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Research Article

Complex Dynamics in One-Dimensional Nonlinear Schrödinger Equations with Stepwise Potential

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We prove the existence and multiplicity of periodic solutions as well as solutions presenting a complex behavior for the one-dimensional nonlinear Schrödinger equation $-\varepsilon^2 u'' + V(x)u = f(u)$, where the potential $V(x)$ approximates a two-step function. The term $f(u)$ generalizes the typical p-power nonlinearity considered by several authors in this context. Our approach is based on some recent developments of the theory of topological horseshoes, in connection with a linked twist maps geometry, which are applied to the discrete dynamics of the Poincaré map. We discuss the periodic and the Neumann boundary conditions. The value of the term $\varepsilon > 0$, although small, can be explicitly estimated.

1. Introduction

In a recent paper [1] we have proved the existence of chaotic dynamics associated with a class of second order nonlinear ODEs of Schrödinger type of the form

$$-u'' - \mu u + g(x) u^3 = 0, \quad (\mu > 0) \tag{1}$$

for $g(x)$ a positive periodic coefficient. The study of such equation was motivated by previous works on some models of Bose-Einstein condensates considered in [2–4].

A more classical form of Nonlinear Schrödinger Equation (NLSE from now on) which has been studied by many authors is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \nu(x)\psi - |\psi|^{p-1}\psi, \quad x \in \mathbb{R}^N, \tag{2}$$

where $\hbar$ denotes the Planck constant, i is the imaginary unit, m is a positive constant, $\nu(x)$ is the potential, and $p > 1$. The search of *stationary waves*, namely, solutions of the form $\psi(t,x) = \exp(i\lambda\hbar^{-1}t)u(x)$, where $\lambda \in \mathbb{R}$ and $u(x)$ is a real valued function, leads to the study of

$$-\frac{\hbar^2}{2m}\Delta u + (\nu(x) + \lambda) u = |u|^{p-1} u. \tag{3}$$

This latter equation, which is usually written as

$$-\varepsilon^2 \Delta u + \lambda u + \nu(x) u = |u|^{p-1} u, \quad \varepsilon > 0 \tag{4}$$

or equivalently (by a standard rescaling) as

$$-\Delta u + \lambda u + \nu(\varepsilon x) u = |u|^{p-1} u, \quad \varepsilon > 0, \tag{5}$$

has motivated a great deal of research from different points of view (see, for instance, [5–11] just to quote a few classical contributions among a very large and constantly increasing literature on the subject). The case of a periodic potential has been considered as well (see [12, 13]).

In various articles, the hypothesis $\lambda > -\inf \nu$ has been assumed. Setting $V(\xi) := \lambda + \nu(\xi)$, this is equivalent to consider the equation

$$\Delta u - V(\varepsilon x) u + |u|^{p-1} u = 0, \quad \varepsilon > 0, \tag{6}$$

with $V(\cdot)$ a positive weight function.

Looking at (6) in one-dimension, we can interpret it as a slowly varying perturbation of the planar Hamiltonian system

$$\begin{aligned} u' &= y, \\ y' &= Vu - |u|^{p-1} u \end{aligned} \tag{7}$$

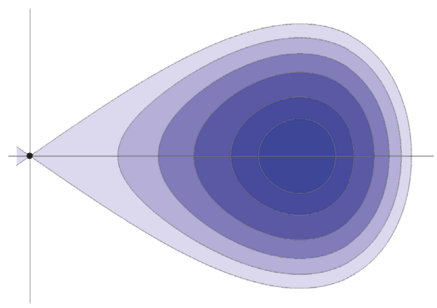

Figure 1: Energy level lines of system (7) in the phase-plane (u, u'), $u \geq 0$, for $V = 1$, and $p = 3$. A darker color represents a lower value of the energy of the orbits.

which presents a hyperbolic equilibrium point (the origin which is a saddle point) with a homoclinic orbit enclosing a region that contains another equilibrium point which is a (local) center (see Figure 1). Similar phase-portraits are common in many different situations and it is known that their perturbations can produce chaotic-like dynamics (see, for instance, [14–18]).

Analogous equations appear in some mathematical models of nonlinear optics derived from Maxwell's equations [19]. For instance, in [20] the study of the propagation of electromagnetic waves in layered media leads to the scalar equation

$$u'' - \lambda u + \epsilon\left(x, \frac{1}{2}u^2\right)u = 0, \quad \lambda > 0, \tag{8}$$

where the dielectric function $\epsilon(x, s)$ takes into account the presence of layers with different refractive indexes. A possible choice of $\epsilon(x, s)$ for three layers (one "internal" and two "external") is given by

$$\epsilon(x, s) = \begin{cases} \epsilon_1(x, s), & |x| \leq d \\ \epsilon_2(s), & |x| > d. \end{cases} \tag{9}$$

For some other typical forms of $\epsilon(x, s)$, see [20] and the references therein. Also this class of equations has been widely investigated in the last decades [20–23].

Finally, we briefly mention another area of research where similar equations arise, that is in the context of wave propagation or stationary solutions for bistable reaction diffusion equations in excitable media (see, for instance, [24–27] and the references therein). In the above quoted papers, a typical one-dimensional model equation takes the form

$$u_t = u_{xx} + g(u, x), \tag{10}$$

where the function $g(u, x)$ is defined piecewise as follows:

$$g(u, x) = \begin{cases} f(u), & -\infty < x \leq x_1, x_2 < x \leq x_3, \dots, x_{2n} < x < +\infty \\ g_0(u, x), & \text{otherwise} \end{cases} \tag{11}$$

(see [26, 27] for different choices of $g_0(s, x)$). The kinetic term is usually taken to be a Nagumo type cubic function $f(s) = s(1 - s)(s - a)$ (with $0 < a < 1/2$) or a McKean's piecewise linear reaction term [28]. Other possible variants concern the equation

$$u_t = D(x)u_{xx} + g(u, x), \tag{12}$$

with $D(x)$ a stepwise function [24]. The case of periodicity in the x-variable has been also considered. With this respect we mention a paper of Keener [29], dealing with the equation

$$u_t = Du_{xx} + \left(1 + g'\left(\frac{x}{L}\right)\right)kf(u) - kau, \tag{13}$$

where $g(x)$ is the so-called 1-periodic "sawtooth function". Analogous investigations have been addressed also in [30, 31].

In the present paper we restrict ourselves to the one-dimensional case and we study a second order nonlinear equation which is related to the models considered above. More in detail, we deal with a class of equations of the form

$$-u'' + V(t)u = q(t)f(u), \tag{14}$$

where, for notational convenience, we consider the independent variable (which usually refers in the above quoted models as a space variable) as a time variable. Such a convention is also motivated by the dynamical systems approach which is followed in the present paper. The choice of introducing two weight functions (that is, $V(t)$ for the linear part of the equation and $q(t)$ for the nonlinear part) is useful in view of dealing with the more general Schrödinger equation

$$-\varepsilon^2\Delta u + \lambda u + \nu(x)u = K(x)|u|^{p-1}u \tag{15}$$

(previously considered in [32]).

The nonlinear term $f(s)$ in (14) is assumed to be a *continuously differentiable function* and is chosen in order to include, as a particular case, the polynomial nonlinearities which usually appear in the context of the NLSEs (see Section 2 for the precise assumptions on $f(s)$). The main hypothesis on the coefficients $V(t)$ and $q(t)$, which are supposed to be nonnegative, is that they are close in the L^1-norm to stepwise functions. Such a particular choice for the shape of the coefficients is mainly motivated by mathematical convenience, as it permits to develop the proofs in a simpler and more transparent way and thus to avoid more complicated technicalities. However, it is interesting to observe that second order equations or, more generally, first order planar systems with piecewise constant coefficients naturally appear in several applications, such as the theory of switching control [33], electric or mechanical systems [34, 35], periodically forced Nagumo equations [36], and biological models subject to seasonal dynamics [37–39], as well as in the context of NLSEs arising in nonlinear optics [23] and in mathematical modelling of structures like crystals or switches in optical fibers [40]. In this connection and as already observed at the beginning of the Introduction, stepwise periodic coefficients have been recently considered also in some (Gross-Pitaevskii) equations describing the phenomenon of Bose-Einstein condensation where the existence of complex dynamics for $u'' + \mu u - g(t)u^3 = 0$ and

$u'' + (\omega - U(t))u - u^3 = 0$ was proved in [1] and [41], respectively. Periodically forced second order nonlinear equations with stepwise coefficients are widely analyzed also in connection with Littlewood's example of unbounded solutions to Duffing equations and its generalizations [42–45]. Finally, we observe that variants of these equations with a stepwise weight function have been considered with respect to the search of multiple "large" solutions, namely, solutions presenting a blow-up phenomenon at the boundary of a given interval (see [46, 47]).

The main part of the paper is devoted to the study of the periodic boundary value problem associated with (14). In doing so, we prove the presence of infinitely many subharmonic solutions and also the existence of solutions with certain oscillatory properties which can reproduce any prescribed sequence of coin-tossing type [48] (see Definition 1 in Section 2). Such chaotic-like solutions are obtained by an application of the theory of *topological horseshoes* [49, 50], in a variant developed in [51, 52]. We will also discuss how the arguments of the proofs can be modified in order to deal with the Neumann and the Dirichlet boundary value problems. In this latter context, the recent years have witnessed a growing interest toward the search of existence and multiplicity results of solutions of

$$-\varepsilon^2 \Delta u + u = u^p, \quad u > 0 \tag{16}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$ with interior and/or boundary peaks [53–56]. With this respect, we stress the fact that our multiplicity results appear to be of completely different nature; they are typically one-dimensional, even if they could be applied to PDEs on thin annular domains of $\mathbb{R}^N$.

For simplicity in the exposition, we will focus our presentation mainly to the study of *positive solutions*. We shall explain how to obtain sign changing solutions with prescribed nodal properties with some illustrative remarks at the end of the article.

