropriate modification of Shelah's construction [57] under CH is shown to produce an example of a group G such that $\mathfrak{Z}_G \neq \mathfrak{M}_G$. A possible explanation can be the somewhat surprising fact that Hesse (maybe unaware of that problem) never claimes a solution, and he never published the solution obtained in his PhD thesis [33].

Here comes Hesse's powerful theorem:

Theorem 11.25 ([33]). *Let λ, κ be infinite cardinal numbers with $cf(\lambda) > \kappa = cf(\kappa)$. Then there exists a group G such that :*

(a) $|G| = \lambda$;

(b) every Hausdorff semigroup topology on G is discrete (in particular, $\mathfrak{M}_G = \mathfrak{P}_G = \delta_G$);

(c) $bd(G) > \kappa$.

Since $\mathrm{bd}(G) > \kappa$ means that $P_\kappa \mathfrak{Z}_G$ (hence $\mathfrak{Z}_G$ too) is non-discrete, while $\mathfrak{M}_G$ is discrete by item (b), one can deduce from the above theorem that there is a large gap between $\mathfrak{M}_G$ and $\mathfrak{Z}_G$. For the sake of convenience we formulate the following corollary giving an explicit negative solution to Markov's first problem.

Corollary 11.26. *For every cardinal λ of uncountable cofinality there exists a group G of size λ such that $\mathfrak{M}_G$ is discrete while $\mathfrak{Z}_G$ is not discrete. In particular, $\mathfrak{Z}_G \neq \mathfrak{M}_G = \mathfrak{P}_G = \delta_G$.*

In particular, there exist groups G such that $\mathfrak{Z}_G \neq \mathfrak{M}_G = \mathfrak{P}_G = \delta_G$ of size ω_1 and $\mathfrak{c}$.

11.2.4 Highly topologizable groups

We conclude with a couple of results on *highly topologizable* groups. Call a group G *highly topologizable* if it admits the maximum number $2^{2^{|G|}}$ of Hausdorff group topologies.

First, we state some Podewski's results about the highly topologizability of ungebunden groups, i.e. groups G with $\mathrm{bd}(G) = |G|$. We remark that his 1977 work [52] was already ready as a preprint in 1972.

Theorem 11.27 ([52])**.** *Let G be a group such that $bd(G) = |G|$. Then G is highly topologizable.*

It turns out that Theorem 11.27 is reversible for countable groups:

Corollary 11.28 ([52])**.** *For a countable group G, the following conditions are equivalent:*

- *G is not $\mathfrak{M}$-discrete (i.e. it is topologizable);*
- *$bd(G) = |G|$;*
- *G is highly topologizable.*

So, for a countable group, it is equivalent to be topologizable, and to be highly topologizable.

Podewski also proved that his Theorem 11.27 applies to abelian groups.

Theorem 11.29 ([52])**.** *If G is an abelian group, then $bd(G) = |G|$. In particular, G is highly topologizable.*

In 1974, Kiltinen directly proved in [35] that infinite abelian groups are highly topologizable; later, Berhanu, Comfort and Reid in [10] strenghtened his result proving that in fact infinite abelian group admit $2^{2^{|G|}}$ *precompact* Hausdorff group topologies.

Now we will consider free groups. Remus first proved in [53] that free groups are highly topologizable, then he himself proved in [54] that every free group G admits $2^{2^{|G|}}$ *precompact* Hausdorff group topologies.

A long series of countable topologizable groups can be found in the following results by Hesse, taken from [32], where the property $\mathrm{bd}(G) = |G|$ was established for the group G. That entails, according to Corollary 11.28, that G is highly topologizable.

Theorem 11.30 ([32]). *If a countable group G has one of the following properties, then G is topologizable:*

1. *G has an infinite normal subgroup N that is an FC-group;*
2. *G contains an infinite solvable normal subgroup;*
3. *G is locally nilpotent.*

Obviously, item 2 implies that every countable solvable group admits a nondiscrete Hausdorff group topology. This partially answers positively a question due to Sharma [56] about the topologizability of infinite solvable groups.

Recall that a group is *locally finite* if every finitely generated subgroup is finite. Then Belyaev [9] proved the following.

Theorem 11.31 ([9]). *Every countably infinite locally finite group is topologizable.*

In particular, a countably infinite locally finite group is highly topologizable by Theorem 11.30.

