Topology and its Applications 252 (2019) 112–144

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue - TOPOSYM 2016

Topological entropy for locally linearly compact vector spaces[☆]

Ilaria Castellano[a,1], Anna Giordano Bruno[b,*]

[a] University of Southampton, Building 54, Salisbury Road, 50171BJ Southampton, UK
[b] Università degli Studi di Udine, Via delle Scienze 206, 33100 Udine, Italy

ARTICLE INFO

Article history:
Received 31 December 2016
Received in revised form 27 September 2018
Accepted 21 November 2018
Available online 23 November 2018

MSC:
primary 15A03, 15A04, 22B05
secondary 20K30, 37A35

Keywords:
Linearly compact vector space
Locally linearly compact vector space
Algebraic entropy
Continuous linear transformation
Continuous endomorphism
Algebraic dynamical system

ABSTRACT

By analogy with the topological entropy for continuous endomorphisms of totally disconnected locally compact groups, we introduce a notion of topological entropy for continuous endomorphisms of locally linearly compact vector spaces. We study the fundamental properties of this entropy and we prove the Addition Theorem, showing that the topological entropy is additive with respect to short exact sequences. By means of Lefschetz Duality, we connect the topological entropy to the algebraic entropy in a so-called Bridge Theorem.

1. Introduction

In [1] Adler, Konheim and McAndrew introduced a notion of topological entropy for continuous self-maps of compact spaces. Later on, in [2], Bowen gave a definition of topological entropy for uniformly continuous self-maps of metric spaces, that was extended by Hood in [31] to uniform spaces. This notion of entropy coincides with the one for compact spaces (when the compact topological space is endowed with the unique uniformity compatible with the topology), and it can be computed for any given continuous endomorphism $\phi : G \to G$ of a topological group G (since ϕ turns out to be uniformly continuous with respect to the

☆ This work was supported by Programma SIR 2014 by MIUR (Project GADYGR, Number RBSI14V2LI, cup G22I15000160008). The authors are members of the "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA - INdAM).
* Corresponding author.
E-mail addresses: ilaria.castellano88@gmail.com (I. Castellano), anna.giordanobruno@uniud.it (A. Giordano Bruno).
1 The first author was partially supported by EPSRC Grant N007328/1 Soluble Groups and Cohomology.

https://doi.org/10.1016/j.topol.2018.11.009
0166-8641/

left uniformity of G). In particular, if G is a totally disconnected locally compact group, by van Dantzig's Theorem (see [42]) the family $\mathcal{B}_{gr}(G) = \{U \leq G \mid U \text{ compact open}\}$ is a neighborhood basis at 0 in G, and the topological entropy of the continuous endomorphism $\phi : G \to G$ can be computed as follows (see [22,28]). For a subset F of G and for every $n \in \mathbb{N}_+$, the *n-th ϕ-cotrajectory* of F is

$$C_n(\phi, F) = F \cap \phi^{-1}F \cap \ldots \cap \phi^{-n+1}F.$$

The *topological entropy of ϕ with respect to $U \in \mathcal{B}_{gr}(G)$* is

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \log[U : C_n(\phi, U)],$$

and the *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) \mid U \in \mathcal{B}_{gr}(G)\}.$$

A fundamental property of the topological entropy is the so-called Addition Theorem: it holds for a topological group G, a continuous endomorphism $\phi : G \to G$ and a closed normal subgroup H of G that is ϕ-invariant (i.e., $\phi H \leq H$), if

$$h_{top}(\phi) = h_{top}(\phi \restriction_H) + h_{top}(\overline{\phi}),$$

where $\overline{\phi} : G/H \to G/H$ is the continuous endomorphism induced by ϕ.

The Addition Theorem for continuous endomorphisms of compact groups was deduced in [21, Theorem 8.3] from the metric case proved in a more general setting in [2, Theorem 19]; the separable case was settled by Yuzvinski in [48]. Recently, in [28], the Addition Theorem was proved for topological automorphisms of totally disconnected locally compact groups; more precisely, taken G a totally disconnected locally compact group, $\phi : G \to G$ a continuous endomorphism and H a closed ϕ-invariant normal subgroup of G, if $\phi \restriction_H$ is surjective and the continuous endomorphism $\overline{\phi} : G/H \to G/H$ induced by ϕ is injective, then

$$h_{top}(\phi) = h_{top}(\phi \restriction_H) + h_{top}(\overline{\phi}).$$

The validity of the Addition Theorem in full generality for continuous endomorphisms of locally compact groups remains an open problem, even in the totally disconnected (abelian) case.

In this paper we introduce a notion of topological entropy ent^* for locally linearly compact vector spaces by analogy with the topological entropy h_{top} for totally disconnected locally compact groups. Recall that a topological vector space V over a discrete field $\mathbb{K}$ was defined in [34] to be *locally linearly compact* if V admits a neighborhood basis at 0 consisting of linearly compact open linear subspaces (see §2.1 for more details and properties). Denote by $\mathcal{B}(V)$ the set of all linearly compact open linear subspaces of V; clearly, V is locally linearly compact if and only if $\mathcal{B}(V)$ is a neighborhood basis at 0.

Definition 1.1. Let V be a locally linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. The *topological entropy of ϕ with respect to $U \in \mathcal{B}(V)$* is

$$H^*(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{U}{C_n(\phi, U)}, \tag{1.1}$$

and the *topological entropy* of ϕ: $V \to V$ is

$$\mathrm{ent}^*(\phi) = \sup\{H^*(\phi, U) \mid U \in \mathcal{B}(V)\}.$$

The limit in (1.1) exists (see Proposition 3.2), and the deep reason for the existence of this limit is that, for every $U \in \mathcal{B}(V)$, its linear subspace $C_2(\phi, U) = U \cap \phi^{-1}U$ (and analogously every $C_n(\phi, U)$) has finite codimension in U (see §3.1, Remark 3.3, and see also Remark 1.3). Moreover, ent^* is always zero on discrete vector spaces (see Corollary 3.9), and it admits all the fundamental properties expected from an entropy function (see §3.2).

One of the main results of the present paper is the Addition Theorem for locally linearly compact vector spaces and their continuous endomorphisms:

Theorem 1.2 *(Addition Theorem). Let V be a locally linearly compact vector space, $\phi : V \to V$ a continuous endomorphism, W a closed ϕ-invariant linear subspace of V and $\overline{\phi} : V/W \to V/W$ the continuous endomorphism induced by ϕ. Then*

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi \restriction_W) + \mathrm{ent}^*(\overline{\phi}).$$

In case V is a locally linearly compact vector space over a discrete finite field $\mathbb{F}$, then V is a totally disconnected locally compact abelian group (see Proposition 2.7(b)) and

$$h_{top}(\phi) = \mathrm{ent}^*(\phi) \cdot \log |\mathbb{F}|$$

(see Proposition 3.11). So, with respect to the general problem of the validity of the Addition Theorem for the topological entropy h_{top} of continuous endomorphisms of locally compact groups, Theorem 1.2 covers the case of those totally disconnected locally compact abelian groups that are also locally linearly compact vector spaces.

To prove the Addition Theorem, we restrict first to the case of continuous endomorphisms of linearly compact vector spaces (see §4.1), and then to topological automorphisms (see §4.2). The technique used for the latter reduction was suggested to us by Simone Virili; in fact, Virili and Salce used it in a different context in [40] giving credit to Gabriel [23]. Finally, in Section 5, we prove the Addition Theorem for topological automorphisms (see Proposition 5.1), so that we can deduce it for all continuous endomorphisms of linearly compact vector spaces (see Proposition 5.2).

A fundamental tool in the proof of the Addition Theorem is the so-called Limit-free Formula (see Proposition 3.23), that permits to compute the topological entropy avoiding the limit in the definition in Equation (1.1). Indeed, taken V a locally linearly compact vector space and $\phi : V \to V$ a continuous endomorphism, for every $U \in \mathcal{B}(V)$ we construct a linearly compact linear subspace U_+ of V (see Definition 3.21) such that U_+ is an open linear subspace of ϕU_+ of finite codimension and

$$H^*(\phi, U) = \dim \frac{\phi U_+}{U_+}.$$

This result is the counterpart of the same formula for the topological entropy h_{top} of continuous endomorphisms of totally disconnected locally compact groups given in [28, Proposition 3.9] (see also [12] for the compact case and [25] for the case of topological automorphisms). Note that a first Limit-free Formula was sketched by Yuzvinski in [48] in the context of the algebraic entropy for endomorphisms of discrete abelian groups; a gap in the formulation was found in [22], and the correct version of this Limit-free Formula was later proved in a slightly more general setting in [12] (and extended in [27, Lemma 5.4] for the intrinsic algebraic entropy of automorphisms of abelian groups).

In [1] Adler, Konheim and McAndrew also sketched a definition of algebraic entropy for endomorphisms of abelian groups, that was later reconsidered by Weiss in [46], and recently by Dikranjan, Goldsmith, Salce and Zanardo for torsion abelian groups in [19]. Later on, using the definitions of algebraic entropy given by Peters in [35,36], the algebraic entropy h_{alg} was extended in several steps (see [15,44]) to continuous endomorphisms of locally compact abelian groups.

In [46] Weiss connected, in a so-called Bridge Theorem, the topological entropy h_{top} of a continuous endomorphism $\phi : G \to G$ of a totally disconnected compact abelian group G to the algebraic entropy h_{alg} of the dual endomorphism $\phi^\wedge : G^\wedge \to G^\wedge$ of the Pontryagin dual $G^\wedge$ of G, by showing that

$$h_{top}(\phi) = h_{alg}(\phi^\wedge).$$

The same connection was given by Peters in [35] for topological automorphisms of metrizable compact abelian groups; moreover, these results were recently extended to continuous endomorphisms of compact abelian groups in [11], to continuous endomorphisms of totally disconnected locally compact abelian groups in [14], and to topological automorphisms of locally compact abelian groups in [43] (in a much more general setting). The problem of the validity of the Bridge Theorem in the general case of continuous endomorphisms of locally compact abelian groups is still open.

In [26] the algebraic dimension entropy $\mathrm{ent}_{\dim}$ was studied for endomorphisms of discrete vector spaces, as a particular interesting case of the algebraic i-entropy ent_i for endomorphisms of modules over a ring R and an invariant i of $\mathrm{Mod}(R)$, introduced in [38] as another generalization of Weiss' algebraic entropy (see also [39]). The algebraic dimension entropy is extended in [5] to locally linearly compact vector spaces, as follows. Let $\phi\colon V \to V$ be a continuous endomorphism of a locally linearly compact vector space V. For every $U \in \mathcal{B}(V)$ and $n \in \mathbb{N}_+$, the *n-th ϕ-trajectory of U* is

$$T_n(\phi, U) = U + \phi U + \ldots + \phi^{n-1}U.$$

The *algebraic entropy of ϕ with respect to U* is

$$H(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{T_n(\phi, U)}{U}, \tag{1.2}$$

and the *algebraic entropy of ϕ* is

$$\mathrm{ent}(\phi) = \sup\{H(\phi, U) \mid U \in \mathcal{B}(V)\}.$$

Remark 1.3. Let $\phi\colon V \to V$ be a continuous endomorphism of a locally linearly compact vector space V. As noted above for the topological entropy, the deep reason for the existence of the limit in Equation (1.2) is the fact that, for $U \in \mathcal{B}(V)$, its linear subspace $U \cap \phi U$ has finite codimension in U, that is, U has finite codimension in $T_2(\phi, U) = U + \phi U$. Furthermore, U has finite codimension in $T_n(\phi, U)$ for every $n \in \mathbb{N}_+$.

This phenomenon was isolated in [18], where, for an endomorphism $\varphi : G \to G$ of an abelian group G, a subgroup N of G is called *φ-inert* if N has finite index in $T_2(\varphi, N) = N + \varphi N$. Consequently, N has finite index in $T_n(\varphi, N)$ for every $n \in \mathbb{N}_+$, and the *intrinsic algebraic entropy of φ with respect to N* can be defined as

$$\widetilde{H}(\varphi, N) = \lim_{n\to\infty} \frac{1}{n} \frac{T_n(\varphi, N)}{N};$$

the *intrinsic algebraic entropy* of φ is

$$\widetilde{\mathrm{ent}}(\varphi) = \sup\{\widetilde{H}(\varphi, N) \mid N \leq G \ \varphi\text{-inert}\}.$$

The concept of *fully inert subgroup* (i.e., a subgroup of G that is φ-inert for every endomorphism $\varphi : G \to G$) was investigated in [17,20,29], while the notion of *inert endomorphism* (i.e., an endomorphism $\varphi : G \to G$ such that N is φ-inert for every subgroup N of G) was deeply studied in [7–9]. For a comprehensive survey on inert properties in group theory see [10].

The second main result of this paper is the following Bridge Theorem, proved in Section 6, connecting the topological entropy ent^* with the algebraic entropy ent from [5] by means of Lefschetz Duality (see §2.2). For a locally linearly compact vector space V and a continuous endomorphism $\phi : V \to V$, we denote by $\widehat{V}$ the dual of V and by $\widehat{\phi} : \widehat{V} \to \widehat{V}$ the dual endomorphism of ϕ with respect to Lefschetz Duality.

Theorem 1.4 *(Bridge Theorem). Let V be a locally linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. Then*

$$\mathrm{ent}^*(\phi) = \mathrm{ent}(\widehat{\phi}).$$

By analogy with the adjoint algebraic entropy for abelian groups from [16], the adjoint dimension entropy was considered in [26]: for a discrete vector space V let $\mathcal{C}(V) = \{N \leq V \mid \dim V/N < \infty\}$; for an endomorphism $\phi : V \to V$, the *adjoint dimension entropy of ϕ with respect to N* is

$$H^*_{\dim}(\phi, N) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{V}{C_n(\phi, N)},$$

and the *adjoint dimension entropy* is

$$\mathrm{ent}^*_{\dim}(\phi) = \sup\{H^*(\phi, N) \mid N \in \mathcal{C}(V)\}.$$

It is known from [26, Theorem 6.12] that, for $\phi^* : V^* \to V^*$ the dual endomorphism of the algebraic dual V^* of V,

$$\mathrm{ent}^*_{\dim}(\phi) = \mathrm{ent}_{\dim}(\phi^*). \tag{1.3}$$

As a consequence, the adjoint dimension entropy $\mathrm{ent}^*_{\dim}$ is proved to take only the values 0 and ∞ (see [26, Corollary 6.16]). So, imitating the same approach used in [24] for the adjoint algebraic entropy, a motivating idea to introduce the topological entropy ent^* in this paper is to "topologize" $\mathrm{ent}^*_{\dim}$ so that it admits all possible values in $\mathbb{N} \cup \{\infty\}$. In fact, if V is a linearly compact vector space and $\phi : V \to V$ a continuous endomorphism, then $\mathcal{B}(V) \subseteq \mathcal{C}(V)$, furthermore $H^*(\phi, U) = H^*_{\dim}(\phi, U)$ for $U \in \mathcal{B}(V)$ (see Lemma 3.10), and so $\mathrm{ent}^*(\phi) \leq \mathrm{ent}^*_{\dim}(\phi)$.