2. Setting of the Problem and Main Results

We consider the second order nonlinear equation (of Schrödinger type)

$$-u'' + V(t)u = f(u) \tag{17}$$

where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable function of the form

$$f(s) = sh(s) \tag{18}$$

with $h : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\begin{gathered} h(0) = 0 \\ \textit{and } h'(s) > 0 \quad \textit{for } s > 0, \\ \textit{with } h(+\infty) = +\infty. \end{gathered} \tag{$*$}$$

As a consequence of $(*)$ it follows that $f(0) = 0$, $f(s) > 0$ for $s > 0$ and

$$\begin{gathered} \lim_{s \to 0^+} \frac{f(s)}{s} = 0, \\ \lim_{s \to +\infty} \frac{f(s)}{s} = +\infty. \end{gathered} \tag{$**$}$$

Since we are looking for *positive solutions*, the actual behavior of $f(s)$ for $s < 0$ will not affect our result. For simplicity, we suppose that f is *odd* (that is, h is *even*). We assume such a symmetry condition also in order to cover the classical example $h(s) = |s|^{p-1}$ with $p > 1$. All the results of the present paper could be proved for a locally Lipschitz continuous function f satisfying $(**)$ and with $h(s) = f(s)/s$ strictly increasing on $]0, +\infty)$ (and strictly decreasing on $(-\infty, 0[$). We prefer to consider the smooth case for simplicity in the presentation.

For the potential $V(t)$ we suppose that $V : \mathbb{R} \longrightarrow \mathbb{R}_0^+ :=]0, +\infty)$ is a T-periodic stepwise function (for some $T > 0$) of the form

$$V(t) := \begin{cases} V_1 & \text{for } t \in [0, T_1[\\ V_2 & \text{for } t \in [T_1, T[\end{cases} \tag{19}$$

with $V_1 \neq V_2$. Writing (17) as the equivalent first order system

$$\begin{aligned} x' &= y \\ y' &= V(t)x - f(x) \end{aligned} \tag{S}$$

in the phase-plane, we can describe the presence of a piecewise constant T-periodic coefficient as follows: the trajectories are governed by the autonomous system

$$\begin{aligned} x' &= y \\ y' &= V_1 x - f(x) \end{aligned} \tag{S_1}$$

in the time interval $[0, T_1[$. At the time $t = T_1$ we have a switching to system

$$\begin{aligned} x' &= y \\ y' &= V_2 x - f(x) \end{aligned} \tag{S_2}$$

which, in turns, rules the motions for a time interval of length

$$T_2 := T - T_1. \tag{20}$$

All this switching behavior is then repeated in a T-periodic fashion.

Recall that, given a first order differential system $z' = Z(t, z)$, its Poincaré map, on a time interval $[t_0, t_1]$, is the function which maps any initial point w_0 to $\zeta(t_1; t_0, w_0)$, where $\zeta(t) := \zeta(t; t_0, w_0)$, is the solution of the differential system satisfying the initial condition $z(t_0) = w_0$. In our setting, it is straightforward to check that the Poincaré map on $[0, T]$ for system (S), that we denote by Φ, can be decomposed as

$$\Phi = \Phi_2 \circ \Phi_1, \tag{21}$$

where Φ_i is the Poincaré map associated with the autonomous system (S_i) along the time interval $[0, T_i]$. Notice that, due to the autonomous nature of the subsystem, the map Φ_2 coincides with the Poincaré map of (S_2) on $[T_1, T]$. The assumptions on $f(s)$ guarantee the global existence of the solutions for all the Cauchy problems and therefore Φ is a global homeomorphism of the plane.

Our goal is to prove the existence of periodic solutions (harmonic and subharmonic) for (17). Following a classical procedure [57], this will be achieved by looking for the fixed points of Φ and its iterates. In our approach we apply some recent results on planar maps which provide not only the existence of fixed points and periodic points, but also the fact that the associated discrete dynamical system is “chaotic”. In the literature one can find several different methods which guarantee the presence of chaos for planar maps or, more generally, for homeomorphisms (or diffeomorphisms) in finite dimensional spaces. Moreover, different definitions of *chaotic dynamics* have been proposed by various authors. For the reader convenience, we recall now the concept of *chaos* that we are going to consider. Although the main definitions and the abstract setting can be presented in the framework of metric spaces, we confine ourselves to the case of homeomorphisms of the plane, which is the situation encountered by dealing with the Poincaré map associated with a planar system.

Definition 1. Let $\Phi : D_\Phi(\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2$ be a homeomorphism and let $\mathscr{D} \subseteq D_\Phi$ be a nonempty set. Assume also that $m \geq 2$ is an integer. We say that Φ *induces chaotic dynamics on m symbols in the set* $\mathscr{D}$ if there exist m nonempty pairwise disjoint compact sets

$$\mathscr{K}_0, \mathscr{K}_1, \ldots, \mathscr{K}_{m-1} \subseteq \mathscr{D} \tag{22}$$

such that, for each two-sided sequence of m symbols

$$(s_i)_{i\in\mathbb{Z}} \in \Sigma_m := \{0, \ldots, m-1\}^{\mathbb{Z}}, \tag{23}$$

there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ with

$$w_i \in \mathscr{K}_{s_i} \text{ and } w_{i+1} = \Phi(w_i), \quad \forall i \in \mathbb{Z} \tag{24}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k–periodic sequence (that is, $s_{i+k} = s_i$, $\forall i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k–periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ satisfying (24).

Note that, as a particular consequence of this definition, we have that for each $i \in \{0, \ldots, m-1\}$ there is at least one fixed point of Φ in $\mathscr{K}_i$. Since Φ is a homeomorphism from

$$\mathscr{K} := \bigcup_{i=0}^{m-1} \mathscr{K}_i \subseteq \mathscr{D} \tag{25}$$

onto its image, it follows also that there exists a nonempty compact set $\Lambda \subseteq \mathscr{K}$ which is invariant for Φ (i.e., $\Phi(\Lambda) = \Lambda$) and such that $\Phi|_\Lambda$ is semiconjugate to the two-sided Bernoulli shift σ on m symbols

$$\begin{gathered} \sigma : \Sigma_m \longrightarrow \Sigma_m, \\ \sigma\left((s_i)_{i\in\mathbb{Z}}\right) = (s_{i+1})_{i\in\mathbb{Z}}, \end{gathered} \tag{26}$$

according to the commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\ \Phi\ } & \Lambda \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma_m & \xrightarrow[\ \sigma\ ]{} & \Sigma_m \end{array} \tag{27}$$

where g is a continuous and surjective function. Moreover, as a consequence of Definition 1 we can take Λ such that it contains as a dense subset the periodic points of Φ and such that the counterimage (by the semiconjugacy g) of any periodic sequence in Σ_m contains a periodic point of Φ (see [58] for the details). As usual, in Σ_m, the set of two-sided sequence of m symbols, we take its standard metric [18] for which Σ_m turns out to be a compact set with the product topology.

We observe that Definition 1 is related to the concept of *chaos in the sense of coin-tossing* [48] and it also implies the presence of chaotic dynamics according to Block and Coppel [59, 60], as well as a positive topological entropy for the map $\Phi|_\Lambda$. Similar examples of complex dynamics for the Poincaré map associated with differential systems have been discussed, e.g., in [61–65], using different methods. See also [1, 31, 66] for recent contributions in this direction.

Now we are in position to state our main result for (17).

Theorem 2. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function of the form (18), with h satisfying $(*)$. Let $V : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ be a T-periodic stepwise function as in (19), such that $V_1 \neq V_2$. Then, there exist a compact set $\mathscr{D} \subset \mathbb{R}_0^+ \times \mathbb{R}$ and, for every integer $m \geq 2$, two positive constants T_1^* and T_2^* such that, if $T_1 > T_1^*$ and $T_2 > T_2^*$, the Poincaré map Φ for system (S) on $[0, T]$ (with $T = T_1 + T_2$) induces chaotic dynamics on m symbols in the set $\mathscr{D}$. Moreover, for the corresponding solutions $(x(t), y(t))$ of (S) we have $x(t) > 0$ for every $t \in \mathbb{R}$.*

The constants T_1^* and T_2^* can be explicitly determined in terms of m and some Abelian integrals depending by V_1, V_2 and $f(x)$ as in formula (41). The set $\mathscr{D}$ is explicitly exhibited in the course of the proof. Indeed, we have $\mathscr{D} := \mathscr{A}$ with $\mathscr{A}$ defined in (54).

The proof is based on a topological technique, named *stretching along the paths* (SAP), which is a variant of the classical Smale’s horseshoe geometry (see [67]). Our approach is closely related to the theory of *topological horseshoes* of Kennedy and Yorke [50] as well as to the concept of *covering relations* introduced by Zgliczyński in [68]. The general theory concerning the “SAP method” has been already exposed in some previous papers (see, for instance, [58] and the references therein). In order to make our paper self-contained, we recall the main notation and the results which are needed for the proof of Theorem 2.

By *path* γ we mean a continuous mapping $\gamma : [t_0, t_1] \longrightarrow \mathbb{R}^2$ and we set $\overline{\gamma} := \gamma([t_0, t_1])$. Without loss of generality we will usually take $[t_0, t_1] = [0, 1]$. By a *sub-path* σ of γ we mean the restriction of γ to a compact subinterval of its domain. An *arc* is the homeomorphic image of the compact interval $[0,1]$. We define an *oriented rectangle* in $\mathbb{R}^2$ as a pair

$$\widehat{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-), \tag{28}$$

where $\mathscr{R} \subseteq \mathbb{R}^2$ is homeomorphic to the unit square $[0,1]^2$ (we usually refer to $\mathscr{R}$ as a *topological rectangle*) and

$$\mathscr{R}^- := \mathscr{R}_l^- \cup \mathscr{R}_r^- \tag{29}$$

is the disjoint union of two disjoint compact arcs $\mathscr{R}_l^-, \mathscr{R}_r^- \subseteq \partial\mathscr{R}$ (which are called the components or sides of $\mathscr{R}^-$). We also denote by $\mathscr{R}^+$ the closure of $\partial\mathscr{R} \setminus (\mathscr{R}_l^- \cup \mathscr{R}_r^-)$ which is the union of two compact arcs $\mathscr{R}_d^+$ and $\mathscr{R}_u^+$. The subscripts l, r, u, d stand, conventionally, for *left, right, up, and down.*

Suppose that $\Phi : D_\Phi (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2$ is a planar homeomorphism of D_Φ onto its image. Let $\widehat{\mathscr{M}} := (\mathscr{M}, \mathscr{M}^-)$ and $\widehat{\mathscr{N}} := (\mathscr{N}, \mathscr{N}^-)$ be oriented rectangles.

