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**FIXED POINTS FOR PLANAR
TWIST-MAPS**

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ABSTRACT

The main topic of this thesis is the study of the existence of fixed points for planar maps defined on topological annuli and satisfying the so-called twist-condition which prescribes that the maps rotate the two boundaries of their domain in opposite direction.

Beginning with a survey about the Poincaré-Birkhoff theorem, which is the most important and classical result on fixed points for planar twist homeomorphism, we present also some more general results for continuous twist maps, achieved by the use of topological “crossing” properties of annular domains.

SOMMARIO

L'argomento principale di questa tesi è lo studio dell'esistenza di punti fissi per mappe definite su anelli topologici, che soddisfino la condizione di twist alle frontiere; si richiede cioè che le mappe in questione ruotino le frontiere dell'anello su cui sono definite in direzioni opposte.

Iniziando con un'esposizione del teorema di Poincaré-Birkhoff – che costituisce il più importante risultato sui punti fissi degli omeomorfismi twist del piano – vengono successivamente esposti alcuni risultati riguardanti mappe twist delle quali si assume solamente la continuità; tali risultati sono stati dimostrati usando alcuni lemmi topologici riguardanti proprietà di “attraversamento” degli anelli.

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INTRODUCTION

Twist maps are a class of continuous applications defined on annular domains, which have the property of rotating the two boundaries of the annulus in opposite directions. The investigation of twist maps is a relevant topic in the study of dynamical systems in two-dimensional manifolds; indeed they naturally appear in a broad number of situations (from KAM theory to the study of some geometrical configurations involving the presence of Smale's horseshoes) and thus they have been widely considered both from the theoretical point of view and for their lead role in various applications, ranging from celestial mechanics to fluid dynamics.

One of the classical and most important examples of a fixed point theorem concerning twist maps on the annulus is the *Poincaré-Birkhoff twist theorem*, also known as Poincaré's last geometric theorem, whose 100th birthday is celebrated just this year. The theorem asserts the existence of at least two fixed points for an area-preserving homeomorphism φ of a closed planar annulus

$$A[a, b] = \{z \in \mathbb{R}^2 : a \leq \|z\| \leq b\}$$

(with $0 < a < b$) onto itself which leaves the inner boundary $A_i = \{z \in A[a, b] : \|z\| = a\}$ and the outer boundary $A_o = \{z \in A[a, b] : \|z\| = b\}$ invariant and rotates A_i and A_o in opposite directions (this is the so-called *twist condition at the boundary*).

The Poincaré-Birkhoff fixed point theorem was stated (and proved in some special cases) by Poincaré in 1912, the year of his death. In 1913 G.D. Birkhoff gave a proof of the existence of one fixed point with an ingenious application of the index of a vector field along a curve. A complete description of Birkhoff's approach, also explaining how to obtain a second fixed point, was afterwards provided in the expository article by Brown and Neumann [14]. The history of this theorem and its generalizations and developments is quite interesting but impossible to summarize in few lines; therefore the first part of this thesis (chapter 2) is entirely devoted to the Poincaré-Birkhoff theorem. Starting from the original words by Poincaré, I made the effort of summing up its one-century-long history, as well as providing a survey of its most important extensions and proofs; a section of the chapter is devoted also to setting out some open problems recently arised. Indeed, after so many years of studies on this topic, some controversial proofs of its generalizations have been settled only recently, making this classical topic still a prolific ground for new discoveries. Finally, in the

last section an application to the problem of the existence of periodic and subharmonic solutions for planar systems of Lotka-Volterra type is investigated.

In chapter 3 the attention is focused on some topological properties of planar regions homeomorphic to the annulus, which represent the domains on which the Poincaré-Birkhoff theorem can be applied. Taking as a base point some results already obtained in the last ten years for rectangular regions, the same results are here transferred into the framework of annular domains. The tools therein developed, which we refer to as *crossing lemmas*, are of great use in order to obtain a fixed point theorem for a class of maps defined on annular regions; these maps are named bend-twist maps and their most interesting feature is the fact that their definition requires weaker hypothesis than the maps which the Poincaré-Birkhoff theorem applies to.

Having the crossing lemmas both for rectangles and for annuli as a starting point, in the next two chapters we apply them to the proof of some fixed point results, making a parallelism between the two frameworks under consideration. In chapter 4 we recall some results about the existence of fixed points for maps defined on regions homeomorphic to the unit square and which are expansive along one direction (we will say that they satisfies the *stretching-along-the-paths property*). As already exposed in [79, 80, 82, 91, 84], this property allows to prove the existence of fixed points, as well as periodic points and chaotic dynamics, in a quite easy way, making use only of elementary tools of planar topology. At the end of the chapter, an application to the pendulum equation is proposed.

Finally, chapter 5 goes back to the study of twist maps of the annulus. More precisely, it deals with the so-called *bend-twist maps*, which are a particular class of twist maps whose radial component changes its sign on the domain. They were first presented by T. Ding in [26], who formulated some fixed point results in the analytic setting. In the present thesis, as well as in the already published work [88], the topological tools developed in chapter 3 allow us to reformulate Ding's results in the more general continuous setting. Indeed we obtain an interesting fixed point theorem for bend-twist maps, which can be applied to situations where the non-conservative behaviour of the system under consideration prevents the possibility of applying the Poincaré-Birkhoff theorem, as explained by the examples in the final section 5.2.

The natural field on which all the techniques herein exposed find useful applications is the study of the existence and multiplicity of periodic solutions for Hamiltonian systems, which represent a classical area of research already widely investigated. Depending on the situation arising, when we are studying the dynamics of a second order ODE (ordinary differential equation), whose Poincaré map enters in

the setting of twist maps, three different approaches to the problem can be used, namely the Poincaré-Birkhoff theorem, the linked-twist maps and the bend-twist maps.

Here we will focus our attention to the case of *nonautonomous planar* Hamiltonian systems of the form

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(t, x, y) \\ \dot{y} = -\frac{\partial H}{\partial x}(t, x, y). \end{cases} \quad (1.1)$$

Such kind of equations are relevant not only for their intrinsic interest from the point of view of the applications, but also because they represent a common ground where several different techniques, ranging from nonlinear analysis (for instance, critical point theory) to the theory of dynamical systems, can compete in order to produce new results.

Here and in what follows we suppose that $H : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function which is T -periodic in its first variable, that is

$$H(t + T, x, y) = H(t, x, y) \quad \forall t, x, y \in \mathbb{R}$$

and sufficiently smooth with respect to x and y in order to guarantee the uniqueness of the solutions for the initial-value problems associated to system (1.1). Some discontinuities in the t -variable may be allowed, provided that the solutions are considered in the Carathéodory sense (see [42]). For instance, as an example of (1.1) we can study (in the phase plane) the periodically perturbed scalar nonlinear second order ODE

$$\ddot{x} + f(x) = p(t), \quad (1.2)$$

or

$$\ddot{x} + p(t)f(x) = 0, \quad (1.3)$$

with $f : \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz function and $p : \mathbb{R} \rightarrow \mathbb{R}$ a T -periodic function with $p \in L^1([0, T])$.

A classical approach to the search of periodic solutions of system (1.1) is the study of the existence and multiplicity of fixed points and periodic points for its Poincaré map. The Poincaré map associated to (1.1) is the function which maps a point $z_0 = (x_0, y_0) \in \mathbb{R}^2$ to the point

$$\Phi(z_0) = (x(T; t_0, z_0), y(T; t_0, z_0)),$$

where $\zeta(t; z_0) = (x(t; t_0, z_0), y(t; t_0, z_0))$ is the solution of (1.1) satisfying the initial condition $\zeta(t_0; z_0) = z_0$. Usually, the natural choice $t_0 = 0$ is made; in that case we use the simplified notation $(x(t; z_0), y(t; z_0)) = (x(t; t_0, z_0), y(t; t_0, z_0))$.

Since we assume the uniqueness of the solutions for the Cauchy problems associated to (1.1), from the fundamental theory of ordinary

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differential equations, we know that Φ is a continuous map, defined on an open subset $\Omega = \text{dom } \Phi \subset \mathbb{R}^2$. Actually Φ is a *homeomorphism* of Ω onto $\Phi(\Omega)$ which is also *orientation-preserving* and *area-preserving* (this latter property follows from Liouville theorem and from the fact that the right-hand side of equation (1.1) is given by a zero-divergence vector field).

The Poincaré-Birkhoff fixed point theorem is an important tool to detect fixed and periodic points for area-preserving homeomorphisms of the plane. In this kind of applications of the Poincaré-Birkhoff theorem usually one has to deal with annular regions homeomorphic to A having inner and outer boundaries not necessarily invariant; therefore we have to use some recent generalizations of the theorem which require that the inner and outer boundaries are strictly star-shaped with respect to some point. The key fact, however, is the possibility to define a suitable lifting of φ to a covering space of the annulus using the standard polar coordinates or some modifications of them, for instance suitably chosen action-angle variables. A natural choice consists in the couple in which the time variable and the energy of the orbit play the role of angle and radius, respectively.

In general, we need to consider annular regions which are not necessarily centered at the origin; to this aim, we introduce the following notation. Given a point $P \in \mathbb{R}^2$, we define

$$A(P) = A[a, b; P] = P + A[a, b]$$

whose inner and outer boundaries will be named as

$$A_i(P) = P + C_a \quad \text{and} \quad A_o(P) = P + C_b.$$

A possible way to verify the twist condition for the Poincaré map of system (1.1) is based on the study of some rotation numbers associated to its solutions which provide some information about the displacement of the angular coordinate.

To begin with, we describe an elementary procedure to introduce some rotation numbers. We fix a point $P = (x_p, y_p)$ and consider a system of polar coordinates around P (typically, we will have $P = (0, 0)$). Suppose that for some $z_0 \in \mathbb{R}^2$, the solution $\zeta(t; z_0) = (x(t; z_0), y(t; z_0))$ satisfies

$$\zeta(t; z_0) \neq P, \quad \forall t \in [0, \tau],$$

for some $\tau > 0$. Passing to the polar coordinates

$$x = x_p + \sqrt{2\rho} \cos \vartheta, \quad y = y_p + \sqrt{2\rho} \sin \vartheta$$

we obtain

$$-\dot{\vartheta}(t) = \frac{\dot{x}(t)(y(t) - y_p) - \dot{y}(t)(x(t) - x_p)}{(x(t) - x_p)^2 + (y(t) - y_p)^2}$$

and thus we can define the number $\text{rot}(t, z_0, P) =$

$$\frac{1}{2\pi} \int_0^t \frac{(y(s) - y_p) \frac{\partial H}{\partial y}(s, x(s), y(s)) + (x(s) - x_p) \frac{\partial H}{\partial x}(s, x(s), y(s))}{(x(s) - x_p)^2 + (y(s) - y_p)^2} ds$$

for $t \in [0, \tau]$ and $(x(t), y(t)) = (x(t; z_0), y(t; z_0))$.

The *rotation number* $\text{rot}(t, z_0, P)$ counts the number of windings of the solution around the point P , in the clockwise sense, during the time interval $[0, t]$. If the above rotation number is defined for $t = mT$ (for some integer $m \geq 1$) and for all the points of the annulus $A(P)$, the twist condition of the Poincaré-Birkhoff theorem for the map $\varphi = \Phi^m$ can be expressed as follows

$$\begin{cases} \text{rot}(mT, z, P) > j & \text{for } z \in A_i(P) \\ \text{rot}(mT, z, P) < j & \text{for } z \in A_o(P) \end{cases} \quad (1.4)$$

(or viceversa), for some $j \in \mathbb{Z}$. The existence of a fixed point for φ (coming from the original version of the theorem or from some of its variants) provides a point z^* in the interior of the annulus which is the initial point of a mT -periodic solution of (1.1) and such that

$$\text{rot}(mT, z^*, P) = j. \quad (1.5)$$

The additional information expressed by relation (1.5) can be exploited in order to obtain multiplicity results or some precise information about the solution. Indeed if there exist two real values $c_1 < c_2$ such that $\text{rot}(mT, z, P) < c_1$ on $A_i(P)$ and $\text{rot}(mT, z, P) > c_2$ on $A_o(P)$, then for every integer $j \in [c_1, c_2] \cap \mathbb{Z}$ there exist at least two mT -periodic solutions of (1.1) which perform j turns around the origin during the time interval $[0, mT]$. In connection with this kind of results, dealing with the existence of infinitely many periodic solutions of the second order ODE $\ddot{x} + f(t, x) = 0$, as well as the existence of subharmonic solutions, we mention the result by Moser and Zehnder in [73, Section 2.10], in which a modification of the Hartmann-Jacobowitz theorem is proved.

The study of twist maps is not only a crucial step in the applications of the Poincaré-Birkhoff theorem to planar Hamiltonian systems. In the past decades a grown interest has been devoted to the study of the so-called *linked twist maps* (from now on abbreviated as LTMs). A typical linked twist map of the plane, as presented by Devaney in [25], can be described as a composition of the form

$$\Psi = \Psi_2^k \circ \Psi_1^\ell,$$

where Ψ_1 and Ψ_2 are twist maps which act on two different annuli $A(P_1)$ and $A(P_2)$, respectively. Usually, one also assume that both Ψ_1 and Ψ_2 perform rotations of angles which are multiple of 2π on the

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boundaries of the annuli; in this way, the maps can be extended as identities outside the annuli. If $A(P_1)$ and $A(P_2)$ cross each other in a proper way, then Ψ has a rich dynamics. The correct crossing of the two annuli $A[a_1, b_1; P_1]$ and $A[a_2, b_2; P_2]$ is usually described by the relations

$$\max\{b_2 - a_1, b_1 - a_2\} < \text{dist}(P_1, P_2) < a_1 + a_2$$

so that linked twist maps can be interpreted as a class of homeomorphisms of the two-disk minus three holes [25] (see figure 1).

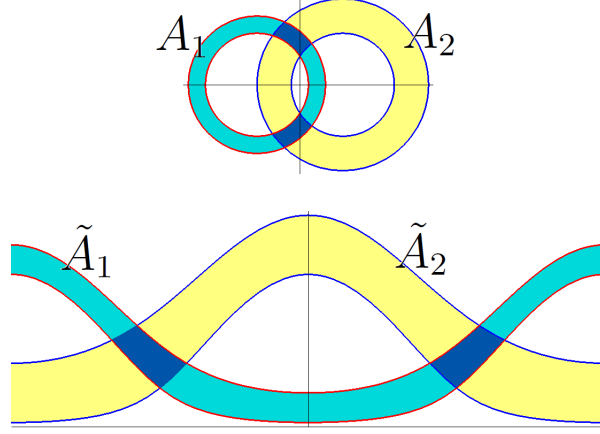


Figure 1: Example of two standard linked annuli A_1 and A_2 . For the figure we have taken $A_i = A[a_i, b_i; P_i]$ with $P_1 = (-5, 0)$, $P_2 = (5, 0)$, $a_1 = 6$, $b_1 = 8$, $a_2 = 6$, $b_2 = 10$. The sets \tilde{A}_1 and \tilde{A}_2 are the annuli A_1 and A_2 viewed from the origin (the scale ratio between the two axes is not respected). Since $(0, 0)$ lies in the intersection of the bounded components of $\mathbb{R}^2 \setminus A_i$ (for $i = 1, 2$), using the usual polar coordinates (ϑ, r) with respect to the origin, we can lift both A_1 and A_2 as 2π -periodic strips bounded between graphs of functions $r = r(\vartheta)$. In this specific case, we have $\tilde{A}_i = \{(\vartheta, r) : x_{P_i} \cos \vartheta + (a_i^2 - x_{P_i}^2 \sin^2 \vartheta)^{1/2} \leq r \leq x_{P_i} \cos \vartheta + (b_i^2 - x_{P_i}^2 \sin^2 \vartheta)^{1/2}\}$, for $P_i = (x_{P_i}, 0)$, $i = 1, 2$.

Examples of LTMs on some manifolds (like the sphere or the torus) have been considered as well (see [103, 104] and the references therein). However, if, instead of annuli of the form $A(P_i)$, we have more general annular regions on which two twist maps act, the linking condition can be more general (see figure 2). LTMs in such more general setting have been recently considered in [65].

A natural way to produce a twist-type Poincaré map associated to (1.1) occurs when the nonautonomous system can be viewed as a perturbation of an autonomous planar system presenting a center-like structure as

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x, y). \end{cases} \quad (1.6)$$

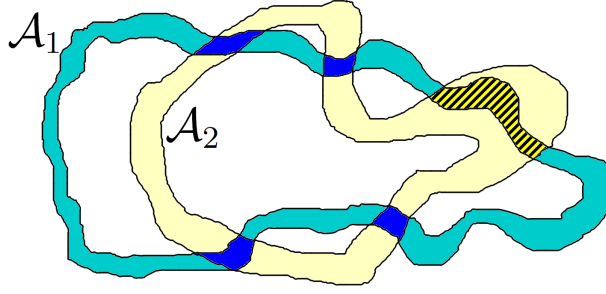


Figure 2: Example of two linked planar topological annuli A_1 and A_2 . Among the five rectangular regions resulting from the intersection of the two annuli, only the four regions in darker color are suitable for a generalized version of LTMs theory as described in [65], while the set painted with zebra stripes does not fit in that framework.

Suppose that there exists a *topological annulus* A (that is a compact subset of \mathbb{R}^2 homeomorphic to a standard annulus $A[a, b]$) which is filled by closed (periodic) orbits of system (1.6). Since the trajectories of (1.6) lie on the level lines of the Hamiltonian, we can parameterize every orbit Γ in A by means of the value $H(\Gamma) = c$. Under mild assumptions on H (H of class \mathcal{C}^1 with $\nabla H(x, y) \neq 0$ for all $(x, y) \in A$, see [49]) it is possible to prove the continuity of the function which maps c into the period τ_c of the closed orbit in A at level c . One can also find a compact interval $[a, b]$ such that the inner and the outer boundaries of A correspond to the level lines $H = a$ and $H = b$ (we can always enter in this situation possibly replacing H with $-H$). In this way, the set A becomes a standard annulus of the form $A = A[a, b]$, with the level of the Hamiltonian playing the role of a radial coordinate. Angular-type coordinates can be introduced using a normalized time along the trajectories, counted from a suitable arc transversal to the annulus (such arc is obtained as a flow-line of the gradient system $\dot{z} = \nabla H(z)$).

If we denote by Φ_H the Poincaré map associated to (1.6), for a fixed time $T > 0$, we can produce a twist condition on A whenever

$$\tau_a \neq \tau_b.$$

Indeed, suppose that $\tau_a < \tau_b$ and fix $m \geq 1$ such that the set

$$Z(m) = \left] \frac{mT}{\tau_b}, \frac{mT}{\tau_a} \right[\cap \mathbb{Z}$$

is nonempty. For each $j \in Z(m)$ the points of the inner boundary A_i of A wind more than j times in the time interval $[0, mT]$. On the other hand, the points of the outer boundary A_o have a number of rotations strictly less than j . This simple observation guarantees that a twist condition analogous to (1.4) holds for Φ_H^m relatively to A . This

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fact will imply a twist condition for Φ^m if the vector field in (1.1) is sufficiently close to that of (1.6) on $[0, mT] \times A$. A recent investigation in this direction, using the Poincaré-Birkhoff fixed point theorem has been performed in [33].

A possible way to produce a linked twist map configuration in the plane is given by a pair of planar autonomous Hamiltonian systems which periodically switch back and forth from one to the other. More precisely, fix $T_1, T_2 > 0$ with

$$T_1 + T_2 = T$$

and consider the systems

$$\begin{cases} \dot{x} = \frac{\partial H_1}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H_1}{\partial x}(x, y) \end{cases} \quad \text{for } t \in [0, T_1[\quad (1.7)$$

and

$$\begin{cases} \dot{x} = \frac{\partial H_2}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H_2}{\partial x}(x, y) \end{cases} \quad \text{for } t \in [T_1, T[, \quad (1.8)$$

repeating then such process in a periodic fashion (for an application to fluid mixing, see [108, Appendix B]). Allowing a discontinuity for $t \equiv 0$ and $t \equiv T_1 \pmod{T}$, the resulting system may be interpreted as a special case of equation (1.1). We enter in the generalized LTMs framework considered in [65] whenever there exist two annular regions A_1 and A_2 filled by periodic orbits of systems (1.7) and (1.8), respectively, and such that A_1 and A_2 link each other in a suitable sense (see figure 2). Moreover, appropriate twist conditions on each of the two annuli should be required.

The situations which enter in the setting of bend-twist maps are intermediate between the twist maps arising in the applications of the Poincaré-Birkhoff theorem and the LTMs; the theory of bend-twist maps is a powerful tool when one has to deal with dissipative systems, whose Poincaré map is not a homeomorphism. An example in this direction will be provided in the last section of this work (see section 5.2).

2 | THE POINCARÉ-BIRKHOFF THEOREM: ONE CENTURY OF RESEARCH

My interest in the Poincaré-Birkhoff theorem is motivated by the contrast between the plainness of its statement and its troubled history, which seems to be widely open also after one century of research on it. The Poincaré-Birkhoff theorem, named also *twist theorem*, is a fixed point result for *area-preserving twist homeomorphisms* of an annulus in the plane \mathbb{R}^2 . It was conjectured by Henri Poincaré (1854-1912) and appeared for the first time in a paper of his [94] in 1912, only few days before his death; indeed it is also known as *Poincaré's last geometric theorem*.¹

Poincaré had been teaching theoretical astronomy and celestial mechanics since 1896; in those years he developed the use of some topological tools (like Kronecker's index) for the search of singular points and limit cycles of differential equations, periodic solutions for the three-body problem and bifurcation of the equilibrium shapes of a rotating fluid. The idea of the theorem we are talking about was in particular motivated by his research on the restricted three-body problem; Poincaré proved the existence of periodic solutions for the three-body problem, in the case in which the masses of the bodies were small. On the other hand, he observed that in the case of big masses, the existence of periodic solutions would have been guaranteed if a particular fixed point result (theorem 2.2) had been true. In [94], Poincaré described his fixed point theorem and exposed some of its possible applications; he also managed to check the validity of his result in many particular cases, but he could not exhibit a general proof; indeed in the first lines of the paper he apologized to the readers for publishing so an incomplete work, saying what follows.

Je n'ai jamais présenté au public un travail aussi inachevé; je crois donc nécessaire d'expliquer en quelques mots les raisons qui m'ont déterminé à le publier, et d'abord celles qui m'avaient engagé à l'entreprendre.

He was sure that the general version of the theorem was true, but he had to leave to anyone else the task of proving it, because he knew that he would not have had enough time to complete his work.

¹ An interesting biographic survey about Poincaré's figure can be found in [69].

2 The Poincaré-Birkhoff theorem: one century of research



Figure 3: Henri Poincaré (1854-1912)

Ma conviction qu'il est toujours vrai s'affermisait de jour en jour, mais je restais incapable de l'asseoir sur des fondements solides.

He said that the result was so important and rich of interesting consequences and possible applications, that he had decided to publish it, although with some reluctance.

D'un autre côté, l'importance du sujet est trop grande (et je chercherai plus loin à la faire comprendre) et l'ensemble des résultats obtenus trop considérable déjà, pour que je me résigne à les laisser définitivement infructueux.

Poincaré's theorem applies to area-preserving homeomorphisms defined on a planar annulus; its main hypothesis is the so-called *twist condition*, which prescribes that the inner boundary of the annulus is moved in the clockwise sense, while the outer one is moved in the counter-clockwise sense (see figure 4).

Before presenting the statement of the theorem, we are giving the precise definition of what we mean as a planar annulus, which will be the main work setting.

Definition 2.1 A set $A \subset \mathbb{R}^2$ is a (non-degenerate) *planar annulus* if

$$A = A[a, b] = \{(x, y) \in \mathbb{R}^2 : a^2 \leq x^2 + y^2 \leq b^2\}$$

for some $0 < a < b$. Its boundary ∂A consists in two disjoint circles $\partial B(0, a)$ and $\partial B(0, b)$ which are called the inner and the outer boundary and denoted by A_i and A_o respectively, so that

$$A_i = aS^1 = C_a \quad \text{and} \quad A_o = bS^1 = C_b.$$

Poincaré's fixed point theorem in its original formulation reads as follows.

Theorem 2.2 *Let $\varphi : A \rightarrow A$ be a homeomorphism of a planar annulus A onto itself which leaves the boundaries invariant and rotates them in different directions (say, it is a twist homeomorphism). If φ is area-preserving, then φ has at least two fixed points in the interior of A .*

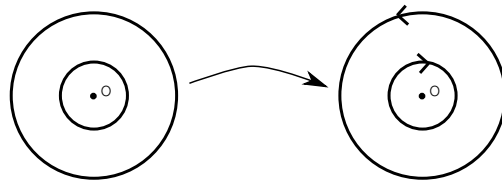


Figure 4: A pictorial description of the twist condition

2.1 COVERINGS AND LIFTINGS

In the statement of Poincaré's theorem we have introduced and used the concept of twist homeomorphism, using an intuitive definition. We need now to make clearer and more precise what we mean by a *twist homeomorphism*. In order to provide a precise definition of this class of applications, we observe that a non-degenerate annulus A is always contained in the holed plane $\mathbb{R}^2 \setminus \{O\} = \mathbb{R}_0^2$; hence we can consider its lifting and move into the setting of the universal covering of \mathbb{R}_0^2 . Although these are standard arguments of algebraic topology, for sake of completeness we are going to present a short survey about the theory of covering spaces.²

Definition 2.3 Let \tilde{X} and X be two topological spaces. A *covering map* is a continuous map $\pi : \tilde{X} \rightarrow X$ such that

- π is surjective
- for every $x \in X$ there exists an open neighbourhood U of x and a corresponding family $\{U_j | j \in J\}$ of open neighbourhoods in \tilde{X} such that

² Definitions and theorems are borrowed from the books [40, 57]

2 The Poincaré-Birkhoff theorem: one century of research

- $\pi^{-1}(U) = \bigcup_{j \in J} U_j$
- $U_k \cap U_\ell = \emptyset$ if $k \neq \ell$
- $\pi|_{U_j} : U_j \rightarrow U$ is a homeomorphism for every $j \in J$.

Definition 2.4 Given a topological space X , we say that the pair (\tilde{X}, π) is a *covering space* of X if

- \tilde{X} is a topological space
- $\pi : \tilde{X} \rightarrow X$ is a covering map.

If \tilde{X} is connected, then we say that the covering is connected. If \tilde{X} is simply connected, then (\tilde{X}, π) is the universal covering of X .

One of the most classical examples of covering maps is the projection in polar coordinates; this is also the covering map we will consider talking about Poincaré's theorem. It is well known that we can introduce a system of polar coordinates (ϑ, r) on \mathbb{R}_0^2 ; if we denote by (x, y) the standard cartesian coordinates of the plane, the change of variables is expressed by the relations $x = r \cos \vartheta, y = r \sin \vartheta$, with $\vartheta \in \mathbb{R}$ and $r \in \mathbb{R}^+$, which allows to define the projection

$$\pi : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}_0^2 \quad \text{such that} \quad (\vartheta, r) \mapsto (x, y)$$

with

$$\pi(\vartheta, r) = (r \cos \vartheta, r \sin \vartheta).$$

In this sense, the infinite strip $\mathbb{R} \times \mathbb{R}^+$ is a covering of the holed plane \mathbb{R}_0^2 , via the covering projection π . Moreover, $(\mathbb{R} \times \mathbb{R}^+, \pi)$ is the universal covering of \mathbb{R}_0^2 .

Definition 2.5 Let (\tilde{X}, π) be a covering space of X and let $A \subset X$ be a subset of X . A map $\vartheta : A \rightarrow \tilde{X}$ is a *local section* if it is continuous and $\pi \circ \vartheta = \text{id}|_A$. If $A = X$, we say that ϑ is a global section.

Recalling definition 2.3, we know that for every point $p \in X$ there exists a family of neighbourhoods $\bigcup U_j \supset \pi^{-1}(p)$ such that for every index $j \in J$ the map

$$\pi|_{U_j} : U_j \rightarrow U$$

is a homeomorphism and, therefore, it is invertible; hence we can define the map

$$\vartheta_j = (\pi|_{U_j})^{-1} : U \rightarrow U_j \subset \tilde{X}$$

and observe that ϑ_j is a local section defined on U . Therefore, for every point $p \in X$ there exists a neighbourhood U on which it is possible to define infinite local sections; on the other hand the existence of a global section is not guaranteed in general.

2.1 Coverings and liftings

Note that, if a global section ϑ exists, the maps π and ϑ are one-to-one continuous maps and therefore every point $p \in X$ has exactly one inverse image under the covering map π which turns out to be a homeomorphism between \tilde{X} and X . The number of inverse images of a point is used to define the degree of the covering.

Definition 2.6 If (\tilde{X}, π) is a connected covering of X such that $\#(\pi^{-1}(p)) = n$ for every $p \in X$, then we say that the covering has *degree* n .

If the degree is greater than 1, then no global sections exist. Going back to the example of \mathbb{R}_0^2 , the covering space $(\mathbb{R} \times \mathbb{R}^+, \pi)$ has infinite degree, because for every $z = (x, y) \in X$ the set

$$\pi^{-1}(z) = \{(\vartheta, r) : \vartheta = \arctan \frac{y}{x} + 2k\pi, r = \|z\|, k \in \mathbb{Z}\}$$

has cardinality equal to \aleph_0 . Hence, in this case a global section does not exist.

The problem of the existence of a section is a particular case of the problem of the lifting.

Definition 2.7 Let $\pi : \tilde{X} \rightarrow X$ be a covering map and let $f : Y \rightarrow X$ be a continuous function. A continuous function $\tilde{f} : Y \rightarrow \tilde{X}$ is a *lifting* of f if

$$\pi \circ \tilde{f} = f.$$

If Y is a connected topological space and \tilde{f}, \tilde{f}' are two liftings of f which coincide on one point $y_0 \in Y$, then $\tilde{f}(y) = \tilde{f}'(y)$ for every $y \in Y$.

Given a covering projection π , the existence of the lifting of a map f is not guaranteed in general.

We are going now to spend some pages on exposing the theorem about the existence of the lifting. To begin with, we introduce some notations. Let X be a topological space and let $x_0 \in X$. We say that ω is a loop with endpoint x_0 if ω is a continuous map

$$\omega : [0, 1] \rightarrow X \quad \text{with} \quad \omega(0) = \omega(1) = x_0.$$

We denote by e_x the loop which is constant in x , that is $e_x(t) = x$ for all $t \in [0, 1]$. Afterwards, we define an equivalence relation on the set of the loops having the same endpoint. We say that two loops ω_1, ω_2 , having p as common endpoint, are homotopically equivalent if there exists a continuous map

$$F : [0, 1] \times [0, 1] \rightarrow X$$

such that

$$F(t, i) = \omega_i(t) \quad \forall t \in [0, 1] \quad \text{and} \quad i = 1, 2$$

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and

$$F(0, j) = F(1, j) = x_0 \quad \forall j \in [0, 1].$$

The map F is a *homotopy* between ω_1 and ω_2 . Using this equivalence relation, we can introduce the equivalence class of a loop ω , denoted by $[\omega]$. The space $\Pi(X, x_0)$, named *fundamental group of X with basepoint x_0* , is the set of the equivalence classes of all the loops having x_0 as endpoint, that is

$$\Pi(X, x_0) = \{[\omega] : \omega(0) = \omega(1) = x_0\}.$$

The set $\Pi(X, x_0)$ is a group with respect to the operation “ \cdot ” defined as

$$[\omega_1] \cdot [\omega_2] = [\omega_1 \cdot \omega_2]$$

where

$$\omega_1 \cdot \omega_2(t) = \begin{cases} \omega_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \omega_2(2t - 1) & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Its identity element corresponds to the class of the constant loop $\mathbf{1} = [e_x]$. Observe that if a topological space X is simply connected, then every loop is homotopic to the constant loop e_x , therefore its fundamental group is trivial, that is

$$\Pi(X, x_0) = \{\mathbf{1}\}.$$

Consider now two topological spaces X, Y and let x_0, y_0 be two points of their; let $f : (X, x_0) \rightarrow (Y, y_0)$ be a continuous function with $f(x_0) = y_0$. Then the map

$$f_* : \Pi(X, x_0) \rightarrow \Pi(Y, y_0) \quad \text{such that} \quad [\omega] \mapsto [f(\omega)]$$

is well-defined. Moreover, f_* is a covariant functor. With all these topological tools, we can now precisely state the problem of the existence of the lifting. Let (\tilde{X}, π) be a covering space of a topological space X , let $\tilde{x}_0 \in \tilde{X}$, $x_0 = \pi(\tilde{x}_0)$ and $y_0 \in Y$ such that $f : Y \rightarrow X$ is a continuous map with $y_0 \mapsto x_0$. Then the existence of the lifting $\tilde{f} : Y \rightarrow \tilde{X}$ corresponds to the commutivity of the diagram below.

$$\begin{array}{ccc} & & (\tilde{X}, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow \pi \\ (Y, y_0) & \xrightarrow{f} & (X, x_0) \end{array}$$

This implies that also the following diagram must commute.

$$\begin{array}{ccc}
 & \Pi(\tilde{X}, \tilde{x}_0) & \\
 \tilde{f}_* \nearrow & \downarrow \pi_* & \\
 \Pi(Y, y_0) & \xrightarrow{f_*} & \Pi(X, x_0)
 \end{array} \tag{2.1}$$

The commutivity of diagram (2.1) requires that

$$f_*(\Pi(Y, y_0)) = (\pi_* \circ \tilde{f}_*)(\Pi(Y, y_0)) \subset \pi_*(\Pi(\tilde{X}, \tilde{x}_0)).$$

This condition is not only necessary, but also sufficient if Y is connected and locally arcwise connected (see [57, theorem 21.2]). Hence the theorem about the existence of the lifting of a map reads as follows.

Theorem 2.8 *Let (\tilde{X}, π) be a covering space of a topological space X and let Y be a connected, locally arcwise connected topological space. Let $f : Y \rightarrow X$ be a continuous function such that $y_0 \mapsto x_0$. Then there exists the lifting \tilde{f} of f to \tilde{X} if and only if*

$$f_*(\Pi(Y, y_0)) \subset \pi_*(\Pi(\tilde{X}, \tilde{x}_0)), \tag{2.2}$$

with $\tilde{x}_0 \in \pi^{-1}(x_0)$.

Consider now the case of an annulus $A = A[a, b]$ and let

$$\tilde{A} \stackrel{\text{def}}{=} \mathbb{R} \times [a, b]$$

be its lifting to the covering space $(\mathbb{R} \times \mathbb{R}^+, \pi)$ of \mathbb{R}_0^2 ; given the infinite strip \tilde{A} , we will denote its boundaries by

$$\tilde{A}_i = \mathbb{R} \times \{a\} \quad \text{and} \quad \tilde{A}_o = \mathbb{R} \times \{b\}.$$

Let φ be a homeomorphism

$$\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\} \quad \text{with} \quad \varphi : (x, y) \mapsto (\varphi_1(x, y), \varphi_2(x, y)).$$

and consider the map

$$g : \tilde{A} \rightarrow \mathbb{R}^2 \setminus \{O\} \quad \text{such that} \quad g(\vartheta, r) = \varphi(\pi(\vartheta, r)).$$

We have to prove that there exists a lifting \tilde{g} of g to $\mathbb{R} \times \mathbb{R}_0^+$ in such a way that the diagram

$$\begin{array}{ccc}
 \tilde{A} & \xrightarrow{\tilde{g}} & \mathbb{R} \times \mathbb{R}_0^+ \\
 \pi \downarrow & \searrow g & \downarrow \pi \\
 A & \xrightarrow{\varphi} & \mathbb{R}^2 \setminus \{O\}
 \end{array} \tag{2.3}$$

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commutes, that is $g = \pi \circ \tilde{g}$. Since $\tilde{A} = \mathbb{R} \times [a, b]$ is a connected, locally connected topological space, theorem 2.8 applies; moreover \tilde{A} , as well as $\mathbb{R} \times \mathbb{R}_0^+$ are simply connected spaces, then their fundamental groups are trivial. Hence condition (2.2) is clearly satisfied.

By this argument we have proved that for every homeomorphism $\varphi : A \rightarrow A$ there exists a lifting $h : \tilde{A} \rightarrow \tilde{A}$. It is now possible to precise the twist condition for a homeomorphism of an annulus, speaking in terms of h .

Definition 2.9 Let $\tilde{A} = \mathbb{R} \times [a, b]$ be the lifting of a planar annulus and let $h : \tilde{A} \rightarrow \tilde{A}$ be a homeomorphism of the form

$$h(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r))$$

where f and s are continuous functions, 2π -periodic in the ϑ -variable. We say that φ is a *twist homeomorphism* if

$$s(\vartheta, a) \cdot s(\vartheta, b) < 0. \quad (2.4)$$

2.2 STATEMENT OF THE THEOREM AND SOME SPECIAL CASES

At this point of the exposition, we have got all the material needed to provide a precise statement of the Poincaré-Birkhoff theorem.

Let $\tilde{A} = \mathbb{R} \times [a, b]$ be the lifting of a planar annulus and let $h : \tilde{A} \rightarrow \tilde{A}$ be a homeomorphism of the form

$$h(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r))$$

where f and s are continuous functions, 2π -periodic in the angle, that is, $h = \tilde{\varphi}$ is the lifting of a homeomorphism $\varphi : A \rightarrow A$. If h

- is area-preserving
- leaves the boundaries invariant, that is $f(\vartheta, a) = a$ and $f(\vartheta, b) = b$
- satisfies the twist condition (2.4)

then h has at least two distinct families of fixed points in the interior of \tilde{A} , which means that φ has at least two distinct fixed points in A . Indeed, due to the periodic behaviour of the involved maps, we can observe that if h has a fixed point (ϑ^*, r^*) , then all the points $(\vartheta^* + 2k\pi, r^*)$ with $k \in \mathbb{Z}$ are fixed points too. Then the theorem ensures the existence of two families of fixed points $\{(\vartheta_1 + 2k\pi, r_1) : k \in \mathbb{Z}\}$ and $\{(\vartheta_2 + 2k\pi, r_2) : k \in \mathbb{Z}\}$ with

$$r_1 \neq r_2 \quad \vee \quad \vartheta_1 - \vartheta_2 \neq 2\ell\pi \quad \forall \ell \in \mathbb{Z}.$$

2.3 Generalizations and efforts of proof

Therefore, when we project the fixed points of h on the annulus via the map π , the two families above defined will generate two points $z_1 \neq z_2$ in the interior of A , which are fixed points for φ .

As the reader can see, Poincaré's theorem has a simple and plain statement; on the other hand its proof is far from being easy and clear.

However there is a special case in which the proof of the theorem is quite simple and can be sketched in few lines. It is sufficient to add the hypothesis of strict monotonicity of the angular variation, that is assume that $h(\vartheta, \cdot)$ is strictly increasing in the radial coordinate for all the values $\vartheta \in \mathbb{R}$. With this auxiliary condition and recalling the twist condition, one obtains the inequalities

$$h(\vartheta, a) < 0 < h(\vartheta, b) \quad \forall \vartheta \in \mathbb{R}$$

then, due to the fact that h is continuous, for every $\vartheta \in \mathbb{R}$ there exists a special value for the ray $\bar{r}(\vartheta)$ such that

$$h(\vartheta, \bar{r}(\vartheta)) = 0;$$

the map $\vartheta \mapsto \bar{r}(\vartheta)$ is continuous and 2π -periodic. Define now the set

$$\Gamma = \{(x, y) \in A : x = \bar{r}(\vartheta) \cos \vartheta, y = \bar{r}(\vartheta) \sin \vartheta\} \quad (2.5)$$

whose points are moved by φ only in the radial direction, by construction. Recalling that the map φ is area-preserving, we conclude that Γ must intersect its image in at least two points, that are two fixed points for φ .