Moreover, if V is a discrete vector space and $\phi : V \to V$ is an endomorphism, then $\widehat{V}$ is linearly compact; it is also known from [5] that $\mathrm{ent}(\phi) = \mathrm{ent}_{\dim}(\phi)$, and so Theorem 1.4 gives the equality

$$\mathrm{ent}_{\dim}(\phi) = \mathrm{ent}^*(\widehat{\phi}), \tag{1.4}$$

that appears to be more natural with respect to that in Equation (1.3).

As a consequence of the Bridge Theorem 1.4 and the Addition Theorem 1.2 for the topological entropy ent^*, we easily obtain the Addition Theorem for the algebraic entropy ent proved in [5] (see Corollary 6.3). Clearly, in the same way, one could deduce Theorem 1.2 from Corollary 6.3 and Theorem 1.4.

We conclude by leaving an open question about the so-called Uniqueness Theorem. Indeed, a Uniqueness Theorem for the topological entropy in the category of compact groups and continuous homomorphisms was proved by Stojanov in [41]. The same result requires a shorter list of axioms restricting to compact abelian groups (see [11, Corollary 3.3]).

We would say that the Uniqueness Theorem holds for the topological entropy ent^* in the category of all locally linearly compact vector spaces over a discrete field $\mathbb{K}$ and their continuous homomorphisms, if ent^* was the unique collection of functions $\mathrm{ent}^*_V : \mathrm{End}(V) \to \mathbb{N} \cup \{\infty\}$, $\phi \mapsto \mathrm{ent}^*(\phi)$, satisfying for every locally linearly compact vector space V over $\mathbb{K}$: Invariance under conjugation (see Proposition 3.15(a)), Continuity for inverse limits (see Proposition 3.15(e)), Addition Theorem, and $\mathrm{ent}^*({}_F\beta) = \dim F$ for any finite-dimensional vector space F over $\mathbb{K}$, where $V = \bigoplus_{n=-\infty}^{0} F \oplus \prod_{n=1}^{\infty} F$ is endowed with the topology inherited from the product topology of $\prod_{n\in\mathbb{Z}} F$, and ${}_F\beta : V \to V$, $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$ is the left Bernoulli shift (see Example 3.16).

Question 1.5. *Does the Uniqueness Theorem hold for the topological entropy* ent^* *in the category of locally linearly compact vector spaces over a discrete field* $\mathbb{K}$*?*

The validity of the Uniqueness Theorem for ent^* in the category of all linearly compact vector spaces over a discrete field $\mathbb{K}$ follows from the Uniqueness Theorem for the dimension entropy $\mathrm{ent}_{\dim}$ in the category of all discrete vector spaces over $\mathbb{K}$ proved in [26] and from Equation (1.4).

It is a pleasure to thank the referee for the careful reading and his/her comments and suggestions, also for a further development of the theory exposed in the present paper.

2. Background on locally linearly compact vector spaces

2.1. Locally linearly compact vector spaces

Fix an arbitrary field $\mathbb{K}$ endowed with the discrete topology. A Hausdorff topological $\mathbb{K}$-vector space V is *linearly topologized* if it admits a neighborhood basis at 0 consisting of linear subspaces of V. Clearly, a linear subspace W of V with the induced topology is still linearly topologized, and the quotient vector space V/W endowed with the quotient topology turns out to be linearly topologized whenever W is also closed in V. A finite-dimensional linearly topologized vector space is necessarily discrete.

A *linear variety* of a linearly topologized vector space V is a coset $v + W$, where W is a linear subspace of V and $v \in V$; the linear variety $v + W$ is *closed* if W is closed in V. Following Lefschetz [34], a *linearly compact space* V is a linearly topologized vector space such that any collection of closed linear varieties of V with the finite intersection property has non-empty intersection. We recall the following known properties that we frequently use in the paper.

Proposition 2.1 *([34, page 78], [32, Propositions 2 and 9],[45, Theorem 28.5]). Let V be a linearly topologized vector space.*

(a) If W is a linearly compact linear subspace of V, then W is closed.
(b) If V is linearly compact and W is a closed linear subspace of V, then W is linearly compact.
(c) If W is another linearly topologized vector space and $\phi : V \to W$ is continuous homomorphism and V is linearly compact, then ϕW is linearly compact as well.
(d) If V is discrete, then V is linearly compact if and only if V has finite dimension.
(e) If W is a closed linear subspace of V, then V is linearly compact if and only if W and V/W are linearly compact.
(f) The direct product of linearly compact vector spaces is linearly compact.
(g) An inverse limit of linearly compact vector spaces is linearly compact.
(h) If V is linearly compact, then V is complete.

The following result ensures that a continuous isomorphism $\phi : V \to W$ of linearly topologized vector spaces is also open whenever V is linearly compact.

Proposition 2.2 *([6, Proposition 1.1(v)]). Let V be a linearly compact vector space and W a linearly topologized vector space. If $\phi\colon V \to W$ is a continuous homomorphism, then $\phi\colon V \to \phi V$ is open. In particular, any continuous bijective homomorphism $\phi\colon V \to W$ is a topological isomorphism.*

Recall that a *linear filter base* $\mathcal{N}$ of a linearly compact vector space V is a non-empty family of linear subspaces of V satisfying

$$\forall U, W \in \mathcal{N},\ \exists Z \in \mathcal{N},\ \text{such that } Z \leq U \cap W.$$

Theorem 2.3 *([45, Theorem 28.20]). Let $\mathcal{N}$ be a linear filter base of a linearly compact vector space V.*

(a) If W is a linearly topologized vector space and $\phi : V \to W$ a continuous homomorphism, then

$$\phi\left(\bigcap_{N\in\mathcal{N}} \overline{N}\right) = \bigcap_{N\in\mathcal{N}} \overline{\phi N}.$$

(b) If each member of $\mathcal{N}$ is closed and if M is a closed linear subspace of V, then

$$\bigcap_{N\in\mathcal{N}} (M+N) = M + \bigcap_{N\in\mathcal{N}} N.$$

Remark 2.4. In [30] Grothendieck introduced axioms for an abelian category $\mathcal{A}$ concerning the existence and some properties of infinite direct sums and products. In particular, we are interested in the following axiom.

(Ab5*) The category $\mathcal{A}$ is complete and if A is an object in $\mathcal{A}$, $\{A_i\}_{i\in I}$ is a lattice of subobjects of A and B is any subobject of A, then

$$\bigcap_{i\in I}(B + A_i) = B + \bigcap_{i\in I} A_i.$$

If the abelian category $\mathcal{A}$ satisfies the axiom (Ab5*), then the inverse limit functor from the category of inverse systems on $\mathcal{A}$ to $\mathcal{A}$ is an exact additive functor (see [30, §1.5, §1.8]).

Now let us consider the complete abelian category ${}_{\mathbb{K}}\mathrm{LC}$ of all linearly compact $\mathbb{K}$-vector spaces. The subobjects of a linearly compact vector space V are the closed linear subspaces of V. Thus, by Theorem 2.3(a), the category ${}_{\mathbb{K}}\mathrm{LC}$ satisfies the axiom (Ab5*), and so the corresponding inverse limit functor is exact.

A topological $\mathbb{K}$-vector space V is *locally linearly compact* (briefly, l.l.c.) if there exists a neighborhood basis at 0 in V consisting of linearly compact open linear subspaces of V (see [34]). In particular, every l.l.c. vector space V is linearly topologized. The structure of l.l.c. vector spaces is described by the following result.

Theorem 2.5 *([34, (27.10), page 79]). A linearly topologized vector space V is l.l.c. if and only if $V \cong_{top} V_c \oplus V_d$, where V_c is a linearly compact linear subspace and V_d is a discrete linear subspace of V. In particular, $V_c \in \mathcal{B}(V)$.*

Thus, every l.l.c. vector space is complete. Moreover, the class of all l.l.c. vector spaces is closed under taking closed linear subspaces, quotient vector spaces modulo closed linear subspaces and extensions.

In view of [5, Proposition 3], for an l.l.c. vector space V and W a closed linear subspace of V,

$$\mathcal{B}(W) = \{U \cap W \mid U \in \mathcal{B}(V)\} \quad \text{and} \quad \mathcal{B}(V/W) = \left\{ \frac{U+W}{W} \mid U \in \mathcal{B}(V) \right\}. \tag{2.1}$$

2.2. Lefschetz Duality

Let V be an l.l.c. vector space and let $\mathrm{CHom}(\mathrm{V}, \mathbb{K})$ be the vector space of all continuous characters $V \to \mathbb{K}$. For a linear subspace A of V, the *annihilator* of A in $\mathrm{CHom}(\mathrm{V}, \mathbb{K})$ is

$$A^\perp = \{\chi \in \mathrm{CHom}(\mathrm{V}, \mathbb{K}) : \chi(A) = 0\}.$$

By [33, 4.(1'), page 86], the continuous characters in $\mathrm{CHom}(\mathrm{V}, \mathbb{K})$ separate the points of V.

We denote by $\widehat{V}$ the vector space $\mathrm{CHom}(\mathrm{V}, \mathrm{K})$ endowed with the topology having the family

$$\{A^\perp \mid A \leq V,\ A \text{ linearly compact}\}$$

as neighborhood basis at 0. The linearly topologized vector space $\widehat{V}$ is an l.l.c. vector space (see [34]). In particular, $\widehat{V}$ is discrete whenever V is linearly compact since $0 = V^\perp$ is open. More generally, V is discrete if and only if $\widehat{V}$ is linearly compact, and V is linearly compact if and only if $\widehat{V}$ is discrete. Moreover, if V has finite dimension, then V is discrete and $\widehat{V}$ is the algebraic dual of V, so $\widehat{V}$ is isomorphic to V.

By Lefschetz Duality, V is canonically isomorphic to $\widehat{\widehat{V}}$; indeed, the canonical map

$$\omega_V : V \to \widehat{\widehat{V}} \text{ such that } \omega_V(v)(\chi) = \chi(v) \quad \forall v \in\ V, \forall \chi \in \widehat{V}, \tag{2.2}$$

is a topological isomorphism. More precisely, denote by $_{\mathbb{K}}\mathrm{LLC}$ the category whose objects are all l.l.c. vector spaces over $\mathbb{K}$ and whose morphisms are the continuous homomorphisms; let

$$\widehat{-} :_{\mathbb{K}} \mathrm{LLC} \to_{\mathbb{K}} \mathrm{LLC}$$

be the duality functor, which is defined on the objects by $V \mapsto \widehat{V}$ and on the morphisms sending $\phi : V \to W$ to $\widehat{\phi} : \widehat{W} \to \widehat{V}$ such that $\phi(\chi) = \chi \circ \phi$ for every $\chi \in \widehat{W}$. Clearly, the biduality functor $\widehat{\widehat{-}}\colon {_{\mathbb{K}}\mathrm{LLC}} \to {_{\mathbb{K}}\mathrm{LLC}}$ is defined by composing $\widehat{-}$ with itself.

Theorem 2.6 *(Lefschetz Duality Theorem). The biduality functor* $\widehat{\widehat{-}}\colon {_{\mathbb{K}}\mathrm{LLC}} \to {_{\mathbb{K}}\mathrm{LLC}}$ *and the identity functor* $\mathrm{id}\colon {_{\mathbb{K}}\mathrm{LLC}} \to {_{\mathbb{K}}\mathrm{LLC}}$ *are naturally isomorphic.*

In particular, the duality functor defines a duality between the subcategory $_{\mathbb{K}}\mathrm{LC}$ of linearly compact vector spaces over $\mathbb{K}$ and the subcategory $_{\mathbb{K}}\mathrm{Vect}$ of discrete vector spaces over $\mathbb{K}$.

We recall that, for a continuous homomorphism $\phi\colon V \to W$ of l.l.c. vector spaces,

(a) if ϕ is injective and it is open onto its image, then $\widehat{\phi}$ is surjective;
(b) if ϕ is surjective, then $\widehat{\phi}$ is injective.

As a consequence of Lefschetz Duality Theorem, every linearly compact vector space is a product of one-dimensional vector spaces (see [34, Theorem 32.1]). Moreover, one can derive the following result (see [5, Proposition 4 and Corollary 1]); note that every compact linearly topologized vector space is linearly compact.

Proposition 2.7. *Let* $\mathbb{K}$ *be a discrete finite field and let* V *be a* $\mathbb{K}$*-vector space.*

(a) If V is linearly compact, then V is compact.
(b) If V is l.l.c., then V is locally compact.

The above proposition implies that, for an l.l.c. vector space V over a discrete finite field, $\mathcal{B}(V)$ is a neighborhood basis at 0 in V consisting of compact open subgroups; so, we obtain the following result.

Corollary 2.8. *An l.l.c. vector space over a discrete finite field is a totally disconnected locally compact abelian group.*

Given an l.l.c. vector space V, let B be a linear subspace of $\widehat{V}$. The *annihilator* of B in V is

$$B^\top = \{x \in V : \chi(x) = 0 \text{ for every } \chi \in B\}.$$

For every linear subspace B of $\widehat{V}$, we have $\omega_V(B^\top) = B^\perp$.

We recall now some known properties of the annihilators (see [34,33]) that we use further on.

Lemma 2.9. *Let V be an l.l.c. vector space and A a linear subspace of V. Then:*

(a) if B is another linear subspace of V and $A \leq B$, then $B^\perp \leq A^\perp$;
(b) $A^\perp = \overline{A}^\perp$;
(c) $A^\perp$ is a closed linear subspace of $\widehat{V}$;
(d) if A is a closed, $(A^\perp)^\top = A$.

Lemma 2.10. *Let V be an l.l.c. vector space and $A_1, \dots, A_n$ linear subspaces of V. Then:*

(a) $\left(\sum_{i=1}^n A_i\right)^\perp = \bigcap_{i=1}^n A_i^\perp$;
(b) $\overline{\sum_{i=1}^n A_i^\perp} \subseteq \left(\bigcap_{i=1}^n A_i\right)^\perp$;
(c) if $A_1, \dots, A_n$ are closed, $\left(\bigcap_{i=1}^n A_i\right)^\perp = \overline{\sum_{i=1}^n A_i^\perp}$;
(d) if $A_1, \dots, A_n$ are linearly compact, $\left(\bigcap_{i=1}^n A_i\right)^\perp = \sum_{i=1}^n A_i^\perp$.

The following is a well-know result concerning l.l.c. vector spaces (see [33, §10.12.(6), §12.1.(1)]).

Remark 2.11. Let V be an l.l.c. vector space and U a closed linear subspace of V, then

$$\widehat{V/U} \cong_{top} U^\perp \text{ and } \widehat{U} \cong_{top} \widehat{V}/U^\perp.$$

The first topological isomorphism is the following. Let $\pi\colon V \to V/U$ be the canonical projection, consider the continuous injective homomorphism $\widehat{\pi}\colon \widehat{V/U} \to \widehat{V}$; noting that $\widehat{\pi}(\widehat{V/U}) = U^\perp$, let

$$\alpha\colon \widehat{V/U} \to U^\perp, \quad \chi \mapsto \widehat{\pi}(\chi), \tag{2.3}$$

that turns out to be a topological isomorphism.

To find explicitly the second topological isomorphism, let $\iota : U \to V$ be the topological embedding of U in V and consider the continuous surjective homomorphism $\widehat{\iota}\colon \widehat{V} \to \widehat{U}$; in particular, $\widehat{\iota}(\chi) = \chi \restriction_U$ for every $\chi \in \widehat{V}$. Since $\ker \widehat{\iota} = U^\perp$, consider the continuous isomorphism induced by $\widehat{\iota}$

$$\beta\colon \widehat{V}/U^\perp \to \widehat{U}, \quad \chi + U^\perp \mapsto \chi \circ \iota, \tag{2.4}$$

that turns out to be a topological isomorphism.

