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Characterized subgroups of the circle group *

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Dedicated to the 70-eth birthday of Hans Weber

Abstract

A subgroup H of the circle group $\mathbb{T}$ is said to be characterized by a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ of integers, if $H = \{x \in \mathbb{T} : u_n x \to 0\}$. The characterized subgroups of $\mathbb{T}$ are known also under the name topologically $\mathbf{u}$-torsion subgroups.

This survey paper is dedicated to characterized subgroups of $\mathbb{T}$: we recall their main properties and collect most of the basic results from the wide bibliography, following, when possible, the historical line, and trying to show the deep roots of this topic in several areas of Mathematics. Due to this universality of the topic, many notions and results were found independently by various authors working unaware of each other, so our effort is also directed towards giving credit to all of them to the best of our knowledge.

We provide also some background on the notions of characterized subgroup and topologically $\mathbf{u}$-torsion subgroup in the general case of topological abelian groups, where they differ very substantially.

1 Introduction

This survey provides a comprehensive outline of the area of the so called characterized subgroups of the circle group, giving also a picture of the general case of characterized subgroups of topological abelian groups (see §7). Our aim is to show the various faces of this topic, inspired by Topological Algebra (topologically torsion elements, the open mapping theorem, etc. – see §2), Number Theory (diophantine approximation and continued fractions – see §4), Analysis (trigonometric series – §6), Descriptive Set Theory (the study of various kinds of thin Borel sets and their hierarchy – see §3), Topology (convergent sequences in precompact group topologies, sequential limit laws, etc. – see §§5,8), etc. Our choice to start with Topological Algebra (i.e., topologically torsion elements) is only explained by the desire to keep the line of the first authors survey [28], although the knowledge accumulated during the years shows that a completely different approach is as natural and possible.

A few words about the term characterized subgroup are in order. As far as subgroups of the circle $\mathbb{T}$ are concerned, characterized subgroups have long been studied under the name topologically $\mathbf{u}$-torsion subgroups (of $\mathbb{T}$, see §2). Since these subgroups are determined (defined, characterized) by a sequence, gradually the term "characterized by a sequence" started to be used since [14] for some time, then gradually it was abbreviated to "characterized" (or, the better version, "characterizable" – see [54]). Both "characterized subgroups" and "topologically $\mathbf{u}$-torsion subgroups" can be successfully defined and used for arbitrary topological abelian groups; we dedicate some attention to these more general versions in §§7,8, trying to show in §8 how big is the divergence between these two approaches; §8 contains also a non-abelian version of "topologically $\mathbf{u}$-torsion element". The non-abelian counterpart

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of "characterized subgroup" of a compact group is largely discussed in the nice survey paper [54], while characterized subgroups of arbitrary topological abelian groups are discussed in the survey paper [33].

This survey is based on (but not limited to) a talk given by the third named author at the 31st Summer Conference on Topology and its Applications in Leicester - August 2nd, 2016.

It is a pleasure to dedicate this survey to Hans Weber, on the occasion of his retirement. Up to some point this determined also our choice of the precise limits of the topic, since our joint papers on characterized subgroups of the circle group with him are the backbone of this survey.

Notation and terminology

We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{N}_+$ the set of positive naturals, and by $\mathbb{P}$ the set of prime numbers. For $n \in \mathbb{N}_+$ and $p \in \mathbb{P}$, $\mathbb{Z}(n)$ denotes the cyclic group of order n and $\mathbb{Z}_p$ the group of p-adic integers. We say that a subset L of $\mathbb{N}$ is *large* if there exists a finite $F \subseteq \mathbb{Z}$ such that $\mathbb{N} \subseteq F + L$.

We denote by $\mathbb{R}$ the reals, and for $x \in \mathbb{R}$ let $\{x\}$ be its fractional part. The Lebesgue measure on $\mathbb{R}$ is denoted by λ. The cardinality of the continuum is $\mathfrak{c}$.

The circle group is $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ written additively $(\mathbb{T}, +)$. We denote by $\varpi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$ the canonical projection and for $r \in \mathbb{R}$ we often write $\bar{r}$ in place of $\varpi(r) = r + \mathbb{Z} \in \mathbb{T}$; let also $\varphi = \varpi \restriction_{[0,1)}\colon [0,1) \to \mathbb{T}$, which is a bijection. We consider on $\mathbb{T}$ the quotient topology of the topology of $\mathbb{R}$ and we denote by μ the (unique) Haar measure on $\mathbb{T}$.

For a topological space (X, τ), we denote by $w(X, \tau)$ its weight.

Let G be a group. The fact that H is a subgroup of G is abbreviated to $H \leq G$ and we denote by $\langle A \rangle$ the subgroup of G generated by a subset A of G. The group G is *divisible* if for every $g \in G$ and $n \in \mathbb{N}$ the equation $x^n = g$ has a solution in G.

Abelian groups will always be written additively. For an abelian group G and $n \in \mathbb{N}_+$, the sets $G[n] = \{x \in G : nx = 0\}$ and $nG = \{nx : x \in G\}$ are subgroups of G.

Let G be an abelian group and p a prime. The *p-torsion subgroup* of G is

$$t_p(G) = \{x \in G : p^n x = 0 \text{ for some } n \in \mathbb{N}\},$$

while the *torsion subgroup* of G is

$$t(G) = \{x \in G : nx = 0 \text{ for some } n \in \mathbb{N}_+\}.$$

For an abelian group G, denote by $\widehat{G}$ its Pontryagin dual.

For a sequence $\mathbf{u}$ in $\mathbb{N}_+$ with $u_0 = 1$, denote by $\mathbf{b}^{\mathbf{u}} = (b_n^{\mathbf{u}})_{n \in \mathbb{N}_+}$ the sequence defined by

$$b_n^{\mathbf{u}} = \frac{u_n}{u_{n-1}}$$

for every $n \in \mathbb{N}_+$. When there is no possibility of confusion we write simply $\mathbf{b} = (b_n)_{n \in \mathbb{N}_+}$.

2 Topologically u-torsion subgroups of $\mathbb{T}$

2.1 Topologically p-torsion and topologically torsion subgroup

The following notions appear in Bracconier [17], Vilenkin [81], Robertson [75], Armacost [3]:

Definition 2.1. [[17, 81]] An element x of an abelian topological group G is:

(a) *topologically p-torsion*, for a prime p, if $p^n x \to 0$;

(b) *topologically torsion* if $n!x \to 0$.

Clearly, a torsion element of a Hausdorff topological abelian group is topologically p-torsion precisely when it is p-torsion.

For a prime p, the topologically p-torsion elements of G form the *topologically p-torsion subgroup* of G, namely

$$t_{\underline{p}}(G) = \{x \in G : p^n x \to 0\}.$$

Similarly, the topologically torsion elements of G form the *topologically torsion subgroup* of G

$$G! = \{x \in G : n!x \to 0\}.$$

Clearly, $t_p(G) \subseteq t_{\underline{p}}(G)$ for every $p \in \mathbb{P}$, and $t(G) \subseteq G!$.

Example 2.2. Armacost [3] observed the following non-trivial facts:

(a) $t_{\underline{p}}(\mathbb{T}) = t_p(\mathbb{T}) = \mathbb{Z}(p^\infty)$ for every $p \in \mathbb{P}$;

(b) $\bar{e} \in \mathbb{T}!$, but $\bar{e} \notin t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, where $e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \in \mathbb{R}$ is the Euler number (sometimes called also Napier's constant).

Item (b) above shows that the topologically torsion subgroup $\mathbb{T}!$ may be much more complicated compared to the torsion subgroup $\mathbb{Q}/\mathbb{Z}$ of $\mathbb{T}$. This is why Armacost [3] posed the following:

Problem 2.3 ([3]). *Describe the subgroup* $\mathbb{T}!$ *of* $\mathbb{T}$.

A solution of this problem was obtained independently by Borel [16], and somewhat earlier by Dikranjan-Prodanov-Stoyanov [43] (although the latter solution was not complete and was completed subsequently in [25] in the sense explained in Remark 2.13). To this end all these authors used the fact that for every $x \in [0,1)$ there exists a unique sequence $(c_n)_{n\in\mathbb{N}_+}$ in $\mathbb{N}$ such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{(n+1)!}, \tag{2.1}$$

with $c_n < n+1$ for every $n \in \mathbb{N}_+$ and $c_n < n$ for infinitely many $n \in \mathbb{N}_+$. A more general property is described in Theorem 2.11.

Theorem 2.4 ([43, 16, 25]). *Let $x \in [0,1)$ as in* (2.1). *Then $\bar{x} \in \mathbb{T}!$ if and only if $\frac{c_n}{n+1} \to 0$ in $\mathbb{T}$.*

2.2 Topologically p-torsion subgroups of a compact group

This subsection is dedicated to the bold question: *how big can be the topologically p-torsion subgroup $t_{\underline{p}}(G)$ of a topological abelian group G?* If G is a discrete torsion-free abelian group, then $t_{\underline{p}}(G) = G! = 0$. This example suggests to impose some (compactness-like) condition on G ensuring non-triviality of the subgroups $t_{\underline{p}}(G)$.

A topological abelian group (G, τ) is:

(a) *totally bounded* if for every non-empty $U \in \tau$, there exists a finite subset F of G such that $G = U + F$;

(b) *precompact* if τ is Hausdorff and totally bounded.

It is well-known that the precompact abelian groups are precisely the topological subgroups of the compact abelian groups. The topological subgroups (up to topological isomorphism) of the locally compact abelian groups are named *locally precompact*.

Using Example 2.2, one can see (see [43, Lemma 4.1.1]) that, for a prime p, a non-torsion element x of a locally precompact abelian group G is topologically p-torsion if and only if $p^n v_n x \to 0$ for every sequence $\mathbf{v} = (v_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$, and this occurs precisely when the cyclic subgroup $\langle x\rangle \cong \mathbb{Z}$ carries the p-adic topology as a topological subgroup of G.

Elements with this (obviously, stronger) property were introduced by Stoyanov [79, 80, 42] (see also [28, 43]) under the name *quasi p-torsion*; they were largely studied in the second half of the seventies and in the eighties of the last century, in connection to the study of groups satisfying the open mapping theorem (the so called *minimal groups*, see [27, 39]). The subgroup of all quasi p-torsion elements of a topological abelian group G is denoted by $td_p(G)$ in [79, 80, 27, 43, 28]. So what was stated above is simply the equality $td_p(G) = t_{\underline{p}}(G)$ for a locally precompact abelian group G.

Moreover, Stoyanov noticed that the sum $wtd(G) := \sum_{p\in\mathbb{P}} td_p(G)$ is always *direct* and called the elements of the subgroup $wtd(G)$ of G *weakly periodic*. Clearly, every torsion element is weakly periodic, so every torsion (in particular, every finite) abelian group consists entirely of weakly periodic elements. Yet, even precompact abelian groups G may have $wtd(G) = 0$ (e.g., every infinite cyclic subgroup of $\mathbb{T}$). However, things change completely for *compact* groups:

Theorem 2.5 ([80])**.** *For every compact abelian group G the subgroup $wtd(G)$ is dense.*

Actually, one can say more, as $wtd(N) = N \cap wtd(G)$ for every closed subgroup N of a topological abelian group G. Hence, the above theorem implies that the subgroup $wtd(G)$ is actually *totally dense*, i.e., it densely intersects all closed subgroups of a compact group G. As a first consequence of this theorem, one can see that $|wtd(G)|$ is "quite close to" $|G|$ when G is compact [80].

