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## Characterized Subgroups of Topological Abelian Groups

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**Abstract:** A subgroup  $H$  of a topological abelian group  $X$  is said to be characterized by a sequence  $\mathbf{v} = (v_n)$  of characters of  $X$  if  $H = \{x \in X : v_n(x) \rightarrow 0 \text{ in } \mathbb{T}\}$ . We study the basic properties of characterized subgroups in the general setting, extending results known in the compact case. For a better description, we isolate various types of characterized subgroups. Moreover, we introduce the relevant class of auto-characterized groups (namely, the groups that are characterized subgroups of themselves by means of a sequence of non-null characters); in the case of locally compact abelian groups, these are proven to be exactly the non-compact ones. As a by-product of our results, we find a complete description of the characterized subgroups of discrete abelian groups.

**Keywords:** characterized subgroup;  $T$ -characterized subgroup;  $T$ -sequence;  $TB$ -sequence; von Neumann radical; convergent sequence; null sequence; abelian group

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### 1. Introduction

For a topological abelian group  $X$ , we denote by  $\widehat{X}$  its dual group, that is the group of all characters of  $X$  (i.e., continuous homomorphisms  $X \rightarrow \mathbb{T}$ ). Following [1], for a sequence of characters  $\mathbf{v} = (v_n) \in \widehat{X}^{\mathbb{N}}$ , let:

$$s_{\mathbf{v}}(X) := \{x \in X : v_n(x) \rightarrow 0\}$$

which is always a subgroup of  $X$ . A subgroup  $H$  of  $X$  is said to be *characterized* if  $H = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} = (v_n) \in \widehat{X}^{\mathbb{N}}$ .

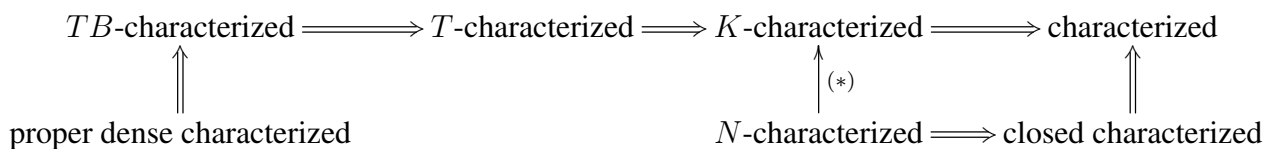
Historically, characterized subgroups were studied exclusively in the case of the circle group  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  (see [2–5]), also in relation to Diophantine approximation, dynamical systems and ergodic theory (see [3,6,7]; one can find more on this topic in the nice survey [8], as well as in the more recent [9–12]). Some general results were then obtained in the case of metrizable compact abelian groups; for example, it is known that every countable subgroup of a metrizable compact abelian group is characterized (see [13, Theorem 1.4] and [14]), and it was pointed out in [14,15] that the metrizability is necessary, as a compact abelian group with a countable characterized subgroup is necessarily metrizable. Only recently, the case of general compact abelian groups was given full attention in [16], and a reduction theorem (to the metrizable case) was obtained.

The few exceptions [8,17,18] only confirm the tendency to study the characterized subgroups of  $\mathbb{T}$  or, more recently, of compact abelian groups. To say the least, even the simplest case of characterized subgroups of *discrete* abelian groups has never been considered in the literature to the best of our knowledge.

The aim of these notes is to develop a general approach to characterized subgroups of arbitrary topological abelian groups, collecting the basic properties so far established in the compact case.

We isolate three special types of characterized subgroups, namely  $T$ -characterized,  $K$ -characterized and  $N$ -characterized subgroups (see Definition 3). Of those,  $T$ -characterized subgroups were introduced by Gabrielyan in [11],  $K$ -characterized subgroups were substantially studied by Kunen and his coauthors in [13,19,20], while  $N$ -characterized subgroups, even if never introduced explicitly, have been frequently used in the theory of duality in topological abelian groups (being nothing else but the annihilators of countable sets of the dual group). One of the advantages of this articulation is the possibility to establish some general permanence properties that fail to be true in the whole class of characterized subgroups, but hold true in some of these subclasses. Moreover, we see that each characterized subgroup is either  $N$ -characterized or coincides with the intersection of an  $N$ -characterized subgroup and a  $K$ -characterized subgroup (see Corollary 3).

Inspired by the notion of  $T$ -characterized subgroup, we introduce also the stronger one of  $TB$ -characterized subgroup (see Definition 4). The following implications hold, and none of them can be reversed in general (see Section 5):



where  $(*)$  holds under the assumption that the subgroup is closed and has infinite index (see Corollary 6).

In Section 6, we introduce the prominent class of auto-characterized groups (see Definition 5). These are the topological abelian groups that are characterized subgroups of themselves by means of a *non-trivial sequence* of characters (see (2)). The fact that compact abelian groups are not auto-characterized is equivalent to the well-known non-trivial fact that the Bohr topology of an infinite discrete abelian group has no non-trivial convergent sequences. Here, we generalize this fact by proving that the property of being non-auto-characterized describes the compact abelian groups within the class

of all locally compact abelian groups (see Theorem 3). Moreover, in the general case, we describe the auto-characterized groups in terms of their Bohr compactification (see Theorem 5).

We study the basic properties of  $K$ - and of  $N$ -characterized subgroups respectively in Sections 7 and 8. For the case of discrete abelian groups, which is considered here for the first time, we give a complete description of characterized subgroups by showing that these are precisely the subgroups of index at most  $\mathfrak{c}$  and that a subgroup is characterized precisely when it is  $K$ - and  $N$ -characterized (see Corollary 16).

In Section 7, we describe when a closed subgroup of infinite index is both  $K$ - and  $N$ -characterized, and we see that this occurs precisely when it is only  $N$ -characterized (see Theorem 6); then, we consider the special case of open subgroups, proving that proper open subgroups of infinite index (respectively, of finite index) are  $K$ -characterized if and only if they are characterized (respectively, auto-characterized) (see Theorems 7 and 8). In particular, no proper open subgroup of a compact abelian group is  $K$ -characterized.

In Section 8, extending a criterion for compact abelian groups given in [16], we show that for locally compact abelian groups one can reduce the study of characterized subgroups to the metrizable case (see Theorem 11). Moreover, we describe the closed characterized subgroups of the locally compact abelian groups by showing that they are precisely the  $N$ -characterized subgroups (see Theorem 12). As a consequence, we add other equivalent conditions to the known fact from [16] that a closed subgroup of a compact abelian group is characterized if and only if it is  $G_\delta$ , namely that the subgroup is  $K$ - and  $N$ -characterized (see Theorem 13).

Section 9 concerns  $T$ -characterized subgroups of compact abelian groups. We establish a criterion to determine when a characterized subgroup of a compact abelian group is not  $T$ -characterized (see Theorem 15), which extends results from [11]. The impact on characterized subgroups of connected compact abelian groups is discussed.

The final Section 10 contains various comments and open problems, both general and specific.

### 1.1. Notation and Terminology

The symbol  $\mathfrak{c}$  is used to denote the cardinality of continuum. The symbols  $\mathbb{Z}$ ,  $\mathbb{P}$ ,  $\mathbb{N}$  and  $\mathbb{N}_+$  are used for the set of integers, the set of primes, the set of non-negative integers and the set of positive integers, respectively. The circle group  $\mathbb{T}$  is identified with the quotient group  $\mathbb{R}/\mathbb{Z}$  of the reals  $\mathbb{R}$  and carries its usual compact topology. Let  $\pi : \mathbb{R} \rightarrow \mathbb{T}$  be the canonical projection; the usual group norm  $\| - \|$  on  $\mathbb{T}$  is defined by  $\|\pi(x)\| = d(x, \mathbb{Z})$  for every  $x \in \mathbb{R}$ . We denote by  $\mathbb{T}_+$  the image of  $[-1/4, 1/4]$  in  $\mathbb{T}$ . If  $m$  is a positive integer,  $G[m] = \{x \in G : mx = 0\}$ , and  $\mathbb{Z}(m)$  is the cyclic group of order  $m$ . Moreover, for  $p \in \mathbb{P}$ , we denote by  $\mathbb{Z}(p^\infty)$  and  $\mathbb{J}_p$ , respectively, the Prüfer group and the  $p$ -adic integers.

We say that an abelian group  $G$  is *torsion* if every element of  $G$  is torsion (*i.e.*, for every  $x \in G$ , there exists  $m \in \mathbb{N}_+$ , such that  $mx = 0$ ). If  $M$  is a subset of  $G$ , then  $\langle M \rangle$  is the smallest subgroup of  $G$  containing  $M$ . We denote by  $\text{Hom}(G, \mathbb{T})$  the group of the homomorphisms  $G \rightarrow \mathbb{T}$ .

For a topological space  $X = (X, \tau)$ , the *weight*  $w(X)$  of  $X$  is the minimum cardinality of a base for  $\tau$ . For a subset  $A$  of  $X$ , we denote by  $\overline{A}^\tau$  the closure of  $A$  in  $(X, \tau)$  (we write only  $\overline{A}$  when there is no possibility of confusion).

A topological abelian group  $X$  is *totally bounded* if for every open subset  $U$  of  $0$  in  $X$ , there exists a finite subset  $F$  of  $X$ , such that  $U + F = X$ . If  $X$  is totally bounded and Hausdorff, we say that  $X$  is *precompact*. We denote by  $\tilde{X}$  the two-sided completion of  $X$ ; in case  $X$  is precompact,  $\tilde{X}$  coincides with the Weil completion.

For a subset  $A$  of  $X$ , the *annihilator* of  $A$  in  $\hat{X}$  is  $A^\perp = \{\chi \in \hat{X} : \chi(A) = \{0\}\}$ , and for a subset  $B$  of  $\hat{X}$ , the *annihilator* of  $B$  in  $X$  is  $B^\perp = \{x \in X : \chi(x) = 0 \text{ for every } \chi \in B\}$ .

We say that a sequence  $\mathbf{v} \in \hat{X}^\mathbb{N}$  is *trivially null* if there exists  $n_0 \in \mathbb{N}$ , such that  $v_n = 0$  for every  $n \geq n_0$ , and we say that  $\mathbf{v}$  is *non-trivial* if it is not trivially null.

## 2. Background on Topological Groups

### 2.1. Basic Definitions

Let  $G$  be an abelian group and  $H$  a subgroup of  $\text{Hom}(G, \mathbb{T})$ . Let  $T_H$  be the weakest group topology on  $G$ , such that all elements of  $H$  are continuous with respect to  $T_H$ ; then  $T_H$  is totally bounded. The other way around, Comfort and Ross proved that any totally bounded group topology is of this type (see [21, Theorem 1.2]).

**Theorem 1.** [21, Theorems 1.2, 1.3 and 1.11, Corollary 1.4] *Let  $G$  be an abelian group and  $H$  a subgroup of  $\text{Hom}(G, \mathbb{T})$ . Then,  $T_H$  is totally bounded and:*

- (a)  $T_H$  is Hausdorff if and only if  $H$  separates the points of  $G$ ;
- (b)  $T_H$  is metrizable if and only if  $H$  is countable.

The following two notions will be often used in the paper.

**Definition 1.** A topological abelian group  $X$  is said to be:

- (i) *maximally almost periodic (MAP)* if  $\hat{X}$  separates the points of  $X$ ;
- (ii) *minimally almost periodic (MinAP)* if  $\hat{X} = \{0\}$ .

We recall that two group topologies  $\tau_1$  and  $\tau_2$  on an abelian group  $X$  are *compatible* if they have the same characters, that is  $(\widehat{X, \tau_1}) = (\widehat{X, \tau_2})$ .

If  $X = (X, \tau)$  is a topological abelian group, denote by  $\tau^+$  its *Bohr topology*, that is the finest totally bounded group topology on  $X$  coarser than  $\tau$  (indeed,  $\tau^+ = T_{\hat{X}}$ ); we denote  $X$  endowed with its Bohr topology also by  $X^+$ , and we call  $\tau^+$  also the *Bohr modification* of  $\tau$ . Clearly,  $\tau$  and  $\tau^+$  are compatible. Moreover,

- (i)  $\tau$  is MAP if and only if  $\tau^+$  is Hausdorff;
- (ii)  $\tau$  is MinAP if and only if  $\tau^+$  is indiscrete.

A subgroup  $H$  of  $(X, \tau)$  is:

- (a) *dually closed* if  $H$  is  $\tau^+$ -closed (or, equivalently,  $X/H$  is MAP);
- (b) *dually embedded* if every  $\chi \in \hat{H}$  can be extended to  $X$ .

Clearly, dually closed implies closed, since  $H \leq \overline{H}^\tau \leq \overline{H}^{\tau^+}$ .

**Fact 1.** *Let  $X$  be a locally compact abelian group. Then:*

- (i) every closed subgroup  $H$  of  $X$  is dually closed, i.e.,  $X/H$  is MAP;
- (ii) in particular, every locally compact abelian group is MAP;
- (iii)  $X$  and  $X^+$  have the same closed subgroups;
- (iv) consequently,  $X$  is separable if and only if  $X^+$  is separable.

For a topological abelian group  $X$  and a subgroup  $L$  of  $X$ , the weak topology  $\sigma(\widehat{X}, L)$  of the dual  $\widehat{X}$  is the totally bounded group topology of  $\widehat{X}$  generated by the elements of  $L$  considered as characters of  $\widehat{X}$ ; namely, for every  $x \in L$ , consider  $\xi_x : \widehat{X} \rightarrow \mathbb{T}$  defined by  $\xi_x(\chi) = \chi(x)$  for every  $\chi \in \widehat{X}$ . A local base of  $\sigma(\widehat{X}, L)$  is given by the finite intersections of the sets  $\xi_x^{-1}(U)$ , where  $x \in L$  and  $U$  is an open neighborhood of 0 in  $\mathbb{T}$ . Clearly, if  $L_1 \leq L_2$ , then  $\sigma(\widehat{X}, L_1) \leq \sigma(\widehat{X}, L_2)$ .

Note that the weak topology  $\sigma(\widehat{X}, X)$  is coarser than the compact-open topology on  $\widehat{X}$ . If  $L$  separates the points of  $\widehat{X}$  (e.g., when  $L$  is dense in  $X$  or when  $L = X$ ), then  $\sigma(\widehat{X}, L)$  is precompact.

**Fact 2.** *If  $X$  is a reflexive topological abelian group, then  $\sigma(\widehat{X}, X)$  coincides with the Bohr topology of  $\widehat{X}$ .*

We recall that a sequence  $\mathbf{v}$  in an abelian group  $G$  is a  $T$ -sequence (respectively,  $TB$ -sequence) if there exists a Hausdorff (respectively, precompact) group topology  $\tau$  on  $G$ , such that  $\mathbf{v}$  is a null sequence in  $(G, \tau)$ .

**Lemma 1.** *Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^\mathbb{N}$ . Then:*

- (i) for a subgroup  $L$  of  $X$ ,  $v_n(x) \rightarrow 0$  in  $\mathbb{T}$  for every  $x \in L$  if and only if  $v_n \rightarrow 0$  in  $\sigma(\widehat{X}, L)$ ;
- (ii) if  $s_{\mathbf{v}}(X)$  is dense in  $X$ , then  $\mathbf{v}$  is a  $TB$ -sequence.