11.3 $\mathfrak{P}$-Hausdorff groups

Definition 11.32 ([16]). *A compact Hausdorff topological group is a* van der Waerden group *if every homomorphism to a compact Hausdorff group is continuous.*

Theorem 11.33 ([65]). *Every compact, connected, semi-simple Lie group is a van der Waerden group.*

For example, Theorem 11.33 applies to the group $SO_3(\mathbb{R})$.

Theorem 11.34 ([14]). *For a compact Hausdorff group (G, τ), the following are equivalent.*

(a) *(G, τ) is van der Waerden;*

(b) *τ is the only (hence, the finest) precompact group topology on G;*

(c) $\mathfrak{P}_G$ *is Hausdorff;*

(d) $\mathfrak{P}_G = \tau$.

By Theorem 11.34, the class of van der Waerden groups is the class of compact $\mathfrak{P}$-Hausdorff groups.

11.3.1 $\mathfrak{P}$-discrete groups

Example 11.35. We collect here some examples of $\mathfrak{P}$-discrete groups that are not $\mathfrak{M}$-discrete, so that $\mathfrak{M}_G \neq \mathfrak{P}_G$.

1. A classical example of a $\mathfrak{P}$-discrete group is the group $SL(2, \mathbb{C})$ (see [48]). Since $SL(2, \mathbb{C})$ is topologizable (by its usual topology induced by $\mathbb{C}^4$), it is not $\mathfrak{M}$-discrete.

2. We shall see in Remark 12.18, item 3, that $\mathfrak{Z}_G = \mathfrak{M}_G \neq \mathfrak{P}_G = \delta_G$ holds for every group G with $S_\omega(X) \leq G \leq S(X)$, where X is an infinite set.

Definition 11.36 ([47])**.** *A group G is called:*

(a) maximally almost periodic*, if the homomorphisms from G to compact groups K separate the points of G (i.e., G admits precompact Hausdorff group topologies);*

(b) minimally almost periodic*, if every homomorphism to a compact group K is trivial.*

So a group is $\mathfrak{P}$-discrete exactly when it is not maximally almost periodic. In particular, examples of $\mathfrak{P}$-discrete groups are provided by all minimally almost periodic groups. Note that $G' = G$ for a minimally almost periodic group.

In the following theorem, we prove that a solvable divisible non-abelian group admits no precompact group topology, although such a group is not minimally almost periodic.

Theorem 11.37. *Every solvable divisible non-abelian group is $\mathfrak{P}$-discrete.*

Proof. Assume that G is a solvable divisible precompact group. We aim to prove that G is abelian.

Let us prove first that its compact completion K is a connected group. Since the connected component of K is the intersection of all open subgroups of K, it will suffice to show that K has no proper open subgroups. Indeed, let N be an open subgroup of K, so that the intersection $G \cap N$ is an open subgroup of G. Since open subgroups in a precompact group have finite index, $n = [G : (G \cap N)]$ is finite. Then $x^n \in G \cap N$ for every $x \in G$. Since G is divisible, for every $g \in G$ there exists $x \in G$ such that $g = x^n$. Hence, $G \subseteq G \cap N$, so that N contains the dense subgroup G. Since N is also closed in K, we deduce that $N = K$.

The next step is to verify that K is also solvable (of the same class as G). Indeed, by the density of G in K and the continuity of the commutator operation, one can easily deduce that G' is dense in K'. Similarly, the n-th commutator subgroup $G^{(n)}$ is dense in $K^{(n)}$. Therefore, $K^{(n)}$ is trivial whenever $G^{(n)}$ is trivial.

Now we prove that the compact connected solvable group K is abelian. As a consequence, G is abelian too. Arguing for a contradiction, assume that $K \neq Z(K)$ is not abelian. By a theorem of Varopoulos [64], the non-trivial quotient $K/Z(K)$ is center-free and isomorphic to a direct product of simple connected compact Lie groups; in particular, $K/Z(K)$ cannot be solvable. This contradicts the fact that the quotient of a solvable group is solvable. □

In the next corollary we see that the converse of Theorem 11.37 holds for countable groups, thus characterizing the solvable divisible countable $\mathfrak{P}$-discrete groups.

Corollary 11.38. *For a solvable divisible countable group G, the following conditions are equivalent:*

(a) G is abelian;

(b) $\mathfrak{M}_G = \mathfrak{P}_G$;

(c) $\mathfrak{P}_G \neq \delta_G$;

(d) $\mathfrak{P}_G$ is Noetherian.

In this case, also $\mathfrak{Z}_G = \mathfrak{M}_G$ is Noetherian.

