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Ciclo XXXI

Qualitative aspects of dynamical systems. Periodic solutions, celestial orbits and persistence

Dottorando
Tobia Dondè

Supervisore
Fabio Zanolin

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Contents

Declaration

I hereby declare that, except where specific reference is made to the work of others, the contents of this thesis are original. The present dissertation is my own work and contains nothing which is the outcome of work done under the supervision of Professor Fabio Zanolin in order to achieve the title of Research Doctorate at the University of Udine.

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Abstract

This thesis addresses several modern problems in the framework of dynamical systems by means of different topological techniques. We obtain original results of qualitative type in each field of application, which we supply with rigorous proofs, comments, possible future developments and in some cases with a localised numerical or quantitative analysis.

Three main research topic are explored: the search of periodic solutions for planar ordinary differential equations systems, the existence and characterisation of parabolic solutions for the planar N-centre problem, and the uniform persistence in eco-epidemiological models together with its consequences in terms of coexistence states.

In the first part of the thesis we provide two different existence and multiplicity results for periodic solutions to a general class of planar systems with sign-changing nonlinearity under a one-sided condition of sublinear type. The first result is obtained via the Poincaré-Birkhoff theorem while the second arises from the theory of bend-twist maps and topological horseshoes. The case of subharmonic solutions is investigated too as well as the presence of symbolic dynamics.

In the second part we take into account the planar generalised N-centre problem. A suitable variational approach lays the groundwork for a topological characterisation of the solutions that describes their interaction with the set of centres, allowing for admissible self-intersections in some admissible sense. The existence of scattering and semibounded solutions is proved.

In the third and last part of the dissertation we move in the field of epidemiology and ecological models in general. A common framework is introduced and the tools already present in literature are first employed in an original strategy in order to provide theoretical support to the numerical-only evidence of persistence in a predator-prey model with one diseased population. Later on, persistence is shown to be a sufficient condition for the existence of a periodic solution in a general class of models, and applications are provided.

Introduction

The theory of dynamical systems can claim two gigantic fathers: Henry Poincaré (1854–1912) and Aleksandr Mikhailovich Lyapunov (1857–1918). Both of them were exceptionally prolific mathematicians, who achieved results whose influence reached well beyond their time, and their interests embraced many a field, so that their names can be found in a wide variety of disciplines to this day.

Poincaré introduced the systematic use of topological methods in the study of differential systems, which at his time mainly arose from celestial mechanics and other natural science fields. His abstract results are of such strength that most part of the literature dealing with fixed points and periodic solutions problems still heavily relies on them. We recall among the others the Poincaré-Miranda and the Poincaré-Birkhoff fixed point Theorems, since we will discuss their consequences on a general class of planar differential systems. Poincaré must also be remembered for his emphasis on the importance of studying periodic solutions, which are nowadays a broadly covered topic with countless applications on the side of nonlinear analysis: for our discussion sake we just point out how crucial this concept becomes in ecology, where seasonality is described through periodic functions.

On the other side, Lyapunov is unanimously recognised as the founder of the modern theory of stability, thanks to his famous local results of stability by first approximation. He also introduced some special functionals, decreasing along the trajectories, that are known today as Lyapunov functions. After more than a century Lyapunov functions are still one of the main tools of physicists and engineers in analysing the stability from a global point of view.

Under the influence of Lyapunov, the idea of using suitably chosen real valued functions to "guide" the behaviour of the solutions has found several successful developments, such as Krasnosel'skiĭ's theory of *guiding functions* [Kra68] or Mawhin's theory of *bounding functions and bounding sets* [Maw74]. Among the applications of the concept of Lyapunov functions, their use in showing the attractivity or repulsiveness of some compact sets finds some important place in view of the study of asymptotic dynamics. Moreover, some pointwise estimates have been extended to integral ones, coming to the handy notion of "averaged Lyapunov functions".
Last but not least, one should recall that Poincaré and Lyapunov themselves are the ancestors of the nowaday very popular and pervasive concept of *chaos*, which, according to some authors, should be regarded as one of the three major achievements of the last century, togeteher with Einstein's relativity and quantum physics (see [Maw94, ADD02]). Indeed, thanks to a "fortunate mistake" Poincaré discovered the chaotic dynamics associated with transversal homoclinic or heteroclinic points. On the other hand, Lyapunov gave rise to the widespread theory of Lyapunov exponents, the principal (and as such sometimes abused) technique for chaos detecting in many practical situations.

In our work we discuss some modern aspects of the theory of dynamical systems which are strongly connected with the studies of Poincaré and Lyapunov. More in detail, we benefit from two categories of tools their work provide: fixed point theorems on one side, generalised Lyapunov functions methods on the other. Quite remarkably, these long-aged techniques offer many answers to recent challenges (see Chapters 1 and 3), while recent challenges can originate from very old problems, well-known also at the time of Poincaré and Lyapunov (see Chapter 2). If needed, another proof of the timelessness of mathematical issues.

Part 1

In the first part of the thesis (Chapter 1) we pursue a general result of existence and multiplicity of periodic solutions for the following class of sign-indefinite nonlinear first order planar systems:

$$\begin{cases} x'(t) = h(y(t)) \\ y'(t) = -a_{\lambda,\mu}(t)g(x(t)). \end{cases}$$

The functions $h, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous, satisfy some linear growth conditions in 0 and at least one among these four conditions holds true:

$$h \text{ is bounded on } \mathbb{R}^{\pm}, \qquad g \text{ is bounded on } \mathbb{R}^{\pm}.$$

The peculiar aspect of this framework resides in asking for a single one-sided condition. This case has seldom been addressed in previous literature (see [Bos11, BZ13] and only some specific results are available, hence the interest in studying such broad class of problems. With no additional assumptions, some fixed point theorems can be applied to the Poincaré map Φ associated to the system in order to obtain periodic solutions. Of course, the sign-changing periodic weight function $a_{\lambda,\mu}$ plays a crucial role through its parameters λ and μ, which rule the magnitude of the positive and negative part of the function, respectively.

A first result is reached through the already mentioned Poincaré-Birkhoff Theorem, originally conjectured by Poincaré in 1912. The annular version of the theorem (refer to [Fon16] for a convenient formulation) can be applied to our system to return *two* distinct fixed points for the Poincaré map Φ, which correspond to two different periodic solutions. The intrinsic structure of the theorem is such that in these situations periodic solutions are provided in couples. However, an important assumption that cannot be removed is that Φ must be well-defined on the whole plane, and this is in general not guaranteed.

If instead we rely on a "bend-twist maps" approach [Din07] which has its core in the Poincaré-Miranda theorem, under some additional conditions on the weight function $a_{\lambda,\mu}$ we are able to attain *four* periodic solutions, which as in the Poincaré-Birkhoff case are topologically different. Still, the global continuability of the solutions is required. Eventually, the Stretching Along the Paths (SAP) technique, based on topological horseshoes (see [KY01, PZ02]), provides four periodic solutions in the same framework, without asking the Poincaré map to be globally defined. This can be viewed as a refinement of the previous

approaches since it also supplies more informations on the solutions, namely the distribution of the zeros along the period.
We treat also the case of subharmonic solutions and we briefly discuss how to prove symbolic dynamics from our theorems. The results obtained in this Chapter have been collected in a recently submitted article [DZ19].

Part 2

In the second part of the thesis (Chapters 2 and 3) we analyse two important applications that fit into the legacy of Poincaré and Lyapunov.
The first one (Chapter 2) is inspired by a classic of celestial mechanics, the N-centre problem, strongly related to the N-body problem for which Poincaré himself proved the non integrability in the case $N = 3$. We take into account the planar generalised N-centre problem, which has been object of recent studies [BDP17] regarding the multiplicity and qualitative description of parabolic (zero-energy) solutions.
Our contribution to the topic is aimed at the enrichment of the solutions classes. Indeed, the previous approaches didn't allow self-intersections of the parabolic orbits, thus obtaining solutions that have a restricted interaction with the set of centres, separating it in two disjoint subsets before leaving with prescribed asymptotic directions (scattering solutions). With a notion of admissible self-intersection adopted first in [Cas17] we are able to find and topologically characterise a wider class of scattering parabolic solutions. Our approach also extend to different types of solutions such as semibounded, periodic and chaotic ones, in the spirit of [FT00].
The achievements of this Chapter are incorporated in the work [DP19], which is undergoing the last refinements.

The second application (Chapter 3) is on the ecological side and deals with the concept of persistence, an important notion in cohabitation models and in epidemiology, when evaluating the "invasion condition" for an infectious disease. For an extensive introduction to persistence and the basic reproduction number R_0 we refer to the beginning of the Chapter itself.
After an overview of some known results in literature concerning semidynamical systems and their asymptotic properties we take into account two predator-

prey epidemiological models, studied before only by numerical means [BH13b]. We prove rigorously the uniform persistence of the associated flows on the positive cone, thus entailing the non extinction of all species. Although the models are described by a three-dimensional ODEs system for which the stability analysis doesn't require any approximation tool, it is nonetheless interesting to see how sometimes theory can justify quantitative evidence and also add some informations thanks to a fitting approach. Indeed, in this case the key tool for persistence checking is an "average Lyapunov functions" theorem stated in [Hut84, Fon88] and slightly adapted from the version proposed in [RMB14]. The theorem returns a more general result which is also able to predict a one-dimensional stable manifold, hard to detect by numerical means. A quantitative counterexample to persistence is provided, and a neat stability analysis is carried on in the Appendixes. These results are partly contained in a paper under revision [Don18].
In the second part of the Chapter we link uniform persistence and existence of periodic solutions via some dissipativity arguments and applying the standard Brouwer fixed point Theorem, which however must be coupled with a topological degree result, the mod p Theorem [ZK71, Ste72], in order to non-trivially link fixed points of iterates of the Poincaré map Φ and fixed points of the map itself. We provide a few applications coming from the literature for which we prove uniform persistence and, accordingly, the existence of a periodic solution. A survey paper on fixed points via uniform persistence is in advance state of preparation [Don19].

Related scientific production

[Don18] T. Dondè. *Uniform persistence in a prey-predator model with a diseased predator*, submitted (2018)
[Don19] T. Dondè. *Periodic solutions of ecological models via uniform persistence*, in preparation (2019)
[DP19] T. Dondè and D. Papini. *A rich family of parabolic solutions for the planar N-centre problem*, in preparation (2019)
[DZ19] T. Dondè and F. Zanolin. *Multiple periodic solutions for one-sided sublinear systems: A refinement of the Poincaré-Birkhoff approach*, submitted (2019)

Chapter 1

Existence, multiplicity and chaos for a class of ODEs with sign-changing nonlinearity

The Poincaré-Birkhoff fixed point Theorem deals with a planar homeomorphism Ψ defined on an annular region A, such that Ψ is area-preserving, leaves the boundary of A invariant and rotates the two components of ∂A in opposite directions (twist condition). Under these assumptions, in 1912 Poincaré conjectured (and proved in some particular cases) the existence of at least two fixed points for Ψ, a result known as "Poincaré's last geometric Theorem". A proof for the existence of at least one fixed point (and actually two in a non-degenerate situation) was obtained by Birkhoff in 1913 [Bir13]. In the subsequent years Birkhoff reconsidered the theorem as well as its possible extensions to a more general setting, for instance, removing the assumption of boundary invariance, or proposing some hypotheses of topological nature instead of the area-preserving condition, thus opening a line of research that is still active today (see for example [Car82, Bon12] and references therein). The skepticism of some mathematicians about the correctness of the proof of the second fixed point motivated Brown and Neumann to present in [BN77] a full detailed proof, adapted from Birkhoff's 1913 paper, in order to eliminate previous possible controversial aspects. Another approach for the proof of the second fixed point has been proposed in [Sla93], coupling [Bir13] with a result for removing fixed points of zero index.

In order to express the twist condition in a more precise manner, the statement of the Poincaré-Birkhoff Theorem is usually presented in terms of the lifted map $\tilde{\Psi}$. Let us first introduce some notation. Let $D(R)$ and $D[R]$ be, respectively, the open and the closed disc of center the origin and radius $R > 0$ in $\mathbb{R}^2$ endowed with the Euclidean norm $||\cdot||$. Let also $C_R := \partial D(R)$. Given $0 < r < R$, we denote by A or $A[r,R]$ the closed annulus $A[r,R] := D[R] \setminus D(r)$. Hence the area-preserving (and orientation-preserving) homeomorphism $\Psi : A \to \Psi(A) = A$ is lifted to a map $\tilde{\Psi} : \tilde{A} \to \tilde{A}$, where $\tilde{A} := \mathbb{R} \times [r,R]$ is the covering space of A via the covering projection $\Pi : (\theta, \rho) \mapsto (\rho\cos\theta, \rho\sin\theta)$ and

$$\tilde{\Psi} : (\theta, \rho) \mapsto (\theta + 2\pi\mathscr{J}(\theta,\rho), \mathscr{R}(\theta,\rho)), \tag{1.0.1}$$

with the functions $\mathscr{J}$ and $\mathscr{R}$ being 2π-periodic in the θ-variable. Then, the classical (1912-1913) Poincaré-Birkhoff fixed point Theorem can be stated as follows (see [BN77]).

Theorem 1.0.1. *Let $\Psi : A \to \Psi(A) = A$ be an area preserving homeomorphism such that the following two conditions are satisfied:*

$(PB1)$ $\mathscr{R}(\theta, r) = r, \quad \mathscr{R}(\theta, R) = R, \quad \forall\, \theta \in \mathbb{R};$

$(PB2)$ $\exists\, j \in \mathbb{Z} : \ (\mathscr{J}(\theta, r) - j)(\mathscr{J}(\theta, R) - j) < 0, \quad \forall\, \theta \in \mathbb{R}.$

Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta,\rho) = j$ for $\Pi(\theta,\rho) = z_i$.

We refer to condition $(PB1)$ as to the "boundary invariance" and we call $(PB2)$ the "twist condition". The function $\mathscr{J}$ can be regarded as a rotation number associated with the points (θ, ρ). In the original formulation of the theorem it is $j = 0$, however any integer j can be considered.

The Poincaré-Birkhoff Theorem is a crucial result in the fields of fixed point theory and dynamical systems, as well as in their applications to differential equations. General presentations can be found in [MZ05, MH92, LC11]. There is a large literature on the subject and certain subtle and delicate points related to some controversial extensions of the theorem have been settled only in recent years (see [Reb97, MUn07, LCW10]). In the applications to the study of periodic non-autonomous planar Hamiltonian systems, the map Ψ is often the Poincaré map (or one of its iterates). In this situation the condition of boundary invariance is usually not satisfied, or very difficult to prove: as a consequence, variants of the Poincaré-Birkhoff Theorem in which the hypothesis $(PB1)$ is not required turn out to be quite useful for the applications (see [DR02] for a general discussion on this topic). As a step in this direction we present the next result, following from W.Y. Ding in [Din82].

Theorem 1.0.2. *Let $\Psi : D[R] \to \Psi(D[R]) \subseteq \mathbb{R}^2$ be an area preserving homeomorphism with $\Psi(0) = 0$ and such that the twist condition $(PB2)$ holds. Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$.*

The proof in [Din82] (see also [DZ92, Appendix]) relies on the Jacobowitz version of the Poincaré-Birkhoff Theorem for a pointed topological disk [Jac76, Jac77] which was corrected in [LCW10], since the result is true for strictly star-shaped pointed disks and not valid in general, as shown by a counterexample in the same article. Another (independent) proof of theorem 1.0.2 was obtained by Rebelo in [Reb97]: the Author brought the proof back to that of theorem 1.0.1 and thus to the "safe" version of Brown and Neumann [BN77]. Other versions of the Poincaré-Birkhoff Theorem giving theorem 1.0.2 as a corollary can be found in [Fra88, Fra06, QT05, Mar13] (see also [FSZ12, Introduction] for a general discussion about these delicate aspects). For Poincaré maps associated with Hamiltonian systems there is a much more general version of the theorem due to Fonda and Ureña in [FUn16, FUn17], which will be recalled later in the dissertation with some more details.

In [Din07, Din12], T.R. Ding proposed a variant of the Poincaré-Birkhoff Theorem by introducing the concept of "bend-twist map". Given a continuous map $\Psi : A \to \Psi(A) \subseteq \mathbb{R}^2 \setminus \{0\}$, which admits a lifting $\tilde{\Psi}$ as in (1.0.1), we define

$$\Upsilon(\theta, \rho) := \mathscr{R}(\theta, \rho) - \rho.$$

We call Ψ a *bend-twist map* if it Ψ satisfies the twist condition and Υ changes its sign on a non-contractible Jordan closed curve Γ contained in the set of points in the interior of A where $\mathscr{J} = j$. The original treatment was given in [Din07] for analytic maps, but there are extensions to continuous maps as well [PZ11, PZ13]. Clearly, the bend-twist map condition is difficult to check in practice, due to the lack of information about the curve Γ (which, in the non-analytic case, may not even be a curve). For this reason, one can rely on the following corollary [Din07, Corollary 7.3] which follows itself from the Poincaré-Miranda Theorem (as observed in [PZ11]).

Theorem 1.0.3. *Let $\Psi : A \to \Psi(A) \subseteq \mathbb{R}^2 \setminus \{0\}$ be a continuous map such that the twist condition $(PB2)$ holds. Suppose that there are two disjoint arcs α, β contained in A, connecting the inner with the outer boundary of the annulus and such that*

$(BT1)$ $\Upsilon > 0$ *on* α *and* $\Upsilon < 0$ *on* β.

Then Ψ has at least two fixed points z_1, z_2 in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$.

A simple variant of the above theorem considers $2n$ pairwise disjoint simple arcs α_i and β_i (for $i = 1, \dots, n$) contained in A and connecting the inner with the outer boundary. We label these arcs in cyclic order so that each β_i is between α_i and α_{i+1} and each α_i is between β_{i-1} and β_i (with $\alpha_{n+1} = \alpha_1$ and $\beta_0 = \beta_n$) and suppose that

(BTn) $\Upsilon > 0$ on α_i and $\Upsilon < 0$ on β_i, for all $i = 1, \dots, n$.

Then Ψ has at least $2n$ fixed points z_i in the interior of A and $\mathscr{J}(\theta, \rho) = j$ for $\Pi(\theta, \rho) = z_i$. These results also apply in the case of a topological annulus (namely, a compact planar set homeomorphic to A) and do not require that Ψ is area-preserving and also the assumption of Ψ being a homeomorphism is not required, as continuity is enough. Moreover, since the fixed points are obtained in regions with index ± 1, the results are robust with respect to small (continuous) perturbations of the map Ψ.

A special case in which condition $(BT1)$ holds is when $\Psi(\alpha) \in D(r)$ and $\Psi(\beta) \in \mathbb{R}^2 \setminus D[R]$, namely, the annulus A, under the action of the map Ψ, is not only twisted, but also strongly stretched, in the sense that there is a portion of the annulus around the curve α which is pulled inward near the origin inside the disc $D(r)$, while there is a portion of the annulus around the curve β which is pushed outside the disc $D[R]$. This special situation where a strong bend and twist occur is reminiscent of the geometry of the Smale horseshoe maps [Sma67, Mos73] and, indeed, we will show how to enter in a variant of the theory of "topological horseshoes" in the sense of Kennedy and Yorke [KY01]. To this aim, we recall a few definitions which are useful for the present setting. By a *topological rectangle* we mean a subset $\mathcal{R}$ of the plane which is homeomorphic to the unit square. Given an arbitrary topological rectangle $\mathcal{R}$ we can define an orientation, by selecting two disjoint compact arcs on its boundary. The union of these arcs is denoted by $\mathcal{R}^-$ and the pair $\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ is called an *oriented rectangle*. Usually the two components of $\mathcal{R}^-$ are labelled as the left and the right sides of $\widehat{\mathcal{R}}$. Given two oriented rectangles $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$, a continuous map Ψ and a compact set $H \subseteq \mathrm{dom}(\Psi) \cap \mathcal{A}$, the notation

$$(H, \Psi) : \widehat{\mathcal{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{B}}$$

means that the following "stretching along the paths" (SAP) property is satisfied: *any path γ, contained in $\mathcal{A}$ and joining the opposite sides of $\mathcal{A}^-$, contains a sub-path σ in H such that the image of σ through Ψ is a path contained in $\mathcal{B}$ which connects the opposite sides of $\mathcal{B}^-$.* We also write $\Psi : \widehat{\mathcal{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{B}}$ when $H = \mathcal{A}$. By a path γ we mean a continuous map defined on a compact interval. When, loosely speaking, we say that a path is contained in a given set we actually refer to its image $\bar{\gamma}$. Sometimes it will be useful to consider a relation of the form

$$\Psi : \widehat{\mathcal{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow^{k} \widehat{\mathcal{B}},$$

for $k \geq 2$ a positive integer, which means that there are at least k compact subsets $H_1, \dots, H_k$ of $\mathcal{A}$ such that $(H_i, \Psi) : \widehat{\mathcal{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{B}}$ for all $i = 1, \dots, k$. From the results in [PZ04a, PZ04b] we have that Ψ has a fixed point in H whenever $(H, \Psi) : \widehat{\mathcal{R}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{R}}$. If for a rectangle $\mathcal{R}$ we have that $\Psi : \widehat{\mathcal{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow^{k} \widehat{\mathcal{B}}$, for $k \geq 2$, then Ψ has at least k fixed points in $\mathcal{R}$. In this latter situation, one can also prove the presence of chaotic-like dynamics of coin-tossing type (this will be briefly discussed later).

The aim of this Chapter is to analyse, under these premises, a class of planar system with periodic coefficients of the form

$$\begin{cases} x' = h(y) \\ y' = -q(t)g(x)\,, \end{cases}$$

which includes the second order scalar equation of Duffing type

$$x'' + q(t)g(x) = 0.$$

The prototypical nonlinearity we consider is a function which changes sign at zero and is bounded only on one-side, such as $g(x) = e^x - 1$. We do not assume that the weight function $q(t)$ is of constant sign, but, for simplicity, we suppose that $q(\cdot)$ has a positive hump followed by a negative one. We prove the presence of periodic solutions coming in pairs (Theorem 1.1.1 in Section 1.1, following the Poincaré-Birkhoff Theorem) or coming in quadruplets (Theorem 1.1.2 in Section 1.1, following bend-twist maps and SAP techniques), the latter depending on the intensity of the negative part of $q(\cdot)$. To this purpose, we shall express the weight function as

$$q(t) = a_{\lambda,\mu}(t) := \lambda a^+(t) - \mu a^-(t)\,, \qquad \lambda, \mu > 0\,,$$

being $a(\cdot)$ a periodic sign changing function.

The discussion unwinds as follows. In Section 1.1 we present our main results (Theorem 1.1.1 and Theorem 1.1.2) for the existence and multiplicity of periodic solutions. In Section 1.2 we provide simplified proofs in the special case of a stepwise weight function: this allows us to highlight the geometric structure underlying the theorems, providing numerical examples and visual interpretations of the tools used. Furthermore, the regions in which fixed points occur via the bend-twist maps and topological horseshoes approaches are explicitly described, while in the general proof of Section 1.3 we are able to determine only the quadrant in which the (chaotic) invariant sets are located. Another advantage of considering this particular framework lies on the fact that a stepwise weight produces a switched system made by two autonomous equations and therefore, in this case, some threshold constants for λ and μ can be explicitly computed. In Section 1.4 we show how to extend our main results to the case of subharmonic solutions. Section 1.5 concludes the paper with a list of some possible applications.

1.1 Framework and results

This Chapter deals with the existence and multiplicity of periodic solutions of sign-indefinite nonlinear first order planar systems of the form

$$\begin{cases} x' = h(y) \\ y' = -a_{\lambda,\mu}(t)g(x). \end{cases} \tag{1.1.1}$$

Throughout the discussion we suppose that $h, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions satisfying the following assumptions:

(C_0)
$$\begin{aligned} &h(0) = 0,\ h(y)y > 0 \text{ for all } y \neq 0 \\ &g(0) = 0,\ g(x)x > 0 \text{ for all } x \neq 0 \\ &h_0 := \liminf_{|y|\to 0} \frac{h(y)}{y} > 0, \quad g_0 := \liminf_{|x|\to 0} \frac{g(x)}{x} > 0. \end{aligned}$$

We will also suppose that *at least one* of the following conditions holds:

(h_-) h is bounded on $\mathbb{R}^-$, (h_+) h is bounded on $\mathbb{R}^+$,

(g_-) g is bounded on $\mathbb{R}^-$, (g_+) g is bounded on $\mathbb{R}^+$.

We also set

$$\mathscr{G}(x) := \int_0^x g(\xi)d\xi, \quad \mathscr{H}(y) := \int_0^y h(\xi)d\xi.$$

Concerning the weight function $a_{\lambda,\mu}(t)$, it is defined starting from a T-periodic sign-changing map $a : \mathbb{R} \to \mathbb{R}$ by setting

$$a_{\lambda,\mu}(t) = \lambda a^+(t) - \mu a^-(t), \qquad \lambda, \mu > 0.$$

As usual, $a^+ := (a + |a|)/2$ is the positive part of $a(\cdot)$ and $a^- := a^+ - a$. Given an interval I, by $a \succ 0$ on I we mean that $a(t) \geq 0$ for almost every $t \in I$ with

$a > 0$ on a subset of I of positive measure. Similarly, $a \prec 0$ on I means that $-a \succ 0$ on I. For sake of simplicity, we suppose that in a period the weight function $a(t)$ has one positive hump followed by one negative hump. The case in which $a(t)$ has several (but finite) changes of sign in a period could be dealt with as well, but will be not treated here.

We hereby note that there is a good amount of recent literature for existence and multiplicity of solutions for "sign-indefinite weight" problems. In hypothesis of sublinearity at ∞ in [BFZ18] existence and multiplicity of positive solutions are proven via the coincidence degree theory; in [BF18] positive subharmonic solutions are found through the Poincaré-Birkhoff approach; [FS18] treats instead a Neumann problem (which however is naturally related to periodic solutions) finding nodal solutions with a shooting technique.