2.3 GENERALIZATIONS AND EFFORTS OF PROOF

In 1913 the first proof of theorem 2.2 appeared in George Birkhoff's paper *Proof of Poincaré's geometric theorem* [7]. The American mathematician George Birkhoff (1884-1944) was an assistant professor at Harvard University and during his career he had been working on many different mathematical topics like asymptotic expansions, boundary value problems, and Sturm-Liouville type problems; his doctoral studies had been widely influenced and guided by Poincaré's works on differential equations and celestial mechanics.

As the author says in the first lines of [7], his proof of Poincaré's theorem is based on methods he had already applied to some questions of similar character and it was said to be "one of the most exciting mathematical events of the era" [1]. The proof exposed in his article takes only few pages and is based on the fixed point index theory. Indeed, Birkhoff proves the existence of one fixed point and eventually the fact that the sum of the indices of all the fixed points is zero. Then his conclusion is based on this sentence:



Figure 5: George Birkhoff, 1884-1944

As Poincaré remarks, the existence of one invariant point implies immediately the existence of a second invariant point.

Of course this is true if the first fixed point has a nonzero index, but Birkhoff’s argument could not be correct in general. Indeed his proof does not precise how to avoid the case in which the first fixed point has zero index. For this reason it has often been said that Birkhoff’s first proof was not complete. However it is interesting to notice that, considering a quite recent work by E.E. Slaminka [101], now we can say that Birkhoff’s argument was correct; more precisely, proving the existence of *one* fixed point is enough for obtaining Poincaré’s conclusion; the conclusion can be achieved using a theorem that shows that it is possible to “remove” isolated fixed points of an area-preserving homeomorphism having index equal to 0. The result we are referring to is the following.

Theorem 2.10 *Let $h : M \rightarrow M$ be an area-preserving homeomorphism of an orientable 2–manifold and let z be an isolated fixed point with index 0, such that there exists a neighbourhood U_z of p with $U_z \cap \text{Fix}(h) = \{z\}$. Then there exists an area-preserving homeomorphism h' such that $h \equiv h'$ on $M \setminus U_z$ and h' has no fixed points on U_z .*

Using Birkhoff’s result, we know that every area-preserving twist homeomorphism of the annulus has at least one fixed point; assume, by contradiction, that there exists a homeomorphism h which has exactly one fixed point z . Clearly, z has index equal to zero. As stated in theorem 2.10, we can construct a homeomorphism h' (which is area-preserving too) which coincides with h on $A \setminus U_z$ (and, therefore, it has no fixed points in $A \setminus U_z$) and which has no fixed points in U_z . Then h' is an area-preserving twist homeomorphism which has no

2.3 Generalizations and efforts of proof

fixed points, in contradiction with Birkhoff's theorem. In light of this theorem, we can now say that also the one in [7] is a complete proof of Poincaré's theorem.

Anyway, in 1925 Birkhoff published a second work [10] on this topic, in which he corrected his previous proof of the theorem, finally removing any doubt about the existence of two distinct fixed points, as he says in his article:

Furthermore the existence of two distinct invariant points is established, whereas the possibility of only a single invariant point has not hitherto been excluded.

In this second article Birkhoff presented also an extension of Poincaré's result, moving towards two different directions. First of all, he weakened the hypothesis on the annular domain, removing the request about the invariance of the outer boundary of the annulus. As he said, the removal of this condition allows to apply the theorem to the problem of proving the existence of infinitely many periodic solutions in a dynamical system with two degrees of freedom. On the other hand, he worked also on the area-preserving condition which appeared to be a strong restriction in its original formulation.

Moreover, Birkhoff's theorem applies to more general domains than the standard annuli, namely to domains that are homeomorphic to a standard annulus. Therefore, we need to introduce a new definition.

Definition 2.11 A set $A \subset \mathbb{R}^2$ is a *generalized (or topological) annulus* if A is homeomorphic to a standard annulus.

Let $A = A[1,2]$ be a planar annulus and let A' be a generalized annulus; then there exists a homeomorphism $\eta : A \rightarrow \eta(A) = A'$. As a consequence of Schoenflies's theorem, the set $\eta(\partial A)$ is independent of the choice of the homeomorphism η . We call the set $\eta(\partial A)$ the *contour of A'* and denote it by $\vartheta A'$. Clearly, for a topological annulus A' embedded in \mathbb{R}^2 , the contour of A' coincides with the boundary of A' . The contour of A' consists into two connected components which are closed arcs (Jordan curves) since they are homeomorphic to S^1 . We call such closed arcs A'_i and A'_o , as in the case of standard annuli. For a planarly embedded topological annulus, they could be chosen as the inner and the outer boundaries of the annulus. In such a special case, the bounded component of $\mathbb{R}^2 \setminus A'$ turns out to be an open simply connected set $D = D(A'_i)$ with

$$\partial D = A'_i \quad \text{and} \quad \text{cl} D = D \cup A'_i$$

homeomorphic to the closed unit disc. For a standard annulus $A = A[a, b]$ we have $D(A_i) = B(0, a)$. On the other hand, in the general

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setting, speaking of inner and outer boundaries is meaningless; yet we keep this terminology. Finally, we define the *interior* of A' as

$$\text{int } A' = A' \setminus \partial A'.$$

According to the notation introduced throughout the chapter, the statement of the theorem presented in [10] can be written as follows.

Theorem 2.12 (Birkhoff) *Let A and A' be two generalized annuli whose common inner boundary is the circle C_a , while their outer boundaries are strictly star-shaped curves; let $\varphi : A \rightarrow A'$ be a homeomorphism which leaves C_a invariant. If φ is a twist homeomorphism, then one of these two alternatives occurs:*

- *there exists an annular region $A'' \subset A$ surrounding C_a such that*

$$\varphi(A'') \subsetneq A'' \tag{2.6}$$

- *φ has at least two fixed points.*

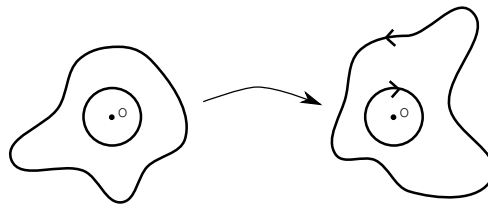


Figure 6: A pictorial description of Birkhoff theorem 2.12

In particular, if the homeomorphism φ is area-preserving, condition (2.6) can not happen and then the existence of two fixed points immediately holds. It follows that Poincaré's theorem can be seen as a corollary of theorem 2.12. The important feature of this statement lies in condition (2.6), which is a much weaker request than the original hypothesis of area-preserving of Poincaré's theorem 2.2. We stress also the fact that the outer boundaries of the annuli are required to be strictly star-shaped curves (see figure 2.12).

Birkhoff was working on this topic for many years, motivated by the interest in finding other versions of the theorem, more suitable for the applications to nonautonomous differential equations; indeed his interest was in proving the existence of fixed points of the Poincaré map associated to systems of ODEs. Moving in this direction and following some remarks made by Poincaré, in [8] he stated a new version of the theorem, dealing with the case in which the domain of the homeomorphism is an infinite annulus.

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Definition 2.13 An *infinite annulus* is a set of the form $A = A[a, \infty[= \{(x, y) : x^2 + y^2 \geq a^2\}$ for some $a > 0$.

Theorem 2.14 Let $A = A[a, \infty)$ be an *infinite annulus* and let $\varphi : A \rightarrow A$ be an *area-preserving homeomorphism* which advances the points on C_a and regresses the points on C_r for all $r \geq R > a$ with a rotation of an angle which is at least $\vartheta_1 > 0$. Then φ has at least two fixed points in A .

Going on with the history of the theorem, during the second half of the twentieth century many authors were motivated to produce extensions of the Poincaré-Birkhoff theorem, trying to generalize the condition of invariance of the annular domain of the homeomorphism under consideration, and in particular of its outer boundary, with the purpose of applying the theorem to the study of the existence of periodic solutions of second order ordinary differential equations of the form

$$x'' + f(t, x) = 0$$

with $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ a continuous and T -periodic function in the t -variable.

Among the large number of works on this subject, we want to mention those by Howard Jacobowitz [52, 53]. In 1976 he published a paper in which he succeeded in extending the Poincaré-Birkhoff theorem to regions that are homeomorphic to a pointed (say holed) disc, developing an idea already formulated by Poincaré. Indeed, in [94] we find the following remark:

Imaginons en effet d'abord que la circonférence extrême intérieure $x = b$ vienne à se réduire à un point, notre couronne circulaire se réduira à un cercle. Si alors sur la circonférence extérieure $x = a$, on a toujours $Y > y$, et dans le voisinage du centre $Y < y$ on inversement; si, de plus, la transformation admet un invariant intégral, il y aura à l'intérieur du cercle au moins deux points inaltérés par la transformation. D'autre part, nous pouvons appliquer les memes principes à une puissance quelconque T^n de la transformation T .

To be more precise, we introduce the following definition.

Definition 2.15 A set $A \subset \mathbb{R}^2$ is a *generalized pointed disc* if it is homeomorphic to the set

$$A(0, r] = B[0, r] \setminus \{O\} \tag{2.7}$$

It is important to stress the fact that Jacobowitz does not impose any condition on the outer boundary of the set since he does not require that it is a star-shaped curve, but only a simple one. In this setting the twist condition expressed as in (2.4) is meaningless and must be replaced with a condition written in terms of a limit computed moving towards the centre of the disc.

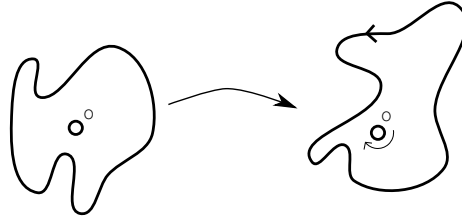


Figure 7: A pictorial description of theorem 2.16

Theorem 2.16 Consider two generalized pointed discs A_1 and A_2 , whose outer boundaries are two simple closed curves Γ_1, Γ_2 . Let $\varphi : A_1 \rightarrow A_2$ be a homeomorphism whose lifting to the space $\mathbb{R} \times \mathbb{R}^+$ has the form $\tilde{\varphi}(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r))$ and satisfies the twist condition

- $s(\vartheta, r) < 0$ on Γ_1
- $\liminf_{r \rightarrow 0} s(\vartheta, r) > 0$.

If φ is area-preserving, then it has at least two distinct fixed points.

Among the papers appeared in those years dealing with the problem of generalizing the hypothesis about the invariance of the annulus, finally we mention the contributes brought by Wei Yue Ding in [29, 30].

In [29] a paper by Ding appeared in *Acta Mathematica Sinica*; in this work he formulates a first version of the theorem considering a generalized annulus bounded by two simple curves Γ_1 and Γ_2 ; this theorem allows the annulus not to be invariant, but adds a condition about the existence of an extension of the homeomorphism to the whole closed disc $\text{cl}D(\Gamma_2)$, guaranteeing the fact that the annulus is not moved too far. More precisely, the theorem asserts what follows.

Theorem 2.17 Let $A = A[a, b]$ be a standard planar annulus and let $\varphi : A \rightarrow \varphi(A) \subset \mathbb{R}^2 \setminus \{O\}$ be an area-preserving twist homeomorphism. If there exists an extension $\varphi_0 : B[0, b] \rightarrow \mathbb{R}^2$ such that $0 \in \varphi_0(B(0, a))$, then φ has at least two fixed points.

The proof is based on the previous result 2.16 by Jacobowitz. In the following year, Ding generalized the result to the case of an annulus bounded by two simple curves. This version of the Poincaré-Birkhoff theorem (see theorem 2.18) is the most general one and also the most useful under the point of view of the applications to the study of second order differential equations.

Theorem 2.18 Let A be a generalized annulus, whose inner boundary is a strictly star-shaped curve Γ_1 , while its outer boundary Γ_2 is a simple closed curve. Let $\varphi : A \rightarrow \varphi(A)$ be a homeomorphism which can be extended to

the all closed disc $\text{cl } D(\Gamma_2)$ surrounded by Γ_2 . If φ is an area-preserving twist homeomorphism such that $0 \in \varphi(D(\Gamma_1))$, then φ has at least two fixed points.

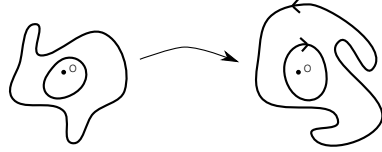


Figure 8: Ding's theorem

When the theorem is applied to the search of periodic solutions of planar Hamiltonian systems, the domains to which the theorem applies typically are sets of the form

$$\mathcal{A} = \{z \in \mathbb{R}^2 : \mathcal{E}(z) \in [a, b]\}$$

which are surrounded by level-lines of an energy function; therefore the boundaries of the domains are strictly star-shaped curves. This fact is crucial, because some counterexamples to Ding's theorem 2.18 have appeared in recent years, showing that the condition of star-shapness for the outer boundary is essential for the proof of the theorem and can not be removed. We will briefly talk about these counterexamples in section 2.7.

The other direction in which generalizations of the Poincaré-Birkhoff theorem were exploited was the one related to the area-preserving condition. Having Birkhoff's theorem 2.12 as a first step, some authors proposed weaker conditions than condition (2.6). In this context we quote the work by Patricia Carter [22], who stated and proved the following theorem.

Theorem 2.19 *If $\varphi : A \rightarrow A$ is a twist homeomorphism of the annulus A which has at most one fixed point, then there exists a simple essential closed curve $C \subset \text{int } A$ which meets its image in at most one point.*

Other recent works have also proved some modified and more general version of the twist condition, see for instance [64, 20, 24, 12, 95]. A further approach to the proof of the Poincaré-Birkhoff theorem shows a connection with the Brouwer plane translation theorem, as exposed by Guillou in [41] and Bonino in [12]. Finally, we mention the possibility of studying the twist theorem under the assumption that a different measure than the Lebesgue one is preserved (see for instance [32, 4]).

For what concerns the proof of the theorem, Birkhoff's proof of theorem 2.2 had not been completely accepted for a long time, especially for what concerned the existence of the second fixed point. Indeed

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the proof was very complex and difficult to understand and it left the mathematical community quite sceptical. Many efforts of obtaining a convincing proof had been performed during the second part of the century and many erroneous proofs were given too. The first proof accepted by the majority of the mathematicians appeared in 1977, in a paper by Morton Brown and Walter D. Neumann [14]. The authors published a very detailed and accurate analysis of Birkhoff’s arguments which clarified any doubt about the existence of two fixed points, as stated by Poincaré more than sixty years before.

2.4 THE PROOF BY BROWN AND NEUMANN

In this section I would like to expose all the details of the proof of Poincaré’s theorem, following the one presented by Brown and Neumann in [14]. As a reference, I used also the more recent work by Dalbono and Rebelo in which a more detailed exposition is provided.

Let $A = A[a, b]$ be a standard annulus in the plane and let $\varphi : A \rightarrow A$ be a homeomorphism satisfying the hypothesis of theorem 2.2. Let $\tilde{A} = \mathbb{R} \times [a, b]$ be the lifting of A to the covering space $\mathbb{R} \times \mathbb{R}_0^+$, whose boundaries are the straight lines

$$\tilde{A}_i = \pi^{-1}(A_i) = \mathbb{R} \times \{a\} \quad \text{and} \quad \tilde{A}_o = \pi^{-1}(A_o) = \mathbb{R} \times \{b\}. \quad (2.8)$$

Assume that the lifting of the homeomorphism $h : \tilde{A} \rightarrow \tilde{A}$ has the following properties:

- $h(\vartheta, a) = (\vartheta + s_2(\vartheta), a)$ for every $\vartheta \in \mathbb{R}$
- $h(\vartheta, b) = (\vartheta - s_1(\vartheta), b)$ for every $\vartheta \in \mathbb{R}$
- $h(\vartheta + 2\pi, r) = h(\vartheta, r) + (2\pi, 0)$ for every $(\vartheta, r) \in \tilde{A}$

where $s_1(\cdot)$ and $s_2(\cdot)$ are continuous, strictly positive and 2π -periodic functions.

Define the sets

$$H_a = \{(\vartheta, r) \in \mathbb{R}^2 : r \leq a\} \quad \text{and} \quad H_b = \{(\vartheta, r) \in \mathbb{R}^2 : r \geq b\},$$

such that $\mathbb{R}^2 = H_a \cup \tilde{A} \cup H_b$. In the following we need to assume that h is extended by continuity to all the plane \mathbb{R}^2 , therefore we set

$$h(\vartheta, r) = \begin{cases} (\vartheta + s_2(\vartheta), r) & \text{on } H_a \\ (\vartheta - s_1(\vartheta), r) & \text{on } H_b. \end{cases} \quad (2.9)$$

Note that when we extend our setting to all the plane \mathbb{R}^2 , speaking about (ϑ, r) as an angular and a radial coordinate loses its original

2.4 The proof by Brown and Neumann

meaning; although we keep this terminology for consistency with the rest of the exposition.

The argument we are going to expose assumes, by contradiction, that h has only *one* family of fixed points. Without loss of generality, we can say that z is a fixed point for h if and only if it has the form

$$z = F_k = (2\pi k, r^*)$$

for some $k \in \mathbb{Z}$ and for a fixed $r^* \in]a, b[$. Define now the sets

$$W_\ell =]\frac{\pi}{2}\ell, \frac{3\pi}{2}\ell[\times \mathbb{R}$$

for every $\ell \in \mathbb{Z}$ (see figure 9) and let W be the union of all the rectangles

$$W = \bigcup_{\ell \in \mathbb{Z}} W_\ell \subset \mathbb{R}^2.$$

From the assumption made on the position of the fixed points, the set $\text{cl } W$ does not contain fixed points of h , then there exists a value $\varepsilon > 0$ such that

$$\|z - h(z)\| > \varepsilon \quad \forall z \in \text{cl } W.$$

By this definition, it holds that

$$\varepsilon < \min\{\min s_1, \min s_2\}. \quad (2.10)$$

We construct now an area-preserving homeomorphism $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which modifies only the radial coordinate of the points of the plane and whose expression is

$$T : (\vartheta, r) \mapsto (\vartheta, r + \frac{\varepsilon}{2}(|\cos \vartheta| - \cos \vartheta)) \quad (2.11)$$

Observe that two cases can occur:

- if $z = (\vartheta, r) \in W$, then $\cos \vartheta < 0$ and $T(\vartheta, r) = (\vartheta, r - \varepsilon \cos \vartheta)$, with $T_2(\vartheta, r) = r - \varepsilon \cos \vartheta > r$
- if $z \notin W$, then $\cos \vartheta \geq 0$ and $T(z) = z$;

this means that T moves upwards all and only the points in W , while the ones in $\mathbb{R}^2 \setminus W$ are left fixed. Moreover, the vertical displacement performed by T is bounded by the value of ε , that is

$$\|T(z) - z\| = |T_2(r, \vartheta) - r| = \varepsilon |\cos \vartheta| \leq \varepsilon \quad \forall z \in \mathbb{R}^2. \quad (2.12)$$

Consider now the composition of the maps $T \circ h$; since both T and h are area-preserving homeomorphisms, their composition is an area-preserving homeomorphism too. We claim that $T \circ h$ has the same fixed points as h . Let z be a fixed point for h , for instance, $z = F_0 =$

2 The Poincaré-Birkhoff theorem: one century of research

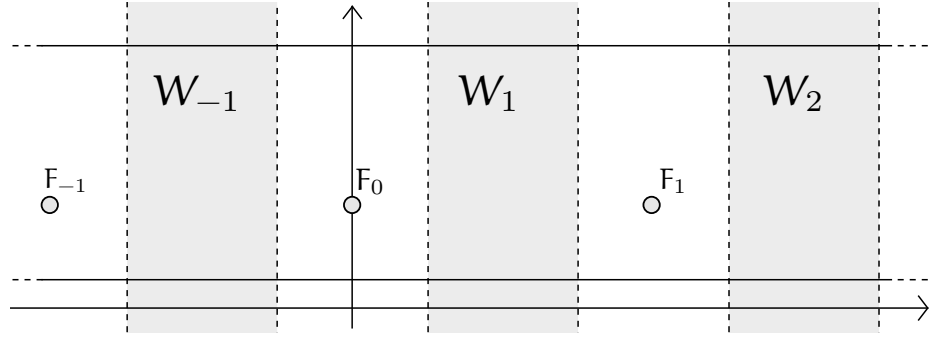


Figure 9: The sets W_ℓ and the fixed points F_k

$(0, r^*)$; then $(T \circ h)(z) = T(z) = z$, because T fixes the points not in W . Then every fixed point of h is a fixed point for $T \circ h$ too. On the other hand, if $h(z) \neq z$, then, by the choice of ε , we recall that $\|z - h(z)\| > \varepsilon$, while $\|T(h(z)) - h(z)\| \leq \varepsilon$ as proved in (2.12); then it must be $z \neq T(h(z))$, since $\|z - T(h(z))\| > 0$. Thanks to these arguments we can conclude that z is a fixed point for h if and only if z is a fixed point for $T \circ h$. In particular, $T \circ h$ has no fixed points in W .

We are going now to introduce a recursive sequence of sets, which do not contain fixed points of h . To begin with, define D_0 as the set of the points of $H_a = \mathbb{R} \times (-\infty, a]$ which are mapped by $T \circ h$ outside H_a ; more precisely,

$$D_0 = H_a \setminus (T \circ h)^{-1}(H_a) \quad (2.13)$$

while the other sets are defined by the recursive relation

$$D_i = (T \circ h)(D_{i-1}) \quad \forall i > 1. \quad (2.14)$$

From the definition of D_0 , it follows that $D_1 \cap D_0 = \emptyset$. Indeed, if there existed a point z in this intersection then from $z \in D_1$ we would get $z = (T \circ h)(z')$ for some $z' \in D_0 \subset H_a$, then $z' \in (T \circ h)^{-1}(H_a)$; on the other hand, since $D_0 \subset H_a$ we would obtain $z \in H_a$ too, in contradiction with definition (2.13) of D_0 . By definitions (2.13) and (2.14), we also deduce that $D_1 \cap H_a = \emptyset$; hence, going on by induction,

$$D_i \cap D_j = \emptyset \quad \forall i \neq j \quad \text{and} \quad D_i \cap H_a = \emptyset \quad \forall i \geq 1.$$

For what concerns the negative indices $i < 0$, we observe that $(T \circ h)^{-1}(H_a) \subset H_a$. Indeed, according to definition (2.9), the map h acts as a horizontal motion on the set H_a , while T^{-1} moves the points downwards. Then, points in H_a (that is points $z = (\vartheta, r)$ with $r \leq a$) still lie in H_a under the action of the negative iterates of $T \circ h$. Hence $D_i \subset H_a$ for every $i < 0$.

In this way we have defined a family $(D_i)_i$ of disjoint sets, which are contained in $\mathbb{R}^2 \setminus H_a$ for $i \geq 1$. We also note that by the definition of D_0 , none of the sets D_i contains fixed points of h .

2.4 The proof by Brown and Neumann

Since the map $T \circ h$ has a periodic behaviour, the sets D_i have a periodic structure. Indeed, if $z = (\vartheta, r) \in D_0$, then, by definition, $z \in H_a$ and $(T \circ h)(z) = (\vartheta + s_2(\vartheta), r') \notin H_a$; according to (2.11), since $r' > a$, then $r' = r - \varepsilon \cos \vartheta$, with $\cos \vartheta < 0$. Consider now the point $z' = (\vartheta + 2k\pi, r) = z + (2k\pi, 0) \in H_a$ and $(T \circ h)(z') = (\vartheta + 2k\pi + s_2(\vartheta + 2k\pi), r - \varepsilon \cos \vartheta) = (T \circ h)(z) + (2k\pi, 0)$, simply using the periodicity of s_2 and of the cosenum. Then $z \in D_0$ if and only if every translated point $z + (2k\pi, 0) \in D_0$. Since $T \circ h$ is a homeomorphism, also all the other sets D_i have the same periodic structure.

We want now to prove that the sets D_i have a positive measure. Consider a point $z = (\vartheta, a) \in W_1 \cap \{r = a\} \subset H_a$; then $(T \circ h)(z) = (\vartheta + s_2(\vartheta), r') \notin H_a$, which means that $z \in D_0$. Since $T \circ h$ is a homeomorphism, we can find an open ball $B(z, \delta) \subset H_a$ which is mapped outside H_a . Then D_0 contains $B(z, \delta) \cap H_a$ and this allows to conclude that $\mu(D_0) > 0$.

The next step consists in proving that there exists an index $n \geq 1$ such that $D_n \cap H_b \neq \emptyset$. If $D_1 \cap H_b \neq \emptyset$ simply take $n = 1$. Otherwise, $D_1 \subset \mathbb{R} \times]a, b[$. Since the sets D_i are periodic and $D_i \subset \{r > a\}$ if $i > 0$, we can project $(D_i)_{i \geq 1}$ in \mathbb{R}_0^2 , using a modified projection map

$$\hat{\pi}(\vartheta, r) = (\sqrt{2r} \cos \vartheta, \sqrt{2r} \sin \vartheta) \quad (2.15)$$

which preserves Lebesgue's measure, in the sense that if $D \subset \mathbb{R}_0^2$ with $\mu(D) < \infty$, then $\mu(D) = \mu(\hat{\pi}(D))$. From the assumption made above, we have $\hat{\pi}(D_1) \subset A[\sqrt{2a}, \sqrt{2b}]$, then we can conclude that

$$0 < \mu(\hat{\pi}(D_1)) < \infty \quad (2.16)$$

and, since $T \circ h$ is an area-preserving homeomorphism,

$$\mu(\hat{\pi}(D_1)) = \mu(\hat{\pi}(D_i)) \quad \forall i \geq 1,$$

where $\hat{\pi}(D_i)$ are disjoint subsets of the infinite annulus $A[a, +\infty)$. Then there exists an index $n > 0$ such that

$$\hat{\pi}(D_n) \cap A_0 \neq \emptyset.$$

Going back to the infinite strip \tilde{A} via $\hat{\pi}^{-1}$, we get $D_n \cap H_b \neq \emptyset$ and, recalling that $D_n \subset (T \circ h)^n(H_a)$ immediately obtain

$$(T \circ h)^n(H_a) \cap H_b \neq \emptyset.$$

Having achieved our purpose, from now on we can forget the sets $(D_k)_k$ since we are not using them anymore.

Let now $z_n = (\vartheta_n, r_n)$ be a point in $(T \circ h)^n(H_a) \cap H_b$, choosed with maximal radial coordinate in such a way that

$$z = (\vartheta, r) \in (T \circ h)^n(H_a) \cap H_b \Rightarrow r \leq r_n; \quad (2.17)$$

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moreover z_n can always be chosen in W_0 . Recalling that $T \circ h$ is non-decreasing with respect to the radial coordinate, we have

$$(T \circ h)^m(H_a) \cap H_b \neq \emptyset \quad \forall m \geq n.$$

We introduce now a sequence of points made by the complete orbit of z_n under the positive and negative iterates of the map, setting

$$z_i = (\vartheta_i, r_i) = (T \circ h)^{i-n}(z_n) \quad \forall i \in \mathbb{Z}; \quad (2.18)$$

this orbit is nontrivial since, from $r_n \geq b$, we immediately deduce that z_n is not a fixed point of h (see (2.9)); therefore $z_0 \in H_a$ while, by definition, $z_n \in H_b$. Moreover, by (2.18), it is easy to see that

$$z_i = (T \circ h)((T \circ h)^{i-1-n}(z_n)) = (T \circ h)(z_{i-1})$$

for every index $i \in \mathbb{Z}$.

Our purpose is to use the sequence of points $(z_k)_k$ in order to construct a curve running from H_a to H_b , avoiding all the fixed points of h and which is a flow-line for the map $T \circ h$, that is which is mapped into itself (except near one of its endpoints). The starting point of such a curve γ is z_{-1} . Let γ_0 be the segment connecting $z_{-1} = (T \circ h)^{-1}(z_0)$ with z_0 and recursively define a sequence of arcs setting

$$\gamma_i = (T \circ h)^i(\gamma_0) \quad \forall i \in \mathbb{Z}.$$

The curve γ is obtained “pasting” together the curves γ_i for the values $i \in \{0, \dots, n\}$ in such a way that

$$\gamma = \gamma_0 \gamma_1 \cdots \gamma_n \quad \text{and} \quad (T \circ h)\gamma = \gamma_1 \gamma_2 \cdots \gamma_{n+1}. \quad (2.19)$$

The endpoints of γ are z_{-1} and z_n which belong to H_a and H_b respectively. Consider now the curve $\gamma\gamma_{n+1}$ joining $z_{-1} \in H_a$ with $z_{n+1} \in H_b$. We are interested in proving two properties of this curve:

- the curve $\gamma\gamma_{n+1}$ is simple
- for every $z = (\vartheta, r) \in \gamma$ we have $r_{-1} \leq r \leq r_{n+1}$.

Lemma 2.20 *The curve $\gamma\gamma_{n+1}$ is simple.*

Proof. Since $T \circ h$ is a homeomorphism, and in particular it is injective, then every subcurve γ_i does not intersect itself; indeed we have

$$\sharp(\gamma_i \cap \gamma_j) = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

To begin with, recall that the set γ_0 is a segment whose endpoints are $z_{-1} = (T \circ h)^{-1}(z_0)$ and $z_0 = (\vartheta_0, r_0)$; γ_0 is contained in H_a , then

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h acts on it as the horizontal translation $(\vartheta, r) \mapsto (\vartheta + s_2(\vartheta), r)$; h is a homeomorphism, then its first component $h_1 : \vartheta \mapsto \vartheta + s_2(\vartheta)$ is a homeomorphism too. More precisely, h_1 is continuous and bijective and, moreover, it is an increasing function. To prove this last feature, as a first step recall that the map T does not act on the angular variable, then the first component of $(T \circ h)$ coincides with h_1 and $(T \circ h)(z_{-1}) = (h_1(\vartheta_{-1}), r_0) = z_0$. Then we have $h_1(\vartheta_{-1}) = \vartheta_0 = \vartheta_{-1} + s_2(\vartheta_{-1}) > \vartheta_{-1}$ since $s_2(\cdot)$ is a positive function. Consider now the restriction of h_1 to the closed interval $[\vartheta_{-1}, \vartheta_0]$; in general, a homeomorphism of a closed real interval is always strictly monotone, then, in this case, we only need to check that $h_1(\vartheta_{-1}) < h_1(\vartheta_0)$. From the definitions, it immediately holds that $h_1(\vartheta_{-1}) = \vartheta_0 < h_1(\vartheta_0)$ and this proves that h_1 is a strictly increasing homeomorphism (see figure 10). Therefore also the inverse function h_1^{-1} is strictly increasing and its expression is $\vartheta \mapsto \vartheta - s_2(\vartheta)$. Then the following relations holds:

- $h_1^{-1}(\vartheta_0) \leq \vartheta \leq \vartheta_0$ for all the points $z = (\vartheta, r) \in \gamma_0$
- $h_1^{-2}(\vartheta_0) \leq \vartheta \leq h_1^{-1}(\vartheta_0)$ for all the points $z = (\vartheta, r) \in \gamma_{-1}$.

More in general, we conclude that

$$\gamma_i \subset V_i \stackrel{\text{def}}{=} \{(\vartheta, r) : h_1^{i-1}(\vartheta_0) \leq \vartheta \leq h_1^i(\vartheta_0), r \in \mathbb{R}\} \quad (2.20)$$

for every $i \leq 0$. Since γ_0 is a segment whose endpoints are z_{-1} and z_0 , by the remarks above exposed, it intersects the boundaries of the strip V_0 only at its endpoints, then, by induction, all the curves γ_i intersect the boundaries of V_i only at their endpoints (see also figure 10). Thus we conclude that for every distinct $i, j \leq 0$, γ_i and γ_j intersect at most in one endpoint. Clearly, the intersection is nonempty if and only if $z_{i+1} = z_j$ or viceversa.

Consider now a pair of arbitrary sets γ_i and γ_j with $i \neq j$; then there exists an integer ℓ such that $(T \circ h)^\ell(\gamma_i) = \gamma_{i'}$ and $(T \circ h)^\ell(\gamma_j) = \gamma_{j'}$,

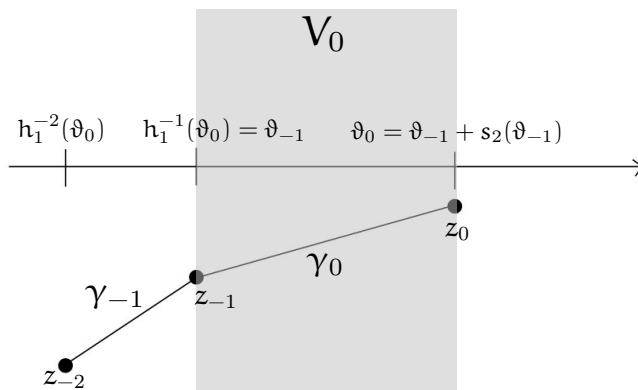


Figure 10: A description of the proof

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with $i', j' < 0$. The argument above ensures that $\gamma_{i'}$ and $\gamma_{j'}$ intersect at most at one endpoint; therefore, since $T \circ h$ is a homeomorphism, the thesis immediately follows. \square

We want now to prove the second property of the curve $\gamma_{\gamma_{n+1}}$, which is expressed by the following lemma.

Lemma 2.21 *For all the points $z = (\vartheta, r) \in \gamma$, we have that*

$$r_{-1} \leq r \leq r_{n+1}.$$

Proof. Because of $T \circ h$ is a homeomorphism, we can prove, by few computations, that $H_a \subset (T \circ h)(H_a)$; indeed, let $z \in H_a$, then the point $z' = (T \circ h)^{-1}(z)$ satisfies $r(z') \leq r(z) \leq a$, because $T \circ h$ is increasing in the radial component. Then $z' \in H_a$ too, hence $z \in (T \circ h)(H_a)$. By induction, we have

$$(T \circ h)^n(H_a) \subset (T \circ h)^m(H_a) \quad \forall m \geq n.$$

Then $\gamma_0 \subset H_a \subset (T \circ h)(H_a) \subset (T \circ h)^n(H_a)$, and by induction

$$\gamma_i \subset (T \circ h)^i(H_a) \subset (T \circ h)^n(H_a) \quad \forall i \in \{0, \dots, n\}.$$

We can conclude that the set γ is contained in $(T \circ h)^n(H_a)$. Let $z = (\vartheta, r) \in \gamma$; then there exists an index $i \in \{0, \dots, n\}$ such that $z \in \gamma_i \subset (T \circ h)^n(H_a)$. Assume, by contradiction, that $r_i > r_{n+1} \geq b$; then $z_i \in (T \circ h)^n(H_a) \cap H_b$ and, by (2.17), we are led to an absurd by $r_i = r_n \leq r_{n+1}$.

The next step consists in proving that $\gamma \subset \mathbb{R} \times [r_{-1}, +\infty[$. First of all, since $z_{-1} = (T \circ h)^{-1}(z_0)$, recalling that the vertical displacement performed by the homeomorphism $T \circ h$ is always positive, we get $r_0 \geq r_{-1}$ and $\gamma_0 \subset \mathbb{R} \times [r_{-1}, +\infty[$. Moreover, since $\gamma_1 = (T \circ h)(\gamma_0)$, using again the property of the map, we get that $r \geq r_{-1}$ for every point in γ_1 . Then, by induction we can easily conclude the proof. \square

Note that from lemma 2.20 it follows that γ does not pass through the fixed points of h . Indeed, assume that z is a fixed point of h and, consequently, a fixed point for $T \circ h$ too; if $z \in \gamma$, then $z \in \gamma_i$ for some index i ; then $z = (T \circ h)(z) \in \gamma_{i+1}$ and this means that $z \in \gamma_i \cap \gamma_{i+1}$. The only possibility is that $z = z_i$, but if z_i is a fixed point, then the whole orbit of equation (2.18) is trivial and coincides with the constant sequence $z_i = z_n \quad \forall i \in \mathbb{Z}$ in contrast with the assumption that z_n is not a fixed point.

The crucial point of the proof of the Poincaré-Birkhoff theorem is the computation of *the index of the curve* γ . For a brief survey about the index of a vector field along a curve, see appendix A. The first step consists in proving the following lemma.

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Lemma 2.22 *Every curve γ running from H_a to H_b and avoiding any fixed point $F_k = (2k\pi, r^*)$ of h has index*

$$i_\gamma(h) \equiv_1 \frac{1}{2};$$

moreover, the index is independent of the curve γ .

Proof. Consider the curve γ whose endpoints are z_{-1} and z_n and recall that $h(z_{-1}) = (\vartheta_{-1} + s_2(\vartheta_{-1}), r_{-1})$ and $h(z_n) = (\vartheta_n - s_1(\vartheta_n), r_n)$. According to definition (A.1), we have $\tilde{\gamma}(0) = (-1, 0)$ and $\tilde{\gamma}(1) = (1, 0)$, therefore, the angle between these two vectors is equal to π . Hence we gain

$$i_\gamma(h) \equiv_1 \frac{\pi}{2\pi} = \frac{1}{2}.$$

We want now to prove that the value of the index is independent of the path. More precisely, let γ_1, γ_2 be two paths going from H_a to H_b and avoiding all the fixed points of h . Denote by z_a^i the starting points of γ_i and by z_b^i the endpoints of γ_i , for $i = 1, 2$. Then construct two other curves, γ_3 and γ_4 , such that

- the support of γ_3 is the segment joining z_b^1 and z_b^2
- the support of γ_4 is the segment joining z_a^2 and z_a^1 .

Since the points of γ_3 and γ_4 lies in H_b and H_a respectively, the homeomorphism h acts on them simply as a horizontal translation; more precisely, $\tilde{\gamma}_3(\cdot) = (-1, 0)$ and $\tilde{\gamma}_4(\cdot) = (1, 0)$ are constant maps, then we conclude that

$$i_{\gamma_3}(h) = i_{\gamma_4}(h) = 0. \quad (2.21)$$

Afterwards we construct a new closed curve $\gamma' = \gamma_1\gamma_3(-\gamma_2)\gamma_4$; using the additivity of the index, we have

$$i_{\gamma'}(h) = i_{\gamma_1}(h) + 0 - i_{\gamma_2}(h) + 0 = i_{\gamma_1}(h) - i_{\gamma_2}(h).$$

Our aim is to prove that γ' has index zero. In order to reach this conclusion, we need that the planar region surrounded by γ' does not contain fixed point of h . Let $\mathcal{F}(h) = \{F_k : k \in \mathbb{Z}\}$ be the set of the fixed points of h and consider the fundamental group of $X = \mathbb{R}^2 \setminus \text{Fix}(h)$ with base in the point z_a^1 and denote it by $\Pi(X, z_a^1)$; the fundamental group, by definition, is the set of all the loops $\sigma : [0, 1] \rightarrow X$ such that $\sigma(0) = \sigma(1) = z_a^1$. Then for every t in $[0, 1]$, the point $\sigma(t)$ is not a fixed point of h . The generators of the fundamental group are all the loops σ such that the region surrounded by their support contains zero or one element of $\text{Fix}(h)$. Since $\gamma' \subset X$, then γ' is homotopic to some generator path and, since the index is invariant under homotopies, it is sufficient to prove that the index of γ' is zero under the assumption that γ' is a generator of $\Pi(X, z_a^1)$.