**Proof.** (i) follows from the definition of  $\sigma(\widehat{X}, L)$ .

(ii) As  $s_{\mathbf{v}}(X)$  is dense in  $X$ , then  $\sigma(\widehat{X}, s_{\mathbf{v}}(X))$  is precompact. By item (i),  $v_n \rightarrow 0$  in  $\sigma(\widehat{X}, s_{\mathbf{v}}(X))$ ; hence,  $\mathbf{v}$  is a  $TB$ -sequence.  $\square$

Let  $G$  be a discrete abelian group. For a sequence  $\mathbf{v} \in G^\mathbb{N}$ , the group topology:

$$\sigma_{\mathbf{v}} := T_{s_{\mathbf{v}}(\widehat{G})} \tag{1}$$

is the finest totally bounded group topology on  $G$ , such that  $\mathbf{v}$  is a null sequence in  $(G, \sigma_{\mathbf{v}})$ .

**Fact 3.** [1, Lemma 3.1, Proposition 3.2] *Let  $G$  be a discrete abelian group and  $\mathbf{v} \in G^\mathbb{N}$ . The following conditions are equivalent:*

- (i)  $\mathbf{v}$  is a  $TB$ -sequence;
- (ii)  $\sigma_{\mathbf{v}}$  is Hausdorff;
- (ii)  $s_{\mathbf{v}}(\widehat{G})$  is dense in  $\widehat{G}$ .

2.2. Useful Folklore Results

We recall the following basic properties that will be used in the paper. Although most of them are well known, we offer proofs for the reader’s convenience.

**Lemma 2.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ . Then:*

- (i)  $\widehat{X/H}$  is algebraically isomorphic to  $H^\perp$ ;
- (ii)  $\widehat{X}/H^\perp$  is algebraically isomorphic to a subgroup of  $\widehat{H}$ .

**Proof.** (i) Let  $\psi : \widehat{X/H} \rightarrow \widehat{X}$  be defined by  $\chi \mapsto \chi \circ \pi$ , where  $\pi : X \rightarrow X/H$  is the canonical projection. Then,  $\psi$  is injective, and its image is  $H^\perp$ .

(ii) Let  $\rho : \widehat{X} \rightarrow \widehat{H}$  be defined by  $\chi \mapsto \chi \upharpoonright_H$ . Then,  $\ker \rho = H^\perp$ , and so, we get the thesis.  $\square$

The following fact follows from the equivalence of items (a) and (e) of [4, Exercise 3.8.25]. Since no proofs are given there, we offer a proof for the reader’s convenience.

**Lemma 3.** *A compact abelian group  $K$  is separable if and only if  $w(K) \leq \mathfrak{c}$ .*

**Proof.** The inequality  $w(K) \leq \mathfrak{c}$  holds for every separable regular topological space  $K$ .

Assume that  $w(K) \leq \mathfrak{c}$ . The discrete abelian group  $X = \widehat{K}$  has size  $|X| = w(K) \leq \mathfrak{c}$ . Consider the embedding  $i : X \rightarrow D(X)$ , where  $D(X)$  is the divisible hull of  $X$ . Then,  $|D(X)| \leq \mathfrak{c}$  and  $D(X) = \bigoplus_{i \in I} D_i$ , for some countable divisible abelian groups  $D_i$  and a set of indices  $I$  with  $|I| \leq \mathfrak{c}$ . Therefore,  $\widehat{i} : \prod_{i \in I} \widehat{D}_i \rightarrow \widehat{X} = K$  is a surjective continuous homomorphism, and each  $\widehat{D}_i$  is a metrizable compact abelian group. By the Hewitt–Marczewski–Pondiczery Theorem, since  $|I| \leq \mathfrak{c}$ , we have that  $\prod_{i \in I} \widehat{D}_i$  is separable; hence,  $K$  is separable, as well.  $\square$

**Lemma 4.** *Let  $X$  be a precompact abelian group. Then, the singleton  $\{0\}$  is  $G_\delta$  if and only if there exists a continuous injection  $X \rightarrow \mathbb{T}^\mathbb{N}$ .*

**Proof.** If there exists a continuous injection  $X \rightarrow \mathbb{T}^\mathbb{N}$ , then  $\{0\}$  is  $G_\delta$  in  $X$ , as it is  $G_\delta$  in  $\mathbb{T}^\mathbb{N}$ .

Assume now that  $\{0\} = \bigcap_{n \in \mathbb{N}} U_n$ , where each  $U_n$  is an open subset of  $X$ , and we can assume that  $U_n$  is in the prebase of the neighborhoods of 0 in  $X$ . Therefore, for every  $n \in \mathbb{N}$ , there exist  $v_n \in \widehat{X}$  and an open neighborhood  $V_n$  of 0 in  $\mathbb{T}$  containing no non-trivial subgroup of  $\mathbb{T}$ , such that  $U_n = v_n^{-1}(V_n)$ . Then,  $\{0\} = \bigcap_{n \in \mathbb{N}} \ker v_n$ . Hence,  $j : X \rightarrow \mathbb{T}^\mathbb{N}$  defined by  $j(x) = (v_n(x))_{n \in \mathbb{N}}$  is a continuous injective homomorphism.  $\square$

**Theorem 2.** *Let  $X$  be a locally compact abelian group. Then,  $X$  is metrizable with  $|X| \leq \mathfrak{c}$  if and only if there exists a continuous injective homomorphism  $X \rightarrow \mathbb{T}^\mathbb{N}$ .*

**Proof.** If there exists a continuous injective homomorphism  $X \rightarrow \mathbb{T}^\mathbb{N}$ , then clearly  $X$  is metrizable and  $|X| \leq |\mathbb{T}^\mathbb{N}| = \mathfrak{c}$ .

Suppose now that  $X$  is metrizable and has cardinality at most  $\mathfrak{c}$ . It is well known (for example, see [22]) that  $X = \mathbb{R}^n \times X_0$ , where  $n \in \mathbb{N}$  and  $X_0$  is a locally compact abelian group admitting an open compact (metrizable) subgroup  $K$ . Clearly, there exist two continuous injective homomorphisms

$j_1 : \mathbb{R}^n \hookrightarrow \mathbb{T}^{\mathbb{N}}$  and  $j_2 : K \hookrightarrow \mathbb{T}^{\mathbb{N}}$ . Therefore,  $j_3 = (j_1, j_2) : \mathbb{R}^n \times K \rightarrow \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \cong \mathbb{T}^{\mathbb{N}}$  is an injective continuous homomorphism, too. Since  $\mathbb{T}^{\mathbb{N}}$  is divisible and  $\mathbb{R}^n \times K$  is open in  $X$ ,  $j_3$  extends continuously to  $\tilde{j}_3 : X \rightarrow \mathbb{T}^{\mathbb{N}}$ . Let  $\pi : X \rightarrow X/(\mathbb{R}^n \times K)$  be the canonical projection. Since  $X/(\mathbb{R}^n \times K)$  is discrete, there exists a continuous injective homomorphism  $j_4 : X/(\mathbb{R}^n \times K) \rightarrow \mathbb{T}^{\mathbb{N}}$ . Let  $\varphi = j_4 \circ \pi : X \rightarrow \mathbb{T}^{\mathbb{N}}$ .

$$\begin{array}{ccc}
 X & & \\
 \pi \downarrow & \searrow \varphi & \\
 X/(\mathbb{R}^n \times K) & \xrightarrow{j_4} & \mathbb{T}^{\mathbb{N}}
 \end{array}$$

Let now  $j : X \rightarrow \mathbb{T}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}} \cong \mathbb{T}^{\mathbb{N}}$  be defined by  $j(x) = (\varphi(x), \tilde{j}_3(x))$  for every  $x \in X$ . Then,  $j$  is continuous, since  $\varphi$  and  $\tilde{j}_3$  are continuous. Moreover,  $j$  is injective, as  $j(x) = 0$  for some  $x \in X$  implies  $\varphi(x) = 0$  and  $\tilde{j}_3(x) = 0$ ; therefore,  $x \in \mathbb{R}^n \times K$ , and so, since  $\tilde{j}_3 \upharpoonright_{\mathbb{R}^n \times K} = j_3$  is injective, one has  $x = 0$ .  $\square$

### 3. General Permanence Properties of Characterized Subgroups

Let  $X$  be a topological abelian group, and denote by  $\mathcal{C}\text{har}(X)$  the family of all subgroups of  $X$  that are characterized.

We start by observing that:

$$\text{if } \mathbf{v} \in \widehat{X}^{\mathbb{N}} \text{ is trivially null, then } X = s_{\mathbf{v}}(X) \tag{2}$$

The following are basic facts on characterized subgroups (see [1,13,16,23]); we give a proof for the reader’s convenience.

**Lemma 5.** *Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Then:*

- (i) *for every subgroup  $J$  of  $X$ ,  $s_{\mathbf{v}^*}(J) = s_{\mathbf{v}}(X) \cap J$ , where  $v_n^* = v_n \upharpoonright_J$  for every  $n \in \mathbb{N}$ ;*
- (ii)  *$s_{\mathbf{v}}(X) = s_{\mathbf{u}}(X)$  if  $\mathbf{u}$  is any permutation of  $\mathbf{v}$ ;*
- (iii)  *$\mathcal{C}\text{har}(X)$  is stable under taking finite intersections;*
- (iv)  *$s_{\mathbf{v}}(X)$  is an  $F_{\sigma\delta}$ -set (i.e., countable intersection of countable unions of closed subsets).*

**Proof.** Items (i) and (ii) are obvious. To prove (iii), if  $\mathbf{u}, \mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , define  $\mathbf{w} = (w_n)$ , where  $w_{2n} = u_n$  and  $w_{2n+1} = v_n$  for every  $n \in \mathbb{N}$ ; hence,  $s_{\mathbf{u}}(X) \cap s_{\mathbf{v}}(X) = s_{\mathbf{w}}(X)$ . To prove (iv), note that  $s_{\mathbf{v}}(X) = \bigcap_m \bigcup_k \bigcap_{n \geq k} S_{n,m}$ , where each  $S_{n,m} = \{x \in X : \|v_n(x)\| \leq \frac{1}{m}\}$  is a closed subset of  $X$ .  $\square$

Now, we prove that, under suitable hypotheses, the relation of being a characterized subgroup is transitive:

**Proposition 1.** *Let  $X$  be a topological abelian group and  $X_0, X_1, X_2$  subgroups of  $X$  with  $X_0 \leq X_1 \leq X_2$  and such that  $X_1$  is dually embedded in  $X_2$ . If  $X_0 \in \mathcal{C}\text{har}(X_1)$  and  $X_1 \in \mathcal{C}\text{har}(X_2)$ , then  $X_0 \in \mathcal{C}\text{har}(X_2)$ .*

**Proof.** Let  $\mathbf{v} \in \widehat{X}_1^{\mathbb{N}}$ , such that  $X_0 = s_{\mathbf{v}}(X_1)$ , and let  $\mathbf{w} \in \widehat{X}_2^{\mathbb{N}}$ , such that  $X_1 = s_{\mathbf{w}}(X_2)$ . As  $X_1$  is dually embedded in  $X_2$ ,  $v_n$  extends to a character  $v_n^*$  of  $X_2$  for every  $n \in \mathbb{N}$ ; so, let  $\mathbf{v}^* = (v_n^*) \in \widehat{X}_2^{\mathbb{N}}$ . Define  $\mathbf{w}^* \in \widehat{X}_2^{\mathbb{N}}$  by letting  $w_{2n}^* = v_n^*$  and  $w_{2n+1}^* = w_n$  for every  $n \in \mathbb{N}$ . Then, by Lemma 5(i),

$$s_{\mathbf{w}^*}(X_2) = s_{\mathbf{v}^*}(X_2) \cap s_{\mathbf{w}}(X_2) = s_{\mathbf{v}^*}(X_2) \cap X_1 = s_{\mathbf{v}}(X_1) = X_0,$$

so  $X_0 \in \mathcal{C}\text{har}(X_2)$ , as required.  $\square$

Clearly, two compatible group topologies have the same characterized subgroups:

**Lemma 6.** *If  $\tau_1$  and  $\tau_2$  are compatible group topologies on an abelian group  $X$ , then  $\mathcal{C}\text{har}(X, \tau_1) = \mathcal{C}\text{har}(X, \tau_2)$ .*

In particular, for a topological abelian group  $(X, \tau)$ , since  $\tau$  and its Bohr modification  $\tau^+$  are compatible,  $\mathcal{C}\text{har}(X, \tau) = \mathcal{C}\text{har}(X, \tau^+)$ .

#### 4. The $\Gamma$ -Radical

**Definition 2.** Let  $X$  be a topological abelian group. For a subset  $\Gamma$  of  $\widehat{X}$ , define the  $\Gamma$ -radical of  $X$  by:

$$n_{\Gamma}(X) := \bigcap_{\chi \in \Gamma} \ker \chi = \Gamma^{\perp}.$$

Clearly,  $n_{\Gamma}(X)$  is a closed subgroup of  $X$ .

The motivation for the choice of the term  $\Gamma$ -radical is the special case  $\Gamma = \widehat{X}$ , when

$$n(X) := n_{\widehat{X}}(X)$$

is usually called the *von Neumann radical* of  $X$ . Then,  $n(X) = \{0\}$  (respectively,  $n(X) = X$ ) precisely when  $\widehat{X}$  separates the points of  $X$  (respectively,  $\widehat{X} = \{0\}$ ); in other words:

- (i)  $X$  is MAP if and only if  $n(X) = \{0\}$ ;
- (ii)  $X$  is MinAP if and only if  $n(X) = X$ .

**Remark 1.** Let  $X$  be a topological abelian group and  $\Gamma$  a subset of  $\widehat{X}$ .

- (i) If  $\Gamma = \emptyset$ , then  $n_{\Gamma}(X) = X$ .
- (ii) If  $\Gamma$  is countable, then  $n_{\Gamma}(X)$  is a characterized subgroup of  $X$  (indeed,  $n_{\Gamma}(X) = s_{\mathbf{v}}(X)$  for  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , such that each character in  $\Gamma$  appears infinitely many times in  $\mathbf{v}$ ).

For a given sequence  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , the *support*  $\Gamma_{\mathbf{v}} = \{v_n : n \in \mathbb{N}\}$  of  $\mathbf{v}$  is the set of all characters appearing in  $\mathbf{v}$ . We abbreviate the notation of the  $\Gamma_{\mathbf{v}}$ -radical by writing:

$$n_{\mathbf{v}}(X) := n_{\Gamma_{\mathbf{v}}}(X),$$

and we call this subgroup the  $\mathbf{v}$ -radical of  $X$ .



**Lemma 7.** Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Then:

- (i)  $n_{\mathbf{v}}(X) \leq s_{\mathbf{v}}(X)$ ;
- (ii)  $n_{\mathbf{v}}(X)$  is dually closed;
- (iii)  $n_{\mathbf{v}}(X)$  is characterized;
- (iv)  $n_{\mathbf{v}}(X)$  is closed, and  $\{0\}$  is  $G_{\delta}$  in  $X/n_{\mathbf{v}}(X)$  (so,  $n_{\mathbf{v}}(X)$  is  $G_{\delta}$ );
- (v)  $[X : n_{\mathbf{v}}(X)] \leq \mathfrak{c}$ .

**Proof.** (i) and (ii) are clear from the definitions, and (iii) follows from Remark 1(ii).

(iv) Let  $\varphi : X \rightarrow \mathbb{T}^{\mathbb{N}}$  be defined by  $\varphi(x) = (v_0(x), \dots, v_n(x), \dots)$  for every  $x \in X$ . Since  $\{0\}$  is  $G_{\delta}$  in  $\mathbb{T}^{\mathbb{N}}$ , we conclude that  $\{0\}$  is  $G_{\delta}$  in  $X/n_{\mathbf{v}}(X)$ . Moreover,  $\ker \varphi = \varphi^{-1}(0) = n_{\mathbf{v}}(X)$ , so  $n_{\mathbf{v}}(X)$  is  $G_{\delta}$  in  $X$ .

