

Menachem Shlossberg*

Balanced and functionally balanced P -groups

<https://doi.org/10.1515/taa-2018-0006>

Received March 30, 2017; accepted June 30, 2017

Abstract: In relation to Itzkowitz's problem [5], we show that a c -bounded P -group is balanced if and only if it is functionally balanced. We prove that for an arbitrary P -group, being functionally balanced is equivalent to being strongly functionally balanced. A special focus is given to the uniform free topological group defined over a uniform P -space. In particular, we show that this group is (functionally) balanced precisely when its subsets B_n , consisting of words of length at most n , are all (resp., functionally) balanced.

Keywords: Itzkowitz's problem; P -group; balanced group; (strongly) functionally balanced group

MSC: 54E15, 22A05

1 Introduction and preliminaries

A topological group G is *balanced* if its left and right uniformities coincide. Recall that the left uniformity \mathcal{L}_G of a topological group G is formed by the sets $U_L := \{(x, y) \in G^2 : x^{-1}y \in U\}$, where U is a neighborhood of the identity element of G . The right uniformity \mathcal{R}_G is defined analogously. A topological group G is called *functionally balanced* [14] in case every bounded left-uniformly continuous real-valued function on G is also right-uniformly continuous. Omitting the term "bounded" we obtain the definition of a *strongly functionally balanced* group. In the sequel we extend these definitions, in a natural way, to include also the symmetric subsets of a topological group (see Definition 2.5). The question of whether every strongly functionally balanced group is balanced was raised by Itzkowitz [5]. This longstanding problem is still open.

Nevertheless, it is known that a functionally balanced group is balanced whenever G is either locally compact [5, 6, 14], metrizable [14] or locally connected [7]. Recall that a topological group is *non-archimedean* if it has a local base at the identity consisting of open subgroups. A strongly functionally balanced non-archimedean group is balanced in case it is \aleph_0 -bounded [4] or strongly functionally generated by the set of all its subspaces of countable o -tightness [16]. For more known results concerning Itzkowitz's problem we refer the reader to the survey paper [2].

The class of all non-archimedean groups contains the class of all P -groups (see Definition 2.1). We prove that a P -group is functionally balanced if and only if it is strongly functionally balanced (Corollary 2.7). This gives a positive answer to [2, Question 3] for P -groups. One of the main results we obtain is that a c -bounded P -group is balanced if and only if it is functionally balanced (Theorem 2.9). So, a negative solution to Itzkowitz's problem cannot be found in the class of c -bounded P -groups.

A uniform space whose uniformity is closed under countable intersection is called a *uniform P -space* (see also Definition 2.1). Such a space is necessarily *non-archimedean*, which means that it possesses a base of equivalence relations (Lemma 2.2). In Section 3 we discuss the coincidence of some universal free objects over the same uniform P -space.

For a free group $F(X)$, over a nonempty set X , we denote by B_n its subset containing all words of length not greater than n . In Section 4 we show that the uniform free topological group $F(X, \mathcal{U})$, over a uniform P -space, is (functionally) balanced if and only if B_n is (resp., functionally) balanced for every $n \in \mathbb{N}$. Hopefully,

*Corresponding Author: **Menachem Shlossberg**; Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, 33100 Udine, Italy, E-mail: menachem.shlossberg@uniud.it

this theorem can be useful in providing a negative solution to Itzkowitz's problem.

Given a symmetric subset B of a topological group G we denote by \mathcal{L}_G^B the trace of the left uniformity \mathcal{L}_G on B . That is, $\varepsilon \in \mathcal{L}_G^B$ if and only if there exists $\delta \in \mathcal{L}_G$ such that $\delta \cap (B \times B) = \varepsilon$. The uniformity \mathcal{R}_G^B is the trace of \mathcal{R}_G . In case $\{A_n\}_{n \in \mathbb{N}}$ is a countable collection of subsets of G , we write $\mathcal{L}_G^n(\mathcal{R}_G^n)$ instead of $\mathcal{L}_G^{A_n}$ (resp., $\mathcal{R}_G^{A_n}$). The *character* of G is the minimum cardinal of a local base at the identity. For a uniform space (X, \mathcal{U}) , the *weight* $w(X, \mathcal{U})$ denotes the minimal cardinality of a base of (X, \mathcal{U}) . For $\varepsilon \in \mathcal{U}$ and $a \in X$ we let $\varepsilon[a] := \{x \in X : (a, x) \in \varepsilon\}$. All topological groups and uniform spaces in this paper are assumed to be Hausdorff. Unless otherwise is stated the uniformity of a topological group G is the two-sided uniformity, that is, the supremum $\mathcal{L}_G \vee \mathcal{R}_G$. Finally, **TGr**, **NA** and **NA_b** denote, respectively, the classes of all topological groups, non-archimedean groups and non-archimedean balanced groups.