A consequence of the above topological isomorphisms is the following relation that we use in the last section.

Lemma 2.12. *Let V be a l.l.c. vector space, and let A, B be closed linear subspaces of V such that $B \leq A$. Then $\widehat{A/B} \cong_{top} B^\perp/A^\perp$.*

Proof. Let $\iota\colon A/B \to V/B$ be the topological embedding and $\beta\colon \widehat{V/B}/(A/B)^\perp \to \widehat{A/B}$ the topological isomorphism given by Equation (2.4). Set $\pi\colon \widehat{V/B} \to \widehat{V/B}/(A/B)^\perp$ be the canonical projection. Since $\widehat{\iota} = \beta \circ \pi$, $\widehat{\iota}$ is an open continuous surjective homomorphism. Let $\alpha : \widehat{V/B} \to B^\perp$ be the topological isomorphism given by Equation (2.3); then $\varphi = \beta \circ \pi \circ \alpha^{-1}$ is an open continuous surjective homomorphism $\varphi : B^\perp \to \widehat{A/B}$.

$$\begin{tikzcd} B^\perp \arrow[r, "\alpha^{-1}"] \arrow[rrr, bend right, "\varphi"'] & \widehat{V/B} \arrow[r, "\pi"] \arrow[rr, bend left, "\widehat{\iota}"] & \frac{\widehat{V/B}}{(A/B)^\perp} \arrow[r, "\beta"] & \widehat{A/B} \end{tikzcd}$$

As $\ker \varphi = A^\perp$, we conclude that $B^\perp/A^\perp \cong_{top} \widehat{A/B}$. $\square$

3. Properties and examples

3.1. Existence of the limit and basic properties

Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. For every $n \in \mathbb{N}_+$, $C_n(\phi, U) \in \mathcal{B}(V)$ by Proposition 2.1(b); hence, $U/C_n(\phi, U)$ has finite dimension by Proposition 2.1(d,e).

Moreover, $C_n(\phi, U) \geq C_{n+1}(\phi, U)$ for every $n \in \mathbb{N}_+$, so we have the following decreasing chain

$$U = C_1(\phi, U) \geq C_2(\phi, U) \geq \dots \geq C_n(\phi, U) \geq C_{n+1}(\phi, U) \geq \dots$$

in $\mathcal{B}(V)$. The largest ϕ-invariant subspace of U, namely,

$$C(\phi, U) = \bigcap_{n \in \mathbb{N}_+} C_n(\phi, U),$$

is the ϕ-*cotrajectory* of U in V; it is a linearly compact linear subspace of V by Proposition 2.1(b).

Lemma 3.1. *Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. For every $n \in \mathbb{N}_+$, $C_n(\phi, U)/C_{n+1}(\phi, U)$ has finite dimension and $C_{n+1}(\phi, U)/C_{n+2}(\phi, U)$ is isomorphic to a linear subspace of $C_n(\phi, U)/C_{n+1}(\phi, U)$.*

Proof. To simplify the notation, let $C_n = C_n(\phi, U)$ for every $n \in \mathbb{N}_+$.

Fix $n \in \mathbb{N}_+$. Since $C_{n+1} \leq C_n$, and C_{n+1} is open while C_n is linearly compact, C_n/C_{n+1} has finite dimension by Proposition 2.1(d,e).

Since $C_{n+2} = C_{n+1} \cap \phi^{-n-1}U$ and $C_{n+1} = U \cap \phi^{-1}C_n$, it follows that

$$\frac{C_{n+1}}{C_{n+2}} \cong \frac{C_{n+1} + \phi^{-n-1}U}{\phi^{-n-1}U} \leq \frac{\phi^{-1}C_n + \phi^{-n-1}U}{\phi^{-n-1}U}.$$

On the other hand, since $C_{n+1} = C_n \cap \phi^{-n}U$,

$$\frac{C_n}{C_{n+1}} \cong \frac{C_n + \phi^{-n}U}{\phi^{-n}U}.$$

Let $\tilde{\phi}$: $V/\phi^{-n-1}U \to V/\phi^{-n}U$ be the injective homomorphism induced by ϕ. Then

$$\tilde{\phi}\left(\frac{\phi^{-1}C_n + \phi^{-n-1}U}{\phi^{-n-1}U}\right) \leq \frac{C_n + \phi^{-n}U}{\phi^{-n}U}$$

and so the thesis follows. □

The following result shows that the limit in the definition of the topological entropy $H^*(\phi, U)$ (see Equation (1.1)) exists and it is a natural number.

Proposition 3.2. *Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. For every $n \in \mathbb{N}_+$, let*

$$\gamma_n = \dim \frac{C_n(\phi, U)}{C_{n+1}(\phi, U)}.$$

Then the sequence of non-negative integers $\{\gamma_n\}_{n\in\mathbb{N}_+}$ is decreasing, hence stationary. Moreover, $H^*(\phi, U) = \gamma$, where γ is the value of the stationary sequence $\{\gamma_n\}_{n\in\mathbb{N}_+}$ for $n \in \mathbb{N}_+$ large enough.

Proof. To simplify the notation, let $C_n = C_n(\phi, U)$ for every $n \in \mathbb{N}_+$. By Lemma 3.1, $\gamma_{n+1} \leq \gamma_n$ for every $n \in \mathbb{N}_+$. Hence, there exist $\gamma \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\gamma_n = \gamma$ for all $n \geq n_0$. Since

$$\frac{U}{C_n} \cong \frac{U/C_{n+1}}{C_n/C_{n+1}},$$

it follows that

$$\dim \frac{U}{C_{n+1}} = \dim \frac{U}{C_n} + \gamma_n,$$

and so, for every $n \in \mathbb{N}$,

$$\dim \frac{U}{C_{n_0+n}} = \dim \frac{U}{C_{n_0}} + n\gamma.$$

Hence, $H^*(\phi, U) = \lim_{n\to\infty} \frac{1}{n}\left(\dim \frac{U}{C_{n_0}} + n\gamma\right) = \gamma$. □

Remark 3.3. As pointed out above and in the introduction, the main property that permits to introduce the topological entropy ent* and to prove the existence of the limit in its definition (see Proposition 3.2) is that for V an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$, the quotient $U/C_n(\phi, U)$ has finite dimension for every $n \in \mathbb{N}_+$. The same property, and then also the result in Proposition 3.2, holds for a wider class of linearly topologized vector spaces that we call locally linearly precompact.

Let V be a linearly topologized vector space over a discrete field $\mathbb{K}$. We say that V is *linearly precompact* if for every open linear subspace U of V there exists a finite dimensional linear subspace F of V such that $U + F = V$ (i.e., V/U has finite dimension). Moreover, V is *locally linearly precompact* if it admits a neighborhood basis at 0 of linearly precompact open linear subspaces. Clearly, a linearly compact vector space is linearly precompact and an l.l.c. vector space is locally linearly precompact.

So, one could study the topological entropy ent^* in the general case of continuous endomorphisms of locally linearly precompact vector spaces. Nevertheless, in this paper we remain in the class of l.l.c. vector spaces, where the stronger condition on the open linear subspaces of being linearly compact ensures a richer theory.

We see now that $H^*(\phi, -)$ is monotone decreasing in the following sense.

Lemma 3.4. *Let V be an l.l.c. vector space, $\phi\colon V \to V$ a continuous endomorphism and $U, U' \in \mathcal{B}(V)$. If $U' \leq U$, then $H^*(\phi, U) \leq H^*(\phi, U')$.*

Proof. Let $U, U' \in \mathcal{B}(V)$ such that $U' \leq U$. As $C_n(\phi, U') \leq C_n(\phi, U)$ for all $n \in \mathbb{N}_+$, from

$$\frac{U'/C_n(\phi, U')}{(U' \cap C_n(\phi, U))/C_n(\phi, U')} \cong \frac{U'}{U' \cap C_n(\phi, U)} \cong \frac{C_n(\phi, U) + U'}{C_n(\phi, U)},$$

since all terms are finite-dimensional, it follows that

$$\dim \frac{U'}{C_n(\phi, U')} \geq \dim \frac{C_n(\phi, U) + U'}{C_n(\phi, U)}$$

Since $C_n(\phi, U) \leq C_n(\phi, U) + U' \leq U$,

$$\begin{aligned} \dim \frac{C_n(\phi, U) + U'}{C_n(\phi, U)} &= \dim \frac{U}{C_n(\phi, U)} - \dim \frac{U}{C_n(\phi, U) + U'} \\ &\geq \dim \frac{U}{C_n(\phi, U)} - \dim \frac{U}{U'}. \end{aligned}$$

Therefore, $\dim \frac{U'}{C_n(\phi, U')} \geq \dim \frac{U}{C_n(\phi, U)} - \dim \frac{U}{U'}$, and so $H^*(\phi, U') \geq H^*(\phi, U)$, since $\dim \frac{U}{U'}$ does not depend on n. $\square$

As a straightforward consequence of Lemma 3.4 it is possible to compute the topological entropy by restricting to any neighborhood basis at 0 in V contained in $\mathcal{B}(V)$:

Corollary 3.5. *Let V be an l.l.c. vector space and $\phi : V \to V$ a continuous endomorphism. If $\mathcal{B} \subseteq \mathcal{B}(V)$ is a neighborhood basis at 0 in V, then*

$$\mathrm{ent}^*(\phi) = \sup\{H^*(\phi, U) \mid U \in \mathcal{B}\}.$$

Consequently, if W is an open linear subspace of V, then $\mathrm{ent}^*(\phi) = \sup\{H^*(\phi, U) \mid U \in \mathcal{B}(W)\}$.

We consider now the case of topological entropy zero. The following result clearly follows from the definitions and from Corollary 3.5, it shows that ent^* vanishes in presence of a neighborhood basis at 0 consisting of *invariant* open linearly compact linear subspaces.

Lemma 3.6. *Let V be an l.l.c. vector space and $\phi : V \to V$ a continuous endomorphism. If $\phi U \leq U$ for every $U \in \mathcal{B}$, where $\mathcal{B} \subseteq \mathcal{B}(V)$ is a neighborhood basis at 0 in V, then $\mathrm{ent}^*(\phi) = 0$.*

Example 3.7. For V an l.l.c. vector space, $\mathrm{ent}^*(id_V) = 0$, where $id_V : V \to V$ is the identity automorphism.

The following result concerns the general case of an endomorphism of zero topological entropy.

Proposition 3.8. *Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. The following conditions are equivalent:*

(a) $H^*(\phi, U) = 0$;
(b) there exists $n \in \mathbb{N}_+$ *such that* $C(\phi, U) = C_n(\phi, U)$;
(c) $C(\phi, U)$ *is open.*

In particular, $\mathrm{ent}^*(\phi) = 0$ *if and only if* $C(\phi, U)$ *is open for all* $U \in \mathcal{B}(V)$.

Proof. (a)⇒(b) Since $\dim \frac{C_n(\phi,U)}{C_{n+1}(\phi,U)} = 0$ eventually by Proposition 3.2, there exists $n_0 \in \mathbb{N}_+$ such that $C_n(\phi, U) = C_{n_0}(\phi, U)$ for every $n \geq n_0$.

(b)⇒(c) is clear since $C_n(\phi, U) \in \mathcal{B}(V)$ for every $n \in \mathbb{N}_+$.

(c)⇒(a) If $C(\phi, U)$ is open, then $U/C(\phi, U)$ has finite dimension by Proposition 2.1(d,e). Therefore, $H^*(\phi, U) \leq \lim_{n\to\infty} \frac{1}{n} \dim \frac{U}{C(\phi,U)} = 0$. □

As a consequence, the topological entropy is always zero on discrete vector spaces, and so in particular on finite-dimensional vector spaces:

Corollary 3.9. *Let $\phi\colon V \to V$ be an endomorphism of a discrete vector space V. Then* $\mathrm{ent}^*(\phi) = 0$.

On the other hand, for linearly compact vector spaces we can simplify the defining formula of the topological entropy as follows. Note that if V is a linearly compact vector space, then $\mathcal{B}(V) = \{U \leq V \mid U \text{ open}\}$; indeed, an open linear subspace of a linearly compact vector space is necessarily linearly compact by Proposition 2.1(b).

Lemma 3.10. *Let V be a linearly compact vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. Then*

$$H^*(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{V}{C_n(\phi, U)}.$$

Proof. Since $U, C_n(\phi, U) \in \mathcal{B}(V)$, we have that V/U and $V/C_n(\phi, U)$ have finite dimension by Proposition 2.1(d,e). Then $\dim \frac{U}{C_n(\phi,U)} = \dim \frac{V}{C_n(\phi,U)} - \dim \frac{V}{U}$ and hence

$$H^*(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \left(\dim \frac{U}{C_n(\phi, U)} - \dim \frac{V}{U} \right) = \lim_{n\to\infty} \frac{1}{n} \dim \frac{V}{C_n(\phi, U)},$$

so we have the thesis. □

By Corollary 2.8, an l.l.c. vector space V over a discrete finite field is in particular a totally disconnected locally compact abelian group. We end this section by relating ent^* to the topological entropy h_{top}.

Proposition 3.11. *Let V be an l.l.c. vector space over a discrete finite field $\mathbb{F}$ and $\phi : V \to V$ a continuous endomorphism. Then*

$$h_{top}(\phi) = \mathrm{ent}^*(\phi) \cdot \log |\mathbb{F}|.$$

Proof. As $\mathcal{B}(V)$ is a local basis at 0 in V contained in $\mathcal{B}_{gr}(V)$, and since also $H_{top}(\phi, -)$ is monotone decreasing (see [13, Remark 4.5.1(b)]), we have that

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) \mid U \in \mathcal{B}(V)\}. \tag{3.1}$$

Now, for every $U \in \mathcal{B}(V)$,

$$\left|\frac{U}{C_n(\phi, U)}\right| = |\mathbb{F}|^{\dim \frac{U}{C_n(\phi,U)}},$$

and so

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{1}{n} \log\left|\frac{U}{C_n(\phi, U)}\right| = \lim_{n\to\infty} \frac{1}{n} \dim \frac{U}{C_n(\phi, U)} \log|\mathbb{F}| = H^*(\phi, U) \cdot \log|\mathbb{F}|.$$

Thus, $h_{top}(\phi) = \mathrm{ent}^*(\phi) \cdot \log|\mathbb{F}|$. □

The previous result points out that, as long as we are dealing with l.l.c. vector spaces over discrete finite fields, the topological entropy ent^* turns out to be a rescaling of the topological entropy h_{top} and the most natural logarithm to compute the topological entropy h_{top} is the one with base $|\mathbb{F}|$.

Remark 3.12. When the discrete field $\mathbb{K}$ is infinite, even if an l.l.c. vector space over $\mathbb{K}$ is no more a locally compact group, one can still compare ent^* with the topological entropy h_{top} for uniformly continuous maps of uniform spaces introduced by Hood [31]. We leave open the problem to find a relation between ent^* and h_{top} in this case. It seems that they are different, at least in the positive values: the Bernoulli shift in this case has infinite topological entropy h_{top}, while ent^* is finite (see Example 3.16 below).