Proof. (a) implies (b) and (d) by Theorem 4.10.

(b) implies (c) by Theorem 11.30, item 2, as $\mathfrak{M}_G \neq \delta_G$.

(c) implies (a) by Theorem 11.37.

(d) trivially implies (a) as G is infinite.

Finally, the last assertion follows by Theorem 4.10. We will see in Remark 11.40 that the property of having $\mathfrak{Z}_G = \mathfrak{M}_G$ Noetherian is strictly weaker than the others, even for groups $G \in \mathscr{N}_2$ (that in particular are solvable). □

Recall the Heisenberg group $H = H(n, K)$ of $n \times n$ unitriangular upper matrixes over the field K studied in §8.

Proposition 11.39. *For every field K with $\operatorname{char} K = 0$, the group $H = H(n, K)$ is $\mathfrak{P}$-discrete, so $\mathfrak{M}_H \neq \mathfrak{P}_H = \delta_H$. In particular, if $\operatorname{char} K = 0$ and K is also countable, then $\mathfrak{Z}_H = \mathfrak{M}_H$ is Noetherian, so does not coincide with $\mathfrak{P}_H = \delta_H$.*

Proof. Since $H \in \mathscr{N}_2$, it is a solvable non-abelian group and we are going to apply Theorem 11.37. To this end, we need to check that H is divisible. Since $Z(H) \cong (K, +)$ is divisible, it suffices to check that $H/Z(H)$ is divisible. This follows from $H/Z(H) \cong (K^n, +) \times (K^n, +)$.

It remains to see that H is not $\mathfrak{M}$-discrete. To this end one may take any non-discrete ring topology on K (every infinite commutative ring has such topologies according to a theorem of Arnautov) and consider the product topology on H. This will be a non-discrete Hausdorff group topology on H.

If K is also countable, then H is $\mathfrak{Z}$-Noetherian by Example 10.2 (a), and $\mathfrak{Z}_H = \mathfrak{M}_H$ by Theorem 4.11. □

Remark 11.40. Now we shall see that the condition of having $\mathfrak{Z}_H = \mathfrak{M}_H$ Noetherian for a group H does not imply none of the properties (a)–(d) of Corollary 11.38, ever for countable divisible groups $H \in \mathscr{N}_2$.

To this end, consider the group $H = H(n, K)$, for a countable field K with $\operatorname{char} K = 0$, and apply Proposition 11.39.

In particular, such groups are $\mathfrak{P}$-discrete, hence $\mathfrak{P}$-Hausdorff, and $\mathfrak{M}$-Noetherian, hence not $\mathfrak{M}$-Hausdorff.

Remark 11.41. Note that when $\operatorname{char} K = 0$, a subgroup S of H is $\mathfrak{P}$-embedded in H if and only it is $\mathfrak{P}$-discrete by Proposition 11.39. In particular, no abelian subgroup of H is $\mathfrak{P}$-embedded in H. For example, $Z(H)$, L, M are not $\mathfrak{P}$-embedded in H.

Motivated by the equality $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ in the abelian case, the question which of these equalities remain true for nilpotent groups was raised in [21, Question 12.1]. Proposition 11.39 and the above example provide a partial answer to this question for a large variety of groups $G \in \mathscr{N}_2$ (see Question 13).

Remark 11.42. In Proposition 11.39 we proved that, given a field K of characteristic 0, the Heisenberg group $H = H(n, K)$ is $\mathfrak{P}$-discrete. In other words, it does not admit precompact Hausdorff group topologies. However, replacing the field K by $\mathbb{Z}$ one obtains a group $H_{\mathbb{Z}}$ that admits a plenty of precompact group topologies, so that now $\mathfrak{P}_{H_{\mathbb{Z}}}$ is not only non-discrete, it is even non-Hausdorff.

If one replaces "precompact" by locally precompact (this means a group that has a locally compact completion), things may change substantially. Indeed, the group $H_{\mathbb{Q}}$ admits plenty of non-discrete locally precompact group topologies (they can be obtained by embedding into the locally compact groups built in [44] via "generalized Heisenberg group" constructions).

Recall also the definition of the group $G_K = K^* \ltimes V$, given a field K and a K-vector space V. See also §§9, 9.1.