(v) Since  $X/n_{\mathbf{v}}(X)$  is algebraically isomorphic to  $\varphi(X) \leq \mathbb{T}^{\mathbb{N}}$  and  $|\mathbb{T}^{\mathbb{N}}| = \mathfrak{c}$ , we conclude that  $[X : n_{\mathbf{v}}(X)] \leq \mathfrak{c}$ .  $\square$

**Remark 2.** Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Then,  $n_{\mathbf{v}}(X)$  is closed and  $G_{\delta}$  in every group topology on  $X$  that makes  $v_n$  continuous for every  $n \in \mathbb{N}$ . In particular,  $n_{\mathbf{v}}(X)$  is closed and  $G_{\delta}$  in every group topology on  $X$  compatible with the topology of  $X$ , so in the Bohr topology of  $X$ .

Lemma 7 gives a bound for the index of the characterized subgroups:

**Corollary 1.** Every characterized subgroup of a topological abelian group  $X$  has index at most  $\mathfrak{c}$ .

**Proof.** Let  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Since  $n_{\mathbf{v}}(X) \leq s_{\mathbf{v}}(X)$  by Lemma 7(i); hence,  $[X : s_{\mathbf{v}}(X)] \leq [X : n_{\mathbf{v}}(X)] \leq \mathfrak{c}$  by Lemma 7(v).  $\square$

The set  $\Gamma_{\mathbf{v}}$  can be partitioned as

$$\Gamma_{\mathbf{v}} = \Gamma_{\mathbf{v}}^{\infty} \dot{\cup} \Gamma_{\mathbf{v}}^0,$$

where:

- (i)  $\Gamma_{\mathbf{v}}^{\infty} := \{v_n \in \Gamma_{\mathbf{v}} : v_n = v_m \text{ for infinitely many } m \in \mathbb{N}\}$ ;
- (ii)  $\Gamma_{\mathbf{v}}^0 := \Gamma_{\mathbf{v}} \setminus \Gamma_{\mathbf{v}}^{\infty}$ .

In other words,  $\Gamma_{\mathbf{v}}^{\infty}$  is the set of all characters appearing infinitely many times in  $\mathbf{v}$ , while each character in its complement  $\Gamma_{\mathbf{v}}^0$  appears finitely many times in  $\mathbf{v}$ . Clearly,  $\mathbf{v}$  is a finitely many-to-one sequence if and only if  $\Gamma_{\mathbf{v}}^{\infty} = \emptyset$ .

In case  $\Gamma_{\mathbf{v}}^{\infty} \neq \emptyset$ , let  $\mathbf{v}^{\infty}$  be the largest subsequence of  $\mathbf{v}$  with  $\Gamma_{\mathbf{v}^{\infty}} = \Gamma_{\mathbf{v}}^{\infty}$ . Then, clearly,  $s_{\mathbf{v}^{\infty}}(X) = n_{\mathbf{v}^{\infty}}(X)$ .

In case  $\Gamma_{\mathbf{v}}^0$  is finite, the subsequence  $\mathbf{v}^{\infty}$  of  $\mathbf{v}$  is cofinite, so  $s_{\mathbf{v}}(X) = s_{\mathbf{v}^{\infty}}(X)$ . In other words, one can safely replace  $\mathbf{v}$  by  $\mathbf{v}^{\infty}$ . This is why from now on, we shall always assume that:

$$\text{either } \Gamma_{\mathbf{v}}^0 = \emptyset \text{ or } \Gamma_{\mathbf{v}}^0 \text{ is infinite.} \tag{3}$$

If  $\Gamma_{\mathbf{v}}^0$  is infinite, we denote by  $\mathbf{v}^0$  the subsequence of  $\mathbf{v}$  such that  $\Gamma_{\mathbf{v}^0} = \Gamma_{\mathbf{v}}^0$ . If  $\Gamma_{\mathbf{v}}^{\infty} \neq \emptyset$ , we have the partition

$$\mathbf{v} = \mathbf{v}^{\infty} \dot{\cup} \mathbf{v}^0$$

of  $\mathbf{v}$  in the two subsequences  $\mathbf{v}^\infty$  and  $\mathbf{v}^0$ . Moreover, always  $\Gamma_{\mathbf{v}^0}^\infty = \emptyset$ , and so, if  $\Gamma_{\mathbf{v}^0}$  is infinite (equivalently,  $\Gamma_{\mathbf{v}^0}^\infty \neq \emptyset$  by (3)),  $\mathbf{v}^0$  is a finitely many-to-one sequence and  $s_{\mathbf{v}^0}(X) = s_{\mathbf{w}}(X)$ , where  $\mathbf{w}$  is a one-to-one subsequence of  $\mathbf{v}^0$  such that  $\Gamma_{\mathbf{w}} = \Gamma_{\mathbf{v}^0} = \Gamma_{\mathbf{v}^0}^\infty$ .

We see now how we can obtain the subgroup of  $X$  characterized by  $\mathbf{v}$  by considering separately the  $\mathbf{v}^\infty$ -radical of  $X$  and the subgroup of  $X$  characterized by  $\mathbf{v}^0$ .

**Lemma 8.** *Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^\mathbb{N}$  satisfying (3).*

- (i) *If  $\Gamma_{\mathbf{v}^0}^\infty = \emptyset$ , then  $\mathbf{v}^\infty = \mathbf{v}$ , so  $\Gamma_{\mathbf{v}^\infty}^\infty \neq \emptyset$  and  $s_{\mathbf{v}}(X) = n_{\mathbf{v}^\infty}(X)$ .*
- (ii) *If  $\Gamma_{\mathbf{v}^0}^\infty$  is infinite and  $\Gamma_{\mathbf{v}^0}^\infty \neq \emptyset$ , then  $s_{\mathbf{v}}(X) = s_{\mathbf{v}^0}(X) \cap n_{\mathbf{v}^\infty}(X)$ .*

**Proof.** (i) Since  $\Gamma_{\mathbf{v}^0}^\infty = \emptyset$ , we have  $s_{\mathbf{v}}(X) = s_{\mathbf{v}^\infty}(X)$ , and as observed above,  $s_{\mathbf{v}^\infty}(X) = n_{\mathbf{v}^\infty}(X)$ .

(ii) Since  $\mathbf{v}^\infty$  and  $\mathbf{v}^0$  are subsequences of  $\mathbf{v}$ , it follows that  $s_{\mathbf{v}}(X) \leq s_{\mathbf{v}^0}(X) \cap n_{\mathbf{v}^\infty}(X)$ . Let now  $x \in s_{\mathbf{v}^0}(X) \cap n_{\mathbf{v}^\infty}(X)$ . Since both  $\mathbf{v}^\infty(x) \rightarrow 0$  and  $\mathbf{v}^0(x) \rightarrow 0$  and since  $\mathbf{v} = \mathbf{v}^\infty \dot{\cup} \mathbf{v}^0$ , we conclude that  $\mathbf{v}(x) \rightarrow 0$ , that is  $x \in s_{\mathbf{v}}(X)$ . This concludes the proof.  $\square$

For  $\mathbf{v} = (v_n) \in \widehat{X}^\mathbb{N}$  and  $m \in \mathbb{N}$ , let

$$\mathbf{v}_{(m)} := (v_n)_{n \geq m}. \tag{4}$$

Note that  $n_{\mathbf{v}_{(m)}}(X) \leq n_{\mathbf{v}_{(m+1)}}(X)$  for every  $m \in \mathbb{N}$ .

### 5. A Hierarchy for Characterized Subgroups

The following definition introduces three specific types of characterized subgroups.

**Definition 3.** Let  $X$  be a topological abelian group. A subgroup  $H$  of  $X$  is:

- (i)  *$T$ -characterized* if  $H = s_{\mathbf{v}}(X)$  where  $\mathbf{v} \in \widehat{X}^\mathbb{N}$  is a non-trivial  $T$ -sequence;
- (ii)  *$K$ -characterized* if  $H = s_{\mathbf{v}}(X)$  for some finitely many-to-one sequence  $\mathbf{v} \in \widehat{X}^\mathbb{N}$  (i.e.,  $\Gamma_{\mathbf{v}^0}^\infty = \emptyset$ );
- (iii)  *$N$ -characterized* if  $H = n_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^\mathbb{N}$ .

In analogy to Definition 3(i), we introduce the following smaller class of characterized subgroups (see also Problem 1).

**Definition 4.** A subgroup  $H$  of a topological abelian group  $X$  is  *$TB$ -characterized* if  $H = s_{\mathbf{v}}(X)$ , where  $\mathbf{v} \in \widehat{X}^\mathbb{N}$  is a non-trivial  $TB$ -sequence.

The  $N$ -characterized subgroups are clearly closed, and they are always characterized as noted above. Every  $TB$ -characterized subgroup is obviously  $T$ -characterized. Moreover, every  $T$ -characterized subgroup is also  $K$ -characterized. Indeed, let  $H = s_{\mathbf{v}}(X)$ , where  $\mathbf{v} \in \widehat{X}^\mathbb{N}$  is a non-trivial  $T$ -sequence, and without loss of generality, assume that  $v_n \neq 0$  for all  $n \in \mathbb{N}$ ; then,  $\Gamma_{\mathbf{v}^0}^\infty = \emptyset$ , that is  $\mathbf{v}$  contains no constant subsequences, and so,  $H$  is  $K$ -characterized.

Furthermore, proper dense characterized subgroups are  $TB$ -characterized by Lemma 1(ii), so also  $T$ -characterized and, in particular,  $K$ -characterized, but they are not  $N$ -characterized

(as  $N$ -characterized subgroups are necessarily closed). We shall see below that closed (even open) subgroups need not be  $K$ -characterized in general. Denote by  $\mathcal{C}\text{har}_K(X)$  (respectively,  $\mathcal{C}\text{har}_N(X)$ ,  $\mathcal{C}\text{har}_T(X)$ ,  $\mathcal{C}\text{har}_{TB}(X)$ ) the family of all  $K$ -characterized (respectively,  $N$ -characterized,  $T$ -characterized,  $TB$ -characterized) subgroups of the topological abelian group  $X$ . Then, we have the following strict inclusions:

$$\mathcal{C}\text{har}_{TB}(X) \subsetneq \mathcal{C}\text{har}_T(X) \subsetneq \mathcal{C}\text{har}_K(X) \subsetneq \mathcal{C}\text{har}(X) \supsetneq \mathcal{C}\text{har}_N(X).$$

We start giving some basic properties that can be proven immediately.

**Corollary 2.** *Let  $X$  be a topological abelian group and  $X_0, X_1, X_2$  subgroups of  $X$  with  $X_0 \leq X_1 \leq X_2$  and such that  $X_1$  is dually embedded in  $X_2$ .*

- (i) *If  $X_0 \in \mathcal{C}\text{har}_K(X_1)$  and  $X_1 \in \mathcal{C}\text{har}_K(X_2)$ , then  $X_0 \in \mathcal{C}\text{har}_K(X_2)$ .*
- (ii) *If  $X_0 \in \mathcal{C}\text{har}_N(X_1)$  and  $X_1 \in \mathcal{C}\text{har}_N(X_2)$ , then  $X_0 \in \mathcal{C}\text{har}_N(X_2)$ .*

**Proof.** (i) It suffices to note that if in the proof of Proposition 1,  $v$  is one-to-one and  $w$  is one-to-one, then  $w^*$  is finitely many-to-one.

(ii) It suffices to note that if in the proof of Proposition 1,  $\Gamma_v = \Gamma_v^\infty$  and  $\Gamma_w = \Gamma_w^\infty$ , then also  $\Gamma_{w^*} = \Gamma_{w^*}^\infty$ .  $\square$

By Lemma 8, we have directly the following:

**Corollary 3.** *Every characterized subgroup of a topological abelian group  $X$  is either  $N$ -characterized or it is the intersection of an  $N$ -characterized subgroup of  $X$  and a  $K$ -characterized subgroup of  $X$ .*

The following stability property is clear for  $N$ -characterized subgroups, while it is not known for characterized subgroups.

**Lemma 9.** *Countable intersections of  $N$ -characterized subgroups are  $N$ -characterized.*

The next correspondence theorem was proven in [16] for characterized subgroups of compact abelian groups.

**Proposition 2.** *Let  $X$  be a topological abelian group and  $F$  a closed subgroup of  $X$ , and let  $\pi : X \rightarrow X/F$  be the canonical projection. If  $H$  is a characterized (respectively,  $K$ -characterized,  $N$ -characterized,  $T$ -characterized) subgroup of  $X/F$ , then  $\pi^{-1}(H)$  is a characterized (respectively,  $K$ -characterized,  $N$ -characterized,  $T$ -characterized) subgroup of  $X$ .*

**Proof.** Let  $\mathbf{u} = (u_n) \in \widehat{X/F}^{\mathbb{N}}$ , and consider  $\widehat{\pi} : \widehat{X/F} \rightarrow \widehat{X}$ . For every  $n \in \mathbb{N}$ , let  $v_n = \widehat{\pi}(u_n) = u_n \circ \pi \in F^\perp \leq \widehat{X}$  and  $\mathbf{v} = (v_n)$ .

(i) Assume that  $H = s_{\mathbf{u}}(X/F)$ . Then,  $\pi^{-1}(H) = s_{\mathbf{v}}(X)$ , as  $x \in s_{\mathbf{v}}(X)$  if and only if  $v_n(x) = u_n(\pi(x)) \rightarrow 0$ , and this occurs precisely when  $\pi(x) \in H$ .

(ii) Assume now that  $H$  is  $K$ -characterized, that is assume that  $H = s_{\mathbf{u}}(X/F)$  and that  $\Gamma_{\mathbf{u}}^\infty = \emptyset$ . By (i),  $\pi^{-1}(H) = s_{\mathbf{v}}(X)$ , and moreover,  $\Gamma_{\mathbf{v}}^\infty = \emptyset$ .

(iii) If  $H$  is  $N$ -characterized, then assume that  $H = n_{\mathbf{u}}(X/F)$ . By (i),  $\pi^{-1}(H) = s_{\mathbf{v}}(X)$ , and moreover,  $\pi^{-1}(H) = n_{\mathbf{v}}(X)$ , since  $v_n(x) = u_n(\pi(x)) = 0$  for every  $n \in \mathbb{N}$  precisely when  $\pi(x) \in H = n_{\mathbf{u}}(X/F)$ .