2 P -groups and uniform P -spaces

Definition 2.1. (see [1], for example) A P -space is a topological space in which the intersection of countably many open sets is still open. A topological group which is a P -space is called a P -group. A *uniform P -space* (X, \mathcal{U}) is a uniform space in which the intersection of countably many elements of \mathcal{U} is again in \mathcal{U} .

Lemma 2.1. [1, Lemma 4.4.1.a] *If G is a P -group, then G is non-archimedean.*

Proof. Let U be a neighborhood of the identity element e . We have to show that U contains an open subgroup H . For every $n \in \mathbb{N}$ there exists a symmetric neighborhood W_n such that $W_n^n \subseteq U$. Since G is a P -group the set $W = \bigcap_{n \in \mathbb{N}} W_n$ is a neighborhood of e . Let H be the subgroup generated by W . Clearly, H is open and $H \subseteq U$. \square

Lemma 2.2. *If (X, \mathcal{U}) is a uniform P -space, then (X, \mathcal{U}) is non-archimedean.*

Proof. Let $\varepsilon \in \mathcal{U}$. We will find an equivalence relation $\delta \in \mathcal{U}$ such that $\delta \subseteq \varepsilon$. For every $n \in \mathbb{N}$ there exists a symmetric entourage $\delta_n \in \mathcal{U}$ such that $\delta_n^n \subseteq \varepsilon$. Since (X, \mathcal{U}) is a uniform P -space, the equivalence relation $\delta = \bigcup_{m \in \mathbb{N}} (\bigcap_{n \in \mathbb{N}} \delta_n)^m \subseteq \varepsilon$ is an element of \mathcal{U} . \square

Definition 2.2. (see [1, 10], for example) Let τ be an infinite cardinal.

1. A topological group G is called τ -bounded if for every neighborhood U of the identity, there exists a set F of cardinality not greater than τ such that $FU = G$.
2. A uniform space (X, \mathcal{U}) is τ -narrow if for every $\varepsilon \in \mathcal{U}$, there exists a set $\{x_\alpha : \alpha < \tau\}$ such that $X = \bigcup_{\alpha < \tau} \varepsilon[x_\alpha]$.

Lemma 2.3. *Let τ be an infinite cardinal. Let G be a topological group in which the intersection of any family of cardinality at most τ of open sets is open. If G is also τ -bounded, then G is balanced.*

Proof. Let H be an open subgroup of G . We will show that there exists a normal open subgroup N of G such that $N \subseteq H$. Let $N = \bigcap_{x \in G} xHx^{-1}$. Clearly, N is a normal subgroup of G and $N \leq H$. We show that N is open. Since G is τ -bounded, there exists a subset $F \subseteq G$ with $|F| \leq \tau$ such that $FH = G$. It is easy to see that $N = \bigcap_{x \in F} xHx^{-1}$. Since $|F| \leq \tau$, this intersection must be open and applying Lemma 2.1 completes the proof. \square

Corollary 2.3. [1, Lemma 4.4.1.b] *An \aleph_0 -bounded P -group is balanced.*

Lemma 2.4. *Let (X, \mathcal{U}) be a uniform P -space. A function $f : (X, \mathcal{U}) \rightarrow \mathbb{R}$ is uniformly continuous if and only if there exists $\varepsilon \in \mathcal{U}$ such that $(x, y) \in \varepsilon \Rightarrow f(x) = f(y)$.*