As the latter remark highlights, it could be meaningful to dedicate more attention to the dependence of the topological entropy ent^* on the choice of the discrete field $\mathbb{K}$:

Problem 3.13. *Let $\mathbb{E}$ be a field extension of $\mathbb{K}$. In which cases there exist functors*

$$\mathrm{Flow}({}_{\mathbb{K}}\mathrm{LC}) \rightleftarrows \mathrm{Flow}({}_{\mathbb{E}}\mathrm{LC}),$$

induced by restriction/extension of scalars? Assume such functors exist, how does ent^* *behave under transportation along those functors?*

Problem 3.13 will be discussed in the forthcoming paper [4].

3.2. Fundamental properties

In this section we list the general properties and examples concerning the topological entropy ent^*.

Lemma 3.14. *Let V be an l.l.c. vector space, $\phi\colon V \to V$ a continuous endomorphism, W a closed ϕ-invariant linear subspace of V and $\overline{\phi}\colon V/W \to V/W$ the continuous endomorphism induced by ϕ. For $n \in \mathbb{N}_+$ and $U \in \mathcal{B}(V)$ and*

$$C_n(\phi\restriction_W, U \cap W) = C_n(\phi, U) \cap W \quad \textit{and} \quad C_n\Big(\overline{\phi}, \frac{U+W}{W}\Big) = \frac{C_n(\phi, U+W)}{W}.$$

Proof. Let $n \in \mathbb{N}_+$ and $U \in \mathcal{B}(V)$. Then $\phi\restriction_W^{-n}(U \cap W) = \phi^{-n}U \cap W$ and

$$\overline{\phi}^{-n}\left(\frac{U+W}{W}\right) = \frac{\phi^{-n}(U+W)+W}{W} = \frac{\phi^{-n}(U+W)}{W}.$$

The thesis follows from these equalities. □

We collect in the next result all the typical properties of an entropy function, that are satisfied by the topological entropy ent^*.

Proposition 3.15. *Let V be an l.l.c. vector space and $\phi\colon V\to V$ a continuous endomorphism.*

(a) (Invariance under conjugation) If $\alpha\colon V\to W$ is a topological automorphism of l.l.c. vector spaces, then

$$\mathrm{ent}^*(\alpha\phi\alpha^{-1})=\mathrm{ent}^*(\phi).$$

(b) (Monotonicity) If W is a closed ϕ-invariant linear subspace of V and $\overline{\phi}\colon V/W\to V/W$ the continuous endomorphism induced by ϕ, then

$$\mathrm{ent}^*(\phi)\geq \mathrm{ent}^*(\phi\restriction_W)+\mathrm{ent}^*(\overline{\phi}).$$

In particular, $\mathrm{ent}^*(\phi)\geq\max\{\mathrm{ent}^*(\phi\restriction_W),\mathrm{ent}^*(\overline{\phi})\}$. (If W is open, $\mathrm{ent}^*(\phi)=\mathrm{ent}^*(\phi\restriction_W)$.)

(c) (Logarithmic law) If $k\in\mathbb{N}$, then $\mathrm{ent}^*(\phi^k)=k\cdot\mathrm{ent}^*(\phi)$.

(d) (weak Addition Theorem) If $V=V_1\times V_2$ for some l.l.c. vector spaces V_1,V_2, and $\phi=\phi_1\times\phi_2:V\to V$ for some continuous endomorphisms $\phi_i\colon V_i\to V_i$, $i=1,2$, then

$$\mathrm{ent}^*(\phi)=\mathrm{ent}^*(\phi_1)+\mathrm{ent}^*(\phi_2).$$

(e) (Continuity on inverse limits) Let $\{W_i\mid i\in I\}$ be a directed system (under inverse inclusion) of closed ϕ-invariant linear subspaces of V. If $V=\varprojlim V/W_i$, then

$$\mathrm{ent}^*(\phi)=\sup_{i\in I}\mathrm{ent}^*(\overline{\phi}_{W_i}),$$

where any $\overline{\phi}_{W_i}\colon V/W_i\to V/W_i$ is the continuous endomorphism induced by ϕ.

Proof. (a) Let $U\in\mathcal{B}(W)$ and $n\in\mathbb{N}_+$. Since $C_n(\alpha\phi\alpha^{-1},U)=\alpha(C_n(\phi,\alpha^{-1}U))$, it follows that

$$\dim\frac{U}{C_n(\alpha\phi\alpha^{-1},U)}=\dim\frac{\alpha(\alpha^{-1}U)}{\alpha C_n(\phi,\alpha^{-1}U)}=\dim\frac{\alpha^{-1}U}{C_n(\phi,\alpha^{-1}U)}.$$

Hence $H^*(\alpha\phi\alpha^{-1},U)=H^*(\phi,\alpha^{-1}U)$. Since α is a topological isomorphism and $U\in\mathcal{B}(W)$ if and only if $\alpha^{-1}U\in\mathcal{B}(V)$, α induces a bijection between $\mathcal{B}(W)$ and $\mathcal{B}(V)$. Thus, $\mathrm{ent}^*(\alpha\phi\alpha^{-1})=\mathrm{ent}^*(\phi)$.

(b) Let $U\in\mathcal{B}(V)$ and $n\in\mathbb{N}_+$. Lemma 3.14 implies that $C_n\left(\overline{\phi},\frac{U+W}{W}\right)=\frac{C_n(\phi,U+W)}{W}$. Since moreover $W\leq C_n(\phi,U+W)\leq U+W$, we have that

$$\frac{U+W}{C_n(\phi,U+W)}=\frac{U+W+C_n(\phi,U+W)}{W+C_n(\phi,U+W)}\cong\frac{U}{(W+C_n(\phi,U+W))\cap U}=\frac{U}{C_n(\phi,U+W)\cap U};$$

hence,

$$\frac{U+W}{W}\Big/C_n\left(\overline{\phi},\tfrac{U+W}{W}\right)\cong\frac{U+W}{C_n(\phi,U+W)}\cong\frac{U}{C_n(\phi,U+W)\cap U}. \tag{3.2}$$

Moreover, by Lemma 3.14,

$$\begin{aligned}\frac{U}{C_n(\phi,U)+U\cap W}&\cong\frac{U}{C_n(\phi,U)}\Big/\frac{C_n(\phi,U)+U\cap W}{C_n(\phi,U)}\\&\cong\frac{U}{C_n(\phi,U)}\Big/\frac{U\cap W}{C_n(\phi,U)\cap U\cap W}\cong\frac{U}{C_n(\phi,U)}\Big/\frac{U\cap W}{C_n(\phi\restriction_W,U\cap W)}.\end{aligned}\tag{3.3}$$

Since $C_n(\phi,U)+U\cap W\le C_n(\phi,U+W)\cap U$, Equations (3.2) and (3.3) yield that

$$\dim\left(\tfrac{U+W}{W}\Big/C_n(\overline{\phi},\tfrac{U+W}{W})\right)=\dim\frac{U}{C_n(\phi,U+W)\cap U}\le\dim\frac{U}{C_n(\phi,U)+U\cap W}$$
$$=\dim\frac{U}{C_n(\phi,U)}-\dim\frac{U}{C_n(\phi\restriction_W,U\cap W)}.$$

By Equation (2.1) we have the thesis.

If W is open, then $\mathcal{B}(W)$ is a neighborhood basis at 0 in V, and so $\mathrm{ent}^*(\phi)=\mathrm{ent}^*(\phi\restriction_W)$ follows by Corollary 3.5.

(c) For $k=0$, $\mathrm{ent}^*(id_V)=0$ by Example 3.7. So let $k\in\mathbb{N}_+$ and $U\in\mathcal{B}(V)$. For every $n\in\mathbb{N}_+$ we have

$$C_{nk}(\phi,U)=C_n(\phi^k,C_k(\phi,U))\quad\text{and}\quad C_n(\phi,C_k(\phi,U))=C_{n+k-1}(\phi,U).\tag{3.4}$$

Let $E=C_k(\phi,U)\in\mathcal{B}(V)$. Hence, by Lemma 3.4 and Equation (3.4),

$$k\cdot H^*(\phi,U)\le k\cdot H^*(\phi,E)=k\cdot\lim_{n\to\infty}\frac{1}{nk}\dim\frac{E}{C_{nk}(\phi,E)}=\lim_{n\to\infty}\frac{1}{n}\dim\frac{E}{C_{(n+1)k-1}(\phi,U)}$$
$$\le\lim_{n\to\infty}\frac{1}{n}\dim\frac{E}{C_{(n+1)k}(\phi,U)}=\lim_{n\to\infty}\frac{1}{n}\dim\frac{E}{C_{n+1}(\phi^k,E)}=H^*(\phi^k,E)\le\mathrm{ent}^*(\phi^k).$$

Consequently, $k\cdot\mathrm{ent}^*(\phi)\le\mathrm{ent}^*(\phi^k)$. Conversely, as $C_{nk}(\phi,U)\le E\le U$, Equation (3.4) together with Lemma 3.4 yields

$$\mathrm{ent}^*(\phi)\ge H^*(\phi,U)=\lim_{n\to\infty}\frac{1}{nk}\dim\frac{U}{C_{nk}(\phi,U)}=\lim_{n\to\infty}\frac{1}{nk}\dim\frac{U}{C_n(\phi^k,E)}$$
$$\ge\lim_{n\to\infty}\frac{1}{nk}\dim\frac{E}{C_n(\phi^k,E)}=\frac{1}{k}\cdot H^*(\phi^k,E)\ge\frac{1}{k}H^*(\phi^k,U).$$

So, $k\cdot\mathrm{ent}^*(\phi)\ge\mathrm{ent}^*(\phi^k)$.

(d) Observe that $\mathcal{B}=\{U_1\times U_2\mid U_i\in\mathcal{B}(V_i),i=1,2\}\subseteq\mathcal{B}(V)$ is neighborhood basis at 0 in V. For $U=U_1\times U_2\in\mathcal{B}$, we have that $C_n(\phi,U)=C_n(\phi_1,U_1)\times C_n(\phi_2,U_2)$ for every $n\in\mathbb{N}_+$; therefore,

$$\frac{U}{C_n(\phi,U)}=\frac{U_1\times U_2}{C_n(\phi_1,U_1)\times C_n(\phi_2,U_2)}\cong\frac{U_1}{C_n(\phi_1,U_1)}\times\frac{U_2}{C_n(\phi_2,U_2)},$$

and so

$$H^*(\phi,U)=\lim_{n\to\infty}\frac{1}{n}\frac{U}{C_n(\phi,U)}=\lim_{n\to\infty}\frac{1}{n}\dim\left(\frac{U_1}{C_n(\phi_1,U_1)}\times\frac{U_2}{C_n(\phi_2,U_2)}\right)=H^*(\phi_1,U_1)+H^*(\phi_2,U_2).$$

By Corollary 3.5, we conclude that $\mathrm{ent}^*(\phi)=\sup\{H^*(\phi,U)\mid U\in\mathcal{B}\}=\mathrm{ent}^*(\phi_1)+\mathrm{ent}^*(\phi_2)$.

(e) By item (b), $\mathrm{ent}^*(\phi)\ge\sup_{i\in I}\mathrm{ent}^*(\overline{\phi}_{W_i})$. Conversely, let $U\in\mathcal{B}(V)$. We claim that there exists $k\in I$ such that $W_k\le U$. In fact, since U is open in V, there exists an open linear subspace A belonging to the canonical neighborhood basis at 0 in $\prod_{i\in I}V/W_i$ such that $A\cap V\le U$. Namely, let $\pi_i\colon V\to V/W_i$ be the canonical projections. Since each V/W_i is linearly topologized by the quotient topology, there exists a finite family $\{U_j\mid j\in J\}$, with $J\subseteq I$, of open linear subspaces of V such that $W_j\le U_j$ for all $j\in J$ and $A=\prod_{i\in I}A_i$, where $A_j=\pi_jU_j$ for $j\in J$ and $A_i=\pi_iV$ for $i\in I\setminus J$. Since $\{W_i\mid i\in I\}$ is directed, there exists $k\in I$ such that $j\le k$ for all $j\in J$. Thus, given the open (and so closed) linear subspace

$$U_k = \bigcap_{j\in J} U_j + W_k \leq U_j$$

for $j \in J$, we have that $\varprojlim \pi_i U_k$ is closed in V since each $\pi_i U_k$ is open (and so closed) in V/W_i. Moreover, U_k is a closed linear subspace of $\varprojlim \pi_i U_k$, thus by [37, Lemma 1.1.7], it follows that

$$U_k = \varprojlim_{i\in I} \pi_i U_k \leq A \cap V \leq U;$$

therefore $W_k \leq U$.

Thus, $H^*(\phi, U) = H^*(\phi, \pi_k U)$ by Lemma 3.14, and hence

$$H^*(\phi, U) \leq \mathrm{ent}^*(\overline{\phi}_{W_k}) \leq \sup_{i\in I} \mathrm{ent}^*(\overline{\phi}_{W_i}),$$

that concludes the proof. □

The following provides the main examples in the theory of entropy functions, that is, the Bernoulli shifts.

Example 3.16.

(a) Let $V = V_d \times V_c$, where

$$V_d = \bigoplus_{n=-\infty}^{0} \mathbb{K} \quad \text{and} \quad V_c = \prod_{n=1}^{\infty} \mathbb{K},$$

be endowed with the topology inherited from the product topology of $\prod_{n\in\mathbb{Z}} \mathbb{K}$. Then V_c is linearly compact and V_d is discrete. In particular, V_c can be identified with $0 \times V_c \in \mathcal{B}(V)$.

The *left Bernoulli shift* is

$${}_{\mathbb{K}}\beta \colon V \to V, \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}.$$

The *right Bernoulli shift* is

$$\beta_{\mathbb{K}} \colon V \to V, \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}.$$

Clearly, $\beta_{\mathbb{K}}$ and ${}_{\mathbb{K}}\beta$ are topological automorphisms of V, and $\beta_{\mathbb{K}}^{-1} = {}_{\mathbb{K}}\beta$.

For every $k \in \mathbb{N}_+$, let $U_k = 0 \times \prod_{n=k}^{\infty} \mathbb{K} \in \mathcal{B}(V)$, and consider $\mathcal{B}_f(V_c) = \{U_k \mid k \in \mathbb{N}_+\} \subseteq \mathcal{B}(V)$. Since $V_c \in \mathcal{B}(V)$, the family $\mathcal{B}_f(V_c)$ is a neighborhood basis at 0 in V contained in $\mathcal{B}(V)$. Thus, by Corollary 3.5, for $\phi \in \{\beta_{\mathbb{K}}, {}_{\mathbb{K}}\beta\}$,

$$\mathrm{ent}^*(\phi) = \sup\{H^*(\phi, U) \mid U \in \mathcal{B}_f(V_c)\}.$$

For every $k \in \mathbb{N}_+$, we have that $\beta_{\mathbb{K}}(U_k) \leq U_k$, so $\mathrm{ent}^*(\beta_{\mathbb{K}}) = 0$ by Lemma 3.6.