Therefore, we suppose that there are t_0 and $T_1 \in]0, T[$ such that

$$(a^*) \qquad a \succ 0 \quad \text{on } [t_0, t_0 + T_1] \quad \text{and} \quad a \prec 0 \quad \text{on } [t_0 + T_1, t_0 + T].$$

Actually, due to the T-periodicity of the weight function, it will be not restrictive to take $t_0 = 0$ and we will assume it for the rest of the paper. Concerning the regularity of the weight function, we suppose that $a(\cdot)$ is continuous (or piecewise-continuous), although from the proofs it will be clear that all the results are still valid for $a^+ \in L^\infty([0, T_1])$ and $a^- \in L^1([T_1, T])$.

The assumptions on h and g allow to consider a broad class of planar systems. We will provide a list of specific applications in Section 1.5. For the reader's convenience, we observe that, for the results to come, we have in mind the model given by the scalar second order equation

$$x'' + a_{\lambda,\mu}(t)(e^x - 1) = 0, \tag{1.1.2}$$

which can be equivalently written as system (1.1.1) with $h(y) = y$ and $g(x) = e^x - 1$ (see Section 1.2).

We denote by Φ the Poincaré map associated with system (1.1.1). Recall that

$$\Phi(z) = \Phi_0^T(z) := (x(T; 0, z), y(T; 0, z)),$$

where $(x(\cdot\,; s, z), y(\cdot\,; s, z))$ is the solution of (1.1.1) satisfying the initial condition $z = (x(s), y(s))$. Since system (1.1.1) has a Hamiltonian structure of the form

$$\begin{cases} x' = \dfrac{\partial \mathbf{H}}{\partial y}(t,x,y) \\ y' = -\dfrac{\partial \mathbf{H}}{\partial x}(t,x,y) \end{cases}$$

for $\mathbf{H}(t,x,y) = a_{\lambda,\mu}(t)\mathscr{G}(x) + \mathscr{H}(y)$, the associated Poincaré map is an area-preserving homeomorphism, defined on a open set $\Omega := \mathrm{dom}\Phi \subseteq \mathbb{R}^2$, with $(0,0) \in \Omega$. Thus a possible method to prove the existence (and multiplicity) of T-periodic solutions can be based on the Poincaré-Birkhoff "twist" fixed point Theorem (refer to [Fon16, Theorem 10.6.1] for a suitable statement in this framework). A standard way to apply this result is to find a suitable annulus around the origin with radii $0 < r_0 < R_0$ such that for some $\mathfrak{a} < \mathfrak{b}$ the twist condition

$$(TC) \qquad \begin{cases} \mathrm{rot}_z(T) > \mathfrak{b}, & \forall\, z \text{ with } ||z|| = r_0 \\ \mathrm{rot}_z(T) < \mathfrak{a}, & \forall\, z \text{ with } ||z|| = R_0 \end{cases}$$

holds, where $\mathrm{rot}_z(T)$ is the rotation number on the interval $[0,T]$ associated with the initial point $z \in \mathbb{R}^2 \setminus \{(0,0)\}$. We recall a possible definition of rot_z for equation (1.1.1), given by the integral formula

$$\mathrm{rot}_z(t_1,t_2) := \frac{1}{2\pi}\int_{t_1}^{t_2} \frac{y(t)h(y(t)) + a_{\lambda,\mu}(t)x(t)g(x(t))}{x^2(t)+y^2(t)}\,dt, \tag{1.1.3}$$

where $(x(t),y(t))$ is the solution of (1.1.1) with $(x(t_1),y(t_1)) = z \neq (0,0)$. For simplicity in the notation, we set

$$\mathrm{rot}_z(T) := \mathrm{rot}_z(0,T).$$

Notice that, due to the assumptions $h(s)s > 0$ and $g(s)s > 0$ for $s \neq 0$, it is convenient to use a formula like (1.1.3) in which the angular displacement is positive when the rotations around the origin are performed in the clockwise sense.

Under these assumptions, the Poincaré-Birkhoff Theorem in the version of [Reb97, Corollary 2] guarantees that for each integer $j \in [\mathfrak{a},\mathfrak{b}]$ there exist *at least two* T-periodic solutions of system (1.1.1), having j as associated rotation number. In virtue of the first condition in (C_0), it turns out that these solutions have precisely $2j$ simple transversal crossings with the y-axis in the interval $[0, T[$ (see, for instance, [BZ13, Theorem 5.1], [MRZ02, Theorem A]). Equivalently, for such periodic solution $(x(t), y(t))$ the x component has precisely $2j$ simple zeros in the interval $[0, T[$.

If we look for mT-periodic solutions, we just consider the m-th iterate of the Poincaré map

$$\Phi^{(m)} = \Phi_0^{mT}$$

and assume the twist condition

$$(TC_m) \qquad \begin{cases} \mathrm{rot}_z(mT) > \mathfrak{b}, & \forall\, z \text{ with } ||z|| = r_0 \\ \mathrm{rot}_z(mT) < \mathfrak{a}, & \forall\, z \text{ with } ||z|| = R_0\,. \end{cases}$$

From this point of view, the Poincaré-Birkhoff is a powerful tool to prove the existence of subharmonic solutions having mT as minimal period. Indeed, if (TC_m) is satisfied with $\mathfrak{a} \leq j \leq \mathfrak{b}$, then what we find are mT-periodic solutions $(x(t), y(t))$ with x having exactly $2j$ simple zeros in $[0, mT[$. In addition, if j and m are relatively prime integers, these solutions cannot be ℓT-periodic for some $\ell = 1, \dots, m-1$. In particular, if (x, y) is one of these mT-periodic solutions, $j = 1$ or $j \geq 2$ is relatively prime with m and T is the minimal period of the weight function $a_{\lambda,\mu}(t)$, then mT will be the minimal period of (x, y).

Clearly, in order to apply this approach the Poincaré map must be defined on the closed disc $D[R_0]$ of center the origin and radius R_0, that is $D[R_0] \subseteq \Omega$. Unfortunately, in general the (forward) global existence of solutions for the initial value problems is not guaranteed. A classical counterexample can be found in [CU67] for the superlinear equation $x'' + q(t)x^{2n+1} = 0$ with $n \geq 1$ where, even for a positive weight $q(t)$, the global existence of the solutions may fail. A typical feature of this class of counterexamples is that solutions presenting a blow-up at some time β^- will make infinitely many winds around the origin as $t \to \beta^-$. It is possible to overcome these difficulties by prescribing the rotation number for large solutions and using some truncation argument on the nonlinearity, as shown in [Har77, FS16]. In our case the boundedness assumption at infinity given by one among $(g^{\pm}), ((h^{\pm})$, prevents such highly oscillatory phenomenon and guarantees the continuability on $[0, T_1]$. The situation is even more complicated in the time intervals where the weight function is negative ([BG71, But76]): unless we impose some growth restrictions on the vector field in (1.1.1), for example that it has at most linear growth at infinity, in general we cannot prevent blow-up phenomena. Another possibility to avoid blow-up is to assume that both (h_-) and (h_+) hold (or, alternatively, both (g_-) and (g_+)): in fact, in this case x' will be bounded, which implies that $x(t)$ is bounded in any compact time-interval and thus, from the second equation in (1.1.1), y' is bounded on compact intervals, too.

With these premises, the following result holds.

Theorem 1.1.1. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous functions satisfying (C_0) and at least one between the four conditions $(h_\pm)$ and $(g_\pm)$. Assume, moreover, the global continuability for the solutions of* (1.1.1)*. Then, for each positive integer k there exists $\Lambda_k > 0$ such that for each $\lambda > \Lambda_k$ and $j = 1, \dots, k$, the system* (1.1.1) *has at least two T-periodic solutions $(x(t), y(t))$ with $x(t)$ having exactly $2j$-zeros in the interval $[0, T[$.*

Notice that in the above result we do not require any condition on the parameter $\mu > 0$. On the other hand, we have to assume the global continuability of the solutions, which in general is not guaranteed. Alternatively, we can exploit the fact that the solutions are globally defined in the interval $[0, T_1]$ where $a \succ 0$ and, instead of asking for the global continuability on $[0, T]$, assume that μ is small enough. Quite the opposite, for the next result we do not require the Poincaré map to be defined on the whole plane, although now the parameter μ plays a crucial role and must be large instead.

Theorem 1.1.2. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous functions satisfying (C_0) and at least one between the four conditions $(h_\pm)$ and $(g_\pm)$. Then, for each positive integer k there exists $\Lambda_k > 0$ such that for each $\lambda > \Lambda_k$ there exists $\mu^* = \mu^*(\lambda)$ such that for each $\mu > \mu^*$ and $j = 1, \dots, k$, the system* (1.1.1) *has at least four T-periodic solutions $(x(t), y(t))$ with $x(t)$ having exactly $2j$-zeros in the interval $[0, T[$.*

The proofs of Theorem 1.1.1 and Theorem 1.1.2 are given in Section 1.3. In the case of Theorem 1.1.2 we will also show how the $2j$ zeros of $x(t)$ are distributed between the intervals $]0,T_1[$ and $]T_1,T[$. Furthermore, we detect the presence of "complex dynamics", in the sense that we can prove the existence of four compact invariant sets where the Poincaré map Φ is semi-conjugate to the Bernoulli shift automorphism on $\ell = \ell_\mu \geq 2$ symbols. A particular feature of Theorem 1.1.2 lies in the fact that such result is robust with respect to small perturbations. In particular, it applies to a perturbed Hamiltonian system of the form

$$\begin{cases} x' = \dfrac{\partial \mathbf{H}}{\partial y}(t,x,y) + F_1(t,x,y,\varepsilon) \\ y' = -\dfrac{\partial \mathbf{H}}{\partial x}(t,x,y) + F_2(t,x,y,\varepsilon) \end{cases} \tag{1.1.4}$$

with $F_1, F_2 \to 0$ as $\varepsilon \to 0$, uniformly in t, and for (x,y) on compact sets. Observe that system (1.1.4) has not necessarily a Hamiltonian structure and therefore it is no more guaranteed that the associated Poincaré map is area-preserving.

In Section 1.4, versions of Theorem 1.1.1 and Theorem 1.1.2 for subharmonic solutions are given. Moreover, in the setting of Theorem 1.1.2, we show that any m-periodic sequence on ℓ symbols can be obtained by a m-periodic point of Φ in each of such compact invariant sets (see Theorem 1.4.2). Figure 1.1 gives evidence of an abundance of subharmonic solutions to a system in the class (1.1.1).

1.2 The stepwise weight case

We focus on the particular case in which $h(y) = y$ and $g \colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function such that

$$g(0) = 0 < g_0, \;\; g(x)x > 0 \;\; \forall\, x \neq 0, \text{ with } g \text{ bounded on } \mathbb{R}^-,$$

so that (C_0) is satisfied along with (g_-). A possible choice could be $g(x) = e^x - 1$, but we stress that we do not ask for g to be unbounded on $\mathbb{R}^+$.

The second order ordinary differential equation originating from (1.1.1) reads

$$x'' + a_{\lambda,\mu}(t) g(x) = 0. \tag{1.2.1}$$

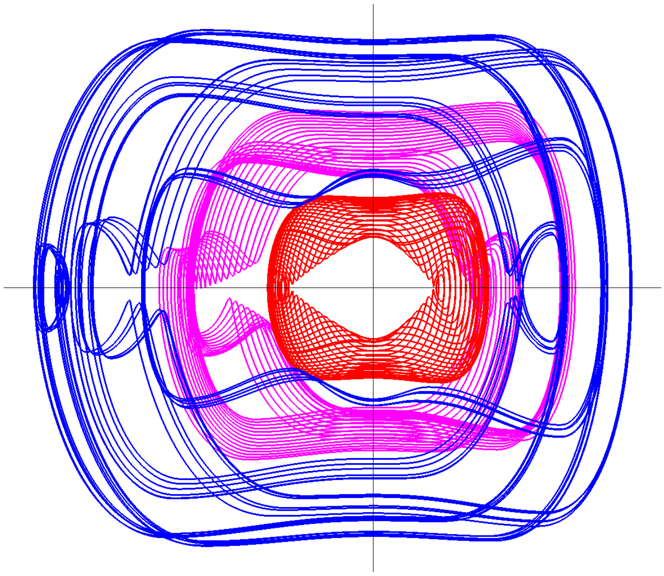

Figure 1.1: A numerical simulation for system (1.1.1). The example is obtained for $h(y) = y$, $g(x) = e^x - 1$ and the weight function $a_{\lambda,\mu}$ where $a(t) = \sin(2\pi t)$, $\lambda = 10$ and $\mu = 2$. The phase-plane portrait is shown for the initial points $(0.4, 0)$, $(0.1, 0.2)$ and $(0.5, 0)$.

In order to illustrate quantitatively the main ideas of the proof we choose a stepwise T-periodic function $a(\cdot)$ which takes value $a(t) = 1$ on an interval of length T_1 and value $a(t) = -1$ on a subsequent interval of length $T_2 = T - T_1$, so that $a_{\lambda,\mu}$ is defined as

$$a_{\lambda,\mu}(t) = \begin{cases} \lambda & \text{for } \ t \in [0, T_1[\\ -\mu & \text{for } \ t \in [T_1, T_1 + T_2[\end{cases} \qquad T_1 + T_2 = T. \tag{1.2.2}$$

With this particular choice of $a(t)$, the planar system associated with (1.2.1) turns out to be a periodic switched system [Bac14]. Such kind of systems are widely studied in control theory.

For our analysis we first take into account the interval of positivity, where

(1.1.1) becomes

$$\begin{cases} x' = y \\ y' = -\lambda g(x). \end{cases} \tag{1.2.3}$$

For this system the origin is a local center, which is global if $\mathscr{G}(x) \to +\infty$ as $x \to \pm\infty$, where $\mathscr{G}(x)$ is the primitive of $g(x)$ such that $\mathscr{G}(0) = 0$. The associated energy function is given by

$$E_1(x, y) := \frac{1}{2}y^2 + \lambda\mathscr{G}(x).$$

For any constant c with $0 < c < \min\{\mathscr{G}(-\infty), \mathscr{G}(+\infty)\}$, the level line of (1.2.3) of positive energy λc is a closed orbit Γ which intersects the x-axis in the phase-plane at two points $(x_-, 0)$ and $(x_+, 0)$ such that

$$x_- < 0 < x_+, \quad \text{and} \quad c = \mathscr{G}(x_-) = \mathscr{G}(x_+) > 0.$$

We call $\tau(c)$ the period of Γ, which is given by

$$\tau(c) = \tau^+(c) + \tau^-(c),$$

where

$$\tau^+(c) := \sqrt{\frac{2}{\lambda}} \int_0^{x_+} \frac{d\xi}{\sqrt{(c - \mathscr{G}(\xi))}}, \quad \tau^-(c) := \sqrt{\frac{2}{\lambda}} \int_{x_-}^0 \frac{d\xi}{\sqrt{(c - \mathscr{G}(\xi))}}$$

The maps $c \mapsto \tau^\pm(c)$ are continuous. To proceed with our discussion, we suppose that $\mathscr{G}(-\infty) \leq \mathscr{G}(+\infty)$ (the other case can be treated symmetrically). Then $\tau^-(c) \to +\infty$ as $c \to \mathscr{G}(-\infty)$, following from the fact that $g(x)/x$ goes to zero as $x \to -\infty$, see [Opi61]. We can couple this result with an estimate near the origin

$$\limsup_{c \to 0^+} \tau(c) \leq 2\pi/\sqrt{\lambda g_0}$$

which follows from classical and elementary arguments.

Proposition 1.2.1. *For each* $\lambda > 0$, *the time-mapping* τ *associated with system* (1.2.3) *is continuous and its range includes the interval* $]2\pi/\sqrt{\lambda g_0}, +\infty[$.

Showing the monotonicity of the whole time-map $\tau(c)$ is in general a difficult task. However, for the exponential case $g(x) = e^x - 1$ this has been proved in [CW86] (see also [Chi87]).

On the interval of negativity of $a_{\lambda,\mu}(t)$, system (1.1.1) becomes

$$\begin{cases} x' = y \\ y' = \mu g(x), \end{cases} \tag{1.2.4}$$

with $g(x)$ as above. For this system the origin is a global saddle with unbounded stable and unstable manifolds, contained in the zero level set of the energy

$$E_2(x, y) := \frac{1}{2}y^2 - \mu\mathscr{G}(x).$$

In the following, given a point $P \in \mathbb{R}^2$, we denote by $\gamma^{\pm}(P)$ and $\gamma(P)$ respectively the positive/negative semiorbit and the orbit for the point P with respect to the (local) dynamical system associated with (1.2.4).

If we start from a point $(0, y_0)$ with $y_0 > 0$ we can explicitly evaluate the blow-up time as follows. First of all we compute the time needed to reach the level $x = \kappa > 0$ along the trajectory of (1.2.4), which is the curve of fixed energy $E_2(x, y) = E_2(0, y_0)$ with $y > 0$. We have

$$y = x' = \sqrt{y_0^2 + 2\mu\mathscr{G}(x)}$$

from which we deduce

$$t = \int_0^{\kappa} \frac{dx}{\sqrt{y_0^2 + 2\mu\mathscr{G}(x)}}.$$

Therefore, the blow-up time is given by

$$T(y_0) = \int_0^{+\infty} \frac{dx}{\sqrt{y_0^2 + 2\mu\mathscr{G}(x)}}.$$

Standard theory guarantees that if the Keller - Osserman condition

$$\int^{+\infty} \frac{dx}{\sqrt{\mathscr{G}(x)}} < +\infty \tag{1.2.5}$$

holds, then the blow-up time is always finite and $T(y_0) \searrow 0$ for $y_0 \nearrow +\infty$. On the other hand, $T(y_0) \nearrow +\infty$ for $y_0 \searrow 0^+$. Hence there exists $\bar{y} > 0$ such that $T(y_0) > T_2$ for $y_0 \in]0, \bar{y}[$ and hence for such points there is no blow-up in $[T_1, T]$.

If we start with null derivative, i.e. from a point $(x_0, 0)$, similar calculations return

$$t = \int_{x_0}^{\kappa} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x_0))}}$$

and, since $\mathscr{G}(x) - \mathscr{G}(x_0) \sim g(x_0)(x - x_0)$ for $|x - x_0| \ll 1$, the improper integral at x_0 is finite. Therefore, the blow-up time is given by

$$T(x_0) = \int_{x_0}^{+\infty} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x_0))}}.$$

If (1.2.5) is satisfied, then the blow-up time is always finite. Moreover, $T(x_0) \to +\infty$ as $x_0 \to 0^+$. A similar but more refined result can be found in [PZ00, Lemma 3].

Now we describe how to obtain Theorem 1.1.1 and Theorem 1.1.2 for system

$$\begin{cases} x' = y \\ y' = -a_{\lambda,\mu}(t) g(x) \end{cases} \tag{1.2.6}$$

in the special case of a T-periodic stepwise function as in (1.2.2). As we already observed, equation (1.2.6) is a periodic switched system and therefore its associated Poincaré map Φ on the interval $[0, T]$ splits as

$$\Phi = \Phi_2 \circ \Phi_1$$

where Φ_1 is the Poincaré map on the interval $[0, T_1]$ associated with system (1.2.3) and Φ_2 is the Poincaré map on the interval $[0, T_2]$ associated with system (1.2.4).

(**I**) *Proof of Theorem 1.1.1 for a stepwise weight.* We start by selecting a closed orbit Γ^0 of (1.2.3) near the origin, at a level energy λc_0, and fix λ sufficiently large, say $\lambda > \Lambda_k$, so that in view of Proposition 1.2.1

$$\tau(c_0) < \frac{T_1}{k+1}. \tag{1.2.7}$$

Next, for the given (fixed) λ, we consider a second energy level λc_1 with $c_1 > c_0$ such that

$$\tau^{-}(c_1) > 2T_2 \tag{1.2.8}$$

and denote by Γ^1 the corresponding closed orbit. Let also

$$\mathcal{A} := \{(x, y) : 2\lambda c_0 \leq E_1(x, y) \leq 2\lambda c_1\}$$

be the planar annular region enclosed between Γ^0 and Γ^1. If we assume that the Poincaré map Φ_2 is defined on $\mathcal{A}$, then the complete Poincaré map Φ associated with system (1.2.6) is a well defined area-preserving homeomorphism of the annulus $\mathcal{A}$ onto its image $\Phi(\mathcal{A}) = \Phi_2(\mathcal{A})$: in fact the annulus is invariant under the action of Φ_1.

During the time interval $[0, T_1]$, each point $z \in \Gamma^0$ performs $\lfloor T_1/\tau(c_0)\rfloor$ complete turns around the origin in the clockwise sense. This implies that

$$\mathrm{rot}_z(0, T_1) \geq \left\lfloor \frac{T_1}{\tau(c_0)} \right\rfloor, \quad \forall\, z \in \Gamma^0 .$$

On the other hand, from [BZ13, Lemma 3.1] we know that

$$\mathrm{rot}_z(T_1, T) = \mathrm{rot}_z(0, T_2) > -\frac{1}{2}, \quad \forall\, z \neq (0, 0) .$$

We conclude that

$$\mathrm{rot}_z(T) > k, \quad \forall\, z \in \Gamma^0 .$$

During the time interval $[0, T_1]$, each point $z \in \Gamma^1$ is unable to complete a full revolution around the origin, because the time needed to cross either the second or the third quadrant is larger than T_1. Using this information in connection to the fact that the first and the third quadrants are positively invariant for the flow associated with (1.2.4), we find that

$$\mathrm{rot}_z(T) < 1, \quad \forall\, z \in \Gamma^1 .$$

The application of the Poincaré-Birkhoff fixed point Theorem guarantees for each $j = 1, \ldots, k$ the existence of at least two fixed points $u_j = (u_x^j, u_y^j)$,

$v_j = (v_x^j, v_y^j)$ of the Poincaré map, with u_j, v_j in the interior of $\mathcal{A}$ and such that $\mathrm{rot}_{u_j}(T) = \mathrm{rot}_{v_j}(T) = j$. This in turns implies the existence of at least two T-periodic solutions of equation (1.2.1) with $x(\cdot)$ having exactly $2j$-zeros in the interval $[0, T]$. □

In this manner, we have proved Theorem 1.1.1 for system (1.2.6) in the special case of a stepwise weight function $a_{\lambda,\mu}$ as in (1.2.2). Notice that no assumption on $\mu > 0$ is required. On the other hand, we have to suppose that Φ_2 is globally defined on $\mathcal{A}$.

Remark 1.2.1. From (1.2.7) and the formulas for the period τ it is clear that assuming T_1 fixed and λ large is equivalent to suppose λ fixed and T_1 large. This also follows from general considerations concerning the fact that equation $x'' + \lambda g(x) = 0$ is equivalent to $u'' + \varepsilon^2 \lambda g(u) = 0$ for $u(\xi) := x(\varepsilon\xi)$.

(**II**) *An intermediate step.* We show how to improve the previous result if we add the condition that μ is sufficiently large. First of all, we take Γ^0 and Γ^1 as before and $\lambda > \Lambda_k$ in order to produce the desired twist for Φ at the boundary of $\mathcal{A}$. Then we observe that the derivative of the energy E_1 along the trajectories of system (1.2.4) is given by $(\lambda + \mu) y g(x)$, so it increases on the first and the third quadrant and decreases on the second and the fourth. Hence, if μ is sufficiently large we can find four arcs $\varphi_i \subseteq \mathcal{A}$, each one in the open i-th quadrant, joining Γ^0 and Γ^1 and such that $\Phi_2(\varphi_i)$ is outside the region bounded by Γ^1 for $i = 1, 3$ and $\Phi_2(\varphi_i)$ is inside the region bounded by Γ^0 for $i = 2, 4$. The corresponding position of $\mathcal{A}$ and $\Phi_2(\mathcal{A})$ is illustrated in Figure 1.2.

At this point we enter in the setting of bend-twist maps. The arcs $\Phi_1^{-1}(\varphi_i)$ divide $\mathcal{A}$ into four regions, homeomorphic to rectangles. The boundary of each of these regions can be split into two opposite sides contained in Γ^0 and Γ^1 and two other opposite sides given by $\Phi_1^{-1}(\varphi_i)$ and $\Phi_1^{-1}(\varphi_{i+1})$ (mod 4). On Γ^0 and Γ^1 we have the previously proved twist condition on the rotation numbers, while on the other two sides we have $E_1(\Phi(P)) > E_1(P)$ for $P \in \Phi_1^{-1}(\varphi_i)$ with $i = 1, 3$ and $E_1(\Phi(P)) < E_1(P)$ for $P \in \Phi_1^{-1}(\varphi_i)$ with $i = 2, 4$. Thus, using the Poincaré-Miranda Theorem, we obtain the existence of at least one fixed point of the global Poincaré map Φ in the interior of each of these regions. In this manner, under an additional hypothesis of the form $\mu > \mu^*(\lambda)$, we improve Theorem 1.1.1 (for system (1.2.6) and again in the special case of a

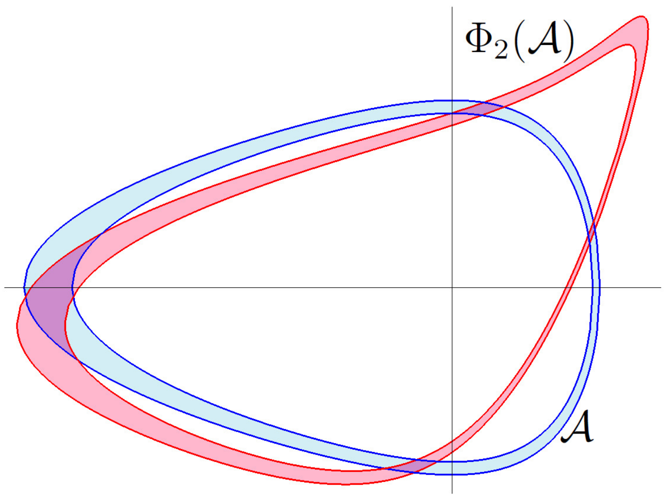

Figure 1.2: A possible configuration of $\mathcal{A}$ and $\Phi_2(\mathcal{A})$. The example is obtained for $g(x) = e^x - 1$, $\lambda = \mu = 0.1$ and $T_2 = 1$. The inner and outer boundary Γ^0 and Γ^1 of the annulus $\mathcal{A}$ are the energy level lines $E_1(x, y) = E_1(2, 0)$ and $E_1(x, y) = E_1(2.1, 0)$. To produce this geometry the value of T_1 is not relevant because the annulus is invariant for system (1.2.3). Since $\tau(c_0) < \tau(c_1)$, to have the desired twist condition we need to assume T_1 large enough.

stepwise weight), finding at least four solutions with a given rotation number j for $j = 1, \dots, k$. On the other hand, we still suppose that Φ_2 is globally defined on $\mathcal{A}$. The version of the bend-twist map theorem that we apply here is robust for small perturbations of the Poincaré map, therefore the result holds also for some non-Hamiltonian systems whose vector field is close to that of (1.2.1).