Proof. The “if” part is trivial for every uniform space (even if it is not a uniform P -space). We prove the “only if” part. Since $f : (X, \mathcal{U}) \rightarrow \mathbb{R}$ is uniformly continuous, for every $n \in \mathbb{N}$ there exists $\varepsilon_n \in \mathcal{U}$ such that $(x, y) \in \varepsilon \Rightarrow |f(x) - f(y)| < \frac{1}{n}$. Let $\varepsilon = \bigcap_{n \in \mathbb{N}} \varepsilon_n$. Since (X, \mathcal{U}) is a uniform P -space, we have $\varepsilon \in \mathcal{U}$. Now, $(x, y) \in \varepsilon$ implies that $f(x) = f(y)$. \square

Corollary 2.4. Let G be a P -group with a symmetric subset B . A function $f : B \rightarrow \mathbb{R}$ is \mathcal{L}_G^B - (\mathcal{R}_G^B) -uniformly continuous if and only if there exists an open subgroup H of G such that f is constant on $B \cap xH$ (resp., $B \cap Hx$) for every $x \in B$.

Proof. Clearly, (B, \mathcal{L}_G^B) ((B, \mathcal{R}_G^B)) is a uniform P -space. Now use Lemma 2.4 and the definition of the left (resp., right) uniformity to conclude the proof. \square

Definition 2.5. We say that a symmetric subset B of a topological group G is:

- *balanced* if the left and right uniformities of G coincide on B .
- *functionally balanced* if every bounded \mathcal{L}_G^B -uniformly continuous function $f : B \rightarrow \mathbb{R}$ is \mathcal{R}_G^B -uniformly continuous.
- *strongly functionally balanced* if every \mathcal{L}_G^B -uniformly continuous function $f : B \rightarrow \mathbb{R}$ is \mathcal{R}_G^B -uniformly continuous.

Theorem 2.6. Let G be a P -group with a symmetric subset B . Then the following assertions are equivalent:

1. B is strongly functionally balanced.
2. B is functionally balanced.
3. If $\varepsilon \in \mathcal{L}_G^B$ is an equivalence relation with at most c equivalence classes, then $\varepsilon \in \mathcal{R}_G^B$.

Proof. (1) \Rightarrow (2) : Trivial.

(2) \Rightarrow (3) : Let $\varepsilon \in \mathcal{L}_G^B$ be an equivalence relation with at most c equivalence classes. It follows that there exists a function $f : B \rightarrow [0, 1] \subseteq \mathbb{R}$ such that $f(x) = f(y)$ if and only if $(x, y) \in \varepsilon$. Clearly, $f : B \rightarrow \mathbb{R}$ is a bounded \mathcal{L}_G^B -uniformly continuous function. Since B is functionally balanced, then $f : B \rightarrow \mathbb{R}$ is also \mathcal{R}_G^B -uniformly continuous. By Corollary 2.4, there exists an open subgroup H of G such that f is constant on $B \cap Hx$ for every $x \in B$. By the definition of the right uniformity, $\{(t, s) \in B^2 \mid ts^{-1} \in H\} \in \mathcal{R}_G^B$. The definition of f implies that $\{(t, s) \in B^2 \mid ts^{-1} \in H\} \subseteq \varepsilon$, and thus $\varepsilon \in \mathcal{R}_G^B$.

(3) \Rightarrow (1): Let $f : B \rightarrow \mathbb{R}$ be a \mathcal{L}_G^B -uniformly continuous function. By Corollary 2.4, there exists an open subgroup H such that f is constant on $B \cap xH$ for every $x \in B$. We have $\{(t, s) \in B^2 \mid t^{-1}s \in H\} \subseteq \varepsilon := \{(t, s) \mid f(t) = f(s)\}$. Hence, $\varepsilon \in \mathcal{L}_G^B$ and clearly ε has at most c equivalence classes. By (3), $\varepsilon = \{(t, s) \mid f(t) = f(s)\} \in \mathcal{R}_G^B$. Therefore, f is \mathcal{R}_G^B -uniformly continuous and we conclude that B is strongly functionally balanced. \square

Letting $B = G$ in Theorem 2.6 we obtain the following:

Corollary 2.7. Let G be a P -group. Then G is functionally balanced if and only if it is strongly functionally balanced.