(iv) If  $H$  is  $T$ -characterized, that is if  $H = s_{\mathbf{u}}(X/F)$  and  $\mathbf{u}$  is a  $T$ -sequence, it remains to verify that  $\mathbf{v}$  is a  $T$ -sequence, as well, since  $\pi^{-1}(H) = s_{\mathbf{v}}(X)$  by (i). Let  $\tau$  be a Hausdorff group topology on  $\widehat{X/F}$ , such that  $u_n \rightarrow 0$  in  $(\widehat{X/F}, \tau)$ . By Lemma 2(i), one can identify  $\widehat{X/F}$  with the subgroup  $F^\perp$  of  $\widehat{X}$  by the algebraic monomorphism  $\psi : \widehat{X/F} \rightarrow \widehat{X}$  defined by  $\chi \mapsto \chi \circ \pi$ . Let  $\tau^*$  be the group topology on  $\widehat{X}$  having as a local base at 0 the open neighborhoods of 0 in  $(\widehat{X/F}, \tau)$ . Then,  $\tau^*$  is a Hausdorff group topology on  $\widehat{X}$  and  $v_n \rightarrow 0$  in  $(\widehat{X}, \tau^*)$ , as  $v_n = \psi(u_n) \in F^\perp$  for every  $n \in \mathbb{N}$  by definition.  $\square$

**Lemma 10.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ , such that  $n(X) \leq H$ . Then,  $H$  is characterized (respectively,  $K$ -characterized,  $N$ -characterized,  $T$ -characterized) if and only if  $H/n(X)$  is characterized (respectively,  $K$ -characterized,  $N$ -characterized,  $T$ -characterized).*

**Proof.** Let  $H = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , and denote by  $\pi : X \rightarrow X/n(X)$  the canonical projection. For every  $n \in \mathbb{N}$ , since  $n(X) \leq \ker v_n$ , the character  $v_n$  factorizes as  $v_n = u_n \circ \pi$ , where  $u_n \in \widehat{X/n(X)}$ . Then,  $H/n(X) = s_{\mathbf{u}}(X/n(X))$ . Vice versa, if  $H/n(X) = s_{\mathbf{u}}(X/n(X))$  for some  $\mathbf{u} \in \widehat{X/n(X)}^{\mathbb{N}}$ , let  $v_n = u_n \circ \pi$  for every  $n \in \mathbb{N}$ . Hence,  $H = s_{\mathbf{v}}(X)$ . Moreover,  $\mathbf{v}$  is a finitely many-to-one sequence if and only if  $\mathbf{u}$  is a finitely many-to-one sequence, so  $H$  is  $K$ -characterized if and only if  $H/n(X)$  is  $K$ -characterized. Similarly,  $\Gamma_{\mathbf{u}}^0$  is finite, precisely when  $\Gamma_{\mathbf{v}}^0$  is finite; hence,  $H$  is  $N$ -characterized if and only if  $H/n(X)$  is  $N$ -characterized.

It remains to check that  $\mathbf{v}$  is a  $T$ -sequence precisely when  $\mathbf{u}$  is a  $T$ -sequence. This follows from the fact that the natural homomorphism  $\widehat{X/n(X)} \rightarrow \widehat{X}$  sending (the members of)  $\mathbf{u}$  to (the members of)  $\mathbf{v}$  is an isomorphism, so certainly preserving the property of being a  $T$ -sequence.  $\square$

The following lemma gives equivalent conditions for a subgroup to be characterized.

**Lemma 11.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H \in \mathcal{C}\text{har}(X)$ ;
- (ii) there exists a closed subgroup  $F$  of  $X$ , such that  $F \leq H$  and  $H/F \in \mathcal{C}\text{har}(X/F)$ ;
- (iii) there exists  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , such that for every closed  $F \leq n_{\mathbf{v}}(X)$ , one has that  $H/F = s_{\mathbf{u}}(X/F)$ , where  $\mathbf{u} = (u_n)$ , and each  $u_n$  is the factorization of  $v_n$  through the canonical projection  $\pi : X \rightarrow X/F$ .

**Proof.** (iii) $\Rightarrow$ (ii) Take  $F = n_{\mathbf{v}}(X)$ .

(ii) $\Rightarrow$ (i) Since  $F \leq H$ , one has  $H = \pi^{-1}(H/F)$ , and one can conclude, by Proposition 2.

(i) $\Rightarrow$ (iii) Let  $H = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$  and  $F \leq n_{\mathbf{v}}(X)$ . Let  $\pi : X \rightarrow X/F$  be the canonical projection. For every  $n \in \mathbb{N}$ , let  $u_n$  be the character of  $X/F$  defined by  $\pi(x) \mapsto v_n(x)$ . Then,  $u_n$  is well defined, since  $F \leq n_{\mathbf{v}}(X) \leq \ker v_n$ . Hence,  $u_n \circ \pi = v_n$  for every  $n \in \mathbb{N}$ , and  $H/F = s_{\mathbf{u}}(X/F)$ . Indeed, for every  $h \in H$ , we have  $u_n(\pi(h)) = v_n(h) \rightarrow 0$ , and hence,  $H/F \leq s_{\mathbf{u}}(X/F)$ . Conversely, if  $\pi(x) \in s_{\mathbf{u}}(X/F)$ , then  $v_n(x) = u_n(\pi(x)) \rightarrow 0$ . Hence,  $x \in H$ , and so,  $\pi(x) \in H/F$ .  $\square$

### 6. Auto-Characterized Groups

The following consequence of [16, Proposition 2.5] motivates the introduction of the notion of auto-characterized group (see Definition 5).

**Proposition 3.** *Let  $X$  be a compact abelian group. Then,  $s_{\mathbf{v}}(X) = X$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$  if and only if the sequence  $\mathbf{v}$  is trivially null.*

**Proof.** It is clear from (2) that  $s_{\mathbf{v}}(X) = X$  if  $\mathbf{v}$  is trivially null. Assume now that  $X = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . By [16, Proposition 2.5], being  $s_{\mathbf{v}}(X)$  compact, there exists  $m \in \mathbb{N}$ , such that  $X = s_{\mathbf{v}}(X) = n_{\mathbf{v}(m)}(X)$ , and so,  $v_n = 0$  for all  $n \geq m$ .  $\square$

If one drops the compactness, then the conclusion of Proposition 3 may fail, as shown in the next example.

**Example 1.**

- (i) Let  $N$  be an infinite countable subgroup of  $\mathbb{T}$ . As mentioned in the Introduction,  $N$  is characterized in  $\mathbb{T}$ ; hence,  $N = s_{\mathbf{v}}(\mathbb{T})$  for a non-trivial sequence  $\mathbf{v} \in \mathbb{Z}^{\mathbb{N}}$ . If  $\mathbf{u} = \mathbf{v} \upharpoonright_N$ , then  $\mathbf{u}$  is non-trivial (since  $N$  is dense in  $\mathbb{T}$ ), and  $s_{\mathbf{u}}(N) = N$ .
- (ii) Let  $X = \mathbb{R}$ , let  $\pi : \mathbb{R} \rightarrow \mathbb{T}$  be the canonical projection and let  $\mathbf{v} = (v_n) \in \widehat{\mathbb{R}}^{\mathbb{N}}$ , such that  $v_0 = 0$  and  $v_n(x) = \pi(\frac{x}{n}) \in \mathbb{T}$  for every  $x \in \mathbb{R}$  and  $n \in \mathbb{N}_+$ . Obviously,  $s_{\mathbf{v}}(\mathbb{R}) = \mathbb{R}$ , even though  $\mathbf{v}$  is non-trivial.
- (iii) Let  $X = \mathbb{Q}_p$ , where  $p$  is a prime. For every  $n \in \mathbb{N}$ , let  $v_n = p^n \in \widehat{\mathbb{Q}_p}$ . Obviously,  $s_{\mathbf{v}}(\mathbb{Q}_p) = \mathbb{Q}_p$ , even though  $\mathbf{v}$  is non-trivial.

Motivated by Proposition 3 and Example 1, we give the following:

**Definition 5.** A topological abelian group  $X$  is *auto-characterized* if  $X = s_{\mathbf{v}}(X)$  for some non-trivial  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ .

Items (ii) and (iii) of Example 1 show that  $\mathbb{R}$  and  $\mathbb{Q}_p$  are auto-characterized.

**Remark 3.** (i) Let  $X$  be an auto-characterized group, so let  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$  be non-trivial and such that  $X = s_{\mathbf{v}}(X)$ . Then, there exists a one-to-one subsequence  $\mathbf{u}$  of  $\mathbf{v}$ , such that  $u_n \neq 0$  for every  $n \in \mathbb{N}$  and  $X = s_{\mathbf{u}}(X)$ .

Indeed, if  $\chi \in \Gamma_{\mathbf{v}}^{\infty}$ , then  $X = s_{\mathbf{v}}(X) \leq \ker \chi$ , and so,  $\chi = 0$ ; therefore,  $\Gamma_{\mathbf{v}}^{\infty}$  is either empty or  $\{0\}$ . Since  $\mathbf{v}$  is non-trivial,  $\Gamma_{\mathbf{v}}^0$  is infinite; hence,  $X = s_{\mathbf{v}}(X) = s_{\mathbf{v}^0}(X)$  by Lemma 8(ii). Let  $\mathbf{u}$  be the one-to-one subsequence of  $\mathbf{v}^0$ , such that  $\Gamma_{\mathbf{u}} = \Gamma_{\mathbf{v}^0}$ ; therefore,  $X = s_{\mathbf{u}}(X)$ .

- (ii) The above item shows that auto-characterized groups are  $K$ -characterized subgroups of themselves. However, one can prove actually that they are  $T$ -characterized subgroups of themselves (indeed,  $TB$ -characterized subgroups of themselves; see [24]).

6.1. Basic Properties of Auto-Characterized Groups

We start by a direct consequence of Lemma 10:

**Lemma 12.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ , such that  $n(X) \leq H$ . Then,  $H$  is auto-characterized if and only if  $H/n(X)$  is auto-characterized.*

The next proposition, describing an auto-characterized group in terms of the null sequences of its dual, follows from the definitions and Lemma 1:

**Proposition 4.** *A topological abelian group  $X$  is auto-characterized if and only if  $(\widehat{X}, \sigma(\widehat{X}, X))$  has a non-trivial null sequence  $\mathbf{v}$  (in such a case,  $X = s_{\mathbf{v}}(X)$ ).*

In the next lemma, we see when a subgroup of an auto-characterized group is auto-characterized, and vice versa.

**Lemma 13.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ .*

- (i) *If  $X$  is auto-characterized and  $H$  is dense in  $X$ , then  $H$  is auto-characterized.*
- (ii) *If  $H$  is auto-characterized and one of the following conditions holds, then  $X$  is auto-characterized:*
  - (a)  *$H$  is a topological direct summand of  $X$ ;*
  - (b)  *$H$  is open and has a finite index.*

**Proof.** (i) Let  $X = s_{\mathbf{v}}(X)$  for  $\mathbf{v} \in \widehat{X}$  with  $v_n \neq 0$  for every  $n \in \mathbb{N}$ , and let  $u_n = v_n \upharpoonright_H \in \widehat{H}$ . Then, each  $u_n$  is non-zero and  $H = s_{\mathbf{u}}(H)$ .

(ii) Let  $H = s_{\mathbf{v}}(H)$  for some  $\mathbf{v} \in \widehat{H}^{\mathbb{N}}$  with  $v_n \neq 0$  for every  $n \in \mathbb{N}$ .

(a) Let  $X = H \times Z$ . For every  $n \in \mathbb{N}$ , let  $u_n$  be the unique character of  $X$  that extends  $v_n$  and such that  $u_n$  vanishes on  $Z$ . Then,  $u_n \neq 0$  for every  $n \in \mathbb{N}$ , and  $X = s_{\mathbf{u}}(X)$ .

(b) Arguing by induction, we can assume without loss of generality that  $[X : H] = p$  is prime. Let  $X = H + \langle x \rangle$  with  $x \notin H$  and  $px \in H$ . If  $px = 0$ , then  $H$  is an open direct summand of  $X$ , so  $H$  is also a topological direct summand of  $X$ ; hence, item (a) applies. Assume now that  $px \neq 0$ , and let  $a_n = v_n(px)$  for every  $n \in \mathbb{N}$ . If  $a_n = 0$  for infinitely many  $n$ , extend  $v_n$  to  $u_n \in \widehat{X}^{\mathbb{N}}$  for those  $n$  by letting  $u_n(x) = 0$ . Then, obviously, the sequence  $\mathbf{u}$  obtained in this way is not trivially null and  $X = s_{\mathbf{u}}(X)$ , so  $X$  is auto-characterized. Assume now that  $a_n \neq 0$  for infinitely many  $n \in \mathbb{N}$ ; for those  $n$ , pick an element  $b_n \in \mathbb{T}$  with  $pb_n = a_n$ , and extend  $v_n$  by letting  $u_n(h + kx) = v_n(h) + kb_n$ . Let  $w_n = pu_n$ . Then,  $w_n(x) = a_n \neq 0$ , so  $\mathbf{w}$  is not trivially null. Moreover,  $X = s_{\mathbf{u}}(X)$  as  $pX \leq H$ .  $\square$

**Lemma 14.** *Let  $X$  be a topological abelian group and  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . If  $F$  is a subgroup of  $X$ , such that  $F \leq s_{\mathbf{v}}(X)$  and  $F$  is not auto-characterized, then  $F \leq n_{\mathbf{v}(m)}(X)$  for some  $m \in \mathbb{N}$ .*

**Proof.** Let  $u_n = v_n \upharpoonright_F$  for every  $n \in \mathbb{N}$  and  $\mathbf{u} = (u_n) \in \widehat{F}^{\mathbb{N}}$ . Then,  $F = s_{\mathbf{u}}(F)$ , so the sequence  $\mathbf{u}$  must be trivially null. Let  $m \in \mathbb{N}$ , such that  $u_n = 0$  for every  $n \geq m$ . Therefore,  $F \leq n_{\mathbf{v}(m)}(X)$ .  $\square$

The following consequence of Lemma 14 is a generalization of Lemma 2.6 in [16], where the group  $X$  is compact.

**Corollary 4.** *Let  $X$  be a topological abelian group,  $F$  and  $H$  subgroups of  $X$ , such that  $F$  is compact and  $F \leq H$ . Then,  $H/F \in \mathcal{C}\text{har}(X/F)$  if and only if  $H \in \mathcal{C}\text{har}(X)$ .*

**Proof.** Denote by  $\pi : X \rightarrow X/F$  the canonical projection. If  $H/F$  is a characterized subgroup of  $X/F$ , then  $H = \pi^{-1}(H/F)$  is a characterized subgroup of  $X$  by Proposition 2. Assume now that  $H = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Since  $F$  is compact, Proposition 3 implies that  $F$  is not auto-characterized. By Lemma 14,  $F$  is contained in  $n_{\mathbf{v}(m)}(X)$  for a sufficiently large  $m \in \mathbb{N}$ . Let  $\pi' : X/F \rightarrow X/n_{\mathbf{v}(m)}(X)$  be the canonical projection and  $q = \pi' \circ \pi$ , then  $q^{-1}(H/n_{\mathbf{v}(m)}(X)) = H = s_{\mathbf{v}}(X) = s_{\mathbf{v}(m)}(X)$ . Therefore, one deduces from Lemma 11 that  $H/n_{\mathbf{v}(m)}(X)$  is a characterized subgroup of  $X/n_{\mathbf{v}(m)}(X)$ . Hence, by Proposition 2,  $H/F = (\pi')^{-1}(H/n_{\mathbf{v}(m)}(X)) \in \mathcal{C}\text{har}(X/F)$ .  $\square$

The argument of the above proof fails in case  $F$  is not compact. For example, take  $F = H = X = N$ , where  $N$  is as in Example 1; then, one cannot conclude that  $\mathbf{v} \upharpoonright_F$  is trivially null and, hence, that  $F$  is contained in  $n_{\mathbf{v}(m)}(X)$ .

### 6.2. Criteria Describing Auto-Characterized Groups

Here, we give two criteria for a group to be auto-characterized. We start below with a criterion for locally compact abelian groups, while a general one, in terms of the Bohr compactification, will be given at the end of the section.

We established in Proposition 3 that no compact abelian group is auto-characterized; now, we prove in Theorem 3 that this property describes the compact abelian groups within the larger class of all locally compact abelian groups. This follows easily from Lemma 13(ii) for the locally compact abelian groups that contain a copy of  $\mathbb{R}$ , while the general case requires the following deeper argument.