Corollary 11.43. *Let K be an algebraically closed field of characteristic 0. Then the group G_K is $\mathfrak{P}$-discrete.*

Proof. One only has to notice that the solvable group G_K is also divisible under the assumptions, in order to apply Theorem 11.37. □

Remark 11.44. Under the hypotheses of Corollary 11.43 on a field K, the subgroup $N = N_K$ is not $\mathfrak{P}$-embedded in $G = G_K$, as $\mathfrak{P}_N = \mathfrak{M}_N = \mathfrak{Z}_N = cof_N$ by Proposition 4.16.

12

Minimal group topologies

Definition 12.1. *A Hausdorff topological group* (G, τ) *is called* minimal *if* τ *is a minimal element in the poset of all Hausdorff group topologies on* G.

In other words, (G, τ) is minimal whenever the following condition holds: if σ is a Hausdorff group topology on G, such that $\sigma \subseteq \tau$, then $\sigma = \tau$.

Proposition 12.2. *Every compact Hausdorff group topology is minimal.*

Example 12.3.

- *For every prime number* p, *the* p*-adic topology on* $\mathbb{Z}$ *is minimal.*
- $\mathbb{Q}$ *does not admit minimal group topologies.*

Minimal group topologies have been widely studied in literature, see for example [23].

A subgroup H of a topological group G is *essential (in G)* if $H \cap N \neq \{e_G\}$ for every closed normal subgroup $N \neq \{e_G\}$ of G.

Theorem 12.4 ([7]). *Let* H *be a dense subgroup of a topological group* G. *Then* H *is minimal if and only if* G *is minimal and* H *is essential in* G.

As an easy consequence of Theorem 12.4, we get the following result for direct products and sums of finite groups.

Corollary 12.5. *Let* $\{G_i \mid i \in I\}$ *be a non-empty family of groups,* $G = \prod_{i \in I} G_i$ *and* $S = \bigoplus_{i \in I} G_i$. *Consider the product topology* $\tau = \prod_{i \in I} \delta_{G_i}$ *on* G *and the induced topology* $\sigma = \tau \restriction_S$ *on* S. *Then:*

1. (G, τ) *is minimal.*
2. *If* G_i *is center-free for every* $i \in I$, *then* (S, σ) *is minimal.*

Proof.

1. The topology τ is minimal, being compact Hausdorff, by Proposition 12.2.
2. Note that S is dense and essential in G. By the previous point, we can apply Theorem 12.4.

□

12.1 Algebraically minimal groups

Definition 12.6. *We will call* algebraically minimal *an infinite group G such that $\mathfrak{M}_G$ is a (necessarily Hausdorff) group topology.*

Note that a group is algebraically minimal exactly when the poset of Hausdorff group topologies on G has a minimum element (i.e. $\mathfrak{M}_G$). In this case, $(G, \mathfrak{M}_G)$ is obviously minimal.

Groups G such that $\mathfrak{M}_G$ is a group topology (so Hausdorff) are quite hard to come by. The $\mathfrak{M}$-discrete groups trivially satisfy this condition, and here we resume some classes of algebraically minimal groups.

Example 12.7. *The following classes of groups are algebraically minimal:*

- *direct products and direct sums of finite, center-free groups, by Theorem 6.55;*
- *permutation groups (see Theorem 12.12);*
- *$\mathfrak{M}$-discrete groups.*

Note that Corollary 11.4 implies that an algebraically minimal group has finite center.

12.1.1 Permutation groups

In what follows, X is an infinite set. For a subgroup $G \leq S(X)$ of the permutation group of X, let $\tau_p(G)$ denote the *point-wise convergence topology of* G. The following classic result was proved by Gaughan in 1967.

Theorem 12.8 ([29])**.** *Let $G = S(X)$. Then $\tau_p(G)$ is contained in every Hausdorff group topology on G.*

In particular, from Theorem 12.8 it follows that $\mathfrak{M}_{S(X)} = \tau_p(S(X))$ is itself a Hausdorff group topology, hence $S(X)$ is algebraically minimal.

Let us see that from Theorem 12.8 it immediately follows Remus' answer to Markov's Third Problem about the connected topologization of groups (however, already solved by Pestov in [51]).

Theorem 12.9 ([55])**.** *If X is an infinite set, then $\mathfrak{M}_{S(X)}$ is totally disconnected, so every Hausdorff group topology on $S(X)$ is totally disconnected.*

Proof. By Theorem 12.8, the family $S_F(X)$, for $F \in [X]^{<\omega}$, is a local base for $\mathfrak{M}_{S(X)}$. As every $S_F(X)$ is a clopen subset of X, the topology $\mathfrak{M}_{S(X)}$ is totally disconnected.