Let $k \in \mathbb{N}_+$ and consider $U_k \in \mathcal{B}_f(V_c)$. As $U_k \leq {}_{\mathbb{K}}\beta(U_k)$, we have that $C_n({}_{\mathbb{K}}\beta, U_k) = \beta_{\mathbb{K}}^{n-1}(U_k) = U_{k+n-1}$ for every $n \in \mathbb{N}_+$. Thus,

$$H^*({}_{\mathbb{K}}\beta, U_k) = \inf_{n\in\mathbb{N}_+} \dim \frac{C_n({}_{\mathbb{K}}\beta, U_k)}{C_{n+1}({}_{\mathbb{K}}\beta, U_k)} = 1,$$

and we conclude that $\mathrm{ent}^*({}_{\mathbb{K}}\beta) = 1$ by Corollary 3.5.

(b) Let now F be a finite dimensional vector space and let $V = V_d \times V_c$, with

$$V_d = \bigoplus_{n=1}^{\infty} F \quad \text{and} \quad V_c = \prod_{n=-\infty}^{0} F,$$

be endowed with the topology inherited from the product topology of $\prod_{n\in\mathbb{Z}} F$, so V_d is discrete and V_c is linearly compact.

The *left Bernoulli shift* is

$${}_F\beta\colon V \to V, \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}},$$

while the *right Bernoulli shift* is

$$\beta_F\colon V \to V, \quad (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}.$$

Clearly, β_F and ${}_F\beta$ are topological automorphisms such that ${}_F\beta^{-1} = \beta_F$.

It is possible, slightly modifying the computations in item (a), to find that

$$\mathrm{ent}({}_F\beta) = \dim F \quad \text{and} \quad \mathrm{ent}(\beta_F) = 0.$$

By the latter example, it is clear that for a topological automorphism $\phi : V \to V$ of an l.l.c. vector space V, in general $\mathrm{ent}^*(\phi^{-1})$ does not coincide with $\mathrm{ent}^*(\phi)$; consequently, property (c) of Proposition 3.15 cannot be extended to any integer $k \in \mathbb{Z}$ by the formula $\mathrm{ent}^*(\phi^k) = |k| \cdot \mathrm{ent}^*(\phi)$. Now, in the last part of this section, we find the precise relation between $\mathrm{ent}^*(\phi^{-1})$ and $\mathrm{ent}^*(\phi)$ and we deduce that equality holds in case V is linearly compact.

Analogously to the classical modulus for topological automorphisms of locally compact groups, we define the *dimension modulus* of V by

$$\Delta_{\dim}\colon \mathrm{Aut}(V) \to \mathbb{Z}, \quad \phi \mapsto \Delta_{\dim}(\phi, U) \quad \text{for } U \in \mathcal{B}(V),$$

where

$$\Delta_{\dim}(\phi, U) = \dim \frac{\phi U}{U \cap \phi U} - \dim \frac{U}{U \cap \phi U}.$$

In the next lemma we verify that this definition is well-posed, in fact it does not depend on the choice of $U \in \mathcal{B}(V)$.

Lemma 3.17. *Let V be an l.l.c. vector space, $\phi\colon V \to V$ a topological automorphism and $U_1, U_2 \in \mathcal{B}(V)$. Then $\Delta_{\dim}(\phi, U_1) = \Delta_{\dim}(\phi, U_2)$.*

Proof. Since $\mathcal{B}(V)$ is closed under taking finite intersections, one can assume $U_1 \leq U_2$ by arguing with $U_1 \cap U_2$. Since $U_1 \cap \phi U_1 \leq \phi U_1 \leq \phi U_2$ and $U_1 \cap \phi U_1 \leq U_1 \leq U_2$, we have

$$\dim \frac{\phi U_1}{U_1 \cap \phi U_1} = \dim \frac{\phi U_2}{U_1 \cap \phi U_1} - \dim \frac{\phi U_2}{\phi U_1},$$
$$\dim \frac{U_1}{U_1 \cap \phi U_1} = \dim \frac{U_2}{U_1 \cap \phi U_1} - \dim \frac{U_2}{U_1}.$$

Since ϕ is an automorphism, $\dim \frac{\phi U_2}{\phi U_1} = \dim \frac{U_2}{U_1}$, and so

$$\Delta(\phi, U_1) = \dim \frac{\phi U_1}{U_1 \cap \phi U_1} - \dim \frac{U_1}{U_1 \cap \phi U_1} = \dim \frac{\phi U_2}{U_1 \cap \phi U_1} - \dim \frac{U_2}{U_1 \cap \phi U_1}.$$

Analogously, since $U_1 \cap \phi U_1 \leq U_2 \cap \phi U_2 \leq \phi U_2$ and $U_1 \cap \phi U_1 \leq U_2 \cap \phi U_2 \leq U_2$,

$$\dim \frac{\phi U_2}{\phi U_1 \cap \phi U_2} = \dim \frac{\phi U_2}{U_1 \cap \phi U_1} - \dim \frac{U_2 \cap \phi U_2}{U_1 \cap \phi U_1},$$
$$\dim \frac{U_2}{U_2 \cap \phi U_2} = \dim \frac{U_2}{U_1 \cap \phi U_1} - \dim \frac{U_2 \cap \phi U_2}{U_1 \cap \phi U_1}.$$

Since ϕ is an automorphism, $\dim \frac{\phi U_2}{\phi U_1} = \dim \frac{U_2}{U_1}$, and so

$$\Delta(\phi, U_2) = \dim \frac{\phi U_2}{U_2 \cap \phi U_2} - \dim \frac{U_2}{U_2 \cap \phi U_2} = \dim \frac{\phi U_2}{U_1 \cap \phi U_1} - \dim \frac{U_2}{U_1 \cap \phi U_1}.$$

Therefore, $\Delta(\phi, U_1) = \Delta(\phi, U_2)$ as required. □

It is clear from the definition that the dimension modulus is always zero for discrete vector spaces. Moreover, it follows from Lemma 3.17 that the dimension modulus is always zero also for linearly compact vector spaces, since one can take V itself as a linearly compact open linear subspace:

Lemma 3.18. *If V is either a discrete or a linearly compact vector space and $\phi : V \to V$ is a topological automorphism, then $\Delta_{\dim}(\phi) = 0$.*

The following result provides the precise relation between $\mathrm{ent}^*(\phi^{-1})$ and $\mathrm{ent}^*(\phi)$, it is inspired by its analogue for totally disconnected locally compact groups provided in [25, Proposition 3.2].

Theorem 3.19. *Let V be an l.l.c. vector space and $\phi : V \to V$ a topological automorphism of V. If $U \in \mathcal{B}(V)$, then $H^*(\phi^{-1}, U) = H^*(\phi, U) - \Delta_{\dim}(\phi)$. Hence,*

$$\mathrm{ent}^*(\phi^{-1}) = \mathrm{ent}^*(\phi) - \Delta_{\dim}(\phi).$$

Proof. Since $C_n(\phi^{-1}, U) = \phi^n C_n(\phi, U)$ and $C_{n+1}(\phi, U) = C_n(\phi, U) \cap \phi^{-1} C_n(\phi, U)$ for every $n \in \mathbb{N}_+$,

$$\dim \frac{C_n(\phi, U)}{C_{n+1}(\phi, U)} = \dim \frac{C_n(\phi, U)}{C_n(\phi, U) \cap \phi^{-1} C_n(\phi, U)}$$
$$= \dim \frac{\phi^{n+1} C_n(\phi, U)}{\phi^{n+1} C_n(\phi, U) \cap \phi^n C_n(\phi, U)} = \dim \frac{\phi C_n(\phi^{-1}, U)}{\phi C_n(\phi^{-1}, U) \cap C_n(\phi^{-1}, U)}.$$

By Proposition 3.2 and Lemma 3.17, it follows that

$$H^*(\phi, U) - H^*(\phi^{-1}, U) = \inf_{n \in \mathbb{N}_+} \left(\dim \frac{C_n(\phi, U)}{C_{n+1}(\phi, U)} - \dim \frac{C_n(\phi^{-1}, U)}{C_{n+1}(\phi^{-1}, U)} \right) =$$
$$= \inf_{n \in \mathbb{N}_+} \left(\dim \frac{\phi C_n(\phi^{-1}, U)}{\phi C_n(\phi^{-1}, U) \cap C_n(\phi^{-1}, U)} - \dim \frac{C_n(\phi^{-1}, U)}{C_n(\phi^{-1}, U) \cap \phi C_n(\phi^{-1}, U)} \right) = \Delta_{\dim}(\phi),$$

since $C_n(\phi^{-1}, U) \in \mathcal{B}(V)$ for all $n \in \mathbb{N}_+$. □

The following is a direct consequence of the above theorem and Lemma 3.18.

Corollary 3.20. *Let V be a linearly compact vector space and $\phi : V \to V$ a topological automorphism. If $U \in \mathcal{B}(V)$, then $H^*(\phi^{-1}, U) = H^*(\phi, U)$. Hence,*

$$\mathrm{ent}^*(\phi^{-1}) = \mathrm{ent}^*(\phi).$$

3.3. Limit-free Formula

This section is devoted to prove Proposition 3.23, which is a formula for the computation of the topological entropy avoiding the limit in the definition. The proof follows the technique used in [28, Proposition 3.9], which was developed by Willis in [47].

Definition 3.21. Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. Let:

- $U_0 = U$;
- $U_{n+1} = U \cap \phi U_n$, for all $n \in \mathbb{N}$;
- $U_+ = \bigcap_{n\in\mathbb{N}} U_n$.

For every $n \in \mathbb{N}_+$, $U_n \geq U_{n+1} \geq U_+$; moreover, each U_n is linearly compact and so is U_+ (see Proposition 2.1(a,b,c)). Furthermore, it is possible to prove by induction that, for every $n \in \mathbb{N}$,

$$U_n = \{u \in U \mid \exists v \in U,\ u = \phi^n(v) \text{ and } \phi^j(v) \in U\ \forall j \in \{0, \dots, n\}\}. \tag{3.5}$$

Since $C_{n+1}(\phi, U) = \{u \in U \mid \phi^k(u) \in U\ \forall k \in \{0, \dots, n\}\}$, it follows that

$$\phi^n C_{n+1}(\phi, U) = U_n \quad \forall n \in \mathbb{N}. \tag{3.6}$$

For every $n \in \mathbb{N}$, $U_n = C_{n+1}(\phi^{-1}, U)$ whenever ϕ is also injective.

In the following result we collect the main properties of the linearly compact subgroup U_+ of an l.l.c. vector space V for $U \in \mathcal{B}(V)$.

Lemma 3.22. *Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. Then:*

(a) U_+ is the largest linear subspace of U such that $U_+ \leq \phi U_+$;
(b) $U_+ = U \cap \phi U_+$;
(c) $\phi U_+/U_+$ has finite dimension.

Proof. (a) Since $U_{n+1} \leq U_n \leq \phi U_{n-1}$ for all $n \in \mathbb{N}_+$, by applying Theorem 2.3(a) to U and the decreasing chain $\{U_n\}_{n\in\mathbb{N}}$, we have

$$U_+ = \bigcap_{n\in\mathbb{N}} U_n \leq \bigcap_{n\in\mathbb{N}} \phi U_n = \phi\Big(\bigcap_{n\in\mathbb{N}} U_n\Big) = \phi U_+.$$

Moreover, for every linear subspace W of V such that $W \leq U$ and $W \leq \phi W$, it is possible to prove by induction that $W \leq U_n$ for all $n \in \mathbb{N}$, and so $W \leq U_+$.

(b) By construction,

$$U \cap \phi U_+ = \bigcap_{n\in\mathbb{N}} (U \cap \phi U_n) = \bigcap_{n\in\mathbb{N}} U_{n+1} = U_+.$$

(c) Since $U_+ = U \cap \phi U_+$ by item (b), U_+ is open in ϕU_+, which is linearly compact. Then $\phi U_+/U_+$ has finite dimension by Proposition 2.1(d,e). □

We are now in position to prove the Limit-free Formula.

Proposition 3.23 *(Limit-free Formula). Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. Then*

$$H^*(\phi, U) = \dim \frac{\phi U_+}{U_+} \tag{3.7}$$

Proof. Let $U \in \mathcal{B}(V)$. For every $n \in \mathbb{N}_+$, let

$$\gamma_n = \dim \frac{C_n(\phi, U)}{C_{n+1}(\phi, U)}.$$

By Proposition 3.2, the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is stationary, and $H^*(\phi, U) = \gamma$ where γ is the value of the stationary sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ for $n \in \mathbb{N}_+$ large enough. Hence, it suffices to prove that

$$\dim \frac{\phi U_+}{U_+} = \gamma. \tag{3.8}$$

Since $\phi U_n + U$ is linearly compact for every $n \in \mathbb{N}$ by Proposition 2.1(f), thus $\dim \frac{\phi U_n + U}{U}$ is finite, being U open, by Proposition 2.1(d,e). Moreover, since $\phi U_n \geq \phi U_{n+1}$ for all $n \in \mathbb{N}$, the sequence of non-negative integers $\left\{\dim \frac{\phi U_n + U}{U}\right\}_{n\in\mathbb{N}}$ is decreasing, and so stationary. Thus, there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\gamma = \dim \frac{C_n(\phi, U)}{C_{n+1}(\phi, U)} \quad \text{and} \quad \phi U_n + U = \phi U_{n_0} + U;$$

since $\{\phi U_n + U\}_{n\in\mathbb{N}}$ is a decreasing chain,

$$\phi U_{n_0} + U = \bigcap_{n\in\mathbb{N}} (\phi U_n + U).$$

Let $m \geq n_0$. Theorem 2.3(b) applied to the linearly compact linear subspace $U + \phi U$ and the descending chain $\{U_n\}_{n\in\mathbb{N}}$ yields

$$\phi U_m + U = \bigcap_{n\in\mathbb{N}} (\phi U_n + U) = \left(\bigcap_{n\in\mathbb{N}} \phi U_n\right) + U = \phi\left(\bigcap_{n\in\mathbb{N}} U_n\right) + U = \phi U_+ + U.$$

As $U_+ = U \cap \phi U_+$ by Lemma 3.22(b), we have

$$\begin{aligned}\dim \frac{\phi U_+}{U_+} &= \dim \frac{\phi U_+}{U \cap \phi U_+} = \dim \frac{U + \phi U_+}{U} \\ &= \dim \frac{U + \phi U_m}{U} = \dim \frac{\phi U_m}{U \cap \phi U_m} = \dim \frac{\phi U_m}{U_{m+1}}.\end{aligned}$$

Equation (3.6) now gives $\phi^{m+1} C_{m+1}(\phi, U) = \phi U_m$, so there exists a surjective homomorphism

$$\varphi : C_{m+1}(\phi, U) \to \phi U_m / U_{m+1}, \quad x \mapsto \phi^{m+1}(x) + U_{m+1},$$

such that $\ker \varphi = C_{m+2}(\phi, U)$. Hence,

$$\frac{\phi U_m}{U_{m+1}} \cong \frac{C_{m+1}(\phi,U)}{C_{m+2}(\phi,U)}.$$

Finally,

$$\dim \frac{\phi U_+}{U_+} = \dim \frac{\phi U_m}{U_{m+1}} = \dim \frac{C_{m+1}(\phi,U)}{C_{m+2}(\phi,U)} = \gamma,$$

as required in Equation (3.8). □

The following useful consequence of the Limit-free Formula is inspired by its analogue in the context of totally disconnected locally compact groups. Here we adapt the proof of [28, Proposition 3.11] to our context for reader's convenience.