As another corollary of the above theorem and of a well-known criterion for minimality of the dense subgroups of a compact group, due to Prodanov and Stephenson (see [43, Corollary 2.5.2]), one obtains a nice connection of topologically p-torsion elements and the open mapping theorem:

Theorem 2.6 ([43, Theorem 4.3.7])**.** *A dense subgroup of a compact abelian group G is minimal if and only if for every prime p every non-trivial closed subgroup of G contains a non-trivial topologically p-torsion element.*

An equivalent "internal" formulation of this theorem is: *a precompact abelian group G is minimal if and only if $t_{\underline{p}}(N) \neq 0$ for every prime p and for every non-trivial closed subgroup N of G.* (Note that $t_{\underline{p}}(N) = N \cap t_{\underline{p}}(G)$ for every subgroup N of G.)

In conclusion, let us recall that the equality $t_{\underline{p}}(G) = td_p(G)$ remains true for all minimal abelian groups, as these groups are precompact, according to the celebrated Prodanov-Stoyanov theorem [43, Theorem 2.2.7].

Theorem 2.5 should be compared with its counterpart concerning (total) density of the torsion part $t(G)$ of a compact abelian group. The compact abelian groups G such that $t(G)$ is totally dense were studied in [41] under the name *exotic tori*. It was shown that exotic tori are finite dimensional and allow for a nice approximation by Lie groups (tori).

2.3 Topologically u-torsion subgroup

The following definition generalizes the ones seen above of topologically p-torsion subgroup and topologically torsion subgroup.

Definition 2.7. [[43, 28]] For a sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ and a topological abelian group G, call an element $x \in G$ *topologically* $\mathbf{u}$*-torsion* if $u_n x \to 0$. The *topologically* $\mathbf{u}$*-torsion subgroup* of G is

$$t_{\mathbf{u}}(G) = \{x \in G : u_n x \to 0\}.$$

We recover the topologically p-torsion subgroup and topologically torsion subgroup as follows:

(a) for $p \in \mathbb{P}$, $t_{\underline{p}}(G) = t_{\mathbf{p}}(G)$, where $\mathbf{p} = (p^n)_{n\in\mathbb{N}}$;

(b) $G! = t_{\mathbf{u}}(G)$, where $\mathbf{u} = ((n+1)!)_{n\in\mathbb{N}}$.

As a matter of fact, the definition of topological $\mathbf{u}$-torsion was given in [43, §4.4.2] only for sequences $\mathbf{u}$ with $u_n|u_{n+1}$ for every $n \in \mathbb{N}$ (see §2.4 and Definition 2.10).

Example 2.8. [[38]] For the torsion subgroup of $\mathbb{T}$, we have that $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z} = t_{\mathbf{u}}(\mathbb{T})$, where $\mathbf{u}$ is the sequence $(1!, 2!, 2\cdot 2!, 3!, 2\cdot 3!, 3\cdot 3!, 4!, \ldots, n!, 2\cdot n!, 3\cdot n!, \ldots, n\cdot n!, (n+1)!, \ldots)$. Similarly, one can characterize arbitrary subgroups of $\mathbb{Q}/\mathbb{Z}$ (see [38]).

This survey is dedicated to the following general problem:

Problem 2.9. *Given a sequence $\mathbf{u}$ in $\mathbb{Z}$, describe the subgroup $t_{\mathbf{u}}(\mathbb{T})$.*

2.4 Arithmetic sequences

Here we see that Problem 2.9 has a complete solution for sequences $\mathbf{u}$ sharing the common property $u_n|u_{n+1}$ for every $n \in \mathbb{N}$, with both sequences $(p^n)_{n\in\mathbb{N}}$ and $((n+1)!)_{n\in\mathbb{N}}$ considered in §2.1.

Definition 2.10. [[43, Chapter 4]] An *arithmetic sequence* is an increasing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_+$, such that $u_0 = 1$ and $u_n|u_{n+1}$ for every $n \in \mathbb{N}$.

The name *arithmetic sequence* was coined later on in [34]. Clearly, an increasing sequence $\mathbf{u}$ in $\mathbb{N}_+$ with $u_0 = 1$ is arithmetic if and only if the sequence of ratios $\mathbf{b^u}$ is in $\mathbb{N}_+$.

Every arithmetic sequence gives rise to a nice representation generalizing (2.1):

Theorem 2.11 ([71])**.** *Let $\mathbf{u}$ be an arithmetic sequence. For every $x \in [0,1)$, there exists a unique sequence $(c_n^{\mathbf{u}}(x))_{n\in\mathbb{N}_+}$ in $\mathbb{N}$ such that*

$$x = \sum_{n=1}^{\infty} \frac{c_n^{\mathbf{u}}(x)}{u_n}, \tag{2.2}$$

with $c_n^{\mathbf{u}}(x) < b_n^{\mathbf{u}}$ for every $n \in \mathbb{N}_+$, and $c_n^{\mathbf{u}}(x) < b_n^{\mathbf{u}} - 1$ for infinitely many $n \in \mathbb{N}_+$.

When no confusion is possible, we shall simply write $c_n^{\mathbf{u}}$ or c_n in place of $c_n^{\mathbf{u}}(x)$. For $x \in [0,1)$, with canonical representation (2.2), let

$$\mathrm{supp}(x) = \{n \in \mathbb{N}_+ : c_n^{\mathbf{u}} \neq 0\} \quad\text{and}\quad \mathrm{supp}_b(x) = \{n \in \mathbb{N}_+ : c_n^{\mathbf{u}} = b_n^{\mathbf{u}} - 1\}.$$

Clearly, $\mathrm{supp}_b(x) \subseteq \mathrm{supp}(x)$ and $\mathrm{supp}_b(x)$ cannot be cofinite by definition.

Theorem 2.12 ([43, 25])**.** *Let $\mathbf{u}$ be an arithmetic sequence and $x \in [0,1)$.*

(a) If $\mathbf{b^u}$ is bounded, then $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$ if and only if $(c_n^{\mathbf{u}}(x))_{n\in\mathbb{N}_+}$ is eventually 0.

(b) If $b_n^{\mathbf{u}} \to +\infty$, then $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$ if and only if $\varpi(\frac{c_n^{\mathbf{u}}(x)}{b_n^{\mathbf{u}}}) \to 0$ in $\mathbb{T}$.

Remark 2.13. Item (b) was not complete in [43, §4.4.2, Theorem], as only the stronger condition "$\frac{c_n^{\mathbf{u}}(x)}{b_n^{\mathbf{u}}} \to 0$ in $\mathbb{R}$" was considered there, missing in this way the elements $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$ with $\varpi(\frac{c_n^{\mathbf{u}}(x)}{b_n^{\mathbf{u}}}) \to 0$ in $\mathbb{T}$, but $\frac{c_n^{\mathbf{u}}(x)}{b_n^{\mathbf{u}}} \not\to 0$ in $\mathbb{R}$. This gap was filled in [25].

Theorem 2.12 is only a corollary of the main result of [25] providing a description of $t_{\mathbf{u}}(\mathbb{T})$ for an arithmetic sequence $\mathbf{u}$. Here we give a consequence.

Corollary 2.14 ([25]). *Let* $\mathbf{u}$ *be an arithmetic sequence. The following conditions are equivalent:*

(a) $\mathbf{b^u}$ *is bounded;*

(b) $t_{\mathbf{u}}(\mathbb{T})$ *is countable;*

(c) $|t_{\mathbf{u}}(\mathbb{T})| < \mathfrak{c}$*;*

(d) $t_{\mathbf{u}}(\mathbb{T})$ *is torsion.*

It was proved much later in [35] that one can add to these four equivalent conditions also "$t_{\mathbf{u}}(\mathbb{T})$ is an F_σ-subgroup". This was inspired by the fact, established earlier by Gabriyelyan [54], that $t_{\mathbf{u}}(\mathbb{T})$ is not an F_σ-subgroup for the sequence $\mathbf{u} = (n+1!)_{n\in\mathbb{N}}$.

Following [34], given an arithmetic sequence $\mathbf{u}$, call an infinite subset A of $\mathbb{N}$:

(a) *b-bounded* if the sequence $\{b_n^{\mathbf{u}} : n \in A\}$ is bounded;

(b) *b-divergent* if the sequence $\{b_n^{\mathbf{u}} : n \in A\}$ diverges to infinity.

Theorem 2.12(a) can be reinforced in the following sharper form:

Corollary 2.15 ([25, Corollary 2.4]). *Let* $\mathbf{u}$ *be an arithmetic sequence and* $x \in [0,1)$*. If* $\mathrm{supp}(x)$ *is b-bounded, then the following conditions are equivalent:*

(a) $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$*;*

(b) $c_n^{\mathbf{u}}(x) = 0$ *for almost all* $n \in \mathbb{N}_+$*;*

(c) $\bar{x}$ *is torsion.*

In 2011 Impieri found a gap in the description given in [25] of $t_{\mathbf{u}}(\mathbb{T})$ for an arbitrary arithmetic sequence $\mathbf{u}$. The following complete description was obtained in [34].

Theorem 2.16 ([34]). *Let* $\mathbf{u}$ *be an arithmetic sequence and let* $x \in [0,1)$*. Then* $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$ *if and only if* $\mathrm{supp}(x)$ *is finite or if* $\mathrm{supp}(x)$ *is infinite and for all* $A \subseteq \mathbb{N}$ *the following conditions hold:*

(a) if A *is b-bounded, then:*

(1) if $A \subset^* \mathrm{supp}(x)$*,* [1] *then* $A+1 \subset^* \mathrm{supp}(x)$*,* $A \subset^* \mathrm{supp}_b(x)$ *and* $\lim_{n\in A} \frac{c_{n+1}^{\mathbf{u}}+1}{b_{n+1}^{\mathbf{u}}} = 1$ *in* $\mathbb{R}$*; moreover, if* $A+1$ *is b-bounded, then* $A+1 \subset^* \mathrm{supp}_b(x)$ *as well;*

(2) if $A \cap \mathrm{supp}(x)$ *is finite, then* $\lim_{n\in A} \frac{c_{n+1}^{\mathbf{u}}}{b_{n+1}^{\mathbf{u}}} = 0$ *in* $\mathbb{R}$*; moreover, if* $A+1$ *is b-bounded, then* $(A+1) \cap \mathrm{supp}(x)$ *is finite as well;*

(b) if A *is b-divergent, then* $\lim_{n\in A} \varpi(\frac{c_n^{\mathbf{u}}}{b_n^{\mathbf{u}}}) = \lim_{n\in A} \varpi(\frac{c_n^{\mathbf{u}}+1}{b_n^{\mathbf{u}}}) = 0$ *in* $\mathbb{T}$*.*

Now we give the counterpart of Corollary 2.15 with "b-bounded" replaced by "b-divergent".

Corollary 2.17 ([34]). *Let* $\mathbf{u}$ *be an arithmetic sequence and suppose that* $x \in [0,1)$ *has b-divergent support. Then* $\bar{x} \in t_{\mathbf{u}}(\mathbb{T})$ *if and only if the following two conditions are satisfied:*

(a) $\lim_{n\in \mathrm{supp}(x)} \varpi\left(\frac{c_n^{\mathbf{u}}}{b_n^{\mathbf{u}}}\right) = 0$ *in* $\mathbb{T}$*; and*

(b) $\lim_{n\in I'} \frac{c_n^{\mathbf{u}}}{b_n^{\mathbf{u}}} = 0$ *in* $\mathbb{R}$ *for every infinite* $I' \subseteq I$ *such that* $I'-1$ *is b-bounded.*

[1]Here $X \subset^* Y$ for subsets of $\mathbb{N}$ means, as usual, that $X \setminus Y$ is finite.