Definition 3. Let $\mathscr{H} \subseteq \mathscr{M} \cap D_\Phi$ be a compact set. We say that $(\mathscr{H}, \Phi)$ *stretches* $\widehat{\mathscr{M}}$ *to* $\widehat{\mathscr{N}}$ *along the paths* and write

$$(\mathscr{H}, \Phi) : \widehat{\mathscr{M}} \multimap\!\!\!\rightarrow \widehat{\mathscr{N}}, \tag{30}$$

if for every path $\gamma : [a,b] \longrightarrow \mathscr{M}$ such that $\gamma(a) \in \mathscr{M}_l^-$ and $\gamma(b) \in \mathscr{M}_r^-$ (or $\gamma(a) \in \mathscr{M}_r^-$ and $\gamma(b) \in \mathscr{M}_l^-$), there exists a subinterval $[t', t''] \subseteq [a,b]$ such that

$$\begin{gathered} \gamma(t) \in \mathscr{H}, \\ \Phi(\gamma(t)) \in \mathscr{N}, \\ \forall t \in [t', t''] \end{gathered} \tag{31}$$

and, moreover, $\Phi(\gamma(t'))$ and $\Phi(\gamma(t''))$ belong to different components of $\mathscr{N}^-$. In the special case in which $\mathscr{H} = \mathscr{M}$, we simply write $\Phi : \widehat{\mathscr{M}} \multimap\!\!\!\rightarrow \widehat{\mathscr{N}}$.

The next result, taken from [69, Theorem 2.1], provides the existence of periodic points and chaotic-like dynamics according to Definition 1, when Φ admits a splitting as in (21).

Theorem 4. *Let $\Phi_1 : D_{\Phi_1} (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2$ and $\Phi_2 : D_{\Phi_2} (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2$ be continuous maps and let $\widehat{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$, $\widehat{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ be oriented rectangles. Suppose that the following conditions are satisfied:*

(i) *there exist $m \geq 2$ pairwise disjoint compact sets $\mathscr{H}_0, \ldots, \mathscr{H}_{m-1} \subseteq \mathscr{A} \cap D_{\Phi_1}$ such that*

$$(\mathscr{H}_i, \Phi_1) : \widehat{\mathscr{A}} \multimap\!\!\!\rightarrow \widehat{\mathscr{B}}, \quad \text{for } i = 0, \ldots, m-1; \tag{32}$$

(ii) *there is a compact set $\mathscr{K} \subseteq \mathscr{B} \cap D_{\Phi_2}$ such that $(\mathscr{K}, \Phi_2) : \widehat{\mathscr{B}} \multimap\!\!\!\rightarrow \widehat{\mathscr{A}}$.*

Then the map $\Phi := \Phi_2 \circ \Phi_1$ induces chaotic dynamics on m symbols in the set

$$\mathscr{H}^* := \bigcup_{j=0,\ldots,m-1} \mathscr{H}'_j, \quad \text{for } \mathscr{H}'_j := \mathscr{H}_j \cap \Phi_1^{-1}(\mathscr{K}). \tag{33}$$

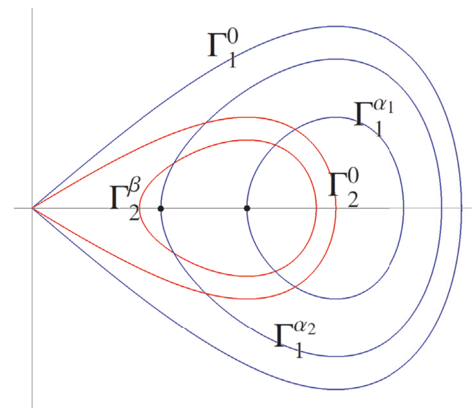

FIGURE 2: The present figure describes our geometric construction. The points on the x-axis marked with a black circle (from right to the left) represent $(a_1, 0)$ and $(a_2, 0)$ on the orbits $\Gamma_1^{\alpha_1}$ and $\Gamma_1^{\alpha_2}$. For this figure we have considered $f(x) = x^3$ and $V_1 = 2$, $V_2 = 1$. For graphical reasons a slightly different x- and y-scaling has been used.

Moreover, for each sequence of m symbols $\mathbf{s} = (s_n)_n \in \{0, \ldots, m-1\}^{\mathbb{N}}$, there exists a compact connected set $\mathscr{C}_{\mathbf{s}} \subseteq \mathscr{H}'_{s_0}$ with $\mathscr{C}_{\mathbf{s}} \cap \mathscr{A}_d^+ \neq \emptyset$ and $\mathscr{C}_{\mathbf{s}} \cap \mathscr{A}_u^+ \neq \emptyset$, such that, for every $w \in \mathscr{C}_{\mathbf{s}}$, there exists a sequence $(y_n)_n$ with $y_0 = w$ and

$$\begin{gathered} y_n \in \mathscr{H}'_{s_n}, \\ \Phi(y_n) = y_{n+1}, \\ \forall n \geq 0. \end{gathered} \tag{34}$$

A dual version of Theorem 4 holds if we interchange the hypotheses on Φ_1 and Φ_2, namely, if we suppose that

(i) there is a compact set $\mathscr{K} \subseteq \mathscr{A} \cap D_{\Phi_1}$ such that $(\mathscr{K}, \Phi_1) : \widehat{\mathscr{A}} \multimap\!\!\!\rightarrow \widehat{\mathscr{B}}$;

(ii) there exist $m \geq 2$ pairwise disjoint compact sets $\mathscr{H}_0, \ldots, \mathscr{H}_{m-1} \subseteq \mathscr{B} \cap D_{\Phi_2}$ such that

$$(\mathscr{H}_i, \Phi_2) : \widehat{\mathscr{B}} \multimap\!\!\!\rightarrow \widehat{\mathscr{A}}, \quad \text{for } i = 0, \ldots, m-1. \tag{35}$$

The corresponding conclusion has to be modified accordingly.

The application of Theorem 4 to Theorem 2 is possible thanks to a *linked twist maps* geometry which appears from the phase-plane analysis of the systems (S_1) and (S_2). The theory of "linked twist maps" regards the case in which a map can be expressed as a composition of two twist maps acting on two annuli crossing each other (see [70–74] for an introduction of the topic and for interesting applications to chaotic mixing). The main argument in the proof of Theorem 2 relies on the construction of two annular regions which cross each other in a suitable manner (see Figures 2 and 3) and such that (Φ_1, Φ_2) acts on them as a linked twist map.

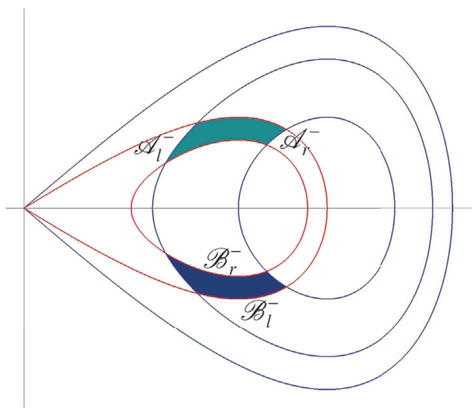

Figure 3: The present figure describes the oriented regions $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ (in lighter and darker colors, respectively). As in Figure 2 we have considered $f(x) = x^3$ and $V_1 = 2$, $V_2 = 1$.

3. Technical Estimates and Proofs

As already observed in Section 2, the motion associated with system (S) is given by a switching in a T-periodic fashion between the orbits of the two autonomous systems (S_1) and (S_2). Such systems have the same qualitative structure and differ only for the value of the V-coefficient. For this reason, we first perform a phase-plane analysis of the planar system

$$\begin{aligned} x' &= y \\ y' &= Vx - f(x) \end{aligned} \tag{36}$$

for $V > 0$ a given parameter. System (36) is a conservative one with associated energy

$$E(x, y) := \frac{1}{2}y^2 + \mathcal{F}_V(x), \tag{37}$$

where

$$\mathcal{F}_V(x) := -\frac{1}{2}Vx^2 + F(x) \quad \text{with } F(x) := \int_0^x f(s)\,ds. \tag{38}$$

As a consequence of $(*)$, there is a unique $x_0 = h^{-1}(V) > 0$ solution of the equation $h(x) = V$. The corresponding equilibrium point $P := (x_0, 0)$ is a center surrounded by a trajectory which represents the homoclinic solution at zero.

The origin and the homoclinic trajectory determine the part of the level line Γ^0 at energy zero contained in the half-plane $x \geq 0$. We denote by $(\overline{x}, 0)$ the intersection point of the homoclinic orbit with the (positive) x-axis. Notice that $\overline{x}$ is the unique (positive) solution of the equation $2F(x)/x^2 = V$. As a consequence of $(*)$, both x_0 and $\overline{x}$, thought as functions of the parameter V, are strictly monotone increasing. Observe also that, for every constant c with

$$0 > c > c_0 := \mathcal{F}_V(x_0), \tag{39}$$

the level line

$$\Gamma^c := \{(x, y) \in \mathbb{R}^+ \times \mathbb{R} : E(x, y) = c\}, \tag{40}$$

is a closed curve which is a (positive) periodic orbit of (36). The period $\tau(c)$ of Γ^c can be computed by the quadrature formula

$$\tau(c) = 2\int_{\alpha(c)}^{\beta(c)} \frac{d\xi}{\sqrt{2(c - \mathcal{F}_V(\xi))}}, \tag{41}$$

where $\alpha(c)$ and $\beta(c)$ are the solutions of the equation $\mathcal{F}_V(x) = c$, with $0 < \alpha(c) < x_0 < \beta(c) < \overline{x}$. Moreover, we have that

$$\begin{aligned} \lim_{c \to c_0^+} \tau(c) &= \frac{2\pi}{\sqrt{\mathcal{F}_V''(x_0)}} = \frac{2\pi}{\sqrt{x_0 h'(x_0)}}, \\ \lim_{c \to 0^-} \tau(c) &= +\infty. \end{aligned} \tag{42}$$

Without further assumptions on $f(x)$ (or, equivalently, on $h(x)$) we cannot guarantee the monotonicity of the time-mapping function $c \longmapsto \tau(c)$. Sufficient conditions ensuring that $\tau(c)$ is strictly increasing can be found in literature. For instance, according to [75], the convexity of the auxiliary function

$$\phi_V(x) := \frac{(\mathcal{F}_V(x) - \mathcal{F}_V(x_0))}{(\mathcal{F}_V'(x))^2} \tag{43}$$

guarantees that $\tau(c)$ is increasing.