Then every Hausdorff group topology on $S(X)$ is totally disconnected, being finer than $\mathfrak{M}_{S(X)}$. $\square$

Corollary 12.10 ([55]). *The group $S(X)$ does not admit any connected Hausdorff group topology, and if $|X| \geq \mathfrak{c}$ then every subgroup $\mathfrak{M}_{S(X)}$-closed has index at least $\mathfrak{c}$.*

Proof. Let $G = S(X)$. Since $\mathfrak{M}_G = \tau_p(G)$ by Theorem 12.8, every proper $\mathfrak{M}_G$-closed subgroup is the intersection of proper $\mathfrak{M}_G$-open subgroups.

So it suffices to ensure that every proper $\mathfrak{M}_G$-open subgroup has index at least $\mathfrak{c}$. Since every such subgroup is contained in a one-point stabilizer $S_{\{x\}}(X)$, for a $x \in X$ (these are maximal subgroups), it suffices to note that $S_{\{x\}}(X)$ has index $|X|$. □

Ten years after Gaughan's Theorem 12.8, Dierolf and Schwanengel (unaware of his result) proved the following:

Theorem 12.11 ([17]). *Let $S_\omega(X) \leq G \leq S(X)$. Then $\tau_p(G)$ is a minimal Hausdorff group topology.*

Note that for every group G, with $S_\omega(X) \leq G \leq S(X)$, one has $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \tau_p(G)$. Although Theorem 12.11 provides new results for groups $S_\omega(X) \leq G \lneq S(X)$, Theorem 12.8 gives a much stronger result for the whole group $S(X)$. That is why Dikranjan conjectured the following.

Conjecture 1 ([40]). *Let $S_\omega(X) \leq G \leq S(X)$. Then $\mathfrak{M}_G = \tau_p(G)$, so G is algebraically minimal.*

The following question was raised by Dikranjan and Shakhmatov (see Theorem 12.8).

Question 10 ([19]). *Does $\mathfrak{M}_{S(X)}$ coincide with $\mathfrak{Z}_{S(X)}$?*

It has recently turned out that Dikranjan's conjecture is true, and Dikranjan-Shakhmatov's question has a positive answer. Recently, it was proved in [4] that $\mathfrak{Z}_G = \mathfrak{M}_G$ is the pointwise convergence topology for all subgroups G of infinite permutation groups $S(X)$, that contain the subgroup $S_\omega(X)$ of all permutations of finite support.

Theorem 12.12 ([4]). *If $S_\omega(X) \leq G \leq S(X)$, then $\mathfrak{Z}''_G \subsetneq \mathfrak{Z}'_G = \mathfrak{Z}_G = \mathfrak{M}_G = \tau_p(G)$. In particular, G is algebraically minimal.*

Remark 12.13. In particular, every group G as in Theorem 12.12 is infinite and $\mathfrak{Z}$-Hausdorff, hence not $\mathfrak{Z}$-Noetherian. Compare this result with Example 10.9.

As a corollary of Theorem 12.12, the same authors have obtained the following answer to another question posed by Dikranjan-Shakhmatov.

Corollary 12.14 ([4]). *The class of groups G satisfying $\mathfrak{Z}_G = \mathfrak{M}_G$ is not closed under taking subgroups.*

Proof. Let H be a group such that $\mathfrak{Z}_H \neq \mathfrak{M}_H$, embed it in $G = S(H)$, and apply Theorem 12.12 to conclude that $\mathfrak{Z}_G = \mathfrak{M}_G$. $\square$

Remark 12.15. With the same idea of Corollary 12.14, one can produce plenty of examples of subgroups that behave badly with respect to Zariski, Markov and Hausdorff embeddings. For example, consider the group $H = \mathbb{Z}$. Then $\mathfrak{Z}_H = \mathfrak{M}_H$ by Theorem 4.11, and this topology is cofinite by Proposition 4.15. On the other hand, embedding H in $G = S(H)$, one has that $\mathfrak{Z}_G\restriction_H = \mathfrak{M}_G\restriction_H$ is a Hausdorff group topology, so that H is neither Zariski, nor Markov, embedded in G. By Proposition 5.10, H is neither Hausdorff embedded, nor super-normal in G.