3 Characterized subgroups of $\mathbb{T}$

Let us warn the reader that this section can be consistently considered as a continuation of the previous one, since the new notion of characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ (given in Definition 3.1), as far as the circle group $\mathbb{T}$ is concerned, is nothing else but the topologically $\mathbf{u}$-torsion subgroup already introduced in the previous section. Nevertheless, both notions can be considered in arbitrary topological abelian groups (as already done in Definition 2.7 for topologically $\mathbf{u}$-torsion subgroups; see §7 for a counterpart of characterized subgroups in the general case). In §8 we offer a comparison showing the striking difference between both notions. Indeed, it turns out that the circle group is, in appropriate sense, the only group where they coincide and produce the same effect (see Theorem 8.7).

3.1 Definition and generalization

The terminology "characterized subgroup" was coined in [14]:

Definition 3.1. [[14]] A subgroup H of $\mathbb{T}$ is *characterized* if $H = t_{\mathbf{u}}(\mathbb{T})$ for some sequence $\mathbf{u}$ in $\mathbb{Z}$. We say also that H is *characterized* by $\mathbf{u}$, that $\mathbf{u}$ *characterizes* H, and that $\mathbf{u}$ is a characterizing sequence for H.

As we mentioned above, for a sequence $\mathbf{u}$ in $\mathbb{Z}$ the subgroup of $\mathbb{T}$ characterized by $\mathbf{u}$ is precisely the topologically $\mathbf{u}$-torsion one. So we offer below also a reformulation of Problem 2.9, using the new term:

Problem 3.2. *Describe the characterized subgroups of* $\mathbb{T}$. *In other words, given a sequence* $\mathbf{u}$ *in* $\mathbb{Z}$, *describe* $t_{\mathbf{u}}(\mathbb{T})$.

One can consider also the "inverse problem": given $H \leq \mathbb{T}$, when is H characterized? In order to attack this problem, one needs to obtain some feedback in the direction of the first one, namely accumulate a reasonable knowledge on the basic properties of the characterized subgroups of $\mathbb{T}$.

3.2 First properties and results

We start giving some basic properties of the characterized subgroups of $\mathbb{T}$.

(a) If H is a finite subgroup of $\mathbb{T}$, then H is characterized. Indeed, the finite subgroups of $\mathbb{T}$ are cyclic, so one can use for example the argument from the initial part of §4.

(b) A sequence $\mathbf{u}$ in $\mathbb{Z}$ characterizes $\mathbb{T}$ if and only if $\mathbf{u}$ is eventually zero.

(c) If a proper subgroup H of $\mathbb{T}$ is characterized, then:

  (1) $H = t_{\mathbf{u}}(\mathbb{T})$ for some strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}_+$;

  (2) $\mu(H) = 0$.

(d) Since $t_{\mathbf{u}}(\mathbb{T}) = \bigcap_{N\geq 2}\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}\left\{x \in \mathbb{T} : \|u_n x\| \leq \frac{1}{N}\right\}$ is a Borel set (more precisely, an $F_{\sigma\delta}$-set), so

$$\text{either } t_{\mathbf{u}}(\mathbb{T}) \text{ is countable or } |t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}.$$

According to (d), every characterized subgroup of K is an $F_{\sigma\delta}$-set. Inspired by a construction of Aaronson-Nadkarni [1], Biró [12] showed that the F_σ-subgroups of $\mathbb{T}$ need not be characterized. His proof is based on the crucial point that he discovered, namely, *the characterized subgroups of* $\mathbb{T}$ *are Polishable* (let us recall that a group is called *Polishable* if it admits a finer Polish group topology; this topology is unique by a result of Solecki [76]). We state now Biró's result; first recall that a nonempty subset K of $\mathbb{T}$ is a *Kronecker set* if K is compact and, for every continuous function $f : K \to \mathbb{T}$ and every $\delta > 0$, there exists $n \in \mathbb{Z}$ such that $\max_{k\in K} \|f(k) - nk\| < \delta$ (i.e., f can be uniformly approximated by characters of $\mathbb{T}$).

Theorem 3.3. *Let K be an uncountable Kronecker set in $\mathbb{T}$. Then the subgroup $\langle K\rangle$ is not Polishable. In particular, $\langle K\rangle$ cannot be characterized.*

Since $\langle K\rangle$ is obviously F_σ, this provides an example of a non-characterized F_σ-subgroup of $\mathbb{T}$, thereby answering a question of the second named author (see also [38], where some special classes of F_σ-subgroups were shown to be characterized).

We recall now a result by Eggleston on the cardinality of the characterized subgroups of $\mathbb{T}$.

Theorem 3.4 ([49])**.** *Let $\mathbf{u}$ be a sequence in $\mathbb{N}_+$.*

(a) If $b_n^{\mathbf{u}} \to +\infty$, then $|t_{\mathbf{u}}(\mathbb{T})| = \mathfrak{c}$.

(b) If $\mathbf{b}^{\mathbf{u}}$ is bounded, then $t_{\mathbf{u}}(\mathbb{T})$ is countable.

This result was proved also in [6], as the authors were unaware of Eggleston's paper.

A simple, yet quite non-trivial, strong sufficient condition for a subgroup to be characterized was found by Borel:

Theorem 3.5 ([15])**.** *All countable subgroups of $\mathbb{T}$ are characterized.*

It was shown by Beiglböck-Steineder-Winkler [11] that $\mathbf{u}$ can be chosen with $\mathbf{b}^{\mathbf{u}}$ bounded, but also arbitrarily fast increasing in the following sense.

Theorem 3.6 ([11, Theorem 4.2])**.** *Let H be a countable subgroup of $\mathbb{T}$ and let $m_1 < m_2 < \dots$ be an (arbitrarily fast) increasing sequence of positive integers. Then there is an increasing characterizing sequence $\mathbf{u}$ with $m_n < u_n$ for all $n \in \mathbb{N}$, such that $H = t_{\mathbf{u}}(\mathbb{T})$.*

Unaware of Borel's theorem, Bíró-Deshouillers-Sós [14] proved that every countable subgroup of $\mathbb{T}$ containing a non-torsion element is characterized (for the missing case of subgroups of $\mathbb{Q}/\mathbb{Z}$ see Example 2.8).

Borel's motivation to study characterized subgroups of $\mathbb{T}$ was the connection to uniform distribution mod 1 of sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is called *uniformly distributed mod* 1 if for all $[a,b] \subseteq [0,1)$,

$$\frac{|\{j \in \{0,\dots,n\} : \{x_j\} \in [a,b]\}|}{n} \longrightarrow a - b.$$

For a sequence $\mathbf{u}$ in $\mathbb{Z}$, let

$$\mathcal{W}_{\mathbf{u}} = \{\beta \in [0,1] : (u_n\beta)_{n\in\mathbb{N}} \text{ is uniformly distributed mod } 1\}.$$

Obviously, $\mathcal{W}_{\mathbf{u}} \subseteq [0,1] \setminus \mathbb{Q}$.

Theorem 3.7 ([83, 64])**.** *Let $\mathbf{u}$ be a strictly increasing sequence in $\mathbb{N}_+$. Then $\lambda(\mathcal{W}_{\mathbf{u}}) = 1$. Moreover, if $u_n = P(n)$ for every $n \in \mathbb{N}$ for some fixed integer valued polynomial $P(x)$, then $\mathcal{W}_{\mathbf{u}} = [0,1] \setminus \mathbb{Q}$.*

As $\mathcal{W}_{\mathbf{u}}$ is properly contained in $[0,1] \setminus \mathbb{Q}$ in general, it makes sense to pay attention also to the "opposite behavior"; in other words, to study the set of those $\beta \in [0,1]$ such that the sequence $(u_n\beta)_{n\in\mathbb{N}}$ is quite far even from being dense mod 1, namely $t_{\mathbf{u}}(\mathbb{T}) = \{\bar\beta \in \mathbb{T} : u_n\bar\beta \to 0\}$.

4 Characterization of the cyclic subgroups of $\mathbb{T}$

In this section we consider the cyclic subgroups of $\mathbb{T}$. We see that they are all characterized by finding appropriate sequences that witness this property. By Theorem 3.5 all countable subgroups of $\mathbb{T}$ are characterized, but in the specific case of cyclic subgroups it is possible to explicitly construct characterizing sequences as shown in [14].

For $\alpha \in \mathbb{R}$ the cyclic subgroup $\langle \bar{\alpha} \rangle$ of $\mathbb{T}$ is finite if and only if α is a rational number, that is, $\alpha = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}_+$. It is not hard to verify that $\langle \bar{\alpha} \rangle = t_{\mathbf{u}}(\mathbb{T})$ for the sequence $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ defined by $u_0 = 1$ and $u_n = qn$ for every $n \in \mathbb{N}$.

4.1 Continued fractions and characterizing sequences

Let now α be an irrational number. Then α has a unique continued fraction expansion

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$$

denoted by $\alpha = [a_0; a_1, a_2, \ldots]$; let $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$.

For every $n \in \mathbb{N}$, let $\mathbf{q} = (q_n)_{n \in \mathbb{N}}$ be the increasing sequence in $\mathbb{N}_+$ of the best approximation denominators of α, with $q_{-1} = 0$ and $q_0 = 1$. This is the sequence defined by

$$[a_0; a_1, \ldots, a_n] = \frac{p_n}{q_n}$$

and

$$q_{n+1} = a_{n+1} q_n + q_{n-1}. \tag{4.1}$$

for every $n \in \mathbb{N}$.

It is well-known that $|q_n \alpha - p_n| < 1/q_{n+1}$, so $q_n \bar{\alpha} \to 0$ in $\mathbb{T}$. In our terms this means that

$$\langle \bar{\alpha} \rangle \subseteq t_{\mathbf{q}}(\mathbb{T}). \tag{4.2}$$

In 1988 Larcher proved that equality holds in Equation (4.2) under the assumption that the sequence $\mathbf{a}$ is bounded. For a relevant example that is covered by Larcher's theorem see Example 4.13 below.

Theorem 4.1 ([65, Theorem 1])**.** *In the above notations, if $\mathbf{a}$ is bounded, then $\langle \bar{\alpha} \rangle = t_{\mathbf{q}}(\mathbb{T})$.*

Larcher showed that in general the inclusion in Equation (4.2) can be strict. This follows also from Theorem 3.4:

Remark 4.2. In the above notations, assume that $a_n \to +\infty$, and consider the sequence $\mathbf{b}^{\mathbf{q}}$. By Equation (4.1), $b^{\mathbf{q}}_{n+1} \geq a_{n+1}$, so $b^{\mathbf{q}}_n \to +\infty$. By Theorem 3.4, $|t_{\mathbf{q}}(\mathbb{T})| = \mathfrak{c}$, so clearly the inclusion in Equation (4.2) is strict.

In case the sequence $\mathbf{a}$ is unbounded, Kraaikamp and Liardet in 1991 proved the following result.

Theorem 4.3 ([63, Theorem 3])**.** *In the above notations, if $\mathbf{a}$ is unbounded, then $|t_{\mathbf{q}}(\mathbb{T})| = \mathfrak{c}$.*

Therefore, they proved that the condition in Larcher's theorem is actually equivalent to the equality in Equation (4.1):

Theorem 4.4 ([63, Proposition])**.** *In the above notations, $\langle \bar{\alpha} \rangle = t_{\mathbf{q}}(\mathbb{T})$ if and only if $\mathbf{a}$ is bounded.*

Let us resume the above results as follows:

Corollary 4.5. *In the above notations, the following conditions are equivalent:*

(a) $\mathbf{a}$ *is bounded;*

(b) $\langle\bar{\alpha}\rangle = t_{\mathbf{q}}(\mathbb{T})$*;*

(c) $t_{\mathbf{q}}(\mathbb{T})$ *is countable;*

(d) $|t_{\mathbf{q}}(\mathbb{T})| < \mathfrak{c}$.