Example 5. Consider the typical nonlinear term $f(x) = |x|^{p-1}x$, with $p > 1$. In this case, condition $(*)$ holds for $h(x) = |x|^{p-1}$ and

$$\mathcal{F}_V(x) = -\frac{1}{2}Vx^2 + \frac{1}{p+1}x^{p+1}, \quad \text{for } x \geq 0. \tag{44}$$

Moreover, we find that

$$\begin{aligned} x_0 &= V^{1/(p-1)}, \\ \overline{x} &= \left(\frac{(p+1)V}{2}\right)^{1/(p-1)}, \\ \mathcal{F}_V''(x_0) &= x_0 h'(x_0) = (p-1)V. \end{aligned} \tag{45}$$

In order to prove the monotonicity of the time-map, via the Chicone theorem in [75] we have to study the sign of auxiliary function $N(x) := (\mathcal{F}_V'(x))^4 \phi_V''(x)$ on the open interval $]0, \overline{x}[$. After performing the required computations and using the change of variable $x = sx_0$, one can see that the sign of $N(x)$ for $0 < x < \overline{x}$ is the same of the expression

$$\begin{aligned} &3 + p(p-7)s^{p-1} + p(2p+1)s^{2p-2} - p(p-2)s^{p+1} \\ &\quad - (2p^2 - 3p + 3)s^{2p} + ps^{3p-1}, \end{aligned} \tag{46}$$

for $0 < s < ((p+1)/2)^{1/(p-1)}$.

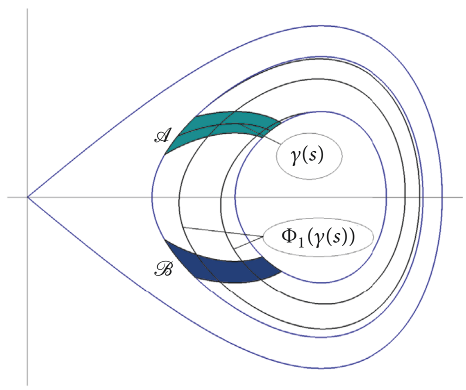

Figure 4: The present figure describes how a path $\gamma(s)$ crossing $\mathscr{A}$ from $\mathscr{A}_l^-$ to $\mathscr{A}_r^-$ is stretched by Φ_1 to a path $\Phi_1(\gamma(s))$ which crosses twice $\mathscr{B}$ from $\mathscr{B}_l^-$ to $\mathscr{B}_r^-$. As in Figure 2 we have considered $f(x) = x^3$ and $V_1 = 2$, $V_2 = 1$. For this example we have taken $T_1 = 50$.

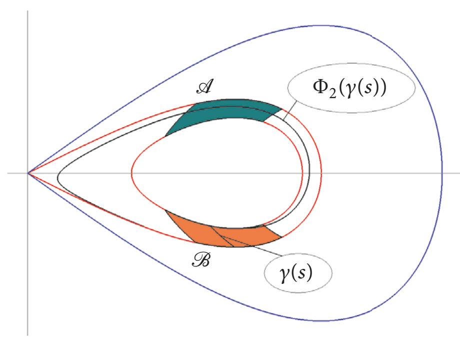

Figure 5: The present figure describes how a path $\gamma(s)$ crossing $\mathscr{B}$ from $\mathscr{B}_l^-$ to $\mathscr{B}_r^-$ is stretched by Φ_2 to a path $\Phi_2(\gamma(s))$ which crosses (once) $\mathscr{A}$ from $\mathscr{A}_l^-$ to $\mathscr{A}_r^-$. As in Figure 2 we have considered $f(x) = x^3$ and $V_1 = 2$, $V_2 = 1$. For this example we have taken $T_2 = 10$.

For instance, if $p = 3$, it is easy to check that the above expression is strictly positive for $s \neq 1$ (which corresponds to $x \neq x_0$) and therefore the time-mapping function $\tau(c)$ is strictly increasing on $]c_0, 0[$. The case $p = 3$ is the model situation that we have chosen in all our illustrative examples of Figures 1–8.

The monotonicity of the period map still holds for an arbitrary $p > 1$. The proof in this case is a more complicated task (see [76, 77]).

Until now we have considered some general properties of the solutions of system (36). As a next step, in order to investigate the dynamics associate to system (S) for a T-periodic potential $V(t)$ defined as in (19), we need to make a comparison between the phase–portraits associated with the autonomous systems (S_1) and (S_2). Keeping the notation just introduced, we set $\mathscr{F}_i := \mathscr{F}_{V_i}$ and denote by E_i the associated energy, for $i = 1, 2$. Accordingly, we indicate by $P_i = (x_{0i}, 0)$ and $(\overline{x}_i, 0)$ the corresponding equilibrium points and the intersection points of the homoclinic orbits with the positive x-axis. Moreover, $\tau_i(c)$ denotes the fundamental period of the closed orbits defined in (41) for the potential functions $\mathscr{F}_i$.

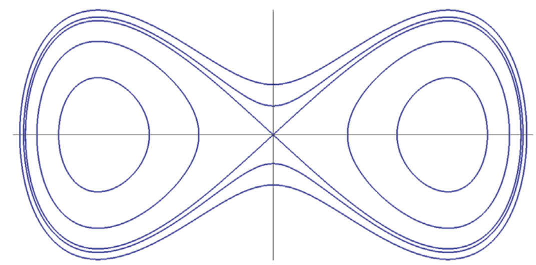

Figure 6: Phase-portrait of (36) for $f(x) = x^3$ and $V = 2$. For graphical reasons a slightly different x- and y-scaling has been used.

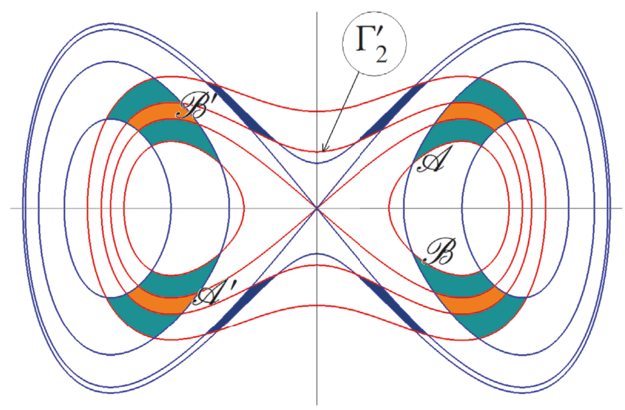

Figure 7: The present figure shows some possible rectangular regions which can be considered for the application of Theorem 4. The sets $\mathscr{A}$ and $\mathscr{B}$ are the same as in Figure 3 and (as we have already proved in Section 3) they can be used to provide a complex dynamics on positive solutions. If we choose, for instance, the sets $\mathscr{A}'$ and $\mathscr{B}'$ we can prove the presence of a complex dynamics generated by solutions which are negative on the time interval $[0, T_1]$ and oscillate in the phase-plane around the point $(-x_{01}, 0)$, and then, in the time interval $[T_1, T]$ oscillate a certain number of times around the origin. More in detail, given any positive integer M, we can produce solutions $u(t)$ of (17) which have precisely $2, 4, \ldots, 2M$ simple zeros in the interval $]T_1, T[$, provided that $T_2 = T - T_1$ is sufficiently large. A lower estimate for T_2 can be easily determined by the knowledge of the period of the closed trajectory Γ_2' which bounds "externally" $\mathscr{A}'$ and $\mathscr{B}'$. Similar remarks can be made by selecting other pairs of topological rectangles among those put in evidence with a color. As in the preceding figures, we have considered $f(x) = x^3$ and $V_1 = 2$, $V_2 = 1$. For graphical reasons a slightly different x- and y-scaling has been used.

Just to fix a case of study, we suppose that

$$V_1 > V_2. \tag{47}$$

The case when $V_1 < V_2$ can be treated in a similar manner. Observe that from (47) it follows that the homoclinic orbit Γ_2^0 of system (S_2) is contained in the part of the right half-plane bounded by the homoclinic orbit Γ_1^0 of system (S_1).

Now we are in position to introduce the rectangular regions (topological rectangles) $\mathscr{A}$ and $\mathscr{B}$ in order to apply Theorem 4. As a first step, we chose a closed trajectory Γ_1^c

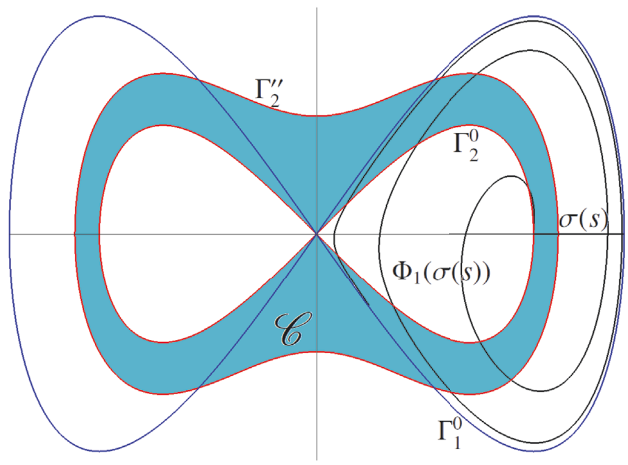

Figure 8: The present figure suggests a possible argument to prove multiplicity of sign changing solutions to the Neumann problem (94). We start with system (S_1) by considering initial points $w_0 = (u_0, 0)$ with $x_{01} < u_0 \leq \overline{x}_1$ (recall that $P_1 = (x_{01}, 0)$ is the positive center of (S_1), while $(\overline{x}_1, 0)$ is the intersection point of the homoclinic orbit Γ_1^0 with the positive x-axis). We parameterize such initial points as an arc $\sigma(s)$. If T_1 is sufficiently large (with a lower bound which can be easily estimated from the equation) we find that $\Phi_1(\sigma(s))$ is a spiral-like curve winding a certain number of times around P_1 and with an end point on Γ_1^0 in the fourth quadrant near the origin. If we fix a (small) positive constant c'' and denote by $\mathscr{C} := \{(x, y) : 0 \leq E_2(x, y) \leq c''\}$ the region between the homoclinic trajectories of (S_2) and the level line $\Gamma_2'' := \Gamma_2^{c''}$, we observe that the points of $\Phi_1(\sigma(s)) \cap \Gamma_2^0$ remain on Γ_2^0 under the action of Φ_2, while the points of $\Phi_1(\sigma(s)) \cap \Gamma_2''$ run around the origin along the periodic orbit Γ_2'' and will perform a certain number of revolutions if T_2 is sufficiently large. More precisely, if we denote by τ'' the period of Γ_2'' and suppose that $T_2/\tau'' > m$, we can find solutions of (94) having precisely $2j$-zeros in the interval $]T_1, T_1 + T_2[$, for every $j = 1, \ldots, m$. As in the preceding figures, we have considered $f(x) = x^3$ and $V_1 = 2, V_2 = 1$. For graphical reasons a slightly different x- and y-scaling has been used.