(**III**) *Proof of Theorem 1.1.2 for a stepwise weight.* We repeat the same construction of (**I**) and choose Γ^0, $\lambda > \Lambda_k$ according to (1.2.7) and Γ^1 so that (1.2.8) is satisfied. Consistently with the previously introduced notation, we take

$$x^1_- < x^0_- < 0 < x^0_+ < x^1_+, \quad \text{with } \mathscr{G}(x^i_-) = \mathscr{G}(x^i_+) = c_i, \; i = 0, 1.$$

Notice that the closed curves Γ^i intersect the coordinate axes at the points $(x^i_\pm, 0)$ and $(0, \pm\sqrt{2\lambda c_i})$. Next we choose $x^\mu_\pm$ and y_0 with

$$x^0_- < x^\mu_- < 0 < x^\mu_+ < x^0_+\,, \quad \text{and } 0 < y_0 < \sqrt{2\lambda c_0}$$

and define the orbits

$$\mathcal{X}_\pm := \gamma(x^\mu_\pm, 0), \quad \mathcal{Y}_\pm := \gamma(0, \pm y_0).$$

Setting

$$\mathcal{T}(\mathcal{X}_\pm) := \pm 2 \int_{x^\mu_\pm}^{x^1_\pm} \frac{dx}{\sqrt{2\mu(\mathscr{G}(x) - \mathscr{G}(x^\mu_\pm))}}\,, \quad \mathcal{T}(\mathcal{Y}) := \int_{x^1_-}^{x^1_+} \frac{dx}{\sqrt{y_0^2 + 2\mu\mathscr{G}(x)}}$$

we tune the values $x^\mu_\pm$, y_0 and μ so that

$$\max\{\mathcal{T}(\mathcal{X}_\pm), \mathcal{T}(\mathcal{Y})\} < T_2\,.$$

Clearly, given the other parameters, we can always choose μ sufficiently large, say $\mu > \mu^*$, so that the above condition is satisfied.

Finally, we introduce the stable and unstable manifolds, W^s and W^u, for the origin as saddle point of system (1.2.4). More precisely, we define the sets

$$\begin{aligned}
W^u_+ &:= \{(x, y) : E_2(x, y) = 0, x > 0, y > 0\},\\
W^s_- &:= \{(x, y) : E_2(x, y) = 0, x < 0, y > 0\},\\
W^u_- &:= \{(x, y) : E_2(x, y) = 0, x < 0, y < 0\},\\
W^s_+ &:= \{(x, y) : E_2(x, y) = 0, x > 0, y < 0\},
\end{aligned}$$

so that $W^s = W^s_- \cup W^s_+$ and $W^u = W^u_- \cup W^u_+$. The resulting configuration is illustrated in Figure 1.3.

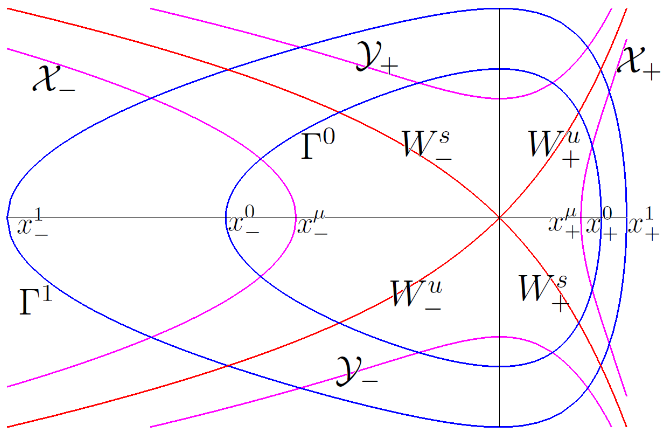

Figure 1.3: The present figure shows the appropriate overlapping of the phase-portraits of systems (1.2.3) and (1.2.4).

The closed trajectories Γ^0, Γ^1 together with $\mathcal{X}_\pm$, $\mathcal{Y}_\pm$, $W^s_\pm$ and $W^u_\pm$ determine eight regions that we denote by $\mathcal{A}_i$ and $\mathcal{B}_i$ for $i = 1, \dots, 4$, as in Figure 1.4.

Each of the regions $\mathcal{A}_i$ and $\mathcal{B}_i$ is homeomorphic to the unit square and thus is a topological rectangle. In this setting, we give an orientation to $\mathcal{A}_i$ by choosing $\mathcal{A}_i^- := \mathcal{A}_i \cap (\Gamma^0 \cup \Gamma^1)$. We take as $\mathcal{B}_i^-$ the closure of $\partial\mathcal{B}_i \setminus (\Gamma^0 \cup \Gamma^1)$.

We can now apply a result in the framework of the theory of topological horseshoes as presented in [PPZ08] and [MRZ10]. Indeed, by the previous choice of $\lambda > \Lambda_k$ we obtain that

$$\Phi_1 : \widehat{\mathcal{A}_i} \Longleftrightarrow\!\!\!\!\!\!\rightarrow^k \widehat{\mathcal{B}_i}\,, \quad \forall\, i = 1, \dots, 4,$$

On the other hand, from $\mu > \mu^*$ it follows that

$$\Phi_2 : \widehat{\mathcal{B}_i} \Longleftrightarrow\!\!\!\!\!\!\rightarrow \widehat{\mathcal{A}_i}\,, \quad \forall\, i = 1, \dots, 4.$$

Then [PPZ08, Theorem 3.1] (see also [MRZ10, Theorem 2.1]) ensures the existence of at least k fixed points for $\Phi = \Phi_2 \circ \Phi_1$ in each of the regions

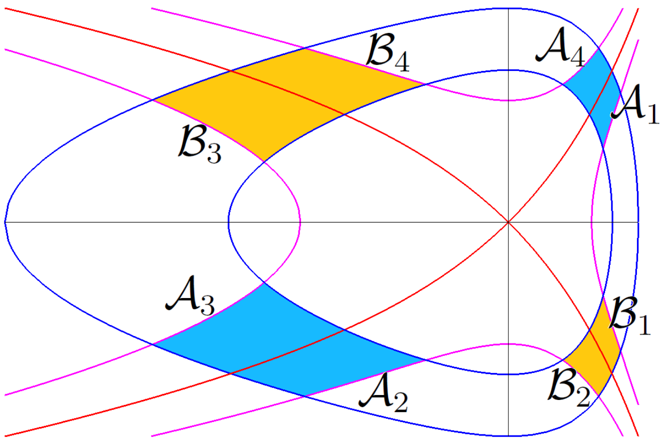

Figure 1.4: The present figure shows the regions $\mathcal{A}_i$ and $\mathcal{B}_i$. We have labelled the regions following a clockwise order, which is useful from the point of view of the dynamics.

$\mathcal{A}_i$. This, in turns, implies the existence of $4k$ T-periodic solutions for system (1.2.6).

Such solutions are topologically different and can be classified as follows: for each $j = 1, \dots, k$ there is a solution (x, y) with

- $(x(0), y(0)) \in \mathcal{A}_1$ with $x(t)$ having $2j$ zeros in $]0, T_1[$ and strictly positive in $[T_1, T]$;
- $(x(0), y(0)) \in \mathcal{A}_2$ with $x(t)$ having $2j - 1$ zeros in $]0, T_1[$ and one zero in $]T_1, T[$;
- $(x(0), y(0)) \in \mathcal{A}_3$ with $x(t)$ having $2j$ zeros in $]0, T_1[$ and strictly negative in $[T_1, T]$;
- $(x(0), y(0)) \in \mathcal{A}_4$ with $x(t)$ having $2j - 1$ zeros in $]0, T_1[$ and one zero in $]T_1, T[$.

In conclusion, for each $j = 1, \dots, k$ we find at least four T-periodic solutions having precisely $2j$-zeros in $[0, T[$. □

Remark 1.2.2. Having assumed that g is bounded on $\mathbb{R}^-$, we can also prove the existence of a T-periodic solution with $(x(0), y(0)) \in \mathcal{A}_3$ and such that $x(t) < 0$ for all $t \in [0, T]$ while $y(t) = x'(t)$ has two zeros in $[0, T[$. Moreover, the results from [MRZ10, PPZ08] guarantee also that each of the regions $\mathcal{A}_i$ contains a compact invariant set where Φ is chaotic in the sense of Block and Coppel (see [AK01]).

We further observe that, for equation (1.2.1) the same results hold if condition (g_-) is relaxed to

$$\lim_{x \to -\infty} \frac{g(x)}{x} = 0. \tag{1.2.9}$$

In this manner we get the same number of four T-periodic solutions as obtained in [BZ13]. However, we stress that, even if the conditions at infinity here and in that work are the same, nevertheless, the assumptions at the origin are completely different. Indeed, in [BZ13] a one-sided superlinear condition in zero, of the form $g'(0^+) = 0$ or $g'(0^-) = 0$ was required. As a consequence, for λ large, one could prove the existence of four T-periodic solutions with prescribed nodal properties which come in pair, namely two "small" and two "large". In our case, if in place of $g_0 > 0$ we assume $g'(0^+) = 0$ or $g'(0^-) = 0$, with the same approach we could prove the existence of eight T-periodic solutions, four "small" and four "large".

We conclude this section by observing that if we want to produce the same results for system (1.1.1), then we cannot replace $(h_\pm)$ or $(g_\pm)$ with a weaker condition of the form of (1.2.9). Indeed, a crucial step in our proof is to have a twist condition, that is, a gap in the period between a fast orbit (like Γ^0) and slow one (like Γ^1). This is no more guaranteed for an autonomous system of the form

$$\begin{cases} x' = h(y) \\ y' = -g(x) \end{cases}$$

if $g(x)$ satisfies a sublinear condition at infinity as (1.2.9). Indeed, the slow decay of g at infinity could be compensated by a fast growth of h at infinity. In [CGMn00] the Authors provide examples of isochronous centers for planar Hamiltonian systems even in the case when one of the two components is sublinear at infinity.

1.3 The general case and proofs

Throughout the section and consistently with Section 1.1, for each $s \in \mathbb{R}$ and $z \in \mathbb{R}^2$ we denote by $(x(\cdot, s, z), y(\cdot; s, z))$ the solution of (1.1.1) satisfying $(x(s), y(s)) = z$ and, for $t \geq s$, we set

$$\Phi_s^t(z) := (x(t, s, z), y(t; s, z)),$$

if the solution is defined on $[s, t]$.

We prove both Theorem 1.1.1 (Theorem 1.3.1) and Theorem 1.1.2 (Theorem 1.3.2) assuming (g_-). The proofs can be easily modified in order to take into account all the other cases, namely (g_+), (h_-) or (h_+). Concerning the T-periodic weight function we suppose for simplicity that $a : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (a^*). More general regularity conditions on $a(\cdot)$ can be considered as well.

Theorem 1.3.1. *Let g, h be locally Lipschitz continuous functions. Assume (C_0), (g_-) and the global continuability of the solutions. For each positive integer k there exists $\Lambda_k > 0$ such that for every $\lambda > \Lambda_k$ and $j = 1, \dots, k$, there are at least two T-periodic solutions for system* (1.1.1) *with x having exactly $2j$-zeros in the interval* $[0, T[$.

Proof. We split the proof into steps in order to reuse some of them for the proof of Theorem 1.3.2.

Step 1. Evaluating the rotation number along the interval $[0, T_1]$ *for small solutions.*
Let $\varepsilon > 0$ be sufficiently small such that $g_0 - \varepsilon > 0$ and $h_0 - \varepsilon > 0$ and take $r^\varepsilon > 0$ such that, by virtue of (C_0),

$$h(\xi)\xi \geq (h_0 - \varepsilon)\xi^2, \quad g(\xi)\xi \geq (g_0 - \varepsilon)\xi^2, \quad \forall\, |\xi| \leq r^\varepsilon.$$

Let $u(t) := (x(t), y(t))$ be a solution of (1.1.1) such that $0 < ||u(t)|| \leq r^\varepsilon$ for all $t \in [0, T_1]$. We consider the modified clockwise rotation number associated with the solution $u(\cdot)$ in the interval $[0, T_1]$ (which is the interval where $a \succ 0$), defined as

$$\mathrm{Rot}^p(u; 0, T_1) := \frac{\sqrt{p}}{2\pi} \int_0^{T_1} \frac{h(y(t))y(t) + \lambda a^+(t) g(x(t))x(t)}{p y^2(t) + x^2(t)}\, dt,$$

where $p > 0$ is a fixed number that will be specified later. The modified rotation number can be traced back to the classical Prüfer transformation and it was successfully applied in [FH93] (see also [Zan96] and the references therein). A systematic use of the modified rotation number in the context of the Poincaré-Birkhoff Theorem, with all the needed technical details, is exhaustively described by Boscaggin in [Bos11]. Here we follow the same approach. The key property of the number $\mathrm{Rot}^p(u)$ is that, when for some p it assumes an integer value, that same value is independent on the choice of p. Moreover, as a consequence of $h(y)y > 0$ for $y \neq 0$, we have that if $\tau_1 < \tau_2$ are two consecutive zeros of $x(\cdot)$, then $\mathrm{Rot}^p(u;\tau_1,\tau_2) = 1/2$ (independently on p). Hence, we can choose suitably the constant p in order to estimate in a simpler way the rotation number. In our case, if we take

$$p := \frac{1}{\lambda}\frac{h_0 - \varepsilon}{(g_0 - \varepsilon)|a^+|_{L^\infty(0,T_1)}},$$

and recall that we are evaluating the rotation number on a "small" solution u so that $h(y(t))y(t) \geq (h_0 - \varepsilon)y^2(t)$ and $g(x(t))x(t) \geq (g_0 - \varepsilon)x^2(t)$, we find

$$\begin{aligned}
\mathrm{Rot}^p(u;0,T_1) &\geq \frac{\sqrt{p}}{2\pi}\int_0^{T_1} \frac{(h_0-\varepsilon)y^2(t) + \lambda a^+(t)(g_0-\varepsilon)x^2(t)}{py^2(t)+x^2(t)}\,dt \\
&= \frac{\lambda(g_0-\varepsilon)\sqrt{p}}{2\pi}\int_0^{T_1} \frac{|a^+|_{L^\infty(0,T_1)}py^2(t) + a^+(t)x^2(t)}{py^2(t)+x^2(t)}\,dt \\
&\geq \frac{\lambda(g_0-\varepsilon)\sqrt{p}}{2\pi}\int_0^{T_1} a^+(t)\,dt = \frac{\sqrt{\lambda}}{2\pi}\kappa(\varepsilon),
\end{aligned}$$

where

$$\kappa(\varepsilon) := \left(\frac{(h_0-\varepsilon)(g_0-\varepsilon)}{|a^+|_{L^\infty(0,T_1)}}\right)^{1/2}\int_0^{T_1} a^+(t)\,dt > 0.$$

Hence, given any positive integer k, we can take

$$\Lambda_k := \left(\frac{2\pi}{\kappa(\varepsilon)}\right)^2 (k+1)^2 \tag{1.3.1}$$

so that, for each $\lambda > \Lambda_k$ we obtain that $\mathrm{Rot}^p(u;0,T_1) > k+1$ and therefore, by [Bos11, Proposition 2.2], $\mathrm{rot}_z(0,T_1) > k+1$, for $z = u(0)$ and rot_z defined in (1.1.3).

Step 2. Evaluating the rotation number along the interval $[0, T]$ *for small solutions.*
Consider now the interval $[T_1, T]$ where $a(t) \leq 0$. In this case, we are in the same situation as in [BZ13, Lemma 3.1] and the corresponding result implies that $\mathrm{rot}_{z_1}(T_1, T) > -1/2$, for $z = u(T_1)$. As a consequence we conclude that if $u(t) := (x(t), y(t))$ is a solution of (1.1.1) such that $0 < ||u(t)|| \leq r^{\varepsilon}$ for all $t \in [0, T_1]$ and $\lambda > \Lambda_k$, then $\mathrm{rot}_z(T) := \mathrm{rot}_z(0, T) > k$ for $z = u(0)$.

Step 3. Consequences of the global continuability.
The global continuability of the solutions implies the fulfillment of the so-called "elastic property" (cf. [Kra68, CMZ90]). In our case, recalling also that non trivial solutions never hit the origin, we obtain

- (i_1) for each $r_1 > 0$ there exists $r_2 \in]0, r_1[$ such that $||z|| \leq r_2$ implies $0 < ||\Phi_0^t(z)|| \leq r_1$, $\forall\, t \in [0, T]$;
- (i_2) for each $R_1 > 0$ there exists $R_2 > R_1$ such that $||z|| \geq R_2$ implies $||\Phi_0^t(z)|| \geq R_1$, $\forall\, t \in [0, T]$

(see [Zan96, Lemma 2]).

Step 4. Rotation numbers for small initial points.
Suppose now that $\varepsilon > 0$ and $\lambda > \Lambda_k$ are chosen as in Step 1 and let $\mu > 0$ be fixed. Using (i_1) in Step 3 we determine a small radius $r_0 = r_0(\varepsilon, \lambda, \mu) > 0$ such that for each initial point $z \in \mathbb{R}^2$ with $||z|| = r_0$ it follows that the solution $u(t) = (x(t), y(t))$ of (1.1.1) with $u(0) = z$ satisfies $0 < ||u(t)|| \leq r^{\varepsilon}$ for all $t \in [0, T_1]$. Hence, by Step 2 we conclude that

$$\mathrm{rot}_z(T) > k, \quad \forall\, z \text{ with } ||z|| = r_0 \,. \tag{1.3.2}$$

Step 5. Evaluating the rotation number along the interval $[0, T]$ *for large solutions.*
Suppose, from now on, that λ and μ are fixed as in Step 4. Let $u(t) = (x(t), y(t))$ be any non trivial solution of (1.1.1) which crosses the third quadrant in the phase-plane. If this happens, we can assume that there is an interval $[\alpha, \beta] \subseteq [0, T]$ such that $x(t) \leq 0$ for all $t \in [\alpha, \beta]$ with $x(\alpha) = 0 = y(\beta)$

and $x(t) < 0$ for all $t \in]\alpha, \beta]$ as well as $y(t) < 0$ for all $t \in [\alpha, \beta[$. Note that from the first equation in (1.1.1) when $y(t) = 0$ also $x'(t) = 0$. Assumption (g_-) implies there is a bound for $g(x)$ when $x \leq 0$, namely $M > 0$ such that $|g(x)| \leq M$ for all $x \leq 0$. Thus, integrating the second equation in system (1.1.1) we get that, for all $t \in [\alpha, \beta]$, the following estimate holds:

$$
\begin{aligned}
|y(t)| &= \left| y(\beta) + \int_\beta^t a_{\lambda,\mu}(s) g(x(s))\, ds \right| \\
&\leq M \int_\alpha^\beta a_{\lambda,\mu}(s)\, ds \leq M \left(\lambda \int_0^{T_1} a^+(t)\, dt + \mu \int_{T_1}^T a^-(t)\, dt \right) =: M_1 .
\end{aligned}
$$

Now, integrating the first equation of the same system we obtain for all $t \in [\alpha, \beta]$ the estimate below:

$$
|x(t)| = \left| x(\alpha) + \int_\alpha^t h(y(s))\, ds \right| \leq M_2 := (\beta - \alpha) \max\{|h(y)|\,;\, |y| \leq M_1\}.
$$

With a similar argument, it is easy to check that the same bounds for $|y(t)|$ and $|x(t)|$ hold if the solution crosses the second quadrant instead of the third one. Hence, any solution that in a time-interval $[\alpha, \beta]$ crosses the third quadrant, or the second quadrant, is such that $|y(t)| \leq M_1$ and $|x(t)| \leq M_2$ for all $t \in [\alpha, \beta]$.

In view of the above estimates and arguing by contradiction, we can then conclude that if the solution u satisfies

$$
||u(t)|| \geq M_3 := 1 + (M_1 + M_2)^{1/2}, \quad \forall\, t \in [0, T],
$$

then for $u(\cdot)$ is impossible to cross the third quadrant and it is also impossible to cross the second one.

Step 6. Rotation numbers for large initial points.
Using (i_2) in Step 3 we determine a large radius $R_0 = R_0(\lambda, \mu) > 0$ such that for each initial point $z \in \mathbb{R}^2$ with $||z|| = R_0$ it follows that the solution $u(t) = (x(t), y(t))$ of (1.1.1) with $u(0) = z$ satisfies $||u(t)|| \geq M_3$ for all $t \in [0, T]$. Hence, by Step 5 we conclude that

$$
\mathrm{rot}_z(T) < 1, \quad \forall\, z \text{ with } ||z|| = R_0 . \tag{1.3.3}
$$

Indeed, if by contradiction $\mathrm{rot}_z(T) \geq 1$, then the solution $u(t)$ of (1.1.1) with $u(0) = z$ must cross at least once one between the third and the second quadrant and this fact is forbidden by the choice of R_0, which implies that $||u(t)|| \geq M_3$ for all $t \in [0, T]$.

Step 7. Applying Poincaré-Birkhoff fixed point Theorem.
At this point we can to conclude the proof. From (1.3.2) and (1.3.3) we have the twist condition (TC) satisfied for $\mathfrak{b} = k$ and $\mathfrak{a} = 1$ and the thesis follows as explained in the introductory discussion preceding the statement of Theorem 1.1.1. □

Remark 1.3.1. In view of the above proof, a few observations are in order.

1. We think that the choice of Λ_k in (1.3.1), although reasonably good, is not the optimal one. One could slightly improve it, by using some comparison argument with the rotation numbers associated with the limiting linear equation

$$x' = h_0 y, \quad y' = -\lambda g_0 a^+(t) x.$$

We do not discuss further this topic in order to avoid too much technical details.

2. In the statement of Theorem 1.3.1 we have explicitly recalled the assumptions on g and h to be locally Lipschitz continuous functions so to have a well defined (single-valued) Poincaré map. With this respect, we should mention that there is a recent version of the Poincaré-Birkhoff Theorem due to Fonda and Ureña [FUn16, FUn17] which, for Hamiltonian systems like (1.1.1), does not require the uniqueness of the solutions for the initial value problems and just the continuability of the solutions on $[0, T]$ is needed. The theorems in [FUn17] apply to higher dimensional Hamiltonian systems as well. For another recent application of such results to planar systems, in which the uniqueness of the solutions of the Cauchy problems is not required, see also [COZ16]. In our case, even if we apply the Fonda-Ureña Theorem, we still need to assume at least an upper bound on $g(x)/x$ and $h(y)/y$ near zero, so to avoid the possibility that a (nontrivial) solution $u(\cdot)$ of (1.1.1) with $u(0) \neq (0, 0)$ may hit the origin at some time $t \in]0, T]$, thus preventing the rotation number to be well defined. See [But78, Section 4] for a detailed discussion of these aspects.

Now we are in position to give the proof of Theorem 1.1.2 (which is presented as Theorem 1.3.2 below). For the next result we do not assume the global continuability of the solutions. Accordingly, both h (at $\pm\infty$) and g (at $+\infty$) may have a superlinear growth.

For the foregoing proof we recall that we denote by $D(R)$ and $D[R]$ the open and the closed disc in $\mathbb{R}^2$ of center the origin and radius $R > 0$. Given $0 < r < R$, we denote by $A[r, R]$ the closed annulus $A[r, R] := D[R] \setminus D(r)$. Let also Q_i for $i = 1, 2, 3, 4$ be the usual quadrants of $\mathbb{R}^2$ counted in the natural counterclockwise sense starting from

$$Q_1 := \{(x, y) : x \geq 0, y \geq 0\}.$$

Theorem 1.3.2. *Let g, h be locally Lipschitz continuous functions satisfying (C_0). Suppose also that (g_-) holds. For each positive integer k there exists $\Lambda_k > 0$, such that for every $\lambda > \Lambda_k$ there exists $\mu^* = \mu^*(\lambda)$ such that for each $\mu > \mu^*$ and $j = 1, \dots, k$, there are at least four T-periodic solutions for system* (1.1.1) *with x having exactly $2j$-zeros in the interval* $[0, T[$.

Proof. For our proof, we will take advantage of some steps already settled in the proof of Theorem 1.3.1.

First of all, we consider system (1.1.1) on the interval $[0, T_1]$, so that the system can be written as

$$x' = h(y), \qquad y' = -\lambda a^+(t)g(x) \tag{1.3.4}$$

and observe that all the solutions of (1.3.4) are globally defined on $[0, T_1]$. To prove this fact, we observe that the sign assumptions on h and g in (C_0) and (g_-) guarantee that (1.3.4) belongs to the class of equations for which the global continuability of the solutions was proved in [DZ96]. Hence our claim is proved.