In [63] there is also a technical criterion (see [63, Theorem 1]) that was applied in 2001 by Bíró-Deshouillers-Sós [14] to set the case when $\mathbf{a}$ is unbounded. This was done by adding elements to the sequence $\mathbf{q}$ to find the sequence $\mathbf{v}_\alpha$ defined by

$$q_0 \leq q_1 < 2q_1 < \ldots < a_2 q_1 < q_2 < 2q_2 < \ldots < a_3 q_2 < q_3 < 2q_3 < \ldots.$$

Clearly, $\mathbf{q} \subseteq \mathbf{v}_\alpha$ and so $t_{\mathbf{v}_\alpha}(\mathbb{T}) \subseteq t_{\mathbf{q}}(\mathbb{T})$.

Theorem 4.6 ([14, Theorem 1*])**.** *In the above notations,* $\langle\bar{\alpha}\rangle = t_{\mathbf{v}_\alpha}(\mathbb{T})$*.*

In [14] one can find three different proofs of the fact that every cyclic subgroup of $\mathbb{T}$ is characterized, but only the one that we mentioned above is constructive.

Among other open problems left in [14], we mention the following problem due to Maharam and Stone:

Problem 4.7 ([14, Problem 3])**.** *Given* $\alpha \in \mathbb{R}$*, find all sequences* $\mathbf{u}$ *in* $\mathbb{N}_+$ *such that* $\langle\bar{\alpha}\rangle = t_{\mathbf{u}}(\mathbb{T})$*.*

In his thesis [68], Marconato considered this problem for sequences $\mathbf{u}$ in $\mathbb{N}_+$ such that $\mathbf{q} \subseteq \mathbf{u} \subseteq \mathbf{v}_\alpha$. This clearly implies that

$$\langle\bar{\alpha}\rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) \subseteq t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{q}}(\mathbb{T}).$$

The aim was to find a condition equivalent to $\langle\bar{\alpha}\rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$.

Theorem 4.8 ([68])**.** *In the above notations, let* $\mathbf{u}$ *be a sequence in* $\mathbb{N}_+$ *such that* $\mathbf{q} \subseteq \mathbf{u} \subseteq \mathbf{v}_\alpha$*. If* $\mathbf{b}^{\mathbf{u}}$ *is bounded, then* $\langle\bar{\alpha}\rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$*.*

We believe that also the converse implication holds true.

Conjecture 4.9 ([68])**.** *In the above notations, if* $\mathbf{u}$ *is a sequence in* $\mathbb{N}_+$ *such that* $\mathbf{q} \subseteq \mathbf{u} \subseteq \mathbf{v}_\alpha$*, then* $\langle\bar{\alpha}\rangle = t_{\mathbf{v}_\alpha}(\mathbb{T}) = t_{\mathbf{u}}(\mathbb{T})$ *if and only if* $\mathbf{b}^{\mathbf{u}}$ *is bounded.*

We recall now two results from [14] that give necessary conditions for a sequence $\mathbf{u}$ in $\mathbb{N}_+$ to satisfy $\langle\bar{\alpha}\rangle \subseteq t_{\mathbf{u}}(\mathbb{T})$ for a given irrational α. In the above notations, by a classical result of Ostrowski, every $m \in \mathbb{N}_+$ has a unique expansion

$$m = \sum_{k=0}^{K} d_k q_k,$$

for some $K \in \mathbb{N}$ and with $0 \leq d_0 < a_1$, $0 \leq d_k \leq a_{k+1}$, and $d_k = a_{k+1}$ implies $d_{k-1} = 0$ for $k \in \mathbb{N}_+$.

For the sequence $\mathbf{u}$ in $\mathbb{N}_+$, for every $n \in \mathbb{N}$ consider the expansion $u_n = \sum_{k=k(n)}^{K(n)} d_k(n) q_k$, where $d_{k(n)}$ is the first non-zero coefficient of the expansion (i.e., $d_{k(n)} > 0$).

Proposition 4.10 ([14, Proposition 1])**.** *In the above notations, if* $\bar{\alpha} \in t_{\mathbf{u}}(\mathbb{T})$*, then* $k(n) \to +\infty$*.*

Let $\mathbf{u}$ be a sequence in $\mathbb{N}_+$ with $u_n \to +\infty$, and let

$$L = \{k \in \mathbb{N} : b_k(n) \neq 0 \text{ for some } n \in \mathbb{N}\}$$

(i.e., $k \in L$ precisely when q_k occurs in the expansion of at least one u_n).

Proposition 4.11 ([14, Proposition 2])**.** *In the above notations, if L is not large, then $t_{\mathbf{u}}(\mathbb{T})$ is uncountable.*

Corollary 4.12. *In the above notations, if $\langle \bar{\alpha} \rangle = t_{\mathbf{u}}(\mathbb{T})$, then L is large.*

4.2 Recurrent sequences

The following example answers [28, Question 3.11] asking to describe the subgroup $t_{\mathbf{f}}(\mathbb{T})$, where $\mathbf{f} = (f_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence $f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 5,\ \ldots$. This was a particular case of the more general form of sequences $\mathbf{u}$ considered in [28], namely those which satisfy

$$u_n | u_{n+1} - u_{n-1} \text{ for all } n \in \mathbb{N}_+ \tag{4.3}$$

(as $\mathbf{f}$ does). A direct alternative proof of the equality (4.4) was provided in [6]. All these authors were unaware of Theorem 4.1 giving an answer to this quest in the case of $\mathbf{f}$ as the next example shows.

Example 4.13. Let ϕ be the Golden ratio, that is, $\phi = \frac{1+\sqrt{5}}{2}$. Then

$$\langle \bar{\phi} \rangle = t_{\mathbf{f}}(\mathbb{T}), \tag{4.4}$$

where $\mathbf{f} = (f_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence $f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ f_3 = 3,\ f_4 = 5,\ \ldots$, by Theorem 4.1. Indeed, $\phi = [1; 1, 1, \ldots]$ and $q_n = f_n$ for every $n \in \mathbb{N}_+$.

Inspired by (4.4) and, more generally, by the class of sequences satisfying (4.3), Barbieri-Dikranjan-Milan-Weber [9] considered sequences $\mathbf{u}$ in $\mathbb{Z}$ verifying an even more general linear recurrence. Here we report only a result in the case of *second-order* linear recurrence, namely: for every $n \geq 2$, $u_n = a_n u_{n-1} + b_n u_{n-2}$, $a_n, b_n \in \mathbb{N}_+$. (Notice that the coefficients a_n, b_n, unlike in Example 4.13, may vary with n.)

Theorem 4.14 ([9])**.** *Let $\mathbf{u}$ be a sequence of second-order linear recurrence in $\mathbb{N}_+$. Then $|t_{\mathbf{u}}(\mathbb{T}))| = \mathfrak{c}$ if and only if $\mathbf{b}^{\mathbf{u}}$ is not bounded.*

This theorem generalizes Theorem 4.3. Indeed, if $\alpha = [a_0; a_1, a_2, \ldots]$ is an irrational and $\mathbf{q}$ is the sequence of the best approximation denominators of α, then for every $n \geq 2$, $q_n = a_n q_{n-1} + q_{n-2}$, $q_0 = 1, q_1 = a_1$. For further results in the case of higher order linear recurrence see [9].

5 Characterized subgroups of $\mathbb{T}$ and precompact group topologies of $\mathbb{Z}$

The following problem was considered by Raczkowski [73, 74] and Barbieri-Dikranjan-Milan-Tonolo-Weber [6, 40]:

For a given sequence $\mathbf{u}$ in $\mathbb{Z}$, does there exist a precompact group topology τ on $\mathbb{Z}$ such that $u_n \xrightarrow{\tau} 0$? If the answer is affirmative, $\mathbf{u}$ is called a TB-sequence (see [40]).

For an abelian group G and a subgroup H of $\widehat{G}$, let T_H be the weakest group topology on G such that all $\chi \in H$ are continuous. We recall the following fundamental theorem by Comfort and Ross on totally bounded and precompact groups topologies on abelian groups.

Theorem 5.1 ([23])**.** *Let G be an abelian group and H a subgroup of $\widehat{G}$. Then:*

(a) T_H is totally bounded and $w(G, T_H) = |H|$;

(b) T_H is Hausdorff if and only if H is dense in $\widehat{G}$;

(c) if τ is a totally bounded group topology on G, then $\tau = T_H$ for some $H \le \widehat{G}$.

For $G = \mathbb{Z}$, we have $\widehat{G} = \mathbb{T}$. In particular:

Corollary 5.2. *The assignment $H \mapsto T_H$ defines a monotone bijective correspondence between infinite subgroups H of $\mathbb{T}$ and precompact group topologies on $\mathbb{Z}$.*

Clearly, a sequence $\mathbf{u}$ in $\mathbb{Z}$ converges to 0 in a totally bounded group topology $\tau = T_H$ on $\mathbb{Z}$ precisely when $H \le t_{\mathbf{u}}(\mathbb{T})$. Therefore, $T_{t_{\mathbf{u}}(\mathbb{T})}$ is the finest totally bounded group topology τ on $\mathbb{Z}$ with $u_n \xrightarrow{\tau} 0$. Here we list some properties of this topology (see also [32] for further properties):

(a) $w(\mathbb{Z}, T_{t_{\mathbf{u}}(\mathbb{T})}) = |t_{\mathbf{u}}(\mathbb{T})|$;

(b) $T_{t_{\mathbf{u}}(\mathbb{T})}$ is precompact if and only if $t_{\mathbf{u}}(\mathbb{T})$ is infinite;

(c) $T_{t_{\mathbf{u}}(\mathbb{T})}$ is metrizable if and only if $|t_{\mathbf{u}}(\mathbb{T})| = \omega$;

(d) $\sigma_{\mathbf{u}}$ is linear if and only if $t_{\mathbf{u}}(\mathbb{T})$ is torsion.

One can find in [21] many examples of totally bounded group topologies $\tau = T_H$ on $\mathbb{Z}$ without any non-trivial convergent sequence. To this end one has to choose the subgroup H of $\mathbb{T}$ such that $u_n \xrightarrow{T_H} 0$ (i.e., $H \le t_{\mathbf{u}}(\mathbb{T})$) never occurs for a non-trivial sequence $\mathbf{u}$ in $\mathbb{Z}$. Non-measurable subgroups H of $\mathbb{T}$ satisfy this condition (for more details see [21] and [73]).

The subgroups H of $\mathbb{T}$ such that $H \le t_{\mathbf{u}}(\mathbb{T})$ for some non-trivial sequence $\mathbf{u}$ satisfy $\mu(H) = 0$, as $\mu(t_{\mathbf{u}}(\mathbb{T})) = 0$. This explains why non-measurable subgroups H of $\mathbb{T}$ do not satisfy $H \le t_{\mathbf{u}}(\mathbb{T})$ for any non-trivial sequence $\mathbf{u}$ in $\mathbb{Z}$.

It was an open question of Raczkowski [74] whether a subgroup H of $\mathbb{T}$ with $\mu(H) = 0$ that is contained in no proper characterized subgroup of $\mathbb{T}$ exists. Such an example was built, under the assumption of Martin Axiom, in [6]. Subsequently, Kunen-Hart [56] gave a proof in ZCF of this fact (see also [57], extending this result on other compact groups beyond the circle $\mathbb{T}$).

As a consequence of Theorem 3.4 one obtains:

Theorem 5.3 ([7])**.** *Let $\mathbf{u}$ be a sequence in $\mathbb{N}_+$.*

(a) If $b^{\mathbf{u}}_n \to +\infty$, then there exists a precompact group topology τ on $\mathbb{Z}$ such that $w(\mathbb{Z}, \tau) = \mathfrak{c}$ and $u_n \xrightarrow{\tau} 0$.