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If γ' is a path which does not surround any fixed point, then it is homotopic to the constant loop and therefore, its index is equal to zero and then $i_{\gamma_1}(h) = i_{\gamma_2}(h)$. As a second case, without loss of generality, we can suppose that γ' is a loop which surrounds $F_0 = (0, r^*)$; for the assumptions made on the position of the points F_k , γ' is homotopic to $\sigma' = \sigma_1\sigma_2\sigma_3\sigma_4$ such that

- σ_1 is the horizontal segment joining $u_1 = (-\pi, r)$ to $u_2 = (\pi, r)$ with $r < a$;
- σ_2 is the vertical segment joining $u_2 = (\pi, r)$ to $u_3 = (\pi, R)$ with $R > b$;
- σ_3 is the horizontal segment joining u_3 to $u_4 = (-\pi, R)$
- σ_4 is the vertical segment joining u_4 to u_1 .

Since σ_1, σ_3 lie in H_a and H_b respectively, their index is zero; since h is a 2π -periodic function in the first variable, then $i_h(\sigma_2) = -i_h(\sigma_4)$. Then, for the additivity of the index, we conclude that

$$i_{\sigma'}(h) = i_{\sigma_2}(h) + i_{\sigma_4}(h) = 0.$$

With this argument, we have proved that

$$i_{\gamma'}(h) = 0 = i_{\gamma_1}(h) - i_{\gamma_2}(h)$$

then the index is independent of the choice of the path. \square

The last step of the proof of Poincaré-Birkhoff theorem consists in the following lemma, whose proof will lots of computations.

Lemma 2.23 *The index $i_\gamma(h)$ is equal to $\frac{1}{2}$.*

Proof. To begin with, we are computing $i_\gamma(T \circ h)$. We recall that the endpoints of γ are the points $z_{-1} \in H_a$ and $z_n \in H_b$ on which the map $T \circ h$ acts in the following way:

$$\begin{aligned} z_{-1} = (\vartheta_{-1}, r_{-1}) &\mapsto z_0 = (\vartheta_{-1} + s_2(\vartheta_{-1}), r_{-1} + \delta_2) \\ z_n = (\vartheta_n, r_n) &\mapsto z_{n+1} = (\vartheta_n - s_1(\vartheta_n), r_n + \delta_1) \end{aligned}$$

with $0 \leq \delta_1, \delta_2 \leq \varepsilon$, for ε as in (2.12). Then the directions $\tilde{\gamma}(0), \tilde{\gamma}(1)$ (cf. (A.1)) can be easily obtained as

$$\tilde{\gamma}(0) = \frac{(s_2(\vartheta_{-1}), \delta_2)}{\|(s_2(\vartheta_{-1}), \delta_2)\|} \quad \text{and} \quad \tilde{\gamma}(1) = \frac{(-s_1(\vartheta_n), \delta_1)}{\|(-s_1(\vartheta_n), \delta_1)\|}. \quad (2.22)$$

By the behaviour of the map $T \circ h$ in the regions H_a and H_b , we know that the angular coordinates of the vectors $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$ satisfy the inequalities $0 \leq \vartheta(0) \leq \frac{\pi}{2}$ and $\frac{\pi}{2} \leq \vartheta(1) \leq \pi$; then, from (2.22) we get

$$\vartheta(0) = \arctan \frac{\delta_2}{s_2(\vartheta_{-1})} \quad \vartheta(1) = \pi - \arctan \frac{\delta_1}{s_1(\vartheta_n)}$$

and, therefore,

$$\Delta\vartheta = \vartheta(1) - \vartheta(0) = \pi - \arctan \frac{\delta_1}{s_1(\vartheta_n)} - \arctan \frac{\delta_2}{s_2(\vartheta_{-1})}.$$

Using the second property of the index, we conclude that

$$i_\gamma(T \circ h) \equiv_1 \frac{\Delta\vartheta}{2\pi} = \frac{1}{2} - \frac{1}{2\pi} \left(\arctan \frac{\delta_1}{s_1(\vartheta_n)} + \arctan \frac{\delta_2}{s_2(\vartheta_{-1})} \right). \quad (2.23)$$

Simply by the definition of ε and recalling equation (2.10), we have the following inequalities

$$0 \leq \delta_1 \leq \varepsilon < \min s_1 \quad \text{and} \quad 0 \leq \delta_2 \leq \varepsilon < \min s_2 \quad (2.24)$$

which allow us to conclude that $\arctan \frac{\delta_1}{s_1(\vartheta_n)}$ and $\arctan \frac{\delta_2}{s_2(\vartheta_{-1})}$ both belong to the interval $[0, \frac{\pi}{4}[$ and therefore

$$\Delta\vartheta \in]\frac{\pi}{2}, \pi]. \quad (2.25)$$

We are now going to prove that in (2.23) we can replace the congruence \equiv_1 with the equality $=$. Let $P : [-1, 0] \rightarrow \mathbb{R}^2$ be a parametrization of γ_0 , such that $P(-1) = z_{-1}$ and $P(0) = z_0$; extend it defining $P(t+1) = (T \circ h)(P(t))$ for every $t \in [-1, n+1]$. The map $P : [-1, n+1] \rightarrow \mathbb{R}^2$ is a parametrization of γ_{n+1} which satisfies $P(i) = z_i$ for every $i = -1, \dots, n+1$ and its restriction

$$P : [-1, n] \rightarrow \mathbb{R}^2$$

is a parametrization of γ .

In order to evaluate the index, we introduce the curve

$$d(t) \stackrel{\text{def}}{=} D(P(t), (T \circ h)(P(t))) = D(P(t), P(t+1)) \quad (2.26)$$

defined from $[-1, n]$ to S^1 (see figure 11); we will use this curve to compute the index of $T \circ h$ thanks to the fact that, by definition, $i_\gamma(T \circ h)$ coincides with the winding number w_d . First of all, we extend $d(\cdot)$ to the interval $[-1, 2n+1]$ setting

$$d_0(t) = \begin{cases} d(t) & \text{if } t \in [-1, n], \\ d(n) & \text{if } t \in [n, 2n+1]. \end{cases} \quad (2.27)$$

Since d_0 is constant on $[n, 2n+1]$, we can use d_0 to evaluate the index along γ instead of d . Our purpose is to write a homotopy between d_0 and the map d_{n+1}

$$d_{n+1}(t) = \begin{cases} D(z_{-1}, P(t+1)) & \text{if } t \in [-1, n], \\ D(P(t-n-1), z_{n+1}) & \text{if } t \in [n, 2n+1]. \end{cases} \quad (2.28)$$

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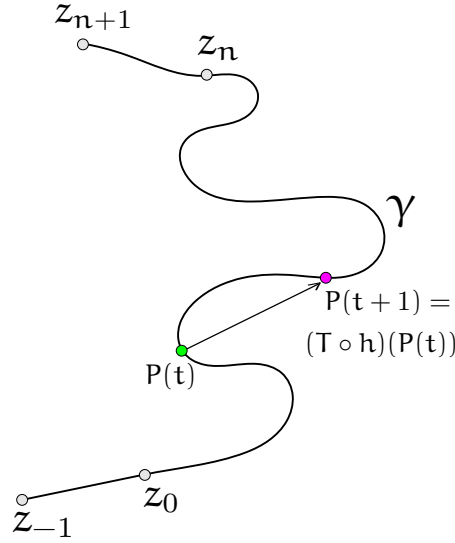


Figure 11: The value $d(t)$, for $t \in [-1, n]$ consists in the normalization of the vector painted in this figure, where $P(t)$ is running from z_{-1} to z_n along γ

For t in $[-1, n]$, this map represents the normalized vector which joins z_{-1} and a point A , where A runs on γ from z_0 to z_{n+1} ; for $t \in [n, 2n + 1]$, the map represents the normalization of the vector which joins a point B with the point z_{n+1} , where B runs on γ from z_{-1} to z_n (see figure 12). At this point we introduce the homotopy

$$d_\lambda(t) = d(t, \lambda) : [-1, 2n + 1] \times [0, n + 2] \rightarrow \mathbb{R}^2.$$

If $0 \leq \lambda \leq n + 1$, we set

$$d_\lambda(t) = \begin{cases} D(z_{-1}, P(t+1)) & \text{if } t \in [-1, \lambda - 1[\\ D(P(t - \lambda), P(t+1)) & \text{if } t \in [\lambda - 1, n[\\ D(P(t - \lambda), P(n+1)) & \text{if } t \in [n, n + \lambda[\\ D(P(n), P(n+1)) & \text{if } t \in [n + \lambda, 2n + 1] \end{cases} \quad (2.29)$$

which coincides with d_0 when $\lambda = 0$ and with d_{n+1} when $\lambda = n + 1$. This mapping is well-defined, since it is always of the form $D(P(t), P(t'))$ with $-1 \leq t < t' \leq n + 1$ and $P(t), P(t') \in \gamma \gamma_{n+1}$, which is a simple curve. The map d_λ , for some $\lambda \in]0, n + 1[$ corresponds to the normalization of a vector moving in the plane as follows:

- on the interval $[-1, \lambda - 1]$, its first endpoint is fixed in z_{-1} , while the second is running from z_0 to z_λ along γ ;
- on the interval $[\lambda - 1, n]$, the first endpoint is running from z_{-1} to $z_{n-\lambda}$, while the second one is running from z_λ to z_{n+1} ;

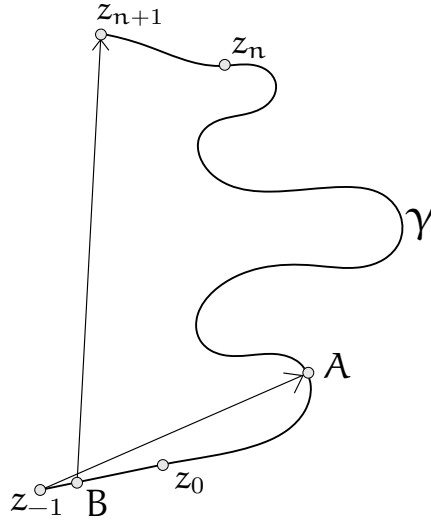


Figure 12: A pictorial description of the map d_{n+1} . During the first time interval, the vector has the first endpoint fixed in z_{-1} , while the other is the point A , running from z_0 to z_n ; on the second interval, the vector has as first endpoint the point B , which runs from z_{-1} to z_n , while the second endpoint is fixed in z_{n+1} .

- on the interval $[n, n + \lambda]$ the first endpoint is running from $z_{n-\lambda}$ to z_n , while the second one is fixed in z_{n+1} ;
- on the interval $[n + \lambda, 2n + 1]$ the endpoints are fixed in z_n and z_{n+1} , respectively.

For what concerns the values $n + 1 \leq \lambda \leq n + 2$, let $P' : [0, n + 1] \rightarrow \mathbb{R}^2$ be a parametrization of the segment whose endpoints are z_0 and z_{n+1} and $P'' : [-1, n] \rightarrow \mathbb{R}^2$ a parametrization of the segment whose endpoints are z_{-1} and z_n . The goal is constructing a homotopy between d_{n+1} and the curve

$$d_{n+2}(t) = \begin{cases} D(z_{-1}, P'(t+1)) & \text{if } t \in [-1, n], \\ D(P''(t-n-1), z_{n+1}) & \text{if } t \in [n, 2n+1]. \end{cases} \quad (2.30)$$

For $t \in [-1, n]$ this curve represents the normalized vector whose endpoints are z_{-1} and a point which runs along the segment $\overrightarrow{z_0 z_{n+1}}$; for $t \in [n, 2n+1]$ it represents the normalized vector whose endpoints are a point which runs along the segment $\overrightarrow{z_{-1} z_n}$ and the point z_{n+1} (see figure 13).

The required homotopy has the form

$$\begin{aligned} d_{n+1+\mu}(t) &= \\ &= \begin{cases} D(z_{-1}, (1-\mu)P(t+1) + \mu P'(t+1)) & \text{if } t \in [-1, n], \\ D((1-\mu)P(t-n-1) + \mu P''(t-n-1), z_{n+1}) & \text{if } t \in [n, 2n+1]. \end{cases} \end{aligned}$$

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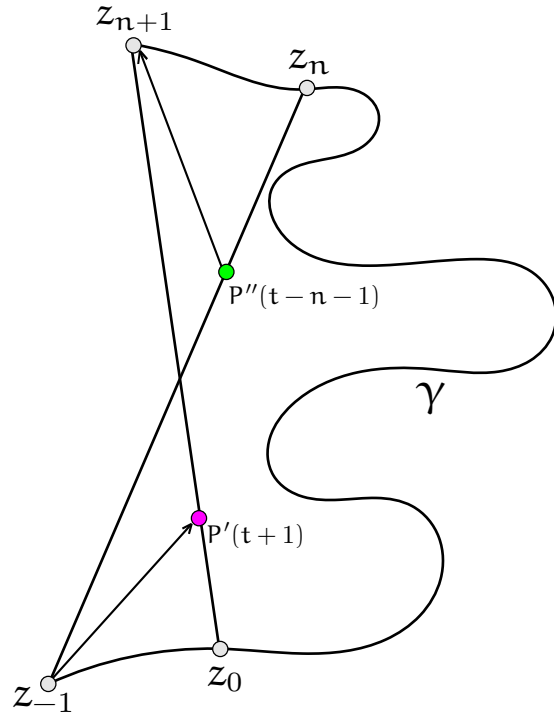


Figure 13: The map d_{n+2} consists in the normalization of a vector which has the first endpoint fixed in z_{-1} and the second one running along P' from z_0 to z_{n+1} during the time interval $[-1, n]$; during the second interval, the first endpoint of the vector is running along P'' from z_{-1} to z_n , while the second endpoint is fixed in z_{n+1} .

for $\mu \in [0, 1]$. Note that when $\mu = 0$ we obtain back the expression of d_{n+1} and when $\mu = 1$ we obtain d_{n+2} , as required. In order to make the definition clearer, consider, for instance, the case $\mu = 1/2$; the behaviour of $d_{n+1+\frac{1}{2}}$ corresponds to the normalization of a vector which moves in the plane as follows:

- during the first time interval, the first endpoint is fixed in z_{-1} , while the second one is the mean point of a segment \overrightarrow{AB} such that A is running from z_0 to z_{n+1} along γ and B is running from z_0 to $P'(n+1) = z_{n+1}$ along the segment $\overrightarrow{z_0 z_{n+1}}$;
- during the second time interval, the first endpoint is the mean point of a segment \overrightarrow{CD} such that C is running from z_{-1} to z_n along γ and D is running from z_{-1} to $P''(n) = z_n$ along the segment $\overrightarrow{z_{-1} z_n}$, while the second endpoint is fixed in z_{n+1} .

Also this homotopy is well-defined. Indeed, assume first by contradiction that a point $Q = (1 - \mu)P(t+1) + \mu P''(t+1)$ coincides with z_{-1}

2.4 The proof by Brown and Neumann

for some $t \in [-1, n]$. By the properties of γ , this can happen only if $\mu = 0$ and $t = -2$ or $\mu = 1$ and $t = -2$. But none of these cases is possible. In the same way it is easy to prove that also the second part of the definition of $d_{n+1+\mu}$ is well-posed.

For the special form of the curve d_{n+2} , we can compute its index. Indeed, the support of d_{n+2} corresponds to the arc on S^1 delimited by the angles

$$D(z_{-1}, z_0) = \arctan \frac{\delta_2}{s_2(z_{-1})} \quad \text{and} \quad D(z_n, z_{n+1}) = \pi - \arctan \frac{\delta_1}{s_1(z_n)}$$

hence, the winding number of d_{n+2} is equal to the right-hand side of equation (2.23) (see figure 13). Since the winding number is invariant under homotopies, we can conclude that

$$\begin{aligned} w(d_{n+2}) &= w(d_0) = i_\gamma(T \circ h) \\ &= \pi - \arctan \frac{\delta_1}{s_1(\vartheta_n)} - \arctan \frac{\delta_2}{s_2(\vartheta_{-1})} \in]\frac{1}{4}, \frac{1}{2}]. \end{aligned} \quad (2.31)$$

As a last step, using (2.31), we can finally compute $i_\gamma(h)$. To this end, we are going to deform the map T and to construct a homotopy between T and the identity map. Indeed, for every $s \in [0, 1]$ define the map $T_s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T_s(\vartheta, r) = (\vartheta, r + s \frac{\varepsilon}{2} (|\cos \vartheta| - \cos \vartheta))$$

such that $T_0 = \text{id}$ and $T_1 = T$. With the same argument previously used for T , we can obtain an estimate for $i_\gamma(T_s \circ h)$

$$i_\gamma(T_s \circ h) \equiv_1 \frac{\Delta \vartheta}{2\pi} = \frac{1}{2} - \frac{1}{2\pi} \left(\arctan \frac{s\delta_1}{s_1(\vartheta_n)} + \arctan \frac{s\delta_2}{s_2(\vartheta_{-1})} \right). \quad (2.32)$$

Observe now that for $s = 1$ the congruence in (2.32) becomes an equality; then, using the invariance of the index, we conclude that it must be an equality also for $s = 0$. This leads to the final formula of the index

$$i_\gamma(h) = i_\gamma(T_0 \circ h) = \frac{1}{2} - 0 = \frac{1}{2}.$$

□

At this point, using the independence of the index on the curve (see lemma 2.22), we have proved that every curve γ which goes from H_a to H_b avoiding all the fixed points of h has index equal to $\frac{1}{2}$.

The absurd will arise showing that there exists another curve with the required properties, but with index different from $\frac{1}{2}$. In order to do this, we can repeat the above argument replacing h with h^{-1} . In this way, the sets H_a and H_b are moved in opposite directions with respect

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to the previous case. Repeating all the argument, we will provide a curve γ' such that $i_{\gamma'}(h^{-1}) = -\frac{1}{2}$. Then from property 4 of the index (see appendix A) we finally gain the required contradiction

$$-\frac{1}{2} = i_{\gamma'}(h^{-1}) = i_{h^{-1}(\gamma')}(h) = i_{\gamma}(h) = \frac{1}{2}$$

which allows to conclude that h has at least two distinct families of fixed points.

2.5 THE PROOF BY BIRKHOFF

As said in section 2.3, considering theorem 2.10, in order to reach the thesis of theorem 2.2 it is sufficient to prove that the homeomorphism φ has at least one fixed point. For this reason, I am going now to reexamine the original proof provided by Birkhoff, following what he exposed in [11, chapter VI].

The framework and the construction are very similar to the ones exposed in section 2.4, therefore many details will be skipped and the exposition will be more concise. Consider the infinite strip $\tilde{A} = \mathbb{R} \times [a, b]$ and an area-preserving homeomorphism $h : \tilde{A} \rightarrow \tilde{A}$ which is periodic in the ϑ -coordinate and leaves the boundaries invariant. Assume that the points of the bottom boundary $\tilde{A}_i = \{(\vartheta, r) : r = a\}$ are moved by h to the right, while the ones of $\tilde{A}_o = \{(\vartheta, r) : r = b\}$ are moved to the left; extend h to the whole plane \mathbb{R}^2 as done in (2.9).

Assume, by contradiction, that h has no fixed points in \tilde{A} . Then there exists a value $\delta > 0$ such that for every $z \in \tilde{A}$

$$\|h(z) - z\| > \delta.$$

Consider an auxiliary transformation of the plane \mathbb{R}^2 defined by

$$T_\varepsilon : (\vartheta, r) \mapsto (\vartheta, r + \varepsilon) \quad (2.33)$$

for a fixed $0 < \varepsilon < \delta$ and the resulting composite map $T_\varepsilon \circ h$ which is area-preserving too and translates \tilde{A} to the strip $\mathbb{R} \times [a + \varepsilon, b + \varepsilon]$. Define as H_ε the narrow strip $\mathbb{R} \times [a, a + \varepsilon]$ and observe that its bottom boundary is moved by $T_\varepsilon \circ h$ into its upper boundary. Moreover, every point of H_ε is moved to a point with radial coordinate $r \geq a + \varepsilon$. Indeed $(T_\varepsilon \circ h)(H_\varepsilon) \subset \{(\vartheta, r) : r \geq a + \varepsilon\}$. Going on applying the homeomorphism $T_\varepsilon \circ h$, we obtain the sequence of closed periodic sets

$$(T_\varepsilon \circ h)^j(H_\varepsilon) \quad (2.34)$$

all having the same area and filling all the set \tilde{A} . Then there exists an index $n > 0$ such that

$$(T_\varepsilon \circ h)^n(H_\varepsilon) \cap \tilde{A}_o \neq \emptyset \quad (2.35)$$

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and a point $z_n \in (T_\varepsilon \circ h)^n(H_\varepsilon)$, with maximal radial coordinate $r_n \geq b$. Define now the orbit of z_n , starting from $z_0 = (T_\varepsilon \circ h)^{-n}(z_n)$, such that

$$z_i = (\vartheta_i, r_i) = (T_\varepsilon \circ h)^{i-n}(z_n) \quad \forall i \geq 0;$$

construct a curve $\gamma = \gamma_1 \cdots \gamma_n \gamma_{n+1}$, running from z_0 to z_{n+1} , where γ_1 is the segment connecting z_0 and z_1 and $\gamma_j = (T_\varepsilon \circ h)(\gamma_{j-1})$ for $j \geq 2$. Due to the behaviour of h on the boundaries of \tilde{A} , the angle $\vartheta(0) = D(z_0, z_1) \in]0, \pi/2[$ and the angle $\vartheta(1) = D(z_n, z_{n+1}) \in]\pi/2, \pi[$ and therefore the index $i_\gamma(T_\varepsilon \circ h) \equiv_1 \Delta\vartheta = \vartheta(1) - \vartheta(0) \in]0, \pi[$. Since $T_\varepsilon \circ h$ has no fixed points in \tilde{A} , we can homotopically deform γ into a curve $\gamma^* = \gamma_1 \sigma \gamma_{n+1}$ where σ is a parametrization of the segment $\overrightarrow{z_1 z_n}$. Then, from the invariance of the index, we have

$$i_\gamma(T_\varepsilon \circ h) = i_{\gamma^*}(T_\varepsilon \circ h) = \Delta\vartheta \in]0, \pi[.$$

When $\varepsilon \rightarrow 0$, we conclude that $i_\gamma(h) = \Delta\vartheta$.

Considering now the inverse map h^{-1} and repeating the construction above, we can find a curve γ' such that

$$i_{\gamma'}(h^{-1}) \in]-\pi, 0[;$$

on the other hand, using properties 1 and 4 of the index,

$$i_{\gamma'}(h^{-1}) = i_{h^{-1}(\gamma')}(h) = i_\gamma(h) \in]0, \pi[$$

in contradiction with what proved above. Then h must have at least one family of fixed points.

2.6 THE CASE OF A HOLED DISC

As already mentioned in section 2.3, in [52] Jacobowitz stated a version of the Poincaré-Birkhoff theorem for a pointed disc whose external boundary is a simple closed curve; however, the proof in [52] is only sketched and many details are missed. In this section, a complete proof of a related result, which is a weaker version of theorem 2.16, will be provided, following closely a work by Rebelo [96], in which the theorem is proved by a direct reduction to the standard Poincaré-Birkhoff theorem.

Theorem 2.24 *Consider a pointed disc $A_1 = A(0, R]$, such that its external boundary is the circumference $\Gamma_1 = C_R$ and let $A_2 = (0, \Gamma_2]$ be a generalized pointed disc whose boundary is a simple closed curve Γ_2 surrounding the origin. Assume that there exists a homeomorphism $\varphi : A_1 \rightarrow A_2$ such that its lifting to the covering space $\mathbb{R} \times \mathbb{R}_0^+$ has the form*

$$h(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r)) \quad (2.36)$$

where s and f are continuous functions, 2π -periodic in ϑ . If

2 The Poincaré-Birkhoff theorem: one century of research

- $s(\vartheta, R) < 0$ for all $\vartheta \in \mathbb{R}$
- $\liminf_{r \rightarrow 0} s(\vartheta, r) > 0$ uniformly in ϑ ,

then φ has at least two fixed points in the interior of A_1 .

Proof. The lifting of A_1 is the infinite strip $\tilde{A}_1 = \mathbb{R} \times]0, R]$, whose outer boundary is the straight line $\tilde{\Gamma}_1 = \{(\vartheta, r) : r = R\}$; the lifting of A_2 is the strip bounded by the x -axis and the curve $\tilde{\Gamma}_2 = \pi^{-1}(\varphi(\Gamma_1))$ which is a simple and periodic curve in $\mathbb{R} \times \mathbb{R}_0^+$.

To begin with, we observe that $h(\tilde{\Gamma}_1) = \tilde{\Gamma}_2$ and $\lim_{r \rightarrow 0} f(\vartheta, r) = 0$, uniformly in $\vartheta \in \mathbb{R}$. From the twist condition, there exists $\varepsilon_1 > 0$ such that

$$0 < \varepsilon_1 < \liminf_{r \rightarrow 0} s(\vartheta, r) \quad (2.37)$$

and there exists $\varepsilon_2 > 0$ such that

$$0 < \varepsilon_2 < -s(\vartheta, R) \quad \forall \vartheta \in \mathbb{R}. \quad (2.38)$$

From (2.37), we can choose a radius $r_2 < R$ such that $s(\vartheta, r_2) > \varepsilon_1$ for all $\vartheta \in \mathbb{R}$; then, moving to the lifting, we obtain the straight line

$$\tilde{C}_{r_2} = \pi^{-1}(C_{r_2}) \subset \text{int } \tilde{A}_1$$

whose image under h is contained in $\text{int } \tilde{A}_2$ and does not intersect $\tilde{\Gamma}_2$. By construction, we also have

$$\varphi(A[r_2, R]) = \text{cl}(D(\Gamma_2)) \setminus D(\varphi(C_{r_2}))$$

or, equivalently, in terms of the lifting $h(\tilde{A}[r_2, R])$ is the strip bounded by $h(\tilde{C}_{r_2})$ and $\tilde{\Gamma}_2$ (see figure 14).

Since $\Gamma_3 \stackrel{\text{def}}{=} \varphi(C_{r_2})$ is contained in the interior of A_2 and therefore $0 \in D(\Gamma_3)$, we can choose a value

$$r_1 \in]0, r_2[\quad \text{such that} \quad C_{r_1} \subset D(\Gamma_3)$$

or, equivalently, \tilde{C}_{r_1} is contained in the interior of the strip whose upper boundary is $\tilde{\Gamma}_3$. Finally, choose $R^* > 0$ such that $\Gamma_2 \subset B(0, R^*)$ and therefore $\tilde{\Gamma}_2$ is contained in the strip whose upper boundary is the line \tilde{C}_{R^*} . Due to the area-preserving condition, we also have $R < R^*$ and therefore $A_1 \subset B(0, R^*)$. In figure 14 all the construction is schematized.

We are going to apply the classical Poincaré-Birkhoff theorem to the annulus $A[r_1, R^*]$ and to a homeomorphism

$$\varphi' : A[r_1, R^*] \rightarrow A[r_1, R^*]$$

which will coincide with φ on $A[r_2, R]$.

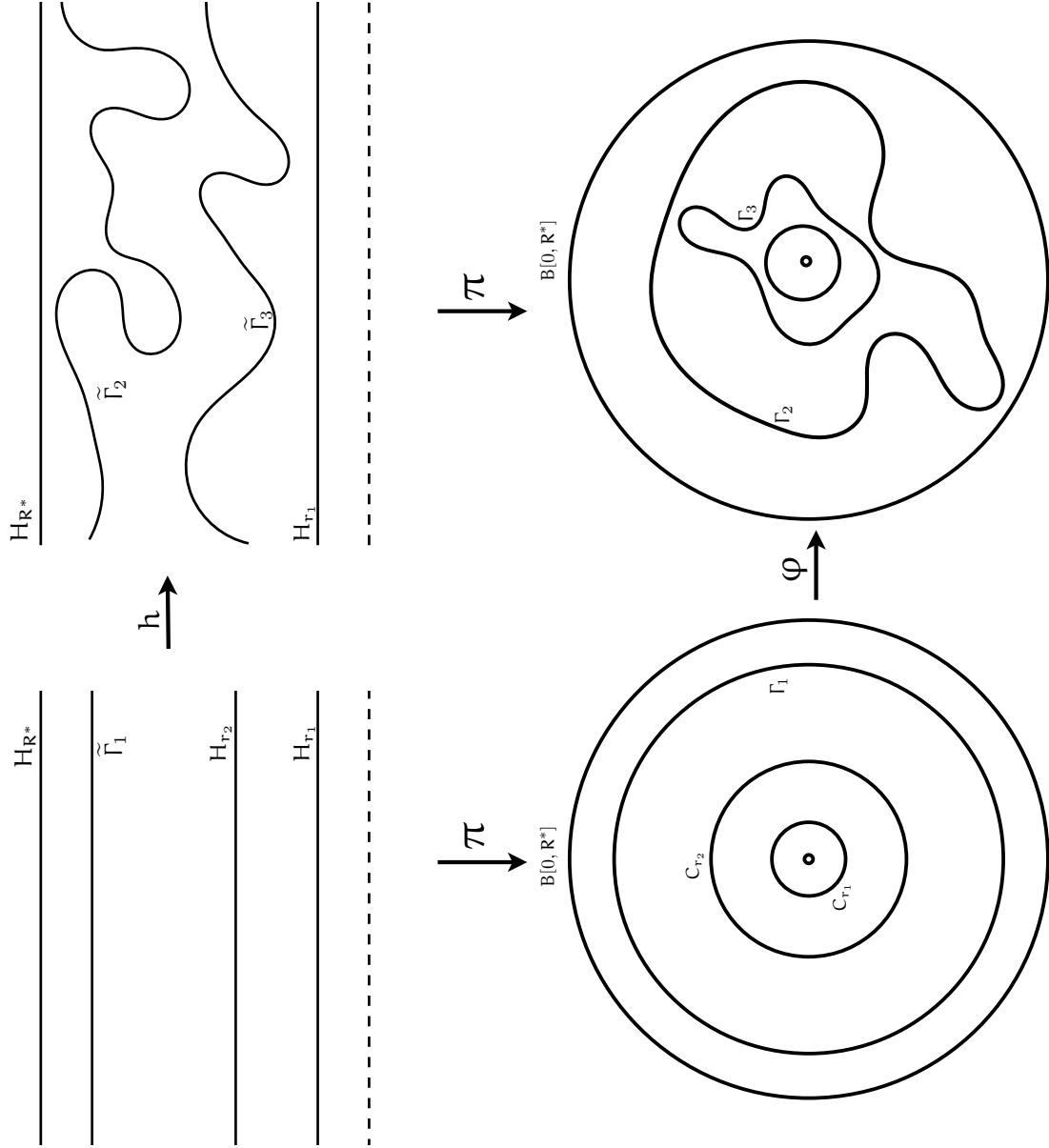


Figure 14: The constructions involved in the proof of theorem 2.24

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Let $\eta_1 : A[r_1, r_2] \rightarrow A[r_1, \Gamma_3]$ be a homeomorphism such that $\eta_1 : C_{r_2} \mapsto \varphi(C_{r_2})$ and leaving the circumference C_{r_1} fixed. Since φ is area-preserving, $\mu(B[0, r_2]) = \mu(\text{cl } D(\Gamma_3))$ and therefore $\mu(A[r_1, r_2]) = \mu(A[r_1, \Gamma_3])$. Then there exists an area-preserving homeomorphism $\eta_1^* : A[r_1, r_2] \rightarrow A[r_1, \Gamma_3]$ which coincides with η_1 on the boundaries of its domain, so that

$$\eta_1^*|_{C_{r_2}} = h \quad \text{and} \quad \eta_1^* : C_{r_1} \mapsto C_{r_1}.$$

This is guaranteed by a result in [71, Chapter 13]. In the same way, there exists a homeomorphism $\eta_2 : A[R, R^*] \rightarrow A[\Gamma_2, R^*]$ which coincides with h on the circumference Γ_1 and leaves C_{R^*} invariant. Using the area-preserving condition for φ , the measure of $A[R, R^*]$ is the same of the one of $A[\Gamma_2, R^*]$ and therefore there exists an area-preserving homeomorphism η_2^* such that $\eta_2^* : A[R, R^*] \rightarrow A[\Gamma_2, R^*]$, with

$$\eta_2^*|_{C_R} = h \quad \text{and} \quad \eta_2^* : C_{R^*} \mapsto C_{R^*}.$$

Define now a new area-preserving homeomorphism φ' , combining φ with η_1^* and η_2^* ; the map φ' is defined on the set $A[r_1, R]$, takes values in $A[r_1, R]$ and is given by

$$\varphi' = \begin{cases} \eta_1^* & \text{on } A[r_1, r_2], \\ \varphi & \text{on } A[r_2, R], \\ \eta_2^* & \text{on } A[R, R^*]. \end{cases} \quad (2.39)$$

Its lifting to $\mathbb{R} \times \mathbb{R}_0^+$ can be expressed as

$$h'(\vartheta, r) = (\vartheta + s^*(\vartheta, r), f^*(\vartheta, r))$$

with s^* and f^* some continuous functions, 2π -periodic in the ϑ -variable. By construction, h' agrees with h on $\tilde{A}[r_2, R]$ and, in particular, on the boundaries \tilde{C}_{r_2} and \tilde{C}_R . From the choice of r_2 , we have that $s^*(\vartheta, r_2) = s(\vartheta, r_2) > \varepsilon_1$ for every $\vartheta \in \mathbb{R}$ and then there exists a value δ_1 such that

$$s^*(\vartheta, r) > \varepsilon_1 \quad \forall r \in]r_2 - \delta_1, r_2] \quad \forall \vartheta \in \mathbb{R};$$

in the same way, since $-s^*(\vartheta, R) = -s(\vartheta, R) > \varepsilon_2$, there exists δ_2 such that

$$-s^*(\vartheta, r) > \varepsilon_2 \quad \forall r \in [R, R + \delta_2[\quad \forall \vartheta \in \mathbb{R}.$$

Let now M be a constant defined as

$$M = \sup\{1 + |s^*(\vartheta, r)| : (\vartheta, r) \in \tilde{A}[r_1, R^*]\}$$

and consider the area-preserving homeomorphism $\psi_1 : A[r_1, R^*] \rightarrow A[r_1, R^*]$ whose lifting is defined by

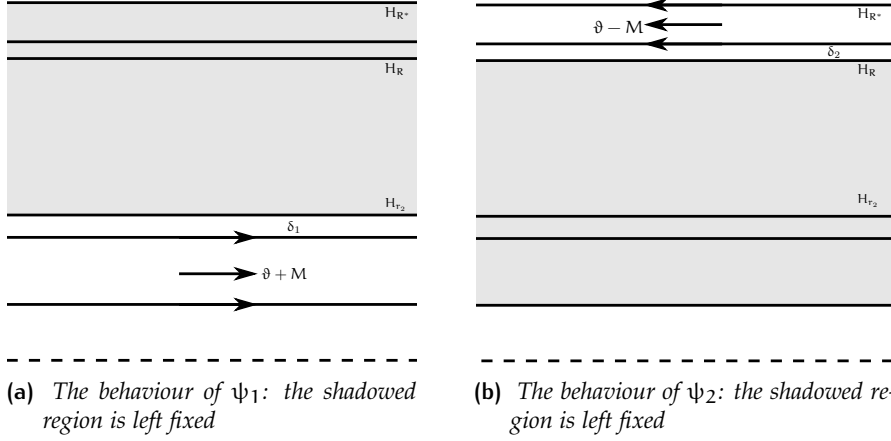
$$\tilde{\psi}_1(\vartheta, r) = (\vartheta + M\xi_1(r), r) \quad (2.40)$$

with $\xi_1 : \mathbb{R} \rightarrow \mathbb{R}$ a positive smooth function such that

2.6 The case of a holed disc

- $\xi_1(r) = 0$ if $r \geq r_2$
- $\xi_1(r) = 1$ if $r \leq r_2 - \delta_1$

and therefore $\psi_1 = \text{id}$ on $A[r_2, R^*]$ (see figure 15a). Similarly, define



$\psi_2 : A[r_1, R^*] \rightarrow A[r_1, R^*]$ whose lifting has the form

$$\tilde{\psi}_2(\vartheta, r) = (\vartheta - M\xi_2(r), r) \quad (2.41)$$

with $\xi_2 : \mathbb{R} \rightarrow \mathbb{R}$ a positive smooth function such that

- $\xi_2(r) = 0$ if $r \leq R$
- $\xi_2(r) = 1$ if $r \geq R + \delta_2$

and therefore $\psi_2 = \text{id}$ on $A[r_1, R]$ (see figure 15b).

Let now $z = (\vartheta, r)$ be a point with $r \in [r_1, r_2]$, then

$$\begin{aligned} (h' \circ \psi_1)(\vartheta, r) &= h(\vartheta + M\xi_1(r), r) \\ &= (\vartheta + M\xi_1(r) + s^*(\vartheta + M\xi_1(r), r), f^*(\vartheta + M\xi_1(r), r)) \\ &= (\vartheta^*, r^*) \end{aligned}$$

with $\vartheta^* > \vartheta$, while, developing similar computations, $(h' \circ \psi_2)(\vartheta, r) = (\vartheta^*, r^*)$ with $\vartheta^* < \vartheta$ if $r \in [R, R^*]$.

Consider now the area-preserving homeomorphism

$$\Psi \stackrel{\text{def}}{=} h' \circ \psi_2 \circ \psi_1 : A[r_1, R^*] \rightarrow A[r_1, R^*]$$

and its lifting

$$\tilde{\Psi}(\vartheta, r) = (\vartheta + s_*(\vartheta, r), f_*(\vartheta, r))$$

with $s_*(\vartheta, r_1) > 0$ and $s^*(\vartheta, R^*) < 0$ for all $\vartheta \in \mathbb{R}$. Then Ψ satisfies all the hypotheses of the Poincaré-Birkhoff theorem and therefore it has two fixed points in $A(r_1, R^*)$. Moreover, since Ψ rotates the sets $A[R, R^*]$ and $A[r_1, r_2]$, the fixed points of Ψ are fixed points for φ . \square

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Theorem 2.24 can be extended to the case in which the external boundary of the pointed disc is a strictly star-shaped curve, surrounding the origin.

Corollary 2.25 *Consider two generalized pointed discs $A_1 = A(0, \Gamma_1]$ and $A_2 = A(0, \Gamma_2]$ whose external boundaries are strictly star-shaped curves which surround the origin. Assume that there exists a homeomorphism $\varphi : A_1 \rightarrow A_2$ such that its lifting to the covering space $\mathbb{R} \times \mathbb{R}_0^+$ has the form*

$$\tilde{\varphi}(\vartheta, r) = h(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r)) \quad (2.42)$$

where s and f are continuous functions, 2π -periodic in ϑ . If

- $s(\vartheta, r) < 0$ for all $(\vartheta, r) \in \Gamma_1$
- $\liminf_{r \rightarrow 0} s(\vartheta, r) > 0$ uniformly in ϑ ,

then φ has at least two fixed points in the interior of A_1 .

Proof. The basic idea of this corollary consists in transforming A_1 into a standard pointed disc, on which we can subsequently apply theorem 2.24. Since Γ_1 is a strictly star-shaped curve, it can be parametrized by a 2π -periodic and continuous function

$$\rho : \mathbb{R} \rightarrow \mathbb{R}_0^+ \quad \text{with} \quad \vartheta \mapsto \rho(\vartheta)$$

such that

$$\tilde{\Gamma}_1 = \{(\vartheta, \rho(\vartheta)) : \vartheta \in \mathbb{R}\}.$$

Defining μ as the mean-value of ρ , that is

$$\mu = \frac{1}{2\pi} \int_0^{2\pi} \rho(s) ds,$$

we can construct an area-preserving homeomorphism $\zeta : \mathbb{R}^2 \setminus \{O\} \rightarrow \mathbb{R}^2 \setminus \{O\}$ whose lifting $\tilde{\zeta}$ transforms $\tilde{\Gamma}_1$ onto \tilde{C}_μ .