12.1.2 When $\mathfrak{M}_G = \mathfrak{P}_G$ for algebraically minimal groups

Theorem 12.16. *Let G be an algebraically minimal group. Then the following conditions are equivalent.*

1. *$\mathfrak{M}_G$ is precompact,*
2. *$\mathfrak{P}_G$ is precompact,*
3. *G is not $\mathfrak{P}$-discrete.*

In this case, $\mathfrak{M}_G = \mathfrak{P}_G$.

Proof. (1) implies (2). If $\mathfrak{M}_G$ is precompact, then $\mathfrak{M}_G = \mathfrak{P}_G$, as $\mathfrak{P}_G$ is the intersection of all precompact Hausdorff group topologies. In particular, $\mathfrak{P}_G$ is precompact too.

(2) implies (3) is obvious, being G infinite.

(3) implies (1) follows from the fact that $\mathfrak{M}_G$ is Hausdorff and $\mathfrak{M}_G \subseteq \mathfrak{P}_G$.

We have also already seen that $\mathfrak{M}_G = \mathfrak{P}_G$ follows by item 1. $\square$

The next result is essentially a reformulation of Theorem 12.16, in which we explicitly state a dichotomy for $\mathfrak{P}_G$, whenever G is an algebraically minimal group.

Remark 12.17. *Let G be an algebraically minimal group.*

(a) *If $\mathfrak{M}_G$ is precompact, then $\mathfrak{M}_G = \mathfrak{P}_G$.*

(b) *If $\mathfrak{M}_G$ is non-precompact, then G is $\mathfrak{P}$-discrete.*

Remark 12.18.
1. The condition $\mathfrak{M}_G = \mathfrak{P}_G$ is weaker than the others in Theorem 12.16: the group G built by Ol′shanskij in Example 11.16 satisfies $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G = \delta_G$.
2. It becomes obviously equivalent to the others if the group is topologizable, i.e. not $\mathfrak{M}$-discrete.

3. By Theorem 12.12, for every G such that $S_\omega(X) \leq G \leq S(X)$, $\mathfrak{M}_G$ is a non-precompact Hausdorff group topology, so G is $\mathfrak{P}$-discrete. Then, $\mathfrak{Z}_G = \mathfrak{M}_G \neq \mathfrak{P}_G = \delta_G$ for such groups (and in particular for $G = S(X)$).

Recall that for a solvable divisible countable group G, the conditions of having $\mathfrak{M}_G = \mathfrak{P}_G$ and of being $\mathfrak{P}$-discrete are equivalent by Corollary 11.38. On the other hand, for an algebraically minimal group G those two conditions are alternative by Remark 12.17. In particular, the following result follows.

Corollary 12.19. *If G is a solvable divisible countable group, then it is not algebraically minimal.*

Proof. Let G be such a group, and assume G to be algebraically minimal. Then Corollary 11.38 and Remark 12.17 give a contradiction. □

13

Diagrams and (non-)implications

In what follows, we consider how some properties are related for infinite groups G.

All the implications in the following diagram are straightforward, and follow from the definitions. We comment the two non-implications below.

$$\begin{tikzcd}
 & & \text{a-min} \arrow[r] \arrow[dd, "/" marking, "(1)"'] & \text{not } \mathfrak{M}\text{-Noetherian} \arrow[dd] \\
\mathfrak{Z}\text{-discrete} \arrow[r] & \mathfrak{M}\text{-discrete} \arrow[ur] \arrow[dr] & & \\
 & & \mathfrak{P}\text{-discrete} \arrow[r] \arrow[uur, "/" marking, "(2)"'] & \text{not } \mathfrak{P}\text{-Noetherian}
\end{tikzcd} \tag{13.1}$$

(1). Consider the direct product $G = \prod_{i \in I} F_i$ of finite center-free groups. By Theorem 6.55 (a), $\mathfrak{M}_G = \mathfrak{P}_G$ is a compact Hausdorff group topology, hence non-discrete.

(2). Consider the group $H = H(1, \mathbb{Q})$, for which $\mathfrak{Z}_H = \mathfrak{M}_H$. By Proposition 8.21, H is $\mathfrak{Z}$-Noetherian, so $\mathfrak{M}$-Noetherian. Moreover, $\mathfrak{Z}_H = \mathfrak{M}_H \neq \mathfrak{P}_H = \delta_H$ by Proposition 11.39.