(b) If $\mathbf{b}^{\mathbf{u}}$ is bounded, then every precompact group topology τ on $\mathbb{Z}$ such that $u_n \xrightarrow{\tau} 0$ is metrizable.

6 Thin sets and characterized subgroups

In this section we recall several results that connect the characterized subgroups of $\mathbb{T}$ with some kind of thin sets in Harmonic Analysis. For a comprehensive survey on thin sets see [19], here we recall only some notions of thin sets related to absolute convergence of trigonometric series, i.e., series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x) \tag{6.1}$$

where $a_n, b_n \in \mathbb{R}$, $n = 0, 1, \dots$ ($b_0 = 0$ for simplicity).

We start by recalling the following result proved independently by Denjoy and Luzin.

Theorem 6.1 (Denjoy - Luzin)**.** *If the trigonometric series* (6.1) *converges absolutely on a set* $A \subseteq [0,1]$ *that is either non-meager or has positive Lebesgue measure, then*

$$\sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty; \tag{6.2}$$

consequently, (6.1) *absolutely converges everywhere.*

Marcinkiewicz in [67] in honor of Niemytzki defined a subset A of $[0,1]$ to be an N*-set* if there is a trigonometric series (6.1) absolutely converging on A failing (6.2) (i.e., not converging absolutely everywhere) (see also [51]). Let $\mathcal{N}$ be the family of all N-sets of $[0,1]$.

In the next result, the first equivalent condition to be an N-set was proved by Salem, then Arbault improved Salem's criterion by replacing the arbitrary reals r_n by the integers n.

Theorem 6.2 (Salem - Arbault)**.** *Let* A *be a subset of* $[0,1]$*. The following conditions are equivalent:*

(a) A *is an* N*-set;*

(b) there exist sequences of reals $(\varrho_n)_{n\in\mathbb{N}_+}$*, with* $\varrho_n \geq 0$ *for every* $n \in \mathbb{N}_+$*, and* $(r_n)_{n\in\mathbb{N}_+}$ *such that* $\sum_{n=1}^{\infty} \varrho_n = \infty$ *and* $\sum_{n=1}^{\infty} \varrho_n \sin \pi r_n x$ *absolutely converges for* $x \in A$*;*

(c) there exists a sequence of reals $(\varrho_n)_{n\in\mathbb{N}_+}$*, with* $\varrho_n \geq 0$ *for every* $n \in \mathbb{N}_+$*, such that* $\sum_{n=1}^{\infty} \varrho_n = \infty$ *and* $\sum_{n=1}^{\infty} \varrho_n \sin \pi n x$ *absolutely converges for* $x \in A$*.*

Replacing the coefficients ϱ_n by coefficients taken from the doubleton $\{0,1\}$, Arbault introduced the following notion, stronger than to be an N-set: a subset A of $[0,1]$ is an N_0*-set* if there is an increasing sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_+$ such that $u_0 = 1$ and $\sum_{n=1}^{\infty} \sin \pi u_n x$ absolutely converges for $x \in A$. Let $\mathcal{N}_0$ be the family of all N_0-sets of $[0,1]$.

Finally, we arrive at the definition that directly connects to characterized subgroups of $\mathbb{T}$, a notion that was considered by Arbault and named in his honour. A subset A of $[0,1]$ is an *Arbault set* (shortly, A*-set*) if there is an increasing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_+$ such that $\sin \pi u_n x \to 0$ for all $x \in A$. Let $\mathcal{A}$ be the family of all A-sets of $[0,1]$. Clearly,

a subset A of $[0,1]$ is an A-set if and only if $\varpi(A) \subseteq t_{\mathbf{u}}(\mathbb{T})$ for some increasing sequence $\mathbf{u}$ in $\mathbb{N}_+$.

In analogy to A-sets, Kahane in [59] defined a subset A of $[0,1]$ to be a *Dirichlet set* (briefly, D*-set*) if there is an increasing sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_+$ such that $\sin \pi u_n x \to 0$ uniformly on A. Let $\mathcal{D}$ be the family of all D-sets of $[0,1]$.

Clearly, a D-set is necessarily an A-set. More precisely we have the following implications:

$$D\text{-set} \Rightarrow N_0\text{-set} \Rightarrow A\text{-set}.$$

Subsets of A-sets are clearly A-sets and subsets of D-sets are clearly D-sets; moreover, A-sets and D-sets have Lebesgue measure zero. The classical Dirichlet Theorem on Diophantine approximation implies that each finite set is a D-set (see [59], for a proof see [18, 8.133]). Moreover, every D-set is contained in a closed perfect D-set (see [48]) .

Following [19], a family $\mathcal{F}$ of subsets of $[0,1]$ is called a *family of thin sets* if $\emptyset \in \mathcal{F}$, $[0,1] \notin \mathcal{F}$ and $\mathcal{F}$ is stable under taking subsets (i.e., if $A \in \mathcal{F}$ and $B \subseteq A$, then $B \in \mathcal{F}$). Moreover, a subset X of $[0,1]$ is $\mathcal{F}$*-permitted* if $A \cup X \in \mathcal{F}$ for every $A \in \mathcal{F}$. Then the family $P(\mathcal{F})$ of all $\mathcal{F}$-permitted subsets of $[0,1]$ is an ideal, and $\mathcal{F}$ is an ideal if and only if $\mathcal{F} = P(\mathcal{F})$.

Clearly, $\mathcal{N}$, $\mathcal{N}_0$, $\mathcal{A}$ and $\mathcal{D}$ are families of thin sets. On the other hand, the families $\mathcal{F}$ of thin sets related to trigonometric serie are often not ideals. For example, by Marcinkiewicz's Theorem there are two perfect D-sets A and B such that $\varpi(A+B)=\mathbb{T}$ and hence $A\cup B\notin\mathcal{A}$ and $A\cup B\notin\mathcal{N}$. Therefore, the families $\mathcal{N}$ and $\mathcal{A}$ of thin sets are not ideals.

This is a reason to look for conditions under which the union of two sets in $\mathcal{F}$ is still in $\mathcal{F}$, and furthermore a reason to look for $\mathcal{F}$-permitted sets. The study of permitted sets was started by Arbault in [2] and independently by Erdös, who proved that every countable subset of $[0,1]$ is $\mathcal{N}$-permitted. Kholshchevnikova proved that every countable subset of $[0,1]$ is also $\mathcal{A}$-permitted (see [61]).

We are mainly interested in A-sets and D-sets in relation to characterized subgroups of $\mathbb{T}$, so we identify $[0,1)$ with $\mathbb{T}$ and we consider A-sets and D-sets of $\mathbb{T}$. More precisely, we identify $[0,1)$ and $\mathbb{T}$ by means of the bijection $\varphi:=\varpi\restriction_{[0,1)}\colon[0,1)\to\mathbb{T}$.

Definition 6.3. A subset A of $\mathbb{T}$ is an *A-set* (respectively, *D-set*) if $\varphi^{-1}(A)\subseteq[0,1]$ is an A-set (respectively, D-set); that is, there exists an increasing sequence $\mathbf{u}\in\mathbb{N}_+^{\mathbb{N}}$ such that $u_nx\to 0$ for every $x\in A$ (respectively, $u_nx\to 0$ uniformly on A).

We still denote by $\mathcal{A}$ and $\mathcal{D}$ respectively the families of all A-sets and all D-sets of $\mathbb{T}$, and we use the terminology $\mathcal{A}$-permitted as above.

6.1 Inclusions of characterized subgroups of $\mathbb{T}$ and $\mathbb{R}$

It is consistent with ZFC that there exists an $\mathcal{A}$-permitted set with cardinality $\mathfrak{c}$ (see [47]). Moreover, Eliaš proved in [47] that the existence of $\mathcal{A}$-permitted sets of size $\mathfrak{c}$ is not provable in ZFC. To do this, he shows that every $\mathcal{A}$-permitted set is perfectly meager, since it is known that the existence of a perfectly meager set of size $\mathfrak{c}$ is not decidable in ZFC.

Motivated by this study on $\mathcal{A}$-permitted sets, Eliaš in [46] studied the problem of when $t_{\mathbf{u}}(\mathbb{T})\subseteq t_{\mathbf{v}}(\mathbb{T})$ for increasing sequences $\mathbf{u},\mathbf{v}$ in $\mathbb{N}_+$ (we always assume that $u_0=v_0=1$). Let

$$\mathcal{S}=\left\{\mathbf{u}\in\mathbb{N}_+^{\mathbb{N}}:\mathbf{u}\text{ increasing},\ b_n^{\mathbf{u}}\to+\infty\right\}.$$

Clearly, if $A\subseteq\mathbb{T}$ is an A-set, then $A\subseteq t_{\mathbf{u}}(\mathbb{T})$ for some $\mathbf{u}\in\mathcal{S}$. This immediately implies the following result, which shows that the special A-sets $t_{\mathbf{u}}(\mathbb{T})$ with $\mathbf{u}\in\mathcal{S}$ are "enough" to establish whether a subset of $\mathbb{T}$ is $\mathcal{A}$-permitted.

Theorem 6.4 ([47, Theorem 1.3])**.** *A set $X\subseteq\mathbb{T}$ is $\mathcal{A}$-permitted if and only if for every $\mathbf{u}\in\mathcal{S}$ there exists $\mathbf{v}\in\mathcal{S}$ such that $X\cup t_{\mathbf{u}}(\mathbb{T})\subseteq t_{\mathbf{v}}(\mathbb{T})$.*

Going back to Eliaš solution to the problem of the inclusion $t_{\mathbf{u}}(\mathbb{T})\subseteq t_{\mathbf{v}}(\mathbb{T})$, for $\mathbf{u},\mathbf{v}\in\mathcal{S}$, we recall first of all that, by [47, Theorem 1.1], for a strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}_+$ and $m\in\mathbb{Z}$ there exists an eventually null sequence $\mathbf{r}$ in $\mathbb{Z}$ (called a *good expansion* of m by $\mathbf{u}$) such that

$$m=\sum_{n\in\mathbb{N}}r_nu_n,\ \text{ where }\left|\sum_{j\leq n}r_ju_j\right|\leq\frac{u_{n+1}}{2}\text{ for every }n\in\mathbb{N}.\tag{6.3}$$

So, if $\mathbf{v}$ is another strictly increasing sequence in $\mathbb{N}_+$, for every $i\in\mathbb{N}$ there exists a good expansion $\mathbf{r}_i$ of v_i by $\mathbf{u}$. Using the approach from [5], consider each $\mathbf{r}_i$ as the i-th row of a countable infinite matrix M; then

$$\mathbf{v}=M\mathbf{u}.$$

Note that M is row-finite, that is, each row $\mathbf{r}_i$ is eventually null. We state now Eliaš' result by using this notation.

Theorem 6.5 ([47, Theorem 1.2]). *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{S}$ *and let* $M = (a_{i,j})_{i,j\in\mathbb{N}}$ *be the row-finite integer matrix such that each row* $\mathbf{r}_i$ *of* A *is a good expansion of* v_i *by* $\mathbf{u}$ *(so* $\mathbf{v} = M\mathbf{u}$*). Then* $t_{\mathbf{u}}(\mathbb{T}) \subseteq t_{\mathbf{v}}(\mathbb{T})$ *if and only if:*

(C) $\mathbf{c}_j$ *is eventually null for each column* $\mathbf{c}_j$ *of* M*;*

(R) there exists $m \in \mathbb{N}$ *such that* $\sum_{j\in\mathbb{N}} |a_{i,j}| \leq m$ *for every* $i \in \mathbb{N}$ *(i.e.,* $\sup_{i\in\mathbb{N}} \|\mathbf{r}_i\|_1 < +\infty$*).*

Theorem 6.5 was strengthened in [5], where characterized subgroups of $\mathbb{R}$ are considered. For a sequence $\mathbf{u}$ in $\mathbb{R}$, let

$$\tau_{\mathbf{u}}(\mathbb{R}) := \{x \in \mathbb{R} : u_n x \to 0 \bmod \mathbb{Z}\}.$$

A subgroup H of $\mathbb{R}$ is *characterized* if $H = \tau_{\mathbf{u}}(\mathbb{R})$ for some sequence $\mathbf{u}$ in $\mathbb{R}$. Note that, if $\mathbf{u}$ is a sequence in $\mathbb{Z}$, then $\tau_{\mathbf{u}}(\mathbb{R}) = \varpi^{-1}(t_{\mathbf{u}}(\mathbb{T}))$.