of system (S_1) which intersects in two distinct points the homoclinic trajectory $\Gamma_2^0 \setminus \{(0, 0)\}$ of system (S_2). From an analytic point of view, this corresponds to solving the system

$$E_2(x, y) = \frac{1}{2}y^2 - \frac{1}{2}V_2 x^2 + F(x) = 0$$
$$E_1(x, y) = \frac{1}{2}y^2 - \frac{1}{2}V_1 x^2 + F(x) = c, \tag{48}$$

for a suitable $c \in]c_{01}, 0[$, with $c_{01} := \mathscr{F}_1(x_{01})$. It is clear that this system has a pair of solutions $(\xi_c, \pm\eta_c)$ with $\xi_c, \eta_c > 0$ if and only if

$$\xi_c := \sqrt{\frac{2|c|}{V_1 - V_2}},$$
$$\mathscr{F}_2(\xi_c) < 0. \tag{49}$$

The latter condition holds if and only if $0 < \xi_c < \overline{x}_2$. We conclude that the desired geometry can be produced if and only if we choose an energy level c for $E_1(x, y)$ such that

$$0 > c > c^* := \mathscr{F}_1(\overline{x}_2) \geq c_{01}. \tag{50}$$

From now on, we suppose to have fixed a constant c satisfying (50). Let us call α_1 such a constant and denote by $(a_1, 0)$ the intersection of the closed orbit $\Gamma_1^{\alpha_1}$ with the positive x-axis which is closer to the origin. Next, we choose a_2 with $0 < a_2 < a_1$ and consider the level line of system (S_1) passing through $(a_2, 0)$. This is the closed orbit $\Gamma_1^{\alpha_2}$, for $\alpha_2 := E_1(a_2, 0)$. For α_2 we further require that

$$\tau_1(\alpha_2) > \tau_1(\alpha_1). \tag{51}$$

We notice that it is always possible to find an open interval $]0, a^*[\subseteq]0, a_1[$ such that for each $a_2 \in]0, a^*[$ the condition (51) holds. This follows from the fact that $\tau_1(c) \longrightarrow +\infty$ as $c \longrightarrow 0^-$. In the special case in which the time-mapping is strictly monotone increasing, one can take an arbitrary $a_2 \in]0, a_1[$. Observe that, by construction, we also have

$$c_{01} \leq c^* < \alpha_1 < \alpha_2 < 0. \tag{52}$$

As a last step, we fix a constant β such that

$$0 > \beta > \max\{E_2(a_1, 0), E_2(a_2, 0)\} \geq c_{02}$$
$$:= \mathscr{F}_2(x_{02}). \tag{53}$$

By this latter choice, the corresponding (periodic) trajectory Γ_2^{β} of system (S_2) intersects both $\Gamma_1^{\alpha_1}$ and $\Gamma_1^{\alpha_2}$ in the region $\{(x, y) : x > 0, E_2(x, y) < 0\}$ bounded by the homoclinic orbit Γ_2^0. Figure 2 illustrates the geometric construction performed above.

Next we define

$$\mathscr{A} := \{(x, y) : x > 0, y > 0, \alpha_1 \leq E_1(x, y) \leq \alpha_2, \beta$$
$$\leq E_2(x, y) \leq 0\} \tag{54}$$

and its specular image $\mathscr{B}$ with respect to the x-axis, namely,

$$\mathscr{B} := \{(x, y) : x > 0, y < 0, \alpha_1 \leq E_1(x, y) \leq \alpha_2, \beta$$
$$\leq E_2(x, y) \leq 0\}. \tag{55}$$

For these regions we select an orientation as follows:

$$\mathscr{A}^- := \mathscr{A}_l^- \cup \mathscr{A}_r^-$$
$$\text{with } \mathscr{A}_l^- := \mathscr{A} \cap \Gamma_1^{\alpha_2}, \ \mathscr{A}_r^- := \mathscr{A} \cap \Gamma_1^{\alpha_1},$$
$$\mathscr{B}^- := \mathscr{B}_l^- \cup \mathscr{B}_r^- \tag{56}$$
$$\text{with } \mathscr{B}_l^- := \mathscr{B} \cap \Gamma_2^0, \ \mathscr{B}_r^- := \mathscr{B} \cap \Gamma_2^{\beta}$$

(see Figure 3).

Now we describe the behavior of the points in the regions $\mathscr{A}$ and $\mathscr{B}$ under the action of the Poincaré maps Φ_1 and Φ_2, respectively.

Suppose that

$$0 \le t \le T_1. \tag{57}$$

For each point $z \in \mathscr{A}$ the solution $\zeta(t;0,z)$ of (S) is indeed a solution of the autonomous conservative system (S_1) and therefore, $E_1(\zeta(t;0,z)) = E_1(z)$, for all $t \in [0,T_1]$. The orbit Γ_1^c for $c := E_1(z)$ is closed trajectory surrounding the equilibrium point $P_1 = (x_{01}, 0)$ of (S_1). Consistently with the previous notation, the period of the points in Γ_1^c is denoted by $\tau_1(c)$. Since all the points of Γ_1^c (for $\alpha_1 \le c \le \alpha_2$) move in the clockwise sense along the orbit under the action of the dynamical system associated with (S_1), it will be convenient to introduce a system of polar coordinates with center at P_1 and take the clockwise orientation as a positive orientation for the angles starting from the positive half-line $\{(x,0) : x > x_{01}\}$. In this manner, we can associate an angular coordinate $\theta(t;z)$ to any solution $\zeta(t;0,z)$ for any initial point $z \in \mathscr{A}$ and $t \in [0,T_1]$.

In such a situation, we find that

$$\frac{d}{dt}\theta(t;z) = \frac{y(t)\,x'(t) - (x(t) - x_{01})\,y'(t)}{(x(t) - x_{01})^2 + y(t)^2} \quad \text{for } (x(t), y(t)) = \zeta(t;0,z). \tag{58}$$

Since $(x(t), y(t))$ is a solution of (S_1) and, moreover, $\alpha_1 \le E_1(x(t), y(t)) \le \alpha_2$ with $x(t) > 0$, for all $t \in [0,T_1]$, we have that $(d/dt)\theta(t;z) > 0$ for all $t \in [0,T_1]$. In fact, by (18) and $(*)$, we obtain

$$\begin{aligned} yx' - (x - x_{01})\,y' &= y^2 - (x - x_{01})(V_1 x - f(x)) \\ &= y^2 \\ &\quad + x(x - x_{01})(h(x) - h(x_{01})) \\ &> 0, \quad \forall x > 0. \end{aligned} \tag{59}$$

We have thus proved that the angle is a strictly increasing function of the time variable. Hence, for any positive integer k we conclude that

$$\theta(t;z) \gtreqless \theta(0,z) + 2k\pi \quad \text{if and only if } t \gtreqless k\tau_1(c) \tag{60}$$

(of course, the above relation holds provided that $t \in [0,T_1]$ and $T_1 > k\tau_1(c)$).

After these preliminary observations, we are now in position to prove the validity of the first condition of Theorem 4 provided that T_1 is large enough.

Let us fix an integer $m \ge 2$ and set

$$T_1^* := \frac{(m+2)\,\tau_1(\alpha_1)\,\tau_1(\alpha_2)}{\tau_1(\alpha_2) - \tau_1(\alpha_1)}. \tag{61}$$

Given T_1^* as above, we also fix $T_1 > T_1^*$ and define

$$\begin{aligned} k_1 &:= \left\lfloor \frac{T_1}{\tau_1(\alpha_1)} \right\rfloor, \\ k_2 &:= \left\lceil \frac{T_1}{\tau_1(\alpha_2)} \right\rceil. \end{aligned} \tag{62}$$

The position (61) and the choice of T_1 imply that

$$\frac{T_1}{\tau_1(\alpha_1)} - \frac{T_1}{\tau_1(\alpha_2)} > m + 2 \tag{63}$$

and hence $k_1 - k_2 > m$.

Now we consider the motion associated with (S_1) for $t \in [0,T_1]$.

First of all, we note that

$$\theta(0,z) \in\,]-\pi, 0[\,, \quad \forall z \in \mathscr{A}. \tag{64}$$

On the other hand, since $\mathscr{A}_r^- \subseteq \Gamma_1^{\alpha_1}$, we know that

$$\theta(T_1,z) \ge \theta(0,z) + 2k_1\pi > (2k_1 - 1)\pi, \quad \forall z \in \mathscr{A}_r^-. \tag{65}$$

Similarly, since $\mathscr{A}_l^- \subseteq \Gamma_1^{\alpha_2}$, we know that

$$\theta(T_1,z) \le \theta(0,z) + 2k_2\pi < 2k_2\pi, \quad \forall z \in \mathscr{A}_l^-. \tag{66}$$

We thus conclude that the range of the angular function $\{\theta(T_1,z) : z \in \mathscr{A}\}$ covers the interval $[2k_2\pi, (2k_1 - 1)\pi] \supseteq [2k_2\pi, (2k_2+1)\pi + 2(m-1)\pi]$. This interval contains m closed disjoint intervals of the form $[2k_2\pi + 2j\pi, (2k_2+1)\pi + 2j\pi]$, for $j = 0, \ldots, m-1$. Hence, if for each nonnegative integer j, we define

$$\begin{aligned} \mathscr{H}_j := \{z \in \mathscr{A} : \theta(T_1,z) \\ \in [2k_2\pi + 2j\pi, (2k_2+1)\pi + 2j\pi]\}, \end{aligned} \tag{67}$$

we obtain m nonempty and pairwise disjoint compact subsets $\mathscr{H}_0, \ldots, \mathscr{H}_{m-1}$ of $\mathscr{A}$.