Consider now the map

$$\zeta \circ \Psi \circ \zeta^{-1} : A(0, \mu] \rightarrow A_3 = \zeta(A_2) \quad (2.43)$$

which is an area-preserving homeomorphism too. It also satisfies the twist-condition (see [96] for all the computations) and therefore we can apply theorem 2.24 to it, since its domain is a standard pointed annulus, as required. Then there exist two fixed points (x_i^*, y_i^*) for $i = 1, 2$ such that

$$\zeta(\Psi(\zeta^{-1}(z_i^*))) = z_i^*$$

and naming $z_i' = \zeta(z_i^*)$ we find

$$\Psi(z_i') = z_i'$$

for $i = 1, 2$. □

2.7 Counterexamples and open problems

In the application of the Poincaré-Birkhoff theorem to differential equations, we often deal with domains whose boundaries are orbits of dynamical systems, which are strictly star-shaped curves, under a topological point of view. In this context, the following corollary turns out to be the most useful version of the Poincaré-Birkhoff theorem.

Corollary 2.26 *Let $A = A[\Gamma_1, \Gamma_2]$ be an annulus with strictly star-shaped boundaries Γ_1, Γ_2 , with $0 \in D(\Gamma_1)$. Let $\varphi : A \rightarrow \varphi(A)$ be an area-preserving twist homeomorphism such that there exists a homeomorphism $\varphi_0 : D(\Gamma_2) \rightarrow \mathbb{R}^2$ with $\varphi_0(0) = 0$ and $\varphi_0|_A = \varphi$. Then φ has two fixed points.*

Proof. The proof is developed applying corollary 2.25 to the holed disc $A(0, \Gamma_2]$. We define an auxiliary homeomorphism h which rotates the set $D(\Gamma_1)$ in a convenient way (see [30, Lemma 2] for the details) and leaves fixed the other points of the plane. In this way, the homeomorphism $h \circ \Psi$ satisfies the hypothesis of the corollary of theorem 2.24 and therefore it has two fixed points.

Eventually observe that h has no fixed point by definition, then Ψ has two fixed points in A as required. \square

Note that the condition $\varphi_0(0) = 0$ can be weakened and replaced with $\varphi_0(0) \in D(\varphi(\Gamma_1))$.

2.7 COUNTEREXAMPLES AND OPEN PROBLEMS

The most used version of the Poincaré-Birkhoff theorem is the one by Ding (theorem 2.18), in which the weaker hypotheses on the shape of the boundaries make its statement more suitable for the applications. In the last years, it has been proved that almost all the assumptions of that theorem are necessary and cannot be removed from the statement. Nevertheless the question about the possibility of removing the condition on the inner boundary, allowing it being a simple curve instead of a star-shaped one, was still an open problem, as remarked by Ding himself in [30]:

The condition is crucial for our proof. However, we doubt of its necessity for the theorem.

On the other hand nobody could success in providing a proof of the theorem which did not use that assumption. This open problem was definitively solved by Martins and Urena in 2007; in [66], they prove that the condition about the star-shapeness of the inner boundary can not be removed, providing an explicit example of an annular domain on which an area-preserving homeomorphism has not fixed points. More in detail, they proved the following theorem.

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Theorem 2.27 *Let $A[\Gamma_1, \Gamma_2]$ be an annular domain whose boundaries are two simple closed curves. Then there exists a C^∞ diffeomorphism $\varphi : A \rightarrow \varphi(A) \subset \mathbb{R}^2 \setminus \{O\}$ such that*

- *it satisfies the twist condition*
- *there exists an extension $\varphi_0 : \text{cl} D(\Gamma_2) \rightarrow \mathbb{R}^2$ with $\varphi_0|_A = \varphi$ and $O \in \varphi_0(D(\Gamma_1))$*
- *φ has no fixed points in A .*

The authors expose also an intuitive argument which explain the basic idea of their paper and the reason for which is not possible to remove that condition without replacing it with some other hypothesis. Indeed, let $\varphi(\vartheta, r) = (\vartheta + s(\vartheta, r), f(\vartheta, r))$ be an area-preserving homeomorphism of the annulus $A[\Gamma_1, \Gamma_2]$ and let Γ be the set of the points with angular displacement equal to zero; it is reasonable to think that in some cases the set Γ can be a Jordan curve. If Γ_1 is not star-shaped, then we can not discard the case in which the inequality $f(\vartheta, r) > r$ can be satisfied on the whole curve Γ (note that this assumption is not possible if Γ_1 is strictly star-shaped, otherwise the condition of area-preserving would be violated). But this immediately implies that φ has not fixed points in A (see [66, figure 2]).

The importance of the explicit example provided in [66] is remarked also by Le Calvez e Wang in [62, Remark 4] where it is said that the proof of the Poincaré-Birkhoff theorem

may fail if none of the loops projects injectively onto S^1 , unlike what is said in [30] and [52].

Indeed the authors conclude their remark constructing of an area-preserving and fixed point free homeomorphism satisfying the twist condition.

Hence in conclusion we can say that the most general version of the Poincaré-Birkhoff theorem that we can obtain is corollary 2.26 where the star-shapeness condition can not be removed.

2.8 AN APPLICATION

In this last section an application of the Poincaré-Birkhoff theorem to the problem of finding subharmonic solutions of a second order ODE is presented.

The system we are going to study is related to the planar nonautonomous Kolmogorov system which describes and models the interaction of two species

$$\begin{cases} p' &= pP(t, p, q) \\ q' &= qQ(t, p, q) \end{cases} \quad (2.44)$$

with P and Q two continuous functions defined on $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ to \mathbb{R} , which are T -periodic in the first variable for some $T > 0$. System (2.44) can be seen as a generalization of Lotka-Volterra prey-predator equations, which can be obtained choosing

$$P(t, p, q) = a - cq \quad Q(t, p, q) = -d + ep$$

with a, c, d, e positive constants. In recent years, the attention has been focused on the case of time-varying coefficients; one of the simplest cases consists in transforming the constants of the Lotka-Volterra model into four functions of the time variable, all periodic of the same period $T > 0$. For instance, we can set $P(t, p, q) = a(t) - b(t)p - c(t)q$, $Q(t, p, q) = -d(t) + g(t)pq$ where all the coefficients are non-negative as in [17] or $P(t, p, q) = a(t) - b(t)p - c(t)q$, $Q(t, p, q) = -d(t) + e(t)p - f(t)q$ with b, c, e, f positive continuous functions as in [63]. In the papers dealing with such kind of applications, conditions about the coefficients have to be imposed in order to obtain the existence of a positive periodic solution, the asymptotic stability of that solution or, more in general, the existence of a compact attractor in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.

On the other hand, if we consider the simpler choice $P(p, t, q) = a(t) - c(t)q$, $Q(t, p, q) = -d(t) + e(t)p$ with a, c, d, e positive and continuous functions having a common period, corresponding to a Lotka-Volterra model with periodic coefficients, then in this case it is not possible to find a compact attractor or the asymptotic stability of a possible periodic solution. In this case, which presents analogies with the periodically perturbed Duffing's equation $u'' + g(u) = e(t)$, some results have been obtained via the Moser twist theorem, the bifurcation theory and by the generalized Poincaré-Birkhoff theorem (see [28] and the references therein).

The application herein presented deals with the generic system

$$\begin{cases} p' &= pP(t, q) \\ q' &= qQ(t, p) \end{cases} \quad (2.45)$$

with the assumptions $p(t) > 0$ and $q(t) > 0$ for every t . Via the change of variable $u = \log p$, $v = \log q$ it is possible to transform system (2.45) into the form

$$\begin{cases} u' &= U(t, v) \\ v' &= V(t, u) \end{cases} \quad (2.46)$$

with $U(t, v) = P(t, e^v)$, $V(t, u) = Q(t, e^u)$, in such a way that there is a one-to-one correspondence between the periodic solutions of the two systems. Thus, we will look for subharmonic solutions of system (2.46), with $U, V : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ two continuous functions, T -periodic in the time variable. The scheme adopted in this kind of applications consists in proving the existence of one periodic solution via some

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topological-degree theorems; then, via a change of variables, proving the existence of subharmonics using the Poincaré-Birkhoff fixed point theorem.

First of all, we state some hypotheses about the boundedness of these functions; indeed we will assume that there exist four continuous and T -periodic functions $\alpha_-, \alpha_+, \beta_-, \beta_+$ such that

- $\alpha_-(t) \leq \liminf_{s \rightarrow -\infty} U(t, s)$ and $\limsup_{s \rightarrow +\infty} U(t, s) \leq \alpha_+(t)$
- $\beta_+(t) \leq \liminf_{s \rightarrow -\infty} V(t, s)$ and $\limsup_{s \rightarrow +\infty} V(t, s) \leq \beta_-(t)$

uniformly in $t \in [0, T]$; moreover there exists a continuous map $\gamma : \mathbb{R} \rightarrow \mathbb{R}_0^+$ such that one of the two conditions

- $-\gamma(t) < U(t, s) < \gamma(t)$
- $-\gamma(t) < V(t, s) < \gamma(t)$

holds for every $t, s \in \mathbb{R}$. Under these assumptions, there exists a periodic solution $(u_0(t), v_0(t))$ of system (2.46) if

$$\begin{aligned} \int_0^T \alpha_+(t) dt < 0 < \int_0^T \alpha_-(t) dt \quad \text{and} \\ \int_0^T \beta_-(t) dt < 0 < \int_0^T \beta_+(t) dt. \end{aligned} \quad (2.47)$$

Moreover, there exists a constant $r_1 > 0$ such that $\|(u_0(t), v_0(t))\| < r_1\sqrt{2}$ for every $t \in \mathbb{R}$. The result follows from the application of Mawhin's continuation theorem [67], which is based on the theory of the Leray-Schauder degree.

In order to apply the Poincaré-Birkhoff theorem and find subharmonic solutions for system (2.46), we also assume conditions guaranteeing the uniqueness of the solutions for the Cauchy problem associated to (2.46). Moreover, we assume that at least one between $-U(t, \cdot)$ and $V(t, \cdot)$ is strictly increasing for every $t \in [0, T]$.

Having to prove the twist condition of the Poincaré-Birkhoff theorem, we need a tool which counts the turns of a solution around the origin. Thus, let $\zeta : I \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a $\mathcal{C}^1(I)$ function defined on an interval I such that $\zeta(t) = (x(t), y(t))$. For every pair $t, s \in I$ we define the usual rotation number along the interval $[s, t]$ denoted by

$$w_\zeta(s, t) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_s^t \frac{y'(\xi)x(\xi) - x'(\xi)y(\xi)}{x(\xi)^2 + y(\xi)^2} d\xi. \quad (2.48)$$

Let now $(u_0(t), v_0(t))$ be the T -periodic solution of system (2.46); perform a change of variables setting $x(t) = u(t) - u_0(t)$ and $y(t) = v(t) - v_0(t)$ thus obtaining the equivalent system

$$\begin{cases} x' = X(t, y) = U(t, v_0(t) + y) - U(t, v_0(t)) \\ y' = Y(t, x) = V(t, u_0(t) + x) - V(t, u_0(t)). \end{cases} \quad (2.49)$$

Just to clarify things, we assume U upper bounded and $V(t, \cdot)$ strictly increasing; then the conditions on the field (X, Y) can be summarized as follows

- $X(t, 0) = Y(t, 0) = 0$ for every $t \in \mathbb{R}$;
- $Y(t, x)x > 0$ for every $t \in \mathbb{R}$ and for every $x \neq 0$ (due to the monotonicity of V);
- there exists a constant $M > r_1 \sqrt{2}$ and there exist four continuous and T -periodic functions k_-, k_+, ℓ_-, ℓ_+ with

$$\int_0^T k_+(t) dt < 0 < \int_0^T k_-(t) dt \quad \text{and} \\ \int_0^T \ell_-(t) dt < 0 < \int_0^T \ell_+(t) dt.$$

such that

- $X(t, s) \geq k_-(t)$ and $Y(t, s) \leq \ell_-(t)$ for every $s \leq -M$ and for every $t \in \mathbb{R}$
- $X(t, s) \leq k_+(t)$ and $Y(t, s) \geq \ell_+(t)$ for every $s \geq M$ and for every $t \in \mathbb{R}$;
- there exists a continuous and T -periodic function $\Gamma(t) > 0$ such that $X(t, s) \leq \Gamma(t)$ for every $t, s \in \mathbb{R}$.

First of all, we denote by \bar{f} the mean-value of a generic function f on the interval $[0, T]$, that is $\bar{f} = \frac{1}{T} \int_0^T f(s) ds$; afterwards, fix two constants $\eta, K \in \mathbb{R}$ such that

$$0 < \eta < \min\{\bar{k}_+, \bar{k}_-, \bar{\ell}_+, \bar{\ell}_-\} \quad (2.50)$$

and

$$K > \max\{\|k_+ - \bar{k}_+\|_1, \|k_- - \bar{k}_-\|_1, \|\ell_+ - \bar{\ell}_+\|_1, \|\ell_- - \bar{\ell}_-\|_1\}, \quad (2.51)$$

where $\|\cdot\|_1$ denotes the standard L^1 norm.

Let z_0 be a point in $\mathbb{R}^2 \setminus \{O\}$ and consider the solution $\zeta(t; z_0) = z(t; z_0) = (x(t; z_0), y(t; z_0))$ of (2.49) having z_0 as initial point. For shortening the notations, we will simply write $z(t) = (x(t), y(t))$, omitting the dependence on the initial point z_0 . Then

$$\zeta(t; z_0) = z(t) = (x(t), y(t)) : I_0 =]\alpha, \omega[\rightarrow \mathbb{R}^2 \setminus \{O\}$$

where I_0 is the maximal interval of definition of the solution; moreover, since the Cauchy problems associated to the system under consideration have an unique solution and recalling that $z(t) \equiv 0$ is a solution of system (2.49), then $z(t) \neq O$ for every $t \in I_0$. Therefore it

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is possible to express the solution $\zeta(t; z_0)$ in polar coordinates, setting $z(t) = r(t)(\cos \vartheta(t), \sin \vartheta(t))$ and, recalling the definition of the rotation number, by few computations we obtain

$$w_{\zeta(\cdot; z_0)}(s, t) = \frac{\vartheta(t) - \vartheta(s)}{2\pi} = \frac{1}{2\pi} \int_s^t \frac{Y(\xi, x)x(\xi) - X(\xi, y)y(\xi)}{\|z(\xi)\|^2} d\xi. \quad (2.52)$$

Then the twist condition of the Poincaré-Birkhoff theorem can be expressed in terms of w_ζ since

$$w_{\zeta(\cdot; z_0)}(0, T) < j \Leftrightarrow \vartheta(T) - \vartheta(0) < 2j\pi \quad (2.53)$$

according to (1.4) and (1.5).

Lemma 2.28 *Due to the assumptions on the vector field (X, Y) , we have that for every $s, t \in I_0$ with $s \geq t$*

$$w_{\zeta(\cdot; z_0)}(t, s) > -1/2. \quad (2.54)$$

Proof. Let $t \in I$ be such that $y(t) = 0$ and $x(t) \neq 0$, then $\dot{\vartheta}(t) = Y(t, x)x(t)/|x(t)|^2 > 0$. Hence, for every $k \in \mathbb{Z}$ the set $S_k = \{(\vartheta, r) : \vartheta > k\pi, r > 0\}$ is positively invariant under the action of (X, Y) , that is if $\zeta(s; z_0) \in S_k$, then $\zeta(t; z_0) \in S_k$ for every $t \geq s$. Then $\vartheta(t) > \vartheta(s) - \pi$ and therefore $w_{\zeta(\cdot; z_0)}(t, s) > -1/2$. \square

We need now to prove some inequalities about the behaviour of the solution $z(t)$. The first property is summarized by the following lemma.

Lemma 2.29 *For every choice of two constants A, L with $M < A \leq L$ if $y(t_1) \geq -A$ for some $t_1 \in I_0$ then there exists a constant $B = B(A, L) > A$ such that if $x(t_1) \geq B$ then there exists a time-value $t_2 > t_1$ at which $x(t_2) = A$ and $y(t_2) \geq L$; moreover $x(t) > A$ for every $t \in [t_1, t_2[$.*

Proof. As a first case, choose $t_1 \in]\alpha, \omega[$ such that

$$y(t_1) \geq L + K \quad \text{and} \quad x(t_1) > A. \quad (2.55)$$

Let $t_3 \leq \omega$ be such that $x(t) > A$ for every $t \in [t_1, t_3[$. Since $x(t) > A > M$, then $Y(t, x(t)) = y'(t) \geq \ell_+(t)$ for every $t \in [t_1, t_3[$, which implies

$$\begin{aligned}
 y(t) &= y(t_1) + \int_{t_1}^t y'(s) ds \geq y(t_1) + \int_{t_1}^t \ell_+(s) ds \\
 &\geq y(t_1) + \int_{t_1}^t (\ell_+(s) - \bar{\ell}_+) ds + \int_{t_1}^t \bar{\ell}_+ ds \\
 &= y(t_1) + \bar{\ell}_+(t - t_1) + \int_{t_1}^t (\ell_+(s) - \bar{\ell}_+) ds \\
 &\geq y(t_1) + \bar{\ell}_+(t - t_1) - \int_0^T |\ell_+(s) - \bar{\ell}_+| ds \\
 &\geq y(t_1) + \eta(t - t_1) - K \geq L + K + \eta(t - t_1) - K \\
 &= L + \eta(t - t_1) \geq L \geq A > M;
 \end{aligned}$$

the chain of inequalities leads to $y(t) > M \forall t \in [t_1, t_3[$, hence $x'(t) \leq k_+(t)$ and, as above,

$$x(t) \leq x(t_1) - \eta(t - t_1) + K \quad \forall t \in [t_1, t_3[. \quad (2.56)$$

Let now $m(t)$ be a T -periodic function such that

$$m(t) \geq \max\{|Y(t, x)| : x \in [A, x(t_1) + K]\}$$

then we also have

$$y(t) \leq y(t_1) + \int_{t_1}^t m(s) ds \quad \forall t \in [t_1, t_3[. \quad (2.57)$$

Inequalities (2.56) and (2.57) mean that, whenever the solution $z(t)$ lies into the strip $[A, x(t_1) + K] \times [L, +\infty[$, it is bounded. Then there can not be a blow-up in the time interval $[t_1, t_3[$; this allows to conclude that $t_3 < \omega$ and $x(t_3) = A$ with $y(t_3) \geq L$. Then the statement of the lemma is proved for every choice of $B = B(A, L) > A$ and for $t_2 = t_3$.

As a second case, we assume $-A \leq y(t_1) < L + K$. Let $[t_1, t_4[$ be the maximal interval on which $x(t) > A$ and $y(t) < L + K$, with $t_4 \leq \omega$. As in the first case, for every $t \in [t_1, t_4[$ we have

$$y(t) \geq y(t_1) + \eta(t - t_1) - K \geq -A + \eta(t - t_1) - K \geq -A - K;$$

define the positive T -periodic function

$$n(t) \geq \max\{|X(t, y)| : y \in [-A - K, L + K]\}.$$

Since $\alpha < t_1 \leq t \leq t_4 \leq \omega$, we can introduce the inequality

$$t_1 + \lfloor \frac{t - t_1}{T} \rfloor T \leq t < t_1 + (\lfloor \frac{t - t_1}{T} \rfloor + 1)T = t_1 + \lfloor \frac{t - t_1}{T} \rfloor T + T$$

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which helps us to split the integral as

$$\begin{aligned} x(t) &\leq x(t_1) + \int_{t_1}^t n(s) ds \leq x(t_1) + \int_{t_1}^{t_1 + \lfloor \frac{t-t_1}{T} \rfloor T + T} n(s) ds \\ &\leq x(t_1) + \lfloor \frac{t-t_1}{T} \rfloor \|n\|_1 + \|n\|_1 \leq x(t_1) + \frac{t-t_1+T}{T} \|n\|_1 \end{aligned}$$

and, on the other side,

$$x(t) \geq x(t_1) - \int_t^{t_1} n(s) ds \geq B - \frac{t-t_1+T}{T} \|n\|_1. \quad (2.58)$$

Then a blow-up can not happen if $z(t)$ lies in $[A, +\infty[\times [-A-K, L+K]$; hence $t_4 < \omega$ and $x(t_4) = A$ or $x(t_4) > A$, $y(t_4) = L+K$.

If $x(t_4) = A$, then

$$\begin{aligned} L+K &\geq y(t_4) \geq y(t_1) + \eta(t_4 - t_1) - K \geq -A - K + \eta(t_4 - t_1) \\ \Rightarrow \eta(t_4 - t_1) &\leq L + A + 2K < 2L + 2K \Rightarrow t_4 - t_1 \leq 2 \frac{L+K}{\eta}. \end{aligned}$$

Choose now the constant $B = B(A, L)$ such that

$$B > A + \|n\|_1 2 \frac{L+K+\eta T}{\eta T} = A + \|n\|_1 k \stackrel{\text{def}}{=} A_1 \quad (2.59)$$

and assume $x(t_1) \geq B$. Then from (2.58)

$$\begin{aligned} x(t_4) &\geq B - \frac{t_4 - t_1 + T}{T} \|n\|_1 > \\ &A + \|n\|_1 \left(k - \frac{t_4 - t_1 + T}{T} \right) \geq A + \|n\|_1 \left(k - 2 \frac{L+K}{\eta T} + 1 \right) = \\ &A + \|n\|_1 (k - (k-2) - 1) = A + \|n\|_1 \geq A \end{aligned}$$

But $x(t_4) > A$ implies $y(t_4) = L+K$, with $x(t) > A$ on $[t_1, t_4[$, which is the first case considered in this proof, with t_4 playing the role of t_1 . Therefore the choice of $B > A_1$ allows to reach the thesis in both cases. \square

Using similar arguments, the following statements can be proved.

Lemma 2.30 *For every $A > M$ and $L \geq A$ there exists a constant $B = B(A, L)$ such that, for every $t_1 \in I_0$ there exists a time $t_2 > t_1$, with $t_2 \in \text{Dom}(\zeta(\cdot; t_0))$ such that*

- if $x(t_1) \geq B$ and $y(t_1) \geq -A$ then there exists t_2 with $x(t_2) = A$ and $y(t_2) \geq L$, and $x(t) > A$ for every $t \in [t_1, t_2[$;
- if $x(t_1) \leq A$ and $y(t_1) \geq B > A$ then there exists t_2 with $x(t_2) \leq -L$ and $y(t_2) = A$, and $y(t) > A$ for every $t \in [t_1, t_2[$;

- if $x(t_1) \leq -B$ and $y(t_1) \leq A$ then there exists t_2 with $x(t_2) = -A$ and $y(t_2) \leq -L$, and $x(t) < -A$ for every $t \in [t_1, t_2[$;
- if $x(t_1) \geq -A$ and $y(t_1) \leq -B$ then there exists t_2 with $x(t_2) \geq L$ and $y(t_2) = -A$, and $y(t) < -A$ for every $t \in [t_1, t_2[$.

Using this lemma we can then prove the following three facts, which will be used in order to obtain the twist condition on one boundary needed for the application of the Poincaré-Birkhoff theorem.

Lemma 2.31 *Let $R > M\sqrt{2}$ and let $R \geq R_1$, then there exist R_2, R_3 with $R_1 < R_2 < R_3$ such that if $z(t)$ is a solution with initial point $z(t_0) \in A(R_1, R_3)$ then the following properties hold:*

- if $\|z(t_0)\| \geq R_2$ and there exists $t_1 > t_0$ such that $\|z(t_1)\| \leq R_1$, then $w_z(t_1, t_0) > 1$ for every $t \geq t_1$;
- if $\|z(t_0)\| \leq R_2$ and there exists $t_1 > t_0$ such that $\|z(t_1)\| \geq R_3$, then $w_z(t_1, t_0) > 1$ for every $t \geq t_1$;
- if $\|z(t)\| \in]R_1, R_3[$ for every $t \geq t_0$, then there exists an integer $m^* = m^*(R_1, R_3) \geq 2$ such that $w_z(t, t_0) > 1$ for every $t \geq t_0 + m^*T$.

The long proof is omitted, since it uses arguments which can be found in [27, 34].

Notice that this lemma also means that any solution which blows up in a finite positive time must perform an infinite number of rotations around the origin. On the other hand, if the third case holds, then the solution is globally defined in the future.

We need now to prove opposite inequalities about $w_z(\cdot, t_0)$, which will provide us with the second part of the twist condition on the boundary of a suitable annulus.

Lemma 2.32 *For each time interval $\tau > 0$ there exists an $S > 0$ sufficiently large such that for every solution z satisfying $x(t_1) \leq -S$ and $x(t_2) \geq S$ for some $t_1 < t_2$, then $t_2 - t_1 > \tau$.*

Hence it is always possible to choose an initial point sufficiently far from the origin such that its rotation around the origin is arbitrarily slow. This also allows to conclude that the solutions are globally defined in \mathbb{R} .

Proof. We have assumed that $x'(t) = X(t, y(t)) \leq \Gamma(t)$; hence

$$x(t_2) - x(t_1) \leq \int_{t_1}^{t_2} \Gamma(s) ds \leq \frac{t_2 - t_1 + T}{T} \|\Gamma\|_1$$

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which holds to the inference $x(t_2) - x(t_1) \rightarrow +\infty \Rightarrow t_2 - t_1 \rightarrow +\infty$. Then for every $\tau > 0$ there exists $S > 0$ such that if $|x(t_2) - x(t_1)| > S$ then $t_2 - t_1 > \tau$. \square

As a last step, we conclude that:

Lemma 2.33 *For every $S > 0$ and for every $m \in \mathbb{N}$ there exist S_1, S_2 with $S < S_1 < S_2$ such that if a solution satisfies $\|z(0)\| \geq S_2$, then $\|z(t)\| \geq S_1$ with $w_\zeta(t, t_0) < 1$ for every $t \in [0, mT]$.*

Proof. Let $S > 0$ and choose $S_1 > S(mT)$, with $S(mT)$ playing the role of S in lemma 2.32 for $\tau = mT$. Define the second radius S_2 as

$$S_2 = 1 + \sup\{\|\zeta(t; s, w)\| : s, t \in [0, mT], w \in B[0, S_1]\} \quad (2.60)$$

and assume, by contradiction, that there exists a point z and $\|z\| \geq S_2$ with $\|\zeta(t^*; 0, z)\| < S_1$. By construction, $\|\zeta(0; 0, z)\| = \|z\| \geq S_2$. If we define $z' = \zeta(t^*; 0, z)$, then $z' \in B(0, S_1)$, but $z = \zeta(0; t^*, z')$ with $\|z\| \geq S_2$, in contradiction with the definition of S_2 . Hence, for every initial point $z \in B^c(0, S_2)$ the solution $\zeta(\cdot; 0, z)$ is in $B^c(0, S_1)$, as required. Note that the global existence of the solution is a crucial assumption for the proof.

Suppose now that the solution $\zeta(\cdot; z_0)$ performs more than one turn around the origin during the time interval $[0, mT]$; then there exist $0 \leq t_1 < t_2 < mT$ such that $\vartheta(t_1) = 0$ and $\vartheta(t_2) = \pi$, with $x(t_1) > S$ and $x(t_2) < -S$. In this case, due to lemma 2.32, we would have $t_2 - t_1 > mT$, a contradiction. Hence $w_\zeta(s, t) < 1$ for every $s, t \in [0, mT]$. \square

Eventually, we can apply the Poincaré-Birkhoff theorem in order to obtain subharmonic solutions. Let $R_1^* \geq R > \max\{M\sqrt{2}, R(M, 1/3)\}$. Then, lemma 2.31 guarantees the existence of R_2^*, R_3^* and $m^* \geq 2$ such that

$$\|z(0)\| = R_2^* \Rightarrow w_\zeta(mT, 0) > 1 \quad \forall m \geq m^*.$$

On the other hand, fix $m \geq m^*$, then lemma 2.33 asserts the existence of S_1, S_2 with $R_3^* < S_1 < S_2$ such that

$$\|z(0)\| = S_2 \Rightarrow w_\zeta(mT, 0) < 1.$$

These are exactly the inequalities we can use as a twist-condition for the Poincaré-Birkhoff theorem. Indeed, let Φ be the Poincaré map associated to system (2.49), such that $\Phi(z) = \zeta(T; z)$. From the standard theory of Hamiltonian systems we know that Φ is an area-preserving homeomorphism of the plane \mathbb{R}^2 such that $\Phi(O) = O$. If we consider its m -th iterate

$$\Phi^m(z) = \zeta(mT; z)$$

we can observe that it satisfies a twist condition on the annulus

$$A = B[0, S_2(m)] \setminus B(R_2^*);$$

indeed, if $z(0) \in A_i$, then $w_\zeta(mT, 0) > 1$ which means that $\Theta(\vartheta_0, r_0) > 0$, while for $z(0) \in A_o$ we have $w_\zeta(mT, 0) < 1$ and therefore $\Theta(\vartheta_0, r_0) < 0$. Then we can apply Ding's version of the Poincaré-Birkhoff theorem and deduce that there exist two fixed point z_1^m, z_2^m for Φ^m in A corresponding to two solutions of system (2.49) which are mT -periodic. Moreover, since $w_{\zeta(\cdot; z_1^m)}(mT, 0) = w_{\zeta(\cdot; z_2^m)}(mT, 0) = 1$, they have mT as their minimal period, which means that they are subharmonic solutions of order m .

Let now $m \geq m^*$ be a fixed integer and let ζ_m be one of the subharmonic solutions of order m obtained above. Since its rotation number is equal to one and recalling lemma 2.31, then $\|\zeta_m(t)\| \geq R_1^*$ for every $t \in \mathbb{R}$. Fix two constants W, W_0 such that $R_1^* < W < W_0$ and apply lemma 2.31 with R_1^*, W, W_0 playing the role of R_1, R_2, R_3 . Then, if $\|\zeta_m(t)\| \in]R_1^*, W_0[$ for every t , then there exists an integer \hat{m} such that $w_{\zeta_m}(mT, 0) > 1$ for every $m \geq \hat{m}$, in contradiction with $w_{\zeta_m}(mT, 0) = 1$.

Hence, for every W_0 there exists $\hat{m} > m^*$ and a time-value $t_0 \in [0, mT]$ at which $\|\zeta_m(t_0)\| \geq W_0$ for all $m \geq \hat{m}$. Fix now an arbitrary $m \geq \hat{m}$. We claim that we cannot have any t_1 with $t_0 < t_1 \leq t_0 + mT$ such that $\|\zeta_m(t_1)\| \leq W$. Indeed, if such a t_1 existed then, according to the first property of lemma (2.31) we would obtain $1 < w_{\zeta_m}(t_0 + mT, t_0) = w_{\zeta_m}(mT, 0) = 1$, a contradiction. In conclusion, for any $W > R_1^*$ there exists $\hat{m} > m^*$ such that $\|\zeta_m(t)\| > W$ for every $t \in \mathbb{R}$ and $m \geq \hat{m}$.

Finally, let ζ_m as above and define

$$(u_m(t), v_m(t)) = \zeta_m(t) + (u_0(t), v_0(t))$$

where (u_0, v_0) is the T -periodic solution of system (2.46) considered at the beginning of this section. Since $\|\zeta_m(t)\| \geq R_1^*$ for every $t \in \mathbb{R}$, with $R_1^* \geq R > \max\{M\sqrt{2}, R(M, 1/3)\}$ and $\|(u_0(t), v_0(t))\| < M$ for every t , then we obtain $w_{(u_m, v_m)}(mT, 0) \in]2/3, 3/4[$. On the other hand w_m is a T -periodic solution of (2.46) and therefore its rotation number must be an integer. Hence, $w_{(u_m, v_m)}(mT, 0) = 1$ which also guarantees that $(u_m(t), v_m(t))$ has mT as minimal period.

3 | CROSSING PROPERTIES FOR TWO CLASSES OF PLANAR SETS

This chapter deals with the problem of finding a compact and connected set crossing a prescribed domain “from one side to another”; in order to do this, we are interested in domains on which it is possible to define a concept of *opposite sides*. The investigation on this topic has as a starting point a *crossing lemma* developed for the case in which the domain is a rectangular region $R \subset \mathbb{R}^2$, homeomorphic to the unit square Q . In [87] we presented an exhaustive exposition of the crossing lemma and its possible applications, ranging from game theory to planar Hamiltonian systems. In some sense, that is a classical result, implicitly used by Poincaré, as well as by Butler and Conley, and rediscovered and applied recently in many different contexts. From the crossing lemma, it is possible to derive a fixed point theorem for continuous functions defined on rectangular domains having the particular property of “stretching” the paths which cross the domain from one side to the other; this theorem leads also to the proof of the existence of periodic points and chaotic dynamics of some second order differential equations (theorems 4.4 and 4.5).

This chapter is beginning recalling some crossing properties for rectangular sets, summarized in section 3.2; afterwards, we will move to two other settings, namely, the study of annular regions (section 3.3) and the case of invariant sets (section 3.4), trying to obtain similar results and showing what we are able to extend to these new frameworks. Analogous problems have been already investigated in the literature for continua of the sphere which are invariant under the antipodal map or continua of fixed points for a twist map in a planar annulus.

3.1 CROSSING PROPERTIES

In 1817 the Czech philosopher and mathematician Bernard Bolzano gave the first proof of the intermediate-value theorem for continuous functions defined on a compact interval $[a, b] \subset \mathbb{R}$; in 1883-1884, H. Poincaré introduced a generalization of that result to the case of continuous vector fields defined on a hypercube in \mathbb{R}^N :

Soient X_1, X_2, \dots, X_N n fonctions continues des n variables x_1, x_2, \dots, x_N . Supposons que X_i soit toujours positif pour $x_i = a_i$ et

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toujours négatif pour $x_i = -a_i$. Il existera au moins un système de valeurs des x qui satisfera aux inégalités

$$-a_1 < x_1 < a_1, -a_2 < x_2 < a_2, \dots, -a_N < x_N < a_N$$

et aux équations

$$X_1 = X_2 = \dots = X_N = 0.$$

This result was published by Poincaré on the *Bulletin Astronomique* in a paper [93] concerning the three-body problem applied to celestial mechanics; he showed that the initial conditions of the periodic solutions of a differential system in \mathbb{R}^N must satisfy the hypothesis of this generalization of the intermediate-value theorem. But his work remained unknown to the most part of the mathematicians.

This theorem is now known as the *Poincaré-Miranda theorem*, due to the fact that the Italian mathematician Carlo Miranda proved its equivalence with the Brouwer fixed point theorem, in 1940. For a continuous vector field defined on a rectangle in \mathbb{R}^2 , its statement reads as follows:

Theorem 3.1 *Let $f = (f_1, f_2) : R = [a_1, a_2] \times [b_1, b_2] \rightarrow \mathbb{R}^2$ be a continuous function such that*

$$\begin{aligned} f_1(a_1, x_2) &\leq 0 \leq f_1(a_2, x_2), \quad \forall x_2 \in [b_1, b_2] \\ f_2(x_1, b_1) &\leq 0 \leq f_2(x_1, b_2), \quad \forall x_1 \in [a_1, a_2], \end{aligned}$$

then there exists a point $z \in R$ such that $f(z) = 0$.

An heuristic proof of this result, as suggested by Poincaré himself in [93], can be described as follows. The “curve” $f_2(x_1, x_2) = 0$ starts at some point of the left side $x_1 = a_1$ and it ends at some point of the right side $x_1 = b_1$. Similarly, the “curve” $f_1(x_1, x_2) = 0$ starts at some point of the lower side $x_2 = a_2$ and it ends at some point of the upper side $x_2 = b_2$. Hence they must intersect at some point of the rectangle. Clearly, in modern language, using in this context the term “curve” is erroneous, but the argument of the proof is safe if we use the fact that the set $f_2(x_1, x_2) = 0$ contains a continuum joining the left to the right side and, similarly, the set $f_1(x_1, x_2) = 0$ contains a continuum joining the lower to the upper side. In order to prove the existence of such continua, one can observe that the set $f_2(x_1, x_2) = 0$ crosses any path from the lower to the upper side of the rectangle (and similarly happens for the set $f_1(x_1, x_2) = 0$ with respect to the paths connecting the left to the right side of the rectangle). In [84] we proved that such “cutting property” for a compact set S guarantees the existence of a compact connected set $C \subset S$ with the same cutting property. Moreover, if C cuts the paths between the lower and the upper sides of a rectangle, then it must intersect the left and the right sides of the rectangle. This

properties are still true for the more general framework of the so-called oriented rectangles, as we are going to expose in section 3.2.

We call such kind of results as *crossing lemmas* and propose new applications to the study of the dynamics of some planar maps. Some important theorems where some forms of these crossing properties are considered, appear in dimension theory with the results of Hurewicz and Wallman[51], in topological games [5, 37], as well as in some proofs of the existence of solutions to nonlinear differential equations [23].

In the applications, typically the set S is a set of solutions of a nonlinear equation depending on a parameter (or equivalently a set of fixed points for a family of parameter-dependent operators). From this point of view, results in this direction, although they may look quite elementary, present a great usefulness in different areas of nonlinear analysis, especially in connection with bifurcation theory (see [2, 18, 23, 97]).

When the operators whose fixed points correspond to the element of S have some special symmetries, it is likely that some of these symmetries are inherited by the set S itself. In this case, it would be desirable to prove that also the continuum $C \subset S$ inherits the symmetries of S .

Our perspective is a little bit different. Namely, we do not assume the knowledge of a map or of an operator (possibly depending on a parameter) whose fixed points are described by the set S . In our approach we consider as a starting point the set S with a generic “symmetry property” (expressed in terms of invariance with respect to a given homeomorphism) and try to develop an analogous crossing lemma which preserves the symmetry. Along this investigation, we will also reconsider some classical properties of continua with the aim to extend them to the invariant setting.

In order to conclude this introduction, we present now some preliminary definitions and results which will be used in the following. Slightly modifying an analogous definition in [6, Definition 2.1] we give the following:

Definition 3.2 Let X be a topological space and let $A, B \subset X$ be two nonempty disjoint sets. Let also $S \subset X$. We say that S *cuts the paths between A and B* if $S \cap \bar{\gamma} \neq \emptyset$, for every path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) \in A$ and $\gamma(1) \in B$.

In order to simplify the statements of the next results, we write

$$S : A \uparrow B$$

to express the fact that S cuts the paths between A and B . For having definition 3.2 meaningful, we implicitly assume that there exists at least a path γ in X connecting A with B (otherwise, we could take $S = \emptyset$, or S any subset of X). Clearly, if a set S satisfies the cutting property

3 Crossing properties for two classes of planar sets

of definition 3.2, then also its closure $\text{cl } S$ cuts the paths between A and B . Therefore, in the sequel and without loss of generality *we usually assume S closed*. Such an assumption is also well-suited for proving the existence of minimal (closed) sets satisfying definition 3.2. Indeed, we can easily prove the following lemma:

Lemma 3.3 *Let X be a topological space and let $A, B \subset X$ two nonempty disjoint sets which are connected by at least one path in X . Let $S \subset X$ be a closed set which cuts the paths between A and B . Then there exists a nonempty, closed set $C \subset S$ which is minimal with respect to the property of cutting the paths between A and B .*

Proof. The proof is a standard application of Zorn's lemma. Let \mathcal{F} be the set of all the nonempty closed subsets F of S such that $F : A \nmid B$, with the elements of \mathcal{F} ordered by inclusion. Clearly, \mathcal{F} is nonempty for at least $S \in \mathcal{F}$. Let $(F_\alpha)_{\alpha \in J}$ be a totally ordered family of subsets of \mathcal{F} . We claim that $F^* = \bigcap_{\alpha \in J} F_\alpha \in \mathcal{F}$. Indeed, let $\gamma : [0, 1] \rightarrow X$ be a path such that $\gamma(0) \in A$ and $\gamma(1) \in B$. The family of compact sets $(\bar{\gamma} \cap F_\alpha)_{\alpha \in J}$ has the finite-intersection property and therefore $\bar{\gamma} \cap \bigcap_{\alpha \in J} F_\alpha = \bar{\gamma} \cap F^* \neq \emptyset$. This proves the claim and the conclusion follows by Zorn's lemma. \square

Due to the above remarks, whenever we speak about a set S such that $S : A \nmid B$, we can assume S closed and minimal.