**Theorem 3.** *If  $X$  is a locally compact abelian group, then  $X$  is auto-characterized if and only if  $X$  is not compact.*

**Proof.** If  $X$  is auto-characterized, then  $X$  is not compact according to Proposition 3. Assume now that  $X$  is not auto-characterized. Then, by Fact 2 and Proposition 4, the dual  $\widehat{X}$  has no non-trivial null sequences in its Bohr topology. However, since  $\widehat{X}$  is locally compact, it has the same null sequences as its Bohr modification  $\widehat{X}^+$ . Therefore,  $\widehat{X}$  is a locally compact group without non-trivial null sequences. We have to conclude that  $X$  is compact.

This follows from the conjunction of several facts. The first one is the deep result that non-discrete locally compact abelian groups have non-trivial null sequences. (This follows, in turn, from that fact that a non-discrete locally compact abelian group either contains a line  $\mathbb{R}$  or an infinite compact subgroup. Since compact groups are dyadic compacts, *i.e.*, continuous images of Cantor cubes, they have non-trivial null sequences.) Now, we can conclude that the locally compact group  $\widehat{X}$  is discrete. It is a well known fact that this implies the compactness of  $X$ .  $\square$

**Remark 4.** An alternative argument to prove that non-discrete locally compact abelian groups have non-trivial null sequences is based on a theorem by Hagler, Gerlits and Efimov (proven independently

also by Efimov in [25]). It states that every infinite compact group  $K$  contains a copy of the Cantor cube  $\{0, 1\}^{w(K)}$ , which obviously has plenty of non-trivial null sequences.

In order to obtain a general criterion describing auto-characterized groups we need another relevant notion in the theory of characterized subgroups:

**Definition 6.** [1] Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ . Let:

$$\mathfrak{g}_X(H) = \bigcap \left\{ s_{\mathbf{v}}(X) : \mathbf{v} \in \widehat{X}^{\mathbb{N}}, H \leq s_{\mathbf{v}}(X) \right\}.$$

A subgroup  $H$  of  $X$  is said to be:

- (i)  $\mathfrak{g}$ -dense if  $\mathfrak{g}_X(H) = X$ ;
- (ii)  $\mathfrak{g}$ -closed if  $\mathfrak{g}_X(H) = H$ .

We write simply  $\mathfrak{g}(H)$  when there is no possibility of confusion. Clearly,  $\mathfrak{g}(H)$  is a subgroup of  $X$  containing  $H$ . Moreover,  $\mathfrak{g}(\{0\})$  is the intersection of all characterized subgroups of  $X$  and  $\mathfrak{g}(\{0\}) \leq n(X)$ .

**Remark 5.** Let  $X = (X, \tau)$  be a topological abelian group and  $H$  a subgroup of  $X$ .

- (i) If  $X_1$  is another topological abelian group and  $\phi : X \rightarrow X_1$  a continuous homomorphism, then  $\phi(\mathfrak{g}_X(H)) \leq \mathfrak{g}_{X_1}(\phi(H))$  (see [1, Proposition 2.6]).
- (ii) Moreover,  $\mathfrak{g}_X(H) \leq \overline{H}^{\tau^+}$ . Indeed,  $\overline{H}^{\tau^+} = \bigcap \{ \ker \chi : \chi \in \widehat{X}, H \leq \ker \chi \}$  and  $\ker \chi = n_{\mathbf{v}}(X) = s_{\mathbf{v}}(X)$  for  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$  with  $\Gamma_{\mathbf{v}} = \Gamma_{\mathbf{v}}^{\infty} = \{ \chi \}$  (i.e.,  $\mathbf{v}$  is the constant sequence given by  $\chi$ ). Item (i) says, in terms of [15,26], that  $\mathfrak{g}$  is a closure operator in the category of topological abelian groups. The inclusion  $\mathfrak{g}_X(H) \leq \overline{H}^{\tau^+}$  says that  $\mathfrak{g}$  is finer than the closure operator defined by  $H \mapsto \overline{H}^{\tau^+}$ .
- (iii) If  $H$  is dually closed, then  $H$  is  $\mathfrak{g}$ -closed by item (ii).
- (iv) If  $(X, \tau)$  is a locally compact abelian group, then every closed subgroup of  $(X, \tau)$  is dually closed, and so, (ii) implies that  $\mathfrak{g}(H) \leq \overline{H}^{\tau}$  for every subgroup  $H$  of  $X$ . Therefore,  $\mathfrak{g}$ -dense subgroups are also dense in this case.
- (v) The inclusion  $\mathfrak{g}(H) \leq \overline{H}$  may fail if the group  $H$  is not MAP (e.g., if  $X$  is MinAP, then  $\mathfrak{g}(H) = X$  for every  $H$ , while  $X$  may have proper closed subgroups).

The next result shows that the auto-characterized precompact abelian groups are exactly the dense non- $\mathfrak{g}$ -dense subgroups of the compact abelian groups.

**Theorem 4.** Let  $X$  be a precompact abelian group. The following conditions are equivalent:

- (i)  $X$  is auto-characterized;
- (ii)  $X$  is not  $\mathfrak{g}$ -dense in its completion  $\widetilde{X}$ .

**Proof.** (ii) $\Rightarrow$ (i) Assume that  $X$  is not  $\mathfrak{g}$ -dense in  $K := \widetilde{X}$ . Then, there exists a sequence  $\mathbf{v} \in \widehat{K}^{\mathbb{N}}$ , such that  $X \leq s_{\mathbf{v}}(K) < K$ . By Proposition 3 (see also Remark 3), we may assume without loss of generality



that  $v_n \neq 0$  for every  $n \in \mathbb{N}$ . Let  $u_n = v_n \upharpoonright_X$  for every  $n \in \mathbb{N}$ . Since  $X$  is dense in  $K$ , clearly  $u_n \neq 0$  for every  $n \in \mathbb{N}$ . Moreover,  $X = s_{\mathbf{u}}(X)$ ; hence,  $X$  is auto-characterized.

(i) $\Rightarrow$ (ii) Suppose that  $X$  is auto-characterized, say  $X = s_{\mathbf{u}}(X)$  for  $\mathbf{u} \in \widehat{X}^{\mathbb{N}}$ , such that  $u_n \neq 0$  for every  $n \in \mathbb{N}$ . For every  $n \in \mathbb{N}$ , let  $v_n \in \widehat{K}$  be the extension of  $u_n$  to  $K$ . Then,  $X \leq s_{\mathbf{v}}(K) < K$  by Proposition 3, so  $X$  is not  $\mathfrak{g}$ -dense in  $K$ .  $\square$

If  $X$  is a MAP abelian group, then  $\tau^+$  is precompact, and the Bohr compactification of  $X$  is  $r_X : X \rightarrow bX$ , where  $bX$  is the completion of  $(X, \tau^+)$  and  $r_X$  is an injective homomorphism. If  $X$  is not MAP, then  $n(X) \neq \{0\}$ . Consider the quotient  $X/n(X)$ , which is a MAP group. Then, take the Bohr compactification  $r_{X/n(X)} : X/n(X) \rightarrow b(X/n(X))$ . The Bohr compactification of  $X$  is  $r_X : X \rightarrow bX$ , where  $bX := b(X/n(X))$  and  $r_X = r_{X/n(X)} \circ \pi$ , where  $\pi : X \rightarrow X/n(X)$  is the canonical projection.

**Corollary 5.** *Let  $X$  be a MAP abelian group. The following conditions are equivalent:*

- (i)  $X$  is auto-characterized;
- (ii)  $X$  is not  $\mathfrak{g}$ -dense in  $bX$ .

**Proof.** Since  $X$  is MAP,  $X$  embeds in  $bX$ . By Lemma 6,  $X$  and  $X^+$  have the same characterized subgroups. Moreover,  $X^+$  is precompact, and by definition,  $bX$  is the completion of  $X^+$ . Then, it suffices to apply Theorem 4.  $\square$

**Theorem 5.** *Let  $X$  be a topological abelian group. The following conditions are equivalent:*

- (i)  $X$  is auto-characterized;
- (ii)  $r_X(X)$  is not  $\mathfrak{g}$ -dense in  $bX$ .

**Proof.** Since  $X$  is auto-characterized precisely when  $X/n(X)$  is auto-characterized by Lemma 12, apply Corollary 5 to conclude.  $\square$

### 7. $K$ -Characterized Subgroups

We start by recalling [23, Lemma 3.19]: if  $X$  is a compact abelian group and  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$  is a one-to-one sequence, then  $s_{\mathbf{v}}(X)$  has Haar measure zero in  $X$ . Since  $K$ -characterized subgroups are characterized by finitely many-to-one sequences (which obviously contain a one-to-one subsequence), this result applies to  $K$ -characterized subgroups and gives the following (formally weaker) result, which will be necessary and more convenient to apply in the current paper:

**Lemma 15.** *If  $X$  is a compact abelian group and  $H \in \mathfrak{Char}_K(X)$ , then  $H$  has Haar measure zero (hence,  $[X : H]$  is infinite). In particular, no open subgroup of  $X$  is  $K$ -characterized.*

Lemma 15 cannot be inverted; take, for example, the constant sequence  $\mathbf{u} = (1)$  in  $\widehat{\mathbb{T}}^{\mathbb{N}}$ .

**Remark 6.** If  $X$  is a connected compact abelian group, then the conclusion of Lemma 15 holds for all non-trivial sequences in  $\widehat{X}$ , since every measurable proper subgroup  $H$  of  $X$  has measure zero (indeed,  $X$  is divisible, so the proper subgroup  $H$  of  $X$  has infinite index; hence, the measure of  $H$  must be 0, as  $X$  has measure 1).

**Example 2.** Here, we provide examples of non-auto-characterized non-compact abelian groups.

- (i) A relatively simple example can be obtained by taking a dense non-measurable subgroup  $X$  of a connected compact abelian group  $K$ . Since we intend to deduce that  $X$  is not auto-characterized by using Theorem 4, we have to check that  $X$  is  $\mathfrak{g}$ -dense in  $K$ . Indeed, every measurable proper subgroup of  $K$  has measure zero as noted in Remark 6; therefore, every proper characterized (hence, every non- $\mathfrak{g}$ -dense) subgroup of  $K$  has measure zero. Therefore,  $X$  is not contained in any proper characterized subgroup of  $K$ , i.e.,  $X$  is  $\mathfrak{g}$ -dense in  $K$ .
- (ii) More sophisticated examples of  $\mathfrak{g}$ -dense subgroups of metrizable compact abelian groups were given in [27] (under the assumption of Martin Axiom) and in [19] (in ZFC). These groups have the additional property of being of measure zero (so that the above elementary argument cannot be used to verify the  $\mathfrak{g}$ -density).

7.1. When Closed Subgroups of Infinite Index are  $K$ -Characterized

The next theorem gives a sufficient condition (see item (iii)) for a closed subgroup of infinite index  $H$  to be  $K$ -characterized. This condition implies, as a by-product, that  $H$  is also  $N$ -characterized.

The easier case of open subgroups will be tackled in Theorem 7, by applying Theorem 6.

**Theorem 6.** *Let  $X$  be a topological abelian group and  $H$  a closed subgroup of  $X$  of infinite index. The following conditions are equivalent:*

- (i)  $H \in \mathfrak{Char}_K(X) \cap \mathfrak{Char}_N(X)$ ;
- (ii)  $H \in \mathfrak{Char}_N(X)$ ;
- (iii)  $X/H$  is MAP, and  $(\widehat{X/H}, \sigma(\widehat{X/H}, X/H))$  is separable.

**Proof.** (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) Since  $H$  is  $N$ -characterized, then  $H$  is dually closed by Lemma 7(ii), that is  $X/H$  is MAP. Let  $H = n_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , and let  $\pi : X \rightarrow X/H$  be the canonical projection. Since  $\ker \pi = H \leq \ker v_n$  for every  $n \in \mathbb{N}$ , one can factorize  $v_n : X \rightarrow \mathbb{T}$  through  $\pi$ , i.e., write  $v_n = \bar{v}_n \circ \pi$  for appropriate  $\bar{v}_n \in \widehat{X/H}$ . It remains to verify that  $D = \{\bar{v}_n : n \in \mathbb{N}\}$  is dense in  $(\widehat{X/H}, \sigma(\widehat{X/H}, X/H))$ . To this end, let  $\bar{y} = \pi(y) \in X/H$ ; if  $\xi_{\bar{y}}(D) = \{0\}$ , then  $\bar{v}_n(\bar{y}) = v_n(y) = 0$ , and so,  $y \in H$ , that is,  $\bar{y} = 0$ .

(iii) $\Rightarrow$ (i) Let  $Y = X/H$ , equipped with the quotient topology. By hypotheses,  $Y$  is infinite and MAP, while  $\widehat{Y}$  is an infinite topological abelian group with a countably infinite dense subgroup  $D$ . According to Proposition 2 applied to the canonical projection  $\pi : X \rightarrow Y$ , it suffices to prove that  $\{0\}$  is a  $K$ -characterized subgroup of  $Y$ . Let  $D = \{v_n : n \in \mathbb{N}\}$  be a one-to-one enumeration of  $D$  and  $\mathbf{v} = (v_n)$ . To prove that  $s_{\mathbf{v}}(Y) = \{0\}$ , we have to show that for every non-zero  $y \in Y$ , there exists a neighborhood  $U$  of 0 in  $\mathbb{T}$ , such that  $v_n(y) \notin U$  for infinitely many  $n \in \mathbb{N}$ . Actually, we show that  $U = \mathbb{T}_+$  works for all non-zero  $y \in Y$ . In fact, for every  $y \in Y \setminus \{0\}$ , one has that  $N_y := \{d(y) : d \in D\} = \{v_n(y) : n \in \mathbb{N}\}$  is a non-trivial subgroup of  $\mathbb{T}$ , as  $Y$  is MAP and  $y \neq 0$ .

Let  $y \in Y \setminus \{0\}$ . If  $N_y$  is infinite, then  $N_y$  is dense in  $\mathbb{T}$ ; so,  $N_y \setminus U$  is infinite, and we are done. Now, consider the case when  $N_y$  is finite. As  $N_y \neq \{0\}$  and  $U$  contains no non-trivial subgroups of  $\mathbb{T}$ , there

exists  $a \in N_y$ , such that  $a \notin U$ . Then, the map  $f_y : D \rightarrow \mathbb{T}$  defined by  $f_y(d) = d(y)$  is a homomorphism with  $f_y(D) = N_y$  finite. Therefore,  $K := \ker f_y$  is a finite-index subgroup of  $D$ , so  $K$  is infinite. Let  $a = v_m(y)$  for some  $m \in \mathbb{N}$ . Then,  $v_m + K = \{d \in D : d(y) = a\}$  is infinite, as well. This means that  $v_n(y) = a \notin U$  for infinitely many  $n$  (namely, those  $n$  for which  $v_n \in v_m + K$ ). Therefore,  $v_n(y) \not\rightarrow 0$ , and so,  $y \notin s_v(Y)$ .

Finally, let us note that the above argument shows also that  $H$  is  $N$ -characterized, as obviously  $H \leq n_v(X)$ .  $\square$

The following is an obvious consequence of Theorem 6.

**Corollary 6.** *Let  $X$  be a topological abelian group and  $H$  a closed subgroup of  $X$  of infinite index. If  $H \in \mathcal{C}\text{har}_N(X)$ , then  $H \in \mathcal{C}\text{har}_K(X)$ .*

Next, we rewrite Theorem 6 in the case of locally compact abelian groups.