Corollary 3.24. *Let V be an l.l.c. vector space and $\phi : V \to V$ a continuous endomorphism. Then*

$$\mathrm{ent}^*(\phi) = \sup\left\{\dim \frac{\phi M}{M} \mid M \leq \phi M \leq V,\ M \textit{ linearly compact},\ \dim \frac{\phi M}{M} < \infty\right\} =: s.$$

Proof. By Proposition 3.23, $\mathrm{ent}^*(\phi) \leq s$. To prove the converse inequality, let M be a linearly compact linear subspace of V such that $M \leq \phi M$ and $\phi M/M$ has finite dimension. By Proposition 2.1(a,d,e), this implies that M is open in ϕM, since M is closed and ϕM is linearly compact, namely, $M \in \mathcal{B}(\phi M)$. By Equation (2.1), there exists $U \in \mathcal{B}(V)$ such that $M = U \cap \phi M$. As $M \leq \phi M$ and $M \leq U$, we deduce that $M \leq U_+$ by Lemma 3.22(a). Therefore $\phi M \leq \phi U_+$, and so

$$\mathrm{ent}^*(\phi) \geq \dim \frac{\phi U_+}{U_+} = \dim \frac{\phi U_+}{U \cap \phi U_+} \geq \dim \frac{\phi U_+ \cap \phi M}{U \cap \phi U_+ \cap \phi M} = \dim \frac{\phi M}{U \cap \phi M} = \dim \frac{\phi M}{M},$$

since $U_+ = U \cap \phi U_+$ by Lemma 3.22(b). Finally, it follows that $s \leq \mathrm{ent}^*(\phi)$. □

4. Reductions for the computation of the topological entropy

In this section we provide two reductions that can be used to simplify the computation of the topological entropy. The first one follows from the Limit-free Formula and shows that it is sufficient to consider linearly compact vector spaces. Once we restrict to linearly compact vector spaces, we make a second reduction to topological automorphisms.

4.1. Reduction to linearly compact vector spaces

Let V be an l.l.c. vector space and $\phi : V \to V$ a continuous endomorphism. By Theorem 2.5 we can assume that $V = V_c \oplus V_d$, where $V_c \in \mathcal{B}(V)$ and consequently V_d is discrete. Let

$$\iota_*\colon V_* \to V, \quad p_*\colon V \to V_*$$

with $* \in \{c, d\}$ be the canonical injections and projections, respectively. Accordingly, we may associate to ϕ the following decomposition

$$\phi = \begin{pmatrix} \phi_{cc} & \phi_{dc} \\ \phi_{cd} & \phi_{dd} \end{pmatrix},$$

where $\phi_{\bullet *} : V_\bullet \to V_*$ is the composition $\phi_{\bullet *} = p_* \circ \phi \circ \iota_\bullet$ for $\bullet, * \in \{c, d\}$. Therefore, $\phi_{\bullet *}$ is continuous being composition of continuous homomorphisms.

Lemma 4.1. *In the above notations, consider* $\phi_{cd}\colon V_c \to V_d$*. Then*

$$\mathrm{Im}(\phi_{cd}) \in \mathcal{B}(V_d) \quad \textit{and} \quad \ker(\phi_{cd}) \in \mathcal{B}(V_c) \subseteq \mathcal{B}(V).$$

Proof. By Proposition 2.1(c), $\mathrm{Im}(\phi_{cd})$ is a linearly compact linear subspace of V_d. For V_d is discrete, $\mathrm{Im}(\phi_{cd})$ has finite dimension by Proposition 2.1(d,e), so $\mathrm{Im}(\phi_{cd}) \in \mathcal{B}(V_d) = \{F \leq V_d \mid \dim F < \infty\}$.

As $\ker(\phi_{cd})$ is a closed linear subspace of V_c, which is linearly compact, $\ker(\phi_{cd})$ is linearly compact as well by Proposition 2.1(b). Thus $V_c/\ker(\phi_{cd}) \cong \mathrm{Im}(\phi_{cd})$ is a finite dimensional linearly compact space, so $V/\ker(\phi_{cd})$ is discrete by Proposition 2.1(d,e). Consequently, $\ker(\phi_{cd})$ is open in V_c, and so $\ker(\phi_{cd}) \in \mathcal{B}(V_c)$. $\square$

We see now that in the above decomposition of ϕ, the unique contribution to the topological entropy of ϕ comes from the "linearly compact component" ϕ_{cc}.

Proposition 4.2. *In the above notations, consider* $\phi_{cc}\colon V_c \to V_c$*. Then* $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi_{cc})$.

Proof. By Lemma 4.1, $K = \ker(\phi_{cd}) \in \mathcal{B}(V_c) \subseteq \mathcal{B}(V)$. Thus, by Corollary 3.5,

$$\mathrm{ent}^*(\phi) = \sup\{H^*(\phi, U) \mid U \in \mathcal{B}(K)\},$$
$$\mathrm{ent}^*(\phi_{cc}) = \sup\{H^*(\phi_{cc}, U) \mid U \in \mathcal{B}(K)\}.$$

For $U \in \mathcal{B}(K)$, as in Definition 3.21, let

$$\begin{aligned} U_0 &= U \quad \text{and} \quad U_0^{cc} = U, \\ U_{n+1} &= U \cap \phi U_n \quad \text{and} \quad U_{n+1}^{cc} = U \cap \phi_{cc} U_n^{cc}, \quad \text{for every } n \in \mathbb{N}, \\ U_+ &= \bigcap_{n\in\mathbb{N}} U_n \quad \text{and} \quad U_+^{cc} = \bigcap_{n\in\mathbb{N}} U_n^{cc}. \end{aligned}$$

Proposition 3.23 implies that

$$\mathrm{ent}^*(\phi) = \sup\left\{\dim \frac{\phi U_+}{U_+} \mid U \in \mathcal{B}(K)\right\},$$
$$\mathrm{ent}^*(\phi_{cc}) = \sup\left\{\dim \frac{\phi_{cc} U_+^{cc}}{U_+^{cc}} \mid U \in \mathcal{B}(K)\right\}.$$

Since $U \leq K = \ker(\phi_{cd}) \leq V_c$, we deduce that $\phi U = \phi_{cc} U \leq V_c$. Then it is possible to prove by induction that $U_n = U_n^{cc}$ for all $n \in \mathbb{N}$; therefore, $U_+ = U_+^{cc}$ and $\phi U_+ = \phi_{cc} U_+^{cc}$. Hence, $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi_{cc})$. $\square$

Remark 4.3. The possibility to reduce the computation of ent^* to the category of linearly compact vector space is a powerful tool. Indeed, the dynamic behaviour of continuous endomorphisms of linearly compact vector spaces is well-understood; see [3].

4.2. Reduction to topological automorphisms

For a continuous endomorphism $\phi : V \to V$ of a linearly compact vector space V, the *surjective core* of ϕ is

$$S_\phi = \bigcap_{n\in\mathbb{N}} \phi^n V.$$

Lemma 4.4. *Let V be a linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. Then:*

(a) S_ϕ is a closed linear subspace of V;
(b) $\phi(S_\phi) = S_\phi$;
(c) $U_+ \leq S_\phi$ for all $U \in \mathcal{B}(V)$.

Proof. (a) is an easy consequence of Proposition 2.1(a,c), while Theorem 2.3(a) implies (b), and item (c) follows by the definition of U_+. □

Thus, S_ϕ is a closed ϕ-stable linear subspace of V, and so $\phi \restriction_{S_\phi}: S_\phi \to S_\phi$ is surjective. The following result shows that $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi \restriction_{S_\phi})$. Moreover, by definition S_ϕ turns out to be the largest closed ϕ-stable linear subspace of V.

Proposition 4.5. *Let V be a linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. Then*

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi \restriction_{S_\phi}).$$

Proof. By Proposition 3.15(b), $\mathrm{ent}^*(\phi) \geq \mathrm{ent}^*(\phi \restriction_{S_\phi})$. To prove the converse inequality, let $U \in \mathcal{B}(V)$. By Lemma 3.22(c) and Lemma 4.4(c), U_+ is linearly compact, $U_+ \leq \phi U_+ \leq S_\phi$ and $\phi U_+/U_+$ has finite dimension. Thus,

$$H^*(\phi, U) = \dim \frac{\phi U_+}{U_+} \leq \mathrm{ent}^*(\phi \restriction_{S_\phi}), \tag{4.1}$$

by Proposition 3.23 and Corollary 3.24. Consequently, $\mathrm{ent}^*(\phi) \leq \mathrm{ent}^*(\phi \restriction_{S_\phi})$. □

Let V be a linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. Let $\mathcal{L}V$ denote the inverse limit $\varprojlim(V_n, \phi)$ of the inverse system $(V_n, \phi)_{n\in\mathbb{N}}$, where $V_n = V$ for all $n \in \mathbb{N}$:

$$\cdots \xrightarrow{\phi} V_n \xrightarrow{\phi} V_{n-1} \xrightarrow{\phi} \cdots \xrightarrow{\phi} V_1 \xrightarrow{\phi} V_0. \tag{4.2}$$

In other words,

$$\mathcal{L}V = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} V_n \mid x_n = \phi(x_{n+1})\ \forall n \in \mathbb{N} \right\}, \tag{4.3}$$

endowed with the topology inherited from the product topology of $\prod_{n\in\mathbb{N}} V_n$. Since $\mathcal{L}V$ is a closed linear subspace of the direct product (see [37, Lemma 1.1.2]), $\mathcal{L}V$ is linearly compact as well by Proposition 2.1(c). Let $\iota : \mathcal{L}V \to \prod_{n\in\mathbb{N}} V_n$ be the canonical embedding.

The natural continuous endomorphism

$$\prod \phi\colon \prod_{n\in\mathbb{N}} V_n \to \prod_{n\in\mathbb{N}} V_n, \quad (x_n)_{n\in\mathbb{N}} \mapsto (\phi(x_n))_{n\in\mathbb{N}}$$

induces a continuous endomorphism $\mathcal{L}\phi : \mathcal{L}V \to \mathcal{L}V$ making the following diagram commute

$$\begin{tikzcd} \prod_{n\in\mathbb{N}} V_n \arrow[r, "\prod \phi"] & \prod_{n\in\mathbb{N}} V_n \\ \mathcal{L}V \arrow[u, "\iota"] \arrow[r, "\mathcal{L}\phi"'] & \mathcal{L}V. \arrow[u, "\iota"'] \end{tikzcd} \tag{4.4}$$

Proposition 4.6. *Let V be a linearly compact vector space and $\phi : V \to V$ a continuous endomorphism. Then $\mathcal{L}\phi\colon \mathcal{L}V \to \mathcal{L}V$ is a topological automorphism.*

Proof. By construction, $\mathcal{L}\phi$ is continuous and injective. Since $\mathcal{L}V$ is linearly compact, it is sufficient to prove that $\mathcal{L}\phi$ is surjective by Proposition 2.2. Let $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{L}V$, that is, $\phi(x_{n+1}) = x_n$ for every $n \in \mathbb{N}$. Clearly, $x = \mathcal{L}\phi((x_{n+1})_{n\in\mathbb{N}})$. □

The next part of this section is devoted to prove that $\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\mathcal{L}\phi)$. To this end, for every $n \in \mathbb{N}$, let $p_n\colon \mathcal{L}V \to V_n$ be the canonical projection given by the usual restriction of the projection $\prod_{n\in\mathbb{N}} V_n \to V_n$.

$$\begin{tikzcd} \mathcal{L}V \arrow[r, "p_n"] \arrow[d, hook, "\iota"'] & V_n \\ \prod_{n\in\mathbb{N}} V_n \arrow[ur] & \end{tikzcd} \tag{4.5}$$

Let also $K_n = \ker p_n$, which is a closed $\mathcal{L}\phi$-invariant linear subspace of $\mathcal{L}V$.

Lemma 4.7. *Let V be a linearly compact vector space, $\phi : V \to V$ a continuous endomorphism and $n \in \mathbb{N}$. Then, in the above notations:*

(a) $p_n(\mathcal{L}V) = S_\phi$;
(b) for the topological isomorphism $\alpha\colon \mathcal{L}V/K_n \to p_n(\mathcal{L}V)$ induced by p_n and the continuous endomorphism $\overline{\mathcal{L}\phi}_n\colon \mathcal{L}V/K_n \to \mathcal{L}V/K_n$ induced by $\mathcal{L}\phi$, one has

$$\overline{\mathcal{L}\phi}_n = \alpha^{-1} \circ \phi\restriction_{S_\phi} \circ\alpha;$$

(c) $\mathrm{ent}^*(\overline{\mathcal{L}\phi}_n) = \mathrm{ent}^*(\phi)$.

Proof. (a) Let $x = (x_n)_{n\in\mathbb{N}} \in \mathcal{L}V$; then $\phi^k(x_{n+k}) = x_n$ for every $k \in \mathbb{N}$, and so $p_n(x) = x_n \in S_\phi$.

Conversely, let $s \in S_\phi$. By Lemma 4.4(b), we can set:

- $x_i = \phi^{n-i}(s) \in S_\phi$ for $i \in \{0, \dots, n\}$;
- $x_i \in S_\phi$ such that $\phi(x_i) = x_{i-1}$ for $i > n$.

Thus, $x = (x_i)_i \in \mathcal{L}V$ and $p_n(x) = s$, as required.

(b) By construction, K_n is a closed $\mathcal{L}\phi$-invariant linear subspace of $\mathcal{L}V$ for every $n \in \mathbb{N}$. Moreover, α is a topological isomorphism by Proposition 2.2. Let $x + K_n \in \mathcal{L}V/K_n$. By (a),

$$\begin{aligned}\alpha^{-1}(\phi\restriction_{S_\phi}(\alpha(x+K_n))) = \alpha^{-1}(\phi\restriction_{S_\phi}(p_n(x))) = \alpha^{-1}(\phi(p_n(x))) = \\ = \alpha^{-1}(p_n(\mathcal{L}\phi(x))) = \mathcal{L}\phi(x) + K_n = \overline{\mathcal{L}\phi}_n(x+K_n).\end{aligned}$$

(c) Since α is a topological isomorphism, (c) is an easy consequence of (b) by Proposition 3.15(a) and Proposition 4.5. □

The next result shows that for the computation of the topological entropy in the case of linearly compact vector spaces one can reduce to topological automorphisms.