In [31], Dolcher studied a similar minimality problem, dealing with closed sets separating two points. The definition of separation is the standard one, that is a set $S \subset X$ separates two points (or, in general, two nonempty sets) if the two points belong to different components of the complement $X \setminus S$. With this respect, we reconsider the following example from [31]. Let $X \subset \mathbb{R}^2$, with the topology of the plane, be defined by

$$X = \{(x, y) : x \geq 0, y = x/n, n \in \mathbb{N}_0\} \cup (\mathbb{R}^+ \times \{0\})$$

and let $A = \{(0, 0)\}$, $B = \{(2, 0)\}$ and $S = \{(x, y) \in X : x = 1\}$. Clearly, $S : A \nmid B$ and S separates A and B in X . As shown in [31] there is no subset of S which is minimal for the property of separating A and B in X . On the other hand, $C = \{(1, 0)\}$ is the minimal subset of S which cuts the paths between A and B . In this sense, we remark the fact that the property of cutting the paths is not equivalent to the property of separating.

In general, if we know that a set S satisfying $S : A \nmid B$, even if minimal with respect to such cutting property, we can say nothing about its connectedness. For an elementary example, one can take $X = S^1$ (with the topology of the plane), $A = \{(-1, 0)\}$, $B = \{(1, 0)\}$ and $S = \{(-1, 0), (1, 0)\}$. In this case, S is a closed set, minimal with respect

to the property of cutting the paths between A and B , but it is not connected. The connectivity of S (or of a minimal subset of it) is, however, an important property for the proof of the existence of fixed points or of zeros for maps in Euclidean spaces. Such connectivity properties have been employed recently in [81, 85, 91] in connection with the theory of *Topological Horseshoes*. In order to recall some main results from the above quoted papers and to propose some further developments, we introduce some main definitions which play a crucial role in our approach.

3.2 GENERALIZED RECTANGLES

Definition 3.4 We say that $J \subset \mathbb{R}^2$ is a *Jordan curve* if it is homeomorphic to $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$. We can equivalently say that J is a Jordan curve if it is the support of a simple closed curve.

Jordan's theorem and Schoenflies's theorem are the most important results about Jordan curves and play a crucial role in our approach; even if their statements are intuitively clear, a rigorous proof is not elementary. The interested reader can find all the details in [71]. We assume these two theorems as a starting point for the exposition.

Theorem 3.5 (Jordan Theorem) *Every Jordan curve J splits the plane in two connected components, of which it is the common boundary.*

Therefore $\mathbb{R}^2 \setminus J = A_i \cup A_e$, where A_i, A_e are open connected sets such that $A_i \cap A_e = \emptyset$ and $\partial A_i = \partial A_e = J$; moreover A_i is a bounded set, while A_e is unbounded.

Theorem 3.6 (Schoenflies Theorem) *Given a Jordan curve J and a homeomorphism $\eta : S^1 \rightarrow J$, there exists a homeomorphism $\tilde{\eta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\tilde{\eta}|_{S^1} = \eta$. As a consequence, we have $\tilde{\eta}(B) = \text{cl } A_i = A_i \cup J$ and $\tilde{\eta}(\mathbb{R}^2 \setminus B) = A_e$.*

Definition 3.7 We say that $\mathcal{D} \subset \mathbb{R}^2$ is a *Jordan domain* if it is homeomorphic to B . Equivalently, $\mathcal{D} = \text{cl } A_i$, where A_i is the internal part of a Jordan curve.

In our approach the sets under consideration are Jordan domains, but we prefer to think about them starting from a planar homeomorphism defined on the unit square Q , instead of the unit disc B . In this way we can more easily introduce a concept of orientation for these sets.

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Definition 3.8 A topological space X is a *generalized rectangle* if it is homeomorphic to the unit square Q .

Given a generalized rectangle X and a homeomorphism $\eta : Q \rightarrow \eta(Q) = X$, the set $\eta(\partial Q)$ is independent of the choice of the homeomorphism η . We call this set the *contour* of X and denote it by ∂X . Clearly, if X is a generalized rectangle embedded in \mathbb{R}^2 , then the contour of X coincides with the standard concept of the boundary ∂X .

We want now to introduce the concept of sides of a rectangle; in particular we are interested in defining *pairs of opposite sides*.

Definition 3.9 An *oriented rectangle* is a pair $(X, X^-) = \tilde{X}$ such that X is a generalized rectangle and $X^- \subset \partial X$ is the union of two disjoint arcs:

$$X^- = X_l^- \cup X_r^-, \quad X_l^- = \eta(\{0\} \times [0, 1]), \quad X_r^- = \eta(\{1\} \times [0, 1]).$$

The sets X_l^- and X_r^- are respectively the left and the right-hand side of the rectangle X . In the same way we define the top and the bottom sides of X as the images of the corresponding sides of Q and denote them by X_t^+ and X_b^+ , with $X^+ = X_b^+ \cup X_t^+ = \partial X \setminus X^-$. We will also write $X = |\tilde{X}|$. When we provide a generalized rectangle of the structure of oriented rectangle, the choice between X^- and X^+ is not relevant; in any way we do the choice, we can always assume that the sequence of the arcs we meet moving along the boundary is "bottom-right-top-left".

Conversely, suppose that $\mathcal{D} \subset \mathbb{R}^2$ is a Jordan domain and J' and J'' two compact disjoint arcs contained in $\partial \mathcal{D}$. Then Schoenflies's theorem ensures the existence of a homeomorphism $\eta : Q \rightarrow \mathcal{D}$ which provides \mathcal{D} with an orientation of its boundary such that $\mathcal{D}^- = J' \cup J''$. In this case we also have that $\mathcal{D}^+ = \partial \mathcal{D} \setminus (J' \cup J'')$ and the order in which we decide to label the "bottom" and "top" parts is irrelevant.

More in general, observe that for any homeomorphism $\eta_1 : Q \rightarrow \eta_1(Q) = X$, defining the oriented rectangle (X, X^-) , there exists a homeomorphism $\eta_2 : Q \rightarrow \eta_2(Q) = X$ such that $\eta_1([0, 1] \times \{0, 1\}) = \eta_2(\{0, 1\} \times [0, 1])$; then, for every oriented rectangle \tilde{X} there exists a "dual" oriented rectangle \tilde{X}' with $|\tilde{X}| = |\tilde{X}'| = X$ and $X^- = X'^+$.

Defined the framework, we can now investigate the existence of sets which cross this type of domains. The next (classical) result guarantees the fact that continua connecting opposite sides of an oriented rectangle must cross each other. Although this seems an obvious fact, its proof requires some work; indeed it can be proved as a consequence of the Jordan curve theorem and using some strong properties of Peano's spaces. The proof is here omitted but the reader can find all its details in [87] and an application to ordinary differential equations in [74].

Lemma 3.10 *Let $\tilde{X} = (X, X^-)$ be an oriented rectangle and let $C_1, C_2 \subset X$ be two closed connected sets such that*

$$C_1 \cap X_l^- \neq \emptyset \neq C_1 \cap X_r^- \quad \text{and} \quad C_2 \cap X_b^+ \neq \emptyset \neq C_2 \cap X_t^+.$$

Then

$$C_1 \cap C_2 \neq \emptyset.$$

Note that the connectedness of C_1 and C_2 is not enough to guarantee the existence of a nonempty intersection (see [39] for a counterexample).

We present now some results about sets separating the opposite sides of an oriented rectangle and show their role in the proof of the existence of fixed points and periodic points for continuous maps defined on such domains. Some of these results can be extended to higher dimension using the topological degree or the fixed point index or other index theories (see [91] and the references therein). Since the applications in the present paper will be all related to planar maps, we prefer to confine ourselves to the use of a more direct tool, Alexander's lemma. Such result, named after J.W. Alexander [3], as shown both in Newman's book [76] and in Sanderson's article [100], is quite useful in proving a broad range of theorems of plane topology. Quoting P.A. Smith [102]

this lemma, the proof of which requires but a few lines, is shown [...] to be one of the sharpest tools in the theory of separation, if skilfully handled.

Results based on applications of Alexander's lemma or to other related theorems in [76] have been fruitfully applied to differential equations by S.P. Hastings [46, 44, 45], J.B. McLeod and J. Serrin [70], R.E.L. Turner [106] and others. For more recent applications see also [61], [54], [97].

The following version of Alexander's lemma will be used in our next results. The proof requires only an elementary modification of the standard one in [47] and therefore is omitted.

Lemma 3.11 *Let $\tilde{X} = (X, X^-)$ be an oriented rectangle and let K_1, K_2 be two closed disjoint subsets of X . Assume that there exist two paths $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$, with $\gamma_1(0), \gamma_2(0) \in X_l^-$ and $\gamma_1(1), \gamma_2(1) \in X_r^-$ such that*

$$\tilde{\gamma}_1 \cap K_1 = \emptyset, \quad \tilde{\gamma}_2 \cap K_2 = \emptyset.$$

Then there exists a path $\gamma : [0, 1] \rightarrow X$, with $\gamma(0) \in X_l^-$ and $\gamma(1) \in X_r^-$ such that $\tilde{\gamma} \cap (K_1 \cup K_2) = \emptyset$.

3 Crossing properties for two classes of planar sets

The next result is a classical and useful consequence of the above lemma (see [76]) and it is usually expressed by the fact that *if a closed set separates the plane, then some component of this set separates the plane* [45, p.131].

Lemma 3.12 *Let $\tilde{X} = (X, X^-)$ be an oriented rectangle and let $S \subset X$ be a closed set such that*

$$S : X_l^- \uparrow X_r^- .$$

Then there exists a compact, connected set $C \subset S$ such that

$$C : X_l^- \uparrow X_r^- .$$

Proof. By lemma 3.3 there exists a closed set $C \subset S$ such that $C : X_l^- \uparrow X_r^-$, with C minimal with respect to the cutting property. Suppose, by contradiction, that C is not connected and let $C_1, C_2 \subset C$ be two closed nonempty disjoint sets with $C_1 \cup C_2 = C$. Since C is minimal and C_1, C_2 are proper subsets of C , there exist two paths γ_1, γ_2 in X which connects X_l^- to X_r^- and such that γ_i avoids C_i (for $i = 1, 2$). Then, by lemma 3.11, there exists a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) \in X_l^-$ and $\gamma(1) \in X_r^-$ with $\tilde{\gamma} \cap C = \emptyset$, contradicting the assumption that $C : X_l^- \uparrow X_r^-$. \square

The cutting property obtained in lemma 3.12 for the continuum C can be equivalently expressed as follows.

Lemma 3.13 *Let $\tilde{X} = (X, X^-)$ be an oriented rectangle and let $C \subset X$ be a closed connected set. Then*

$$C : X_l^- \uparrow X_r^-$$

if and only if

$$C \cap X_b^+ \neq \emptyset \neq C \cap X_t^+ .$$

Proof. If $C : X_l^- \uparrow X_r^-$, then, necessarily, C must cut the upper and the lower sides of \tilde{X} which are the images of particular paths connecting X_l^- to X_r^- . On the other hand, if $\gamma : [0, 1] \rightarrow X$ is any path with $\gamma(0) \in X_l^-$ and $\gamma(1) \in X_r^-$, then $\tilde{\gamma}$ and C are two continua connecting the opposite sides of the oriented rectangle and therefore, $\tilde{\gamma} \cap C \neq \emptyset$ by lemma 3.10. This proves that $C : X_l^- \uparrow X_r^-$. \square

There are some interesting connections between these pure topological theorems and some results about combinatorial games. For instance, lemma 3.13, as well as lemma 3.10 can be seen as a continuous version of the so-called Hex theorem asserting that the game Hex can not end in a tie [37, 87].

The combination of lemma 3.12 and lemma 3.13 gives the *crossing lemma* for rectangular regions, asserting that *if a closed set intersects all*

the paths from the left to the right side of an oriented rectangle, then it contains a continuum connecting the two other sides. Note also that such a continuum can be taken as irreducible between X_b^+ and X_t^+ (by using some classical results from [59] and [2]). See [87] for a recent survey on this subject and its connections with various different results of plane topology, as well as for a different proof based on Whyburn's lemma.

At this point we have all the tools for formalizing Poincaré's comment to the proof of the planar case of the Poincaré-Miranda theorem. The next result is a version of the Poincaré-Miranda theorem for oriented rectangles. Such theorem ensures the existence of a zero for a continuous vector field defined on a hypercube of \mathbb{R}^n under the assumptions that the i -th component of the vector field changes its sign on the i -th opposite faces of the hypercube. It was first stated by Poincaré in 1883-1884 in [92, 93]. In 1940, C. Miranda published a simple proof of the equivalence between this theorem and the Brouwer fixed point theorem. For recent comments about this result, see [13, 58, 68, 87].

There are several different approaches to prove this classical result, especially in the two-dimensional case. We propose a proof which is based on some elementary concepts introduced above and which is also in the spirit of Poincaré's own description of his result in the planar case.

Lemma 3.14 *Let $\tilde{X} = (X, X^-)$ be an oriented rectangle and let $f = (f_1, f_2) : X \rightarrow \mathbb{R}^2$ be a continuous function such that*

$$f_1 \leq 0 \text{ on } X_l^-, \quad f_1 \geq 0 \text{ on } X_r^-, \quad \text{and} \quad f_2 \leq 0 \text{ on } X_b^+, \quad f_2 \geq 0 \text{ on } X_t^+$$

(or vice-versa). Then, there exists $w \in X$ such that $f(w) = (0, 0)$.

Proof. Let $S_i = \{z \in X : f_i(z) = 0\}$ be the set of the zeros of f_i , for $i = 1, 2$ and consider two paths $\gamma_i : [0, 1] \rightarrow X$ such that

$$\gamma_1(0) \in X_l^-, \quad \gamma_1(1) \in X_r^-, \quad \gamma_2(0) \in X_b^+, \quad \gamma_2(1) \in X_t^+.$$

Introduce now the composite maps computing f_i along γ_i ,

$$\hat{f}_i = f_i \circ \gamma_i : [0, 1] \rightarrow \mathbb{R} \quad \text{for} \quad i = 1, 2.$$

From the assumptions on f , we observe that $\hat{f}_i(0) \leq 0$ and $\hat{f}_i(1) \geq 0$ for $i = 1, 2$. Hence we can apply the intermediate-value theorem to \hat{f}_1 and \hat{f}_2 and conclude that there exist $t_1^*, t_2^* \in [0, 1]$ such that

$$\begin{aligned} \hat{f}_1(t_1^*) = f_1(\gamma_1(t_1^*)) = 0 &\Rightarrow \gamma_1(t_1^*) \in S_1 \\ \hat{f}_2(t_2^*) = f_2(\gamma_2(t_2^*)) = 0 &\Rightarrow \gamma_2(t_2^*) \in S_2. \end{aligned}$$

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This argument proves that $S_1 : X_l^- \uparrow X_r^-$ and then, from lemma 3.12 it contains a compact connected set C_1 with the same cutting property. In the same way, since $S_2 : X_b^+ \uparrow X_t^+$, then there exists a continuum C_2 which cuts the paths between the top and the bottom sides of X .

As a last step, applying lemma 3.13 to the sets C_i , we can conclude that $C_1 \cap X_b^+ \neq \emptyset \neq C_1 \cap X_t^+$ and $C_2 \cap X_l^- \neq \emptyset \neq C_2 \cap X_r^-$ (recall that the way in which we choose to label the sides of X is purely conventional). Lemma 3.10 guarantees that there exists a point $w \in C_1 \cap C_2 \subset S_1 \cap S_2$, that is a point in which both the components of f vanish. Hence $f(w) = (0, 0)$, as required.

Another way to conclude the proof is applying Bolzano's theorem to $f_2|_{C_1}$. Since $f_2 \leq 0$ on $C_1 \cap X_b^+$ and $f_2 \geq 0$ on $C_1 \cap X_t^+$, there exists a point $w \in C_1$ such that $f_2(w) = 0$. \square

3.3 ANNULAR REGIONS

In this section, we are moving our attention from generalized rectangles to planar annuli, trying to develop analogous results of section 3.2 in this new setting. Our aim now is to reconsider the results obtained for topological rectangles and adapt them to a form which may be better suited to deal with the new setting of topological annuli in which the role of the left and the right sides of the rectangle will be played by the inner and the outer boundaries of the annulus. We begin with a version of Alexander's lemma which reads as follows.

Lemma 3.15 *Let X be a topological annulus and let K_1, K_2 be closed disjoint subsets of X . Assume that there exist two paths $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$, with $\gamma_1(0), \gamma_2(0) \in X_i$ and $\gamma_1(1), \gamma_2(1) \in X_o$ such that*

$$\bar{\gamma}_1 \cap K_1 = \emptyset, \quad \bar{\gamma}_2 \cap K_2 = \emptyset.$$

Then there exists a path $\gamma : [0, 1] \rightarrow X$, with $\gamma(0) \in X_i$ and $\gamma(1) \in X_o$ such that $\bar{\gamma} \cap (K_1 \cup K_2) = \emptyset$.

In the setting of the rectangles we were interested in finding subsets which link the sides belonging to one of the pairs X^- or X^+ . In the annulus we can produce only one pair of *sides*, that is the couple (X_i, X_o) ; the crossing properties in the other direction will be translated in the request that a set *turns around* the whole annulus. More precisely we want to find subsets of X which are essentially embedded in X , according to the next definition.

Definition 3.16 A set $C \subset X$ is *essentially embedded* in X if the inclusion

$$i_C : C \rightarrow X, \quad i_C(x) = x, \quad \forall x \in C$$

is not homotopic to a constant map.

We can obtain a new version of the crossing lemma, analogous to lemma 3.13. The result is a corollary of Borsuk's separation theorem [50, Theorem 6-47] adapted to our situation. We give a proof, for completeness, following [50].

Lemma 3.17 *Let X be a topological annulus and let $S \subset X$ be a closed set. Then S is essentially embedded in X if and only if*

$$S : X_i \not\downarrow X_o .$$

Proof. Up to a homeomorphism defining the annulus X , we can assume $X = A[a, b]$ with

$$0 < a < b < 1.$$

In this case, $X_i = \partial B(0, a)$ and $X_o = \partial B(0, b)$.

Suppose that $S : X_i \not\downarrow X_o$ and let $\mathcal{C}(0)$ be the connected component of $\mathbb{R}^2 \setminus S$ containing the origin. By the assumption, $\mathcal{C}(0) \cup S$ is closed and

$$B[0, a] \subset \mathcal{C}(0) \cup S \subset B[0, b].$$

Assume, by contradiction, that S is not essentially embedded in X and therefore the inclusion $i_S : S \rightarrow X$ is homotopic in X to a constant map, say $X \ni x \mapsto p$, for all $x \in X$, for a suitable point $p \in X$. It follows immediately that the map $f : S \rightarrow S^1$ defined by $x \mapsto x/\|x\|$ is homotopic to a constant, that is *inessential*. Then, by [50, Theorem 4-5], there exists a continuous and inessential extension \tilde{f} of f with \tilde{f} defined on $\mathcal{C}(0) \cup S$. We define now the map $r : B[0, 1] \rightarrow S^1$ by

$$r(x) = \begin{cases} \tilde{f}(x), & \text{for } x \in \mathcal{C}(0) \cup S \\ x/\|x\|, & \text{for } x \notin \mathcal{C}(0) \cup S \end{cases}$$

which is continuous. We are led to a contradiction since $r(\cdot)$ is a retraction of $B[0, 1]$ onto S^1 .

Suppose now that S is essentially embedded in X and also assume, by contradiction, that there exists a path $\gamma : [0, 1] \rightarrow X$ with $\|\gamma(0)\| = a$, $\|\gamma(1)\| = b$ and such that $\gamma(t) \notin S$, for all $t \in [0, 1]$. Passing to the covering space $H = \mathbb{R} \times [a, b]$ of $X = A[a, b]$, the path γ lifts to a family of paths $\tilde{\gamma}_n : [0, 1] \rightarrow H$ with $\tilde{\gamma}_n(t) = \tilde{\gamma}_0(t) + (2n\pi, 0)$ and such that $\tilde{\gamma}_n(t) \cap \pi^{-1}(S) = \emptyset$, for all $t \in [0, 1]$ and every $n \in \mathbb{Z}$. We can replace $\tilde{\gamma}_0$ (as well as all its copies) with a one-to-one continuous map defining an arc $\Gamma_0 \subset H$ connecting the lines $\rho = a$ and $\rho = b$ and avoiding $\pi^{-1}(S)$. Without loss of generality we can also assume that Γ_0 intersects the line $\{\rho = a\}$ exactly at one point and the same happens with respect to $\{\rho = b\}$. Let $\Gamma_0 \cap \{\rho = a\} = \{P_a\}$ and $\Gamma_0 \cap \{\rho = b\} = \{P_b\}$ and $\Gamma_1 = (2\pi n, 0) + \Gamma_0$ for some $n > 1$ in order to have $\Gamma_0 \cap \Gamma_1 = \emptyset$. In order to simplify the

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notation, assume $n = 1$. Let \mathcal{J} be the Jordan curve obtained by joining (in the counterclockwise sense) the point P_a to $(2\pi, 0) + P_a$ along the line $\{\rho = a\}$, the point $(2\pi, 0) + P_a$ to $(2\pi, 0) + P_b$ along Γ_1 , the point $(2\pi, 0) + P_b$ to P_b along the line $\{\rho = b\}$ and, finally, the point P_b to P_a along Γ_0 . The curve \mathcal{J} is the boundary of an open bounded domain D with $\text{cl}D = D \cup \mathcal{J}$ homeomorphic to the unit square Q . Roughly speaking, $\text{cl}D$ is the set of all the points of the strip H between Γ_0 and Γ_1 , with the boundary arcs included. Let $\eta : Q \rightarrow \eta(Q) = \text{cl}D$ be a homeomorphism mapping the left side of Q to Γ_0 , the lower side of Q to the segment $\{P_a + (\vartheta, 0) : \vartheta \in [0, 2\pi]\}$, the right side of Q to Γ_1 and the upper side of Q to the segment $\{P_b + (\vartheta, 0) : \vartheta \in [0, 2\pi]\}$. By the construction of the topological rectangle $\text{cl}D$ and since $\Gamma_0 \cap \pi^{-1}(S) = \emptyset$, we have that the set

$$S' = \pi^{-1}(S) \cap \eta([0, 1] \times [0, 1]) \subset \text{cl}D$$

is mapped homeomorphically onto S by the covering projection π . Now we choose $\varepsilon \in]0, 1/2[$ sufficiently small such that

$$S' \subset \pi^{-1}(S) \cap \eta([\varepsilon, 1 - \varepsilon] \times [0, 1])$$

and we also introduce the set

$$B = \pi(\eta([\varepsilon, 1 - \varepsilon] \times [0, 1])).$$

By construction, the set B is a topological rectangle contained in $A[a, b]$ and containing the set S . The continuous map

$$(z, \lambda) \mapsto \pi(\eta((1 - \lambda)z + \lambda(1/2, 1/2)))$$

defined on $([\varepsilon, 1 - \varepsilon] \times [0, 1]) \times [0, 1]$ when restricted to $S \times [0, 1]$ provides a homotopy between the identity i_S and a constant map. This contradicts the assumption that S is essentially embedded in $A[a, b]$. \square

In this context too we are able to recover a result of *minimality* for the crossing set; indeed the version of lemma 3.12 for a topological annulus reads as follows.

Lemma 3.18 *Let X be a topological annulus and let $S \subset X$ be a closed set such that*

$$S : X_i \uparrow X_o .$$

Then there exists a compact, connected set $C \subset S$ such that

$$C : X_i \uparrow X_o$$

(and, therefore, C is essentially embedded in X).

Proof. The proof follows the same argument of the one of lemma 3.12, nevertheless we give the details for completeness. By lemma 3.3 there exists a closed set $C \subset S$ such that $C : X_i \dashv X_o$, with C minimal with respect to the cutting property. Suppose, by contradiction, that C is not connected and let $C_1, C_2 \subset C$ be two closed nonempty disjoint sets with $C_1 \cup C_2 = C$. Since C is minimal and C_1, C_2 are proper subsets of C , there exist two paths γ_1, γ_2 in X which connects X_i to X_o and such that γ_i avoids C_i (for $i = 1, 2$). Then, by lemma 3.15, there exists a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) \in X_i$ and $\gamma(1) \in X_o$ with $\bar{\gamma} \cap C = \emptyset$, contradicting the assumption that $C : X_i \dashv X_o$. The continuum C is also essentially embedded in X by lemma 3.17. \square

The result in lemma 3.18 has been proved using a minimality argument. In some cases, the minimality of the set C may be useful for the proof of some topological properties of the continuum. An example in this direction is given in the next lemma.

Lemma 3.19 *Let X be a topological annulus and let $C \subset X$ be a compact connected set which is minimal with respect to the property of cutting all the paths in X from X_i to X_o . Let $f : C \rightarrow \mathbb{R}$ be a nonconstant continuous function. Then, for every $k \in]\min f(C), \max f(C)[$ there exist at least two points $w, z \in C$ with $w \neq z$ such that $f(w) = f(z) = k$.*

Proof. Without loss of generality, we can assume that $k = 0$ and f changes its sign on C . The existence of at least a zero for $f|_C$ follows from Bolzano's theorem. Suppose, by contradiction, that there is only one point, say z , in C such that $f(z) = 0$. Consider the two nonempty compact sets $K_1 = \{x \in C : f(x) \leq 0\}$ and $K_2 = \{x \in C : f(x) \geq 0\}$. By the assumption, we have that $\{z\} = K_1 \cap K_2$ and $K_1 \neq C$ as well as $K_2 \neq C$. By the assumption of minimality of C , it follows that there exists a path γ_1 connecting X_i to X_o in X and avoiding K_1 and, similarly, there exists a path γ_2 connecting X_i to X_o in X and avoiding K_2 . From lemma 3.15 we know that there exists a path γ in X connecting X_i to X_o and avoiding $C = K_1 \cup K_2$. This contradicts the cutting property of C . \square

We observe that the existence of two solutions is not guaranteed if the minimality of the set C is not assumed (see [88, Example 2.8]).

In the case of topological rectangles, there is a complete symmetry between the fact that a set cuts the paths between a given pair of opposite sides or it cuts the paths connecting a complementary pair of opposite sides. Thus once we have achieved a result as lemma 3.13, also its dual version, involving the other pair of sides, is guaranteed. In the case of topological annuli, the situation is different. We have just proved a result which express the fact that a compact set which crosses all the paths from the inner to the outer boundary must contain a continuum which nontrivially winds around the annulus. A

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dual result should express the fact that if a compact set intersects all the nontrivial loops of the annulus, then it must contain a continuum joining the inner and the outer boundaries of the annulus. This is precisely the content of the next lemma. To this end, we first recall some basic facts from homotopy theory. Let $\omega : I \rightarrow X$ be a *loop*, that is a continuous path such that $\omega(0) = \omega(1)$. We say that ω is (homotopically) trivial in X if it is homotopic in X to the constant loop $e_{x_0} : I \rightarrow x_0$ with $x_0 = \omega(0) = \omega(1)$. Since a loop in X (up to a change in the parameter) may be also seen as a continuous map $\omega : S^1 \rightarrow X$, triviality of ω can be also expressed by the fact that there is a continuous extension $\alpha : B[0, 1] \rightarrow X$ with $\alpha_{S^1} = \omega$. We say that a loop ω in X is *nontrivial* if it is not homotopically trivial in X .

Lemma 3.20 *Let X be a topological annulus and let $S \subset X$ be a closed set such that $S \cap \bar{\omega} \neq \emptyset$ for each nontrivial loop ω in X . Then there exists a compact, connected set $C \subset S$ such that*

$$C \cap X_i \neq \emptyset \neq C \cap X_o. \quad (3.1)$$

Proof. Without loss of generality (up to a homeomorphism) we suppose that $X = A[a, b]$ with $0 < a < b$, so that $X_i = aS^1$ and $X_o = bS^1$. By the assumption of crossing the nontrivial loops, we know that $S_i = S \cap X_i \neq \emptyset$ and also $S_o = S \cap X_o \neq \emptyset$.

Suppose, by contradiction, that S does not contain any compact connected set C satisfying (3.1). Then, by the Kuratowski-Whyburn lemma [2, 59], it follows that S splits as the disjoint union of two compact sets S', S'' with $S' \supset S_i$ and $S'' \supset S_o$. We pass now to the covering space $H = \mathbb{R} \times [a, b]$ of $X = A[a, b]$ and consider the closed subsets of H

$$W' = \pi^{-1}(S') \cup \mathbb{R} \times \{a\} \quad \text{and} \quad W'' = \pi^{-1}(S'') \cup \mathbb{R} \times \{b\}.$$

By definition, $\pi(W' \cup W'') \supset S$. Moreover, $W' \cap W'' = \emptyset$ and both sets are invariant with respect to the translation $(\vartheta, r) \mapsto (\vartheta + 2\pi, r)$. We define

$$\delta = \text{dist}(W', W'') = \inf\{\|w' - w''\| : w' \in W', w'' \in W''\}.$$

It is clear that $\delta > 0$ and it is actually a minimum (this follows from a standard compactness argument, using the periodicity). Then we define the two closed ε -tubular neighborhoods of W' and W'' as $W'[\varepsilon] = \{z = (\vartheta, r) \in H : \text{dist}(z, W') \leq \varepsilon\}$ and $W''[\varepsilon] = \{z = (\vartheta, r) \in H : \text{dist}(z, W'') \leq \varepsilon\}$, for

$$0 < \varepsilon \leq \delta/3.$$

We modify now a pigeonhole argument used in the proof of Theorem 1 in [5] as follows. Let us fix a positive integer $N > (b - a)/\varepsilon$ and consider the rectangle

$$\mathcal{R} = [0, 2N\pi] \times [a, b].$$

The sets

$$K_1 = W'[\varepsilon] \cap \mathcal{R} \quad \text{and} \quad K_2 = W''[\varepsilon] \cap \mathcal{R}$$

are closed and disjoint. The lower edge $[0, 2N\pi] \times \{a\}$ is the image of a path connecting the left to the right side of \mathcal{R} and avoiding the set K_2 . Similarly, the upper edge $[0, 2N\pi] \times \{b\}$ corresponds to a path connecting the left to the right side of \mathcal{R} and avoiding the set K_1 . Alexander's lemma (lemma 3.11) guarantees the existence of a path $\gamma(t) = (\vartheta(t), r(t)) : I = [0, 1] \rightarrow \mathcal{R}$ with $\vartheta(0) = 0$ and $\vartheta(1) = 2N\pi$, such that $\gamma(t) \notin K_1 \cup K_2$ for all $t \in I$. For each $i = 0, \dots, N$, let

$$t_i = \min\{t \in I : \vartheta(t) = 2i\pi\},$$

so that

$$0 = t_0 < t_1 < \dots < t_N \leq 1$$

and the $N + 1$ -tuple of points $(r(t_0), r(t_1), \dots, r(t_N))$ in $]a, b[$ is well defined. Clearly, by the choice of N such that $N\varepsilon > (b - a)$, there exists at least a pair of points (t_j, t_k) with $j < k$ such that $|r(t_j) - r(t_k)| < \varepsilon$. Since $\varepsilon \leq \delta/3$ and

$$\min_{t \in I} \{\text{dist}(\gamma(t), W' \cup W'')\} > \delta/3,$$

we conclude that the segment joining $(2k\pi, r(t_j))$ and $(2k\pi, r(t_k))$ does not intersect the set $W' \cup W''$ (of course, such statement is trivial if $r(t_j) = r(t_k)$). We can now define the path

$$\tilde{\omega}(s) = \begin{cases} \gamma(t_j + 2s(t_k - t_j)), & \text{for } 0 \leq s \leq 1/2 \\ (2k\pi, r(t_k) + (2s - 1)(r(t_j) - r(t_k))), & \text{for } 1/2 \leq s \leq 1 \end{cases}$$

which takes values in $\mathcal{R} \setminus (W' \cup W'')$ and

$$\tilde{\omega}(0) = (2j\pi, r(t_j)), \quad \tilde{\omega}(1) = (2k\pi, r(t_j)).$$

Hence the projection $\omega = \pi \circ \tilde{\omega} : I \rightarrow X$ is a nontrivial loop in X (in fact, it corresponds to $k - j \in \mathbb{Z} \setminus \{0\}$ in the fundamental group of X) and, by construction, $\omega(t) \notin S, \forall t \in I$. This contradicts the hypothesis and hence the conclusion follows. \square

Our last result can be seen as a continuous version of the no-tie theorem for Hex game on the annulus, which has been proved being equivalent to the Poincaré-Birkhoff theorem. See [5] for a discrete version of this result.

Lemma 3.21 *Let X be a topological annulus and let $C_1, C_2 \subset X$ be closed connected sets such that*

$$C_1 \cap X_i \neq \emptyset \neq C_1 \cap X_o \quad \text{and} \quad C_2 \text{ is essentially embedded in } X.$$

Then

$$C_1 \cap C_2 \neq \emptyset.$$

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Proof. Assume, by contradiction that $C_1 \cap C_2 = \emptyset$, and let $\text{dist}(C_1, C_2) = \delta > 0$. In a $\delta/2$ -neighbourhood of C_1 we can find the image of a path $\gamma : [0, 1] \rightarrow X$ with $\gamma(0) \in X_i$ and $\gamma(1) \in X_o$. By construction, $\gamma(t) \notin C_2$, for all $t \in [0, 1]$. This proves that it is not true that $C_2 : X_i \uparrow X_o$ and therefore (by lemma 3.17) C_2 is not essentially embedded in X , thus contradicting one of our assumptions. \square

3.4 A CROSSING LEMMA FOR INVARIANT SETS

The lifting of a planar annulus $A[a, b]$ is a set which is invariant under the translation $(\vartheta, r) \mapsto (\vartheta + 2\pi, r)$. The question arising now is whether the crossing results obtained in the previous section hold also for sets which are invariant under a generic homeomorphism h .

Let X be a topological space and let $h : X \rightarrow X$ be a homeomorphism. Our first result is a version of lemma 3.3 for h -invariant sets. Indeed, we have

Lemma 3.22 *Let $A, B \subset X$ two nonempty disjoint sets which are connected by at least one path in X . Let $S \subset X$ be a closed set which satisfies $S : A \uparrow B$ and is invariant for h . Then there exists a nonempty closed set $C \subset S$ which is minimal with respect to the property of cutting the paths between A and B and invariant for h .*

Proof. Let \mathcal{F} be the set of all the nonempty closed subsets F of S such that $F : A \uparrow B$ and $h(F) = F$, with the elements of \mathcal{F} ordered by inclusion. Let $(F_\alpha)_{\alpha \in J}$ be a totally ordered subset of \mathcal{F} and define $F^* = \bigcap_{\alpha \in J} F_\alpha$. From $h(F_\alpha) = F_\alpha$ for all $\alpha \in J$, it follows that $h(F^*) = F^*$. The proof that $F^* : A \uparrow B$ is the same as that of lemma 3.3. Thus we obtain $F^* \in \mathcal{F}$ and the conclusion follows from Zorn's lemma. \square

The purpose consists in developing a result analogous to lemma 3.12 in the frame of h -invariant sets. Note that if in lemma 3.12 we have a continuum $C : X_l^- \uparrow X_r^-$ which is also h -invariant, then by lemma 3.13 we also have an h -invariant continuum intersecting X_b^+ and X_t^+ .

Before moving to the class of generalized and oriented rectangles, we consider for one moment a planar rectangle $R = [a, b] \times [c, d]$ oriented in the standard way and suppose that $h : R \rightarrow R$ is a *homeomorphism*. Let also $S \subset R$ be a compact set which intersects all the paths in R joining R_l^- to R_r^- and such that $h(S) = S$. We are looking for the existence of a continuum $C \subset S$ with $C : R_l^- \uparrow R_r^-$ and $h(C) = C$. It is not difficult to see that, in general, the answer is negative, as shown by the following elementary example.

3.4 A crossing lemma for invariant sets

Example 3.23 Let $R = [-2, 2] \times [0, 1]$ and $S = \{(\pm 1, y) : y \in [0, 1]\}$. Clearly, S has the property of intersecting all the paths in R joining the left edge to the right edge. Moreover, consistently with lemma 3.12, S contains two continua $\{-1\} \times [0, 1]$ and $\{1\} \times [0, 1]$ connecting the lower and the upper sides of R . However, if we take as a homeomorphism $h(x, y) = (-x, y)$, that is the symmetry with respect to the y -axis, then $h(S) = S$ but there is no connected subset of S which is invariant under h . \triangleleft

Observe that in example 3.23 the set S cannot be split as the union of two disjoint closed (nonempty) invariant subsets; this is equivalent to saying that S is not *invariantly connected*, according to [60, Definition 4.2]. Then, if we are allowed to replace the word *connected* with *invariantly connected*, we can get a full extension of lemma 3.12, as follows.

Lemma 3.24 Let $\tilde{X} = (X, X^-)$ be an oriented rectangle, let $h : X \rightarrow X$ be a homeomorphism and let $S \subset X$ be a closed set such that

$$S : X_l^- \uparrow X_r^- \quad \text{and} \quad h(S) = S.$$

Then there exists a compact, invariantly connected set $C \subset S$ such that

$$C : X_l^- \uparrow X_r^- .$$

Proof. By lemma 3.22 there exists a closed set $C \subset S$ such that $C : X_l^- \uparrow X_r^-$, with C invariant for h and minimal with respect to the cutting property. Suppose, by contradiction, that C is not invariantly connected and let $C_1, C_2 \subset C$ be two closed nonempty disjoint sets with $C_1 \cup C_2 = C$ and $h(C_i) = C_i$ for $i = 1, 2$. Now we conclude as in the proof of lemma 3.12. Indeed, since C is minimal and C_1, C_2 are proper subsets of C , there exist two paths γ_1, γ_2 in X which connects X_l^- to X_r^- and such that γ_i avoids C_i (for $i = 1, 2$). Then, by lemma 3.11, there exists a path $\gamma : I \rightarrow X$ with $\gamma(0) \in X_l^-$ and $\gamma(1) \in X_r^-$ with $\bar{\gamma} \cap C = \emptyset$, contradicting the assumption that $C : X_l^- \uparrow X_r^-$. \square

In order to achieve the connectedness of the set C , we propose a partial extension of lemma 3.12 with a further assumption on h which prevents the possibility of a situation like the one described in example 3.23. For simplicity in the exposition, we confine ourselves to the case of a planar rectangle. Note that here we are not requiring h to be a homeomorphism.

Lemma 3.25 (*h-Invariant Crossing lemma*) Let $R = [a, b] \times [c, d]$ be an oriented rectangle and let $h : R \rightarrow R$ be a continuous map such that

$$h(R_b^+) \subset R_b^+, \quad h(R_t^+) \subset R_t^+ . \quad (3.2)$$

Suppose that there exists a path $\sigma : I \rightarrow R$ with $\sigma(0) \in R_l^-$ and $\sigma(1) \in R_r^-$ such that

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- $\forall t \in I, \exists s \geq t : h(\sigma(t)) = \sigma(s)$.