**Corollary 7.** *Let  $X$  be a locally compact abelian group and  $H$  a subgroup of  $X$ . Then,  $H \in \mathcal{C}\text{har}_N(X)$  if and only if  $H$  is closed and  $\widehat{X/H}$  is separable.*

**Proof.** As both conditions imply that  $H$  is closed, we assume without loss of generality that  $H$  is closed. Since  $X/H$  and  $\widehat{X/H}$  are locally compact abelian groups,  $X/H$  is MAP, and the Bohr topology on  $\widehat{X/H}$  coincides with  $\sigma(\widehat{X/H}, X/H)$  by Fact 2, so the separability of  $\widehat{X/H}$  is equivalent to the separability of  $\widehat{X/H}^+$  by Fact 1. If  $H$  has a infinite index in  $X$ , apply Theorem 6 to conclude. If  $H$  has a finite index in  $X$ , then the equivalence is trivially satisfied; indeed,  $H$  is a finite intersection of kernels of characters, so it is  $N$ -characterized, and  $\widehat{X/H}$  is finite, so separable.  $\square$

As a consequence of Theorem 6, we find a sufficient condition for an open subgroup of infinite index to be  $K$ -characterized:

**Theorem 7.** *Let  $X$  be a topological abelian group and  $H$  an open subgroup of  $X$  of infinite index. Then, the following conditions are equivalent:*

- (i)  $H \in \mathcal{C}\text{har}(X)$ ;
- (ii)  $H \in \mathcal{C}\text{har}_K(X)$ ;
- (iii)  $[X : H] \leq \mathfrak{c}$ ;
- (iv)  $\widehat{X/H}$  is separable.

**Proof.** (ii) $\Rightarrow$ (i) is clear, and (i) $\Rightarrow$ (iii) is Corollary 1.

(iii) $\Rightarrow$ (iv) Since  $\widehat{X/H}$  is a compact abelian group of weight at most  $\mathfrak{c}$ , it is separable by Lemma 3.

(iv) $\Rightarrow$ (ii) As  $[X : H]$  is infinite, we can apply Theorem 6 to conclude that  $H$  is  $K$ -characterized.  $\square$

The following is another direct consequence of Theorem 6.

**Corollary 8.** *If  $X$  is a metrizable compact abelian group, then every closed non-open subgroup of  $X$  is  $K$ -characterized.*

7.2. When Closed Subgroups of Finite Index are  $K$ -Characterized

We start by giving the following useful technical lemma.

**Lemma 16.** *Let  $X$  be a topological abelian group and  $H$  an open subgroup of  $X$ , such that  $X = H + \langle x \rangle$  for some  $x \in X \setminus H$ . If  $H$  is auto-characterized, then  $H \in \mathcal{Char}_K(X)$ .*

**Proof.** Let  $\mathbf{u} \in \widehat{H}^{\mathbb{N}}$ , such that  $H = s_{\mathbf{u}}(H)$ . By Remark 3, we can assume that  $\mathbf{u}$  is one-to-one and that  $u_n \neq 0$  for every  $n \in \mathbb{N}$ .

Assume first that  $H \cap \langle x \rangle = \{0\}$ . If  $o(x)$  is infinite, then fix an irrational number  $\alpha \in \mathbb{R}$ , and for every  $n \in \mathbb{N}$ , let  $v_n(x) = \alpha + \mathbb{Z}$  and  $v_n(h) = u_n(h)$  for every  $h \in H$ . If  $o(x) = k$  is finite, then for every  $n \in \mathbb{N}$ , let  $v_n(x) = \frac{1}{k} + \mathbb{Z}$  and  $v_n(h) = u_n(h)$  for every  $h \in H$ . In both cases, it is straightforward to prove that  $H = s_{\mathbf{v}}(X)$ . Moreover, since  $\mathbf{u}$  is one-to-one, then also  $\mathbf{v}$  is one-to-one.

Suppose now that  $H \cap \langle x \rangle = \langle mx \rangle$  for some  $m \in \mathbb{N}$ , with  $m \geq 1$ . As  $x \notin H$ , one has  $m \geq 2$ .

For every  $n \in \mathbb{N}$ , let  $a_n = u_n(mx) \in \mathbb{T}$ . Since  $u_n(mx) \rightarrow 0$ , there exists  $n_0$ , such that  $\|a_n\| < \frac{1}{m^2}$  for every  $n \geq n_0$ . As  $s_{\mathbf{u}_{(n_0)}}(H) = H$ , we shall assume for simplicity that  $\|a_n\| < \frac{1}{m^2}$  every  $n \in \mathbb{N}$ .

**Claim 1.** For every  $a \in \mathbb{T}$  with  $\|a\| < \frac{1}{m^2}$ , there exists  $b \in \mathbb{T}$ , such that

$$mb = a \text{ and } \|kb\| > \frac{1}{m^2} \text{ for every } k \in \mathbb{N}, 1 \leq k < m. \tag{5}$$

**Proof.** We tackle the problem in  $\mathbb{R}$ , that is identifying  $\mathbb{T}$  with  $[0, 1)$ . First, assume that  $0 \leq a < \frac{1}{m^2}$ , and let:

$$b = \frac{a}{m} + \frac{1}{m}$$

Then,  $mb = a + 1 \equiv_{\mathbb{Z}} a$  and  $\frac{1}{m} \leq b \leq \frac{2}{m}$ . Let now  $k \in \mathbb{N}$  with  $1 \leq k \leq m - 1$ , then:

$$\frac{1}{m^2} < \frac{k}{m} \leq kb = k\frac{a}{m} + \frac{k}{m} < \frac{m-1}{m^2} + \frac{m-1}{m} = 1 - \frac{1}{m^2} \tag{6}$$

Therefore,  $\|kb\| > \frac{1}{m^2}$ . This establishes condition (5) in the current case.

It remains to consider the case  $\frac{m^2-1}{m^2} < a < 1$ . Let  $a^* = 1 - a$ , i.e.,  $a^* = -a$  in  $\mathbb{T}$ . Then, obviously,  $\|a^*\| < \frac{1}{m^2}$  and  $0 \leq a^* < \frac{1}{m^2}$ . Hence, by the above case applied to  $a^*$ , there exists  $b^* \in \mathbb{T}$  satisfying condition (5) with  $-a$  in place of  $a$  (i.e.,  $mb^* = -a$ ). Let  $b = -b^* \in \mathbb{T}$ . Then, condition (5) holds true for  $b$  and  $a$ , as  $\|k(-b)\| = \|kb\|$  for every  $k \in \mathbb{N}$  with  $1 \leq k < m$ .  $\square$

For every  $n \in \mathbb{N}$ , apply Claim 1 to  $a_n$  to get  $b_n$  as in Equation (5), then define  $v_n : X \rightarrow \mathbb{T}$  by letting  $v_n(x) = b_n$  for every  $x \in X \setminus H$  and  $v_n(h) = u_n(h)$  for every  $h \in H$ . As  $u_n(mx) = v_n(mx) = mv_n(x) = mb_n = a_n$ , this definition is correct. Moreover, since  $H$  is open in  $X$ ,  $v_n \in \widehat{X}$ . Since  $\mathbf{u}$  is one-to-one, then  $\mathbf{v} = (v_n)$  is one-to-one, too.

We show that

$$v_n(kx) \rightarrow 0 \text{ for } k \in \mathbb{N} \text{ if and only if } k \in m\mathbb{Z}. \tag{7}$$

In fact, if  $k = k'm$  for some  $k' \in \mathbb{N}$ ,

$$v_n(kx) = k'v_n(mx) = k'a_n \rightarrow 0.$$

The other way around, assume that  $k = k'm + r$ , where  $k' \in \mathbb{N}$  and  $1 \leq r \leq m - 1$ . Then,

$$v_n(kx) = k'v_n(mx) + rv_n(x) = k'a_n + rb_n \not\rightarrow 0$$

since  $k'a_n \rightarrow 0$  and  $\|rb_n\| \geq \frac{1}{m^2}$  by Equation (5).

We deduce finally that  $H = s_v(X)$ . Indeed,  $H = s_u(H) \leq s_v(X)$ , so it remains to prove that  $s_v(X) \leq H$ . To this end, let  $y \in X \setminus H$ , that is  $y = h + kx$  for some  $h \in H$  and  $k \in \mathbb{N}$  with  $1 \leq k < m$ . Then,

$$v_n(y) = v_n(h + kx) = u_n(h) + v_n(kx).$$

Since  $h \in H = s_u(H)$ , that is,  $u_n(h) \rightarrow 0$ , while  $v_n(kx) \not\rightarrow 0$  by condition (7), we conclude that  $v_n(y) \not\rightarrow 0$ , that is,  $y \notin s_v(X)$ . Hence,  $H = s_v(X)$ .  $\square$

Every open finite-index subgroup is a finite intersection of kernels of characters, so it is  $N$ -characterized. In the next theorem we describe when a proper open finite-index subgroup is  $K$ -characterized.

**Theorem 8.** *Let  $X$  be a topological abelian group and  $H$  a proper open subgroup of  $X$  of finite index. Then,  $H \in \mathcal{C}\text{har}_K(X)$  if and only if  $H$  is auto-characterized.*

**Proof.** Assume that  $H \in \mathcal{C}\text{har}_K(X)$ . We can write  $H = s_u(X)$  for  $u \in \widehat{X}^{\mathbb{N}}$  one-to-one. Let  $v_n = u_n \upharpoonright_H$  for every  $n \in \mathbb{N}$ . Then, the map  $u_n \mapsto v_n$  is finitely many-to-one, as  $X/H$  is finite. Therefore,  $v = (v_n)$  is finitely many-to-one. Obviously,  $H = s_v(H)$ , so  $H$  is auto-characterized.

Now, assume that  $H$  is auto-characterized. Since  $H$  has finite index in  $X$ , there exist  $x_1, \dots, x_n \in X$ , such that  $X = H + \langle x_1, \dots, x_n \rangle$  and that, letting  $X_i := H + \langle x_1, \dots, x_i \rangle$  for  $i = 1, \dots, n$  and  $X_0 := H$ , the subgroup  $X_{i-1}$  is a proper subgroup of  $X_i$  for  $i = 1, \dots, n$ . We shall prove by induction on  $i = 1, \dots, n$ , that

$$H \in \mathcal{C}\text{har}_K(X_i). \tag{8}$$

As  $X = X_n$ , this will give  $H \in \mathcal{C}\text{har}_K(X)$ , as desired.

Before starting the induction, we note that according to Lemma 13(ii), all subgroups  $X_i$ , for  $i = 1, \dots, n$ , are auto-characterized, as each  $X_{i-1}$  is open in  $X_i$ . For  $i = 1$ , the assertion in condition (8) follows from Lemma 16. Assume that  $1 < i \leq n$  and condition (8) holds true for  $i - 1$ , i.e.,  $H \in \mathcal{C}\text{har}_K(X_{i-1})$ . Since  $X_{i-1}$  is open in  $X_i$ , again Lemma 16 applied to  $X_i = X_{i-1} + \langle x_i \rangle$  gives that  $X_{i-1} \in \mathcal{C}\text{har}_K(X_i)$ . As  $H \in \mathcal{C}\text{har}_K(X_{i-1})$  by our inductive hypothesis, we conclude with Corollary 2(i) that  $H \in \mathcal{C}\text{har}_K(X_i)$ .  $\square$

### 7.3. Further Results on $K$ -Characterized Subgroups

The next corollary resolves an open question from [28]:

**Corollary 9.** *Let  $X$  be an infinite discrete abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H \in \mathcal{C}\text{har}(X)$ ;
- (ii)  $H \in \mathcal{C}\text{har}_K(X)$ ;

(iii)  $[X : H] \leq c$ .

**Proof.** (ii) $\Rightarrow$ (i) is clear, and (i) $\Rightarrow$ (iii) is Corollary 1.

(iii) $\Rightarrow$ (ii) If  $[X : H]$  is infinite, then  $H \in \mathcal{C}\text{har}(X)$  by Theorem 7. Therefore, assume that  $[X : H]$  is finite, then  $H$  is infinite, and hence,  $H$  is auto-characterized by Theorem 3; therefore,  $H \in \mathcal{C}\text{har}_K(X)$  by Theorem 8.  $\square$

We give now sufficient conditions for a non-closed characterized subgroup to be  $K$ -characterized.

**Theorem 9.** Let  $X$  be a topological abelian group and  $H \in \mathcal{C}\text{har}(X)$  a non-closed subgroup of  $X$ , such that:

- (i)  $X/\overline{H}$  is MAP, and  $(\widehat{X/\overline{H}}, \sigma(\widehat{X/\overline{H}}, X/\overline{H}))$  is separable;
- (ii) if  $1 < [X : \overline{H}] < \omega$ , then  $\overline{H}$  is auto-characterized.

Then,  $H \in \mathcal{C}\text{har}_K(X)$ .

**Proof.** As  $H \neq \overline{H}$  is dense in  $\overline{H}$  and obviously  $H \in \mathcal{C}\text{har}(\overline{H})$ , we deduce that  $H \in \mathcal{C}\text{har}_K(\overline{H})$ , as dense characterized subgroups are  $TB$ -characterized by Lemma 1(ii). If  $\overline{H} = X$ , we are done. Therefore, assume that  $\overline{H}$  is proper.

Our aim now is to apply Corollary 2, so we need to check that  $\overline{H} \in \mathcal{C}\text{har}_K(X)$ . If  $\overline{H}$  has finite index in  $X$ , then  $\overline{H}$  is auto-characterized by hypothesis, and so, Theorem 8 yields  $\overline{H} \in \mathcal{C}\text{har}_K(X)$ . If  $\overline{H}$  has infinite index in  $X$ , then  $\overline{H} \in \mathcal{C}\text{har}_K(X)$  by Theorem 6.  $\square$

**Corollary 10.** Let  $X$  be a divisible topological abelian group and  $H \in \mathcal{C}\text{har}(X)$  a non-closed subgroup of  $X$ , such that  $\overline{H}$  is dually closed. If  $(\widehat{X/\overline{H}}, \sigma(\widehat{X/\overline{H}}, X/\overline{H}))$  is separable, then  $H \in \mathcal{C}\text{har}_K(X)$ . In particular,  $H \in \mathcal{C}\text{har}_K(X)$  whenever  $\widehat{X/\overline{H}}$  is separable.

**Proof.** The first part of our hypothesis entails that  $X/\overline{H}$  is MAP. Moreover, divisible topological abelian groups have no proper closed subgroup of finite index. Therefore, the first assertion follows directly from Theorem 9.

The topology  $\sigma(\widehat{X/\overline{H}}, X/\overline{H})$  of the dual  $\widehat{X/\overline{H}}$  is coarser than the compact-open topology of  $\widehat{X/\overline{H}}$ , so that the separability of  $\widehat{X/\overline{H}}$  yields the separability of  $(\widehat{X/\overline{H}}, \sigma(\widehat{X/\overline{H}}, X/\overline{H}))$ . Hence, the second assertion can be deduced from the first one.  $\square$

In the case of connected locally compact abelian groups, one obtains the following stronger conclusion:

**Corollary 11.** Let  $X$  be a connected locally compact abelian group. Then:

$$\mathcal{C}\text{har}_K(X) = \begin{cases} \mathcal{C}\text{har}(X), & \text{if } X \text{ is not compact} \\ \mathcal{C}\text{har}(X) \setminus \{X\}, & \text{if } X \text{ is compact.} \end{cases}$$

**Proof.** The group  $X$  is divisible, as connected locally compact abelian groups are divisible.

Let  $H \in \mathcal{C}\text{har}(X)$ . Our next aim will be to check that  $\widehat{X/\overline{H}}$  is separable. Indeed, if  $H = s_{\mathbf{v}}(X)$  for  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , then one has the chain of subgroups  $n_{\mathbf{v}}(X) \leq H \leq \overline{H} \leq X$ . If  $\overline{H} = X$ , then  $\widehat{X/\overline{H}}$  is trivially

separable. Otherwise,  $\overline{H}$  has infinite index in  $X$ , since  $X/\overline{H}$  is divisible, so  $\widehat{X/n_{\mathbf{v}}(X)}$  is separable by Corollary 7; since  $\overline{H}$  contains  $n_{\mathbf{v}}(X)$ , the quotient group  $X/\overline{H}$  is a quotient group of  $X/n_{\mathbf{v}}(X)$ . Therefore,  $\widehat{X/\overline{H}}$  is isomorphic to a subgroup of the separable group  $\widehat{X/n_{\mathbf{v}}(X)}$ , so  $\widehat{X/\overline{H}}$  is separable, as well (see [29]). Furthermore,  $\overline{H}$  is dually closed (so  $X/\overline{H}$  is MAP) by Fact 1.