Now, we repeat the same computations as in the Steps 1-3-4-5-6 of the preceding proof (with only minor modifications, since now we work on $[0, T_1]$ instead of $[0, T]$) and, having fixed $\lambda > \Lambda_k$ (with the same constants Λ_k as in (1.3.1)), we are in the following setting:

(TC_*) *There are constants* $r_0 = r_0(\varepsilon, \lambda)$ *and* $R_0 = R_0(\lambda)$*, with* $0 < r_0 < R_0$*, such that*
$rot_z(T_1) > k + 1,\ \forall\, z : ||z|| = r_0\,;\quad rot_z(T_1) < 1,\ \forall\, z : ||z|| = R_0\,.$

By a classical compactness argument following by the global continuability of the solutions of (1.3.4) we can determine two positive constants $\mathfrak{s}_0 = \mathfrak{s}_0(\lambda, r_0, R_0)$ and $\mathfrak{S}_0 = \mathfrak{S}_0(\lambda, r_0, R_0)$ with

$$0 < \mathfrak{s}_0 < r_0 < R_0 < \mathfrak{S}_0\,,$$

such that

$$\mathfrak{s}_0 \leq ||\Phi_0^t(z)|| \leq \mathfrak{S}_0\,,\quad \forall\, t \in [0, T_1]\,,\quad \forall\, z :\ r_0 \leq ||z|| \leq R_0\,. \tag{1.3.5}$$

We introduce now the following sets, which are all annular sectors and hence topological rectangles according to the terminology of the Introduction.

$$\mathcal{P}_1 := A[r_0, R_0] \cap Q_1\,,\quad \mathcal{P}_2 := A[r_0, R_0] \cap Q_3\,,$$

$$\mathcal{M}_1 := A[\mathfrak{s}_0, \mathfrak{S}_0] \cap Q_4\,,\quad \mathcal{M}_2 := A[\mathfrak{s}_0, \mathfrak{S}_0] \cap Q_2\,.$$

To each of these sets we give an *orientation*, by selecting a set $[\cdot]^-$ which is the union of two disjoint arcs of its boundary, as follows.

$$\mathcal{P}_i^- := \mathcal{P}_i \cap \partial A[r_0, R_0],\quad \widehat{\mathcal{P}_i} := (\mathcal{P}_i, \mathcal{P}_i^-),\quad i = 1, 2,$$

$$\mathcal{M}_i^- := \mathcal{M}_i \cap \{(x, y) : xy = 0\},\quad \widehat{\mathcal{M}_i} := (\mathcal{M}_i, \mathcal{M}_i^-),\quad i = 1, 2.$$

To conclude the proof, we show that for each integer $j = 1, \ldots, k$ and $i = 1, 2$ there is a pair of compact disjoint sets $H'_{i,j}\,, H''_{i,j} \subseteq \mathcal{P}_i$ such that

$$(H, \Phi) : \widehat{\mathcal{P}_i} \Leftrightarrow\!\!\!\!\!\rightarrow \widehat{\mathcal{P}_i} \quad i = 1, 2, \tag{1.3.6}$$

where H stands for $H'_{i,j}$ or $H''_{i,j}$ and $\Phi := \Phi_0^T$. Along the proof we will also check that the $4k$ sets $H'_{i,j}$ and $H''_{i,j}$ for $i = 1, 2$ and $j = 1, \ldots, k$ are pairwise disjoint. A fixed point theorem introduced in [PZ02] and recalled at the beginning of the Chapter (see [PZ04a, Theorem 3.9] for the precise formulation which is needed in the present situation) ensures the existence of at least a fixed point for the Poincaré map Φ in each of the $4k$ sets $H'_{i,j}$ and $H''_{i,j}$.

To prove (1.3.6) we proceed with two steps. First we show that, for any fixed $j \in \{1, \dots k\}$ there is a compact set $H'_{1,j} \subseteq \mathcal{P}_1$ such that

$$(H'_{1,j}, \Phi_0^{T_1}) : \widehat{\mathcal{P}_1} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{M}_1} \tag{1.3.7}$$

and another compact set $H''_{1,j} \subseteq \mathcal{P}_1$ such that

$$(H''_{1,j}, \Phi_0^{T_1}) : \widehat{\mathcal{P}_1} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{M}_2}\,. \tag{1.3.8}$$

In the same manner we also prove that there are disjoint compact sets $H'_{2,j}$, $H''_{2,j} \subseteq \mathcal{P}_2$ such that

$$(H'_{2,j}, \Phi_0^{T_1}) : \widehat{\mathcal{P}_2} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{M}_2} \tag{1.3.9}$$

and

$$(H''_{2,j}, \Phi_0^{T_1}) : \widehat{\mathcal{P}_2} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{M}_1}\,. \tag{1.3.10}$$

Next, we prove that

$$\Phi_{T_1}^{T} : \widehat{\mathcal{M}_i} \Bumpeq\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{P}_{\boldsymbol{\ell}}}, \quad \forall\, i = 1, 2,\ \forall\, \boldsymbol{\ell} = 1, 2. \tag{1.3.11}$$

Clearly, once all the above relations have been verified, we obtain (1.3.6), using the composition $\Phi = \Phi_{T_1}^{T} \circ \Phi_0^{T_1}$ and counting correctly all the possible combinations.

Proof of (1.3.7). We choose a system of polar coordinates (θ, ρ) starting at the positive y-axis and counting the positive rotations in the clockwise sense, so that, for $z \neq 0$ and $t \in [0, T_1]$, $\theta(t, z)$ denotes the angular coordinate associated with the solution $u = (x, y)$ of (1.3.4) with $u(0) = z$. For $z \in \mathcal{P}_1$ we already know that $u(t) \in A[\mathfrak{s}_0, \mathfrak{S}_0]$ for all $t \in [0, T_1]$. For any fixed $j \in \{1, \dots, k\}$ we define

$$H'_{1,j} := \{z \in \mathcal{P}_1 \,:\, \theta(T_1, z) \in [(\pi/2) + 2j\pi, \pi + 2j\pi]\}.$$

Note that an initial point $z \in \mathcal{P}_1$ belongs to $H'_{1,j}$ if and only if $\Phi_0^{T_1}(z) \in \mathcal{M}_1$ with $x(\cdot)$ having precisely $2j$ zeros in the interval $]0, T_1[$. Then it is clear that $H'_{1,j_1} \cap H'_{1,j_2} = \emptyset$ for $j_1 \neq j_2$.

Let $\gamma : [a_0, a_1] \to \mathcal{P}_1$ be a continuous map such that $||\gamma(a_0)|| = r_0$ and $||\gamma(a_1)|| = R_0$, that is γ is a path contained in (with values in) $\mathcal{P}_1$ and meeting the opposite sides of $\mathcal{P}_1^-$. For simplicity in the notation, we set

$$\vartheta(t, \xi) := \theta(t, \gamma(\xi)).$$

As we have previously observed $\Phi_0^t(\gamma(\xi)) \in A[\mathfrak{s}_0, \mathfrak{S}_0]$ for all $t \in [0, T_1]$ and $\xi \in [a_0, a_1]$. From property (TC_*) we have that

$$\vartheta(T_1, a_0) > \vartheta(0, a_0) + 2(k+1)\pi \geq 2(k+1)\pi \geq 2(j+1)\pi,$$

while

$$\vartheta(T_1, a_1) < \vartheta(0, a_0) + 2\pi \leq \frac{\pi}{2} + 2\pi.$$

The continuity of the map $\varphi : [a_0, a_1] \ni \xi \mapsto \vartheta(T_1, \xi)$ implies that the range of φ covers the interval $[(\pi/2) + 2j\pi, \pi + 2j\pi]$. Therefore there exist b_0, b_1 with $a_0 < b_0 < b_1 < a_1$ such that $\vartheta(T_1, b_0) = \pi + 2j\pi$, $\vartheta(T_1, b_1) = (\pi/2) + 2j\pi$ and

$$\vartheta(T_1, \xi) \in \left[\frac{\pi}{2} + 2j\pi, \pi + 2j\pi\right], \quad \forall\, \xi \in [b_0, b_1].$$

If we denote by σ the restriction of the path γ to the subinterval $[b_0, b_1]$, we have that σ has values in $H'_{1,j}$ and, moreover the path $\Phi_0^{T_1} \circ \sigma$ has values in $\mathcal{M}_1$ and connects the two components of $\mathcal{M}_1^-$. Thus the validity of (1.3.7) is checked.

Proof of (1.3.8)*.* This is only a minor variant of the preceding proof and, with the same setting and notation as above, we just define

$$H''_{1,j} := \{z \in \mathcal{P}_1 \,:\, \theta(T_1, z) \in [(3\pi/2) + 2(j-1)\pi, 2j\pi]\}.$$

An initial point $z \in \mathcal{P}_1$ belongs to $H''_{1,j}$ if and only if $\Phi_0^{T_1}(z) \in \mathcal{M}_2$ with $x(\cdot)$ having exactly $2j-1$ zeros in the interval $]0, T_1[$. Then it is clear that $H''_{1,j_1} \cap H''_{1,j_2} = \emptyset$ for $j_1 \neq j_2$. Moreover, it is also evident that all the sets H' and H'' are pairwise disjoint. The rest of the proof follows the same steps as the previous one, with minor modifications and using again the crucial property (TC_*).

Proof of (1.3.9) *and* (1.3.10). Here we follow a symmetric argument by defining a family of pairwise disjoint compact subsets of $\mathcal{P}_2$ as

$$H'_{2,j} := \{z \in \mathcal{P}_2 \,:\, \theta(T_1, z) \in [(3\pi/2) + 2j\pi, 2(j+1)\pi]\}$$

and

$$H''_{2,j} := \{z \in \mathcal{P}_2 \,:\, \theta(T_1, z) \in [(\pi/2) + 2j\pi, \pi + 2j\pi]\},$$

respectively. Note that we consider an initial point $z \in \mathcal{P}_2$ with an associated angle $\theta(0, z) \in [\pi, (3/2)\pi]$. The rest of the proof is a mere repetition of the arguments presented above.

Proof of (1.3.11). We have four conditions to check, but it is clear that it will be sufficient to prove

$$\Phi_{T_1}^{T} : \widehat{\mathcal{M}_1} \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{P}_1} \ \text{ and } \ \Phi_{T_1}^{T} : \widehat{\mathcal{M}_1} \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow \widehat{\mathcal{P}_2}\,,$$

the other case being symmetric. In this situation, we have only to repeat step by step the argument described in [PZ04b, pp. 85-86] where a similar situation is taken into account. There is however a substantial difference between the equation considered in [PZ04b] and our equation, that in the interval $[T_1, T]$ can be written as

$$x' = h(y), \qquad y' = \mu a^{-}(t) g(x).$$

Indeed, in our case we cannot exclude that some solutions are not globally defined on $[T_1, T]$, due to the possibility of blow-up phenomena in some quadrants. To overcome this difficulty, we follow an usual truncation argument. More precisely, recalling the "large" constant $\mathfrak{S}_0$ introduced in (1.3.5), we define the truncated functions

$$\hat{g}(x) := \begin{cases} g(-\mathfrak{S}_0), & \forall\, x \leq -\mathfrak{S}_0 \\ g(x), & \forall\, x \in [-\mathfrak{S}_0, \mathfrak{S}_0] \\ g(\mathfrak{S}_0), & \forall\, x \geq \mathfrak{S}_0 \end{cases}$$

and

$$\hat{h}(y) := \begin{cases} h(-\mathfrak{S}_0) + y + \mathfrak{S}_0, & \forall\, y \leq -\mathfrak{S}_0 \\ h(y), & \forall\, y \in [-\mathfrak{S}_0, \mathfrak{S}_0] \\ h(\mathfrak{S}_0) + y - \mathfrak{S}_0, & \forall\, y \geq \mathfrak{S}_0 \end{cases}$$

which are locally Lipschitz continuous with g bounded and h having a linear growth. Now the uniqueness and the global existence of the solutions in the interval $[T_1, T]$ for the solutions of the truncated system

$$x' = \hat{h}(y), \qquad y' = \mu a^{-}(t)\hat{g}(x) \tag{1.3.12}$$

is guaranteed. Apart for few minor details we can closely follow the proof of [PZ04b, Theorem 3.1] for the interval when the weight function is negative. Of course, now the result will be valid for the Poincaré map $\hat{\Phi}_{T_1}^{T}$ associated with system (1.3.12) for the time-interval $[T_1, T]$. If we treat with such technique the Poincaré map $\hat{\Phi}_{T_1}^{T}$, we can prove that any path γ in $\mathcal{M}_1$ joining the two sides of $\mathcal{M}_1^{-}$ contains a sub-path σ with $\bar{\sigma} \subseteq \mathcal{M}_1^{-}$ such that $\hat{\Phi}_{T_1}^{t}(\bar{\sigma}) \subset D[0, \mathfrak{S}_0]$ for all $t \in [T_1, T]$ and such that $\hat{\Phi}_{T_1} \circ \sigma$ is a path in $\mathcal{P}_1$ joining the opposite sides of $\mathcal{P}_1^{-}$. This in turn implies

$$\hat{\Phi}_{T_1}^{T} : \widehat{\mathcal{M}_1} \Leftrightarrow\!\!\!\!\to \widehat{\mathcal{P}_1}.$$

On the other hand, the condition $\hat{\Phi}_{T_1}^{t}(\bar{\sigma}) \subset D[0, \mathfrak{S}_0]$ for all $t \in [T_1, T]$ implies that $\hat{\Phi}_{T_1}^{t}(\bar{\sigma}) = \Phi_{T_1}^{t}(\bar{\sigma})$ for all $t \in [T_1, T]$ and therefore we have that

$$\Phi_{T_1}^{T} : \widehat{\mathcal{M}_1} \Leftrightarrow\!\!\!\!\to \widehat{\mathcal{P}_1}.$$

All the other instances of (1.3.11) can be verified in the same manner. This completes the proof of the theorem. $\square$

As a byproduct of the method of proof we have adopted, we are able to classify the nodal properties of the four T-periodic solutions as follows.

Proposition 1.3.1. *Let g, h be locally Lipschitz continuous functions satisfying (C_0) and at least one between the four conditions $(h_{\pm})$ and $(g_{\pm})$. For each positive integer k there exists $\Lambda_k > 0$, such that for every $\lambda > \Lambda_k$ there exists $\mu^* = \mu^*(\lambda)$ such that for each $\mu > \mu^*$ and $j = 1, \dots, k$, there are at least four T-periodic solutions for system* (1.1.1) *which can be classified as follows:*

- *one solution with $x(0) > 0, x'(0) > 0$ and with $x(\cdot)$ having exactly $2j$ zeros in $]0, T_1[$ and no zeros in $[T_1, T]$;*

- *one solution with* $x(0) > 0, x'(0) > 0$ *and with* $x(\cdot)$ *having exactly* $2j-1$ *zeros in* $]0, T_1[$ *and one zero in* $[T_1, T]$;
- *one solution with* $x(0) < 0, x'(0) < 0$ *and with* $x(\cdot)$ *having exactly* $2j$*-zeros in* $]0, T_1[$ *and no zeros in* $[T_1, T]$;
- *one solution with* $x(0) < 0, x'(0) < 0$ *and with* $x(\cdot)$ *having exactly* $2j-1$ *zeros in* $]0, T_1[$ *and one zero in* $[T_1, T]$.

Remark 1.3.2. We present some observations related to the proof of Theorem 1.3.2.

1. In view of the results in [PZ04b] and the notation recalled in the Introduction, our proof implies that

$$\Phi_0^T : \widehat{\mathcal{P}_i} \Leftrightarrow^{2k} \widehat{\mathcal{P}_i} \quad i = 1, 2.$$

As a consequence, there are two compact invariant sets contained in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, where Φ_0^T induces chaotic dynamics on $2k$ symbols. Also, for any periodic sequence of symbols subharmonic solutions associated with that periodic sequence do exist (see [MPZ09]).

2. In principle, our approach could be extended to differential systems in which the weight function displays a finite number of positive humps separated by negative ones. Although the feasibility of this study is quite clear, we have not pursued this line of research for sake of conciseness.

3. A comparison between Theorem 1.3.1 and Theorem 1.3.2 suggests that it could be interesting to present examples of differential systems in which there are exactly two (respectively four) T-periodic solutions with given nodal properties and then discuss the change in the number of solutions using μ as a bifurcation parameter.

1.4 Subharmonic solutions

In this Section we briefly discuss how to adapt the proofs of Theorem 1.1.1 and Theorem 1.1.2 to obtain subharmonic solutions. Throughout the Section we suppose that $a : \mathbb{R} \to \mathbb{R}$ is a continuous T-periodic weight function satisfying (a^*). As before, more general regularity conditions on $a(\cdot)$ can be considered.

Speaking of subharmonic solutions we must observe that if $u = (x, y)$ is a mT-periodic solution of (1.1.1), then also $u_i(\cdot) := u(\cdot - iT)$ is a mT-periodic solution for all $i = 1, \dots, m-1$. Such solutions, although distinct, are considered to belong to the same periodicity class.

First of all, we look at the proof of Theorem 1.3.1 and give estimates on the rotation numbers on the interval $[0, mT]$ for some integer $m \geq 2$. Iterating the argument in Step 1-4 and taking a smaller r_0 if necessary, we can prove that for $\lambda > \Lambda_k$ as in (1.3.1), we obtain

$$\mathrm{rot}_z(mT) > mk, \quad \forall\, z \text{ with } \|z\| = r_0\,.$$

Repeating the computation in Step 5-6 for the interval $[0, mT]$ and taking a larger R_0 if necessary, we get

$$\mathrm{rot}_z(mT) < 1, \quad \forall\, z \text{ with } \|z\| = R_0\,.$$

In this manner we have condition (TC_m) satisfied for $\mathfrak{b} = mk$ and $\mathfrak{a} = 1$. Now, if we fix an integer $j \in \{1, \dots, mk\}$ which is relatively prime with m, we obtain at least two mT-periodic solutions of system (1.1.1) with $x(\cdot)$ having exactly $2j$ simple zeros in the interval $[0, mT[$. As m and j are coprime numbers, these solutions cannot be ℓT-periodic for some $\ell \in \{1, \dots, m-1\}$. Since, by (a^*), T is the minimal period of $a(\cdot)$, we conclude that mT is the minimal period of the solution (x, y) of (1.1.1) (see [DZ93, DZ96] for previous related results and how to prove the minimality of the period via the information about the rotation number). In this manner, the following result is proved.

Theorem 1.4.1. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous functions satisfying (C_0) and at least one between the four conditions $(h_\pm)$ and $(g_\pm)$. Assume, moreover, the global continuability for the solutions of (1.1.1). Let $m \geq 2$ be a fixed integer. Then, for each positive integer k there exists $\Lambda_k > 0$ such that for each $\lambda > \Lambda_k$ and $j = 1, \dots, mk$, with j relatively prime with m, the system (1.1.1) has at least two periodic solutions $(x(t), y(t))$ of minimal period mT, not belonging to the same periodicity class.*

Also for Theorem 1.4.1 the same observation as in Remark 1.3.1.2 applies, namely, using Fonda-Ureña version of the Poincaré-Birkhoff Theorem, we can remove the local Lipschitz condition outside the origin.

Looking for an extension of Theorem 1.1.2 to the case of subharmonic solutions, in view of Remark 1.3.2.1 and [PZ04b, MPZ09], the condition

$$\Phi_0^T : \widehat{\mathcal{P}_i} \Leftrightarrow\!\!\to^{2k} \widehat{\mathcal{P}_i} \quad i = 1, 2$$

implies the following property with respect to periodic solutions. Let $\mathcal{P} = \mathcal{P}_i$ for $i = 1, 2$. There exists $2k$ pairwise disjoint compact sets $S_1, \dots S_{2k} \subseteq \mathcal{P}$ such that for each m-periodic two-sided sequence $\xi := (\xi_n)_{n\in\mathbb{Z}}$, with $\xi_n \in \{1, \dots, 2k\}$ for all $n \in \mathbb{Z}$, there exists a fixed point z^* of Φ_0^{mT} such that $\Phi_0^{nT}(z^*) \in S_{\xi_n}, \forall\, n \in \mathbb{Z}$. In this case we say that the trajectory associated with the initial point z^* follows the periodic *itinerary* $(\dots, S_{\xi_0}, \dots, S_{\xi_n}, \dots)$.

From this observation, the following result holds.

Theorem 1.4.2. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous functions satisfying (C_0) and at least one between the four conditions $(h_\pm)$ and $(g_\pm)$. Let $m \geq 2$ be a fixed integer. Then, for each positive integer k there exists $\Lambda_k > 0$ such that for each $\lambda > \Lambda_k$ there exists $\mu^* = \mu^*(\lambda)$ such that for each $\mu > \mu^*$ the following property holds: given any periodic two-sided sequence $(\xi_n)_{n\in\mathbb{Z}}$, with $\xi_n \in \{1, \dots, 2k\}$ for all $n \in \mathbb{Z}$ and with minimal period m, the system (1.1.1) has at least two periodic solutions $(x(t), y(t))$ of minimal period mT, following an itinerary of sets associated with $(\xi_n)_{n\in\mathbb{Z}}$ and not belonging to the same periodicity class.*

A simple comparison of the two theorems shows that when m grows, the number of different subhamonics found by Theorem 1.4.2 largely exceeds the number of those obtained by Theorem 1.4.1. The number of m-order subharmonics can be precisely determined by a combinatorial formula coming from the study of "aperiodic necklaces" [Fel18].

1.5 Some examples

We propose a few examples of equations, coming from the literature, which fit into the framework of our theorems.

As a first case, we consider the following Duffing type equation with relativistic acceleration

$$(\varphi(u'))' + a_{\lambda,\mu}(t)g(u) = 0, \tag{1.5.1}$$

where

$$\varphi(s) = \frac{s}{\sqrt{1-s^2}}.$$

A variant of (1.5.1) in the form of $(\varphi(u'))' + g(t,u) = 0$ is considered in [BG13] where pairs of periodic solutions with prescribed nodal properties are found by means of the Poincaré-Birkhoff Theorem for a function $g(t,u)$ having at most linear growth in u at infinity. Equation (1.5.1) can be equivalently written as

$$\begin{cases} x' = \varphi^{-1}(y) \\ y' = -a_{\lambda,\mu}(t)g(x) \end{cases}$$

which is the same as system (1.1.1) with $h = \varphi^{-1}$. In this case, since φ^{-1} is bounded, both (h_-) and (h_+) are satisfied and the global existence of the solutions is guaranteed. Moreover all the other assumptions required for h in (C_0) are satisfied with $h_0 = 1$. Hence all the previous results apply to (1.5.1) once we assume that (C_0) holds for g. Notice that we do not need any growth assumption on g.

Our second example is inspired by the work of Le and Schmitt [LS95] where the authors proved the existence of T-periodic solutions for the second order equation

$$u'' + k(t)e^u = p(t), \tag{1.5.2}$$

with k, p T-periodic functions, with k changing sign, p with zero mean value and such that

$$\int_0^T k(t)e^{u_0(t)}dt < 0,$$

where $u_0(t)$ denotes a T-periodic solution of $u'' = p(t)$. If we call $\tilde{u}(t)$ the T-periodic solution of (1.5.2) whose existence is guaranteed by [LS95, Remark 6.4], and set

$$u(t) = x(t) + \tilde{u}(t),$$

then (1.5.2) is transformed to the equivalent equation

$$x'' + q(t)(e^x - 1) = 0$$

with

$$q(t) := k(t)e^{\tilde{u}(t)}$$

which changes sign if and only if $k(t)$ changes sign. Thus we enter in the setting of (1.1.2) and Theorem 1.1.2 can be applied. Note that in general, in an interval where $k < 0$ we may have blow-up of the solutions in the first quadrant of the phase-plane and thus the Poincaré map cannot be defined on a whole (large) annulus surrounding the origin. Clearly, in order to apply our theorem, we will just need to adapt our conditions on the weight $a_{\lambda,\mu}(t)$ to the coefficients $k(t)$ and $p(t)$.

As a last example, we consider a model adapted from the classical Lotka-Volterra predator-prey system. We take into account the system

$$\begin{cases} P' = P(-c_1(t) + d(t)N) \\ N' = N(c_2(t) - b(t)P) \end{cases}$$

which represents the dynamics of a prey population $N(t) > 0$ under the effect of a predator population $P(t) > 0$. Notice that the order in which the two equations appear in the above system is not the usual one but it is convenient so to enter the setting of system (1.1.1). All the coefficients b, c_1, c_2, d are continuous and T-periodic functions. In [DZ96] the existence of infinitely many subharmonic solutions was proved under the assumption that $c_1(t)$ and $c_2(t)$ have positive average and $b(t) \succ 0$, $d(t) \succ 0$ in $[0, T]$. Extensions to higher dimensional systems have been recently obtained in [FT] under similar sign conditions on the coefficients. Some results about the stability of the solutions for this model are obtained as a special case in [LGOT96]. For the main Lotka-Volterra model thereby proposed the search of subharmonics solutions has been recently addressed [LGMH18] under some specific assumptions on the averages of $b(t)$ and $d(t)$, in a framework that can be regarded as a counterpart to ours.

Our aim now is to discuss the case in which $b(t)$ may be negative on some subinterval. We perform a change of variables setting $u = \log P$ and $v = \log N$

and obtain the new system

$$\begin{cases} u' = -c_1(t) + d(t)e^v \\ v' = c_2(t) - b(t)e^u. \end{cases}$$

Suppose now a T-periodic solution $(\tilde{u}(t), \tilde{v}(t))$ is given. With a further change of variables we can write the generic solutions as

$$\begin{aligned} u(t) &= x(t) + \tilde{u}(t) \\ v(t) &= y(t) + \tilde{v}(t) \end{aligned}$$

and by substitution in the previous system we arrive at

$$\begin{cases} x' = d(t)e^{\tilde{v}(t)}(e^y - 1) \\ y' = -b(t)e^{\tilde{u}(t)}(e^x - 1)\,. \end{cases}$$

At this point, we can adapt the coefficients in order to enter in the frame of system (1.1.1). More precisely, we suppose that $d(t)e^{\tilde{v}(t)} \equiv \text{constant} = D > 0$, so that $h(y) := D(e^y - 1)$, $g(x) := e^x - 1$ and set

$$q(t) := b(t)e^{\tilde{u}(t)},$$

which changes sign if and only if $b(t)$ changes sign. Thus we enter in the setting of (1.1.2) and Theorem 1.1.2 can be applied. The same remark of the previous case holds. In particular, in order to apply our theorem, we need to translate our conditions on the weight $a_{\lambda,\mu}(t)$ to the coefficients $b(t)$.