Theorem 4.8. *Let V be a linearly compact vector space and $\phi\colon V \to V$ a continuous endomorphism. Then*

$$\mathrm{ent}^*(\mathcal{L}\phi) = \mathrm{ent}^*(\phi).$$

Proof. Since $\bigcap_{n\in\mathbb{N}} K_n = 0$, we have that the canonical map $\rho\colon \mathcal{L}V \to \varprojlim \mathcal{L}V/K_n$ is an injective continuous homomorphism of linearly compact spaces. Since $\rho(\mathcal{L}V)$ is linearly compact, then ρ is also surjective (see [37, Lemma 1.1.7]). By Proposition 2.2, we conclude that

$$\mathcal{L}V \cong_{top} \varprojlim \mathcal{L}V/K_n. \tag{4.6}$$

By Proposition 3.15(a), the latter identification preserves the topological entropy. Now Proposition 3.15(a,e) and Lemma 4.7(c) give $\mathrm{ent}^*(\mathcal{L}\phi) = \sup_{n\in\mathbb{N}} \mathrm{ent}^*(\overline{\mathcal{L}\phi}_n) = \mathrm{ent}^*(\phi)$. □

A *flow* in the category ${}_{\mathbb{K}}\mathrm{LC}$ is a pair (V,ϕ), where V is a linearly compact vector space and $\phi\colon V \to V$ is a continuous endomorphism. If (V,ϕ) and (W,ψ) are flows in ${}_{\mathbb{K}}\mathrm{LC}$, then a morphism of flows from (V,ϕ) to (W,ψ) is a continuous homomorphism $h\colon V \to W$ such that $h\circ\phi = \psi\circ h$. We let $\mathrm{Flow}({}_{\mathbb{K}}\mathrm{LC})$ denote the resulting category of flows in ${}_{\mathbb{K}}\mathrm{LC}$. Clearly, it is well-defined a functor

$$\mathcal{L}\colon \mathrm{Flow}({}_{\mathbb{K}}\mathrm{LC}) \to \mathrm{Flow}({}_{\mathbb{K}}\mathrm{LC}) \tag{4.7}$$

given by $\mathcal{L}(V,\phi) = (\mathcal{L}V, \mathcal{L}\phi)$ and $\mathcal{L}(h)\colon \mathcal{L}V \to \mathcal{L}W$ is the continuous homomorphism induced by the following morphism of inverse systems

$$\begin{tikzcd}
\cdots \arrow[r] & V_n \arrow[r, "\phi"] \arrow[d, "h"] & \cdots \arrow[r] & V_1 \arrow[r, "\phi"] \arrow[d, "h"] & V_0 \arrow[d, "h"] \\
\cdots \arrow[r] & W_n \arrow[r, "\psi"] & \cdots \arrow[r] & W_1 \arrow[r, "\psi"] & W_0
\end{tikzcd}$$

namely $\mathcal{L}h((v_n)_{n\in\mathbb{N}}) = (h(v_n))_{n\in\mathbb{N}} \in \mathcal{L}W$ for every $(v_n)_{n\in\mathbb{N}} \in \mathcal{L}V$.

We conclude this section by showing that the functor $\mathcal{L}$ preserves the topological extensions in the sense of Proposition 4.9. Let $\phi\colon V \to V$ be a continuous endomorphism of a linearly compact space V. For a closed ϕ-invariant linear subspace W of V, consider the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r] \arrow[d, "\phi\restriction_W"] & V \arrow[r] \arrow[d, "\phi"] & V/W \arrow[r] \arrow[d, "\overline{\phi}"] & 0 \\
0 \arrow[r] & W \arrow[r] & V \arrow[r] & V/W \arrow[r] & 0
\end{tikzcd} \tag{4.8}$$

Thus, one constructs as above the following exact sequence of inverse systems of linearly compact spaces (see (4.2))

$$0 \longrightarrow (W_n, \phi\restriction_W)_{n\in\mathbb{N}} \longrightarrow (V_n,\phi)_{n\in\mathbb{N}} \longrightarrow (V_n/W_n, \overline{\phi})_{n\in\mathbb{N}}, \longrightarrow 0 \tag{4.9}$$

where $V = V_n$ and $W = W_n$ for every $n \in \mathbb{N}$. Denote by

$$\mathcal{L}W = \varprojlim(W_n, \phi\restriction_W), \quad \mathcal{L}V = \varprojlim(V_n,\phi) \quad \text{and} \quad \mathcal{L}(V/W) = \varprojlim(V_n/W_n, \overline{\phi})$$

the corresponding inverse limits. Since the inverse limit functor in ${}_{\mathbb{K}}\mathrm{LC}$ is exact (see Remark 2.4) we have the short exact sequence in ${}_{\mathbb{K}}\mathrm{LC}$ (see [37, pages 4-5])

$$0 \longrightarrow \mathcal{L}W \longrightarrow \mathcal{L}V \longrightarrow \mathcal{L}(V/W) \longrightarrow 0. \tag{4.10}$$

In order to simplify the notation, we regard $\mathcal{L}W$ as a closed linear subspace of $\mathcal{L}V$ and, since $\mathcal{L}(V/W) \cong_{top} \mathcal{L}V/\mathcal{L}W$, we identify $\mathcal{L}(V/W)$ with $\mathcal{L}V/\mathcal{L}W$. Since the topological entropy is invariant under conjugation by Proposition 3.15(a), it one easily verifies the latter identification preserves the topological entropy.

The linear subspace $\mathcal{L}W$ turns out to be closed and $\mathcal{L}\phi$-invariant in $\mathcal{L}V$, therefore the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}W \arrow[r] \arrow[d, "(\mathcal{L}\phi)\restriction_{\mathcal{L}W}"'] & \mathcal{L}V \arrow[r] \arrow[d, "\mathcal{L}\phi"] & \mathcal{L}(V/W) \arrow[r] \arrow[d, "\overline{\mathcal{L}\phi}"] & 0 \\ 0 \arrow[r] & \mathcal{L}W \arrow[r] & \mathcal{L}V \arrow[r] & \mathcal{L}(V/W) \arrow[r] & 0 \end{tikzcd} \tag{4.11}$$

commutes, where $\overline{\mathcal{L}\phi}\colon \mathcal{L}(V/W) \to \mathcal{L}(V/W)$ is the continuous endomorphism induced by $\mathcal{L}\phi$.

On the other hand, by restriction in view of Equation (4.4) we have the commutative diagrams

$$\begin{tikzcd} \prod_{n\in\mathbb{N}} W_n \arrow[r, "\prod \phi\restriction_W"] & \prod_{n\in\mathbb{N}} W_n \\ \mathcal{L}W \arrow[u, "\iota'"] \arrow[r, "\mathcal{L}(\phi\restriction_W)"'] & \mathcal{L}W, \arrow[u, "\iota'"'] \end{tikzcd} \qquad \begin{tikzcd} \prod_{n\in\mathbb{N}}(V_n/W_n) \arrow[r, "\prod \overline{\phi}"] & \prod_{n\in\mathbb{N}}(V_n/W_n) \\ \mathcal{L}(V/W) \arrow[u, "\iota''"] \arrow[r, "\mathcal{L}\overline{\phi}"'] & \mathcal{L}(V/W), \arrow[u, "\iota''"'] \end{tikzcd} \tag{4.12}$$

where $\mathcal{L}(\phi\restriction_W)$ and $\mathcal{L}\overline{\phi}$ are both topological automorphisms by Proposition 4.6.

We see now how the functor $\mathcal{L}$ behaves under taking closed invariant linear subspaces and quotient vector spaces.

Proposition 4.9. *Let V be a linearly compact vector space, $\phi: V \to V$ a continuous endomorphism and W a closed ϕ-invariant linear subspace of V. Then*

$$(\mathcal{L}\phi)\restriction_{\mathcal{L}W} = \mathcal{L}(\phi\restriction_W) \quad \text{and} \quad \overline{\mathcal{L}\phi} = \mathcal{L}\overline{\phi},$$

where $\overline{\phi}: V/W \to V/W$ is induced by ϕ and $\overline{\mathcal{L}\phi}: \mathcal{L}V/\mathcal{L}W \to \mathcal{L}V/\mathcal{L}W$ is induced by $\mathcal{L}\phi$. In particular, $(\mathcal{L}\phi)\restriction_{\mathcal{L}W}$ and $\overline{\mathcal{L}\phi}$ are topological automorphisms. Moreover,

$$\mathrm{ent}^*(\phi\restriction_W) = \mathrm{ent}^*((\mathcal{L}\phi)\restriction_{\mathcal{L}W}) \quad \text{and} \quad \mathrm{ent}^*(\overline{\phi}) = \mathrm{ent}^*(\overline{\mathcal{L}\phi}).$$

Proof. Let $(x_n)_{n\in\mathbb{N}} \in \mathcal{L}W$, that is, $\phi\restriction_W(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$. As $W \le V$ and $\mathcal{L}W \le \mathcal{L}V$,

$$\mathcal{L}(\phi\restriction_W)((x_n)_{n\in\mathbb{N}}) = (\phi\restriction_W(x_n))_{n\in\mathbb{N}} = (\phi(x_n))_{n\in\mathbb{N}} = (\mathcal{L}\phi)\restriction_{\mathcal{L}W}((x_n)_{n\in\mathbb{N}}).$$

Now let $(x_n + W)_n \in \mathcal{L}(V/W)$. As $\mathcal{L}(V/W) \cong \mathcal{L}V/\mathcal{L}W$,

$$\mathcal{L}\overline{\phi}((x_n+W)_{n\in\mathbb{N}}) = (\phi(x_n)+W)_{n\in\mathbb{N}} \cong (\phi(x_n))_{n\in\mathbb{N}} + \mathcal{L}W = \mathcal{L}\phi((x_n)_{n\in\mathbb{N}}) + \mathcal{L}W = \overline{\mathcal{L}\phi}((x_n)_{n\in\mathbb{N}} + \mathcal{L}W).$$

By Proposition 4.6, it follows that $(\mathcal{L}\phi)\restriction_{\mathcal{L}W}$ and $\overline{\mathcal{L}\phi}$ are a topological automorphisms. Clearly, one deduces that $\mathcal{L}W$ is a $\mathcal{L}\phi$-stable linear subspace of $\mathcal{L}V$.

The last assertion follows from Theorem 4.8. □

5. Addition Theorem

This section is devoted to prove the Addition Theorem for the topological entropy (see Theorem 1.2). We start by proving it for topological automorphisms.

Proposition 5.1. *Let V be an l.l.c. vector space, $\phi\colon V \to V$ a topological automorphism, W a closed ϕ-stable linear subspace of V and $\overline{\phi} : V/W \to V/W$ the topological automorphism induced by ϕ. Then*

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}).$$

Proof. By Proposition 3.15(b), we have $\mathrm{ent}^*(\phi) \geq \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi})$. To prove the converse inequality, let M be a linearly compact linear subspace of V such that $M \leq \phi M$ and $\phi M/M$ has finite dimension. Then

$$\frac{\phi M}{M + (\phi M \cap W)} \cong \frac{\phi M}{M} \Big/ \frac{M+(\phi M \cap W)}{M}.$$

Since $\frac{M+(\phi M\cap W)}{M} \cong \frac{\phi M \cap W}{\phi M \cap W \cap M} = \frac{\phi M \cap W}{M \cap W}$, we have that

$$\dim \frac{\phi M}{M} = \dim \frac{\phi M \cap W}{M \cap W} + \dim \frac{\phi M}{M + \phi M \cap W}. \tag{5.1}$$

Since W is closed and M is linearly compact, $M \cap W$ is linearly compact by Proposition 2.1(a). Moreover, $\phi(M \cap W) = \phi M \cap \phi W = \phi M \cap W$, so $M \cap W$ has finite codimension in $\phi(M \cap W)$. Corollary 3.24 yields

$$\dim \frac{\phi M \cap W}{M \cap W} \leq \mathrm{ent}^*(\phi\restriction_W). \tag{5.2}$$

On the other hand, $\pi M = \frac{M+W}{W}$ is linearly compact in V/W by Proposition 2.1(c). By the modular law, since $M \leq \phi M$,

$$M + (\phi M \cap W) = \phi M \cap (M + W).$$

Therefore,

$$\frac{\phi M}{M + (\phi M \cap W)} = \frac{\phi M}{\phi M \cap (M+W)} \cong \frac{\phi M + W}{M + W}.$$

Moreover,

$$\pi(M) = \frac{M+W}{W} \leq \frac{\phi M + W}{W} = \overline{\phi}(\pi M).$$

Then

$$\dim \frac{\phi M}{M + \phi M \cap W} = \dim \frac{\overline{\phi}(\pi M)}{\pi M},$$

and so, by Corollary 3.24,

$$\dim \frac{\phi M}{M + \phi M \cap W} \leq \mathrm{ent}^*(\overline{\phi}). \tag{5.3}$$

By Equations (5.1), (5.2) and (5.3), we conclude that

$$\dim \frac{\phi M}{M} \leq \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}),$$

so Corollary 3.24 yields the required inequality $\mathrm{ent}^*(\phi) \leq \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi})$. □

A second step towards the proof of the Addition Theorem consists in proving it for linearly compact vector spaces:

Proposition 5.2. *Let V be a linearly compact vector space, $\phi\colon V \to V$ a continuous endomorphism, W a closed ϕ-invariant linear subspace of V and $\overline{\phi}\colon V/W \to V/W$ the continuous endomorphism induced by ϕ. Then*

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}).$$

Proof. Consider the following short exact sequence of flows in ${}_{\mathbb{K}}\mathrm{LC}$

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "\phi\restriction_W"'] & V \arrow[r] \arrow[d, "\phi"] & V/W \arrow[r] \arrow[d, "\overline{\phi}"] & 0 \\ 0 \arrow[r] & W \arrow[r] & V \arrow[r] & V/W \arrow[r] & 0. \end{tikzcd}$$

By applying the functor $\mathcal{L}$ (see (4.7) and (4.11)), we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}W \arrow[r] \arrow[d, "(\mathcal{L}\phi)\restriction_{\mathcal{L}W}"'] & \mathcal{L}V \arrow[r] \arrow[d, "\mathcal{L}\phi"] & \mathcal{L}(V/W) \arrow[r] \arrow[d, "\overline{\mathcal{L}\phi}"] & 0 \\ 0 \arrow[r] & \mathcal{L}W \arrow[r] & \mathcal{L}V \arrow[r] & \mathcal{L}(V/W) \arrow[r] & 0, \end{tikzcd}$$

where $\mathcal{L}\phi : \mathcal{L}V \to \mathcal{L}V$ is a topological automorphism by Proposition 4.6, and $\mathcal{L}W$ is a closed $\mathcal{L}\phi$-stable linear subspace of $\mathcal{L}$ by Proposition 4.9. Therefore,

$$\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\mathcal{L}\phi) = \mathrm{ent}^*(\mathcal{L}(\phi\restriction_W)) + \mathrm{ent}^*(\overline{\mathcal{L}\phi}) = \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}),$$

by Proposition 5.1, Theorem 4.8 and Proposition 4.9. □

We are now in position to prove the general statement of the Addition Theorem.