Characterized subgroups of $\mathbb{R}$ were studied in relation to uniform distribution of sequences modulo $\mathbb{Z}$ by Kuipers and Niederreiter in the book [64], where it is proved in [64, Theorem 7.8] that $\tau_{\mathbf{u}}(\mathbb{R})$ has Lebesgue measure zero, if $\mathbf{u}$ is a sequence in $\mathbb{R}$ not converging to 0 in $\mathbb{R}$ (they give credit to Schoenberg for this result, see [78]). Moreover, Borel proved in [15, Proposition 2] that if $H = \tau_{\mathbf{v}}(\mathbb{R})$ is a non-trivial proper characterized subgroup of $\mathbb{R}$, then there exist $\gamma \in \mathbb{R}$ and a strictly increasing sequence $\mathbf{u}$ in $\mathbb{N}_+$ such that $\gamma H = \tau_{\mathbf{u}}(\mathbb{R})$. This underlines the strict relation between characterized subgroups of $\mathbb{R}$ and characterized subgroups of $\mathbb{T}$. Borel proved also that every countable subgroup of $\mathbb{R}$ is characterized and left open the general question of a full description of the characterized subgroups of $\mathbb{R}$.

In [5] the characterized subgroups $\tau_{\mathbf{u}}(\mathbb{R})$ of $\mathbb{R}$ are always considered under the assumption that $\mathbf{u}$ is in $\mathbb{R} \setminus \{0\}$ and $|b_n^{\mathbf{u}}| \to +\infty$, (where $b_n^{\mathbf{u}} = \frac{u_n}{u_{n-1}}$ for $n \in \mathbb{N}_+$ and $u_0 = 1$). Thus, the cardinality of $\tau_{\mathbf{u}}(\mathbb{R})$ is $\mathfrak{c}$. Moreover, we can always assume that such sequences are in $\mathbb{R}_+$, since for any sequence $\mathbf{w}$ in $\mathbb{R}$ we have $\tau_{\mathbf{w}}(\mathbb{R}) = \tau_{|\mathbf{w}|}(\mathbb{R})$ where $|\mathbf{w}| := (|w_n|)_{n\in\mathbb{N}}$.

Theorem 6.6 ([5, Theorem 1.1]). *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be sequences in* $\mathbb{R}_+$ *such that* $v_n = \alpha_n u_n$ *for every* $n \in \mathbb{N}$*. If* $\alpha_n \to +\infty$ *and* $\alpha_n \leq \kappa b_{n+1}^{\mathbf{u}}$ *eventually for some* $\kappa < 1$*, then* $\tau_{\mathbf{u}}(\mathbb{R}) \not\subseteq \tau_{\mathbf{v}}(\mathbb{R})$*.*

The relation $v_n = \alpha_n u_n$ for $n \in \mathbb{N}_+$ can be written as $\mathbf{v} = M\mathbf{u}$, where M is an infinite diagonal matrix with the values α_n on the diagonal. In [5], this suggested to consider the situation when $\mathbf{u}$ and $\mathbf{v}$ are sequences in $\mathbb{R}$ such that

$$\mathbf{v} = M\mathbf{u},$$

where $M = (a_{i,j})_{i,j\in\mathbb{N}}$ is a row-finite infinite real matrix.

The following result extends Theorem 6.5 to a more general setting. Indeed, under the assumptions of Theorem 6.5 the condition (6.4) is satisfied with $\kappa = \frac{1}{2}$, hence Theorem 6.5 follows directly from Theorem 6.7 and Equation (6.3).

Theorem 6.7 ([5, Corollary 1.3]). *Let* $\mathbf{u}$ *be a sequence in* $\mathbb{R} \setminus \{0\}$ *such that* $|b_n^{\mathbf{u}}| \to +\infty$*. Let* $M = (a_{i,j})_{i,j\in\mathbb{N}}$ *be a row-finite integer matrix such that there exists* $0 < \kappa < 1$ *with*

$$\left|\sum_{j\leq n} a_{i,j} u_j\right| \leq \kappa \cdot |u_{n+1}| \text{ for every } n, i \in \mathbb{N}. \tag{6.4}$$

If $\mathbf{v} = M\mathbf{u}$*, then* $\tau_{\mathbf{u}}(\mathbb{R}) \subseteq \tau_{\mathbf{v}}(\mathbb{R})$ *if and only if (C) and (R) hold.*

The following question was left open in [5]. For two subgroups H, K of an infinite group G say that H is *almost contained* in K if $[H : K \cap H]$ is finite. Similarly, say that H is *weakly contained* in K if $[H : K \cap H]$ is at most countable.

Question 6.8 ([5, Question 1.4]). *Do the characterizations for inclusion given in Theorem 6.6 and Theorem 6.7 remain true also for almost inclusion or for weak inclusion in the above sense?*

6.2 Dirichlet sets of $\mathbb{T}$

In 1981, Erdös, Kunen and Mauldin proved that if P is a non-empty perfect set of $\mathbb{R}$, then there exists a perfect set M of $\mathbb{R}$ of Lebesgue measure zero such that $P + M = \mathbb{R}$ (see [50]). Eliaš improved this result:

Theorem 6.9 ([48, Theorem 3.2]). *For every perfect subset P of $\mathbb{T}$, there exists a perfect D-set D of $\mathbb{T}$ such that $P + D = \mathbb{T}$.*

This answers in the negative the problem of the existence of a perfect $\mathcal{N}$-permitted set from [2].

For an arithmetic sequence $\mathbf{u}$ and for $L \subseteq \mathbb{N}$, using the representation given by Theorem 2.11, in [4] the following set was introduced

$$K_L^{\mathbf{u}} := \{x \in \mathbb{T} : \mathrm{supp}_{\mathbf{u}}(\varphi^{-1}(x)) \subseteq L\},$$

imitating the same set for the specific sequence $(2^n)_{n\in\mathbb{N}}$ defined in [67] (see also [19]). Clearly, $0 \in K_L^{\mathbf{u}}$ since $\mathrm{supp}_{\mathbf{u}}(0) = \emptyset$.

We assume the subset L of $\mathbb{N}$ to be infinite and non-cofinite, since we are interested in those sets $K_L^{\mathbf{u}}$ that are D-sets; indeed, finite $K_L^{\mathbf{u}}$ are always D-sets and $K_L^{\mathbf{u}}$ D-set implies L non-cofinite. Under these assumptions on L, the set $K_L^{\mathbf{u}}$ is closed and perfect.

In [4] a characterization is given of those $K_L^{\mathbf{u}}$ that are D-sets (see Theorem 6.10 below). To state it we need some technical details. Since L is an infinite non-cofinite subset of $\mathbb{N}$, one has a partition $L = \bigcup_{n\in\mathbb{N}_+} L_n$, where each $L_n = \{m_n^L, \dots, M_n^L\}$ is a finite set of consecutive naturals and the consecutive intervals L_n and L_{n+1} are not adjacent (i.e., $M_n^L < m_{n+1}^L - 1$). For every $n \in \mathbb{N}_+$, let $G_n = \{M_n^L + 1, \dots, m_{n+1}^L - 1\}$ be the non-empty set "between" L_n and L_{n+1}, and let

$$\widetilde{b}_n^L := \prod_{i\in G_n} b_i^{\mathbf{u}} = \frac{u_{m_{n+1}^L - 1}}{u_{M_n^L}}.$$

Theorem 6.10 ([4, Theorem 1.7]). *If $\mathbf{u}$ is an arithmetic sequence and L is an infinite non-cofinite subset of $\mathbb{N}$, then $K_L^{\mathbf{u}}$ is a D-set precisely when $\sup_{n\in\mathbb{N}} \widetilde{b}_n^L = +\infty$.*

To better understand this result, recall that an infinite non-cofinite subset L of $\mathbb{N}$ is large if and only if the sequence $(|G_n^L|)_{n\in\mathbb{N}}$ is bounded. First, if L is non-large, then $K_L^{\mathbf{u}}$ is a D-set. Moreover, a consequence of Theorem 6.10 is that, when $\{b_n^{\mathbf{u}} : n \in \mathbb{N} \setminus L\}$ is not bounded, $K_L^{\mathbf{u}}$ is always a D-set; on the other hand, when $\mathbf{u}$ is q-bounded, $K_L^{\mathbf{u}}$ is a D-set if and only if L is non-large.

So, taking an infinite non-cofinite subset L of $\mathbb{N}$ which is non-large and with $G := \mathbb{N} \setminus L$ non-large, we obtain that $K_L^{\mathbf{u}}$ and $K_G^{\mathbf{u}}$ are D-sets; for $\mathbf{u} = (2^n)_{n\in\mathbb{N}}$, this is due to Marcinkiewicz (see [67]). Since we have also that

$$\mathbb{T} = K_L^{\mathbf{u}} + K_G^{\mathbf{u}}, \tag{6.5}$$

we can state the following result, related to Erdös - Kunen - Mauldin Theorem (see Theorem 6.9 above).

Theorem 6.11. *The circle group $\mathbb{T}$ can be written as the sum of two closed perfect D-sets (which have necessarily Haar measure zero).*

Since in the above notations, $K_L^{\mathbf{u}}$ and $K_G^{\mathbf{u}}$ are D-sets, there exist $\mathbf{v}$, $\mathbf{w}$ subsequences of $\mathbf{u}$ such that $\mathbf{v}$, $\mathbf{w}$ "witness" respectively that $K_L^{\mathbf{u}}$, $K_G^{\mathbf{u}}$ are D-sets, so in particular $K_L^{\mathbf{u}} \subseteq t_{\mathbf{v}}(\mathbb{T})$ and $K_G^{\mathbf{u}} \subseteq t_{\mathbf{w}}(\mathbb{T})$. Then Equation 6.5 gives also that $\mathbb{T} = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$, and therefore we can state the following result; recall that a subgroup of $\mathbb{T}$ is *a-characterized* if it is characterized by an arithmetic sequence.

Corollary 6.12 ([4, Theorem 1.12]). *The circle group $\mathbb{T}$ can be written as the sum of two proper a-characterized subgroups.*

This answers a problem from [6], that now we discuss in detail. A subgroup H of $\mathbb{T}$ is *factorizable* (respectively, *a-factorizable*), if $H = t_{\mathbf{v}}(\mathbb{T}) + t_{\mathbf{w}}(\mathbb{T})$ with proper characterized (respectively, a-characterized) subgroups $t_{\mathbf{v}}(\mathbb{T})$ and $t_{\mathbf{w}}(\mathbb{T})$ (see [4]). Clearly, a-factorizable implies factorizable.

Problem 6.13 ([6, Question 5.1]). *For sequences $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{Z}$, describe $t_{\mathbf{u}}(\mathbb{T}) + t_{\mathbf{v}}(\mathbb{T})$ in terms of $\mathbf{u}$ and $\mathbf{v}$.*

(a) When is a given factorizable subgroup of $\mathbb{T}$ characterized?