Let $\gamma : [0,1] \longrightarrow \mathscr{A}$ be a (continuous) path such that $\gamma(0) \in \mathscr{A}_l^-$ and $\gamma(1) \in \mathscr{A}_r^-$ and consider the path $[0,1] \ni s \longmapsto \Phi_1(\gamma(s))$. Passing to the polar coordinates we have that

$$\theta(0,\gamma(s)) \in\,]-\pi, 0[\,, \quad \forall s \in [0,1] \tag{68}$$

and, moreover,

$$\begin{aligned} &\theta(T_1,\gamma(0)) < 2k_2\pi, \\ &\theta(T_1,\gamma(1)) > (2k_1 - 1)\pi \\ &\qquad > (2k_2+1)\pi + 2(m-1)\pi. \end{aligned} \tag{69}$$

Hence, for every $j = 0, \ldots, m-1$ there exists an interval $[s_j', s_j''] \subseteq [0,1]$ such that $\gamma(s) \in \mathscr{H}_j$ for all $s \in [s_j', s_j'']$, and, moreover,

$$\begin{aligned} \theta\left(T_1, \gamma\left(s_j'\right)\right) &= 2k_2\pi + 2j\pi, \\ \theta\left(T_1, \gamma\left(s_j''\right)\right) &= (2k_2+1)\pi + 2j\pi. \end{aligned} \tag{70}$$

Using the fact that $\alpha_1 \le E_1(\Phi_1(\gamma(s))) \le \alpha_2$ for every $s \in [0,1]$ (by the invariance of the annular region bounded by $\Gamma_1^{\alpha_1}$ and $\Gamma_1^{\alpha_2}$ with respect to system (S_1)), we conclude that the set $\{\Phi_1(\gamma(s)) : s \in [s_j', s_j'']\}$ crosses the region $\mathscr{B}$. Hence, by the continuity of the map $[s_j', s_j''] \ni s \longmapsto \Phi_1(\gamma(s))$, we can find a subinterval $[t_j', t_j''] \subseteq [s_j', s_j'']$ such that

$$\Phi_1(\gamma(s)) \in \mathscr{B}, \quad \forall s \in \left[t_j', t_j''\right] \tag{71}$$

and, moreover,

$$\Phi_1\left(\gamma\left(t_j'\right)\right) \in \mathscr{B}_l^-, \quad \Phi_1\left(\gamma\left(t_j''\right)\right) \in \mathscr{B}_r^-. \tag{72}$$

We also know that

$$\gamma(s) \in \mathscr{H}_j, \quad \forall s \in \left[t_j', t_j''\right]. \tag{73}$$

According to Definition 3 we conclude that

$$(\mathscr{H}_j, \Phi_1) : \widehat{\mathscr{A}} \multimap \widehat{\mathscr{B}}, \quad \text{for } j = 0, \ldots, m-1 \tag{74}$$

and thus the first condition in Theorem 4 is fulfilled (see Figure 4 for a graphical description of this step in the proof).

After the time T_1 we switch to equation (S_2). As previously observed, due to the autonomous nature of the system, the study of the solutions in the time interval $[T_1, T]$ is equivalent to the study for $0 \le t \le T_2$. Now we set

$$T_2^* := \tau_2(\beta) \tag{75}$$

and fix $T_2 > T_2^*$.

Arguing in a similar manner as before, we introduce a system of polar coordinates with center at the point $P_2 = (x_{02}, 0)$ and take the clockwise orientation as a positive orientation for the angles starting from the positive half-line $\{(x,0) : x > x_{02}\}$. In this manner, we can associate an angular coordinate $\vartheta(t;z)$ to any solution (S_2) with initial point $z \in \mathscr{B}$ and $t \in [0, T_2]$. Repeating the same argument as above, one can see that the angle is a strictly increasing function of the time variable and, moreover, for any positive integer k

$$\vartheta(t;z) \gtreqless \vartheta(0,z) + 2k\pi \quad \text{if and only if } t \gtreqless k\tau_2(c) \tag{76}$$

(of course, the above relation holds provided that $t \in [0, T_2]$ and $T_2 > k\tau_2(c)$ for $c \in [\beta, 0]$).

Let $\gamma : [0,1] \longrightarrow \mathscr{B}$ be a (continuous) path such that $\gamma(0) \in \mathscr{B}_l^-$ and $\gamma(1) \in \mathscr{B}_r^-$ and consider the path $[0,1] \ni s \longmapsto \Phi_2(\gamma(s))$. Passing to the polar coordinates we have that

$$\vartheta(0, \gamma(s)) \in \,]0, \pi[\,, \quad \forall s \in [0,1] \tag{77}$$

and, moreover,

$$\vartheta(T_2, \gamma(0)) < \pi, \quad \vartheta(T_2, \gamma(1)) > 2\pi. \tag{78}$$

Indeed, the point $\gamma(0) \in \mathscr{B}_l^-$ lies on the homoclinic trajectory Γ_2^0 and therefore $\Phi_2(\gamma(0))$ remains in the fourth quadrant. On the other hand, the point $\gamma(1) \in \mathscr{B}_r^-$ lies on the periodic orbit Γ_2^β and, since $T_2 > \tau_2(\beta)$ (which is the period of Γ_2^β), it follows that $\gamma(1)$ makes at least one turn around the center P_2 during the time interval $[0, T_2]$.

Hence, there exists an interval $[s', s''] \subseteq [0,1]$ such that

$$\vartheta\left(T_2, \gamma\left(s'\right)\right) = \pi, \quad \vartheta\left(T_2, \gamma\left(s''\right)\right) = 2\pi. \tag{79}$$

Using the fact that $\beta \le E_2(\Phi_2(\gamma(s))) \le 0$ for every $s \in [0,1]$ (by the invariance of the region bounded by Γ_2^β and Γ_2^0 with respect to system (S_2)), we conclude that the set $\{\Phi_2(\gamma(s)) : s \in [s', s'']\}$ crosses the region $\mathscr{A}$. Hence, by the continuity of the map $[s', s''] \ni s \longmapsto \Phi_2(\gamma(s))$, we can find a subinterval $[t', t''] \subseteq [s', s'']$ such that

$$\Phi_2(\gamma(s)) \in \mathscr{A}, \quad \forall s \in \left[t', t''\right] \tag{80}$$

and, moreover,

$$\Phi_2\left(\gamma\left(t'\right)\right) \in \mathscr{A}_l^-, \quad \Phi_2\left(\gamma\left(t''\right)\right) \in \mathscr{A}_r^-. \tag{81}$$

According to Definition 3 we conclude that

$$(\mathscr{B}, \Phi_2) : \widehat{\mathscr{B}} \multimap \widehat{\mathscr{A}} \tag{82}$$

and thus the second condition in Theorem 4 is fulfilled for $\mathscr{K} := \mathscr{B}$ (see Figure 5 for a graphical description of this last step in the proof).

Then, Theorem 4 applies and the proof of Theorem 2 is complete (with respect to the "chaotic part"), providing the existence of complex dynamics on m symbols for the Poincaré map Φ (of system (S)) on the compact set $\mathscr{A}$, according to Definition 1.

Regarding the fact that all the solutions we find via Theorem 4 are always positive in the x-variable, we need only to observe that, in the first step of the proof concerning the property

$$(\mathscr{H}_i, \Phi_1) : \widehat{\mathscr{A}} \multimap \widehat{\mathscr{B}}, \tag{83}$$

we have also found that $x(t) > 0$ for all $t \in [0, T_1]$ when $(x(0), y(0)) \in \mathscr{H}_i$. Moreover, by construction, it follows that $x(t) > 0$ for all $t \in [0, T_2]$ when $(x(0), y(0)) \in \mathscr{B}$. This concludes the proof.

Remark 6. From the proof of Theorem 2 (and the geometric construction involving $\mathscr{A}$ and $\mathscr{B}$) it follows that our result is stable with respect to small perturbations of the coefficients. Indeed, from the fundamental theory of ODEs we know that, for any fixed time interval $[0, T]$, a "small" perturbation of the coefficients in the L^1-norm on $[0, \mathrm{T}]$ produces a "small" perturbation on the Poincaré map (in the sense of the continuous dependence of the solutions from the data). With this respect, the following result holds.

Theorem 7. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $V : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ be as in Theorem 2, with $V_1 \ne V_2$. Given $m \ge 2$ and T_1^* and T_2^* according to Theorem 2, let us fix $T_1 > T_1^*$ and $T_2 > T_2^*$ and let $T := T_1 + T_2$. Then, there is $\varepsilon = \varepsilon(m, T_1, T_2) > 0$ such that equation*

$$-u'' + a(t)u = f(u), \tag{84}$$

has infinitely many periodic (subharmonic) solutions as well as solutions presenting a complex dynamics, with $u(t) > 0$, *provided that $a(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is a (measurable) T-periodic function satisfying*

$$\int_0^T |a(t) - V(t)|\,dt < \varepsilon. \tag{85}$$

In Theorem 7 the solutions are considered in the Carathéodory sense [78] (when $a(t)$ is only measurable). On the other hand, we can also take an arbitrarily smooth function $a(t)$ which approximates the step function $V(t)$, provided that (85) is satisfied. The stability of Theorem 2 with respect to small perturbations of the Poincaré map is not confined to the coefficient of the nonlinearity. For instance, we can obtain the same result for a perturbed equation of the form

$$-u'' + cu' + a(t)u = q(t) f(u) + e(t, u, u'), \tag{86}$$

provided that c, $\int_0^T |a(t) - V(t)|dt$ and $|e(\cdots)|$ are sufficiently small and $q(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is a T-periodic function with $\int_0^T |q(t) - 1|dt$ small, too. Finally, we stress that T_1^* and T_2^* can be easily computed and are not necessarily "large" (see, for instance, the examples considered in Figures 4-5).

4. Related Results

In the previous sections we have discussed the presence of chaotic-like dynamics (including the existence of infinitely many subharmonic solutions) for (17), by assuming that the period is large enough. From this point of view, our results can be interpreted in line with analogous theorem on Hamiltonian systems with slowly varying coefficients (see [14, 16]). On the other hand, via a simple change of variable, we can apply our results to a typical Schrödinger equation of the form

$$-\varepsilon^2 u'' + \mathscr{V}(t)u = f(u), \tag{87}$$

for $\mathscr{V} : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ a periodic stepwise function of fixed period $\mathscr{T} > 0$, of the form

$$\mathscr{V}(t) := \begin{cases} \mathscr{V}_1 & \text{for } t \in [0, \mathscr{T}_1[\\ \mathscr{V}_2 & \text{for } t \in [\mathscr{T}_1, \mathscr{T}[\end{cases} \tag{88}$$

with $\mathscr{V}_1 \neq \mathscr{V}_2$ and $0 < \mathscr{T}_1 < \mathscr{T}$. As before, we also set $\mathscr{T}_2 := \mathscr{T} - \mathscr{T}_1$. Writing (87) as an equivalent first order system

$$\begin{aligned} u' &= \varepsilon^{-1} v \\ v' &= \varepsilon^{-1}(\mathscr{V}(t)u - f(u)) \end{aligned} \tag{89}$$

in the phase-plane, we obtain the following result.