Assume that there exists a compact set $S \subset R$ which cuts the paths between R_l^- and R_r^- and satisfies

$$h(S) \subset S.$$

Then there exists a compact connected set $C \subset S$ such that

- $h(C) = C$
- $C \cap R_b^+ \neq \emptyset \neq C \cap R_t^+$
- C cuts the paths between R_l^- and R_r^- .

Proof. As a first step, lemma 3.12 and lemma 3.13 guarantee the existence of a continuum in S joining the lower and the upper sides of the rectangle and cutting the paths from the left to the right side in R . We call such a continuum C^0 . Note that C^0 is not necessarily invariant.

In order to obtain an invariant set, define a sequence of continua

$$C^{i+1} = h(C^i), \forall i \geq 0.$$

Since $h(S) \subset S$, we also know that $C^i \subset S, \forall i \geq 0$. By the cutting property of C^0

$$C^0 \cap \bar{\sigma} \neq \emptyset,$$

so that there exists $t_0 \in I$ with $\sigma(t_0) \in C^0$. Clearly, $h(\sigma(t_0)) \in C^1$. On the other hand, by the hypothesis on σ , there exists $t_1 \in I$ such that $t_1 \geq t_0$ and $h(\sigma(t_0)) = \sigma(t_1) \in C^1$. Going on by induction and using step by step the hypothesis on σ , we obtain a monotone sequence $t_0 \leq t_1 \leq \dots \leq t_i \leq t_{i+1} \leq \dots$ in $[0, 1]$ such that $\sigma(t_i) \in C^i, \forall i \geq 0$. Let $t_i \nearrow t^* \in [0, 1]$. Then, passing to the limit in $\sigma(\cdot)$ and $h \circ \sigma$, we obtain

$$h(\sigma(t^*)) = \sigma(t^*) \in \text{Li } C^i$$

and therefore, by a classical result from [59, Theorem 6, Ch.5, §47, II],

$$C = \text{Ls } C^i$$

is a nonempty continuum. Recall that $z \in C$ if and only if there exists a sequence z_k with $z_k \in C^{i_k}$ for $(i_k)_k$ an increasing sequence of indexes such that $z_k \rightarrow z$. Then $h(z_k) \rightarrow h(z)$, with $h(z_k) \in C^{1+i_k}$ and therefore $h(z) \in C$. Thus we have proved that $h(C) \subset C$.

Conversely, for $z \in C$ and $(z_k)_k \rightarrow z$, as above, fix $k \geq 2$ and, from $z_k \in C^{i_k} = h(C^{-1+i_k})$, take $y_k \in C^{-1+i_k}$ such that $h(y_k) = z_k$. By compactness, $(y_k)_k$ has a convergent subsequence, which can be named $(y_{k_n})_n \rightarrow w$. By definition, $w \in C$. On the other hand, $h(w) = \lim h(y_{k_n}) = \lim z_{k_n} = z$. This proves that $h(C) = C$. Since $C^0 \cap R_b^+ \neq \emptyset$, we have

$$\emptyset \neq h(C^0) \cap R_b^+ \subset h(C^0) \cap h(R_b^+) \subset C^1 \cap R_b^+.$$

3.4 A crossing lemma for invariant sets

Then, by induction, we obtain

$$C^i \cap R_b^+ \neq \emptyset, \quad \forall i \geq 0$$

and, by compactness, we conclude that $C \cap R_b^+ \neq \emptyset$. The fact that $C \cap R_t^+ \neq \emptyset$ is proved in the same way. Having proved that C is a continuum intersecting the horizontal edges of the rectangle, we conclude that it cuts all the paths between R_l^- and R_r^- . \square

Note that the same result holds true (with an obvious modification in the proof) if we replace condition (3.2) with

$$h(R_b^+) \subset R_t^+, \quad h(R_t^+) \subset R_b^+. \quad (3.3)$$

By the assumption $h(R_b^+) \subset R_t^+$ in equation (3.2), we have

$$h(x, c) = (f(x), c), \quad \forall x \in [a, b],$$

where $f : [a, b] \rightarrow [a, b]$ is a suitable continuous function. It easily follows that if f is monotone nondecreasing, then the path $\sigma(t) = (a + t(b - a), c)$ satisfies the hypothesis of lemma 3.25. A similar observation holds for R_t^+ . Hence, we easily obtain the following corollary.

Corollary 3.26 *Let $\tilde{R} = (R, R^-)$ be an oriented rectangle, as in lemma 3.25, and let $h : R \rightarrow R$ be a continuous map satisfying relations (3.2); suppose also that at least one between $h|_{R_b^+}$ and $h|_{R_t^+}$ is monotone nondecreasing. Assume that there exists a compact set $S \subset R$ which cuts the paths between R_l^- and R_r^- and satisfies $h(S) \subset S$. Then the same conclusions of lemma 3.25 hold.*

Notice that in example 3.23 the function h is decreasing along both the horizontal edges of the rectangle. A trivial case of a continuous map which is monotone nondecreasing along the horizontal lines is the identity. In such a case, corollary 3.26 reduces to the crossing lemma of section 3.2.

A useful property of continua connecting two compact disjoint sets is the minimality, often named as irreducibility [2]. Indeed, a stronger version of lemma 3.12 holds, guaranteeing the existence of compact connected set $C \subset S$, irreducible between the left and the right sides of the domain. (see [2, Proposition 3]).

Then, as a next step, we look for the existence of irreducible invariant continua in this new setting.

Lemma 3.27 *Let X be a compact Hausdorff space and let $h : X \rightarrow X$ be a homeomorphism. Assume $A, B \subset X$ are closed disjoint sets, invariant for h and let $C \subset X$ be a continuum such that*

$$h(C) = C, \quad \text{and} \quad C \cap A \neq \emptyset \neq C \cap B. \quad (3.4)$$

Then there exists $\mathcal{E} \subset C$ satisfying (3.4) and minimal with respect to such property.

3 Crossing properties for two classes of planar sets

Proof. Let \mathcal{F} be the family of all the continua $\mathcal{C} \subset C$ which are invariant for the homeomorphism h and which intersect both A and B , with \mathcal{F} ordered by inclusion. The family \mathcal{F} is nonempty since $C \in \mathcal{F}$. The existence of a minimal sub-continuum of C satisfying (3.4), will be ensured by Zorn's lemma.

Given any chain $(C_j)_j$ in \mathcal{F} , we observe that $\bigcap_{j \in \mathcal{J}} C_j \neq \emptyset$ for every finite subset of indices \mathcal{J} . Hence, by the compactness of X , $C^* = \bigcap C_j$ is nonempty. Moreover C^* is compact and intersects both A and B . Using the fact that h is a homeomorphism, we obtain the invariance of C^* . Thus, if we prove that C^* is connected, we will get $C^* \in \mathcal{F}$ and Zorn's lemma will allow to conclude the proof.

Assume, by contradiction, that C^* is not connected. Then there exist C', C'' nonempty compact sets such that $C^* = C' \cup C''$ and $C' \cap C'' = \emptyset$. Then there are also two open disjoint sets A', A'' with $A' \supset C'$ and $A'' \supset C''$. We claim that there exists an index j^* such that the set $C_{j^*} \subset A' \cup A''$. Otherwise, it would happen that $C_j \not\subset A' \cup A''$ which would imply that $D_j = C_j \setminus (A' \cup A'') \neq \emptyset, \forall j \in \mathcal{J}$. The family $(D_j)_{j \in \mathcal{J}}$ is a family of closed sets with the finite-intersection property and therefore we obtain that

$$\exists \hat{x} \in \bigcap_{j \in \mathcal{J}} D_j \subset X \setminus (A' \cup A'').$$

This is in contradiction to $\bigcap_{j \in \mathcal{J}} D_j \subset \bigcap_{j \in \mathcal{J}} C_j = C^* \subset A' \cup A''$, then there exists some $C_{j^*} \subset A' \cup A''$.

Let us now define $\mathcal{J}^* = \{j \in \mathcal{J} : C_j \subset C_{j^*}\}$ and observe that

$$\bigcap_{j \in \mathcal{J}^*} C_j = \bigcap_{j \in \mathcal{J}} C_j = C^* \quad \text{and} \quad C_j \subset A' \cup A'', \forall j \in \mathcal{J}^*.$$

If $\mathcal{J}' = \{j \in \mathcal{J}^* : C_j \cap A' \neq \emptyset\}$ and $\mathcal{J}'' = \{j \in \mathcal{J}^* : C_j \cap A'' \neq \emptyset\}$ then $\mathcal{J}^* = \mathcal{J}' \cup \mathcal{J}''$ and, moreover, both \mathcal{J}' and \mathcal{J}'' are nonempty sets. Otherwise, if $\mathcal{J}' = \emptyset$, then $C_j \subset A''$ for all $j \in \mathcal{J}^*$, from which we derive that $C_j \subset X \setminus A'$ which is a closed set. Then $C^* \subset X \setminus A'$ that means $C^* \cap A' = \emptyset$, which is an absurd.

Hence we have proved that $\mathcal{J}' \neq \emptyset \neq \mathcal{J}''$ and thus there are two indices $j' \in \mathcal{J}'$ and $j'' \in \mathcal{J}''$ such that $C_{j'} \cap A' \neq \emptyset \neq C_{j''} \cap A''$. Consider the set $\hat{C} = C_{j'} \cup C_{j''} \subset C_{j^*}$. By definition, $\hat{C} \cap A' \neq \emptyset \neq \hat{C} \cap A''$. On the other hand, we know that $\hat{C} = C_{j'}$ or $\hat{C} = C_{j''}$ and thus we have found a disconnection of an element of the chain, leading to a contradiction. \square

3.5 AN APPLICATION

As a conclusion of this chapter, a possible application of the results exposed in section 3.3 is presented, consisting in an example inspired

by a model arising from the theory of fluid mixing considered by Kennedy and Yorke in [55]. Following [55, Section 2], we consider a planar map $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is the composition of a squeeze map J_λ and a rotation map R_{ϑ_0} . In [55] these two maps are defined as follows:

$$J_\lambda(x, y) = (\lambda x, y/\lambda), \quad \text{with } \lambda > 1.$$

In order to define the map R_{ϑ_0} we pass to the polar coordinates (ϑ, ρ) and require that R_{ϑ_0} is a counterclockwise rotation which leaves invariant all the concentric circumferences $\rho > 0$, keeps still all the points of the plane with $\rho \geq 1$ and satisfies

$$\lim_{\rho \rightarrow 0^+} \Theta(\vartheta, \rho) = \vartheta_0 > 0,$$

where $\Theta(\vartheta, \rho)$ is the angular displacement performed by R_{ϑ_0} on the point $z = (\rho \cos \vartheta, \rho \sin \vartheta)$.

In [55], under the hypotheses that R_{ϑ_0} is a diffeomorphism and that $\vartheta_0 > \frac{\pi}{2}$, with ϑ_0 not an odd multiple of $\pi/2$, the authors prove the existence of a Smale horseshoe if $\lambda > 0$ is sufficiently large. In particular, they prove that there exists an invariant Cantor set on which the map $F = R_{\vartheta_0} \circ J_\lambda$ is conjugate to an m -shift.

We are going to prove a result which, although not so sharp like that in [55], makes use of weaker conditions. To be more specific, from now on, the following assumptions will be made.

Let $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a continuous map with

$$J(x, y) = (J_1(x, y), J_2(x, y))$$

which satisfies the following properties.

- $J(0) = 0$, $J(z) \neq 0$ for $z \neq 0$ and $J(Q_i) \subset Q_i$ for $i = 1, \dots, 4$, where Q_i denotes the i -th closed quadrant of the plane;
- $J_2(x, 0) = 0$ and $J_1(x, 0) = a(x)x$, with $a(x) > 1$ for $x \neq 0$;
- $J_1(0, y) = 0$ and $J_2(0, y) = b(y)y$, with $0 < b(y) < 1$ for $y \neq 0$.

Plainly speaking, J leaves the quadrants invariant and moves the points of the x -axis away from the origin, while it pushes the points of the y -axis toward the origin. In the sequel it will be convenient to express the map J (restricted to $\mathbb{R}^2 \setminus \{O\}$) via its lifting to the covering space $\mathbb{R} \times \mathbb{R}_0^+$ as

$$\tilde{J} : (\vartheta, \rho) \mapsto (\vartheta', \rho'), \quad \vartheta' = \vartheta + \Theta_J(\vartheta, \rho), \quad \rho' = \mathcal{R}_J(\vartheta, \rho),$$

with Θ_J and \mathcal{R}_J continuous functions which are 2π -periodic in the ϑ -variable. Note that the assumption $J(Q_i) \subset Q_i$ for $i = 1, \dots, 4$, reflects to the fact that

$$|\Theta_J(\vartheta, \rho)| \leq \frac{\pi}{2}. \quad (3.5)$$

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As a second map, we consider a continuous counterclockwise rotation R around the origin such that $R(0) = 0$ and $R(z) \neq 0$ for $z \neq 0$. We also express R (restricted to $\mathbb{R}^2 \setminus \{0\}$) by means of polar coordinates through its lifting

$$\tilde{R} : (\vartheta, \rho) \mapsto (\vartheta', \rho'), \quad \vartheta' = \vartheta + \Theta_R(\vartheta, \rho), \quad \rho' = \mathcal{R}_R(\vartheta, \rho), \quad (3.6)$$

with Θ_R and \mathcal{R}_R continuous functions which are 2π -periodic in the ϑ -variable and assume the following conditions.

There exist $r_0 \in]0, 1[$ and ϑ_0 such that

- $\Theta_R(\vartheta, r_0) \geq \vartheta_0$, $\Theta_R(\vartheta, 1) = 0$, $\forall \vartheta \in \mathbb{R}$;
- $\mathcal{R}_R(\vartheta, \rho) = \rho$, $\forall \vartheta \in \mathbb{R}$ and $\rho \in [r_0, 1]$.

According to the above hypotheses, the map R leaves invariant the circumferences of center the origin and radius $\rho \in [r_0, 1]$. Moreover, the points with $\rho = r_0$ are rotated in the counterclockwise sense by an angle larger or equal to ϑ_0 , while the points with $\rho = 1$ are kept still.

Under the above assumptions on J and R , the following result holds.

Theorem 3.28 *Let R be a homeomorphism of the annulus $\mathcal{A} = A[r_0, 1]$ onto itself and suppose also that $\vartheta_0 > 2\pi + \frac{\pi}{2}$. Then the map $\Psi = J \circ R$ has at least four fixed points in the interior of the annulus. Such result is stable with respect to small continuous perturbations of the map Ψ .*

Proof. Let us denote \mathcal{A}_i and \mathcal{A}_o the inner and outer boundaries of \mathcal{A} . We also restrict the map Ψ to the annulus \mathcal{A} and consider its lifting $\tilde{\Psi}$ to the covering space $\mathbb{R} \times [r_0, 1]$ as

$$\tilde{\Psi} : (\vartheta, \rho) \mapsto (\vartheta'', \rho''), \quad \vartheta'' = \vartheta + \Theta_\Psi(\vartheta, \rho), \quad \rho'' = \mathcal{R}_\Psi(\vartheta, \rho),$$

with Θ_Ψ and \mathcal{R}_Ψ continuous functions which are 2π -periodic in the ϑ -variable. By the above positions for \tilde{J} and \tilde{R} we have that

$$\vartheta'' = \vartheta' + \Theta_J(\vartheta', \rho'), \quad \text{with } \vartheta' = \vartheta + \Theta_R(\vartheta, \rho), \quad \rho' = \mathcal{R}_R(\vartheta, \rho).$$

We also introduce a set $S \subset \mathcal{A}$ defined as

$$S = \pi(\{(\vartheta, \rho) : \Theta_\Psi(\vartheta, \rho) = 2\pi\}),$$

where π is the standard covering projection associated to the polar coordinates. The set S is a compact subset of the annulus consisting of the points which are rotated by an angle of exactly 2π under the action of Ψ .

Suppose that $\gamma : I \rightarrow \mathcal{A}$ is a path with $\gamma(0) \in \mathcal{A}_i$ and $\gamma(1) \in \mathcal{A}_o$. We express the points of $\gamma(t)$ in polar coordinates as

$$\gamma(t) = (\rho(t) \cos \vartheta(t), \rho(t) \sin \vartheta(t))$$

and consider the angular displacement for the map R along the points of $\gamma(t)$, using the function

$$\omega_\gamma(t) : I \ni t \mapsto \Theta_R(\vartheta(t), \rho(t)).$$

By the assumptions, $\omega_\gamma(0) > 2\pi + \frac{\pi}{2}$ and $\omega_\gamma(1) = 0$. Now we apply the map J to the points of $R(\gamma(t))$. Using condition (3.5) we have that the angular displacement Θ_Ψ along the curve $\gamma(t)$ can be expressed as

$$\Theta_\Psi(\vartheta(t), \rho(t)) = \omega_\gamma(t) + \Delta(t),$$

where $\Delta(t)$ is a continuous function satisfying

$$|\Delta(t)| \leq \frac{\pi}{2}, \quad \forall t \in [0, 1].$$

Recalling the properties of ω_γ we find

$$\Theta_\Psi(\vartheta(0), \rho(0)) > 2\pi, \quad \Theta_\Psi(\vartheta(1), \rho(1)) \leq \frac{\pi}{2} < 2\pi$$

and therefore, by the continuity of γ we can conclude that $\bar{\gamma} \cap S \neq \emptyset$. Now we can apply lemma 3.18 which ensures the existence of a compact connected set $C \subset S$ which is essentially embedded into \mathcal{A} . It is also clear that C (as well as S) is contained in the interior of \mathcal{A} .

Let us consider now the intersection of \mathcal{A} with the first quadrant Q_1 . The boundary of such intersection consists of two segments $L_1 = [r_0, 1] \times \{0\}$, $L_2 = \{0\} \times [r_0, 1]$ and two arcs $C_1 = r_0 S^1 \cap Q_1$, $C_2 = S^1 \cap Q_1$. We also define

$$\mathcal{B} = R^{-1}(\mathcal{A} \cap Q_1).$$

The set \mathcal{B} is a topological rectangle for which we give an orientation by setting

$$\mathcal{B}_l^- = R^{-1}(L_1), \quad \mathcal{B}_r^- = R^{-1}(L_2), \quad \mathcal{B}_t^+ = R^{-1}(C_1), \quad \mathcal{B}_b^+ = R^{-1}(C_2).$$

By the assumptions, we see that

$$\|\Psi(z)\| > \|z\|, \quad \forall z \in \mathcal{B}_l^- \quad \text{and} \quad \|\Psi(z)\| < \|z\|, \quad \forall z \in \mathcal{B}_r^-.$$

Hence, on each path with values in \mathcal{B} connecting \mathcal{B}_l^- with \mathcal{B}_r^- there is some point where $\|\Psi(z)\| = \|z\|$. lemma 3.13 ensures the existence of a continuum $C' \subset \mathcal{B}$ with $C' \cap \mathcal{B}_b^+ \neq \emptyset$ and $C' \cap \mathcal{B}_t^+ \neq \emptyset$. We are now in position to apply lemma 3.21 which guarantees that $C' \cap C \neq \emptyset$. By definition of C and C' we conclude that any point $w \in C' \cap C$ is a fixed point for Ψ with $\Psi(w) = w \in \text{int}(Q_1 \cap \mathcal{A})$. Repeating the same argument for the other quadrants we find the remaining three fixed points. \square

Following the proof it is clear that if $\vartheta_0 > 2j\pi + \frac{\pi}{2}$, for some positive integer j , then there are at least $4j$ fixed points.

For a different application of classical separation results to the existence of fixed points and periodic points to planar maps arising from ordinary differential equations, we refer also to [88].

4 | FIXED POINT RESULTS FOR RECTANGULAR REGIONS

In chapter 3 some topological lemmas about the existence of continua crossing rectangular and annular regions from one side to another have been presented. Now it is time to explain how those tools can be exploited in order to obtain fixed point theorems, which are of great use for proving the existence of periodic solutions of planar Hamiltonian systems.

This chapter is devoted to recalling some results about rectangular domains, while chapter 5 will deal with some fixed point theorem for maps defined on annular domains. The results which are recollected in the present chapter are not new; indeed they were first presented in [80] and afterwards further developed in [85, 90]; a survey on these topics can also be found in [88]. Nevertheless, for reader's convenience and in order to make this thesis more complete and self-contained, all the most relevant results will be herein recollected.

4.1 A FIXED POINT THEOREM

The main result treated in this section is a fixed point theorem for continuous maps defined on generalized rectangles, which are expansive along some direction. The proof of this theorem has as a key point the crossing lemma which we talked about in section 3.2 and it depends also on the concept of *stretching along the paths* (SAP property) that we are now going to define.

Definition 4.1 Let $\tilde{A} = (A, A^-)$ and $\tilde{B} = (B, B^-)$ be two oriented rectangles and let $K \subset A$ be a nonempty compact set. Suppose also that $f : K \rightarrow \mathbb{R}^2$ is a continuous map. We say that the pair (K, f) *stretches A to B along the paths* and write

$$(K, f) : \tilde{A} \rightleftarrows \tilde{B},$$

if, for every path $\gamma : [t_0, t_1] \rightarrow A$, with $\gamma(t_0)$ and $\gamma(t_1)$ belonging to different components of A^- , there exists a subinterval $[s_0, s_1] \subset [t_0, t_1]$ such that

$$\gamma(t) \in K \quad \text{and} \quad f(\gamma(t)) \in B \quad \forall t \in [s_0, s_1]$$

with $f(\gamma(s_0))$ and $f(\gamma(s_1))$ belonging to different components of B^- . If $K = A$, we simply write $f : \tilde{A} \rightleftarrows \tilde{B}$, instead of $(A, f) : \tilde{A} \rightleftarrows \tilde{B}$.

4 Fixed point results for rectangular regions

If it is more convenient, without loss of generality we can assume $f(K) \subset B$. Indeed, $(K, f) : \tilde{A} \rightleftarrows \tilde{B}$, if and only if $(K', f) : \tilde{A} \rightleftarrows \tilde{B}$, with $K' = K \cap f^{-1}(B)$. However we also stress that $f : \tilde{A} \rightleftarrows \tilde{B}$, *does not imply* that $f(A) \subset B$; if we know that $f : \tilde{A} \rightleftarrows \tilde{B}$, we can only infer that $(H, f) : \tilde{A} \rightleftarrows \tilde{B}$, for $H = f^{-1}(B)$. In general, for a continuous map $f : A \rightarrow \mathbb{R}^2$, it holds that if $(K, f) : \tilde{A} \rightleftarrows \tilde{B}$, for a suitable set $K \subset A$, then $(H, f) : \tilde{A} \rightleftarrows \tilde{B}$, for any compact set H such that $K \cap f^{-1}(B) \subset H \subset A$.

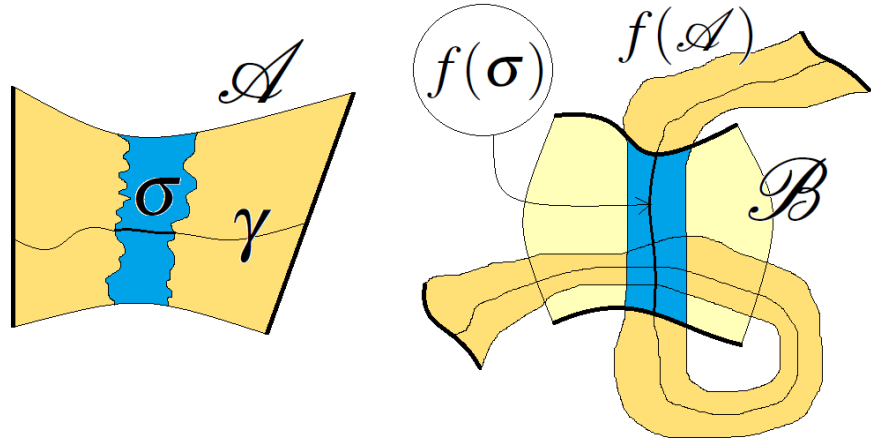


Figure 15: A pictorial description of the SAP property: A map f transforms a generalized rectangle A to a snake-like set $f(A)$ which crosses the generalized rectangle B . Both A and B are oriented by putting in evidence with bold lines their $[\cdot]^-$ -sets. A path γ in A connecting the two components of A^- contains a sub-path σ such that $f(\sigma)$ is contained in B and connects the two components of B^- . For a suitable compact set $K \subset A$ (for instance, the part of A indicated in figure with a darker color), we have that $(K, f) : \tilde{A} \rightleftarrows \tilde{B}$.

Now we are in position to present the fixed point theorem. Its proof was already given in some preceding papers (see [82, 85]) but it will be here repeated in order to show to the reader the role played by the crossing lemma.

Theorem 4.2 *Let $\tilde{R} = (R, R^-)$ be an oriented rectangle and let $f : H \rightarrow \mathbb{R}^2$ be a continuous map defined on a compact set $H \subset R$. Assume that*

$$(H, f) : \tilde{R} \rightleftarrows \tilde{R}.$$

Then there exists a point $w \in H$ such that $f(w) = w$.

Proof. Without loss of generality, we suppose that $f(H) \subset R$. Let $\eta : Q \rightarrow \eta(Q) = R$ be a homeomorphism which provides R with the structure of oriented rectangle and let $K = \eta^{-1}(H)$, $\varphi = \eta^{-1} \circ f \circ \eta$. By the assumptions, $\varphi : K \rightarrow Q$ and $(K, \varphi) : \tilde{Q} \rightleftarrows \tilde{Q}$, where $\tilde{Q} = (Q, Q^-)$.

4.1 A fixed point theorem

For $\varphi = (\varphi_1, \varphi_2)$ observe that $0 \leq \varphi_2(x, y) \leq 1$, for all $(x, y) \in K$ and, moreover, let $S \subset Q$, be the compact set defined by

$$S = \{(x, y) \in K : x - \varphi_1(x, y) = 0\}.$$

Let $\gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow Q$ be a path joining the bottom and the top sides of Q , that is $\gamma_1(0) = 0$ and $\gamma_1(1) = 1$. Since $(K, \varphi) : \tilde{Q} \xrightarrow{\cong} \tilde{Q}$, there exists $[s_0, s_1] \subset [0, 1]$ with $\gamma(t) \in K$ and $\varphi(\gamma(t)) \in Q$ such that $t \in [s_0, s_1]$ and, moreover, $\varphi_1(\gamma(s_0)) = 0$, $\varphi_1(\gamma(s_1)) = 1$. Then, by Bolzano's theorem, the map $[s_0, s_1] \ni t \mapsto \gamma_1(t) - \varphi_1(\gamma_1(t), \gamma_2(t))$ vanishes at some point $t^* \in [s_0, s_1]$, with $\gamma(t^*) \in K$. We have thus proved that $S \cap \tilde{\gamma} \neq \emptyset$ for each path γ with values in Q and joining Q_1^- with Q_1^+ . Now the crossing lemma guarantees that S contains a continuum C which intersects Q_b^+ and Q_t^+ at some points, say $p = (p_1, 0)$ and $q = (q_1, 1)$, respectively. Evaluating $\psi(x, y) = y - \varphi_2(x, y)$ along C , we have that $\psi(p) = -\varphi_2(p) \leq 0$ and $\psi(q) = 1 - \varphi_2(q) \geq 0$. Therefore, there exists $z \in C$ such that $\psi(z) = 0$. By the definition of S and the inclusions $C \subset S \subset K$, we conclude that $\varphi(z) = z$ and, finally, $f(w) = w$, for $w = \eta(z) \in H$. \square

In theorem 4.2 we have required f to be defined and continuous only on $H \subset R$. Using Tietze theorem, it is also possible to assume $f : R \rightarrow \mathbb{R}^2$ continuous. In any case, the behaviour of f outside H , as well as possible discontinuities of f in $R \setminus H$, do not effect the result.

An immediate consequence of definition 4.1 is the fact that the SAP property is preserved by the composition of maps, as stated in the following lemma.

Lemma 4.3 *Let $\ell \in \mathbb{N}$ be a fixed index, with $\ell \geq 3$ and consider the family of oriented rectangles $\tilde{A}_i = (A_i, A_i^-)$ for $i = 1, \dots, \ell$; for $i = 1, \dots, \ell - 1$ let $f_i : K_i \rightarrow \mathbb{R}^2$ be continuous maps defined on the compact sets $K_i \subset A_i$. Let K be the (compact) subset of K_1 where the map $f = f_{\ell-1} \circ \dots \circ f_2 \circ f_1$ is defined and such that $f_1(x) \in K_2$, $(f_2 \circ f_1)(x) \in K_3, \dots, (f_{j-1} \circ \dots \circ f_2 \circ f_1)(x) \in K_j, \dots, (f_{\ell-2} \circ \dots \circ f_2 \circ f_1)(x) \in K_{\ell-1}$, $\forall x \in K$. If*

$$(K_i, f_i) : \tilde{A}_i \xrightarrow{\cong} \tilde{A}_{i+1} \quad \forall i = 1, \dots, \ell - 1$$

then

$$(K, f) : \tilde{A}_1 \xrightarrow{\cong} \tilde{A}_\ell.$$

Then, from theorem 4.2 and lemma 4.3, the following result holds.

Theorem 4.4 *Let $\tilde{R} = (R, R^-)$ be an oriented rectangle, let H_0, H_1, \dots, H_{m-1} be $m \geq 2$ nonempty compact and pairwise disjoint subsets of R , and let $f : \mathcal{H} = \bigcup_{j=0}^{m-1} H_j \rightarrow \mathbb{R}^2$ be a continuous map. Assume that*

$$(H_j, f) : \tilde{R} \xrightarrow{\cong} \tilde{R} \quad \forall j = 0, 1, \dots, m - 1. \quad (4.1)$$

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Then, for any k -periodic sequence $(s_i)_{i \in \mathbb{Z}} \in \{0, 1, \dots, m-1\}^{\mathbb{Z}}$, with $k \geq 1$, there exists at least one two-sided sequence $(w_i)_{i \in \mathbb{Z}}$ such that

$$w_i \in H_{s_i} \quad \text{and} \quad w_{i+1} = f(w_i) \quad \forall i \in \mathbb{Z},$$

with $w_{i+k} = w_i, \forall i \in \mathbb{Z}$.

On the other hand, lemma 4.3 leads to the following result.

Theorem 4.5 *Let $\tilde{R} = (R, R^-)$ be an oriented rectangle, let H_0, H_1, \dots, H_{m-1} be $m \geq 2$ nonempty compact and pairwise disjoint subsets of R , and let $f : \mathcal{H} = \bigcup_{j=0}^{m-1} H_j \rightarrow \mathbb{R}^2$ be a continuous map. Assume that (4.1) holds. Then, for any sequence $\xi = (s_i)_{i \in \mathbb{N}} \in \{0, 1, \dots, m-1\}^{\mathbb{N}}$, there exists a continuum*

$$\mathcal{C}_\xi \in H_{s_0}, \quad \text{with} \quad \mathcal{C}_\xi \cap R_b^+ \neq \emptyset, \quad \mathcal{C}_\xi \cap R_t^+ \neq \emptyset,$$

such that, for each $w \in \mathcal{C}_\xi$, the sequence

$$w_0 = w, \quad w_{i+1} = f(w_i), \quad \forall i \in \mathbb{N},$$

satisfies

$$w_i \in H_{s_i} \quad \forall i \in \mathbb{N}.$$

From one-sided sequences it is possible to get two-sided sequences via a diagonal argument (see [82, theorem 2.2]). Thus, as a corollary of theorem 4.4 and theorem 4.5 one easily obtains the existence of chaotic dynamics.

Theorem 4.6 *Let $\tilde{R} = (R, R^-)$ be an oriented rectangle, let H_0, H_1, \dots, H_{m-1} be $m \geq 2$ nonempty compact and pairwise disjoint subsets of R and let $f : \mathcal{H} = \bigcup_{j=0}^{m-1} H_j \rightarrow \mathbb{R}^2$ be a continuous map. If (4.1) holds, then f induces chaotic dynamics on m symbols on H , in the sense that for every two-sided sequence of m symbols $(s_i)_{i \in \mathbb{Z}} \in \Sigma^m$ there exists a point $z \in H_{s_0}$ such that*

$$\varphi^i(z) \in H_{s_i} \quad \forall i \in \mathbb{Z}. \quad (4.2)$$

Moreover, if $(s_i)_{i \in \mathbb{Z}}$ is a k -periodic sequence (for some $k \geq 1$) then there exists a point z satisfying (4.2) and which is also a k -periodic point.

The definition of chaos which we refer to is the so-called *coin-tossing chaos* asserting that for every possible sequence of outcomes of an m -sided coin, there exists a point $z \in \mathcal{H}$ which is able to reproduce that sequence jumping among the sets H_{s_i} . In the applications, the map f will be the Poincaré map of some planar system; hence, the possibility of reproduce the outcomes of a periodic sequence will provide us with the existence of a periodic point for every possible period $k \geq 1$.

Theorem 4.6 is strongly related to a result by Kennedy and Yorke [56, theorem 1]. Indeed the assumption of the *stretching* $(H_j, f) : \tilde{R} \rightleftarrows \tilde{R}$, for $j = 0, 1, \dots, m - 1$ corresponds to a *horseshoe hypothesis* with a crossing number $M \geq m$, assumed by Kennedy and Yorke. Their theorem ensures the existence of a closed invariant set $R_I \subset R$ for f such that $f|_{R_I}$ is semiconjugate to a one-sided M -shift (respectively, semiconjugate to a two-sided M -shift if f is one-to-one). The results in [56] hold in the more general setting of mappings defined on locally connected compact sets of a separable metric space; however, the special geometry for our simplified setting allows us to draw as a further conclusion with respect to [56, theorem 1] also the information about the existence of periodic points for f and therefore to harmonic and subharmonic solutions in the framework of Poincaré maps.

In the applications of theorem 4.6 there is often a natural splitting of the map f as

$$f = \psi \circ \varphi. \quad (4.3)$$

For instance, if f is the Poincaré map associated to a planar differential system, it may be natural to decompose f as two (or more than two) Poincaré maps corresponding to some peculiar behaviours of the system in different time intervals. In [84, 85] we introduced a corollary of theorem 4.6 dealing with this situation which will be also used in the application presented in the next section. For sake of simplicity, we confine ourselves to the case $m = 2$.

Corollary 4.7 *Let $\tilde{M} = (M, M^-)$ and $\tilde{N} = (N, N^-)$ be a pair of oriented rectangles, and let H_0 and H_1 be two nonempty compact and disjoint subsets of M . Let also $\varphi : H = H_0 \cup H_1 \rightarrow \mathbb{R}^2$ and $\psi : N \rightarrow \mathbb{R}^2$ be continuous maps. Assume the conditions*

- $(H_j, \varphi) : \tilde{M} \rightleftarrows \tilde{N}$ for $j = 0, 1$
- $\psi : \tilde{N} \rightleftarrows \tilde{M}$.

Then $f = \psi \circ \varphi$ induces chaotic dynamics on two symbols in H .

Corollary 4.7 has been applied in [86] in connection to the theory of the linked twist maps. For extensions of this theory to higher dimensions, see [83], [91, 90] and [99].

4.2 AN APPLICATION

In the second part of the chapter a possible application of the topological theorems to the search of periodic solutions and chaotic-like dynamics associated to a second order scalar ODEs is presented. The results are obtained applying the theoretical theorems above exposed

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to the associated Poincaré map. More in detail, we focus our attention to the pendulum equation with a moving support. As explained in [43, Ch.8], a pendulum equation with a harmonically moving support is equivalent to a pendulum with a stationary support in a space with a periodically varying constant of gravity. Accordingly, mechanical systems of this kind are modelled by a second order equation of the form

$$u'' + w(t) \sin u = 0, \quad (4.4)$$

or, equivalently, by the first order system in the phase plane $(x, y) = (u, u')$

$$\begin{cases} x' = y \\ y' = -w(t) \sin x, \end{cases} \quad (4.5)$$

where the weight $w(t)$ is a periodic function of period $T > 0$. Following a classical approach (see [43]) one is usually led to study the linearized equation

$$\begin{cases} x' = y \\ y' = -w(t)x, \end{cases} \quad (4.6)$$

which represents a reasonable approximation of (4.5) in the case of *small solutions*. In this special case we have to study a Hill equation, which, for a general $w(t)$, represents still a nontrivial task. In [43, p. 344] Den Hartog suggests to consider a simplified form of (4.6) by assuming a squarewave weight function. For recent results about the Hill equation with stepwise coefficients we refer also to [78, 77, 38] and the references therein. Following the same suggestion, we are going to analyze the global dynamics of system (4.5) in the simplified case in which $w(t)$ is a stepwise function. In [15] and in [16] the case in which $w(t) > 0$ for all $t \in \mathbb{R}$ and the case in which $w(t)$ changes its sign were discussed, respectively; here we consider the case in which the weight may vanish during some time interval. This results has already appeared in [87].

Physically, for $u(t) = \vartheta(t)$, which is the angle between the rod and the vertical line pointing downward, this corresponds to a model in which the pendulum winds around its pivot with constant angular speed $\vartheta'(t) = \text{constant} = \vartheta_0$ and without the effect of a gravity field for some time interval, coming back to the usual oscillation mode under the effect of a constant gravity field for a subsequent time interval. We also assume that the switching between these two oscillatory modes occurs in a T -periodic fashion. In conclusion, we suppose that the weight $w : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic function and there are $T_0, T_1 \in]0, T[$ with

$$T_0 + T_1 = T,$$

such that

$$w(t) = \begin{cases} K & \text{for } 0 \leq t < T_0 \\ 0 & \text{for } T_0 \leq t < T. \end{cases} \quad (4.7)$$

We assume $K > 0$ (it is easy to check that the case $K < 0$, which corresponds to the so-called inverted pendulum, can be treated with minor modifications in our forthcoming analysis).

Equation (4.5) with a weight function as in (4.7) can be viewed as a superposition of the equations

$$\begin{cases} x' = y \\ y' = -K \sin x, \end{cases} \quad (4.8)$$

and

$$\begin{cases} x' = y \\ y' = 0 \end{cases} \quad (4.9)$$

the first acting on an interval of length T_0 and the second one on an interval of length T_1 . As a last but crucial remark, we notice that system (4.5), as well as (4.8) and (4.9) is studied in the *cylindrical phase space*, namely, we assume

$$(x_1, y) \equiv (x_2, y) \quad \text{for} \quad \frac{x_2 - x_1}{2\pi} \in \mathbb{Z}.$$

This last remark, however, will not be used in the proof of theorem 4.8 below; however, it turns out to be useful in view of extending our theorem to more general situations (see remark 4.1).

The Poincaré map $\Phi : z \mapsto \zeta(T, z)$ associated to system (4.5) can be splitted as

$$\Phi = \Phi_1 \circ \Phi_0 \quad (4.10)$$

where Φ_0 is the Poincaré map associated to the classical pendulum equation (4.8) for the time interval $[0, T_0]$, while Φ_1 is the Poincaré map associated to equation (4.8) for the time interval $[0, T_1]$. By a direct integration of the equation, Φ_1 can be easily described as the shift $\Phi_1(x, y) = (x + T_1 y, y)$.