Proof of Theorem 1.2. Let $V_c \in \mathcal{B}(V)$ and $W_c = W \cap V_c$; then $W_c \in \mathcal{B}(W)$. By Theorem 2.5, there exists a discrete linear subspace $W_d \leq W$ such that $W = W_c \oplus W_d$. Let $V_d \leq V$ such that $V = V_c \oplus V_d$ and $W_d \leq V_d$. Clearly, V_d is a discrete subspace of V, since V_c is open and $V_c \cap V_d = 0$. By construction, the diagram

$$\begin{tikzcd}
0 \arrow[r] & W_c \arrow[r, "\iota_c^W"] \arrow[d, hook] \arrow[rrr, bend left, "(\phi\restriction_W)_{cc}"] & W \arrow[r, "\phi\restriction_W"] & W \arrow[r, "p_c^W"] & W_c \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & V_c \arrow[r, "\iota_c^V"] \arrow[rrr, bend right, "\phi_{cc}"'] & V \arrow[r, "\phi"] & V \arrow[r, "p_c^V"] & V_c \arrow[r] & 0
\end{tikzcd}$$

commutes, where $\iota_c^W, \iota_c^V, p_c^W, p_c^V$ are the canonical injections and projections of W and V, respectively. This yields that W_c is a closed ϕ_{cc}-invariant subspace of V_c and that

$$(\phi\restriction_W)_{cc} = \phi_{cc}\restriction_{W_c}. \tag{5.4}$$

Now, let $\pi\colon V \to V/W$ be the canonical projection and let $\overline{V} = V/W$. Let $\overline{V}_c = \pi(V_c)$ and $\overline{V}_d = \pi(V_d)$; then $\overline{V}_c \in \mathcal{B}(\overline{V})$ since $\overline{V}_c$ is open and it is linearly compact by Proposition 2.1(c), while $\overline{V}_d$ is discrete; moreover, $\overline{V} = \overline{V}_c \oplus \overline{V}_d$. The canonical continuous isomorphism $\alpha\colon V_c/W_c \to \overline{V}_c$ is a topological isomorphism by Proposition 2.2; it makes the following diagram commute

$$\begin{tikzcd}
\overline{V}_c \arrow[r, "\iota_c^{\overline{V}}"] \arrow[rrr, bend left, "\overline{\phi}_{cc}"] & \overline{V} \arrow[r, "\overline{\phi}"] & \overline{V} \arrow[r, "p_c^{\overline{V}}"] & \overline{V}_c \\
V_c/W_c \arrow[u, "\alpha"] \arrow[rrr, "\overline{\phi_{cc}}"] & & & V_c/W_c. \arrow[u, "\alpha"']
\end{tikzcd}$$

In other words, $\alpha^{-1}\overline{\phi}_{cc}\alpha = \overline{\phi_{cc}}$ and so, by Proposition 3.15(a),

$$\mathrm{ent}^*(\overline{\phi}_{cc}) = \mathrm{ent}^*(\overline{\phi_{cc}}). \tag{5.5}$$

Finally, by using Equations (5.4) and (5.5), we obtain

$$\begin{aligned}
\mathrm{ent}^*(\phi) = \mathrm{ent}^*(\phi_{cc}) &= \mathrm{ent}^*(\phi_{cc}\restriction_{W_c}) + \mathrm{ent}^*(\overline{\phi_{cc}}) \\
&= \mathrm{ent}^*((\phi\restriction_W)_{cc}) + \mathrm{ent}^*(\overline{\phi}_{cc}) = \mathrm{ent}^*(\phi\restriction_W) + \mathrm{ent}^*(\overline{\phi}),
\end{aligned}$$

by Proposition 4.2 and Proposition 5.2. □

6. Bridge Theorem

This section is devoted to prove that the topological entropy of a continuous endomorphism coincides with the algebraic entropy of the dual endomorphism with respect to the Lefschetz Duality.

Lemma 6.1. *Let V be a l.l.c. vector space. Then $U \in \mathcal{B}(V)$ if and only if $U^\perp \in \mathcal{B}(\widehat{V})$.*

Proof. Let $U \in \mathcal{B}(V)$. Since U is linearly compact, $U^\perp$ is open by definition. Conversely, assume that $U^\perp$ is open. Since $U^\perp = \overline{U}^\perp$ by Lemma 2.9(b), we may assume that U is closed. Since $U^\perp$ is open in $\widehat{V}$, there exists a linearly compact linear subspace W of V such that $W^\perp \leq U^\perp$. Since U and W are closed linear subspaces of V, $U \leq W$ by Lemma 2.9(a,d). Thus U is a closed linear subspace of the linearly compact space W, and clearly U is linearly compact by Proposition 2.1(a).

Finally, since U is open in V if and only if the quotient V/U is discrete, Remark 2.11 implies that U is open in V if and only if $U^\perp \cong_{top} \widehat{V/U}$ is linearly compact. □

Lemma 6.2. *Let V be a l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and $U \in \mathcal{B}(V)$. Then $(\phi^{-n}(U))^\perp = (\widehat{\phi})^n(U^\perp)$ for every $n \in \mathbb{N}_+$.*

Proof. We prove the result for $n = 1$, that is,

$$(\phi^{-1}(U))^\perp = \widehat{\phi}(U^\perp). \tag{6.1}$$

The proof for $n > 1$ follows easily from this case noting that $(\widehat{\phi})^n = \widehat{(\phi^n)}$.

Let $W = U^\perp$; then $W \in \mathcal{B}(\widehat{V})$ by Lemma 6.1 and $U = W^\top$ by Lemma 2.9(d). We prove that

$$\phi^{-1}(W^\top) = (\widehat{\phi}(W))^\top, \tag{6.2}$$

that is equivalent to Equation (6.1) by Lemma 2.9(d). So let $x \in \phi^{-1}(W^\top)$; equivalently, $\phi(x) \in W^\top$, that is $\chi(\phi(x)) = 0$ for every $\chi \in W$. This occurs precisely when $\widehat{\phi}(\chi)(x) = 0$ for every $\chi \in W$, if and only if $x \in (\widehat{\phi}(W))^\top$. This chain of equivalences proves Equation (6.2). □

By applying the previous lemmas, we can now give a proof to the Bridge Theorem.

Proof of Theorem 1.4. Let $U \in \mathcal{B}(V)$; so, $U^\perp \in \mathcal{B}(\widehat{V})$ by Lemma 6.1. For $n \in \mathbb{N}_+$, it follows from Lemma 2.10 and Lemma 6.2 that

$$C_n(\phi, U)^\perp = T_n(\widehat{\phi}, U^\perp).$$

Hence, in view of Lemma 2.12, $U/C_n(\phi, U) \cong \widehat{U/C_n(\phi, U)} \cong T_n(\widehat{\phi}, U^\perp)/U^\perp$, and so

$$\dim \frac{U}{C_n(\phi, U)} = \dim \frac{T_n(\widehat{\phi}, U^\perp)}{U^\perp}.$$

Therefore, $H^*(\phi, U) = H(\widehat{\phi}, U^\perp)$. By Lemma 6.1, we can conclude that $\mathrm{ent}^*(\phi) = \mathrm{ent}(\widehat{\phi})$. □

As a consequence of the Addition Theorem for the topological entropy ent^* and the Bridge Theorem, we deduce now the Addition Theorem for the algebraic entropy ent proved in [5].

Corollary 6.3. *Let V be an l.l.c. vector space, $\phi : V \to V$ a continuous endomorphism and W a closed ϕ-invariant linear subspace of V. Then*

$$\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_W) + \mathrm{ent}(\overline{\phi}).$$

Proof. Since $W^\perp$ is a closed $\widehat{\phi}$-invariant linear subspace of $\widehat{V}$, consider the topological isomorphisms $\alpha : \widehat{V/W} \to W^\perp$ and $\beta : \widehat{V}/W^\perp \to \widehat{W}$ given by Equations (2.3) and (2.4), respectively. It is possible to verify that the following diagrams commute.

$$\begin{tikzcd} W^\perp \arrow[r, "\widehat{\phi}\restriction_{W^\perp}"] & W^\perp \\ \widehat{V/W} \arrow[u, "\alpha"] \arrow[r, "\widehat{\overline{\phi}}"'] & \widehat{V/W} \arrow[u, "\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} \widehat{V}/W^\perp \arrow[r, "\overline{\widehat{\phi}}"] \arrow[d, "\beta"'] & \widehat{V}/W^\perp \arrow[d, "\beta"] \\ \widehat{W} \arrow[r, "\widehat{\phi\restriction_W}"'] & \widehat{W} \end{tikzcd} \tag{6.3}$$

By Theorem 1.4 and Proposition 3.15(a),

$$\mathrm{ent}(\phi) = \mathrm{ent}^*(\widehat{\phi}), \quad \mathrm{ent}(\phi\restriction_W) = \mathrm{ent}^*(\widehat{\phi\restriction_W}) = \mathrm{ent}^*(\overline{\widehat{\phi}}) \quad \text{and} \quad \mathrm{ent}(\overline{\phi}) = \mathrm{ent}^*(\widehat{\overline{\phi}}) = \mathrm{ent}^*(\widehat{\phi}\restriction_{W^\perp}).$$

Then Theorem 1.2 yields

$$\mathrm{ent}(\phi) = \mathrm{ent}^*(\widehat{\phi}) = \mathrm{ent}^*(\overline{\widehat{\phi}}) + \mathrm{ent}^*(\widehat{\phi}\restriction_{W^\perp}) = \mathrm{ent}(\phi\restriction_W) + \mathrm{ent}(\overline{\phi}),$$

and this concludes the proof. □

Alternatively, one can deduce the Addition Theorem for the topological entropy ent* from the Addition Theorem for the algebraic entropy ent and the Bridge Theorem.

References

[1] R.L. Adler, A.G. Konheim, M.H. McAndrew, Topological entropy, Trans. Am. Math. Soc. 114 (1965) 309–319.
[2] R. Bowen, Entropy for group endomorphisms and homogeneous spaces, Trans. Am. Math. Soc. 153 (1971) 401–414.
[3] I. Castellano, The corank of a flow over the category of linearly compact vector space, available at https://arxiv.org/abs/1803.04018.
[4] I. Castellano, Topological entropy for locally linearly compact vector spaces and field extensions, available at https://arxiv.org/abs/1809.02144.
[5] I. Castellano, A. Giordano Bruno, Algebraic entropy for locally linearly compact vector spaces, in: Rings, Polynomials, and Modules, Springer, Cham, 2017, pp. 103–127.
[6] J.F. Conn, Non-Abelian Minimal Closed Ideals of Transitive Lie Algebras, Princeton University Press, 2014.
[7] U. Dardano, S. Rinauro, Inertial automorphisms of an abelian group, Rend. Semin. Mat. Univ. Padova 127 (2012) 213–233.
[8] U. Dardano, S. Rinauro, On the ring of inertial endomorphisms of an abelian group, Ric. Mat. 63 (2014) 103–115.
[9] U. Dardano, S. Rinauro, Inertial endomorphisms of an abelian group, Ann. Mat. Pura Appl. 195 (2016) 219–234.
[10] U. Dardano, D. Dikranjan, S. Rinauro, Inertial properties in groups, Int. J. Group Theory 7 (2018) 17–62.
[11] D. Dikranjan, A. Giordano Bruno, The connection between topological and algebraic entropy, Topol. Appl. 159 (2012) 2980–2989.
[12] D. Dikranjan, A. Giordano Bruno, Limit free computation of entropy, Rend. Ist. Mat. Univ. Trieste 44 (2012) 297–312.
[13] D. Dikranjan, A. Giordano Bruno, Topological entropy and algebraic entropy for group endomorphisms, in: Proceedings ICTA2011, Islamabad, Pakistan July 4–10 2011, Cambridge Scientific Publishers, 2012, pp. 133–214.
[14] D. Dikranjan, A. Giordano Bruno, The Bridge theorem for totally disconnected LCA groups, Topol. Appl. 169 (2014) 21–32.
[15] D. Dikranjan, A. Giordano Bruno, Entropy on abelian groups, Adv. Math. 298 (2016) 612–653.
[16] D. Dikranjan, A. Giordano Bruno, L. Salce, Adjoint algebraic entropy, J. Algebra 324 (2010) 442–463.
[17] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, Fully inert subgroups of divisible Abelian groups, J. Group Theory 16 (2013) 915–939.
[18] D. Dikranjan, A. Giordano Bruno, L. Salce, S. Virili, Intrinsic algebraic entropy, J. Pure Appl. Algebra 219 (2015) 2933–2961.
[19] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, Algebraic entropy for abelian groups, Trans. Am. Math. Soc. 361 (2009) 3401–3434.
[20] D. Dikranjan, L. Salce, P. Zanardo, Fully inert subgroups of free Abelian groups, Period. Math. Hung. 69 (2014) 69–78.
[21] D. Dikranjan, M. Sanchis, Bowen's entropy for endomorphisms of totally bounded abelian groups, in: Descriptive Topology and Functional Analysis, in: Springer Proc. Math. Stat., vol. 80, 2014, pp. 143–162.
[22] D. Dikranjan, M. Sanchis, S. Virili, New and old facts about entropy on uniform spaces and topological groups, Topol. Appl. 159 (2012) 1916–1942.
[23] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. Fr. 90 (1962).
[24] A. Giordano Bruno, Adjoint entropy vs Topological entropy, Topol. Appl. 159 (2012) 2404–2419.
[25] A. Giordano Bruno, Topological entropy for automorphisms of totally disconnected locally compact groups, Topol. Proc. 45 (2015) 175–187.
[26] A. Giordano Bruno, L. Salce, A soft introduction to algebraic entropy, Arab. J. Math. 1 (2012) 69–87.

[27] A. Giordano Bruno, S. Virili, On the algebraic Yuzvinski formula, Topol. Algebra Appl. 3 (2015) 86–103.
[28] A. Giordano Bruno, S. Virili, Topological entropy in totally disconnected locally compact groups, Ergod. Theory Dyn. Syst. 37 (7) (2017) 2163–2186.
[29] B. Goldsmith, L. Salce, P. Zanardo, Fully inert subgroups of Abelian p-groups, J. Algebra 419 (2014) 332–349.
[30] A. Grothendieck, Sur quelques points d'algébre homologique, Tohoku Math. J. (2) 9 (2) (1957) 119–183.
[31] B.M. Hood, Topological entropy and uniform spaces, J. Lond. Math. Soc. 8 (1974) 633–641.
[32] E. Matlis, Injective modules over Noetherian rings, Pac. J. Math. 8 (1958) 511–528.
[33] G. Köthe, Topological Vector Spaces, Springer, Berlin, Heidelberg, 1983.
[34] S. Lefschetz, Algebraic Topology, vol. 27, American Mathematical Soc., 1942.
[35] J. Peters, Entropy on discrete abelian groups, Adv. Math. 33 (1979) 1–13.
[36] J. Peters, Entropy of automorphisms on LCA groups, Pac. J. Math. 96 (1981) 475–488.
[37] L. Ribes, P. Zalesskii, Profinite Groups, Springer, Berlin, Heidelberg, 2000.
[38] L. Salce, P. Zanardo, A general notion of algebraic entropy and the rank-entropy, Forum Math. 21 (2009) 579–599.
[39] L. Salce, P. Vámos, S. Virili, Length functions, multiplicities and algebraic entropy, Forum Math. 25 (2013) 255–282.
[40] L. Salce, S. Virili, The Addition Theorem for algebraic entropies induced by non-discrete length functions, Forum Math. 28 (2016) 1143–1157.
[41] L.N. Stoyanov, Uniqueness of topological entropy for endomorphisms on compact groups, Boll. Unione Mat. Ital., B (7) 1 (1987) 829–847.
[42] D. van Dantzig, Studien over topologische Algebra, Dissertation, Amsterdam, 1931.
[43] S. Virili, Algebraic and topological entropy of group actions, submitted.
[44] S. Virili, Entropy for endomorphisms of LCA groups, Topol. Appl. 159 (2012) 2546–2556.
[45] S. Warner, Topological Rings, vol. 178, Elsevier, 1993.
[46] M.D. Weiss, Algebraic and other entropies of group endomorphisms, Theory Comput. Syst. 8 (1974) 243–248.
[47] G.A. Willis, The scale and tidy subgroups for endomorphisms of totally disconnected locally compact groups, Math. Ann. 361 (2015) 403–442.
[48] S. Yuzvinski, Metric properties of endomorphisms of compact groups, Izv. Acad. Nauk SSSR, Ser. Mat. 29 (1965) 1295–1328 (in Russian). English translation: Transl. Am. Math. Soc. 66 (1968) 63–98.