Theorem 8. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function of the form (18), with h satisfying $(*)$. Let $\mathscr{V} : \mathbb{R} \longrightarrow \mathbb{R}_0^+$ be a $\mathscr{T}$-periodic stepwise function as in (88). Then, there exist a compact set $\mathscr{D} \subset \mathbb{R}_0^+ \times \mathbb{R}$ and, for every integer $m \geq 2$, a constant $\varepsilon_m^* > 0$ such that, for every $\varepsilon \in]0, \varepsilon_m^*[$ the Poincaré map Ψ^ε associated with (89) on $[0, \mathscr{T}]$ induces chaotic dynamics on m symbols in the set $\mathscr{D}$. Moreover, for the corresponding solutions $(u(t), v(t))$ of (89) we have $u(t) > 0$ for every $t \in \mathbb{R}$.*

The constant ε_m^* can be explicitly determined in terms of m and $\mathscr{T}_1, \mathscr{T}_2$.

Proof. As in the proof of Theorem 2, we suppose

$$0 < \mathscr{V}_2 < \mathscr{V}_1 \tag{90}$$

(the treatment of the other situation is completely similar and thus is omitted).

The change of variables $t := \varepsilon s$ and $x(s) := u(\varepsilon s)$, $y(s) := v(\varepsilon s)$, transforms system (89) to the equivalent first order system

$$\begin{aligned} x' &= y \\ y' &= V(s)x - f(x) \end{aligned} \tag{91}$$

for $V(s) := \mathscr{V}(\varepsilon s)$ a stepwise periodic function of period $T := \mathscr{T}/\varepsilon$. By (88) and setting $T_1 := \mathscr{T}_1/\varepsilon$ and $T_2 := \mathscr{T}_2/\varepsilon$, it follows that $V(s) = V_1 := \mathscr{V}_1$ on $[0, T_1[$ and $V(s) = V_2 := \mathscr{V}_2$ on $[T_1, T[$ with $T = T_1 + T_2$. Notice that if we denote by Φ the Poincaré map associated with system (91) on $[0, T]$, then it follows that $\Psi^\varepsilon = \Phi$ (one can easily check this fact, because $(x(T), y(T)) = (u(\mathscr{T}), v(\mathscr{T}))$).

Now, for (91) we can apply Theorem 2. In particular, through the proof of that result in Section 3, we find a compact region $\mathscr{D} \subseteq \mathbb{R}_0^+ \times \mathbb{R}$ and two constants T_1^* and T_2^* such that the chaotic dynamics for Φ (according to Definition 1) is ensured provided that $T_1 > T_1^*$ and $T_2 > T_2^*$. The lower estimates on T_1 and T_2 transfer to an upper bound for ε, so that if

$$0 < \varepsilon < \varepsilon_m^* := \min\left\{\frac{\mathscr{T}_1}{T_1^*}, \frac{\mathscr{T}_2}{T_2^*}\right\}, \tag{92}$$

then the chaotic dynamics for Ψ^ε on the same set $\mathscr{D}$ is guaranteed. In particular, recalling the definition of T_1^* in (61) and T_2^* in (75), we derive a precise estimate to ε_m^*. For instance, if we take $V_1 = 2$, $V_2 = 1$ and $f(x) = x^3$ (like in the example of Figure 2) then we know that $\varepsilon_2^* \geq \min\{\mathscr{T}_1/50, \mathscr{T}_2/10\}$ and therefore, if we choose $\varepsilon < \min\{\mathscr{T}_1/50, \mathscr{T}_2/10\}$, the conclusion of Theorem 8 holds. □

We observe that, similarly to Remark 6, the stability of the result with respect to small perturbations of the coefficients is guaranteed, too.

5. Boundary Value Problems on Finite Intervals: Positive Solutions

We start this section by briefly describing how the method applied in the proof of Theorem 2 can be adapted to obtain multiplicity results for the Neumann boundary value problem

$$\begin{aligned} -\varepsilon^2 u'' + \mathscr{V}(t)u &= f(u) \\ u'(0) = u'(T) &= 0 \end{aligned} \tag{93}$$

(see (87)). Our aim is to find multiple positive solutions for (93), where the number of the solutions becomes arbitrarily large as $\varepsilon \longrightarrow 0^+$. Actually, such kind of result can be obtained by a variant of Theorem 2, via a change of variables as in the proof of Theorem 8. With this respect, we study the equivalent problem

$$-u'' + V(t)u = f(u)$$
$$u'(0) = u'(T) = 0 \tag{94}$$

(see (17)) and look for multiplicity results, where the number of the solutions increases as the time-interval length grows.

As in Section 2 we suppose that $V : [0,T] \longrightarrow \mathbb{R}_0^+$ is a stepwise function of the form (19) for $V_1 > V_2$ (the case in which $V_1 < V_2$ can be treated in a similar manner). The assumptions on $f(x)$ are the same as in Section 2.

Following the argument of the proof of Theorem 2 we consider the Poincaré map Φ associated with the planar system (S) as well as its components Φ_i associated with systems (S_i). The difference with respect to the proof of Theorem 2, consists into the fact that this time we look for initial points $w_0 := (u_0, 0)$, with $u_0 > 0$ such that the second component of $\Phi(w_0) := \zeta(T; 0, w_0)$ vanishes. In other words, we apply a *shooting method*, looking for a solution which departs at time $t = 0$ from a point on the positive x-axis and hits again the (positive) x-axis at the time $t = T$, with $x(t) > 0$ for all $t \in [0,T]$.

We repeat step by step (keeping the same notation) the geometrical construction in the proof of Theorem 2. In particular, as before, we choose the closed orbits $\Gamma_1^{\alpha_1}$ and $\Gamma_1^{\alpha_2}$ of system (S_1) and Γ_2^{β} of system (S_2) in order to produce the regions $\mathscr{A}$ and $\mathscr{B}$ in the phase-plane. Recall also that $\tau_1(\alpha_2) > \tau_1(\alpha_1)$ (see (51)). We have already indicated by $(a_i, 0)$ the intersection point of $\Gamma_1^{\alpha_i}$ with the positive x-axis which is closer to the origin. We introduce now also the second intersection point of $\Gamma_1^{\alpha_i}$ with the positive x-axis, which will be denoted by $(b_i, 0)$. Clearly we have

$$0 < a_2 < a_1 < x_{01} < b_1 < b_2 \tag{95}$$

(compared also with Figure 2).

We produce the solutions of (94) by shooting from initial points $w_0 = (u_0, 0)$ with

$$u_0 \in [a_2, a_1] \ \text{ or } u_0 \in [b_1, b_2]. \tag{96}$$

Just to fix one case for our discussion, let us assume that the former of the above alternative occurs. More precisely, we shall develop the following argument that we first briefly describe in an heuristic manner for the reader's convenience. We start from an initial point in the segment $[a_2, a_1] \times \{0\}$ of the phase-plane and apply the Poincaré map Φ_1 for a time T_1 sufficiently long so that the image of such segment crosses at least m times the set $\mathscr{B}$. Then we switch to the Poincaré map Φ_2 and apply it for a time T_2 sufficiently long so that the above arcs crossing $\mathscr{B}$ will be transformed (by Φ_2) to some curves winding around the point P_2 and crossing at least k times the x-axis. Putting all together these facts we conclude that there are at least $m \times k$ solutions to (94). We present now the technical justification, by slightly modifying the argument of the proof of Theorem 2.

As before, we represent the solutions of (S_1) in polar coordinates with respect to the center $P_1 = (x_{01}, 0)$, using a clockwise orientation for the angular coordinate. In this case, instead of taking $z \in \mathscr{A}$, we have $z \in [a_2, a_1] \times \{0\}$ and therefore $\theta(0, z) = -\pi$. Accordingly, we replace the path $\gamma : [0,1] \longrightarrow \mathscr{A}$ with the map which parameterizes the segment $[a_2, a_1] \times \{0\}$, namely we take

$$\gamma(s) := (s, 0), \quad \text{for } s \in [a_2, a_1]. \tag{97}$$

As a consequence, for

$$c(s) := E_1(\gamma(s)), \quad s \in [a_2, a_1], \tag{98}$$

we have that

$$\theta(t, \gamma(s)) \gtreqless (k-1)\pi \quad \text{if and only if } t \gtreqless k\frac{\tau_1(c(s))}{2}. \tag{99}$$

For an integer $m \geq 2$, we define T_1^* as in (61) and fix $T_1 > T_1^*$. Then, repeating the same argument of the proof of Theorem 2, we define the integers k_1 and k_2 and obtain

$$\theta(T_1, \gamma(a_1)) = \theta(T_1, (a_1, 0)) \geq (2k_1 - 1)\pi \tag{100}$$

and

$$\theta(T_1, \gamma(a_2)) = \theta(T_1, (a_2, 0)) < 2k_2\pi \tag{101}$$

(which, in this case, is an obvious choice). As a consequence we find m pairwise disjoint intervals $[t'_j, t''_j] \subseteq [a_2, a_1]$ for $j = 0, \ldots, m-1$ such that

$$\Phi_1((s,0)) \in \mathscr{B}, \quad \forall, s \in \left[t'_j, t''_j\right] \tag{102}$$

and, moreover,

$$\Phi_1\left(\left(t'_j, 0\right)\right) \in \mathscr{B}_l^-,$$
$$\Phi_1\left(\left(t''_j, 0\right)\right) \in \mathscr{B}_r^-. \tag{103}$$

After the time T_1 we switch to equation (S_2). As remarked before, the study of the solutions in the time interval $[T_1, T]$ is equivalent to the study for $0 \leq t \leq T_2$, where, for the moment, T_2 is not yet fixed.