Recall the following result of Hernández:

Theorem 2.8. [4, Theorem 2] Let G be a non-archimedean \aleph_0 -bounded topological group. Then G is balanced if and only if it is strongly functionally balanced.

In case the non-archimedean group is a P -group it suffices to require c -boundedness, as it follows from the following theorem.

Theorem 2.9. Let G be a c -bounded P -group. Then G is balanced if and only if it is functionally balanced.

Proof. If H is a subgroup of index at most c , then $\varepsilon := \{(t, s) \mid t^{-1}s \in H\}$ has at most c equivalence classes. So, in case G is a c -bounded P -group, condition (3) of Theorem 2.6 is equivalent to the coincidence of the left and right uniformities. This completes the proof. \square

Remark 2.5.

- (a) Theorem 2.9 means that a negative solution to Itzkowitz’s problem cannot be found in the class of c -bounded P -groups.
- (b) Theorem 2.9 can be viewed also as a corollary of Theorem 2.6. The latter plays an important role in proving Theorem 4.4. As pointed out by the referee, Theorem 2.9 admits a much shorter proof. Indeed, one can take an open subgroup U of G and a map $f : G \rightarrow [0, 1]$, so that f is constant on each xU and injective on distinct cosets. Since f is left uniformly continuous and G is a functionally balanced P -group, there is an open subgroup V such that f is constant on each Vx . Then Vx is contained in xU and G is balanced.

3 Coincidence of free objects

Definition 3.1. [8, Definition 3.1] Let Ω be a subclass of \mathbf{TGr} and (X, \mathcal{U}) be a uniform space. By an Ω -free topological group of (X, \mathcal{U}) we mean a pair $(F_\Omega(X, \mathcal{U}), i)$, where $F_\Omega(X, \mathcal{U}) \in \Omega$ and $i : X \rightarrow F_\Omega(X, \mathcal{U})$ is a uniform map satisfying the following universal property. For every uniformly continuous map $\varphi : (X, \mathcal{U}) \rightarrow G$, where $G \in \Omega$, there exists a unique continuous homomorphism $\Phi : F_\Omega(X, \mathcal{U}) \rightarrow G$ for which the following diagram commutes:

$$\begin{array}{ccc} (X, \mathcal{U}) & \xrightarrow{i} & F_\Omega(X, \mathcal{U}) \\ & \searrow \varphi & \downarrow \Phi \\ & & G \end{array}$$

For $\Omega = \mathbf{TGr}$ the universal object $F_\Omega(X, \mathcal{U})$ is the *uniform free topological group* of (X, \mathcal{U}) . This group was invented by Nakayama [9] and studied, among others, by Numella [11] and Pestov [12, 13]. In particular, Pestov described its topology.

Let (X, \mathcal{U}) be a non-archimedean uniform space.

- 1. For $\Omega = \mathbf{NA}$ we obtain the *free non-archimedean group* $F_{\mathbf{NA}}$.
- 2. In case $\Omega = \mathbf{NA}_b$, the universal object is the *free non-archimedean balanced group* $F_{\mathbf{NA}}^b$.

These groups were defined and studied by Megrelishvili and the author in [8].

We collect some known results from [8, 12]. Denote by j_2 the mapping $(x, y) \mapsto x^{-1}y$ from X^2 to $F(X)$ and by j_2^* the mapping $(x, y) \mapsto xy^{-1}$.

Definition 3.2. [3, Chapter 4] If P is a group and $(V_n)_{n \in \mathbb{N}}$ a sequence of subsets of P , define

$$[(V_n)] := \bigcup_{n \in \mathbb{N}} \bigcup_{\pi \in S_n} V_{\pi(1)} V_{\pi(2)} \cdots V_{\pi(n)}.$$

Remark 3.1. [8, Remark 4.3] Note that if $(V_n)_{n \in \mathbb{N}}$ is a constant sequence such that

$$V_1 = V_2 = \cdots = V_n = \cdots = V,$$

then $[(V_n)] = \bigcup_{n \in \mathbb{N}} V^n$. In this case we write $[V]$ instead of $[(V_n)]$. It is easy to see that if $V = V^{-1}$, then $[V]$ is simply the subgroup generated by V .

Definition 3.3.