As $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$, if $\mathfrak{Z}_G = \delta_G$, then $\mathfrak{Z}_G = \mathfrak{M}_G = \delta_G$ trivially is a group topology. In the same way, as $\mathfrak{M}_G \subseteq \mathfrak{P}_G$, if $\mathfrak{M}_G = \delta_G$, then $\mathfrak{M}_G = \mathfrak{P}_G = \delta_G$ is a group topology. If $\mathfrak{P}_G = \delta_G$, then it is a Hausdorff group topology, hence non-Noetherian. This proves the vertical arrows and the horizontal arrows in the first row.

$$
\begin{tikzcd}
\mathfrak{Z}\text{-discrete} \arrow[rr] \arrow[dd] & & \mathfrak{M}\text{-discrete} \arrow[rr] \arrow[dl, "/"{marking}, "(b)"'] \arrow[dd] \arrow[dr] & & \mathfrak{P}\text{-discrete} \arrow[dd] \\
& \mathfrak{Z}_G = \mathfrak{M}_G \arrow[rr, bend left=10, "/"{marking}, "(d)"] \arrow[dddrrr, bend right, "/"{marking}, "(c)"] & & \mathfrak{M}_G = \mathfrak{P}_G \arrow[ddr, "/"{marking}, "(c)"] & \\
\mathfrak{Z}_G \text{ group topology} \arrow[rr, "(a)"'] \arrow[ur, "(a)"] & & \mathfrak{M}_G \text{ group topology} \arrow[rr, "(e)"'] \arrow[ur, "(e)\ \mathfrak{M}_G \text{ precomp.}"'] \arrow[uurr, bend right, "(e)\ \mathfrak{M}_G \text{ not precomp.}"] & & \mathfrak{P}_G \text{ group topology} \arrow[d] \\
& & & & \text{not } \mathfrak{P}\text{-Noetherian}
\end{tikzcd}
$$

(a). As $\mathfrak{Z}_G \subseteq \mathfrak{M}_G$, if $\mathfrak{Z}_G$ is a group topology, then it is Hausdorff (as $\mathfrak{Z}_G$ always is T_1) and $\mathfrak{Z}_G = \mathfrak{M}_G$ by definition of $\mathfrak{M}_G$.

(b). Recall Hesse'e Examples.

(c). Recall that $\mathfrak{Z}_G = \mathfrak{M}_G = \mathfrak{P}_G$ is Noetherian for every abelian group G, by Theorem 4.10.

(d). See the counter-example produced to show the non-implication (2) in diagram (13.1).

(e). If $\mathfrak{M}_G$ is a group topology, then Remark 12.17 applies, giving that either $\mathfrak{M}_G$ is precompact and $\mathfrak{M}_G = \mathfrak{P}_G$, or $\mathfrak{M}_G$ is non-precompact and $\mathfrak{P}_G = \delta_G$. In both cases, $\mathfrak{P}_G$ is a group topology.

The horizontal arrows in the first and second row of the following diagram have already been discussed in the diagram above, as well as the vertical arrows in the first column.

The non-implications (1) and (2) were already in diagram (13.1), and they have already been discussed there.

All the other implications are straightforward, and follow from the inclusions $\mathfrak{Z}_G \subseteq \mathfrak{M}_G \subseteq \mathfrak{P}_G$ and the definitions of discrete, Hausdorff, Noetherian, group, topology.

$$
\begin{tikzcd}
\mathfrak{Z}\text{-discrete} \arrow[r] \arrow[d] & \mathfrak{M}\text{-discrete} \arrow[r] \arrow[d] & \mathfrak{P}\text{-discrete} \arrow[r, leftrightarrow] \arrow[d] \arrow[ddl, "/" marking, "(2)"] & \text{not } \mathfrak{P}\text{-precompact} \\
\mathfrak{Z}_G \text{ group topology} \arrow[r] \arrow[d] & \mathfrak{M}_G \text{ group topology} \arrow[r] \arrow[d] \arrow[ur, "/" marking, "(1)"] & \mathfrak{P}_G \text{ group topology} \arrow[d] & \\
\mathfrak{Z}\text{-Hausdorff} \arrow[r] \arrow[d] & \mathfrak{M}\text{-Hausdorff} \arrow[r] \arrow[d] & \mathfrak{P}\text{-Hausdorff} \arrow[d] & \\
\text{not } \mathfrak{Z}\text{-Noetherian} \arrow[r] & \text{not } \mathfrak{M}\text{-Noetherian} \arrow[r] & \text{not } \mathfrak{P}\text{-Noetherian} &
\end{tikzcd}
$$

14
Open questions

Question 11 ([20, 25]). *Let H be a (normal) subgroup of a group G. If H is Zariski embedded in G, must it also be Markov embedded in G?*

Obviously, Question 11 has a positive answer if $\mathfrak{Z}_G = \mathfrak{M}_G$, by Fact 5.12 (b).