We introduce another system of polar coordinates with center at the point $P_2 = (x_{02}, 0)$ and take the clockwise orientation as a positive orientation for the angles starting from the positive half-line $\{(x, 0) : x > x_{02}\}$. In this manner, we can associate an angular coordinate $\vartheta(t; z)$ to any solution (S_2) with initial point $z \in \mathscr{B}$ and $t \in [0, T_2]$. In order to have the condition $u'(T) = 0$ satisfied, we look for solutions (in the phase-plane) such that $y(T_2) = 0$. In terms of these new angular coordinates, this corresponds to the condition $\vartheta(T_2, z) = k\pi$ for some positive integer k.

We have already proved that the angle is a strictly increasing function of the time variable. Moreover, as a consequence of (60) and the symmetry of the orbits with respect to the x-axis, we find that, for any positive integer k and for any $z \in \mathscr{B}$,

$$\vartheta(t; z) > k\pi \quad \text{whenever } t > k\frac{\tau_2(c)}{2} \tag{104}$$

(of course, the above relation holds provided that $t \in [0, T_2]$ and $T_2 > k\tau_2(c)/2$ for $c \in [\beta, 0]$).

Now we are ready to introduce the constant T_2^* which will be defined as follows. Let us fix a positive integer k and set

$$T_2^* = T_2^*(k) := k\frac{\tau_2(\beta)}{2} \tag{105}$$

and fix $T_2 > T_2^*$.

The point $\Phi_1((t'_j, 0)) \in \mathscr{B}_l^-$ lies on the level line Γ_2^0 of the homoclinic trajectory and therefore, $\vartheta(T_2; \Phi_1((t'_j, 0))) < \pi$. On the other hand, $\Phi_1((t''_j, 0)) \in \mathscr{B}_r^-$ lies on the level line Γ_2^β of the closed trajectory of period $\tau_2(\beta)$ and therefore we have $\vartheta(T_2; \Phi_1((t''_j, 0))) > k\pi$. Hence, we can conclude that the arc $\{\Phi_1((s, 0)) : s \in [t'_j, t''_j]\}$, which connect in $\mathscr{B}$ the opposite sides $\mathscr{B}_l^-$ and $\mathscr{B}_r^-$, is mapped by Φ_2 to an arc which crosses at least k times the x-axis. In other words, for each integer $i \in \{1, \ldots, k\}$ there exists one point

$$z_j^i \in \left\{\Phi_1((s, 0)) : s \in \left[t'_j, t''_j\right]\right\} \tag{106}$$

such that $\vartheta(T_2; z_j^i) = i\pi$. The corresponding solution $u(t)$ has precisely i-zeros of the derivative in the interval $]0, T_2]$.

Finally, recalling that the $m \times k$ points $z_j^i \in \mathscr{B}$ are images through Φ_1 of corresponding points in the interval $[a_2, a_1] \times \{0\}$, allows us to conclude with the following claim.

Theorem 9. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function of the form (18), with h satisfying $(*)$. Let $V : [0, T] \longrightarrow \mathbb{R}_0^+$ be a stepwise function as in (19), and such that $V_1 > V_2 > 0$. Then, for every pair of integers (m, k) with $m \geq 2$ and $k \geq 1$, there exist two positive constants T_1^* and T_2^* such that, if $T_1 > T_1^*$ and $T_2 > T_2^*$, the problem (94) has at least $m \times k$ positive solutions on $[0, T]$ (with $T = T_1 + T_2$). More precisely, for each $i = 1, \ldots, k$ there are at least m solutions of (94) with $u'(T_1) < 0$ and such that $u'(t)$ has precisely i-zeros in $[T_1, T]$.*

The solutions that we have produced are only those obtained by shooting from $[a_2, a_1] \times \{0\}$, achieving the set $\mathscr{B}$ at the time T_1 and coming back at the x-axis at the time T. With obvious changes in the argument, we could produce solutions which are in $\mathscr{A}$ at the time T_1. Moreover, we could also start from the segment $[b_1, b_2] \times \{0\}$ and reach the region $\mathscr{B}$ (or, respectively, $\mathscr{A}$) at the time T_1. Therefore, with the same technique, we can produce *four* different classes of $m \times k$ solutions. Finally, we can consider (by suitably modifying the same approach) also the case $V_2 > V_1 > 0$.

A variant of Theorem 9 for problem (93) with ε sufficiently small can be also obtained via the same change of variables as in the proof of Theorem 8.

In order to obtain *positive* solutions for the Dirichlet (two-point) boundary value problem

$$\begin{gathered} -\varepsilon^2 u'' + \mathscr{V}(t) u = f(u) \\ u(0) = u(T) = 0 \end{gathered} \tag{107}$$

or (after a suitable rescaling)

$$\begin{gathered} -u'' + V(t) u = f(u) \\ u(0) = u(T) = 0 \end{gathered} \tag{108}$$

a natural implementation of the shooting approach considered for the periodic and the Neumann problem consists into shooting from the positive y-axis of the phase-plane. More precisely, we consider the initial condition

$$(x(0), y(0)) = w_0 = (0, r), \quad \text{with } r > 0 \tag{109}$$

for system (S) and look for a suitable value of the parameter r such that

$$\Phi(w_0) := (x(T; 0, w_0), y(T; 0, w_0)) \tag{110}$$

satisfies

$$x(T; 0, w_0) = 0, \quad \text{with } x(t) > 0 \ \forall t \in]0, T[. \tag{111}$$

In such a situation, for a weight function with only two steps, we do not obtain multiplicity results (in general). However, one could easily provide a mechanism for multiple solutions by taking a weight function $V(t)$ with three steps defined as follows:

$$V(t) := \begin{cases} V_1 & \text{for } t \in [0, T_1[\\ V_2 & \text{for } t \in [T_1, T_1 + T_2[\\ V_3 & \text{for } t \in [T_1 + T_2, T[\end{cases} \tag{112}$$

with $T = T_1 + T_2 + T_3$ and V_1, V_2, V_3 positive constants with

$$V_2 > \max\{V_1, V_3\}. \tag{113}$$

In fact, under these assumptions, we can find solutions of (108) which are positive on $]0, T[$ and oscillate a certain (prescribed) number of times in the phase-plane around the point $P_2 = (x_{02}, 0)$, during the time interval $[T_1, T_1 + T_2]$. The number of such solutions can be larger than a preassigned number m, provided that $T_2 > T_2^*$ for T_2^* a sufficiently large time (depending on m). An analogous conclusion can be derived for problem (107).

As a consequence of the technique we can also prove the stability of the multiplicity results with respect to small perturbations of the coefficients.

6. Final Remarks

6.1. Sign Changing Solutions. In the previous sections we have focused our attention only to the search of positive solutions. We stress the fact that the same approach works well in order to produce *nodal solutions*, that is solutions with a prescribed number of zeros in a given interval. For the sake of simplicity in the exposition and consistently with the classical case $f(u) = |u|^{p-1}u$, we confine ourselves to the case of an *odd* nonlinear term in the equation. If $f(s) = sh(s)$ is not odd, we can still adapt the same argument with minor efforts.

The assumption (18), with $h(s)$ an *even* function satisfying $(*)$, implies that the phase–portrait of (36) is similar to that of Figure 1 with a mirror symmetry with respect to the y-axis (see Figure 6).

Now we consider again (17) with a periodic stepwise weight function $V(t)$ as in (19), with $V_1 \neq V_2$. Actually, in order to enter in the same situation analyzed in Section 3, we assume

$$V_1 > V_2 > 0. \tag{114}$$

As previously observed, the dynamics associated with system (S) can be described as that of a couple of switching systems of the form (S_1) and (S_2). This time we are interested also in sign changing solutions and therefore we exploit the properties of the trajectories in the negative half-plane, too. If we overlap the energy level lines of the two autonomous systems we have the possibility of applying our abstract result (namely, Theorem 4) to different choices of rectangular regions $\mathscr{A}$ and $\mathscr{B}$ which are determined through the intersections of the level lines of the two systems. An illustrative example of possible choices of topological rectangles is given in Figure 7 where we have put in evidence some sets which are suitable for the application of Theorem 4.

As a consequence, we can find different rectangular regions where chaotic dynamics occur. The corresponding solutions can have different qualitative behaviors on the intervals $]jT, jT + T_1[$ and/or $]jT + T_1, (j+1)T[$, depending on the regions where we apply Theorem 4. For instance, we can find solutions with prescribed nodal properties on the intervals $]jT, jT + T_1[$ and $]jT + T_1, (j+1)T[$, as well as solutions which are positive (or negative) on $]jT, jT+T_1[$ and have a prescribed number of zeros on $]jT + T_1, (j+1)T[$.

For the search of sign changing solutions to the Neumann problems (93) or (94), we suppose (114) and apply a shooting method by selecting initial points on the x-axis of the phase-plane. A possible argument is illustrated in Figure 8.

Finally, for the Dirichlet problem (108), where again we suppose that $V(t)$ is step function like in (19), we can prove the existence of multiple sign changing solutions by shooting from the y-axis in the phase-plane and suitably adapting the argument described above for the Neumann problem.

6.2. Remarks on the Weight Functions. Throughout the paper we have focused our analysis to an equation of the form (17), namely $-u'' + V(t)u = f(u)$. We point out that the same technique works for the case of equation

$$-u'' + Vu = q(t)\, f(u), \tag{115}$$

with $q(\cdot)$ a positive stepwise function. This follows from the elementary observation that the equation $-y'' + ay = bf(y)$, with constant coefficients $a, b > 0$, is equivalent to $-u'' + (a/b)u = f(u)$, for $u(t) := y(b^{-1/2}t)$. Hence, if we take $q(t)$ as a step function on an interval $[0, T_q]$ (with $q(t)$ possibly periodic), we find that (115) turns out to be equivalent to (17). For the Dirichlet problem, existence, multiplicity and stability of positive solutions to equations of the form (115) have been obtained in [79] in the more general PDEs case for $q(t)$ a sign changing weight function. In our setting, we suppose $q(t) > 0$ for every t.

Finally, we observe that a simple adaptation of our approach (as described in Sections 2 and 3 can be applied to (14) provided that both $V(t)$ and $q(t)$ are close to stepwise functions.

Data Availability

No data were used to support this study.

Disclosure

Preliminary results from this paper have been presented at the "Workshop on Nonlinear Partial Differential Equations", June 19-20, 2013, at the Universidad Complutense de Madrid.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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