If  $H$  is not closed, Corollary 10 gives that  $H \in \mathcal{C}\text{har}_K(X)$ . If  $H$  is a proper closed subgroup of  $X$ , then  $H$  has infinite index, as  $X/H$  is divisible, so  $H \in \mathcal{C}\text{har}_K(X)$  by Theorem 6. This proves the inclusion  $\mathcal{C}\text{har}(X) \setminus \{X\} \subseteq \mathcal{C}\text{har}_K(X) \setminus \{X\}$ , which, along with the obvious inclusion  $\mathcal{C}\text{har}_K(X) \subseteq \mathcal{C}\text{har}(X)$ , proves the equality  $\mathcal{C}\text{har}(X) \setminus \{X\} = \mathcal{C}\text{har}_K(X) \setminus \{X\}$ .

It remains to consider the (closed) subgroup  $H = X$ , which obviously belongs to  $\mathcal{C}\text{har}(X)$ . If  $X$  is compact, then  $X \notin \mathcal{C}\text{har}_K(X)$ , by Lemma 15, so  $\mathcal{C}\text{har}_K(X) = \mathcal{C}\text{har}(X) \setminus \{X\}$ . If  $X$  is not compact, then  $H = X \in \mathcal{C}\text{har}_K(X)$  by Theorem 3 and Remark 3(ii). Hence,  $\mathcal{C}\text{har}_K(X) = \mathcal{C}\text{har}(X)$  in this case.  $\square$

In particular, the above corollary yields  $\mathcal{C}\text{har}_K(\mathbb{T}) = \mathcal{C}\text{har}(\mathbb{T}) \setminus \{\mathbb{T}\}$  and  $\mathcal{C}\text{har}_K(\mathbb{R}) = \mathcal{C}\text{har}(\mathbb{R})$ .

**Remark 7.** As we shall see in Corollary 18, connectedness is necessary in this corollary.

### 8. $N$ -Characterized Subgroups

The following consequence of Lemma 14 gives a sufficient condition for a characterized subgroup to be  $N$ -characterized:

**Corollary 12.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ , which is not auto-characterized. If  $H \in \mathcal{C}\text{har}(X)$ , then  $H \in \mathcal{C}\text{har}_N(X)$ .*

**Proof.** Let  $H = s_{\mathbf{v}}(X)$  for some  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Then,  $H \leq n_{\mathbf{v}(m)}(X)$  for some  $m \in \mathbb{N}$  by Lemma 14. Since  $H = s_{\mathbf{v}}(X) = s_{\mathbf{v}(m)}(X) \geq n_{\mathbf{v}(m)}(X)$ , we deduce that  $H = n_{\mathbf{v}(m)}(X)$ .  $\square$

Here comes an easy criterion establishing when a subgroup is  $N$ -characterized.

**Theorem 10.** *Let  $X$  be a topological abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H$  is closed, and there exists a continuous injection  $X/H \rightarrow \mathbb{T}^{\mathbb{N}}$ ;
- (ii)  $H \in \mathcal{C}\text{har}_N(X)$ ;
- (iii)  $H$  is closed, and  $\{0\}$  is  $G_{\delta}$  in  $(X/H)^+$ .

**Proof.** (i) $\Rightarrow$ (ii) Suppose that there exists a continuous injection  $j : X/H \rightarrow \mathbb{T}^{\mathbb{N}}$ . Let  $\pi : X \rightarrow X/H$  be the canonical projection. For every  $n \in \mathbb{N}$ , let  $p_n : \mathbb{T}^{\mathbb{N}} \rightarrow \mathbb{T}$  be  $n$ -th projection, and let  $v_n = p_n \circ j \circ \pi$ .

$$\begin{array}{ccc}
 X/H & \xrightarrow{j} & \mathbb{T}^{\mathbb{N}} \\
 \pi \uparrow & & \downarrow p_n \\
 X & \xrightarrow{v_n} & \mathbb{T}
 \end{array}$$

Therefore,  $v_n \in \widehat{X}^{\mathbb{N}}$  for every  $n \in \mathbb{N}$  and  $H = n_{\mathbf{v}}(X)$ , where  $\mathbf{v} = (v_n)$ .

(ii)⇒(i) Let  $H = n_{\mathbf{v}}(X)$  for  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ . Let  $\pi : X \rightarrow X/n_{\mathbf{v}}(X)$  be the canonical projection, and define  $j : X/n_{\mathbf{v}}(X) \rightarrow \mathbb{T}^{\mathbb{N}}$  by  $j(\pi(x)) = (v_n(x))_{n \in \mathbb{N}}$  for every  $x \in X$ . Since  $n_{\mathbf{v}}(X) = \bigcap_{n \in \mathbb{N}} \ker v_n$ , then  $j$  is well defined and injective. Moreover,  $j$  is continuous.

Finally, (i) and (iii) are obviously equivalent. □

The above criterion simplifies in the case of open subgroups:

**Corollary 13.** *Let  $X$  be a topological abelian group and  $H$  an open subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}(X)$ ;
- (ii)  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}_N(X)$ ;
- (iii)  $[X : H] \leq \mathfrak{c}$ .

**Proof.** (ii)⇒(i) is obvious, and (i)⇒(iii) is Corollary 1.

(iii)⇒(ii) Since  $H$  is open,  $X/H$  is discrete. By hypothesis  $|X/H| \leq \mathfrak{c}$ , so there exists a continuous injection  $X/H \rightarrow \mathbb{T}^{\mathbb{N}}$ . Hence,  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}_N(X)$  by Theorem 10. □

The next is another consequence of Theorem 10.

**Corollary 14.** *Let  $X$  be a metrizable precompact abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H$  is closed;
- (ii)  $H$  is closed and  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}(X)$ ;
- (iii)  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}_N(X)$ .

**Proof.** (iii)⇒(ii) and (ii)⇒(i) are clear.

(i)⇒(iii) Since  $X$  is metrizable,  $X/H$  is metrizable, as well, and so,  $\{0\}$  is  $G_{\delta}$  in  $X/H$ . By Lemma 4, there exists a continuous injective homomorphism  $X/H \rightarrow \mathbb{T}^{\mathbb{N}}$ . Therefore,  $H$  is  $N$ -characterized by Theorem 10. □

According to Theorem 6, one can add to the equivalent conditions in Corollaries 13 and 14 also “ $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}_K(X)$ ”, in case  $[X : H] \geq \omega$ .

In the sequel, we consider the case of locally compact abelian groups. The following theorem was proven in Theorem B of [16] for compact abelian groups.

**Theorem 11.** *Let  $H$  be a locally compact abelian group and  $H$  a subgroup of  $X$ . Then,  $H \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}(X)$  if and only if  $H$  contains a closed  $G_{\delta}$ -subgroup  $K$  of  $X$ , such that  $H/K \in \mathcal{C}\mathcal{h}\mathcal{a}\mathcal{r}(X/K)$ , where  $X/K$  is a metrizable locally compact abelian group.*

**Proof.** The equivalence follows from Lemma 11, since the subgroup  $n_{\mathbf{v}}(X)$  is closed and  $G_{\delta}$  for every  $\mathbf{v} \in \widehat{X}^{\mathbb{N}}$ , by Lemma 7(iv). Since  $K$  is closed and  $G_{\delta}$ ,  $X/K$  is a metrizable locally compact abelian group. □



By Lemma 7,  $n_v(X)$  is always closed and characterized. Theorem 12 describes the closed characterized subgroups of the locally compact abelian groups  $X$  by showing that these are precisely the  $N$ -characterized subgroups of  $X$ .

**Theorem 12.** *Let  $X$  be a locally compact abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H$  is closed and  $H \in \mathcal{C}\text{har}(X)$ ;
- (ii)  $H \in \mathcal{C}\text{har}_N(X)$ ;
- (iii)  $H$  is closed and  $G_\delta$  in the Bohr topology;
- (iv)  $H$  is closed,  $G_\delta$  and  $[X : H] \leq \mathfrak{c}$ .
- (v)  $H$  is closed, and  $\widehat{X/H}$  is separable.

**Proof.** (iii) $\Rightarrow$ (ii) follows from Theorem 10, (ii) $\Rightarrow$ (i) by Lemma 7, and (ii) $\Leftrightarrow$ (v) is Corollary 7.

(iv) $\Rightarrow$ (iii) The group  $X/H$  is locally compact, metrizable and has cardinality at most  $\mathfrak{c}$ ; therefore, by Theorem 2, there exists a continuous injective homomorphism  $j : X/H \rightarrow \mathbb{T}^{\mathbb{N}}$ . Then, Theorem 10 gives the thesis.

(i) $\Rightarrow$ (iv) Let  $H = s_v(X)$  for  $v \in \widehat{X}^{\mathbb{N}}$ , and let  $\pi : X \rightarrow X/n_v(X)$  be the canonical projection. By Corollary 1,  $[X : H] \leq \mathfrak{c}$ . By Lemma 7,  $n_v(X) \leq s_v(X)$ , and  $n_v(X)$  is closed and  $G_\delta$ . Then,  $X/n_v(X)$  is a metrizable locally compact abelian group, and by hypothesis,  $s_v(X)$  is closed. Therefore,  $s_v(X)/n_v(X)$  is closed and, hence,  $G_\delta$  in  $X/n_v(X)$ . Therefore,  $s_v(X) = \pi^{-1}(s_v(X)/n_v(X))$  is closed and  $G_\delta$  in  $X$ .  $\square$

We see now the following result from [16] as a consequence of Theorem 12.

**Corollary 15.** [16, Theorem A] *Let  $X$  be a compact abelian group and  $H$  a closed subgroup of  $X$ . Then,  $H \in \mathcal{C}\text{har}(X)$  if and only if  $H$  is  $G_\delta$ .*

**Proof.** If  $H$  is characterized, then  $H$  is  $G_\delta$  by Theorem 12. *Vice versa*, assume that  $H$  is  $G_\delta$ . Then,  $X/H$  is compact and metrizable, hence  $|X/H| \leq \mathfrak{c}$ . Therefore, again Theorem 12 implies that  $H$  is characterized.  $\square$

In the following theorem, we use that the  $G_\delta$ -subgroups of a locally compact abelian group are always closed (see [28] (Theorem A.2.14)).

**Theorem 13.** *Let  $X$  be a compact abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H \in \mathcal{C}\text{har}_K(X)$  and  $H$  is closed;
- (ii)  $H$  is  $G_\delta$  and non-open;
- (iii)  $H \in \mathcal{C}\text{har}_N(X)$  and  $H$  is non-open;
- (iv)  $H \in \mathcal{C}\text{har}(X)$  and  $H$  is closed and non-open.

**Proof.** (i) $\Rightarrow$ (iv) Since Lemma 15 implies that  $H$  is non-open, (iv) $\Leftrightarrow$ (iii) by Theorem 12, and (iv) $\Leftrightarrow$ (ii) by Corollary 15.

(ii)⇒(i) Since  $X/H$  is a metrizable compact non-discrete (hence infinite) abelian group,  $\{0\}$  is closed and non-open in  $X/H$ ; hence,  $\{0\}$  is  $K$ -characterized in  $X/H$  by Corollary 8. Therefore,  $H$  is  $K$ -characterized by Proposition 2.  $\square$

This theorem generalizes Corollary 15 as it implies that, for a closed non-open subgroup  $H$  of a compact abelian group  $X$ , one has:

$$H \text{ is } G_\delta \Leftrightarrow H \in \mathfrak{Char}(X) \Leftrightarrow H \in \mathfrak{Char}_N(X) \Leftrightarrow H \in \mathfrak{Char}_K(X).$$

The following immediate consequence of Corollaries 13 and 9 shows that for a discrete abelian group all characterized subgroups are  $N$ -characterized.

**Corollary 16.** *Let  $X$  be an infinite discrete abelian group and  $H$  a subgroup of  $X$ . The following conditions are equivalent:*

- (i)  $H \in \mathfrak{Char}(X)$ ;
- (ii)  $H \in \mathfrak{Char}_N(X)$ ;
- (iii)  $H \in \mathfrak{Char}_K(X)$ ;
- (iv)  $[X : H] \leq \mathfrak{c}$ .

### 9. $T$ -Characterized Closed Subgroups of Compact Abelian Groups

In [30, Theorem 4], Gabrielyan observed that if  $\mathbf{u}$  is a  $T$ -sequence of an infinite countable abelian group  $G$ , then (see formula (1) for the definition of  $\sigma_{\mathbf{v}}$ )

$$n(G, \sigma_{\mathbf{v}}) \cong s_{\mathbf{v}}(\widehat{G_d})^\perp \text{ algebraically,}$$

where  $G_d$  denotes the abelian group  $G$  endowed with the discrete topology. Therefore, the following fact is an immediate corollary of this result.

**Fact 4.** *Let  $\mathbf{v}$  be a  $T$ -sequence of an infinite countable abelian group  $G$ . Then:*

- (i)  $(G, \sigma_{\mathbf{v}})$  is MAP if and only if  $\mathbf{v}$  is a  $TB$ -sequence;
- (ii)  $(G, \sigma_{\mathbf{v}})$  is MinAP if and only if  $s_{\mathbf{v}}(\widehat{G_d}) = \{0\}$  and  $G = \langle \mathbf{v} \rangle$ .

Recall that a topological abelian group  $X$  is *almost maximally almost periodic (AMAP)* if  $n(X)$  is finite.

**Remark 8.** In relation to Fact 4, Lukács in [31] found a  $T$ -sequence in  $\mathbb{Z}(p^\infty)$  that is not a  $TB$ -sequence, providing in this way an example of a non-trivial AMAP group. More precisely, he found a characterizing sequence  $\mathbf{v}$  for  $p^m\mathbb{J}_p \leq \mathbb{J}_p$  for a fixed  $m \in \mathbb{N}_+$ , i.e.,  $s_{\mathbf{v}}(\mathbb{J}_p) = p^m\mathbb{J}_p$ . In this way, being  $\mathbb{J}_p/p^m\mathbb{J}_p$  finite, then

$$s_{\mathbf{v}}(\mathbb{J}_p)^\perp = n(\mathbb{Z}(p^\infty), \sigma_{\mathbf{v}}) \neq \{0\} \text{ is finite.}$$

Therefore,  $(\mathbb{J}_p, \sigma_{\mathbf{v}})$  is AMAP. Further results in this direction were obtained by Nguyen in [32]. Finally, Gabrielyan in [33] proved that an abelian group  $G$  admits an AMAP group topology if and only if  $G$  has non-trivial torsion elements.

The following theorem, due to Gabrielyan, links the notions of  $T$ -characterized subgroup and MinAP topology.