Recall that if K is a field with $\operatorname{char} K = 0$, the nilpotent group $H(n, K)$ is $\mathfrak{P}$-discrete by Proposition 11.39, but not $\mathfrak{Z}$-discrete (being $\mathfrak{Z}$-Noetherian by Example 10.2 (a)).

Question 12 ([21, 25]). *If G is a nilpotent group, does $\mathfrak{Z}_G = \mathfrak{M}_G$ necessarily hold?*

Question 13 ([24, Question 7.4]). *Does there exist a group G such that $\mathfrak{Z}_G \neq \mathfrak{M}_G \neq \mathfrak{P}_G$? Can it be chosen solvable?*

For the following question, see Proposition 9.2 and Corollary 9.3.

Question 14 ([24, Question 7.5]). *Let K be a field. Does the conclusion of Proposition 9.2 holds, also in the general case (i.e., if $\operatorname{char} K = p > 0$ and K is an algebraic extension of $\mathbb{F}_p$)?*

Corollary 9.7 leaves open the following:

Question 15 ([25, Question 8.11]). *If (†) fails (i.e. $\operatorname{char} K = p$ and K is an algebraic extension of $\mathbb{F}_p$), is N_K Hausdorff embedded in G_K?*

Of course, here $\dim_K V < \infty$ and K is infinite (as the case of infinite $\dim_K V$ is covered by Corollary 9.10).

Recall that if G_K is countable, then N_K is Markov embedded in G_K by Remark 9.19 (c). On the other hand, if K is uncountable, then (†) obviously holds, so N_K is not Hausdorff embedded in G_K by Corollary 9.8. But one can still ask:

Question 16 ([24, Question 7.7]). *If G_K is not countable, is N_K Markov embedded in G_K?*

A positive answer to Question 16 would give that N_K is both Zariski and Markov embedded, but not Hausdorff embedded in G_K, as for the countable case.

A negative one would give that N_K is Zariski embedded, but not Markov embedded in G_K. In particular, the Zariski and Markov topologies of G_K would differ (on N_K) by Fact 5.12 (b).

We have no example of a $\mathfrak{Z}$-Noetherian group that is not $\mathfrak{M}$-Noetherian. This is why we ask the following question.

Question 17. Is a $\mathfrak{Z}$-Noetherian group necessarily $\mathfrak{M}$-Noetherian?

Obviously, a negative answer to Question 17 requires a $\mathfrak{Z}$-Noetherian group G such that $\mathfrak{Z}_G \neq \mathfrak{M}_G$. We do not even know if such groups exist, so we ask also the following question.

Question 18. Does the equality $\mathfrak{Z}_G = \mathfrak{M}_G$ hold true for a $\mathfrak{Z}$-Noetherian group G? What about an $\mathfrak{M}$-Noetherian group G?

In particular, a positive answer to the first part of Question 18 will provide a positive answer to Question 17.

We do not know if $\mathfrak{Z}$-compactness implies $\mathfrak{M}$-compactness, so we ask the following (see also Question 17):

Question 19. Is every $\mathfrak{Z}$-compact group also a $\mathfrak{M}$-compact group?

Finally, compare the question below with Question 18.

Question 20. Does the equality $\mathfrak{M}_G = \mathfrak{Z}_G$ hold true for a $\mathfrak{Z}$-compact group G? What about an $\mathfrak{M}$-compact group G?

Question 21. If $\mathfrak{M}_G$ is a Hausdorff topology on a group G, is it necessarily a group topology?

Next we report some still open questions and unsolved problems from [20] and [25].

Problem 3 ([20, 25])**.** *Describe the class of groups G such that G is Markov (respectively, Hausdorff) embedded in every group that contains G as a normal subgroup.*

Question 22 ([25])**.** *Let G be an infinite group. Is $\mathfrak{Z}_G$ (resp. $\mathfrak{M}_G$, $\mathfrak{P}_G$) a group topology, if it is Hausdorff?*

We conclude this final chapter with a conjecture on groups in $\mathscr{N}_2$. Recall that when G is abelian, then $\mathfrak{Z}_G$ is Hausdorff if and only if G is finite.

Conjecture 2. *If $G \in \mathscr{N}_2$ is infinite, then G is not $\mathfrak{Z}$-Hausdorff.*

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