- 1. [12] For every $\psi \in \mathcal{U}^{F(X)}$ let

$$V_\psi := \bigcup_{w \in F(X)} w(j_2(\psi(w)) \cup j_2^*(\psi(w)))w^{-1}.$$

2. [8, Definition 4.9.2] As a particular case in which every ψ is a constant function we obtain the set

$$\tilde{\varepsilon} := \bigcup_{w \in F(X)} w(j_2(\varepsilon) \cup j_2^*(\varepsilon))w^{-1}.$$

Theorem 3.4.

1. (Pestov [12, Theorem 2]) Let (X, \mathcal{U}) be a uniform space. The set $\{[(V_{\psi_n})]\}$, where $\{\psi_n\}$ extends over the family of all possible sequences of elements from $\mathcal{U}^{F(X)}$, is a local base at the identity element of the uniform free topological group $F(X, \mathcal{U})$.
2. [8, Theorem 4.13] Assume that (X, \mathcal{U}) is a non-archimedean uniform space. Then,
 - (a) The set $\{[\mathcal{V}_\psi] : \psi \in \mathcal{U}^{F(X)}\}$ is a local base at the identity element of $F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U})$, the uniform free non-archimedean group.
 - (b) The family (of normal subgroups) $\{[\tilde{\varepsilon}] : \varepsilon \in \mathcal{B}\}$ is a local base at the identity element of $F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$, the uniform free non-archimedean balanced group.

Remark 3.2. Let (X, \mathcal{U}) be a non-archimedean uniform space. By the universal properties of the universal objects it is clear that:

1. the topology of $F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U})$ is the maximal non-archimedean group topology on $F(X)$ that is coarser than the topology of $F(X, \mathcal{U})$.
2. the topology of $F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$ is the maximal non-archimedean balanced group topology on $F(X)$ that is coarser than the topology of $F(X, \mathcal{U})$.

In particular, if $F(X, \mathcal{U})$ is non-archimedean, then $F(X, \mathcal{U})$ coincides with $F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U})$. If $F(X, \mathcal{U})$ is also balanced, then these groups coincide also with $F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$.

Theorem 3.5. Let (X, \mathcal{U}) be a uniform space. Suppose that there exists an infinite cardinal τ such that $\bigcap_{i \in I} \varepsilon_i \in \mathcal{U}$ for any family of entourages $\{\varepsilon_i : i \in I\} \subseteq \mathcal{U}$ with $|I| \leq \tau$. Then,

1. the intersection of any family of cardinality at most τ of open subsets of $F(X, \mathcal{U})$ is open. In particular, $F(X, \mathcal{U})$ is a P -group and we have $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U})$.
2. if the uniform space (X, \mathcal{U}) is also τ -narrow then

$$F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U}).$$

Proof. (1) : Let I be an arbitrary set with $|I| \leq \tau$. For every $i \in I$ let $\{\psi_n^i\}$ be a sequence of elements from $\mathcal{U}^{F(X)}$ (see Theorem 3.4.1). We define a function φ as follows. For every $w \in F(X)$ let $\varphi(w) = \bigcap_{i \in I, n \in \mathbb{N}} \psi_n^i(w)$. By our assumption on the cardinal τ , we have $\varphi \in \mathcal{U}^{F(X)}$.

Clearly, $\mathcal{V}_\varphi \subseteq V_{\psi_n^i} \forall i \in I, \forall n \in \mathbb{N}$. It follows that $[\mathcal{V}_\varphi] \subseteq \bigcap_{i \in I} [(V_{\psi_n^i})]$. By [12, Theorem 2] (see also Theorem 3.4.1) and Remark 3.1, $[\mathcal{V}_\varphi]$ is a neighborhood of the identity of $F(X, \mathcal{U})$. It follows that the intersection of any family of cardinality at most τ of open subsets of $F(X, \mathcal{U})$ is open. Therefore, $F(X, \mathcal{U})$ is a P -group. By Lemma 2.1 and Remark 3.2, $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U})$.