(b) When is a given characterized subgroup of $\mathbb{T}$ factorizable? In particular, is $\mathbb{T}$ factorizable?

Consider the counterpart of these questions for a-characterized and a-factorizable subgroups of $\mathbb{T}$ as well.

First of all, Corollary 6.12 states that $\mathbb{T}$ is a-factorizable, answering the last part of Problem 6.13(b). Moreover we have the following description of all countable subgroups of $\mathbb{T}$ (that are all characterized as recalled above) that are factorizable and a-factorizable.

Theorem 6.14. *Let H be a countable subgroup of $\mathbb{T}$. Then:*

(a) H is factorizable if and only if H is non-cocyclic;

(b) H is a-factorizable if and only if H is a-characterized and non-cocyclic.

Since this result solves completely the countable case, we are left with the uncountable characterized subgroups of $\mathbb{T}$. In this case we find a family of uncountable a-characterized subgroups of $\mathbb{T}$ that are a-factorizable:

Theorem 6.15 ([4, Theorem 1.14]). *Let $\mathbf{u}$ be an arithmetic sequence in $\mathbb{N}$ such that $b_n^{\mathbf{u}} \to +\infty$. Then the a-characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

We conjecture that the condition $b_n^{\mathbf{u}} \to +\infty$ can be replaced by the weaker one $\sup_{n\in\mathbb{N}_+} b_n^{\mathbf{u}} = +\infty$:

Conjecture 6.16. *Let $\mathbf{u}$ be an arithmetic sequence in $\mathbb{N}$ that is not q-bounded. Then the a-characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ is a-factorizable.*

The following general problem remains open.

Problem 6.17. *Describe the uncountable (a-)factorizable (a-)characterized subgroups.*

The following question is left open as well.

Question 6.18. *Suppose that an (a-)characterized subgroup $t_{\mathbf{u}}(\mathbb{T})$ is factorizable. Is it true that it is also a-factorizable?*

As a consequence of the results above, a countable factorizable subgroup H of $\mathbb{T}$ is a-factorizable precisely when H is a-characterized.

7 Characterized subgroups of topological abelian groups

The following notion appeared first in [28] and later also in Dikranjan-Milan-Tonolo [40].

Definition 7.1. [[40]] For G a topological abelian group and $\mathbf{u}$ a sequence in $\widehat{G}$, let

$$s_{\mathbf{u}}(G) = \{x \in G : u_n(x) \to 0\}.$$

A subgroup H of G is said to be *characterized by a sequence* $\mathbf{u}$ *in* $\widehat{G}$ if $H = s_{\mathbf{u}}(G)$. When the sequence $\mathbf{u}$ is not relevant, we shortly say that H *is characterized.*

At the first stages the name *basic g-closed subgroups* was coined (e.g., in Dikranjan-Milan-Tonolo [40] and later also in [11]). Formally the name "characterized" started to appear only later on, to replace the terminology "has a characterizing sequence" (or set) used in [56, 57, 38].

If $G = \mathbb{T}$, then we can identify $\widehat{\mathbb{T}} = \mathbb{Z}$, so $t_{\mathbf{u}}(\mathbb{T}) = s_{\mathbf{u}}(\mathbb{T})$ for a sequence $\mathbf{u}$ in $\mathbb{Z}$.

Note that $s_{\mathbf{u}}(G) = \bigcap_{N\geq 2}\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}\left\{x \in G : \|u_n(x)\| \leq \frac{1}{N}\right\}$. Then, if K is a compact abelian group, then:

(a) $s_{\mathbf{u}}(K)$ is a Borel set, and so either $s_{\mathbf{u}}(K)$ is countable or $|s_{\mathbf{u}}(K)| = \mathfrak{c}$;

(b) $\mu(s_{\mathbf{u}}(K)) = 0$ if $\mathbf{u}$ is faithfully indexed (see [22, Lemma 3.10] and see [74] for locally compact abelian groups).

For G an abelian group and $H \leq G$, let

$$\mathfrak{g}_G(H) = \bigcap\{s_{\mathbf{u}}(G) : \mathbf{u} \in \widehat{G}^{\mathbb{N}},\ H \leq s_{\mathbf{u}}(G)\};$$

H is $\mathfrak{g}$-*closed* if $H = \mathfrak{g}_G(H)$.

It is proved in [40] that a topological abelian group G is maximally almost periodic if and only if every cyclic subgroup of G is $\mathfrak{g}$-closed.

Resolving [40, Problem 5.1], Lukács [66], Beiglböck-Steineder-Winkler [11, Theorem 3.6] and Dikranjan-Kunen [38, Corollary 1.9] independently and simultaneously proved the following

Theorem 7.2 ([66, 11, 38])**.** *All countable subgroups of a compact abelian group are* $\mathfrak{g}$*-closed.*

The question of whether countable subgroups of arbitrary compact metrizable abelian groups are characterized was raised in [14] (in a somewhat implicit form, as the notion of characterized subgroup was not introduced at that stage). Dikranjan-Kunen [38] and Beiglböck-Steineder-Winkler [11], independently and using different techniques, answered this question:

Theorem 7.3 ([38, 11])**.** *If K is a compact metrizable abelian group, then every countable subgroup of K is characterized.*

Every characterized subgroup of K is $F_{\sigma\delta}$. Inspired by Biró's Theorem 3.3, Gabriyelyan [53, Theorem 2] proved that if K is an uncountable Kronecker set of an infinite compact metrizable abelian group X, then $\langle K\rangle$ is not Polishable. In particular, $\langle K\rangle$ cannot be characterized.

Theorem 7.4 ([31])**.** *Every G_δ-subgroup of a compact abelian group K is characterized.*

Gabriyelyan [53, Corollary 1] proved that every characterized subgroup H of a compact metrizable abelian group K admits a finer Polish (and locally quasi-convex) group topology.

It turned out that this necessary condition can be reinforced so that it becomes sufficient (although not always necessary) as follows. According to Gabriyelyan [54], a group G is *characterizable* if there is a compact metrizable abelian group X and a continuous monomorphism $p : G \to X$ with dense image such that $p(G)$ is a characterized subgroup of X; G is called *strongly characterizable* if for every compact metrizable abelian group X and every continuous monomorphism $p : G \to X$ with dense image the subgroup $p(G)$ is a characterized subgroup of X. It was shown by Gabriyelyan [54] that every second countable locally compact abelian group is characterizable, he asked if every second countable locally compact abelian group is strongly characterizable. Negro [70] positively answered Gabriyelyan's problem. In particular, this shows that a dense subgroup of a compact metrizable abelian group admitting a finer locally compact group topology (i.e., having locally compact associated Polish topology) is necessarily characterized. In [37], the second named author and Impieri proved that when $X = \mathbb{T}$, the proper characterized subgroups with locally compact associated Polish topology are exactly the countable subgroups, and $\mathbb{T}$ is the only compact abelian group with that property.

Protasov-Zelenyuk [72] introduced the concept of a *T-sequence* in an abelian group G – this is a non-trivial sequence $\mathbf{u}$ in G that converges to 0 in some Hausdorff group topology on G.

Definition 7.5. [[6, 40]] A non-trivial sequence $\mathbf{u}$ in an abelian group G is a *TB-sequence* if there exists a precompact topology τ on G such that $u_n \xrightarrow{\tau} 0$.

Protasov-Zelenyuk [72] gave a complete characterization of T-sequences. A counterpart of this for TB-sequences was obtained by Dikranjan-Milan-Tonolo [40]:

Theorem 7.6 ([40]). *Let $\mathbf{u}$ be a non-trivial sequence in an abelian group G.*

(a) If τ is a totally bounded group topology on G, then $\tau = T_H$ for some $H \leq \widehat{G}$ and $u_n \xrightarrow{\tau} 0$ if and only if $H \leq s_{\mathbf{u}}(\widehat{G})$.

(b) $T_{s_{\mathbf{u}}(\widehat{G})}$ is the finest totally bounded group topology τ on G with $u_n \xrightarrow{\tau} 0$.

(c) $\mathbf{u}$ is a TB-sequence if and only if $s_{\mathbf{u}}(\widehat{G})$ is dense in $\widehat{G}$.

In [40, Theorem 3.3] one can also find a description of the subsequences of the sequence $1/p^n$ in $\mathbb{Z}(p^\infty)$ that are TB-sequences.

For other results on characterized subgroups and related topics see the recent papers [38, 31, 32, 33, 35, 36, 37, 52, 53, 54, 55, 70].

8 Topologically u-torsion subgroup vs characterized subgroups

8.1 Separation axioms via sequential limit laws

Here we give an alternative approach to topologically torsion elements and subgroups (given so far in Definition 2.7), better adapted for not necessarily abelian groups. This is why, we use only in this section multiplicative notation for groups. In particular, we denote by e_G the neutral element of a group G.

Taylor [77] introduced appropriately *limit laws* in order to describe varieties of topological algebras in parallel with Birkhoff's theorem describing varieties of universal algebras by means of identities. Analogously, for every sequence $\mathbf{u} = (u_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$, a *sequential limit law* depending on $\mathbf{u}$ was defined by Kopperman-Mislove-Morris-Nickolas-Pestov-Svetlichny [62, 69]: a topological group G is said to *satisfy the sequential limit law* $\mathbf{u}$ if

$$x^{u_n} \to e_G \tag{8.1}$$

for all elements x of G. More generally, when G does not satisfy the limit law $\mathbf{u}$ one can still consider the *subset* $t_{\mathbf{u}}(G)$ of all elements $x \in G$ that satisfy (8.1). When G is abelian, then $t_{\mathbf{u}}(G)$ is a subgroup of G, coinciding with the subgroup already defined in Definition 2.7 (this is why we use the same notation).

Following [24, 26, 28], for every element x of a topological group G one can consider the set $S_G(x)$ of all sequential limit laws satisfied by x, i.e.,

$$S_G(x) = \left\{ \mathbf{u} \in \mathbb{Z}^{\mathbb{N}} : \lim_{n\to\infty} x^{u_n} = e_G \right\}.$$

It is easy to see that $S_G(x)$ is a subgroup of $\mathbb{Z}^{\mathbb{N}}$. When no confusion is possible we shall omit the subscript $_G$ in $S_G(x)$. Let

$$\mathfrak{t}(x) = \{y \in G : S(x) \subseteq S(y)\} = \bigcap \{t_{\mathbf{u}}(G) : \mathbf{u} \in S(x)\}.$$

i.e., this is the set of all elements y of G that satisfy all sequential limit laws valid for the element x.

Similarly, for a subgroup (or just a subset) H of G we set

$$S_G(H) = \{\mathbf{u} \in \mathbb{Z}^{\mathbb{N}} : H = t_{\mathbf{u}}(\mathbb{T})\} \leq \mathbb{Z}^{\mathbb{N}} \quad \text{and} \quad \mathfrak{t}(H) = \bigcap \{t_{\mathbf{u}}(G) : \mathbf{u} \in S_G(H)\}.$$

We say that H is $\mathfrak{t}$-*closed* (respectively, $\mathfrak{t}$-*dense*), if $\mathfrak{t}(H) = H$ (respectively, $\mathfrak{t}(H) = G$). In case $H = t_{\mathbf{u}}(G)$ for some $\mathbf{u}$ (necessarily belonging to $S_G(H)$), we say that H is *basic* $\mathfrak{t}$-*closed*.

If $\langle x \rangle = \langle y \rangle$, then obviously $S(x) = S(y)$ (more precisely, $\langle y \rangle \leq \langle x \rangle$ implies $S(x) \subseteq S(y)$). The question of whether the converse is also true leads one to introduce the following classes of topological groups (see [30]):

(a) $\mathfrak{G}_1$ – all topological groups G such that $S(x) = S(y)$ implies $\langle x \rangle = \langle y \rangle$, for every pair of elements $x, y \in G$.