Chapter 2

Rich families of parabolic solutions for the planar N-centre problem

The N-centre problem is an old and well-known problem in Celestial Mechanics. It addresses the motion of a null-mass particle which is subjected to the attractive field generated by N fixed heavy bodies. The problem is a first step approximation of the more complex N-body problem, in which the relative motion of N moving bodies is studied: in fact, a circular $N+1$ body problem in a rotating coordinate system can be reduced to the N-centre problem by neglecting some of the acting forces. However, the N-centre problem displays many interesting features on its own and has recently raised a good number of works on which this Chapter is inspired, [ST13, BDT17, BDP17, Cas17].

We take into account the planar generalised N-centre problem

$$\ddot{x} = \nabla U(x), \qquad x \in \mathbb{R}^2 \setminus \Sigma \tag{2.0.1}$$

where $\Sigma = \{c_1, \ldots, c_N\}$ is the discrete set of the centres c_i with mass $m_i > 0$ for $i = 1, \ldots, N$. The potential U is of the following form:

$$U(x) = \sum_{i=1}^{N} \frac{m_i}{\alpha |x - c_i|^\alpha} + W(x), \qquad \alpha \in [1, 2)\,,$$

where $W(x)$ is a sufficiently regular function that will be characterised later. When $\alpha = 1$ and $W(x) \equiv 0$ we fall into the classical Newtonian case. When $N = 1$ the famous Keplerian problem arises, which itself generates quite complicated dynamics and has been extensively studied also recently by means of perturbation theory, [FG17, FG18]. For $N \leq 2$ the problem is fully integrable, while for $N \geq 3$ it has been proven to be not integrable [Bol84].

The case we consider, $\alpha \in [1, 2)$, is commonly addressed as *weak forces* case (often contemplating also $\alpha \in [0, 2)$), opposite to the *strong forces* case where $\alpha > 2$. The main reason under this choice is of topological nature and will be cleared in the following, when we also point out some differences in the methods dealing with one case or the other.

We are interested in parabolic solutions of (2.0.1), i.e. solutions which satisfy the zero energy condition

$$\frac{|\dot{x}(t)|^2}{2} = U(x(t)) \qquad \forall t \in \mathbb{R}. \tag{2.0.2}$$

In [BDP17] the authors proved the existence of parabolic scattering solutions for the planar generalised N-centre problem. By scattering solution we mean a solution with prescribed asymptotic directions and prescribed behaviour around the centres. Some technical assumptions are required on the potential, which however include the standard case

$$\ddot{x} = -\sum_{i=1}^{N} \frac{m_i(x - c_i)}{|x - c_i|^{\alpha+2}}, \qquad x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\}$$

with $m_i > 0$ for $i = 1, \dots, N$ and $\alpha \in [1, 2)$. The interaction with the centres is prescribed in the sense that, for all non trivial partitions of the centres set $\Sigma = \{c_1, \dots, c_N\}$ in two subsets, a scattering solution exists which separates Σ according to the chosen partition and is self-intersection free. This approach however does not allow for self-intersections of the solution orbits.

Aim of this Chapter is, starting from the same setting, to prove existence of a richer family of scattering solutions which may display self-intersections in a sense inspired by the recent work [Cas17], which deals with positive energies solutions. We explicitly characterise the solutions by means of their topologically-prescribed interaction with the centres, while in other works ([BK17]) no qualitative descriptions of the solutions is pursued. To complete the discussion it is proven in the spirit of [FT00] (where instead the two-centres strong forces case is taken into account) the existence of multiple solutions which are semibounded.

2.1 Variational framework

Let $\Sigma = \{c_1, \dots, c_N\} \subset \mathbb{R}^2$ for $N \geq 2$. Without loss of generality from now on we suppose

$$\sum_{i=1}^{N} c_i = 0.$$

We are interested in finding solutions of (2.0.1) with the potential $U : \mathbb{R}^2 \setminus \Sigma \to \mathbb{R}$ satisfying the following conditions, coming from the setting in [BDP17]:

a) $U \in C^\infty(\mathbb{R}^2 \setminus \Sigma)$ and $U(x) > 0$ for all $x \in \mathbb{R}^2 \setminus \Sigma$;

b) $\exists \alpha \in [1, 2)$ such that close to Σ it holds

$$U(x) = \frac{m_i}{\alpha |x - c_i|^\alpha} + U_i(x) \qquad i = 1, \dots, N \tag{2.1.1}$$

with $m_i > 0$ and $U_i \in C^\infty(\mathbb{R}^2 \setminus (\Sigma \setminus \{c_i\})$;

c) for $|x|$ large enough it holds

$$U(x) = \frac{m}{\alpha |x|^\alpha} + W(x) \tag{2.1.2}$$

with $m > 0$ and

$$W(x) = O\left(\frac{1}{|x|^\beta}\right), \quad \nabla W(x) = O\left(\frac{1}{|x|^{\beta+1}}\right) \qquad \text{for } |x| \to +\infty \tag{2.1.3}$$

for some $\beta > \alpha/2 + 1$.

Potential estimates and the virial identity From (2.1.2) we can deduce that there exist $K > \sup_i |c_i|$ and some constants $C_-, C_+ > 0$ such that the following estimates hold for $|x| \geq K$:

$$2|W(x)| + |\nabla W(x) \cdot x| \leq \frac{(2-\alpha)m}{2\alpha} \frac{1}{|x|^\alpha} \tag{2.1.4}$$

$$\frac{C_-}{|x|^\alpha} \leq U(x) \leq \frac{C_+}{|x|^\alpha}\,. \tag{2.1.5}$$

For a parabolic solution of (2.0.1) it also holds the so-called "virial identity":

$$\frac{d^2}{dt^2}\left(\frac{|x(t)|^2}{2}\right) = 2U(x(t)) + \nabla U(x(t)) \cdot x(t). \tag{2.1.6}$$

Now, it follows from (2.1.4) that $\frac{d^2}{dt^2}|x(t)|^2$ is strictly positive for $|x| \geq K$ and, henceforth, that if a solution leaves either forward or backward in time the ball of radius K it cannot enter the ball again.

Homotopy classes Let now $K > 0$ be the one for which the estimates (2.1.4) and (2.1.5) hold true. In particular $\Sigma \subset B_K(0)$.

Let now $H^1(-1,1) = H^1([-1,1];\mathbb{R}^2)$ be the usual Sobolev space and consider also its subspace $H^1_p(-1,1) = \{u \in H^1(-1,1): \ u(-1) = u(1)\}$.

Referring to [FT00] define

$$\widehat{\Lambda} = \{u \in H^1_p(-1,1): \ u(t) \notin \Sigma \ \forall t \in (-1,1)\}$$
$$\widehat{\Gamma}(p_1,p_2) = \{u \in H^1(-1,1): \ u(t) \notin \Sigma \ \forall t \in (-1,1), \ u(-1) = p_1, u(1) = p_2\}.$$

We equip $\widehat{\Lambda}$ and $\widehat{\Gamma}(p_1,p_2)$ with the fixed endpoints homotopy relation, namely for $u_1, u_2 \in \widehat{\Gamma}(p_1,p_2)$ we say that u_1 is *homotopic* to u_2 and write $u_1 \sim u_2$ if and only if there exists a continuous function $H : [-1,1]\times[0,1] \to \mathbb{R}^2$ such that

$$H(t,0) = u_1(t), \ H(t,1) = u_2(t), \ H(0,s) = p_1, \ H(1,s) = p_2 \quad \forall s \in [0,1]$$

and if $t \in (-1,1)$ then $H(t,s) \in \mathbb{R}^2 \setminus \Sigma$ for all $s \in [0,1]$. In the case $u_1, u_2 \in \widehat{\Lambda}$ we consider instead the usual loops homotopy in $\mathbb{R}^2 \setminus \Sigma$.
From now on we restrict $\widehat{\Lambda}$ taking into account only non null-homotopic loops.

Maupertuis functional and properties The Maupertuis functional is defined on either $\widehat{\Lambda}$ or $\widehat{\Gamma}(p_1, p_2)$ as

$$\mathcal{M}(u) = \int_{-1}^{1} |\dot{u}(t)|^2 dt \int_{-1}^{1} U(u(t)) dt. \tag{2.1.7}$$

$\mathcal{M}$ is always nonnegative. We recall some of the properties of $\mathcal{M}$ giving references or a sketch of their proof.

Lemma 2.1.1. *The Maupertuis functional $\mathcal{M}$ is lower semicontinuous and coercive.*

Proof. The lower semicontinuity follows from standard arguments, whereas coercivity is not trivial due to the assumption (2.1.2).

Let u_n be a sequence such that

$$\|u_n\|_{H^1} \to +\infty\,. \tag{2.1.8}$$

For the case $u_n \in \widehat{\Gamma}(p_1, p_2)$ we drive a similar argument as in [BDP17], which with some modifications works also for $u_n \in \widehat{\Lambda}$ and

$$\min_{t\in[-1,1]} |u_n(t)| < +\infty\,.$$

We set ourselves in the first case, assuming that $u_n \in \widehat{\Gamma}(p_1, p_2)$, that (2.1.8) holds and reasoning by contradiction, by imposing that $\mathcal{M}(u_n)$ is bounded. The condition on the norm easily implies that $\int_{-1}^{1} |\dot{u}_n(t)| dt \to +\infty$, from which it follows that the sequence

$$\delta_n := \int_{-1}^{1} U(u_n(t)) dt$$

is infinitesimal and that there exists a sequence $\{t_n\} \in [-1, 1]$ such that

$$U(u_n(t_n)) \leq \frac{\delta_n}{2}.$$

Since U attains a positive minimum on every compact set, it must also be $|u_n(t_n)| \to +\infty$. The estimate (2.1.5) allows us to write

$$\delta_n \geq 2U(u_n(t_n)) \geq \frac{2C_-}{|u_n(t_n)|^\alpha}$$

for n large enough. Some calculations lead to

$$\int_{-1}^{1} |\dot{u}_n(t)|^2 dt \geq \left(\frac{C_-}{\delta_n}\right)^{2/\alpha}$$

which allow us to conclude

$$\mathcal{M}(u_n) \geq \left(\frac{C_-}{\delta_n}\right)^{2/\alpha} \delta_n = C_-^{2/\alpha} \delta_n^{1-2/\alpha} \to +\infty$$

obtaining a contradiction.

As for the case $u_n \in \widehat{\Lambda}$, if $\min_{t\in[-1,1]} |u_n(t)|$ is bounded then the previous argument can be adjusted to prove coercivity. For $u_n \in \widehat{\Gamma}(p_1, p_2)$ the above condition on the norm implies that

$$\int_{-1}^{1} |\dot{u}_n(t)| dt \to +\infty \tag{2.1.9}$$

and this can easily be seen using Hölder's inequality

$$|u_n - p_1| = \left|\int_{-1}^{t} \dot{u}_n(s) ds\right| \leq \int_{-1}^{1} |\dot{u}_n(t)| dt \leq \sqrt{2} \|\dot{u}_n\|_{L^2},$$

evaluating

$$\|u_n - p_1\|_{L^2}^2 = \int_{-1}^{1} |u_n(t) - p_1|^2 dt \leq 4\|\dot{u}_n\|_{L^2}^2$$

so to obtain

$$\|u_n\|_{L^2} \leq 2\|\dot{u}_n\|_{L^2} + \sqrt{2}|p_1|$$

and finally getting

$$\|u_n\|_{H^1} \leq 3\|\dot{u}_n\|_{L^2} + \sqrt{2}|p_1|.$$

From (2.1.9) it follows

$$\delta_n := \int_{-1}^{1} U(u_n(t))dt \to 0^+.$$

Following the proof of the first part of Lemma 4.2 in [BDT17] there exists a sequence $\{t_n\} \in [-1, 1]$ such that

$$U(u_n(t_n)) \leq \frac{\delta_n}{2}.$$

Since U attains a positive minimum on every compact set, it must also be $|u_n(t_n)| \to +\infty$. The estimate (2.1.5) allows us to write

$$\delta_n \geq 2U(u_n(t_n)) \geq \frac{2C_-}{|u_n(t_n)|^\alpha}$$

for n large enough: hence

$$\begin{aligned}\int_{-1}^{1} |\dot{u}_n(t)|^2 dt \geq \int_{-1}^{t_n} |\dot{u}_n(t)|^2 dt &\geq \frac{1}{2}\left(\int_{-1}^{t_n} |\dot{u}_n(t)| dt\right)^2 \\ &\geq \frac{1}{2}\left|\int_{-1}^{t_n} \dot{u}_n(t) dt\right|^2 = \frac{1}{2}|u_n(t_n) - p_1|^2 \\ &\geq \frac{1}{2}||u_n(t_n)| - |p_1||^2.\end{aligned}$$

Since $|u_n(t_n)| \to +\infty$ for any $c > 0$ there is n large enough so that $c|u_n(t_n)| - |p_1| > 0$: then

$$\begin{aligned}|u_n(t_n)| - |p_1| &\geq \left(\frac{2C_-}{\delta_n}\right)^{1/\alpha} - |p_1| = \\ &= \left(\frac{C_-}{\delta_n}\right)^{1/\alpha} + \left(\left(1 - \frac{1}{2^{1/\alpha}}\right)\left(\frac{2C_-}{\delta_n}\right)^{1/\alpha} - |p_1|\right) \\ &\geq \left(\frac{C_-}{\delta_n}\right)^{1/\alpha}\end{aligned}$$

which gives

$$\int_{-1}^{1} |\dot{u}_n(t)|^2 dt \geq \left(\frac{C_-}{\delta_n}\right)^{2/\alpha}$$

and lead us to conclude

$$\mathcal{M}(u_n) \geq \left(\frac{C_-}{\delta_n}\right)^{2/\alpha} \delta_n = C_-^{2/\alpha} \delta_n^{1-2/\alpha} \to +\infty$$

which is a contradiction.

Additional problems arise if $u_n \in \widehat{\Lambda}$ but instead

$$\min_{t\in[-1,1]} |u_n(t)| \to +\infty,$$

i.e. we consider orbits that do not interact with the centres but simply turn around Σ (possibly many times). The estimate (2.1.5) gives for n large enough

$$U(u_n) \geq \frac{C_-}{|u_n|^\alpha} \to 0.$$

Since u_n is not null-homotopic in $\mathbb{R}^2 \setminus \Sigma$ there exist s_n, t_n such that $-1 \leq s_n < t_n \leq 1$,

$$-\pi < \arg u_n(t) < \pi \qquad \forall t \in (s_n, t_n)$$

and

$$\arg u_n(s_n^+) = -\pi, \qquad \arg u_n(t_n^-) = \pi \qquad \text{or viceversa.}$$

We define then $v_n := u_{n|[s_n,t_n]}$.

$$\begin{aligned}
\mathcal{M}(u_n) &= \int_{-1}^{1} |\dot{u}_n(t)|^2 dt \int_{-1}^{1} U(u_n(t)) dt \\
&\geq \int_{-1}^{1} |\dot{u}_n(t)|^2 dt \int_{-1}^{1} \frac{C_-}{|u_n(t)|^\alpha} dt \\
&\geq C_-^{1/2} \left(\int_{-1}^{1} \frac{|\dot{u}_n(t)|}{|u_n|^{\alpha/2}} dt \right)^2 \\
&\geq C_-^{1/2} \left(\int_{s_n}^{t_n} \frac{|\dot{u}_n(t)|}{|u_n|^{\alpha/2}} dt \right)^2 \\
&= C_-^{1/2} \left(\int_{v_n} \frac{1}{|z|^{\alpha/2}} |dz| \right)^2 \\
&\geq C_-^{1/2} \left| \int_{v_n} \frac{1}{z^{\alpha/2}} dz \right|^2
\end{aligned}$$

using Hölder inequality and curvilinear complex integral notation, where we define

$$z^{\alpha/2} := \exp_{\mathbb{C}} \left\{ \frac{\alpha}{2} \operatorname{Log} z \right\} \qquad \text{for } z \in \mathbb{C} \setminus \{0\}$$

with the principal logarithm given by

$$\operatorname{Log} z = \log |z| + i \arg(z) \qquad \arg(z) \in]-\pi, \pi].$$

Going on with the estimates above

$$\begin{aligned}
\mathcal{M}(u_n) &\geq C_-^{1/2} \left| \int_{v_n} \frac{1}{z^{\alpha/2}} dz \right|^2 \\
&\geq C_-^{1/2} \left| \frac{2}{2-\alpha} \left(u_n(t_n)^{1-\alpha/2} - u_n(s_n)^{1-\alpha/2} \right) \right|^2 \\
&\geq \frac{4C_-^{1/2}}{(2-\alpha)^2} \left| |u_n(t_n)|^{1-\alpha/2} e^{i\pi(1-\alpha/2)} - |u_n(s_n)|^{1-\alpha/2} e^{-i\pi(1-\alpha/2)} \right|^2 .
\end{aligned}$$

Here we need the following geometric result, which comes from the simple fact that the longest side in a obtuse triangle is the one opposite to the obtuse angle.

Lemma 2.1.2. *For* $0 < \theta \le \pi/2$ *and* $R_1, R_2 > 0$ *it holds*

$$|R_1 e^{i\theta} - R_2 e^{-i\theta}| \ge 2\min(R_1, R_2)\sin\theta.$$

The proof is contained in the following Figure. Define

$$w_1 = \min(R_1, R_2)e^{i\theta}, \qquad w_2 = \max(R_1, R_2)e^{i\theta} :$$

then

$$|w_1 - \bar{w}_2| = |\bar{w}_1 - w_2| \ge |w_1 - \bar{w}_1| = 2\min(R_1, R_2)\sin\theta.$$

In our case $\theta_1 = \theta_2 = \pi(1 - \alpha/2)$ and

$$\min(R_1, R_2) = \min(|u_n(t_n)|^{1-\alpha/2}, |u_n(s_n)|^{1-\alpha/2}) \ge \min|u_n|^{1-\alpha/2}$$

and so we obtain

$$\mathcal{M}(u_n) \ge \frac{16C_-^{1/2}}{(2-\alpha)^2}\sin^2(\pi(1-\alpha/2))(\min|u_n|)^{2-\alpha}$$

but letting

$$\|u_n\|_{H^1} \to +\infty$$

it is clear that $\mathcal{M}(u_n) \to \infty$ as well, hence we conclude. □

The main feature of the Maupertuis functional is the link between its minimisers and the solutions of (2.0.1).

Lemma 2.1.3. *For each critical point* u *of* $\mathcal{M}$ *we have that*

$$x(t) = u\left(\frac{t}{\omega}\right), \qquad \omega = \left(\frac{\int_{-1}^{1}|\dot{u}(t)|^2 dt}{2\int_{-1}^{1}U(u(t))dt}\right)^{1/2}$$

is a smooth zero energy solution of (2.0.1) *on the interval* $[-\omega, +\omega]$.

Proof. Refer to [ST13, Theorem B.1]. □

We will make use of the connection between $\mathcal{M}$ and another functional, the *action* functional, given by

$$\mathcal{A}_{[-1,1]}(u) = \int_{-1}^{1} \left(\frac{1}{2} |\dot{u}(t)| + U(u(t)) \right) dt. \tag{2.1.10}$$

It can be shown that for any $u \in \widehat{\Gamma}(p_1, p_2)$ or $u \in \widehat{\Lambda}$ it holds

$$\sqrt{2\mathcal{M}(u)} = \inf_{\Theta>0} \mathcal{A}_{[-\Theta,\Theta]}(u(\cdot/\Theta))$$

and in particular for every minimiser of $\mathcal{M}$ it holds

$$\sqrt{2\mathcal{M}(u)} = \mathcal{A}_{[-\omega,\omega]}(x)$$

where u and x are linked as in the previous Lemma.

Minimisers and properties We recall some variational and topological properties of the Maupertuis minimisers that will be used later on.

Lemma 2.1.4. *a) $\mathcal{M}$ admits a minimiser in the weak H^1 closure of any homotopy class inside $\widehat{\Gamma}(p_1, p_2)$ or $\widehat{\Lambda}$.*

b) $\mathcal{M}$ is independent of linear time rescaling, hence linear riparametrisations of minimisers are still minimisers.

c) Let u be a minimiser of $\mathcal{M}$ in some homotopy class inside either $\widehat{\Lambda}$ or $\widehat{\Gamma}(p_1, p_2)$. Suppose u is collision-free and let $[a, b] \subset [-1, 1]$ and $q_1 = u(a), q_2 = u(b)$. Then the restriction

$$\bar{u}(t) = u\left(\frac{1}{2}((1-t)a + (1+t)b)\right)$$

minimises $\mathcal{M}$ in the homotopy class $[u_{|[a,b]}] \subset \widehat{\Gamma}(q_1, q_2)$.

d) If u and v are collision-free minimisers of $\mathcal{M}$ respectively in $\widehat{\Gamma}(p_1, p_2)$ and $\widehat{\Gamma}(p_2, p_3)$ then the concatenation

$$u\#v(t) = \begin{cases} u(2t+1) & t \in [-1, 0) \\ v(2t-1) & t \in [0, 1] \end{cases}$$

is a minimiser of $\mathcal{M}$ in its homotopy class inside $\widehat{\Gamma}(p_1, p_3)$.

Proof. For *a)* we give below a simple example on how a minimiser can be built thanks to the choice of α. *b)* follows from the definition of $\mathcal{M}$. For *c)* and *d)* we refer to [Cas17, Lemma 3.2].

Let $A = [u]$ be a homotopy class contained in $\widehat{\Gamma}(p_1, p_2)$. Let $\bar{A}$ be its weak H^1 closure and call

$$\tilde{A} = \{v \in \bar{A} : \ \mathcal{M}(v) < +\infty\}.$$

We build an explicit element in $\tilde{A}$. We note that blow up problems may arise only in the proximity of Σ, so we choose $p_1 = c_1$ and define

$$v(t) := c_1 + \left(\frac{1+t}{2}\right)^{\sigma} (p_2 - c_1).$$

Setting $M = |p_2 - c_1|$ and $\eta = (p_2 - c_1)/|p_2 - c_1|$, it holds

$$\dot{v}(t) = \frac{M\sigma}{2}\left(\frac{1+t}{2}\right)^{\sigma-1}\eta\,,$$

and $\dot{v}$ belongs to $L^2([-1,1])$ if and only if $\sigma > 1/2$. Recalling (2.1.1) we compute

$$\begin{aligned}
\mathcal{M}(v) &= \frac{M\sigma}{2}\int_{-1}^{1}\left|\frac{1+t}{2}\right|^{2\sigma-2} dt \int_{-1}^{1} \frac{m_1}{M\alpha\,|c_1 + ((1+t)/2)^{\sigma}(p_2 - c_1)|^{\alpha}} dt \\
&\quad + \int_{-1}^{1} U_1(x(t))dt \\
&\leq C + \frac{M\sigma m_1}{2\alpha}\int_{-1}^{1}\left(\frac{1+t}{2}\right)^{2\sigma-2-\alpha\sigma} dt
\end{aligned}$$

which is bounded if and only if $\sigma < 1/(2-\alpha)$. Hence choosing $\sigma \in]1/2, 1/(2-\alpha)[$ we have $\mathcal{M}(v) < +\infty$. A similar example can built when $[u] \subset \widehat{\Lambda}$. □

Remark 2.1.1. We point out that for $\alpha > 2$ the existence of a minimiser in each homotopy class is not guaranteed. On the other side, in that case there cannot be collision minimisers, hence each minimiser gives rise to an entire classical solution.

On the topological side we can immediately exclude some self-intersection behaviours of minimisers.

Lemma 2.1.5. *Let u be a minimiser of $\mathcal{M}$ in any homotopy class of $\widehat{\Gamma}(p_1, p_2)$ or $\widehat{\Lambda}$. Then*

1. *it cannot exist a null-homotopic subloop of u (1-gon), i.e. $[a, b] \subset [-1, 1]$ such that $u(a) = u(b)$ and $u_{|[a,b]}$ is null-homotopic;*

2. *it cannot exist a null-homotopic double self-intersection of u (2-gon), i.e. $I_1, I_2 \subset [-1, 1]$ such that $I_1 \cap I_2 = \emptyset$, u identifies the endpoints of I_1 and I_2 and $u_{|I_1} \# u_{|I_2}$ is null-homotopic.*

Proof. We refer to [Cas17, Proposition 4.6]. In general we rule out tangential self-intersections thanks to Lemma 2.1.3 and the uniqueness of the solution of the Cauchy problem associated to (2.0.1). For transversal self-intersections, here the main argument is the possibility to build a non-smooth minimiser: in the first case it is $u_{|[-1,a]} \# u_{|[b,1]}$ while in the second case it can be

$$u_{|[-1,a_2]} \# u^{-1}_{|[b_1,a_1]} \# u_{|[b_2,1]}$$

where we supposed $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, $-1 < a_1 < b_1 < a_2 < b_2 < 1$ and $u(a_1) = u(b_2)$, $u(b_1) = u(a_2)$. We briefly wrote $u^{-1}(t) = u(-t)$. □

Lemma 2.1.6. *Let $\{u_n\}_n$ be a sequence of minimisers of $\mathcal{M}$ and suppose it converges uniformly on compact sets to a limit u_∞. Then u_∞ inherits the minimising properties of the previous Lemmas on any compact set on which it is collision-free.*

Proof. Refer to [FT00, Lemma 1.5]. □

2.2 Topological characterisation

Let

$$\xi_1, \xi_2 \in \mathbb{S}^1, \qquad \xi_1 \neq \xi_2$$

be two given angles. For each fixed $R > K$ let $\gamma_R \in \widehat{\Gamma}(c_1, R\xi_1)$ be a self-intersection free path and be u_R a minimiser of $\mathcal{M}$ in the weak H^1 closure of the homotopy class $[\gamma_R]$. The existence of u_R is granted by Lemma 2.1.4. Call x_R the parabolic solution of (2.0.1) obtained after a time rescaling of u_R (Lemma 2.1.3), which is collision-free thanks to [ST13, Theorem 4.12]. In passing to the limit for $R \to +\infty$ we must check that

1. the (weak) limit Π_0 exists and corresponds to a generalised solution of (2.0.1)

2. Π_0 preserves the asymptotic direction ξ_1, i.e.

$$\lim_{t \to +\infty} \frac{\Pi_0(t)}{|\Pi_0(t)|} = \xi_1$$

3. Π_0 is collision-free.