(2): It is known that the universal morphism $i : (X, \mathcal{U}) \rightarrow F(X, \mathcal{U})$ is a uniform embedding and that $i(X)$ algebraically generates $F(X, \mathcal{U})$. Since (X, \mathcal{U}) is τ -narrow, we obtain by [1, Theorem 5.1.19] (see also [1, Exercise 5.1.a]) that $F(X, \mathcal{U})$ is τ -bounded. By item (1) and Lemma 2.3, we conclude that the non-archimedean group $F(X, \mathcal{U})$ is also balanced and so we have $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$. \square

Omitting the τ -narrowness assumption from Theorem 3.5.2, we obtain the following counterexample.

Example 3.3. By [10, Example 3.14], for every cardinal $\tau > \aleph_1$, there exists a Hausdorff uniform P -space such that $w(X, \mathcal{U}) = \aleph_1 < \tau < \chi(F(X, \mathcal{U}))$. In view of Theorem 3.5.1 and [8, Theorem 4.16.1], we have $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}(X, \mathcal{U}) \neq F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$.

As corollaries we obtain the following two results of Nickolas and Tkachenko.

Corollary 3.6. [10, Lemma 3.12] If (X, \mathcal{U}) is an \aleph_0 -narrow uniform P -space, then the group $F(X, \mathcal{U})$ has a base at the identity consisting of open normal subgroups.

Proof. By Theorem 3.5.2, $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$. Now use item (b) of Theorem 3.4.2. \square

Corollary 3.7. [10, Theorem 3.13] If (X, \mathcal{U}) is an \aleph_0 -narrow uniform P -space, then $\chi(F(X, \mathcal{U})) = w(X, \mathcal{U})$.

Proof. If (X, \mathcal{U}) is an \aleph_0 -narrow uniform P -space, then by Theorem 3.5.2 we have $F(X, \mathcal{U}) = F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})$. On the other hand, by [8, Theorem 4.16.1], we have $\chi(F_{\mathcal{N}, \mathcal{A}}^b(X, \mathcal{U})) = w(X, \mathcal{U})$. We conclude that $\chi(F(X, \mathcal{U})) = w(X, \mathcal{U})$. \square

4 The subsets B_n

Theorem 4.1. Suppose that τ is a topological P -group topology on a free group $F(X)$ and μ is either the right, left or two-sided uniformity of $(F(X), \tau)$. Then, $\varepsilon \in \mu$ if (and only if) $\varepsilon \cap (B_n \times B_n) \in \mu \cap (B_n \times B_n) \forall n \in \mathbb{N}$.

Proof. It is clear that if $(F(X), \tau)$ is a P -group, then $(F(X), \mu)$ is a uniform P -space. Let us assume that $\varepsilon \cap (B_n \times B_n) \in \mu \cap (B_n \times B_n) \forall n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ there exists $\delta_n \in \mathcal{U}$ such that $\varepsilon \cap (B_n \times B_n) = \delta_n \cap (B_n \times B_n)$. Since $(F(X), \mu)$ is a uniform P -space, we have $\delta = \bigcap_{n \in \mathbb{N}} \delta_n \in \mu$. Hence,

$$\delta = \bigcup_{n \in \mathbb{N}} (\delta \cap (B_n \times B_n)) \subseteq \bigcup_{n \in \mathbb{N}} (\delta_n \cap (B_n \times B_n)) = \bigcup_{n \in \mathbb{N}} (\varepsilon \cap (B_n \times B_n)) = \varepsilon,$$

and we conclude that $\varepsilon \in \mu$. \square

Corollary 4.2. Let (X, \mathcal{U}) be a uniform P -space and μ be either the right, left or two-sided uniformity of $F(X, \mathcal{U})$. Then, $\varepsilon \in \mu$ if (and only if)

$$\varepsilon \cap (B_n \times B_n) \in \mu \cap (B_n \times B_n) \forall n \in \mathbb{N}.$$

Proof. By Theorem 3.5, $F(X, \mathcal{U})$ is a P -group. Now the proof follows from Theorem 4.1. \square

Corollary 4.3. Let (X, \mathcal{U}) be a uniform P -space. Then, $F(X, \mathcal{U})$ is balanced if and only if B_n is balanced for every $n \in \mathbb{N}$.