We describe now the main steps of the proof of the presence of chaotic dynamics for equation (4.4), using corollary 4.7. To this aim, first of all we recall some basic facts about the phase plane analysis of (4.8), which corresponds to the nonlinear simple pendulum equation

$$x'' + K \sin x = 0, \quad K > 0.$$

Equation (4.8) is a simple example of a first order planar Hamiltonian system

$$\begin{cases} x' = y \\ y' = -g(x), \end{cases} \quad (4.11)$$

with $g : \mathbb{R} \rightarrow \mathbb{R}$ a locally Lipschitz continuous function. The orbits associated to (4.11) lie on the level lines of the energy function

$$E(x, y) = \frac{1}{2}y^2 + G(x), \quad \text{with} \quad G(x) = \int_0^x g(s) ds.$$

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For the pendulum equation we have

$$G(x) = K(1 - \cos x)$$

and the well known phase portrait shown in figure 16 below. We set

$$d = 2\sqrt{K}$$

and consider, for each $\mu \in]0, d]$, the set

$$\Gamma^\mu = \{(x, y) \in [-\pi, \pi] \times \mathbb{R} : E(x, y) = \frac{1}{2}\mu^2\}.$$

For each $0 < \mu < d$, the set Γ^μ is a closed curve surrounding the origin and intersecting the x -axis at the points $(\pm \arccos(1 - \mu^2/2K), 0)$ and the y -axis at the points $(0, \pm\mu)$. Actually, Γ^μ is a periodic orbit which is run in the clockwise sense and its period, denoted by τ_μ , can be expressed by means of an elliptic integral (see [42, pp. 180-181]). The time-map $e \mapsto \tau_\mu$ is a strictly increasing function with

$$\lim_{\mu \rightarrow 0^+} \tau_\mu = \frac{2\pi}{\sqrt{K}} \quad \text{and} \quad \lim_{\mu \rightarrow d^-} \tau_\mu = +\infty$$

(see [105, Figure 14]). On the other hand, for $\mu = d$, the level set Γ^d is the union of four orbits which are the two equilibrium points $(-\pi, 0)$ and $(\pi, 0)$ (which coincide each other in the cylindrical phase space and correspond to the unstable equilibrium position of the pendulum) and the two connecting orbits

$$L^+ = \{(x, y) \in]-\pi, \pi[\times \mathbb{R} : E(x, y) = 2K, y > 0\},$$

$$L^- = \{(x, y) \in]-\pi, \pi[\times \mathbb{R} : E(x, y) = 2K, y < 0\}.$$

The line L^+ is the orbit through $(0, d)$ which connects $(-\pi, 0)$ (for $t \rightarrow -\infty$) to $(\pi, 0)$ (for $t \rightarrow +\infty$) in the upper half-plane, while L^- is the orbit through $(0, -d)$ which connects $(\pi, 0)$ (for $t \rightarrow -\infty$) to $(-\pi, 0)$ (for $t \rightarrow +\infty$), in the lower half-plane.

We are ready now to define two generalized rectangles \mathcal{M} and \mathcal{N} and choose a suitable orientation for each of them (see Figure 17) in order to apply corollary 4.7. To this end, we fix two numbers b, c with

$$0 < b < c < d$$

and consider (in the upper half plane) the intersection of the region

$$\mathcal{W} = \{(x, y) \in [-\pi, \pi] \times \mathbb{R} : \frac{1}{2}c^2 \leq E(x, y) \leq \frac{1}{2}d^2\}$$

with the strip

$$\mathcal{S} = \mathbb{R} \times [0, b]$$

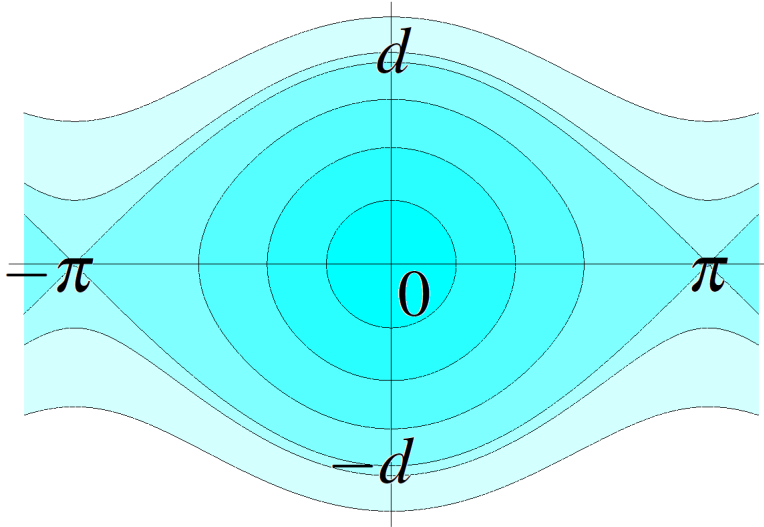


Figure 16: Energy level lines for equation (4.8) in the phase plane. The two separatrices (heteroclinic connections) connecting the unstable equilibria (saddle points) $(-\pi, 0)$ and $(\pi, 0)$ intersect the vertical axis at $(0, d)$ (in the upper half-plane) and $(0, -d)$ (in the lower half-plane), respectively, for $d = 2\sqrt{K}$.

This intersection is made by two disjoint sets which are topological rectangles. We call \mathcal{M} the component of $\mathcal{W} \cap \mathcal{S}$ contained in the right half-plane and we call \mathcal{N} the symmetric one with respect to the y -axis. One can easily find a homeomorphism mapping the unit square onto \mathcal{M} . Indeed, the function

$$h : (\mu, y) \mapsto (\arccos(1 - (2K)^{-1}(\mu^2 - y^2)), y)$$

maps the rectangle $[c, d] \times [0, b]$ homeomorphically onto \mathcal{M} and from this it is a simple task to obtain the desired homeomorphism defined on Q onto \mathcal{M} . Having checked that \mathcal{M} is a topological rectangle, we have that also \mathcal{N} is a topological rectangle, using the symmetry $(x, y) \mapsto (-x, y)$ transforming \mathcal{M} into \mathcal{N} .

Observe that \mathcal{W} is an invariant set for system (4.8); indeed, each point $z_0 = (x_0, y_0) \in \mathcal{W}$ belongs to the energy level line Γ^{μ_0} with

$$\mu_0 = 2\sqrt{E(x_0, y_0)} \in [c, d]$$

and the solution of (4.8) with initial point z_0 lies on Γ^{μ_0} . In particular, for each $z_0 \in \mathcal{M}$, we can represent the solution $(x(t), y(t))$ of (4.8) with $(x(0), y(0)) = z_0$ in polar coordinates, so that

$$x(t) = \rho(t, z_0) \cos \vartheta(t, z_0), \quad y(t) = \rho(t, z_0) \sin \vartheta(t, z_0),$$

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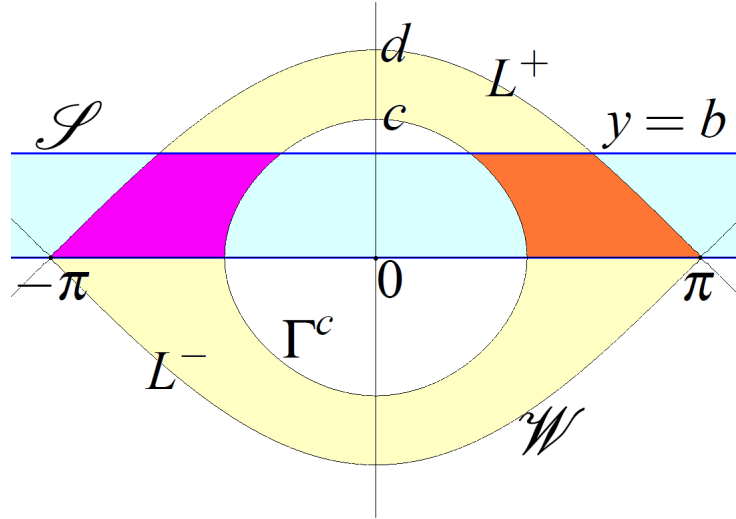


Figure 17: The set \mathcal{W} is the part of the strip $[-\pi, \pi] \times \mathbb{R}$ between the energy level lines Γ^c and Γ^d . The intersection of \mathcal{W} with the strip \mathcal{S} produces two rectangular regions (generalized rectangles), painted with a darker color. The set \mathcal{N} is the component of the intersection with $x < 0$, while \mathcal{M} is the component of the intersection with $x > 0$. The sets \mathcal{M} and \mathcal{N} are symmetric with respect to the y -axis.

The angular function $\vartheta(t, z_0)$ is well defined, continuous with respect to $(t, z_0) \in \mathbb{R} \times \mathcal{M}$ and satisfies $\vartheta(0, z_0) \in [0, \pi/2]$ (since \mathcal{M} is contained in the first quadrant). It is easy to check that $t \mapsto \vartheta(t, z_0)$ is a strictly decreasing function provided that $z_0 \neq (\pi, 0)$. As we have already observed, for $\mu_0 \in [c, d[$, we know that Γ^{μ_0} is a periodic orbit of period τ_{μ_0} which is run in the clockwise sense. Hence, if we take any initial point $z_0 \in \mathcal{M} - \Gamma^d$, we conclude that z_0 is a periodic point of system (4.8) of period τ_{μ_0} and therefore, for j a nonnegative integer,

$$\vartheta(t, z_0) - \vartheta(0, z_0) \lesseqgtr -2j\pi \quad \text{if and only if} \quad t \gtrless j\tau_{\mu_0}$$

(remember that the motion associated to (4.8) occurs in the clockwise sense and therefore the angle decreases when the time increases).

As \mathcal{W} is invariant for Φ_0 , similarly, the strip \mathcal{S} is invariant for Φ_1 . In this case, under the effect of (4.9), all the points of \mathcal{N} which belong also to the x -axis are rest points, while all the other points in \mathcal{N} (with $y > 0$) move from the left to the right along the lines $y = \text{constant} = y_0$ with constant speed $x'(t) = y_0 > 0$.

For \mathcal{M} and \mathcal{N} we consider now the following orientations:

$$\mathcal{M}_l^- = \mathcal{M} \cap \Gamma^c, \quad \mathcal{M}_r^- = \mathcal{M} \cap \Gamma^d = \mathcal{M} \cap L^+ \cup \{(\pi, 0)\}.$$

$$\mathcal{N}_l^- = \mathcal{N} \cap [-\pi, 0] \times \{0\}, \quad \mathcal{N}_r^- = \mathcal{N} \cap [-\pi, 0] \times \{b\}.$$

Step 1. Stretching the paths from \mathcal{M} to \mathcal{N} by Φ_0 . Assume that T_0 is fixed with

$$T_0 \geq 2\tau_c.$$

Consider any initial point $z_0 \in \mathcal{M}_r^-$. Since $\mathcal{M}_r^- \subset L^+ \cup \{(\pi, 0)\}$, which is an invariant set, we have that, for every $t \in [0, T_0]$, the solution of (4.8) with $(x(0), y(0)) = z_0$ belongs to \mathcal{M}_r^- . Hence

$$\vartheta(T_0, z_0) \geq 0, \quad \forall z_0 \in \mathcal{M}_r^-.$$

On the other hand, if $z_0 \in \mathcal{M}_l^-$, then

$$\vartheta(T_0, z_0) \leq \vartheta(0, z_0) - 4\pi < \frac{\pi}{2} - 4\pi = -3\pi - \frac{\pi}{2}.$$

Now we define the compact sets

$$H_0 = \{z \in \mathcal{M} : \Phi_0(z) \in \mathcal{N} \text{ and } \vartheta(T_1, z) \in [-3\pi/2, -\pi]\}$$

and

$$H_1 = \{z \in \mathcal{M} : \Phi_0(z) \in \mathcal{N} \text{ and } \vartheta(T_1, z) \in [-7\pi/2, -3\pi]\}.$$

Using the fact that the angular coordinates of the points of \mathcal{N} belong to the intervals $]\pi/2 + 2k\pi, \pi + 2k\pi[$ (for $k \in \mathbb{Z}$) and using the above angular estimates, we conclude that H_0 and H_1 are both nonempty and, moreover, $H_0 \cap H_1 = \emptyset$.

Note that a point $z \in \mathcal{M}$ belongs to H_0 ($j = 0, 1$) if and only if the solution $(x(t), y(t))$ of (4.8) with $(x(0), y(0)) = z$ is such that $(x(T_0), y(T_0)) \in \mathcal{N}$ and, moreover, $x(0) > x(T_0)$ with $x(t)$ having exactly $2j + 1$ simple zeros in $]0, T_0[$ where it changes its sign with $x' \neq 0$.

Let $\gamma : [0, 1] \ni s \mapsto \gamma(s) \in \mathcal{M}$ be a continuous curve with $\gamma(0) \in \mathcal{M}_l^-$ and $\gamma(1) \in \mathcal{M}_r^-$. By the previous estimates, we know that

$$\vartheta(T_0, \gamma(0)) < -3\pi - \frac{\pi}{2} \quad \text{and} \quad \vartheta(T_0, \gamma(1)) \geq 0.$$

By a continuity argument, we can find two subintervals $[s'_0, s''_0]$ and $[s'_1, s''_1]$ of $[0, 1]$, with

$$0 < s'_1 < s''_1 < s'_0 < s''_0 < 1,$$

such that

$$\begin{aligned} \gamma(s) &\in H_j, \quad \forall s \in [s'_j, s''_j], \quad j = 0, 1, \\ \Phi_0(\gamma(s)) &\in \mathcal{N}, \quad \forall s \in [s'_1, s''_1] \cup [s'_0, s''_0], \\ \Phi_0(\gamma(s'_j)) &\in \mathcal{N}_r^-, \quad \Phi_0(\gamma(s''_j)) \in \mathcal{N}_l^-, \quad \text{for } j = 0, 1. \end{aligned}$$

We have thus proved that condition (i) of Corollary 4.7 holds for $\varphi = \Phi_0$. \square

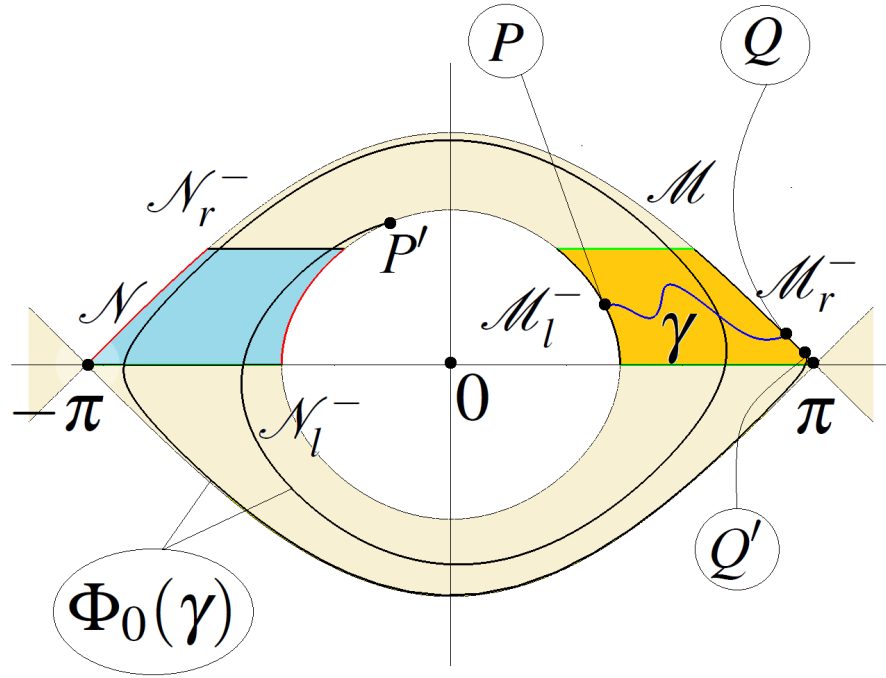


Figure 18: A graphical illustration of the property $\Phi_0 : \tilde{\mathcal{M}} \rightleftharpoons \tilde{\mathcal{N}}$ with crossing number larger than or equal to two. The path γ in \mathcal{M} joining a point $P \in \mathcal{M}_l^-$ to a point $Q \in \mathcal{M}_r^-$ is transformed by Φ_0 into a path $\Phi_0(\gamma)$ joining P' to Q' . If the time T_0 is sufficiently large, the path $\Phi_0(\gamma)$ will make a certain number of windings around the origin and will cross the set \mathcal{N} at least twice.

Step 2. Stretching the paths from \mathcal{N} to \mathcal{M} by Φ_1 . Assume that

$$T_1 \geq \frac{2\pi}{b}.$$

By the simple form of Φ_1 we immediately see that

$$\Phi_1(z) = z \quad \forall z \in \mathcal{N}_l^-$$

and

$$x_1 > \pi, \text{ for } (x_1, b) = \Phi_1(z) \text{ with } z = (x, b) \in \mathcal{N}_r^-.$$

Let $\gamma : [0, 1] \ni s \mapsto \gamma(s) \in \mathcal{N}$ be a continuous curve with $\gamma(0) \in \mathcal{N}_l^-$ and $\gamma(1) \in \mathcal{N}_r^-$. Hence, for $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, and $\Phi_1(\gamma(s)) = (\sigma_1(s), \sigma_2(s))$, we have that

$$\sigma_1(0) = \gamma_1(0) < 0, \quad \sigma_1(1) > \pi/2$$

and

$$\sigma_2(s) = \gamma_2(s) \in [0, b], \forall s \in [0, 1].$$

By a continuity argument, we can find a subinterval $[s', s'']$ of $[0, 1]$ such that $\Phi_1(\gamma(s)) \in \mathcal{M}$ for all $s \in [s', s'']$ and

$$\Phi_1(\gamma(s')) \in \mathcal{M}_l^-, \quad \Phi_1(\gamma(s'')) \in \mathcal{M}_r^-.$$

We have thus proved the second condition of corollary 4.7 for $\psi = \Phi_1$. \square

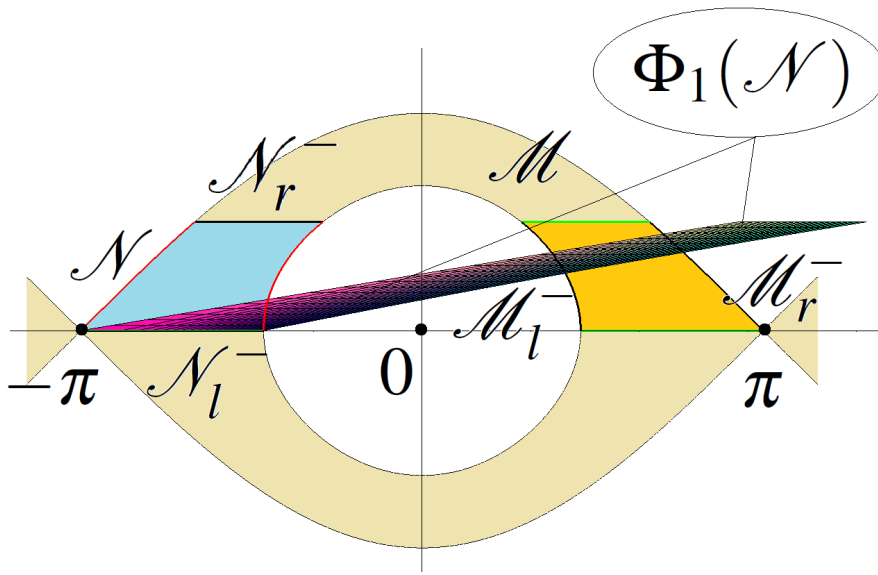


Figure 19: A graphical illustration of the property $\Phi_1 : \tilde{\mathcal{N}} \rightleftharpoons \tilde{\mathcal{M}}$. In this case, the SAP property is evident from the way in which $\Phi_1(\mathcal{N})$ goes across \mathcal{M} .

In conclusion, using corollary 4.7 we have proved the following theorem.

Theorem 4.8 *Let $w(t)$ be a T -periodic stepwise function defined as in (4.7). Fix two constants b, c with*

$$0 < b < c < d = 2\sqrt{K}$$

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and let τ_c be the fundamental period of the periodic orbit of the pendulum equation $x'' + K \sin x = 0$ with $x(0) = 0$ and $x'(0) = c$. Then, for

$$T_0 \geq 2\tau_c \quad \text{and} \quad T_1 \geq \frac{2\pi}{b}, \quad T = T_0 + T_1,$$

equation (4.4) exhibits chaotic dynamics on two symbols. The precise behaviour of the chaotic-like solutions can be described as follows.

There exists a (nonempty) compact set Λ which is contained in the first quadrant of the phase plane which is invariant for the Poincaré map Φ associated to (4.5) and such that $\Phi|_{\Lambda}$ is semiconjugate (via a continuous map g) to the two-sided Bernoulli shift on two symbols. In particular, for any sequence $\xi = (s_i)_i \in \{0, 1\}^{\mathbb{Z}}$ there exists a point $z \in g^{-1}(\xi) \in \Lambda$ such that the solution $x(t)$ of (4.4) with $(x(0), x'(0)) = z$ has precisely $2s_i + 1$ simple zeros in $]iT, T_0 + iT[$ and exactly one zero in $]T_0 + iT, (i + 1)T[$. Moreover, if the sequence of symbols $\xi = (s_i)_i$ is k -periodic, then there exists a $z \in g^{-1}(\xi)$ which is a k -periodic point for Φ in Λ and, consequently, the solution $x(t)$ is kT -periodic.

By the oddness of the sin function, it is clear that there is another family of chaotic solutions with initial points belonging to an invariant set (for the Poincaré map) contained in the third quadrant of the phase plane.

A careful checking of the proof will convince the reader that the argument is stable with respect to small perturbations on all the coefficients of the equation. The same observation was already employed in the previous papers [15, 16] where Corollary 4.7 was applied to the pendulum equation (under different conditions on the weight coefficient). The mathematical details which justify this assertion about the robustness of our result are fully developed in [86] and we refer to [86, pp. 900-902] for a description how to slightly modify the sets \mathcal{M} and \mathcal{N} in order to make the proof valid also in presence of small perturbations. Thus the following result holds true as well.

Theorem 4.9 *Let $w(t)$ be a T -periodic stepwise function defined as in (4.7). Fix two constants b, c with $0 < b < c < d = 2\sqrt{K}$ and let τ_c be the fundamental period of the periodic orbit of the pendulum equation $x'' + K \sin x = 0$ with $x(0) = 0$ and $x'(0) = c$. Then, for T_0 and T_1 fixed and satisfying*

$$T_0 > 2\tau_c \quad \text{and} \quad T_1 > \frac{2\pi}{b}, \quad T = T_0 + T_1,$$

there exists $\varepsilon > 0$ such that for every T -periodic L^1_{loc} functions $q(t)$ and $p(t)$ satisfying

$$\int_0^T |q(t) - w(t)| dt < \varepsilon, \quad \int_0^T |p(t)| dt < \varepsilon$$

and for every κ with $|\kappa| < \varepsilon$, the pendulum equation

$$u'' + \kappa u' + q(t) \sin u = p(t)$$

exhibits chaotic dynamics on two symbols.

Remark 4.1 Here is a list of possible directions toward which our results (theorem 4.8 and theorem 4.9) can be easily extended.

- If we work in the cylindrical phase plane and take T_1 sufficiently large we can easily find conditions in order to have that $\Phi_1(\mathcal{N})$ crosses multiple times the set $\mathcal{M} \pmod{2\pi}$. In this manner, and with a minimal expense in the computations needed for the proof, we can prove the presence of even more complicated dynamics (namely on a larger set of symbols) in which the classical oscillations of the pendulum around the equilibrium position $u(t) = \vartheta(t) = 0$ alternate with a certain number of full revolutions.
- In theorem 4.9, due to the particular form of $w(t)$ in (4.7), we assume that $\int_0^{T_1} |q(t)| dt < \varepsilon$. Actually, it is not difficult to get a more precise and better upper bound in terms of the L^1 -norm of $q(\cdot)$ in $[0, T_1]$ so that our perturbative argument works.
- We have confined ourselves to the study of the nonlinear equation

$$u'' + w(t)g(u) = 0,$$

for $g(x) = \sin x$, motivated by the study of a pendulum type equation with moving support. It is possible to adapt our argument to some more general functions $g(x)$ (see [15, 16] for a similar treatment in the cases when $w(t)$ is of constant sign or $w(t)$ is a sign-changing weight).

The geometry for the application of Corollary 4.7 is that of the composition of a twist map acting on a topological annulus (the set \mathcal{W} in our proof) with a squeezing and stretching map on a strip (the set \mathcal{S} in our proof). Such kind of geometry, unlike the case of the *linked twist maps* [25, 107, 104], requires only one of the two mappings twisting the boundaries of an annulus. The same kind of geometrical configuration as the one considered in our example was proposed in an abstract setting in [84]. Concrete examples of ODEs presenting such kind of geometry have been obtained by Ruiz-Herrera in [98] dealing with population dynamics models. Some geometric configurations which are topologically equivalent (in the sense that an annulus is crossed by a topological strip) have been considered in [79, 85, 109, 110] and in 1997 by Kennedy and Yorke [55] dealing with a problem of turbulent fluid dynamics.

5

BEND-TWIST MAPS

Our study for this section of the thesis is motivated by a recent approach considered by T. Ding in [26, Chapter 7] for the proof of the Poincaré-Birkhoff theorem for analytic functions. In the same chapter, a concept of *bend-twist map* is introduced. Roughly speaking, analytic bend-twist maps are those analytic twist maps in which the radial displacement $\|\varphi(z)\| - \|z\|$ changes its sign on a Jordan closed curve which is non-contractible in the annulus and where the angular displacement vanishes. Our goal is to extend Ding's definition to a pure topological setting and obtain some fixed point theorems for *continuous* bend-twist maps. The results do not require any regularity on the maps involved. Moreover, we do not assume hypotheses like homeomorphism, area-preserving or invariance of the boundaries and, as an additional feature, some of our results are stable under small perturbations. These facts, in principle, suggest the possibility to produce some new applications to planar differential systems which are not conservative. Our main existence theorem (see theorem 5.6) follows from the Borsuk separation theorem and Alexander's lemma which we have extensively applied in a recent paper [89]. The result partially extends Ding's theorem to the non-analytic setting. The main difference between theorem 5.6 and the corresponding theorem in [26] lies on the fact that we obtain at least one fixed point, whence two fixed points are given in [26]. On the other hand, we show, by a simple example, that only one fixed point may occur in some situations. Both in our case and in Ding's, the main hypothesis for the bend-twist theorem is a rather abstract one. Hence some more applicable corollaries, in the line of [26], are provided (see theorem 5.10 and corollary 5.11). In a final section we outline an application of our results to the periodic problem for some nonlinear ordinary differential equations.

In this chapter we are going to reconsider, in a purely topological framework, the concept of *bend-twist map* introduced by Tongren Ding in [26], using the crossing lemmas developed in chapter 3 for annular regions and managing to obtain some interesting fixed-points results. The basic setting in [26] consists in a pair of starlike planar annuli A and A^* with $A \subset A^*$ and a continuous map $f : A \rightarrow A^*$. Without loss of generality (via a translation of the origin), one can always assume that $0 = (0, 0)$ belongs to the open set $D(A^*)$. Accordingly, our basic setting can be described as follows.

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Let $A \subset \mathbb{R}^2$ be a topological annulus (embedded in the plane) with $0 \in D(A)$. Passing to the covering space $\mathbb{R} \times \mathbb{R}_0^+$ we have that the inner and outer boundaries of A are lifted to the lines $J_i = \pi^{-1}(A_i)$ and $J_o = \pi^{-1}(A_o)$ which are periodic in the sense that $(\vartheta, r) \in J$ if and only if $(\vartheta + 2\pi, r) \in J$, for $J = J_i, J_o$. In [26] the boundaries are assumed to be starlike, that is both J_i and J_o are graphs of 2π -periodic functions $\lambda_i, \lambda_o : \mathbb{R} \rightarrow \mathbb{R}_0^+$, $\vartheta \mapsto r = \lambda(\vartheta)$ (for $\lambda = \lambda_i, \lambda_o$) with $\lambda_i(\vartheta) < \lambda_o(\vartheta)$, for all $\vartheta \in \mathbb{R}$. The condition about the strictly star-shapeness of the boundaries of A is crucial for entering in the setting of the Poincaré-Birkhoff theorem (see [26, 62, 66, 96]). However, it is not assumed in this chapter unless when explicitly required.

Let $\varphi = (\varphi_1, \varphi_2) : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a continuous map and consider its lifting $\tilde{\varphi}$ defined on $\tilde{A} = \pi^{-1}(A)$ to $\mathbb{R} \times \mathbb{R}_0^+$ such that

$$\varphi \circ \pi = \pi \circ \tilde{\varphi}.$$

By definition, given a lifting $\tilde{\varphi}$ of φ , all the other liftings of φ are of the form

$$(\vartheta, r) \mapsto \tilde{\varphi}(\vartheta, r) + (2k\pi, 0),$$

for some $k \in \mathbb{Z}$. We assume that $\tilde{\varphi}$ can be expressed as

$$\tilde{\varphi} : (\vartheta, r) \mapsto (\vartheta + \Theta(\vartheta, r), R(\vartheta, r)), \quad (5.1)$$

where Θ, R are continuous real-valued functions defined on $\pi^{-1}(A)$ and 2π -periodic in the ϑ -variable. We also introduce an auxiliary function Υ giving the radial displacement

$$\Upsilon(\vartheta, r) = R(\vartheta, r) - r \quad \text{for } (\vartheta, r) \in \tilde{A}$$

which will play a key role in the definition of bend-twist maps. Observe that, instead of using the polar coordinates, we can equivalently express Υ on the points of A as

$$\Upsilon(z) = \|\varphi(z)\| - \|z\| \quad \text{with } z \in A.$$

In the same way, also the angular displacement Θ can be referred directly to the points of the annulus A since $\Theta(\vartheta, r)$ has the same value at every point $(\vartheta, r) \in \pi^{-1}(z)$. This allows to define

$$\Theta(z) = \Theta(\vartheta, r) \quad \text{for } z = \pi(\vartheta, r).$$

In some applications (for instance to some planar maps associated to ordinary differential equations), the number $\Theta(\vartheta, r)$ has the meaning of a *rotation number* associated to a given trajectory departing from the point $\pi(\vartheta, r)$. In particular, observe that any solution $(\tilde{\vartheta}, \tilde{r}) \in \tilde{A}$ of the system

$$\begin{cases} \Theta(\vartheta, r) = 2\ell\pi \\ \Upsilon(\vartheta, r) = 0 \end{cases} \quad (5.2)$$

determines a fixed point $\bar{z} = (\bar{x}, \bar{y}) = \pi(\bar{\vartheta}, \bar{r}) \in A$ of the map φ . Every fixed point is “tagged” with the integer ℓ computing the rotations performed by the fixed point around the origin. This is an important information associated to \bar{z} in the sense that, once we have fixed Θ in order to express $\tilde{\varphi}$ as in (5.1), then solutions of (5.2) for different values of $\ell \in \mathbb{Z}$ determine different fixed points of φ . In other words, if $(\bar{\vartheta}_1, \bar{r}_1)$ and $(\bar{\vartheta}_2, \bar{r}_2)$ are solutions of (5.2) for $\ell = \ell_1$ and $\ell = \ell_2$ respectively, with

$$\ell_1 \neq \ell_2,$$

then their projection on A are distinct points, that is

$$\bar{z}_1 = \pi(\bar{\vartheta}_1, \bar{r}_1) \neq \bar{z}_2 = \pi(\bar{\vartheta}_2, \bar{r}_2).$$

Indeed, if, by contradiction, $\bar{z}_1 = \bar{z}_2$, then $\bar{r}_1 = \bar{r}_2$ and $\bar{\vartheta}_2 = \bar{\vartheta}_1 + 2m\pi$ for some $m \in \mathbb{Z}$. Hence, by the 2π -periodicity of $\Theta(\cdot, r)$, we have

$$2\pi\ell_2 = \Theta(\bar{\vartheta}_2, \bar{r}_2) = \Theta(\bar{\vartheta}_1 + 2m\pi, \bar{r}_1) = \Theta(\bar{\vartheta}_1, \bar{r}_1) = 2\pi\ell_1,$$

a contradiction.

Conversely, one can easily check that any fixed point $\bar{z} \in A$ of the map φ lifts to a discrete periodic set

$$\pi^{-1}(\bar{z}) = \{(\bar{\vartheta} + 2\ell\pi, \bar{r}) : \ell \in \mathbb{Z}\},$$

for which there exists an integer $\ell = \ell_{\bar{z}}$ such that each point $(\bar{\vartheta}, \bar{r}) \in \pi^{-1}(\bar{z})$ is a solution of (5.2) with the same value of $\ell = \ell_{\bar{z}}$.

Looking for a solution of system (5.2), an usual assumption on the map $\tilde{\varphi}$ is the so-called *twist condition at the boundaries*, which is one of the main hypotheses of the Poincaré-Birkhoff fixed point theorem, as widely exposed in chapter 2. In our setting, the twist condition is expressed as follows.

Definition 5.1 We say that $\tilde{\varphi}$ satisfies the *twist condition* if

$$\begin{cases} \Theta(\vartheta, r) < 2j\pi, & \text{for } (\vartheta, r) \in J_i \\ \Theta(\vartheta, r) > 2j\pi, & \text{for } (\vartheta, r) \in J_o \end{cases} \quad (5.3)$$

(or viceversa), for some $j \in \mathbb{Z}$.

If we prefer to express the twist condition directly on φ , we will write

$$\Theta < 2j\pi \text{ on } A_i \quad \text{and} \quad \Theta > 2j\pi \text{ on } A_o \quad (5.4)$$

(or viceversa).

In order to introduce the concept of bend-twist maps we recall a (wrong) attempt of proving the Poincaré-Birkhoff theorem (see chapter 2, theorem 2.2), as described by M. Wilson in a letter to Birkhoff [9]:¹

¹ We put in *Italic* the original words by Wilson. The notation is the original one and to make it compatible with that of the present paper we have to notice that the lifting of

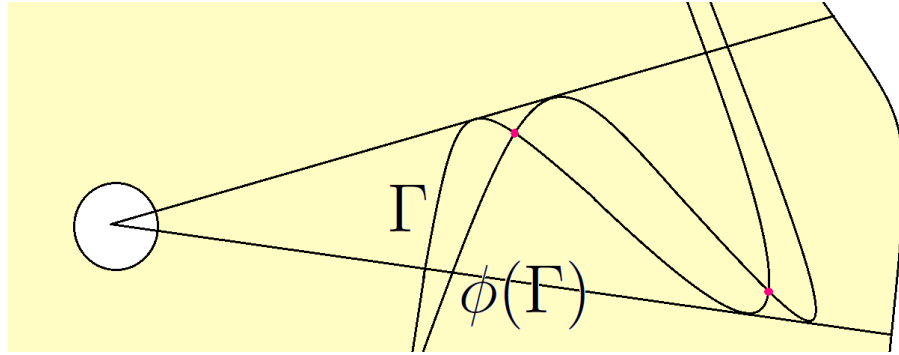


Figure 20: A sketch of the problem in Wilson's argument. We depict a sector of an annular domain in which there is a portion of a non star-shaped curve Γ where $\Theta = 0$. The points of Γ are moved radially to the points of $\phi(\Gamma)$ with preservation of the area. The points in $\Gamma \cap \phi(\Gamma)$ are not fixed points for ϕ . A similar situation is described by Martins and Ureña in [66, Figure 1-2].

"Won't you bother with finding out what ridiculous error there is in this simple thing that occurred to me yesterday ? [...] The set of the points of the annulus with $\varphi' - \varphi = 0$ may be of great complexity containing ovals or ovals within ovals in the ring. But, as this set is closed and cannot be traversed by any continuous curve from the inner to the outer circles without being cut in at least one point, such set must include at least one continuous curve circling around the ring. [...] Now, upon this curve, the shift $r' - r$ is continuous and could not be always positive or always negative without shrinking said curve or expanding it, contrary to the supposed invariance of areas or integrals. Hence, there must be at least two points for which $r' = r$ as well as $\varphi' = \varphi$."

The gap in this argument is not only in the fact that the set of the points of the annulus where $\Theta = 0$ may not contain a "curve" (this perhaps is not the serious mistake), but even in the case in which there is actually a simple closed curve $\Gamma \subset A[a, b]$ included in the set where $\Theta = 0$, with Γ encircling A_i , the points of $\phi(\Gamma) \cap \Gamma$ (which are supposed to exist by the area-preserving assumption) are not necessarily fixed points for ϕ . Indeed, if Γ is not star-shaped, one could have that $\Upsilon > 0$ (or $\Upsilon < 0$) along Γ and, at the same time, $\phi(\Gamma) \cap \Gamma \neq \emptyset$.

Of course, if we were able to prove that the radial displacement function Υ vanishes at some points of the locus $\Theta = 0$, then we would find fixed points for ϕ (making the above wrong argument meaningful). From this point of view, the study of the structure of the sets of points where $\Theta = 0$, may give useful information for the search of fixed points of ϕ . Such approach was considered, for instance, by G.R.

ϕ considered in [9] is expressed as a map $(r, \varphi) \mapsto (r', \varphi')$. Thus our condition $\Theta = 0$ corresponds to $\varphi' - \varphi = 0$ in [9]. We also remark that the twist condition is assumed in [9] with $j = 0$ (like in the original version of Poincaré-Birkhoff theorem).

Morris in [72] who proved the existence of infinitely many periodic solutions of minimal period $2m\pi$ (for each positive integer m), for the forced superlinear equation

$$\ddot{x} + 2x^3 = p(t),$$

where $p(t)$ is a smooth function with least period 2π and mean value zero. For his proof, Morris considered the problem of the existence of fixed points for the area-preserving homeomorphism of the plane

$$T^m: (a, b) \rightarrow (a', b') = (x(2m\pi; a, b), \dot{x}(2m\pi; a, b)),$$

where $x(t; a, b)$ is the solution of the differential equation such that

$$x(0; a, b) = a, \quad \dot{x}(0; a, b) = b.$$

In [72], starlike Jordan curves around the origin C were constructed such that each point $P \in C$ is mapped to $T^m P$ on the same ray OP (see also [21] for a description of Morris result in comparison to other different approaches).

In [26] Tongren Ding considers the case of a topological annulus \mathcal{A} embedded in the plane having as its boundaries two simple closed curves which are starlike with respect to the origin. It is assumed that there exists an *analytic* function $f: \mathcal{A} \rightarrow \mathcal{A}^*$, with \mathcal{A}^* another starlike annulus with $\mathcal{A} \subset \mathcal{A}^*$ and, moreover, that f satisfies the twist condition (5.3). It is also observed that the set Ω_f of the points in \mathcal{A} where $\Theta = 2j\pi$ contains at least a Jordan curve Γ which is not contractible in \mathcal{A} . The function f is called a *bend-twist map* if there exists a Jordan curve $\Gamma \subset \Omega_f$, with Γ non contractible in \mathcal{A} , such that Υ changes its sign on Γ . Then, Ding's theorem is stated in [26, Theorem 7.2, p.188] as follows.

Theorem 5.2 *Let $f: \mathcal{A} \rightarrow \mathcal{A}^*$ be an analytic bend-twist map. Then it has at least two distinct fixed points in \mathcal{A} .*

Note that in Ding's theorem, the assumptions that f is area-preserving and leaves the annulus invariant are not needed. This represents a strong improvement of the hypotheses required for the Poincaré-Birkhoff twist theorem. On the other hand, the assumption that a given function is a bend-twist map does not seem easy to be checked in the applications. The following corollary (see [26, Corollary 7.3, p.188]) comes in our help providing more explicit conditions for the applicability of the abstract result.

Corollary 5.3 *Let $f: \mathcal{A} \rightarrow \mathcal{A}^*$ be an analytic twist map. If there are two disjoint continuous curves Γ_1 and Γ_2 in \mathcal{A} , connecting respectively the inner and the outer boundaries of \mathcal{A} and such that $\Upsilon < 0$ on Γ_1 and $\Upsilon > 0$ on Γ_2 , then f is a bend-twist map on \mathcal{A} and therefore it has at least two distinct fixed points.*

5.1 MAIN RESULTS

Our aim now is to reformulate the above results in a general topological setting in order to obtain a version of Theorem 5.2 and Corollary 5.3 for general, and not necessarily analytic, maps.