The arguments for 1–3 follow the ones in [BDP17] and we will detail them more in the proof of Theorem 2.3.1.

We have thus built a branch Π_0 that joins c_1 to infinity without crossing Σ. With a iterative procedure we now build $\Pi_1, \Pi_2, \dots, \Pi_{N-1}$ as the parabolic solutions associated with minimisers of $\mathcal{M}$ in the homotopy classes of self-intersection free paths belonging to

$$\Gamma_i = \left\{ u \in \widehat{\Gamma}(c_i, c_{i+1}) : \; u(]-1,1]) \cap \bigcup_{j=0}^{i-1} \Pi_j = \emptyset \right\}, \qquad i = 1, 2, \dots, N-2$$

respectively. Note that Γ_i is never empty as

$$\mathbb{R}^2 \setminus \bigcup_{j=0}^{i-1} \Pi_j$$

is path-connected: moreover, it holds

$$\Pi_i \cap \bigcup_{j=0}^{i-1} \Pi_j = \{c_i\}, \qquad i = 1, 2, \dots, N-1$$

Eventually, we build Π_N in the same fashion as Π_0 but taking into account

$$\gamma_R \in \left\{ u \in \widehat{\Gamma}(c_N, R\xi_2) : \; u(]-1,1]) \cap \bigcup_{j=0}^{i-1} \Pi_j = \emptyset \right\}$$

and we call

$$\Pi = \bigcup_{i=0}^{N} \Pi_i.$$

Letters and words

Definition 2.2.1. A *word* is a sequence of letters

$$z = (l_i)_{i\in\mathcal{I}}, \qquad l_i \in \{0, 1, \ldots, N\} \; \forall i \in \mathcal{I} \subseteq \mathbb{Z}.$$

z is finite if $|\mathcal{I}| < +\infty$, semi-infinite if $\mathcal{I} = \mathbb{N} \setminus \{0\}$ or $\mathcal{I} = \mathbb{Z} \setminus \mathbb{N}$ and infinite if $\mathcal{I} = \mathbb{Z}$. A word is *non redundant* if for each $i, j \in \mathcal{I}$ such that $|i - j| = 1$ it holds $l_i \neq l_j$, i.e. it does not contain consecutive identical letters.

In the following, for a finite word z we denote by z^n the juxtaposition of n copies of z and by z^∞ the limit of z^n for $n \to \infty$. z^{-1} denotes the reverse word of z, which in the case of finite and infinite words simply means the specular word while in the case of semi-infinite words it also changes the unbounded side: if z is unbounded on the right ($\mathcal{I} = \mathbb{N}$) then z^{-1} is unbounded on the left ($\mathcal{I} = \mathbb{Z} \setminus \mathbb{N}$) and they read

$$z = l_1 l_2 \ldots l_n \ldots, \qquad z^{-1} = \ldots l_{-n} \ldots l_{-2} l_{-1}.$$

Definition 2.2.2. A smooth path γ in $\mathbb{R}^2 \setminus \Sigma$ is said to *realise* the word z if $\{t : \gamma(t) \in \Pi\} = \{t_i\}_{i\in\mathcal{I}}$ is such that $t_i < t_j$ for $i < j$ and

$$\gamma(t_i) \in \Pi_{l_i} \quad \forall i = 1, \ldots, n$$

with transversal intersection.

Note that non redundant words identify the homotopy class of γ in $\mathbb{R}^2 \setminus \Sigma$.

Remark 2.2.1. The definition features a discrete set of intersection instants because in the following γ will be a minimiser of the Maupertuis functional in some homotopy class and as such the intersections with Π will always be transversal. We will prove that the intersection instants with any Π_i with $1 \leq i \leq N$ cannot accumulate thanks to the uniqueness of the solution for the Cauchy problem associated to our problem, while outside a large ball containing Σ, say for $|x| \geq K$, the gravitational field is bounded and so the orbits cannot rotate fast enough to generate an accumulation point for the intersection instants.

The next definition is crucial.

Definition 2.2.3. A non redundant word z is *admissible* if either

$$|l_i - l_j| \geq 2 \qquad \forall i, j \in I \text{ such that } |i - j| = 1$$

or, for $a, b, b+1, c \in \{0, 1, \dots, N\}$,

1. if z contains the string $ab(b+1)c$ then

$$c < a < b \quad \text{or} \quad a < b < b+1 < c \quad \text{or} \quad b+1 < c < a \tag{2.2.1}$$

2. if z is a finite or semi-infinite word and begins with $b(b+1)c$ then

$$b + 1 < c$$

3. if z is a finite or semi-infinite word and ends with $ab(b+1)$ then

$$a < b$$

4. z^{-1} satisfies $1 - 3$.

Remark 2.2.2. Note that 2-3 can be viewed as particular cases of 1 choosing respectively $a = -1$ and $c = -1$ in (2.2.1). 4 means that admissible words can be read also in the opposite sense.

The admissibility of words and hence of homotopy classes find motivation in the work of Castelli [Cas17], in which paths that display a subloop around one single centre are not allowed. As we will see, when $1 \leq \alpha < 2$ the restriction to admissible classes is unavoidable in order to obtain collision-free solutions, which can be easily ruled out in the case $\alpha > 2$ (strong-forces case) as done in [FT00].

Our statement is more articulate being given directly on the symbols with which we characterise the solutions. The following Lemma clarifies the connection between the two definitions.

Lemma 2.2.1. *Let u be a collision-free minimiser of $\mathcal{M}$ in some homotopy class in $\widehat{\Gamma}(p_1, p_2)$ or $\widehat{\Lambda}$ which realises an admissible word z. Then any subloop of u must enclose at least two centres.*

Proof. We argue by contradiction. If u turns around a single centre then z must contain a string of the type $b(b+1)$ or $(b+1)b$. We then check point 1-3 of the admissible word definition. If we fall in the case of point 2, i.e. z begins with $b(b+1)c$, it is clear that to close the loop it must be $c \leq b$, but this contradicts the admissibility of z (note that $c \neq b+1$ because z is non redundant and there are no other cases as 1-gons and 2-gons cannot occur thank to Lemma 2.1.5). If instead we are in the case of point 3, i.e. z ends with $ab(b+1)$, then the loop closes only if $a \geq b+1$, which again is not allowed. Finally, if the string is of the type of point 1, $ab(b+1)c$, then we have two ways to close the loop around a single center:

- if $a < b$ then it must be $a \leq c \leq b$ but the first and second of (2.2.1) rule out this possibility;
- if $a > b+1$ then either $c \leq b$ or $c \geq a$, not allowed respectively by the second and third of (2.2.1).

We are finished as z cannot be admissible if u admits a subloop enclosing only one centre. $\square$

2.3 Main results and proofs

By scattering solutions we mean solutions with prescribed asymptotic directions and following a prescribed path around the centers.

Theorem 2.3.1. *Given*

$$\xi_-, \xi_+ \in \mathbb{S}^1, \qquad \xi_- \neq \xi_+$$

and a finite admissible word z of odd length such that $0 < l_i < N$ for at least one letter, there exists a zero energy solution x of (2.0.1) *which realises z and satisfies*

$$\lim_{t \to \pm\infty} |x(t)| = +\infty, \qquad \lim_{t \to \pm\infty} \frac{x(t)}{|x(t)|} = \xi_\pm. \tag{2.3.1}$$

We ask for at least one letter belonging to $\{1, 2, \dots, N-1\}$ to avoid trivial parabolic solutions such as the keplerian ones in the case $N = 1$.

Remark 2.3.1. This theorem can be viewed as a generalisation of the result obtained in [BDP17]. There, no self-intersections were allowed and the scattering solutions were separating Σ in two given subsets.

Once the partition $\mathcal{P} \subset \Sigma, \mathcal{P} \neq \emptyset, \Sigma$ and the asymptotic directions $\xi_- \neq \xi_+$ are given, it is always possible to build Π such that the centres contained in $\mathcal{P}$, say, $\{c_1, \dots, c_k\}$, are joined first and the centres not contained in $\mathcal{P}$, say, $\{c_{k+1}, \dots, c_N\}$, come after. Then the word realised by the desired scattering solution is simply $z = k$.

Remark 2.3.2. Π can be viewed as a singular scattering solution with asymptotic directions ξ_1, ξ_2.

Proof. The proof follows closely the scheme proposed in [BDP17, Sections 3.2–4]: scattering solutions are obtained via a limiting procedure of fixed endpoints problems (Bolza problems). To pass to the limit we must first prove that there is convergence in a suitable sense; subsequently we must check that the limit is a classical parabolic solution that preserves the asymptotic directions ξ_-, ξ_+ and realises z.

Given $\xi_- \neq \xi_+$ we can write

$$\xi_\pm = e^{i\theta_\pm}$$

with, say, $0 \leq \theta_- < \theta_+ < 2\pi$. Now fix $\theta_1 \in (\theta_-, \theta_+)$ and $\theta_2 \in [0, 2\pi) \setminus [\theta_-, \theta_+]$ and build Π as in Section 2.2 with asymptotic directions

$$\xi_1 = e^{i\theta_1}, \qquad \xi_2 = e^{i\theta_2}.$$

Let now $R > K$ be fixed and $\gamma_R \in \widehat{\Gamma}(R\xi_-, R\xi_+)$ be any path that realises z. Call u_R a minimiser for $\mathcal{M}$ in the homotopy class of γ_R and let $x_R : [-\omega_R, +\omega_R] \to \mathbb{R}^2$ be the parabolic solution obtained after a time rescaling of u_R. x_R is collision-free thanks to [ST13, Theorem 4.12] (note that (2.1.1) is crucial). We need to prove that the action functional

$$\mathcal{A}_{[-1,1]}(u) = \int_{-1}^{1} \left(\frac{1}{2}|\dot{u}(t)| + U(u(t)) \right) dt \tag{2.3.2}$$

evaluated on x_R inside the ball of radius R is uniformly bounded with respect to R, namely

$$\limsup_{R \to +\infty} \mathcal{A}_{[t_R^-, t_R^+]}(x_R) < +\infty \tag{2.3.3}$$

where $t_R^- < t_R^+$ are the unique instants for which $|x_R(t_R^\pm)| = K$. This is done exactly as in [BDP17, Section 3.2], the only difference being in an arbitrary path $\gamma \in \widehat{\Gamma}(K\xi_-, K\xi_+)$ which in our case may display self-intersections but surely preserves the boundedness of its action. From (2.3.3) we deduce after brief calculations that $\|x\|_{H^1_{loc}}$ is uniformly bounded: hence, x_R converges weakly in $H^1_{loc}(\mathbb{R})$ and uniformly on compact sets to a generalised parabolic solution x_∞ of (2.0.1).

The proof that x_∞ inherits the asymptotic directions $\xi_\pm$ is carried on in [BDP17, Section 3.3]. To show that x_∞ is collision-free the usual Levi-Civita regularisation works for the case $\alpha = 1$, whereas if $\alpha \in (1, 2)$ the argument of [BDP17, Section 3.4] can be adapted as well because of our choice of admissible classes: we recall it briefly. If

$$2\delta := \min_{i \neq j} |c_i - c_j|$$

then each x_R cannot self-intersect inside $B_\delta(c_i)$ for all i, otherwise it would contradict either the admissibility of z (loop around a centre), the regularity of minimisers (empty loop with transversal intersection) or the uniqueness of solutions for the Cauchy problem associated to (2.0.1) (empty loop with tangential intersection). $\square$

The result on semibounded solutions is given in the case of a semi-infinite word z unbounded on the right ($I = \mathbb{N} \setminus \{0\}$). A symmetrical result holds if z is unbounded on the left ($I = \mathbb{Z} \setminus \mathbb{N}$) and the proof can be carried on similarly.

Theorem 2.3.2. *Given $\xi_- \in \mathbb{S}^1$ and a semi-infinite admissible word z such that $0 < l_i < N$ for at least one letter, there exists a zero energy solution x of* (2.0.1) *which realises z and satisfies*

$$\lim_{t\to-\infty} |x(t)| = +\infty, \qquad \limsup_{t\to+\infty} |x(t)| < +\infty, \qquad \lim_{t\to-\infty} \frac{x(t)}{|x(t)|} = \xi_-. \tag{2.3.4}$$

Proof. Let

$$z = l_1 l_2 l_3 \ldots = (l_i)_{i=1}^{\infty}$$

be a semi-infinite admissible word and fix $\xi_\pm \in \mathbb{S}^1$, $\xi_- \neq \xi_+$. Let ξ_1, ξ_2 be two "intermediate" directions and build Π as in the scattering case. Now for all fixed $n \in \mathbb{N}$ and $R > K$ let $x_{n,R}$ denote a parabolic solutions related to the homotopy class of $\gamma \in \widehat{\Gamma}(R\xi_-, R\xi_+)$ where γ realises $[z]_n$, i.e. the truncation of z up to the first $2n+1$ letters. Suppose for simplicity sake that

$$0 < l_1, l_{2n+1} < N,$$

otherwise take into account the greatest odd length string of $[z]_n$ such that the first and last letter belong to $\{1, 2, \ldots, N-1\}$.

We fix R and for now we drop the label. After a suitable time rescaling let

$$|x_n(0)| = K = |x_n(T_n)|$$

and

$$0 < t_1^n < t_2^n < \ldots < t_{2n+1}^n < T_n$$

be the intersection instants with Π, namely

$$x_n(t_i^n) \in \Pi_{l_i}.$$

Since Π_{l_i} is compact for all $i = 1, 2, \dots, N-1$ there exists p_i such that up to a subsequence

$$x_n(t_i^n) \to p_i \in \Pi_{l_i} \qquad \text{for } n \to +\infty.$$

Consider now two consecutive letters, for simplicity sake l_1 and l_2. If $|l_1 - l_2| \geq 2$ then, called

$$\delta^* = \min\left\{|x-y| : \ x \in \Pi_{l_i}, \ y \in \Pi_{l_j}, \ |l_i - l_j| \geq 2\right\},$$

it clearly holds

$$|p_1 - p_2| \geq \delta^*.$$

In passing to the limit for $n \to +\infty$ it must be

$$x_{n|[t_1^n, t_2^n]} \to x_{|[t_1, t_2]}$$

with $x(t_1) = p_1$, $x(t_2) = p_2$. Indeed, if after suitable time rescaling $x_{|[t_1,t_2]}$ is a minimiser of $\mathcal{M}$, then it holds

$$t_2 - t_1 \geq \mu > 0$$

where μ depends possibly on δ^* and $\mathcal{M}(x)$ but is independent of n. This follows from

$$\mathcal{A}_{[t_1,t_2]}(x) \leq 2\int_{t_1}^{t_2} U(x(t))dt \leq (t_2 - t_1)\sup U(x)$$

If instead $|l_1 - l_2| = 1$ then it may happen that $p_1 = p_2 = c_{\max\{l_1, l_2\}}$. In this case however we can be sure that $p_3 \neq p_1$ because of the admissibility of $[z]_n$. More generally, for all $2 \leq i \leq 2n$ it cannot happen

$$p_{i-1} = p_i = p_{i+1}$$

and henceforth it must be

$$\max\{t_i - t_{i-1}, t_{i+1} - t_i\} \geq \mu > 0.$$

It easily follows that, upon proving the limit $x(t)$ is a local minimiser for $\mathcal{M}$,

$$T_n \geq \sum_{i=1}^{2n}(t_{i+1}^n - t_i^n) \to +\infty \qquad \text{for } n \to +\infty$$

and hence x remains trapped inside the ball B_K.

We devote the rest of the proof to show that $x_{n,R} \to x$ weakly in H^1_{loc} and uniformly on compact sets by means of a uniform bound on the action (2.1.10) inside the ball B_K with respect to both n and R.

Since ∂B_K we call p_0 the limit of $x_{n,R}(0)$ for $n \to +\infty$. We fix $\bar{n} \in \mathbb{N}$ and a path $\bar{\gamma}$ such that

$$\bar{\gamma}(-1) = p_0, \qquad \bar{\gamma}(+1) = p_{2\bar{n}}$$

and $\bar{\gamma}$ realises $[z]_{\bar{n}-1}$. For any $n > \bar{n}$ we define on $[-1,1]$

$$\begin{aligned}
\eta_1(t) &= R\exp\left(\arg(x_{n,R}(0)) + \frac{t+1}{2}(\arg(p_0) - \arg(x_{n,R}(0))\right) \\
\eta_2(t) &= \Pi_{l_{2\bar{n}}|[a,b]}\left(a + \frac{t+1}{2}(b-a)\right)
\end{aligned}$$

where $\Pi_{l_{2\bar{n}}}(a) = p_{2\bar{n}}$ and $\Pi_{l_{2\bar{n}}}(b) = x_{n,R}(t_{2\bar{n}}^n)$. We then call

$$\zeta(t) = \begin{cases} \eta_1(t+2) & -3 \leq t < -1 \\ \bar{\gamma}(t) & -1 \leq t < 1 \\ \eta_2(t-2) & 1 \leq t \leq 3 \end{cases}$$

and by $\tilde{\zeta}(t) = \zeta(t/3)$.

We can now proceed to evaluate the action for $x_{n,R}$ over the fixed time interval $[0, t_{2\bar{n}}^n]$. First, by the zero energy equation we have

$$\mathcal{A}_{[0,t_{2\bar{n}}^n]}(x_{n,R}) = \sqrt{2\mathcal{M}(u_{n,R})}$$

where $u_{n,R}$ is a $[-1,1]$ time rescale of $x_{n,R|[0,t_{2\bar{n}}^n]}$. By minimality

$$\sqrt{2\mathcal{M}(u_{n,R})} \leq \sqrt{2\mathcal{M}(\tilde{\zeta})} = \inf_{\theta>0} \mathcal{A}_{[-\theta,\theta]}(\tilde{\zeta}(t/\theta))$$

and taking advantage of the linearity of the action functional and its independence of time translations

$$\inf_{\theta>0} \mathcal{A}_{[-\theta,\theta]}(\tilde{\zeta}(t/\theta)) \leq \mathcal{A}_{[-3,3]}(\zeta) = \mathcal{A}_{[-1,1]}(\eta_1) + \mathcal{A}_{[-1,1]}(\bar{\gamma}) + \mathcal{A}_{[-1,1]}(\eta_2).$$

Recalling the definition of η_2 it is easy to infer

$$\mathcal{A}_{[-1,1]}(\eta_2) \leq \mathcal{A}_{[a,b]}(\Pi_{l_{2\bar{n}}|[a,b]}) \leq \mathcal{A}(\Pi_{l_{2\bar{n}}}) = M_2$$

which is constant and bounded being $\Pi_{l_{2\bar{n}}}$ a parabolic solution of (2.0.1). As for $\bar{\gamma}$, being smooth and bounded away from $\Sigma \setminus \{c_{l_{2\bar{n}}}, c_{l_{2\bar{n}+1}}\}$ it must be

$$\mathcal{A}_{[-1,1]}(\bar{\gamma}) \leq M_0$$

and finally, since η_1 is as well smooth and bounded away from Σ it also holds

$$\mathcal{A}_{[-1,1]}(\eta_1) \leq M_1.$$

Therefore, for all $n > \bar{n}$ it holds

$$\mathcal{A}_{[0,t^n_{2\bar{n}}]}(x_{n,R}) \leq M_0 + M_1 + M_2 < +\infty$$

from which we deduce that $\|x_{n,R}\|$ is bounded in H^1_{loc} and hence we have uniform convergence on compact sets. $\square$

Chapter 3

Persistence in ecological systems and periodic solutions

The concept of persistence has undergone a huge increase of popularity throughout biomathematics in the last two or three decades. From epidemiological models to competing-cooperative species ones, the coexistence of all populations is a key property often investigated, and the conditions for its validity or its failure are increasingly refined in order to apply to more and more models, both of theoretical or applied origin, and described by differential equations of all kinds.

In this Chapter we focus our attention to the mathematical interpretation of time-persistence in the context of models described via ordinary differential equations, thus evaluating the asymptotic behaviour of the models and in doing so entering the vast framework of semidynamical systems. Though other types of persistence have recently arisen much interest (e.g. spatial persistence and pattern formation) our discussion is nonetheless actual as a large part of literature deals with such problems, with many numerical studies that have still to find rigorous explanations on the mathematical side. We point out that persistence comes to be a weak assumption when looking at the asymptotic dynamics: for such it is commonly taken into account its stronger counterpart, uniform persistence, which asks for extinction to be avoided also asymptotically.

Historically speaking there have been two main mathematical approaches to persistence, with a peak of works at the end of the 80s, also if the basis of the discussion were laid before [FW77, FW84]. Usually addressed as "flow on the boundary" and "Lyapunov-like function" ones, they both share the focus on the behaviour of the model near the boundary of the domain, which usually embodies the concept of extinction. The "flow on the boundary" approach, which finds its core result in the well-known work [BFW86], analyses the flow restricted on the boundary and by a suitable decomposition generates conditions to check on the flow that assure the global persistence: see [Hof89] and references therein for a concise but effective presentation of this approach. The 'Lyapunov-like function" key feature is in displaying a generalised Lyapunov function that increases along the orbits, thus guaranteeing the repulsiveness of the boundary: the first result on this side is given [Hut84] and improved on the point of view of applicability in [Fon88], but many more generalisations have been proposed also in recent times, and we recall especially [RMB14] as it will be our main source in Section 3.2. The two approaches find some sort of unification in [Hof89], being combined through the notion of Morse decomposition of the boundary: hypotheses of one approach or the other can be checked as needed on each set of the decomposition, allowing a greater feasibility. For a thorough account on these two approaches we suggest the survey [Wal91].

A feature that has granted persistence such exposure is its connection with a very important quantity in ecoepidemiology, the basic reproduction number R_0, a crucial tool in studying biological and ecological models and especially epidemics. Introduced in the well-known Kermack-McKendrick model, it has found many applications and has been generalised in many ways, one for all [DHM90]. For a complete historical account we refer to the interesting book [Bac11]. A first interpretation sees R_0 as the number of secondary cases produced by an infectious individual in a completely susceptible population. When the model displays periodic coefficients or delays, the meaning of R_0 is less immediate and can enclose more specific informations on the model, see for example [BAD12]. Even in the case of autonomous models R_0 can be deceiving if we follow the simplistic definition given above: a striking example is given in

[BH13b], which will be discussed in detail in Section 3.2.

Many works have highlighted the connection between persistence and R_0: the role of R_0 as an invasion marker motivates its natural appearances in survival or extinction conditions such as persistence. Without claiming to be complete we enumerate some general results on the side of biology and epidemics which are close to our framework and which we will recall afterwards in our discussion: [MR03, RMB12, GR16].

Another connection can be found between persistence and the existence of periodic solutions when the model features periodic coefficients. Many techniques have been applied, mainly based on the topological degree theory and fixed point theorems, to obtain some general existence results. Our dissertation focuses on [Fer90], in which these results are given in the context of periodic processes, which include semidynamical systems. We stress that in general conditions like uniform persistence and permanence do not necessarily imply the existence of periodic solutions, and we provide a counterexample in Section 3.3. Other works [Zan92, BZ98] underline the association between fixed points of the Poincaré map and persistence, and both properties have extensively studied: we recall three papers that are near in spirit to the discussion and deal with predator-prey systems [Kir89, LGOT96, Tin01].

The aim of this Chapter is double. We introduce some definitions and tools in the framework of semidynamical systems and locally compact metric spaces, although we point out that concepts like uniform persistence or permanence can be defined in much more general frameworks, as done in [FG15]. We choose to present them in such fashion for the applications' sake. In Section 3.2 we analyse two autonomous models which have been studied numerically, providing a neat theoretical result of uniform persistence: we complete the discussion in the Appendixes where a full stability analysis is performed. In Section 3.3 we start from uniform persistence to provide a general existence theorem for periodic solutions in the more general context of periodic processes, following the line of [Fer90] and applying standard fixed point results as well as some degree theory. Our approach is close to the one exposed in [Zha95]. The theorem and the persistence tools are then applied to three different models coming from the biological area to obtain a periodic solution as well as uniform persistence.

3.1 Theoretical framework

Let $\mathbb{R}_+ = [0, +\infty[$ denote from now on the non negative half line and $(\mathcal{X}, d)$ be a generic locally compact metric space. Given a closed subset $X \subseteq \mathcal{X}$ let π be a semidynamical system defined on X, meaning a continuous map

$$\pi : X \times \mathbb{R}_+ \to X$$

such that

$$\begin{array}{ll} i) & \pi(x,0) = x \\ ii) & \pi(\pi(x,t),s) = \pi(x,t+s) \end{array} \qquad \forall x \in X,\ \forall s,t \geq 0.$$

We introduce concepts and results in the framework of continuous semidynamical systems as it is the most natural when dealing with applications coming from ecology: a clear example is given in Section 3.2. All of the following can be extended smoothly to dynamical systems, even in the case of local (semi)dynamical systems, taking advantage of the metric space setting and results such as Vinograd's Theorem (see [Car72]). Section 3.3 will be developed in the framework of ω-periodic processes, which better describe flows coming from periodic coefficients ODEs: although we will provide extensions of definitions and results note that a discrete semidynamical system can be associated with each ω-periodic process, hence again we can extend the following contents, provided some care in treating objects no more continuous.

We fix some notation that will be used throughout the chapter. The ω-limit of a point $x \in X$ is defined as

$$\omega(x) := \{y \in X :\ \exists 0 < t_n \nearrow +\infty \text{ such that } \pi(x,t_n) \to y\}.$$

We will assume that π is *dissipative*, i.e. for each $x \in X$ it holds $\omega(x) \neq \emptyset$ and the set

$$\bigcup_{x \in X} \omega(x)$$

is precompact in X (i.e. has compact closure in X).