Theorem 4.4. Let (X, \mathcal{U}) be a uniform P -space. The following are equivalent:

1. B_n is strongly functionally balanced for every $n \in \mathbb{N}$.
2. B_n is functionally balanced for every $n \in \mathbb{N}$.
3. $F(X, \mathcal{U})$ is functionally balanced.
4. $F(X, \mathcal{U})$ is strongly functionally balanced.

Proof. $G := F(X, \mathcal{U})$ is a P -group by Theorem 3.5. So, the implications (1) \iff (2) and (3) \iff (4) can be derived from Theorem 2.6.

(2) \implies (3) : Using Theorem 2.6, it suffices to show that if $\varepsilon \in \mathcal{L}_G$ with at most c equivalence classes, then $\varepsilon \in \mathcal{R}_G$. For such an ε it is clear that $\varepsilon_n := \varepsilon \cap (B_n \times B_n)$ has at most c equivalence classes in B_n . It follows from our assumption (2) and from Theorem 2.6 (with $B = B_n$) that $\varepsilon_n \in \mathcal{R}_G^n$. Corollary 4.2 implies that $\varepsilon \in \mathcal{R}_G$, as needed.

(3) \implies (2) : Suppose that $F(X, \mathcal{U})$ is functionally balanced and fix an arbitrary $n \in \mathbb{N}$. We show that B_n is functionally balanced. Let $\varepsilon_n := \varepsilon \cap (B_n \times B_n) \in \mathcal{L}_G^n$ be an equivalence relation with at most c equivalence classes in B_n , where $\varepsilon \in \mathcal{L}_G$ is an equivalence relation on $F(X)$. Let

$$\delta := \varepsilon \cup ((F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n])).$$

We will show that δ has the following properties:

- (a) δ is an equivalence relation with $\varepsilon \subseteq \delta$.
- (b) $\delta_n = \delta \cap (B_n \times B_n) = \varepsilon_n$.
- (c) δ has at most c equivalence classes in $F(X)$.

We prove the nontrivial part of (a). Namely, the transitivity of δ . Let $(x, y), (y, z) \in \delta$. If $(x, y), (y, z) \in \varepsilon$ or $(x, y), (y, z) \in (F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n])$ the assertion is trivial. So, without loss of generality assume that $(x, y) \in \varepsilon$ and $(y, z) \in (F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n])$. Since $(x, y) \in \varepsilon$ and $y \in (F(X) \setminus \varepsilon[B_n])$, it follows that $x \in (F(X) \setminus \varepsilon[B_n])$. Since z is also in $(F(X) \setminus \varepsilon[B_n])$, we have $(x, z) \in (F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n]) \subseteq \delta$.

To see that property (b) is satisfied, first observe that $B_n \subseteq \varepsilon[B_n]$. Therefore,

$$\begin{aligned} \delta_n &= \delta \cap (B_n \times B_n) = (\varepsilon \cup ((F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n]))) \cap (B_n \times B_n) = \\ &= (\varepsilon \cap (B_n \times B_n)) \cup ((F(X) \setminus \varepsilon[B_n]) \times (F(X) \setminus \varepsilon[B_n])) \cap (B_n \times B_n) = \\ &= (\varepsilon \cap (B_n \times B_n)) \cup \emptyset = \varepsilon_n. \end{aligned}$$

To prove (c) combine the following two observations. On the one hand, the fact that ε_n has at most c equivalence classes in B_n , implies that there exist at most c equivalence classes $\delta[x]$, where $x \in \varepsilon[B_n]$. On the other hand, by the definition of δ , the number of equivalence classes $\delta[x]$, with $x \notin \varepsilon[B_n]$ is less than two. So, δ has at most c equivalence classes in $F(X)$.

Now, using (a) and (c) together with our assumption that $F(X, \mathcal{U})$ is functionally balanced, we obtain by Theorem 2.6 that $\delta \in \mathcal{R}_G$. Finally, we use property (b) and Corollary 4.2 to conclude that $\varepsilon_n \in \mathcal{R}_G^n$. \square

Acknowledgments: I would like to thank A.M. Brodsky, L. Jennings, M. Megrelishvili and the referee for their useful suggestions. This work is supported by Programma SIR 2014 by MIUR, project GADYGR, number RBSI14V2LI, cup G22I15000160008, and also by a grant of Israel Science Foundation (ISF 668/13).

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