Let $A \subset \mathbb{R}^2$ be a topological annulus (embedded in the plane) with $0 \in D(A)$ and let $\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a continuous map admitting a lifting of the form (5.1). For every $j \in \mathbb{Z}$, let us introduce the set of the points of A which are rotated by φ of an angle of $2j\pi$, denoted by

$$\Omega_\varphi^j = \{(r \cos \vartheta, r \sin \vartheta) : \Theta(\vartheta, r) = 2j\pi\}.$$

Lemma 5.4 *Let φ satisfy the twist condition (5.3) for some $j \in \mathbb{Z}$. Then the set Ω_φ^j contains a compact connected set C^j which is essentially embedded in A and $C^j : A_i \uparrow A_o$.*

Proof. Our claim is an immediate consequence of Lemma 3.18 once that we have checked that $\Omega_\varphi^j : A_i \uparrow A_o$. This latter property follows from the continuity of Θ and the twist condition. Indeed, if $\gamma : [0, 1] \rightarrow A$ is a path with $\gamma(0) \in A_i$ and $\gamma(1) \in A_o$, then $\Theta(\gamma(t)) - 2j\pi$ must vanish somewhere. \square

This result corresponds to [26, Lemma 7.2, p.185] for a general φ . The Jordan curve $\Gamma \subset \Omega_f$ considered in [26] in the analytic case is now replaced by the essentially embedded continuum $C^j \subset \Omega_\varphi^j$. Following [26] we can now give the next definition.

Definition 5.5 Let $\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a continuous map (admitting a lifting of the form (5.1)) which satisfies the twist condition (5.3), for some $j \in \mathbb{Z}$. We say that φ is a *bend-twist map in A* if there exists a compact connected set $C^j \subset \Omega_\varphi^j$ with C^j essentially embedded in A and such that Υ changes its sign on C^j .

As a consequence of this definition, the following theorem, a version of theorem 5.2 for mappings which are not necessarily analytic, holds.

Theorem 5.6 *Let $\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a bend-twist map. Then there exists a fixed point $z \in \text{int } A$ with $\Theta(z) = 2j\pi$.*

The proof is an obvious consequence of the connectedness of C^j . Observe that, if we were able to prove that C^j is a Jordan curve, then, as in [26], the existence of at least two fixed points could be ensured.

In general, and in contrast with theorem 5.2, we cannot hope to have more than one fixed point as shown by the following example which refers to a standard planar annulus $A = A[a, b]$.

Example 5.7 Let $c = (a + b)/2$ and consider the set

$$C = \{(\vartheta, r) : r = c + \varepsilon_1 \sin^2(\vartheta/2)\} \cup \{(2k\pi, r) : r \in [c - \delta, c + \delta], k \in \mathbb{Z}\},$$

with $0 < \varepsilon_1, \delta < (b - a)/4$. The angular map Θ in $\pi^{-1}(A)$ is defined as

$$\Theta(\vartheta, r) = \begin{cases} -\frac{\text{dist}(z, C)}{\text{dist}(z, C) + \text{dist}(z, J_i)} & \text{for } z = (\vartheta, r), \text{ with } r < c + \varepsilon_1 \sin^2(\vartheta/2) \\ \frac{\text{dist}(z, C)}{\text{dist}(z, C) + \text{dist}(z, J_o)} & \text{for } z = (\vartheta, r), \text{ with } r \geq c + \varepsilon_1 \sin^2(\vartheta/2) \end{cases}$$

while, for the radial map R , we set

$$R(\vartheta, r) = r + \varepsilon_2(r - a)(r - c)(r - b),$$

with $\varepsilon_2 > 0$ and sufficiently small in order to have $a \leq R(\vartheta, r) \leq b$, for all (ϑ, r) . The functions Θ and R define by (5.1) a continuous map $\tilde{\varphi} : \mathbb{R} \times [a, b] \rightarrow \mathbb{R} \times [a, b]$ and, projecting by π , a map $\varphi : A[a, b] \rightarrow A[a, b]$. It is easy to check that φ leaves the boundaries of the annulus invariant and satisfies the twist condition (5.3) with $j = 0$. The set Ω_φ^0 is the image of C through π . In accordance with Lemma 5.4 we can take $C^0 = \Omega_\varphi^0$. The function Υ vanishes on the circumferences $r = a$, $r = b$ and $r = c$ and, moreover, it is negative on the open annulus $A(a, c)$ and positive on $A(c, b)$. Hence it changes its sign on C^0 . However, φ has a unique fixed point in $A[a, b]$ which is $F = (c, 0)$ (see figure 21). \triangleleft

Perhaps the set C^0 in example 5.7 is not completely satisfactory. Indeed, although it represents a compact connected set which cuts all the paths between A_i and A_o , it is not minimal with respect to this property. One could suppose that if we modify Definition 5.5 by considering only minimal compact subsets of Ω_φ^j which are essentially embedded in A , then we could provide the existence of at least two fixed points for φ , as in Ding's theorem. Though, we have preferred to give a definition avoiding the concept of minimality because the existence of minimal sets will be only guaranteed by Zorn's lemma and, moreover, such sets could be quite pathological and thus intractable from the point of view of the applications. Nevertheless, the following improvement of the result can be proved.

Theorem 5.8 Let $C \subset A = A[a, b]$ be a compact connected set which is minimal with respect to the property of cutting all the paths in A from A_i to A_o . Let $f : C \rightarrow \mathbb{R}$ be a continuous function such that f changes sign on C . Then there exist at least two points $z_1, z_2 \in C$ with $z_1 \neq z_2$ and such that $f(z_1) = f(z_2) = 0$.

Proof. The existence of at least a zero for $f|_C$ follows from Bolzano's theorem. Suppose, by contradiction, that there exists only one point

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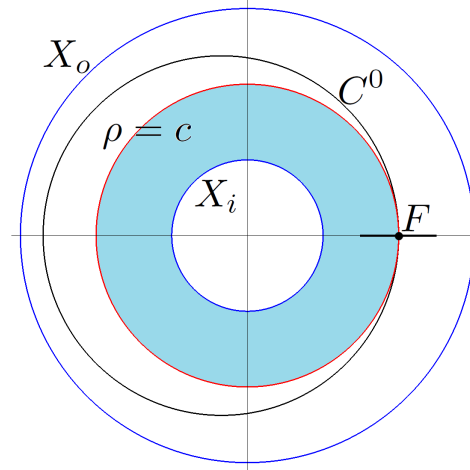


Figure 21: A description of the geometry in Example 5.7. The set C^0 made of the points of the annulus $A[a, b]$ where $\Theta = 0$, is the union of a closed curve (contained in the part of the annulus between $r = c$ and A_o) and a small segment $[c - \delta, c + \delta] \times \{0\}$. The function Υ vanishes at $r = a, c, b$, it is negative for $a < r < c$ and positive for $c < r < b$ (we have painted with a darker color the part of the annulus where $\Upsilon < 0$). The point F is the unique fixed point of φ since $\{F\} = C^0 \cap \{\Upsilon = 0\}$.

$z \in C$ such that $f(z) = 0$ and consider the two nonempty compact sets $K_1 = \{x \in C : f(x) \leq 0\}$ and $K_2 = \{x \in C : f(x) \geq 0\}$. By the assumption, we have that $\{z\} = K_1 \cap K_2$ and $K_1 \neq C$ as well as $K_2 \neq C$. By the minimality of C , it follows that there exists a path γ_1 connecting A_i to A_o in A and avoiding K_1 and, similarly, there exists a path γ_2 connecting A_i to A_o in X and avoiding K_2 . From a version of Alexander's lemma in the plane (see Newman and Kallipoliti-Papasoglu) we know that there exists a path γ in A connecting A_i to A_o and avoiding $C = K_1 \cup K_2$. This contradicts the cutting property of C . \square

The proof is based on a version of Alexander's lemma which we rewrite below.

Lemma 5.9 (Alexander's lemma) *Let K_1, K_2 be closed sets on the plane such that $K_1 \cap K_2$ is a connected set (possibly empty) and at least one between K_1 and K_2 is bounded. Let $x, y \in \mathbb{R}^2 \setminus (K_1 \cup K_2)$. If there is a path joining x and y in $\mathbb{R}^2 \setminus K_1$ and a path joining x and y in $\mathbb{R}^2 \setminus K_2$, then there is a path joining x and y in $\mathbb{R}^2 \setminus (K_1 \cup K_2)$.*

The crucial problem in this approach consists in the fact that finding the minimal set C could not be possible in practical cases. Although, we are able to gain the existence of two fixed points also by the follow-

ing corollary, in which stronger, but easier to check, assumptions on Υ are required, as happens in corollary 5.3.

Theorem 5.10 *Let $\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a continuous map (admitting a lifting of the form (5.1)) which satisfies the twist condition (5.3), for some $j \in \mathbb{Z}$. If there are two disjoint arcs Γ_1 and Γ_2 in A , both connecting A_i with A_o in A and such that $\Upsilon < 0$ on Γ_1 and $\Upsilon > 0$ on Γ_2 , then φ has at least two distinct fixed points in $\text{int } A$ with $\Theta = 2j\pi$.*

Proof. Our argument is reminiscent of a similar one used in the proof of a bifurcation result in [49]. Without loss of generality (up to a homeomorphism), we can suppose that $A = A[a, b]$. We also suppose (passing possibly to a sub-arc) that each Γ_n intersects A_i and A_o exactly in one point, respectively. Let also P_a^n and P_b^n be the intersection points of Γ_n with the circumferences $r = a$ and $r = b$, respectively (for $n = 1, 2$). Let C'_a be the arc of A_i from P_a^1 to P_a^2 and let C''_a be the arc of A_i from P_a^2 to P_a^1 (in the counterclockwise sense). Similarly (again in the counterclockwise sense), we determine two arcs C'_b and C''_b on A_o . The Jordan curves obtained by joining $C'_a, \Gamma_2, C'_b, \Gamma_1$ and $C''_a, \Gamma_1, C''_b, \Gamma_2$ bound two generalized rectangles R_1 and R_2 . We claim that in the interior of R_n ($n = 1, 2$) there exists at least one fixed point for φ having j as associated rotation number. We prove the claim for R_1 , since the proof for R_2 is exactly the same.

First of all, by the covering projection π , we lift the set R_1 to the strip

$$\pi^{-1}(A) = \mathbb{R} \times [a, b]$$

and observe that $\pi^{-1}(R_1)$ can be written as

$$\pi^{-1}(R_1) = \mathcal{R} + (2m\pi, 0),$$

with \mathcal{R} a generalized rectangle contained in the strip and such that its boundary projects homeomorphically onto ∂R_1 by π . As observed above, R_1 is the compact region of the plane bounded by the Jordan curve $C'_a, \Gamma_2, C'_b, \Gamma_1$. By the Schoenflies theorem [71] we can choose a homeomorphism $\eta : [0, 1]^2 \rightarrow \mathcal{R}$ in such a way that

$$\begin{aligned} (\pi \circ \eta)([0, 1] \times \{0\}) &= C'_a, & (\pi \circ \eta)([0, 1] \times \{1\}) &= C'_b, \\ (\pi \circ \eta)(\{0\} \times [0, 1]) &= \Gamma_2, & (\pi \circ \eta)(\{1\} \times [0, 1]) &= \Gamma_1. \end{aligned}$$

The vector field

$$f = (f_1, f_2) : [0, 1]^2 \rightarrow \mathbb{R}^2,$$

defined by

$$f(x, y) = (\Upsilon(\eta(x, y)), \Theta(\eta(x, y)) - 2j\pi)$$

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is such that

$$\begin{aligned} f_1(0, y) &= \Upsilon(\vartheta, r) && \text{with } \pi(\vartheta, r) \in \Gamma_2, \quad \forall y \in [0, 1], \\ f_1(1, y) &= \Upsilon(\vartheta, r) && \text{with } \pi(\vartheta, r) \in \Gamma_1, \quad \forall y \in [0, 1], \\ f_2(x, 0) &= \Theta(\vartheta, r) - 2j\pi && \text{with } \pi(\vartheta, r) \in C'_a, \quad \forall x \in [0, 1], \\ f_2(x, 1) &= \Theta(\vartheta, r) - 2j\pi && \text{with } \pi(\vartheta, r) \in C'_b, \quad \forall x \in [0, 1]. \end{aligned}$$

Thus, by the assumptions on Θ and Υ , we find that

$$\begin{aligned} f_1(0, y) &> 0 > f_1(1, y), \quad \forall y \in [0, 1] \\ f_2(x, 0) &< 0 < f_2(x, 1), \quad \forall x \in [0, 1]. \end{aligned} \tag{5.5}$$

The above (strict) inequalities imply that we are in the setting of a two-dimensional version of the Poincaré-Miranda theorem and that

$$\deg(f,]0, 1[^2, 0) = -1, \tag{5.6}$$

where “deg” denotes Brouwer’s degree. Therefore there exists at least one point (x^*, y^*) such that $f(x^*, y^*) = 0$. This, in turns, implies the existence of a fixed point $(\vartheta^*, r^*) = \eta(x^*, y^*) \in \text{int } \mathcal{R}$ such that $\pi(\vartheta^*, r^*)$ is a fixed point of φ in the interior of R_1 and such that $\Theta = 2j\pi$. \square

With the same argument of the proof of theorem 5.10, the next result can be obtained.

Corollary 5.11 *Let $\varphi : A \rightarrow \mathbb{R}^2 \setminus \{O\}$ be a continuous map (admitting a lifting of the form (5.1)) which satisfies the twist condition (5.3), for some $j \in \mathbb{Z}$. Assume that there exist $2k$ disjoint arcs ($k \geq 1$) connecting A_i with A_o in A . We label these arcs in a cyclic order $\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots, \Gamma_{2k}, \Gamma_{2k+1} = \Gamma_1$ and assume that*

$$\Upsilon < 0 \text{ on } \Gamma_n \text{ for } n \text{ odd, } \quad \Upsilon > 0 \text{ on } \Gamma_n \text{ for } n \text{ even}$$

(or viceversa). Then φ has at least $2k$ distinct fixed points in $\text{int } A$, all the fixed points with $\Theta = 2j\pi$.

Observe that theorem 5.10, as well as corollary 5.11, are stable with respect to small continuous perturbations of the map φ . This follows from the fact that equality (5.6) is true for any function f satisfying the strict inequalities (5.5). Thus, if we perturb the function φ with a new continuous map ψ with $\|\psi - \varphi\|_\infty \leq \varepsilon$ on A for $\varepsilon > 0$ sufficiently small, we have that the twist condition and the conditions on Υ on Γ_1 and Γ_2 are satisfied also for ψ , and hence we get fixed points for ψ as well.

On the other hand, both theorem 5.6 and theorem 5.2 are not stable even in case of arbitrarily small perturbations, which can make the fixed point disappear. In order to show this possibility, the following example can be considered.

Example 5.12 Let $A = A[a, b]$ be a planar annulus with $a = \frac{1}{2}$ and $b = 5$. We consider an angular map Θ in $\pi^{-1}(A)$ as $\Theta(\vartheta, r) = (r - 3)^2(r - 1)$, while, for the radial map, we take $R(\vartheta, r) = r + r^2(\cos^2 \vartheta + 4 \sin^2 \vartheta) - 16$. The functions Θ and R define by (5.1) a continuous map $\tilde{\varphi} : \tilde{A} \rightarrow \tilde{A}$ and, projecting by π , a map $\varphi : A[a, b] \rightarrow A[a, b]$. It is easy to check that φ satisfies the twist condition (5.3) with $j = 0$. The set Ω_φ^0 is the union of the circumferences S^1 and $3S^1$. The function Υ vanishes on the ellipse $x^2 + 4y^2 = 16$. According to definition 5.5, the map φ is a bend-twist map as Υ changes its sign on $3S^1$. Indeed φ has exactly four fixed points which are the intersections of the ellipse with the circumference $3S^1$. However, for any $\varepsilon > 0$ sufficiently small, the map $\varphi_\varepsilon = M_\varepsilon \circ \varphi$ (where $M_\varepsilon(z) = ze^{i\varepsilon}$) is a rotation of a small angle ε has no fixed points in A . The reason is that the set $3S^1$ disappears after an arbitrary small perturbation for $\varepsilon > 0$, while the set S^1 is stable (in the sense that it continues into a nearby closed Jordan curve) but it is not suitable for the bend-twist map theorem since Υ has constant sign on it. \triangleleft

Up to now we have presented all our results in terms of liftings of planar maps given by the standard covering projection π in polar coordinates. In this way we can make a simpler comparison with other results, like the Poincaré-Birkhoff fixed point theorem and the Ding's analytic bend-twist maps theorem, which are usually expressed in the same framework. It is clear, however, that our approach works exactly the same also if different covering projections are used. For instance, in the applications to planar systems which are a perturbation of the first order Hamiltonian system

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x, y) \\ \dot{y} = -\frac{\partial H}{\partial x}(x, y), \end{cases} \quad (5.7)$$

if we have an annulus filled by periodic orbits of (5.7), it could be convenient to choose as a radial coordinate the number \mathcal{E} expressing the level of the Hamiltonian and as an angular coordinate a normalized time of the corresponding orbit at level \mathcal{E} . We are going to use this remark for the application in the next section (see [49, 65] for some analogous cases).

5.2 AN APPLICATION

It appears that the presence of bend-twist maps associated to planar differential equations is ubiquitous. This does not mean that proving their existence in concrete equations is a simple task. It is a common belief that periodic solutions obtained for planar Hamiltonian systems

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via the Poincaré-Birkhoff fixed point theorem are not preserved by arbitrarily small perturbations which destroy the Hamiltonian structure of the equations. A typical example occurs when we add a small friction to a conservative system of the form

$$\ddot{x} + f(x) = 0, \quad (5.8)$$

passing to

$$\ddot{x} + \varepsilon \dot{x} + f(x) = 0. \quad (5.9)$$

In general, for any continuous f and each continuous function $\delta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\delta(s)s > 0$ for all $s \neq 0$, the only possible periodic solutions of

$$\ddot{x} + \delta(\dot{x}) + f(x) = 0$$

are the constant ones, corresponding to the zeros of f (if any).

For (5.8) one can easily find conditions on $f(x)$ guaranteeing the existence of an annulus in the phase-plane filled by periodic orbits of the equivalent first order Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x). \end{cases} \quad (5.10)$$

To present a specific example, let us assume that there exists an open interval $I =]a, b[$ with $-\infty \leq a < 0 < b \leq +\infty$ such that $f : I \rightarrow \mathbb{R}$ is locally Lipschitz continuous with $f(0) = 0$ and

$$f(s)s > 0, \quad \forall s \in I \setminus \{0\}. \quad (5.11)$$

The corresponding potential function

$$F(x) = \int_0^x f(s) ds,$$

is strictly decreasing on $]a, 0]$ and strictly increasing on $[0, b[$. Hence, for every constant c with

$$0 < c < C = \min\{F(a^+), F(b^-)\},$$

the energy level line E^c defined by

$$E(x, y) = c, \quad \text{for } E(x, y) = \frac{1}{2}y^2 + F(x), \quad x \in I,$$

is a closed periodic orbit surrounding the origin. We denote by τ_c the fundamental period of E^c . By the above assumptions it turns out that the map $c \mapsto \tau_c$ is continuous (see, for instance [49, (v) page 83] where such result is proved in a more general situation).

In this setting we propose an application of the Poincaré-Birkhoff twist theorem and the bend-twist maps theorem to equations which are small perturbations of (5.8).

To begin with, we suppose that there exist two levels c_1 and c_2 such that

$$\tau_1 < \tau_2,$$

for $\tau_i = \tau_{c_i}$. For convenience in the next exposition, we also suppose that

$$0 < c_1 < c_2 < C.$$

The case in which $c_2 < c_1$ can be treated analogously. The planar annulus

$$\mathcal{A} = \{(x, y) \in I \times \mathbb{R} : c_1 \leq E(x, y) \leq c_2\} \quad (5.12)$$

is filled by periodic trajectories whose period varies continuously with the parameter c . In particular the inner boundary \mathcal{A}_i and the outer boundary \mathcal{A}_o of the annulus are the energy level lines E^{c_1} and E^{c_2} , respectively.

Consider the level line E^c with $c_1 \leq c \leq c_2$. By (5.11) it follows that E^c is strictly star-shaped around the origin. Hence, for every angle ϑ , the line

$$L_\vartheta = \{(r \cos \vartheta, r \sin \vartheta) : r > 0\} \quad (5.13)$$

intersects the curve E^c exactly in one point. From this fact, we can immediately obtain another covering projection map onto the annulus which is equivalent to the projection in polar coordinates π . In this way, we can describe the points of \mathcal{A} by means of pairs (ϑ, \mathcal{E}) , where, for each point $z \in \mathcal{A}$, we have that ϑ is the usual angle in polar coordinates and $\mathcal{E} = E(z)$.

The continuity of the map $c \mapsto \tau_c$ implies that for every T with

$$\tau_1 < T < \tau_2, \quad (5.14)$$

equation (5.8) has at least one T -periodic solution $\hat{x}(\cdot)$, where for

$$\hat{c} := E(\hat{x}(0), \dot{\hat{x}}(0)),$$

we have that $\tau_{\hat{c}} = T$. Actually, due to the autonomous nature of the system, there is at least a continuum of periodic solutions given by the shifts in time of \hat{x} , that is the functions $\hat{x}_\vartheta(\cdot)$, with $\hat{x}_\vartheta(t) = \hat{x}(t + \vartheta)$. From the point of view of the Poincaré map, which is the map

$$\Phi : z \mapsto \varphi(T, z),$$

where $\varphi(\cdot, z)$ is the solution of (5.10) with $\varphi(0, z) = z$, we have that Φ has a continuum of fixed points which are all the points of the closed curve $E^{\hat{c}}$. The uniqueness of the periodic trajectory is not guaranteed in general (unless we assume some further conditions, like the strict monotonicity of the period with respect to c). In this autonomous case, as we have observed above, an arbitrarily small perturbation destroying the Hamiltonian structure of the equation may have the effect that the nontrivial T -periodic solutions disappear.

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As a next step, we consider a perturbation of equation (5.8) in the form of

$$\ddot{x} + (1 + w(t))f(x) = 0, \quad (5.15)$$

where $w : \mathbb{R} \rightarrow \mathbb{R}$ is a T -periodic function. For our purposes, only weak regularity assumptions on $w(\cdot)$ are needed. For instance, we can suppose that $w \in L^1(0, T)$ and consider the solutions of (5.15) in the generalized (Carathéodory) sense (see [42]). In this case, by the theorem of continuous dependence of the solutions in the Carathéodory setting, the Poincaré map associated to the planar system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(1 + w(t))f(x) \end{cases} \quad (5.16)$$

is well defined on \mathcal{A} if $w(t)$ is sufficiently small in the L^1 -norm on $[0, T]$. Then the following theorem holds.

Theorem 5.13 *Assume (5.14). Then there exists $\varepsilon > 0$ such that for each $w(\cdot)$ with $|w|_{L^1(0, T)} < \varepsilon$ equation (5.15) has at least two T -periodic solutions with initial value in \mathcal{A} , for \mathcal{A} defined in (5.12).*

Theorem 5.13 is substantially a variant of a result of Buttazzoni and Fonda [19]. The proof follows a version of the Poincaré-Birkhoff fixed point theorem due to W. Ding [30] which applies to an area-preserving twist homeomorphism of a planar annulus with star-shaped boundaries. To be more precise, it should be remarked that recently the counterexamples in [66] and in [62] have shown that the theorem fails for annular domains with non star-shaped boundaries as already recalled in section 2.7. Here we use a result by Rebelo [96, Corollary 2] which holds for an area-preserving homeomorphism of the plane Ψ such that $\Psi(0) = 0$ and with Ψ satisfying a twist condition on the boundary of a starlike annulus surrounding the origin.

We give a sketch of the proof of Theorem 5.13 for the reader's convenience.

Proof. If we denote by $\psi(\cdot, z) = (\psi_1(\cdot, z), \psi_2(\cdot, z))$ the solution of (5.16) with $\psi(0, z) = z$ and by Ψ the corresponding Poincaré map

$$\Psi(z) = \psi(T, z), \quad (5.17)$$

we have that Ψ is defined on the set

$$\mathcal{D} = \{(x, y) \in I \times \mathbb{R} : E(x, y) \leq c_2\}$$

(if $|w|_{L^1(0, T)}$ is sufficiently small) as an area-preserving homeomorphism of \mathcal{D} onto $\Psi(\mathcal{D})$ with $\Psi(0) = 0$ and $\psi(t, z) \neq 0$, for all $t \in [0, T]$

and $z \in \mathcal{A}$. Passing to the polar coordinates we can determine an angular function $\vartheta(t, z)$ so that

$$\psi(t, z) = \|\psi(t, z)\|(\cos(\vartheta(t, z)), \sin(\vartheta(t, z))).$$

It turns out that, in terms of the lifting $\tilde{\Psi}$ associated to Ψ (compare to (5.1)), we have that

$$\tilde{\Psi}(\vartheta, r) = (\vartheta + \Theta(\vartheta, r), R(\vartheta, r)) \quad (5.18)$$

with

$$R(\vartheta, r) = \|\Psi(z)\|$$

and

$$\Theta(\vartheta, r) = \vartheta(0, z) - \vartheta(T, z) = \int_0^T \frac{(1 + w(t))f(\psi_1(t, z))\psi_1(t, z) + \psi_2^2(t, z)}{\|\psi(t, z)\|^2} dt,$$

for $z = (r \cos \vartheta, r \sin \vartheta)$ (see [111] for the details). Assumption (5.14) for system (5.10) which now is viewed as a comparison system for (5.16) implies that if the perturbation $w(\cdot)$ is sufficiently small, then $\Theta > 2\pi$ on \mathcal{A}_i and $\Theta < 2\pi$ on \mathcal{A}_o and thus the twist condition (5.3) holds for $j = 1$.

Finally, using the fact that \mathcal{A}_i and \mathcal{A}_o are strictly star-shaped with respect to the origin with $\Psi(0) = 0$, we can apply W. Ding's version of the Poincaré-Birkhoff theorem [30, 96] and the existence of at least two distinct fixed points for Ψ in the interior of \mathcal{A} is ensured. \square

A natural question that now can arise is whether such (nontrivial) T -periodic solutions would persist if a sufficiently small perturbation destroying the Hamiltonian structure of the equation was performed. In the abstract setting of the Poincaré-Birkhoff theorem an answer can be found in the papers by Neumann [75] and Franks [35, 36] according to which if we have a finite number of fixed points then there are also fixed points with nonzero index. Actually, in [75, Theorem 2.1], the more general situation that the set of fixed points does not separate the boundaries is considered as well. In such cases, a standard application of the fixed point index theory (or the topological degree theory for maps of the plane) guarantees the persistence of fixed points for maps which are close to the Poincaré map and hence the existence of nontrivial T -periodic solutions also for sufficiently small perturbations of equation (5.15) holds. From this point of view, we could say that the bend-twist map theorem, in the form of theorem 5.10 provides an effective criterion to prove the persistence of periodic solutions under small perturbations. In order to show an example of equation (5.15) to which our result can be applied, we consider a special form of the

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T -periodic weight $w(t)$. For simplicity in the exposition we confine ourselves to the case of a continuous and T -periodic function $w : \mathbb{R} \rightarrow \mathbb{R}$ such that there is an interval $]t_0, t_1[\subset [0, T]$ such that

$$w(t) > 0, \forall t \in]t_0, t_1[\quad \text{and} \quad w(t) = 0, \forall t \in [0, T] \setminus]t_0, t_1[. \quad (5.19)$$

By the continuity of $w(\cdot)$ we can get the following corollary of theorem 5.13 where the smallness of w in the L^1 -norm is expressed in terms of δ_0 .

Corollary 5.14 *Assume (5.14) and let $w(\cdot)$ be a continuous and T -periodic function satisfying (5.19). Then there exists $\delta_0 > 0$ such that if*

$$t_1 - t_0 < \delta_0,$$

equation (5.15) has at least two T -periodic solutions with initial value in \mathcal{A} .

In comparison to this result obtained via the Poincaré-Birkhoff fixed point theorem, using Corollary 5.11 we can obtain the following.

Theorem 5.15 *Assume (5.14) and let $w(\cdot)$ be a continuous and T -periodic function satisfying (5.19). Then there exists $\delta_1 > 0$ such that if*

$$t_1 - t_0 < \delta_1,$$

equation (5.15) has at least four T -periodic solutions with initial value in \mathcal{A} . Moreover, the result is stable with respect to small perturbations. In particular, for the equation

$$\ddot{x} + \varepsilon \dot{x} + (1 + w(t))f(x) = 0, \quad (5.20)$$

there are at least four T -periodic solutions with initial value in the annulus \mathcal{A} if ε is sufficiently small.

Proof. Without loss of generality (via a time-shift leading to an equivalent equation), we can suppose that

$$w(t) = 0, \forall t \in [0, T - \delta] \quad \text{and} \quad w(t) > 0, \forall t \in]T - \delta, T[,$$

where we have set

$$\delta = t_1 - t_0.$$

To begin with, we consider the Poincaré map Ψ on the annulus \mathcal{A} , with Ψ defined as in (5.17). Passing to the polar coordinates and following the same argument as in the proof of theorem 5.13, we find a constant δ_0 such that if $\delta < \delta_0$ then $\Theta > 2\pi$ on \mathcal{A}_i and $\Theta < 2\pi$ on \mathcal{A}_o and hence the twist condition (5.3) holds for $j = 1$.

In order to check the validity of the condition on the map Υ , it is convenient to enter in the setting of the modified polar coordinates

(ϑ, ε) , instead of the standard polar coordinates (ϑ, r) . In this case, we can express the function Υ as

$$\Upsilon(z) = E(\Psi(z)) - E(z). \quad (5.21)$$

We split now the map Ψ as

$$\Psi = \Psi_2 \circ \Psi_1$$

with Ψ_1 and Ψ_2 defined as

$$\Psi_1(z) = \varphi(T - \delta, z),$$

where $\varphi(\cdot, z)$ is the solution of the autonomous system (5.10) with $\varphi(0, z) = z$ and

$$\Psi_2(z) = \psi(T; T - \delta, z),$$

where $\psi(\cdot; T - \delta, z)$ is the solution of system (5.16) which departs from the point z at the time $T - \delta$. Performing this splitting we have also used the fact that system (5.16) coincides with the autonomous system (5.10) on $[0, T - \delta]$. Hence we have

$$E(\Psi_1(z)) = E(z), \quad \forall z \in \mathcal{A}. \quad (5.22)$$

Let us consider now a solution $\psi(t) = (\psi_1(t), \psi_2(t))$ of (5.16) and evaluate the energy E along such solution. We obtain

$$\begin{aligned} \frac{d}{dt} E(\psi_1(t), \psi_2(t)) &= \psi_2'(t)\psi_2(t) + f(\psi_1(t))\psi_1'(t) \\ &= -(1 + w(t))f(\psi_1(t))\psi_2(t) + f(\psi_1(t))\psi_2(t) \\ &= -w(t)f(\psi_1(t))\psi_2(t). \end{aligned}$$

For $t \in]T - \delta, T[$ we have that $w(t) > 0$ and therefore the energy evaluated on a solution for the time interval $[T - \delta, T]$ is decreasing as long as the solution remains in the first or in the third quadrant and it is increasing as long as the solution remains in the second or in the fourth quadrant.

Let $\alpha \in]0, \pi/2[$ be a fixed angle (the smaller α we take, the larger δ_1 will be allowed). Recalling the definition of L_ϑ in (5.13), let Λ_1 be the intersection of the line $L_{\frac{\pi}{2}-\alpha}$ with the annulus \mathcal{A} . We are interested in the motion of the points of Λ_1 under the action of Ψ_2 . Since $\frac{d}{dt}\vartheta(t) > 0$, the points of Λ_1 move in the clockwise sense and therefore they remain in the first quadrant if δ is sufficiently small. Hence $\frac{d}{dt} E(\psi_1(t), \psi_2(t))$ is negative for $t \in [T - \delta, T]$ when $(\psi_1(0), \psi_2(0)) \in \Lambda_1$. This proves that

$$E(\Psi_2(z)) - E(z) < 0, \quad \forall z \in \Lambda_1.$$

Arguing in the same way, we have that

$$E(\Psi_2(z)) - E(z) > 0, \quad \forall z \in \Lambda_2 = L_{\pi-\alpha} \cap \mathcal{A},$$

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$$E(\Psi_2(z)) - E(z) < 0, \quad \forall z \in \Lambda_3 = L_{\frac{3\pi}{2}-\alpha} \cap \mathcal{A},$$

$$E(\Psi_2(z)) - E(z) > 0, \quad \forall z \in \Lambda_4 = L_{2\pi-\alpha} \cap \mathcal{A}.$$

All these relations hold provided that δ is chosen suitably small (say $\delta < \delta_1$) so that the solutions of (5.16) which depart at the time $T - \delta$ from Λ_i , remain in the same quadrant of Λ_i for all $t \in [T - \delta, T]$. In order to make such argument more precise we can evaluate the angular displacements and choose $\delta > 0$ such that

$$\int_{T-\delta}^T \frac{(1+w(t))f(\psi_1(t,z))\psi_1(t,z) + \psi_2^2(t,z)}{\|\psi(t,z)\|^2} dt < \frac{\pi}{2} - \alpha, \quad (5.23)$$

holds for all $z \in \Lambda_i$ ($i = 1, \dots, 4$).

Finally, recalling (5.22) and the definition of Υ in the (ϑ, ε) -coordinates given in (5.21) and setting

$$\Gamma_i = \Psi_1^{-1}(\Lambda_i), \quad i = 1, \dots, 4,$$

we conclude that $\Upsilon < 0$ on Γ_i for i odd and $\Upsilon > 0$ on Γ_i for i even. The thesis is thus achieved using Corollary 5.11. \square

An analysis of the proof and of inequality (5.23) shows that our argument is still valid if we take $w(t) = \chi_{[t_0, t_1]} W(t)$, where $W(\cdot)$ is a fixed positive function in $L^1([0, T])$.

Clearly, the same result holds also for equation (5.20) which can be viewed as a perturbation of (5.15). Of course, for such an application we exploit also the fact that in Corollary 5.11 no area-preserving type hypothesis is required. The smallness of ε will depend on the smallness of δ_1 .

We have achieved our result for a very special form of the weight function w . A natural question concerns which kind of shape for a T -periodic coefficient $q(t)$ may be required in order to obtain a similar result for equation

$$\ddot{x} + q(t)f(x) = 0.$$

Generally speaking our argument may work (modulo technical difficulties) whenever we can split the behaviour of the solutions of the equivalent system in the phase-plane into two regimes, depending by a different shape of $q(t)$ in two subintervals of its domain. In at least one of these regimes, we need to have a control of the trajectories and prove that they do not go too far from an annular region described by the level lines of an associated autonomous system. In the other regime, we need to show that there are at least *some* trajectories which are, in some sense, transverse to the annulus (and move into opposite directions). A different application of our technique has already been exposed in section 3.5 where we have considered a model of fluid mixing which is reminiscent to the case in which $q(t)$ changes its sign.

5.2 An application

A theorem about the existence of four solutions in this setting appears rather unusual (with respect to corollary 5.14 and other analogous results following from the Poincaré-Birkhoff twist theorem). For previous multiplicity results in a completely different setting (namely the Floquet problem for a superlinear equation), see [48].

A | INDEX OF A VECTOR FIELD ALONG A CURVE: DEFINITION AND PROPERTIES

We recall here some definitions and basic properties of the fixed point index. They are basic well-known facts, which are summarized in the sequel for completeness.

Let u and v be two distinct points in the plane \mathbb{R}^2 ; the *direction* from u to v is the normalized vector

$$D(u, v) = \frac{u - v}{\|u - v\|}$$

which can be seen as a point in S^1 . If $h : X \rightarrow \mathbb{R}^2 \setminus \{0\}$ is a map which has no fixed points on a curve $\gamma : [0, 1] \rightarrow X$, then we can compute the *index of h along γ* which is denoted by $i_\gamma(h)$ and represents the total rotation performed by the vector $D(z, h(z))$ when z moves along the curve γ . Since h has no fixed points on γ , the direction

$$\tilde{\gamma}(t) \stackrel{\text{def}}{=} D(\gamma(t), h(\gamma(t))) = \frac{\gamma(t) - h(\gamma(t))}{\|\gamma(t) - h(\gamma(t))\|} \quad (\text{A.1})$$

is a well-defined vector of \mathbb{R}^2 which lies on S^1 . Therefore there exists a continuous function $\vartheta(\cdot)$ such that it can be expressed by the polar coordinates

$$\tilde{\gamma}(t) = (\cos \vartheta(t), \sin \vartheta(t)) \quad \forall t \in [0, 1]. \quad (\text{A.2})$$

The *index of h along γ* is defined as

$$i_\gamma(h) = \frac{\vartheta(1) - \vartheta(0)}{2\pi}. \quad (\text{A.3})$$

Notice that the index is well-defined, since it is independent of the choice of the angular function $\vartheta(\cdot)$. Indeed, if $\vartheta_1(\cdot)$ and $\vartheta_2(\cdot)$ are two different functions both satisfying relation (A.2), then necessarily $\vartheta_1(\cdot) = \vartheta_2(\cdot) + 2k\pi$ for some $k \in \mathbb{Z}$, then $\vartheta_1(1) - \vartheta_1(0) = \vartheta_2(1) - \vartheta_2(0)$. The index of a map along a curve satisfies some basic properties.

1. Let $h_\lambda : X \rightarrow \mathbb{R}^2$ be a family of homeomorphism such that $\lambda \mapsto h_\lambda$ is a continuous map, and let $\gamma_\lambda : [0, 1] \rightarrow X$ be a family of paths such that $\lambda \mapsto \gamma_\lambda$ is a continuous map, then, if every $h_\lambda(\gamma_\lambda(t)) \neq \gamma_\lambda(t)$ for every λ and for every $t \in [0, 1]$, then the index of h_λ along γ_λ is constant with respect to the parameter λ . This property is called homotopy invariance.

A Index of a vector field along a curve: definition and properties

2. The index is congruent modulo 1 to $\frac{1}{2\pi}$ times the angle between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$
3. If $\gamma = \gamma_1\gamma_2$ is a path obtained by pasting two different paths, that is $\gamma|_{[a,c]} = \gamma_1$ and $\gamma|_{[c,b]} = \gamma_2$, with suitable $a < c < b$, then $i_\gamma(h) = i_{\gamma_1}(h) + i_{\gamma_2}(h)$.
4. The index $i_\gamma(h)$ coincides with $i_{h(\gamma)}(h^{-1})$.

B | NOTATIONS

- $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the sets of integer, real and complex numbers respectively; $\mathbb{N} = \{0, 1, \dots, n, \dots\}$ is the set of the nonnegative integers, while \mathbb{N}^* is the set $\{1, 2, \dots, n, \dots\}$
- $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$, $\mathbb{R}^+ = [0, +\infty[$, $\mathbb{R}_0^+ =]0, +\infty[$
- $\|\cdot\|$ the euclidean norm in \mathbb{R}^n , $\|\cdot\|_p$ is the norm in L^p
- $Q = [0, 1] \times [0, 1]$ is the unit square in \mathbb{R}^2
- $B[P, r] = \{z \in \mathbb{R}^2 : \|P - z\| \leq r\}$ is the closed ball of centre P and radius r , while the open one is $B(P, r) = \{z \in \mathbb{R}^2 : \|P - z\| < r\}$
- $C_r = \partial B(0, r) = B[P, r] \setminus B(P, r)$
- $S^1 = \partial B(0, 1)$
- $A[a, b] = B[0, b] \setminus B(0, a)$ is a standard planar annulus whose interior is $A(a, b) = \text{int } A[a, b]$

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