Remark 3.1.1. We hereby note that a ω-limit set does not need to be closed, even in the case of a dissipative semidynamical system. Consider the Duffing equation in the phase-plane:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - 2x^3. \end{cases}$$

The energy is given by $E(x, y) = y^2/2 - x^2/2 + x^4/2$, so that for each $c \geq 0$ the energy level set $E_c := \{(x, y) \in \mathbb{R}^2 : \ E(x, y) = c\}$ corresponds to a closed orbit. If we the define π as the flow of the system on

$$X := \{(x, y) \in \mathbb{R}^2 : \ E(x, y) \leq c\}$$

for some $c > 0$ then being X compact π is dissipative but it is easy to see that

$$\omega(X) = X \setminus (E_0 \setminus \{(0, 0)\}) \, ,$$

which is not closed. $\square$

The stable manifold of a subset $Y \subseteq X$ is given by

$$W^s(Y) := \{x \in X : \ \omega(x) \subseteq Y\}.$$

Some other concepts are introduced in order to better unfold the discussion.

Definition 3.1.1. Let $Y \subseteq X$. Y is

- *forward invariant* with respect to π if and only if

$$\pi(Y, t) \subseteq Y \quad \forall \, t \geq 0 \, ;$$

- *isolated* in the sense of [Con78] if it is forward invariant and there exists a neighbourhood $\mathcal{N}(Y)$ of Y such that Y is the maximal forward invariant set in $\mathcal{N}(Y)$;

- a *uniform repeller* with respect to π if Y is compact and if it exists $\eta > 0$ such that

$$\liminf_{t \to +\infty} d(Y, \pi(x, t)) \geq \eta \qquad \forall \, x \in X \setminus Y.$$

The semidynamical systems π is called

- *uniformly persistent* if ∂X is a uniform repeller;
- *permanent* if π is dissipative and uniformly persistent.

□

Permanence implies the existence of a compact forward invariant attractor $K \subset \operatorname{int} X$ such that $d(K, \partial X) > 0$ (see [Con78, Chapter II.5] or [Hut84, Theorem 2.2], recapped later in Theorem 3.3.1).

In this preliminary Section we give some criteria for the global uniform persistence of π. All the following results come from the two approaches to persistence illustrated before, the "flow on the boundary" and the "Lyapunov-like function" ones. In enunciating them we provide here and there some slight extensions but the main goal of our work is to apply them in an original way, combining these criteria and checking them on suitable partitions of the boundary and in a carefully chosen order.

On the "flow on the boundary" side, the key feature of [Hof89] is splitting the repulsive set into a Morse decomposition, allowing to check some necessary and sufficient conditions for uniform repulsiveness on each Morse set instead of the whole repeller.

Definition 3.1.2. Let Σ be a closed subset of X. A *Morse decomposition* for Σ is a finite collection $\{M_1, M_2, \ldots, M_n\}$ of pairwise disjoint, compact, forward invariant sets such that for each $x \in \Sigma$ either $x \in M_i$ or $\omega(x) \subseteq M_i$ for some $i = 1, 2, \ldots, n$.

Our first repelling condition, illustrated in [BFW86], is stated in [Hof89] as follows.

Theorem 3.1.1. *Let Σ be a closed forward invariant subset of X. Given a Morse decomposition $\{M_1, M_2, \ldots, M_n\}$ of Σ such that each M_i is isolated in X, a necessary and sufficient condition for Σ being a uniform repeller is that*

$$W^s(M_i) \subseteq \Sigma \qquad \forall\, i = 1, 2, \ldots, n.$$

This quite powerful result allows to check the repelling conditions not on the whole Σ but on the Morse sets instead, which by definition are the ω-limits. In the case of an ODEs system these usually coincide with the equilibria on the boundary of the domain and are quite easy to compute: the difficult task is to evaluate the stable manifolds of such equilibria, which in general may be quite complicated objects.

On the "Lyapunov-like function" side, [Fon88] provides many useful criteria to determine whether a set is a uniform repeller or not. One sufficient condition is given when the set Σ is not forward invariant.

Theorem 3.1.2. *Let $\Sigma \subset X$ be compact and $X \setminus \Sigma$ be forward invariant. If for any $x \in \Sigma$ there exists $t_x > 0$ such that*

$$\pi(x, t_x) \in X \setminus \Sigma$$

then Σ is a uniform repeller.

We note immediately that if $\Sigma = \partial X$ then a straightforward consequence of the theorem is that when the flow computed against the normal outward vector to the boundary returns a negative quantity then the system is uniformly persistent.

The main result of this second approach to persistence was first enunciated in [Hut84] and later successfully improven in [Fon88] on the side of applicability.

Theorem 3.1.3. *Let $\Sigma \subset X$ be compact and such that $X \setminus \Sigma$ is forward invariant: suppose moreover that it exists a continuous function $\Lambda : X \to \mathbb{R}_+$ satisfying*

i) $\Lambda(x) = 0$ *if and only if* $x \in \Sigma$ *;*

ii) there exists a lower semicontinuous and bounded below function $\psi : X \to \mathbb{R}$ and $\alpha \in [0, 1]$ such that

$$\begin{aligned} &a) \quad \dot{\Lambda}(x) \geq \Lambda^{\alpha}(x)\psi(x) \qquad \forall x \in X \setminus \Sigma \\ &b) \quad \sup_{t>0} \int_0^t \psi(\pi(x, s))ds > 0 \qquad \forall\, x \in \overline{\omega(\Sigma)}. \end{aligned}$$

Then Σ is a uniform repeller.

The derivative that appears in *a)* denotes differentiation along the orbits or Dini derivative (see for instance [LaS76]). In the following applications, when Λ is differentiable on a open subset of $\mathbb{R}^N$ containing X and the semidynamical system π is the flow of an ODE of the type $x' = f(x)$, we can write

$$\dot{\Lambda}(x) = \langle D\Lambda(x), f(x)\rangle.$$

Note that this result is derived from a more general abstract theorem in [Fon88], in which the differentiabily of Λ along the orbits is not required and the statement gives necessary and sufficient conditions for uniform repulsiveness. We point out that some useful discussions on this work and a minor correction of the proof are carried on in [Tel07].

Theorem 3.1.3 can bump into some issues when dealing with models of various genesis due to the assumption of Σ being compact. Many versions of this result have been proposed: in [Hof89] a similar result is given for Σ closed and forward invariant, which again can cause some problems as in general the boundary is not wholly forward invariant. In Section 3.2 we give a very useful generalisation that is illustrated in [RMB14], where the hypotheses are weakened with a payback in terms of dissipativity of the semidynamical system π: this result is based on [Fon88, Theorem 1] and we add a characterisation close in spirit to Theorem 3.1.3.

We conclude this brief introductory Section by pointing out that all these three sufficient conditions for uniform persistence (along with many others that we do not recall here because not needed) can be combined in some clever way when dealing with models coming from the literature, as in general each of them fail on the whole boundary. This is our aim in Section 3.2, where an iterative procedure is applied, and Section 3.3, where instead a juxtaposing scheme is used in the applications.

3.2 An unusual autonomous prey-predator model with a diseased species

In [BH13b] the Authors illustrate two models displaying some interesting asymptotic behaviours with peculiar consequences on the basic reproduction number and its reliability in predicting the evolution of the systems. Among others two models are studied, featuring a prey and a predator population with an infectious disease affecting one or the other. It is shown by numerical means that under some choices of the parameters the basic reproduction number R_0^* associated with the non trivial disease-free stationary equilibrium fails in its role as a marker for the spread of the disease. Indeed, as under those hypotheses the equilibrium is not stable, it seems quite a natural assumption that an analysis focused on such point may not reflect the asymptotic dynamics of the system. Instead, the stable limit cycle bifurcating from the unstable equilibrium may be the right place to investigate for the long-term behaviour. To the best of our knowledge no formal explanation for this insight has been proposed yet, and that is the aim of this Section.

The interesting feature of the systems we take into account is that although they are autonomous the disease-free dynamics is ruled by a limit cycle, which acts much like a periodic perturbation of the systems themselves. Under this point of view the newly defined $\overline{R_0}$, an average quantity evaluated over the period of the limit cycle, is close to the one introduced in [BG06] and [WZ08].

We make use of the mathematical theory of persistence introduced before to reach a result that includes and justifies some of the numerical evidence in [BH13b], pointing out also some differences and counterexamples as well as some open problems. After a theoretical preamble we analyse the model and its flow on the boundary and test the conditions for uniform persistence mainly given in [RMB14]. We discuss the results and their connections with the original work. In Appendix A we deal with the analysis of the disease-free model, for which existence and stability of the equilibria are illustrated, while in Appendix B a full stability analysis is carried on for the two models taken into account.

We fit in the framework illustrated in the previous Section, thus having a semidynamical system π defined on a closed subset X of a locally compact metric space. As anticipated, we provide a handy version of Theorem 3.1.3 as exposed in [RMB14], adding a slight modification in order to derive suitable conditions for the models we take into account.

Theorem 3.2.1. *Let Σ be a closed subset of X such that $X \setminus \Sigma$ is forward invariant. Suppose X admits a compact global attractor K in its interior and that there exist a closed neighbourhood V of Σ and a continuous function $\Lambda : X \to \mathbb{R}_+$ such that*

i) $\Lambda(x) = 0$ *if and only if* $x \in \Sigma$ *;*

ii) $\forall\, x \in V \setminus \Sigma \quad \exists\, t_x > 0 : \quad \Lambda(\pi(x, t_x)) > \Lambda(x)$*.*

Then Σ is a uniform repeller. If moreover Σ is forward invariant then condition ii) can be replaced with

ii') it exists a continuous function $\psi : X \to \mathbb{R}$, bounded below and such that

a) $\dot{\Lambda}(x) \geq \Lambda(x)\psi(x) \qquad \forall\, x \in V \setminus \Sigma$ *;*

b) $\sup_{t>0} \int_0^t \psi(\pi(x, s))ds > 0 \quad \forall\, x \in \overline{\omega(\Sigma)}$*.*

Proof. The only thing that needs proof is that *ii')* implies *ii)* when Σ is forward invariant: the result is otherwise proven in [RMB14] in the same framework. We use the proof scheme illustrated in [Hut84] and [Fon88].

The first step is to prove that given *b)* the average condition holds true for all $x \in \Sigma$. Since ψ is continuous, the map $y \mapsto \sup_{t>0} \int_0^t \psi(\pi(y, s))ds$ is lower semicontinuous. Being closed and contained in the compact global attractor K the set $\overline{\omega(\Sigma)}$ is compact, thus condition *ii')* and the permanence of sign guarantee that there exist $\delta > 0$ and an open neighbourhood W_δ of $\overline{\omega(\Sigma)}$ such that

$$\sup_{t>0} \int_0^t \psi(\pi(y, s))ds > \delta \quad \forall\, y \in W_\delta.$$

Fixed $x \in \Sigma$, by definition of $\overline{\omega(\Sigma)}$ there exists t_0 such that $\pi(x, t) \in W_\delta$ for all $t \geq t_0$, so let us call $x_1 := \pi(x, t_0)$. By definition $x_1 \in W_\delta$ and by the previous inequality there exists t_1 such that $\int_0^{t_1} \psi(\pi(x_1, s))ds > \delta$. If we call

$x_2 := \pi(x_1, t_1) = \pi(x, t_0 + t_1)$ we can iterate the procedure and come at last to

$$\begin{aligned}
\sup_{t>0} \int_0^t \psi(\pi(x,s))ds &\geq \int_0^{t_0} \psi(\pi(x,s))ds + \sum_{k=1}^{n} \int_{t_{k-1}}^{t_{k-1}+t_k} \psi(\pi(x,s))ds \\
&\geq \int_0^{t_0} \psi(\pi(x,s))ds + \sum_{k=1}^{n} \int_0^{t_k} \psi(\pi(x_k,s))ds \\
&> \int_0^{t_0} \psi(\pi(x,s))ds + n\delta.
\end{aligned}$$

By choosing n large enough we get that *b)* holds for any $x \in \Sigma$.

Now, condition *b)* and the continuity of ψ implies that there exists an open neighbourhood U of Σ such that

$$\forall\, x \in U \quad \exists\, t_x > 0 : \quad \int_0^{t_x} \psi(\pi(x,s))ds > 0.$$

Applying condition *a)* on a closed neighbourhood V' of Σ contained in $V \cap U$ returns

$$0 < \int_0^{t_x} \psi(\pi(x,s))ds \leq \int_0^{t_x} \frac{\dot{\Lambda}(\pi(x,s))}{\Lambda(\pi(x,s))} ds = \log \frac{\Lambda(\pi(x,t_x))}{\Lambda(x)}$$

for all $x \in V' \setminus \Sigma$, which of course implies *ii)*. $\square$

We remark the fact that the second of the last chain of inequalities follows from hypothesis *a)* even in the case of a general Dini derivative, thanks to some general differential inequalities for which we refer to [LL69].

3.2.1 Models and boundary flow analysis

The model proposed in [BH13b] for the case of a prey-predator population with a diseased predator is described by the following non linear autonomous system:

$$\begin{cases} \dot{N}(t) = rN(t)(1 - N(t)) - \dfrac{N(t)(S(t) + I(t))}{h + N(t)} \\ \dot{S}(t) = \dfrac{N(t)(S(t) + I(t))}{h + N(t)} - mS(t) - \beta S(t)I(t) \\ \dot{I}(t) = \beta S(t)I(t) - (m + \mu)I(t). \end{cases} \tag{3.2.1}$$

$N(t)$ denotes the prey population at time $t \geq 0$ while the total predator population $P(t)$ is split between susceptible individuals $S(t)$ and infected individuals $I(t)$. The constants that appear in the model are positive and denote:

- r the logistic growth rate of the prey in absence of predator;
- h the half-saturation constant of the Holling type II functional response for the predator;
- m the exponential decay rate of the predator in absence of prey;
- μ the mortality increase due to the disease;
- β the disease transmissibility.

Likewise, we introduce the model accounting for the case of a diseased prey.

$$\begin{cases} \dot{S}(t) = r(S(t) + I(t))(1 - S(t)) - \dfrac{S(t)P(t)}{h + S(t) + I(t)} - \beta S(t)I(t) \\ \dot{I}(t) = \beta S(t)I(t) - \dfrac{I(t)P(t)}{h + S(t) + I(t)} - (\mu + r(S(t) + I(t)))I(t) \\ \dot{P}(t) = \dfrac{(S(t) + I(t))P(t)}{h + S(t) + I(t)} - mP(t). \end{cases} \tag{3.2.2}$$

The coefficients have the same meaning as before but now is the total prey population $N(t)$ to be split between susceptible prey $S(t)$ and infected prey

$I(t)$, while the predator $P(t)$ is as a whole. Note that the logistic term for the prey is split between the first and the second equation, which put together read

$$\begin{aligned}\dot{N}(t) &= \frac{d}{dt}(S+I)(t)\\ &= r(S(t)+I(t))(1-(S(t)+I(t))) - \frac{(S(t)+I(t))P(t)}{h+(S(t)+I(t))} - \mu I(t),\end{aligned}$$

where indeed we spot the global logistic term, the Holling-type II functional response for the whole prey population and an additional mortality due to the disease. This particular splitting choice may reflect certain intraspecific dynamics within the prey, for example the diseased population may not enter in competition for resources with the susceptible one (a possible cause could be segregation).

From now on we will drop the time dependence when no confusion arises. We immediately note that the models share a common disease-free predator-prey system, given by

$$\begin{cases} \dot{N} = rN(1-N) - \dfrac{NS}{h+N} \\ \dot{S} = \dfrac{NS}{h+N} - mS\,, \end{cases} \tag{3.2.3}$$

whose analysis is carried on in details in Appendix A.

To fit both models within the theoretical framework exposed before we set $\mathcal{X} = \mathbb{R}^3$ and

$$X := \{(x,y,z) \in \mathcal{X} : x, y, z \geq 0\} = \mathbb{R}^3_+$$

is the positive cone, which is closed with respect to $\mathcal{X}$. With some abuse of notation let $\pi(\xi, t) : X \times \mathbb{R}_+ \to X$ denote the flow, i.e. the solution of (3.2.1) or (3.2.2) at time t with initial conditions given by $\xi = (x_0, y_0, z_0)$. The semidynamical system π is dissipative in both cases as we can prove that for $k > 0$ large enough the compact set

$$X_k := \{(x,y,z) \in \mathbb{R}^3_+ : x+y+z \leq k\}$$

is forward invariant with respect to π and the points in $\Sigma_k := \{(x,y,z) \in X_k : x+y+z = k\}$ are sent by π into $X_k \setminus \Sigma_k$. We perform the calculations for the diseased predator model but the lower bound that we reach for k holds also for the dissipativity of the diseased prey model. We compute the flow against the outward normal vector to Σ_k and simple calculations return

$$(\dot{N}, \dot{S}, \dot{I}) \cdot (1, 1, 1) = rN(1 - N) - (mS + (m + \mu)I)$$

which is always negative if $N \geq 1$ and for $0 < N < 1$ it is straightforward to see that

$$rN(1 - N) - (mS + (m + \mu)I) \leq \frac{r}{4} - mk + m\,,$$

which is negative if k satisfies

$$k > 1 + \frac{r}{4m}\,. \tag{3.2.4}$$

This lower bound ensures the forward invariance of X_k, the uniform repulsiveness of $\Sigma_k \subset X_k$ for each k thanks to Theorem 3.1.2, and thus the global dissipativity of π.

Diseased predator boundary flow Fixed k as in (3.2.4) we now restrict our analysis to X_k and move to the study of the flow on its boundary ∂X_k. For convenience sake we split the boundary in three faces (note that in regards of persistence the face Σ_k is of no interest any more):

$$\begin{aligned} \partial_x X_k &= \{(x, y, z) \in X_k : \; x = 0\} \\ \partial_y X_k &= \{(x, y, z) \in X_k : \; y = 0\} \\ \partial_z X_k &= \{(x, y, z) \in X_k : \; z = 0\} \end{aligned}$$

The axes are forward invariant. The predator-only axes follow the dynamics of the predator-only face $\partial_x X_k$, which is forward invariant as well and is ruled by an exponential decay of rate m, as it is easy to see when taking into account $S + I$:

$$\frac{d(S + I)}{dt} = -m(S + I) - \mu I \leq -m(S + I).$$

Thus the whole face (axes included) is exponentially attracted to the origin.

In the absence of predators the prey has a logistic growth of rate r,

$$\dot{N} = rN(1-N),$$

so the origin and $(1,0,0)$ are fixed points of π on the prey-only axis, the first unstable and the second stable with respect to the axis direction.

For the detailed study of the flow on $\partial_z X_k$ (disease-free case) we refer to Appendix A.

When there are no susceptible predators ($\partial_y X_k$) we have $\dot{S} = (NI)/(h+N) > 0$, hence the face (axes excluded) is mapped through π into $X_k \setminus \partial_y X_k$. A scheme of the global boundary flow is shown in Figure 3.1.

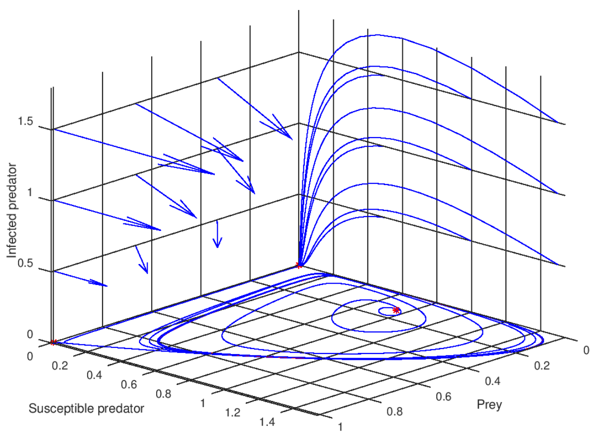

Figure 3.1: The diseased predator model boundary flow. We used the parameters values of [BH13b], namely $\mu = 0.5, r = 2, h = m = 0.3$ and $\beta = 1.3$. Orbits are displayed for the forward invariant faces $\partial_x X_1$ and $\partial_z X_1$ while for $\partial_y X_1$ we plotted the flow vectors. Equilibria are highlighted in red.

The equilibria lying on the boundary are the origin, the prey-only logistic equilibrium $(1,0,0)$ and the disease-free non trivial equilibrium $(N^*, P^*, 0)$. Under the crucial hypothesis

$$m < \frac{1-h}{1+h}, \tag{3.2.5}$$

a limit cycle γ^* arises in the disease-free face $\partial_z X_k$ (see Appendix A). Appendix B is appointed to the stability analysis of such points and orbit.

Diseased prey boundary flow Again, let k be fixed as in (3.2.4) and let us focus our attention on X_k. Here also we split the boundary into three faces,

$$\begin{aligned}\partial_x X_k &= \{(x,y,z) \in X_k : \ x = 0\}\\ \partial_y X_k &= \{(x,y,z) \in X_k : \ y = 0\}\\ \partial_z X_k &= \{(x,y,z) \in X_k : \ z = 0\},\end{aligned}$$

but note that here the disease-free face is $\partial_y X_k$. The face is forward invariant and the system reduces to (3.2.3), which is analysed in Appendix A.

The infected prey axis $\{(x,y,z) \in X : \ x = z = 0\}$ is not forward invariant as $\dot{S} = rI > 0$, hence points on this axis are sent by the flow π into the prey-only face $\partial_z X_k$. This behaviour extends to the whole face $\partial_x X_k$, which is sent by π into $X_k \setminus \partial_x X_k$.

The main difference from the diseased predator model is to be found on the forward invariant prey-only face $\partial_z X_k$. Here the leading equations are

$$\begin{cases} \dot{S} = r(S+I)(1-S) - \beta SI \\ \dot{I} = (\beta S - (\mu + r(S+I)))I\,, \end{cases} \tag{3.2.6}$$

which under the condition $\mu < \beta - r$ give rise to a non trivial equilibrium point $(S^\#, I^\#, 0)$, stable within the face. A schematic boundary flow for the system (3.2.2) is illustrated in Figure 3.2.

The equilibria on the boundary are the origin, the prey-only logistic equilibrium $(1,0,0)$, the same disease-free non trivial equilibrium $(N^*, 0, P^*)$, the new equilibrium on the prey-face $(S^\#, I^\#, 0)$ and again the limit cycle γ^* in the disease-free face $\partial_y X_k$ when (3.2.5) holds.

3.2.2 Persistence results

In this paragraph we reach the conditions needed for the global uniform persistence of systems (3.2.1) and (3.2.2) by proving repeatedly the repulsiveness of

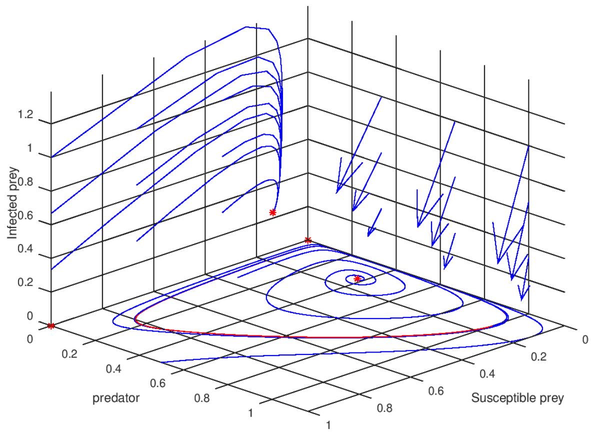

Figure 3.2: The diseased prey model boundary flow. Parameters are $\mu = r = 1$, $h = m = 0.3$ and $\beta = 10$. The equilibria are highlighted in red.

elements of a suitable decomposition of ∂X_k, removing them one by one and restricting the analysis to the remaining part of the boundary. The need for this iterative procedure will be explained in subsequent discussion.

Theorem 3.2.2. *Let system* (3.2.1) *be defined on the positive cone* $\mathbb{R}^3_+$ *with associated flow* π *and let the following hold:*

a) $m < \dfrac{1-h}{1+h}$

b) $\beta P^* > m + \mu$

c) for all $(N_0, S_0, 0) \in \gamma^*$

$$\sup_{t>0} \int_0^t (\beta S(s) - (m+\mu))ds > 0 \qquad \textit{for } S(0) = S_0\,. \tag{3.2.7}$$

Then π *is uniformly persistent on* $\mathbb{R}^3_+$.

Regarding the diseased prey model, the theorem reads as follows.

Theorem 3.2.3. *Let* (3.2.2) *be defined on the positive cone* $\mathbb{R}^3_+$ *with associated flow* π *and let the following hold:*

a) $m < \dfrac{1-h}{1+h}$

b) $\beta N^* > r + \mu$

c) *for all* $(S_0, 0, P_0) \in \gamma^*$

$$\sup_{t>0} \int_0^t (\beta S(s) - (r+\mu))ds > 0 \qquad \text{for } S(0) = S_0 \, .$$

d) $S^{\#} + I^{\#} > \dfrac{mh}{1-m}$.

Then π *is uniformly persistent on* $\mathbb{R}^3_+$.

Proof of Theorem 3.2.2. By the previous calculations π is dissipative, hence for a fixed k_1 satisfying (3.2.4) each orbit $\pi(\xi, t)$ enters definitively the compact set

$$X_1 = \{(x, y, z) \in \mathbb{R}^3_+ : \; x + y + z \le k_1\} \, ,$$

so for now we restrict our analysis to this set. Hypothesis a) of Theorem 3.2.2 guarantees the instability of the stationary equilibrium $(N^*, P^*, 0)$ as well as the existence and stability of a limit cycle γ^* in $\partial_z X_1$ (note that the computation of the lower bound on k guarantees $\gamma^* \subset X_1$).