(b) $\mathfrak{G}_2$ – all topological groups G such that $S(x) \subseteq S(y)$ implies $\langle y \rangle \leq \langle x \rangle$ (i.e., $y \in \mathfrak{t}(x)$ implies $y \in \langle x \rangle$ for every pair of elements $x, y \in G$);

(c) $\mathfrak{G}_3$ – all topological groups G such that for every cyclic subgroup $\langle x \rangle$ of G there exists $\mathbf{u} \in S(x)$ such that $\langle x \rangle = t_{\mathbf{u}}(G)$.

It is easy to see that $\mathfrak{G}_1 \supseteq \mathfrak{G}_2 \supseteq \mathfrak{G}_3$ and that $G \in \mathfrak{G}_2$ (respectively, $G \in \mathfrak{G}_3$) if and only if every cyclic subgroup of G is $\mathfrak{t}$-closed (respectively, characterized).

As pointed out in [30], one can consider the above three classes as an appropriate form of "separation axioms" for topological groups.

Let us specify these notions in the case $G = \mathbb{T}$. The fact that all countable subgroups of $\mathbb{T}$ are $\mathfrak{t}$-closed (in particular, $\mathbb{T} \in \mathfrak{G}_2$) was exposed by the second named author at the Summer Conference on General Topology and Its Applications in New York in 2001 (see [28]). The stronger property $\mathbb{T} \in \mathfrak{G}_3$ is nothing else but Theorem 3.5. So, following also the terminology of §3, the equality $\langle x \rangle = t_{\mathbf{u}}(G)$ for a sequence (sequential limit law) $\mathbf{u}$ in (c) means that $\mathbf{u}$ *characterizes* the cyclic subgroup $\langle x \rangle$. Furthermore, $\mathbf{u}$ is a *characterizing sequence* of a subgroup H of $\mathbb{T}$ if and only if H is a basic $\mathfrak{t}$-closed subgroup of $\mathbb{T}$. The correspondence $H \mapsto S_G(H)$ defines a Galois correspondence between subgroups of $\mathbb{T}$ and subgroups of $\mathbb{Z}^{\mathbb{N}}$ (see [24, 28, 40]). Let us reformulate Problem 4.7, faced also by Di Santo [24], in the new terms:

Problem 8.1 ([24], Maharam-Stone 2001)**.** *Given a characterized subgroup H of $\mathbb{T}$, describe the subgroup $S_G(H)$ of $\mathbb{Z}^{\mathbb{N}}$.*

Towards a (partial) solution of this problem we give below a number of properties of these subgroups $S_G(H)$ in the general case.

First of all, one can easily reduce the study of $S_G(H)$ to that of $S_G(x)$ for single elements $x \in G$ by noticing that $S_G(H) = \bigcap_{x\in H} S_G(x)$. Actually a much "smaller" intersection will do; in fact, if $H = \langle x_i : i \in I\rangle$, then $S_G(H) = \bigcap_{i\in I} S_G(x_i)$.

Lemma 8.2 ([26]). *Let G, H be topological groups and $f : G \to H$ a continuous homomorphism. Then for $x \in G$ and $\mathbf{u} \in \mathbb{Z}^{\mathbb{N}}$,*

$$S(x) \subseteq S(f(x)) \quad \textit{and} \quad f(t_{\mathbf{u}}(G)) \subseteq t_{\mathbf{u}}(H).$$

The proof of the above lemma uses only the continuity of the restriction $f\restriction_{\langle x\rangle}$. Actually, one can prove that if the groups G and H are metrizable, then the inclusion $S(x) \subseteq S(f(x))$ *is equivalent* to the continuity of the restriction $f\restriction_{\langle x\rangle}$. Indeed, $S(x) \leqslant S(y)$ precisely when f sends convergent to 0 sequences in $\langle x\rangle$ to convergent to 0 sequences in $\langle y\rangle$, i.e., $f\restriction_{\langle x\rangle}$ is continuous. This leads to:

Lemma 8.3 ([26]). *Let G be an abelian metrizable group and $x, y \in G$. Then the following conditions are equivalent:*

(a) $S(x) \leqslant S(y)$;

(b) $y \in \mathfrak{t}(x)$ (in other words, $y \in \bigcap\{t_{\mathbf{u}}(G) : \mathbf{u} \in S(x)\}$);

(c) there exists a continuous homomorphism $f : \langle x\rangle \to \langle y\rangle$ with $f(x) = y$.

This provides a simple description of the $\mathfrak{t}$-closure of a cyclic subgroup in the metric case:

Corollary 8.4. *Let G be an abelian metrizable group and let $x \in G$. Then*

$$\mathfrak{t}(x) = \{y \in G : \exists\ f : \langle x\rangle \to G \textit{ continuous homomorphism},\ f(x) = y\}.$$

In the sequel we provide two series of examples from [26] clarifying the properties of the class $\mathfrak{G}_1$.

Example 8.5. [26]

(a) Let $B_p = \langle x_p\rangle$ be a cyclic p-group for every p prime and let $B_P = \prod_{p\in P} B_p$, where P is a non-empty set of prime numbers. Then, for every $a = (a_p)_{p\in P} \in B_P$, the following conditions are equivalent:

   (1) $ord(a_p) = ord(x_p)$ for every $p \in P$;

   (2) $a \in B_P$ is a topological generator of B_P;

   (3) $\mathfrak{t}(a) = B_P$.

   If some of these equivalent conditions holds, then $B_P \notin \mathfrak{G}_1$ whenever P is infinite (as $S(a) = S(b)$ for any pair a, b of topological generators of B_P).

(b) If $G \in \mathfrak{G}_1$, then G does not contain copies of $\mathbb{Z}(p)^2$ for every prime p. Indeed, it suffices to see that the group $\mathbb{Z}(p)^2 \notin \mathfrak{G}_1$ for every prime p. In fact, if a is a non-zero element of $\mathbb{Z}(p)$, then the elements $x = (a, 0)$ and $y = (0, a)$ of $\mathbb{Z}(p)^2$ have $S(x) = S(y)$, but $\langle x\rangle \neq \langle y\rangle$. This proves that every finite abelian subgroup of any group $G \in \mathfrak{G}_1$ must be cyclic.

The next example introduces a new class that is "transversal" to the classes $\mathfrak{G}_i$:

Example 8.6. [26] In every topological abelian group G the subgroups $\{0\}$ and G are always $\mathfrak{t}$-closed. Denote by $\mathfrak{D}$ the class of all topological abelian groups having no proper $\mathfrak{g}$-closed subgroups. Clearly, $G \in \mathfrak{D} \cap \mathfrak{G}_1$ if and only if $G \cong \mathbb{Z}(p)$ for some $p \in \mathbb{P}$. In (a)–(d) we provide a huge series of groups in $\mathfrak{D}$. This creates a lot of further examples of groups that do not belong to $\mathfrak{G}_1$.

(a) For every prime p, the compact group $\mathbb{Z}_p$ of p-adic integers belongs to $\mathfrak{D}$ (hence $\mathbb{Z}_p$ does not belong to $\mathfrak{G}_1$). To see that also the locally compact group $\mathbb{Q}_p$ of p-adic numbers belongs to $\mathfrak{D}$ for every prime p note that if $Aut(G)$ acts transitively in $G \setminus \{0\}$, then $G \in \mathfrak{D}$ by Lemma 8.4.

(b) Let G be a compact abelian group. Then $G \in \mathfrak{D}$ if and only if there exist $p \in \mathbb{P}$ and a cardinal α such that either $G = \mathbb{Z}_p^\alpha$ (if G torsion-free) or $G = \mathbb{Z}(p)^\alpha$. This characterization can be extended in an appropriate way to the general case of locally compact abelian groups (see [28]).

(c) If G is a topological group without non-trivial convergent sequences then $S(x) = \mathcal{Z}_0$ (where $\mathcal{Z}_0$ is the subgroup of $\mathbb{Z}^\mathbb{N}$ of sequences $\mathbf{u}$ with $u_n = 0$ for almost all $n \in \mathbb{N}$) and $\mathfrak{t}(x) = G$ for every non-torsion $x \in G$. In particular, every torsion-free abelian group belongs to $\mathfrak{D}$ when equipped with its maximal precompact topology.

(d) For every non-zero $r \in \mathbb{R}$, $S(r) = \mathcal{Z}_0$. Consequently, $S(H) = \mathcal{Z}_0$ and $\mathfrak{t}(H) = \mathbb{R}$ for every subgroup $H \neq 0$ of $\mathbb{R}$. This means that $\mathbb{R} \in \mathfrak{D}$. In particular, $\mathbb{R} \notin \mathfrak{G}_1$.

8.2 A characterization of $\mathbb{T}$ via the Galois correspondence generated by topological torsion

8.2.1 The non-discrete case

The next theorem from [26] gives a characterization of the non-discrete locally compact groups in which all cyclic (countable) subgroups are $\mathfrak{t}$-closed:

Theorem 8.7 ([26])**.** *Let G be a non-discrete locally compact group. Then the following conditions are equivalent:*

(a) every countable subgroup of G is $\mathfrak{t}$-closed;

(b) every cyclic subgroup of G is $\mathfrak{t}$-closed (i.e., $G \in \mathfrak{G}_2$);

(c) $G \in \mathfrak{G}_1$;

(d) $G \cong \mathbb{T}$.

This theorem, along with Theorem 3.5, shows that for a non-discrete locally compact group G all three properties – $G \in \mathfrak{G}_1$, $G \in \mathfrak{G}_2$ and $G \in \mathfrak{G}_3$ – are equivalent and determine G up to topological isomorphism, namely $G \cong \mathbb{T}$. The proof uses the various cases of Example 8.5 and 8.6 in order to rule out, one-by-one, the possibilities excluded by the theorem. It should be emphasized that the group G is not supposed to be abelian in this theorem.

An analogous characterization can be given for the Cartesian powers $\mathbb{Z}_p^\alpha$ of the p-adic integers (as the torsion-free compact abelian groups having no proper $\mathfrak{t}$-closed subgroups – see Example 8.6(b)).

8.2.2 The discrete case

In the sequel we make use of the obvious fact that if G is an infinite discrete cyclic group, then G does not belong to $\mathfrak{G}_1$.

Lemma 8.8 ([26])**.** *Let G be a group.*

(a) If G is torsion, then for $i = 1, 2, 3$ the group G equipped with some Hausdorff topology belongs to $\mathfrak{G}_i$ if and only if $G \in \mathfrak{G}_i$ as a discrete group.

(b) If $G \in \mathfrak{G}_2$ is discrete, then G is torsion.

In order to characterize the discrete groups in the class $\mathfrak{G}_1$ and $G \in \mathfrak{G}_2$ we need the following:

Lemma 8.9 ([26])**.** *Let G be an abelian torsion group. Then the following conditions are equivalent:*

(a) G belongs to $\mathfrak{G}_1$;

(b) $G \in \mathfrak{G}_2$;

(c) $|C_1| = |C_2|$ implies that $C_1 = C_2$ for every pair of cyclic subgroups C_1, C_2 of G.

The following theorem from [26] provides the missing counterpart of Theorem 8.7 in the discrete case.

Theorem 8.10 ([26])**.** *Let G be a group. Then the following conditions are equivalent:*

(a) every subgroup of the discrete group G is $\mathfrak{t}$-closed;

(b) G is torsion and $(G, \tau) \in \mathfrak{G}_2$, for some Hausdorff group topology τ;

(c) G is torsion and $(G, \tau) \in \mathfrak{G}_1$, for some Hausdorff group topology τ;

(d) G is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$;

(e) $G \in \mathfrak{G}_3$ when equipped with the discrete topology.

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