**Theorem 14.** [11] *Let  $X$  be a compact abelian group and  $H$  a closed subgroup of  $X$ . Then,  $H \in \mathcal{Char}_T(X)$  if and only if  $H$  is  $G_\delta$  and  $H^\perp$  carries a MinAP topology.*

Following [4, §4], for a topological abelian group  $X$  and a prime number  $p$ , we denote by  $T_p(X)$  the closure of the subgroup  $X_p = \{x \in X : p^n x \rightarrow 0\}$ . In case  $X$  is compact, one can prove that

$$T_p(X) = \{mX : m \in \mathbb{N}_+, (m, p) = 1\}. \tag{9}$$

In particular,  $T_p(X)$  contains the connected component  $c(X)$  of  $X$ . More precisely, if  $X/c(X) = \prod_{p \in \mathbb{P}} (X/c(X))_p$  is the topologically primary decomposition of the totally disconnected compact abelian group  $X/c(X)$ , then:

$$T_p(X)/c(X) \cong (X/c(X))_p = T_p(X/c(X))$$

Following [34], we say that  $d \in \mathbb{N}$  is a *proper divisor* of  $n \in \mathbb{N}$ , provided that  $d \notin \{0, n\}$  and  $dm = n$  for some  $m \in \mathbb{N}$ . Note that, according to our definition, each  $d \in \mathbb{N} \setminus \{0\}$  is a proper divisor of zero.

**Definition 7.** Let  $G$  be an abelian group.

- (i) For  $n \in \mathbb{N}$  the group  $G$  is said to be of *exponent*  $n$  (denoted by  $exp(G)$ ) if  $nG = \{0\}$ , but  $dG \neq \{0\}$  for every proper divisor  $d$  of  $n$ . We say that  $G$  is *bounded* if  $exp(G) > 0$  and, otherwise, that  $G$  is *unbounded*.
- (ii) [35] If  $G$  is bounded, the *essential order*  $eo(G)$  of  $G$  is the smallest  $n \in \mathbb{N}_+$ , such that  $nG$  is finite. If  $G$  is unbounded, we define  $eo(G) = 0$ .

In the next theorem, we aim to give a detailed description of the closed characterized subgroups  $H$  of  $X$  that are not  $T$ -characterized. As stated in Corollary 15, a closed subgroup  $H$  of a compact abelian group  $X$  is characterized if and only if  $H$  is  $G_\delta$  (i.e.,  $X/H$  is metrizable). This explains the blanket condition imposed on  $H$  to be a  $G_\delta$ -subgroup of  $X$ .

**Theorem 15.** *For a compact abelian group  $X$  and a  $G_\delta$ -subgroup  $H$  of  $X$ , the following conditions are equivalent:*

- (i)  $H \notin \mathcal{Char}_T(X)$ ;
- (ii)  $H^\perp$  does not admit a MinAP group topology;
- (iii) there exists  $m \in \mathbb{N}$ , such that  $m(X/H)$  is finite and non-trivial;
- (iv)  $eo(X/H) < exp(X/H)$ ;
- (v) there exists a finite set  $P$  of primes, so that:
  - (a)  $T_q(X) \leq H$  for all  $q \in \mathbb{P} \setminus P$ ,
  - (b) for every  $p \in P$  there exist  $k_p \in \mathbb{N}$  with  $p^{k_p} T_p(X) \leq H$ ,
  - (c) there exists  $p_0 \in P$ , such that  $p_0^{k_{p_0}-1} T_{p_0}(X) \not\leq H$  and  $p_0^{k_{p_0}-1} T_{p_0}(X) \cap H$  has finite index in  $p_0^{k_{p_0}} T_{p_0}(X)$ ;

(vi) there exists a finite set  $P$  of primes, so that  $X/H \cong \prod_{p \in P} K_p$ , where each  $K_p$  is a compact  $p$ -group, and there exist some  $p_0 \in P$  and  $k \in \mathbb{N}$ , such that  $p_0^k K_{p_0}$  is finite and non-trivial.

**Proof.** (i) $\Leftrightarrow$ (ii) is Theorem 14, and (iii) $\Leftrightarrow$ (iv) is clear from the definition.

(ii) $\Leftrightarrow$ (iii) The main theorem in [36] states that an abelian group  $G$  does not admit a MinAP group topology precisely when there exists  $m \in \mathbb{N}_+$ , such that  $mG$  is finite and non-trivial. In our case,  $G = H^\perp$  is topologically isomorphic to  $\widehat{X/H}$ , so  $G$  does not admit a MinAP group topology if and only if  $m\widehat{X/H} = \widehat{m(X/H)}$  is finite and non-trivial, and this occurs precisely when  $m(X/H)$  is finite and non-trivial.

(vi) $\Rightarrow$ (v) Write  $X/H \cong \prod_{p \in P} K_p$ , where each  $K_p$  is a compact  $p$ -group. Let  $p^{k_p} = \exp(K_p)$  for every  $p \in P$ , and let  $p_0^k K_{p_0}$  be finite and non-trivial for  $p_0 \in P$  and  $k \in \mathbb{N}$ .

Obviously, all  $q \in \mathbb{P} \setminus P$  are coprime to  $m = \exp(X/H)$ . As  $mX \leq H$ , we deduce from the equality (9) that

$$T_q(X) \leq H, \text{ for all } q \in \mathbb{P} \setminus P. \tag{10}$$

This proves (a). From (10), we deduce that  $c(X) \leq H$ .

The quotient groups  $X' = X/c(X)$  and  $H' = H/c(X)$  are totally disconnected; hence,  $X' = \prod_{p \in \mathbb{P}} X'_p$  and  $H' = \prod_{p \in \mathbb{P}} H'_p$ . Here,

$$X'_p = T_p(X)/c(X) \text{ and } H'_p = T_p(H)/c(X) \text{ for every } p \in \mathbb{P}. \tag{11}$$

Furthermore,  $X' = \prod_{p \in P} X'_p \times \prod_{q \in \mathbb{P} \setminus P} X'_q$ . From (10), we deduce that  $X'_q \leq H'_q$  for all  $q \in \mathbb{P} \setminus P$ . Therefore,  $\prod_{q \in \mathbb{P} \setminus P} X'_q \leq H'$  and  $H' = \prod_{p \in P} H'_p \times \prod_{q \in \mathbb{P} \setminus P} X'_q$ . Hence,  $X/H = \prod_{p \in P} X'_p/H'_p$ , and consequently,  $X'_p/H'_p \cong K_p$  for all  $p \in P$ . Thus,  $p^{k_p} X'_p \leq H'_p$  for all  $p \in P$ . Equivalently,  $p^{k_p} T_p(X) \leq H$  for  $p \in P$ . This proves (b).

As  $p_0^k K_{p_0}$  is finite and non-trivial, we deduce that  $k < k_{p_0}$ . Therefore,  $p_0^{k_{p_0}-1} K_{p_0}$  is still finite and non-trivial. Hence,  $p_0^{k_{p_0}-1} X'_{p_0} \not\leq H'_{p_0}$ , and so,

$$p_0^{k_{p_0}-1} T_{p_0}(X) \not\leq H.$$

To prove the second assertion in (c), note that the finiteness of  $p_0^k K_{p_0}$  yields that:

$$p_i^{k_{p_0}-1} (X'_{p_0}/H'_{p_0}) = (p_0^{k_{p_0}-1} X'_{p_0} + H'_{p_0})/H'_{p_0} \cong p_0^{k_{p_0}-1} X'_{p_0}/(H'_{p_0} \cap p_0^{k_{p_0}-1} X'_{p_0})$$

is finite. Hence, from (11), we deduce that  $T_{p_0}(H) \cap p_0^{k_{p_0}-1} T_{p_0}(X)$  has finite index in  $p_0^{k_{p_0}-1} T_{p_0}(X)$ . Therefore,

$$H \cap p_0^{k_{p_0}-1} T_{p_0}(X) = \cap T_{p_0}(H) \cap p_0^{k_{p_0}-1} T_{p_0}(X)$$

has finite index in  $p_0^{k_{p_0}-1} T_{p_0}(X)$ .

(v) $\Rightarrow$ (iii) Let  $m'$  be the product of all  $p^{k_p}$  when  $p$  runs over  $P$ , and let  $m = m'/p$ . Then, an argument similar to the above argument shows that  $m(X/H) \neq \{0\}$  is finite.  $\square$

**Corollary 17.** Let  $X$  be a compact abelian group and  $H$  a closed subgroup of  $X$  that does not contain the connected component  $c(X)$  of  $X$ . Then,  $H \in \mathcal{Char}_T(X)$  if and only if  $H \in \mathcal{Char}(X)$ .

**Proof.** Clearly,  $H$   $T$ -characterized implies  $H$  characterized. Therefore, assume that  $H$  is characterized. Then,  $H$  is  $G_\delta$  in  $X$  by Corollary 15. Let  $\pi : X \rightarrow X/H$  be the canonical projection. Then,  $\pi(c(G))$  is a non-trivial connected subgroup of  $X/H$ ; hence,  $X/H$  is unbounded, as its connected component  $c(X/H)$  is a non-trivial divisible subgroup. According to Theorem 15,  $H$  is  $T$ -characterized.  $\square$

By Corollary 17, for a connected compact abelian group  $X$  and  $H$  a closed subgroup of  $X$ ,

$$H \in \mathcal{C}\text{har}_T(X) \Leftrightarrow H \in \mathcal{C}\text{har}(X) \setminus \{X\}. \tag{12}$$

This result was obtained in [11]; actually, the following more precise form holds (the equivalence (i) $\Leftrightarrow$ (ii) is proven in [11, Theorem 1.14]):

**Corollary 18.** *For a compact abelian group  $X$ , the following conditions are equivalent:*

- (i)  $X$  is connected;
- (ii)  $H \in \mathcal{C}\text{har}_T(X) \Leftrightarrow H \in \mathcal{C}\text{har}(X) \setminus \{X\}$  for every closed subgroup  $H$  of  $G$ ;
- (iii)  $H \in \mathcal{C}\text{har}_K(X) \Leftrightarrow H \in \mathcal{C}\text{har}(X) \setminus \{X\}$  for every closed subgroup  $H$  of  $G$ .

**Proof.** (i) $\Rightarrow$ (ii) by (12), and (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Assume that  $X$  is not connected. Then,  $X$  has a proper open subgroup  $H$ , as the connected component of  $X$  is an intersection of clopen subgroups (see [22]). Then,  $H \in \mathcal{C}\text{har}(X) \setminus \{X\}$ , but  $H \notin \mathcal{C}\text{har}_K(X)$  by Lemma 15.  $\square$

Obviously, each one of the above equivalent conditions implies that  $H \in \mathcal{C}\text{har}_T(X) \Leftrightarrow H \in \mathcal{C}\text{har}_K(X)$  for every closed subgroup  $H$  of a compact abelian group  $X$ . To see that in general the latter property is strictly weaker than  $H \in \mathcal{C}\text{har}_T(X)$ , consider the group  $X = \mathbb{Z}(3) \times \mathbb{Z}(2)^\mathbb{N}$ . Then, for the closed subgroup  $H = \{0\}$  of  $X$ , one has  $H \in \mathcal{C}\text{har}_K(X)$  by Theorem 13, while  $H \notin \mathcal{C}\text{har}_T(X)$  by Theorem 15, as  $3X$  is finite and non-trivial (moreover,  $\widehat{X}$  does not admit any MinAP group topology, as noticed by Remus; see [37]).

### 10. Final Comments and Open Questions

In this section, we collect various open questions arising throughout the paper.

For a topological abelian group  $X$  and  $\mathbf{v} \in \widehat{X}^\mathbb{N}$ , we defined for each  $m \in \mathbb{N}$  the closed subgroup  $n_{\mathbf{v}(m)}(X)$  in (4). Since these subgroups are contained one in each other, the increasing union  $\mathbf{F}_{\mathbf{v}}(X) := \bigcup_{m \in \mathbb{N}} n_{\mathbf{v}(m)}(X)$  is an  $F_\sigma$ -subgroup of  $X$ , and it is contained in  $s_{\mathbf{v}}(X)$ . We do not know whether  $\mathbf{F}_{\mathbf{v}}(X)$  is characterized or not:

**Question 1.** For a topological abelian group  $X$  and  $\mathbf{v} \in \widehat{X}^\mathbb{N}$ , is  $\mathbf{F}_{\mathbf{v}}(X)$  a characterized subgroup of  $X$ ?

This question is motivated by [16, Theorem 1.11], where it is proven that under some additional restraint, the union of a countably infinite increasing chain of closed characterized subgroups of a metrizable compact abelian group is still characterized. On the other hand, it is known that every characterized subgroup is  $F_{\sigma\delta}$  (see Lemma 5(iv)) and that the characterized subgroup need not be  $F_\sigma$ .

In analogy to  $T$ -characterized subgroups, we have introduced here the notion of the  $TB$ -characterized subgroup (see Definition 4). In relation to what is already known for  $T$ -characterized subgroups, one could consider the following general problem.

**Problem 1.** Study the  $TB$ -characterized subgroups of topological abelian groups.

Next comes a more precise question on the properties of  $T$ - and  $TB$ -characterized subgroups. In fact, we do not know whether the counterpart of Corollary 2 is true for  $T$ - and  $TB$ -characterized subgroups:

**Question 2.** Let  $X$  be a topological abelian group and  $X_0, X_1, X_2$  subgroups of  $X$  with  $X_0 \leq X_1 \leq X_2$ , such that  $X_1$  is dually embedded in  $X_2$ .

- (i) If  $X_0 \in \mathcal{Char}_T(X_1)$  and  $X_1 \in \mathcal{Char}_T(X_2)$ , is then  $X_0 \in \mathcal{Char}_T(X_2)$ ?
- (ii) If  $X_0 \in \mathcal{Char}_{TB}(X_1)$  and  $X_1 \in \mathcal{Char}_{TB}(X_2)$ , is then  $X_0 \in \mathcal{Char}_{TB}(X_2)$ ?

A full description of open  $K$ -characterized subgroups is given in Theorems 7 and 8, while Theorem 6 describes the closed  $K$ -characterized subgroups of infinite index that are also  $N$ -characterized. Moreover,  $N$ -characterized closed subgroups of infinite index are  $K$ -characterized. This leaves open the following general problem and question.

**Problem 2.** For a topological abelian group  $X$ , describe  $\mathcal{Char}_K(X)$ .

**Question 3.** Let  $X$  be a topological abelian group. Can one add “ $H \in \mathcal{Char}_K(X)$ ” as an equivalent condition in Theorem 6? Equivalently, does there exist a closed subgroup  $H$  of  $X$ , such that  $H \in \mathcal{Char}_K(X) \setminus \mathcal{Char}_N(X)$ ?

In Theorem 8, we have seen in particular that a proper open finite-index subgroup  $H$  of a topological abelian group  $X$  is auto-characterized precisely when  $H \in \mathcal{Char}_K(X)$ . We do not know whether also the stronger condition  $H \in \mathcal{Char}_T(X)$  is equivalent:

**Question 4.** Let  $H$  be a topological abelian group and  $H$  an open subgroup of  $X$  of finite index. Does  $H \in \mathcal{Char}_T(X)$  whenever  $H$  is auto-characterized? What about the case when  $H$  is a topological direct summand of  $X$ ?

By looking at Theorem 10 and Corollary 14, the following natural question arises:

**Question 5.** Are the closed  $G_\delta$ -subgroups of a precompact abelian groups always  $N$ -characterized?

This amounts to asking whether there exists a continuous injection from  $X/F$  into  $\mathbb{T}^{\mathbb{N}}$  for every closed  $G_\delta$  subgroup  $F$  of a precompact abelian group  $X$ ; in other words, we are asking for a generalization of Lemma 4.

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### Author Contributions

Dikran Dkranjan, Anna Giordano Bruno and Daniele Impieri contributed equally to this work.

### Conflicts of Interest

The authors declare no conflict of interest.

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