Let us now call

$$\Sigma_1 := \{(x, y, z) \in X_1 : \; x = 0\}.$$

Σ_1 is a closed and invariant subset of X_1. The origin $M_1 = \{(0, 0, 0)\}$ forms a Morse decomposition for Σ_1 being compact, forward invariant and attracting the whole face so that $\omega(\Sigma_1) = M_1$. Now, $W^s(M_1) \cap X_1 = \Sigma_1$ (refer to Appendix B for the stability analysis): the saddle behaviour of the origin guarantees that it is isolated in X_1 and we can hence apply Theorem 3.1.1 to obtain the uniform repulsiveness of Σ_1 in X_1. Coupling this with the dissipativity proven before returns the permanence of π on $X_1 \setminus \Sigma_1$, that is the existence of a compact invariant attractor $K_1 \subset X_1 \backslash \Sigma_1$ such that $d(K_1, \Sigma_1) > 0$ (see [Hut84, Theorem 2.2] or [Con78, Chapter II.5]).

We can now analyse our flow π on

$$X_2 := X_1 \setminus \Sigma_1$$

which is an open subset of X_1, hence a locally compact metric space. Setting

$$\Sigma_2 := \{(x, y, z) \in X_2 : \ y = z = 0\}$$

we see that Σ_2 is forward invariant and closed with respect to X_2. $M_2 = \{(1, 0, 0)\}$ attracts all the points in Σ_2 and is indeed a Morse decomposition for such set. Again, by the stability analysis $W^s(M_2) \cap X_2 = \Sigma_2$ and hence Theorem 3.1.1 holds, giving uniform repulsiveness of Σ_2 and a compact invariant attractor K_2 inside $X_2 \setminus \Sigma_2$ such that $d(K_2, \Sigma_2) > 0$.

Now set

$$X_3 := X_2 \setminus \Sigma_2 \,,$$

which is again a locally compact metric space. We define

$$\Sigma_3 := \{(x, y, z) \in X_3 : \ y = 0\}.$$

Σ_3 is closed in X_3 and $X_3 \setminus \Sigma_3$ is forward invariant. Note that Σ_3 itself is not forward invariant, thus some classical results such as Theorem 3.1.1 do not apply. Recall that by the previous step X_3 contains a compact global attractor K_2. The set

$$V := \left\{(x, y, z) \in X_3 : \ y \leq \frac{xz}{(h + x)(m + \beta z)}\right\}$$

is a closed neighbourhood of Σ_3 and on V we define the real-valued function

$$\Lambda(x, y, z) := y.$$

By definition $\Lambda(x,y,z)=0$ if and only if $(x,y,z)\in\Sigma_3$ and if $(x_0,y_0,z_0)\in V\setminus\Sigma_3$ then

$$\begin{aligned}\dot{\Lambda}(x_0,y_0,z_0)&=\frac{x_0z_0}{h+x_0}+y_0\left(\frac{x_0}{h+x_0}-m-\beta z_0\right)\\&>\frac{x_0z_0}{h+x_0}-y_0(m+\beta z_0)\\&\geq\frac{x_0z_0}{h+x_0}-\frac{x_0z_0}{(h+x_0)(m+\beta z_0)}(m+\beta z_0)=0\end{aligned}$$

since we are in V. Hence, Λ increases along orbits that have starting points in $V\setminus\Sigma_3$, which means

$$\forall\,\xi\in V\setminus\Sigma_3\quad\exists\,t_\xi>0:\quad\Lambda(\pi(\xi,t_\xi))>\Lambda(\xi).$$

We are under the hypotheses i) and ii) of Theorem 3.2.1 and we obtain that Σ_3 is a uniform repeller in X_3. As before, this leads to the existence of a compact global attractor $K_3\subseteq K_1$ inside $X_3\setminus\Sigma_3$ such that $d(K_3,\Sigma_3)>0$.

Eventually, call

$$X_4:=X_3\setminus\Sigma_3$$

and

$$\Sigma_4:=\{(x,y,z)\in X_4:\ z=0\}.$$

$\Sigma_4\subset X_4$ is closed and globally invariant thanks to the Kolmogorov structure of the third equation of (3.2.1): by the uniqueness of the Cauchy problem associated with (3.2.1) this implies that $X_4\setminus\Sigma_4$ is forward invariant. By the previous step X_4 contains a compact global attractor K_3. On X_4 we define the non negative function

$$\Lambda(x,y,z):=z$$

and we note that $\Lambda(x,y,z)=0$ if and only if $(x,y,z)\in\Sigma_4$. Setting $V=X_4$ it holds

$$\dot{\Lambda}(x,y,z)=\dot{I}=I(\beta S-(m+\mu))=\Lambda(x,y,z)\psi(x,y,z)$$

with $\psi(x, y, z) = \beta y - (m + \mu)$. In order to satisfy hypothesis ii') of Theorem 3.2.1 we need to check that

$$\sup_{t>0} \int_0^t \psi(\pi(\xi, s))ds > 0 \qquad \forall \xi \in \overline{\omega(\Sigma_4)}.$$

The set $\overline{\omega(\Sigma_4)}$ is the union of $M_3 = \{(N^*, P^*, 0)\}$, the non trivial stationary equilibrium, and $M_4 = \gamma^*$, the stable limit cycle arising from the disease-free model. In the first case the above integral condition reads $\beta P^* - m + \mu > 0$ which is exactly hypothesis b) of Theorem 3.2.2 and is equivalent to

$$R_0^* > 1$$

where

$$R_0^* := \frac{\beta P^*}{m + \mu}.$$

As for M_4, the integral condition is equivalent to hypothesis c) of the Theorem. We hence proved the uniform persistence of π in X_1.

To end the proof it suffices to remember that for all k_1 satisfying the lower bound (3.2.4) any orbit entering X_1 is thereby trapped, and that the argument is irrespective of the choice of k_1. We are done. □

Proof of Theorem 3.2.2. This proof follows the scheme of the previous one, so let us skip some formal details such as the dissipativity argument and focus on the dynamics inside

$$X_1 = \{(x, y, z) \in \mathbb{R}^3_+ : \ x + y + z \leq k_1\},$$

with k_1 satisfying (3.2.4). We set

$$\Sigma_1 := \{(x, y, z) \in X_1 : \ x = y = 0\}.$$

Σ_1 is closed, forward invariant and $M_1 = \{(0, 0, 0)\}$ is a Morse decomposition for Σ_1. From the stability analysis we get $W^s(M_1) \cap X_1 = \Sigma_1$ and so Theorem 3.1.1 applies and returns a compact global attractor $K_1 \subset X_1 \setminus \Sigma_1$ such that $d(K_1, \Sigma_1) > 0$.

Now, as before

$$X_2 := X_1 \setminus \Sigma_1$$

and we choose

$$\Sigma_2 := \{(x, y, z) \in X_2 : \; x = 0\}\,.$$

X_2 is a locally compact metric space and Σ_2 is closed with respect to X_2 but not forward invariant. In order to apply Theorem 3.2.1 we recall the attractor K_1 inside X_2 and choose as the generalised Lyapunov function

$$\Lambda(x, y, z) := x\,,$$

which is non negative and identically zero only on Σ_2. We can identify a closed neighbourhood of Σ_2 on which the function Λ increases along the orbits, namely

$$V := \left\{(x, y, z) \in X_2 : \; x \le \frac{ry}{rk_1 + \frac{z}{y+1} + \beta y}\right\} :$$

it holds indeed for any $(x_0, y_0, z_0) \in V \setminus \Sigma_2$

$$\begin{aligned}\dot{\Lambda}(x_0, y_0, z_0) &= r(x_0 + y_0) - x_0 \left(r(x_0 + y_0) + \frac{z_0}{x_0 + y_0 + 1} + \beta y_0 \right) \\ &> ry_0 - x_0 \left(rk_1 + \frac{z_0}{y_0 + 1} + \beta y_0 \right) \ge 0.\end{aligned}$$

As shown before this implies condition $ii)$ of Theorem 3.2.1, which delivers another attractor $K_2 \subset X_2 \setminus \Sigma_2$ such that $d(K_2, \Sigma_2) > 0$.

For the third step we take $X_3 := X_2 \setminus \Sigma_2$ and

$$\Sigma_3 := \{(x, y, z) \in X_3 : \; z = 0\}\,.$$

$\Sigma_3 \subset X_3$ is closed and forward invariant. We are in condition to apply both Theorem 3.1.1 and Theorem 3.2.1. In the first case we must evaluate the stable manifolds of the two equilibria contained in X_3, $M_2 = \{(1, 0, 0)\}$ and $M_5 = \{(S^{\#}, I^{\#}, 0)\}$, whose union gives the ω-limit of the face: $\omega(X_3) = M_2 \cup M_5$. From the stability analysis (see Appendix B)

$$W^s(M_2) \cap X_3 = \{(x, y, z) \in X_3 : \; y = 0\} \subset \Sigma_3$$

while the computation for M_5 is more involved but the third eigenvalue is known and is positive if and only if hypothesis d) of Theorem 3.2.3 is satisfied. If we want to apply Theorem 3.2.1 instead we choose

$$\Lambda(x, y, z) := z$$

and the whole X_3 as the closed neighbourhood of Σ_3. It holds

$$\dot{\Lambda}(x, y, z) = \dot{P} = P\left(\frac{S + I}{h + S + I} - m\right) = \Lambda(x, y, z)\psi(x, y, z)$$

with $\psi(x, y, z) = (x + y)/(h + x + y) - m$. Condition ii') is satisfied if $\psi(M_2), \psi(M_5) > 0$, and the first descends from hypothesis a) while the second coincides with hypothesis d). We gain a compact attractor K_3 in $\subset X_3 \setminus \Sigma_3$.

The last step is similar to the one in the diseased predator case as the last face that remains to analyse is the disease-free one, that is writing $X_4 = X_3 \setminus \Sigma_3$ and choosing

$$\Sigma_4 := \{(x, y, z) \in X_4 : \ y = 0\}\,.$$

Here we apply Theorem 3.2.1 with the incidence of infected individuals as the generalised Lyapunov function:

$$\Lambda(x, y, z) := \frac{y}{x + y}.$$

Recalling $N = S + I$ and evaluating the time derivative of Λ

$$\dot{\Lambda} = \frac{d}{dt}\left(\frac{I}{N}\right) = \frac{\dot{I}}{N} - \left(\frac{I}{N}\right)\frac{\dot{N}}{N}\,.$$

Some calculations lead to

$$\frac{d}{dt}\left(\frac{I}{N}\right) = \frac{I}{N}\left(\beta S - \mu - r + \mu\frac{I}{N}\right)\,,$$

hence $\psi(x, y, z) = \beta x - \mu - r + \mu y/(x + y)$. Now, $\omega(\Sigma_4) = M_3 \cup M_4$ with $M_3 = \{(N^*, 0, P^*)\}$ and $M_4 = \gamma^*$, thus the condition ii') of Theorem 3.2.1 reads in the first case

$$\beta N^* - (\mu + r) > 0\,,$$

which is equivalent to

$$R_0^* = \frac{\beta N^*}{\mu + r} > 1 ,$$

while on the limit cycle it becomes exactly hypothesis c). Global uniform persistence is proven. □

3.2.3 Discussion

Condition (3.2.5) causes a shift of stability in the disease-free plane, from the non trivial stationary equilibrium (N^*, P^*) to the limit cycle γ^*. The natural thought would be that, since the dynamics approaches the limit cycle instead of the equilibrium point, a repelling condition should be given on γ^*. This heuristic hypothesis proves indeed true: condition (3.2.7) is the key assumption for uniform persistence, and is given on the points of the limit cycle itself. A further evidence that the focus should be adjusted on the limit cycle instead of the unstable equilibrium comes from the flow on the boundary point of view, where the equilibria form in both models a Morse decomposition of ∂X_1 which is acyclic and ends precisely with the limit cycle:

$$M_1 \to M_2 \to M_4 \leftarrow M_3 .$$

The additional equilibrium point M_5 in the diseased prey model does not enter this chain: actually, it is an independent (trivial) chain itself.

These observations may lead to the conclusion that no condition is needed on (N^*, P^*). This proves indeed wrong: in both theorems we still require $R_0^* > 1$ and this hypothesis[1] cannot be avoided in view of the last step of the iterative scheme illustrated above and the stability analysis carried on in the Appendixes. In fact, $R_0^* - 1$ is concordant with the third eigenvalue of the Jacobian evaluated in the non trivial equilibrium point (see Appendix B): if negative, a one dimensional stable manifold arises that has non empty intersection with the interior of X_1, thus meaning that initialising system (3.2.1) - respectively, (3.2.2) - with

[1] Note that the definition of R_0^* changes between the models: hypotheses c) of Theorems 3.2.2 and 3.2.3 are quite different, both in the use of P^* and N^* respectively and in the parameters involved, m for the predator and r for the prey.

points in $\operatorname{int} X_1 \cap W^s(\{(N^*, P^*, 0)\})$ - respectively, $\operatorname{int} X_1 \cap W^s(\{(N^*, 0, P^*)\})$ - results in the asymptotic extinction of the disease.

An example of this phenomenon is provided by the system

$$\begin{cases} \dot{x} = y - \varepsilon\left(\dfrac{x^3}{3} - x\right) \\ \dot{y} = -x \\ \dot{z} = \delta\varphi(x^2 + y^2)z \end{cases} \tag{3.2.8}$$

with $\varepsilon, \delta > 0$ and $\varphi(s) = \max\{\min\{s - 2, 0.5\}, -1\}$. A van der Pol equation in the Liénard plane (x, y) is coupled with a radius-dependent height z which takes into account the two well-known ω-limits of the equation, the unstable origin and a stable limit cycle on the plane that for $0 < \varepsilon \ll 1$ is close to the circumference $x^2 + y^2 = 4$. The result is shown in Figure 3.3. The z-axis is the stable one dimensional manifold of the origin: starting arbitrarily close but not on it leads the orbit to approach the (x, y)-plane so that the effect of the instability of the origin pushes the orbit towards the limit cycle. The choice of φ causes the z-component to increase near the limit cycle, so that condition (3.2.7) holds but uniform persistence fails due to the z-axis being exponentially attracted to the origin.

In analysing numerically the systems (3.2.1) and (3.2.2) the Authors provide in [BH13b] an elegant notion of average basic reproduction number $\overline{R_0}$, based on the limit cycle and similar in structure to R_0^*: for the diseased predator it reads

$$\overline{R_0} := \frac{\beta\overline{P}}{m + \mu}, \tag{3.2.9}$$

with $\overline{P} = \int_0^T P(t)dt$ where T is the period of the limit cycle and $P(0) = P(T) \in \gamma^*$. Similarly, for the diseased prey we have

$$\overline{R_0} := \frac{\beta\overline{N}}{r + \mu}.$$

Their condition for the persistence of the disease becomes $\overline{R_0} > 1$, which of course implies hypotheses c) of both our theorems. We can thus say that

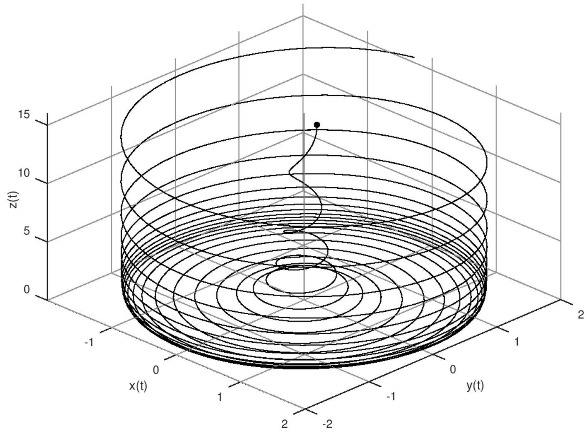

Figure 3.3: An orbit of system (3.2.8) starting close to the z-axis, moving away from the unstable origin in proximity of the (x, y)-plane and approaching the limit cycle, which is repulsive in the z-direction. Parameters: $\varepsilon = \delta = 0.1$, $x_0 = y_0 = 0.1$ and $z_0 = 15$. The time span is $[0, 150]$

Theorem 3.2.2 and Theorem 3.2.2 are rigorous theoretical results that support the numerical evidence provided thereby. We also reach some more degree of generality due to our specific condition on the limit cycle, which is independent of its period.

The definition of $\overline{R_0}$ is reminescent of the studies on the meaning and the computation of the basic reproduction number in seasonal models as found in Bacaër (see [BG06, BAD12]). In [BG06, Section 5] a quantity is derived that generalises the notion of R_0 to the periodic coefficients case and indeed, as we pointed out at the beginning of the Section, the stable limit cycle asymptotically drives the autonomous system as a sort of periodic perturbation. The link between formula (14) of Bacaër's work and hypothesis c) of our theorems however remains subtle, as the former deals with averages of the periodic coefficients while the latter gives conditions on the points of the limit cycle: a further investigation in this sense could reveal interesting connections.

It is important to stress that although $R_0^* > 1$ alone does not guarantee uniform persistence, as shown in [BH13b] where the disease extinguishes in some situations where $\overline{R_0} < 1 < R_0^*$, it is still a necessary condition in order to avoid the presence of an internal one dimensional stable manifold for the stationary equilibrium. This assumption is not required by [BH13b]: of course, in that case persistence holds almost everywhere as only a one dimensional set of initial values leads the systems to the asymptotic extinction of the disease; moreover, such set is known theoretically from its tangent line but is hard to compute precisely or to spot by numerical means. Nevertheless it is interesting to note how the necessary condition $R_0^* > 1$ arises both from the stability analysis and hypothesis ii') of Theorem 3.2.1.

Quite remarkably, hypothesis d) of Theorem 3.2.3 is nowhere to be found in [BH13b]. This can be explained with the fact that the Authors focus on the persistence of the disease but not necessarily of all populations. The equilibrium $(S^\#, I^\#, 0)$ can cause the extinction of the predator if not prevented by the mentioned condition, which can be derived either from the stability analysis (imposing the positivity of the third eigenvalue, see Appendix B) or from Theorem 3.2.1, as we did in the main dissertation. In principle one could argue that hypothesis d) may be implied by the other hypotheses as it does not clearly display the dependence on the parameters. We show by a numerical example that this is not the case. Changing slightly the coefficients given in [BH13b] we can fit in a situation where (3.2.5) and $\overline{R_0} > 1$ hold but hypothesis d) fails and the predator reach extinction approaching the equilibrium $(S^\#, I^\#, 0)$. The chosen parameters are

$$h = 0.1, \quad m = 0.81, \quad \beta = 10, \quad \mu = r = 1\,,$$

coupling a strong disease transmissibility with a high predator mortality. Evaluating numerically the eigenvalues of $J(S^\#, I^\#, 0)$ we find

$$\lambda_{1,2} = -1.03586 \pm 0.64794i, \qquad \lambda_3 = \frac{S^\# + I^\#}{h + S^\# + I^\#} - m = -0.02270$$

and Figure 3.4 shows a numerical evidence of extinction in the predator-free equilibrium $(S^\#, I^\#, 0)$. If we alter the value of m the behaviour can change

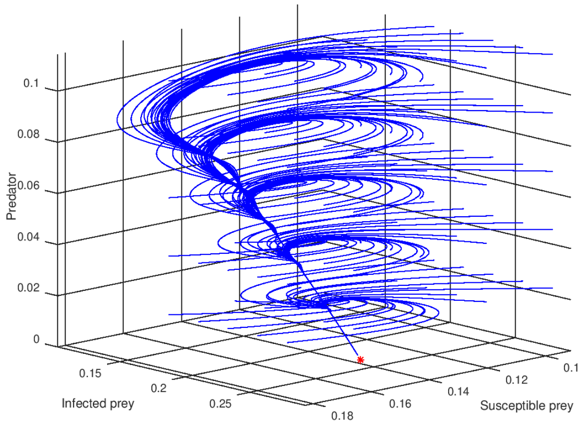

Figure 3.4: A grid of point close to the $(S^{\#}, I^{\#}, 0)$ equilibrium is attracted towards it, resulting in the extinction of the predator.

dramatically: from some point on the orbits move upwards and this is a consequence of the repulsiveness of the boundary, see Figure 3.5.

An interesting optimality question arises from the comparison between the hypotheses c) of the theorems and $\overline{R_0} > 1$, that is which is the first time $\tau > 0$ to realise

$$\int_0^\tau \psi(\pi(x,s))ds > 0 \qquad \forall x \in \gamma^*.$$

For large values of β there is numerical evidence that $\tau < T$, as we would expect being the limit cycle independent of β and being easier for the disease to survive with a high infection rate. However, the dependence on the other parameters is not trivial and may display some interesting behaviours which are difficult to investigate theoretically, as little is known about the limit cycle and its properties.

The proof of Theorems 3.2.2 and 3.2.2 is original in its iterative application of Theorems 3.1.1 and 3.2.1 to some carefully chosen decomposition of the boundary, progressively removing those sets for which uniform repulsiveness

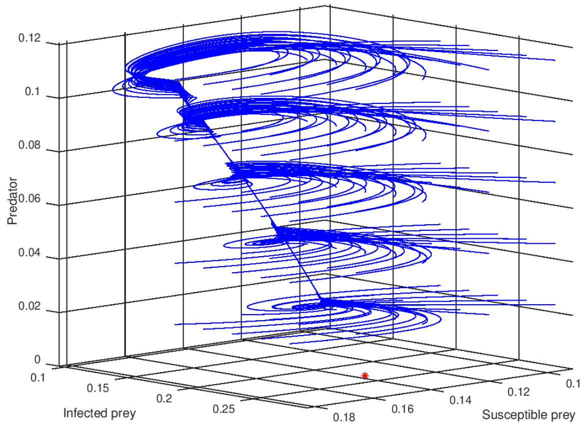

Figure 3.5: The same simulation with $m = 0.75$. The threshold value for the sign-change of the third eigenvalue is around $m = 0.7873$.

has been proven. This approach is not redundant, in the sense that standard techniques does not apply to such a heterogeneous boundary, which is neither globally forward invariant nor all weakly repulsive. In this setting, the results in [Hut84] and [Fon88] cannot be used on the whole boundary. Even the powerful unifying tool [Hof89, Corollary 4] fails on the disease-free face as all the equilibria (Morse sets) are involved but $\Lambda(x, y, z) = z$ does not satisfy the hypotheses of the Corollary in the origin and in the prey-only equilibrium. To avoid these points we should restrict Λ to a subset of the disease-free face which is disjoint from the x-axis, but then we lose either the forward invariance or the closure of such subset. It is not clear if a different generalised Lyapunov function could be defined: it should take into account how each equilibrium has its own instability direction, as shown in Appendix B. This is also the reason why Theorem 3.2.1 itself cannot be applied on the whole X_1.

We draw the attention on the abstract problem of generalising such an iterative technique. One issue is intrinsic and lies in the choice of the partition and in which order to remove its sets from the boundary. We speculate that this process could follow the acyclic chain of Morse sets, as it is in our case: there are however non trivial issues that could arise, especially when the chain is more involved than the one we showed at the beginning of this paragraph. Another question is if Theorem 3.2.2 could be derived following other approaches. System (3.2.1) falls under the class of models (23) of [GR16]: we speculate that some of the arguments could be adapted in order to prove the persistence of the disease even if the results therein are given for interactions of competitive or cooperative type, which is not our case.

Another open question of theoretical nature is to find the reverse of Theorems 3.2.2 and 3.2.3. A hint is given by the fact that Theorem 3.1.3 is a sufficient condition for uniform repulsiveness but it comes from a more general result, which also include the necessary condition.

We conclude by pointing out some possible further extensions of this work. We dealt with the two main models proposed in [BH13b] but some extensions are also thereby analysed, for example the case when both populations share the disease (cross-infection) or when the disease influences the density dependence of the infected population. It may be interesting to investigate if the techniques we adopted here can also apply to those models so to reach similar persistence results. Another interesting possibility is to allow the disease transmissibility β to vary in time, for example choosing a periodic function describing seasonality: this may link again to the previously cited works [RMB12, GR16] and to the next Section, in which we discuss the existence of periodic solutions in persistent models.

3.3 Periodic solutions of ecological models via uniform persistence

The aim of this Section is to prove the existence of periodic solutions once uniform persistence is given. Such a result allows to deploy the conditions for uniform persistence described before in order to obtain not only persistence or permanence but also at least one periodic solution. This is done in the framework of ω-periodic processes, which we are going to illustrate thoroughly. We follow the line of [Fer90], in which the Author extended to ω-periodic processes the persistence results of the "Lyapunov-like function" approach ([Hut84, Fon88]), obtaining some existence results for periodic solutions which however are briefly justified: we plan to clarify and investigate some of the blind spots. A similar approach is carried on in [Zha95]: the same Author extended the "flow on the boundary" approach to persistence ([BFW86, Hof89]) some years later in [Zha01].

Some fixed point theorems are used as well as results on the side of degree theory. dissipativity is crucial in order to obtain a compact global attractor on which apply such tools. The theoretical result is exploited to prove uniform persistence and existence of a periodic solution in three different applications coming from the ecological area: a simple Kolmogorov equation, the SIR model for infectious diseases (achieving here a stronger result than the one in [Kat14]) and a model describing the seasonal phytoplankton blooming [HBOS05].

3.3.1 A general fixed point theorem

Let again fit in the theoretical framework of the beginning: $(\mathcal{X}, d)$ is a locally compact metric space and X is a closed subset. We denote by u a process defined on X, i.e. a continuous map

$$u : \mathbb{R} \times X \times \mathbb{R}_+ \to X$$

such that

$$\begin{array}{ll} i) & u(\sigma, x, 0) = x \\ ii) & u(\sigma, x, s+t) = u(\sigma + s, u(\sigma, x, s), t) \end{array} \qquad \forall x \in X,\ \forall \sigma \in \mathbb{R},\ \forall s, t \geq 0.$$

A process is *autonomous* if it is independent of the first variable σ. We dealt with autonomous processes in the previous Sections as they coincide with continuous semidynamical systems by setting

$$\pi(x, t) := u(0, x, t).$$

Here we are interested in non trivial dependence from the first variable and in particular in ω-*periodic* processes, i.e. processes that are periodic of period ω

with respect to σ. In this case it remains defined a discrete dynamical system $\{\Phi^k\}_{k\in\mathbb{N}}$ associated with u,

$$\Phi^k(x) := u(0, x, k\omega). \tag{3.3.1}$$

We immediately note that when u is the flow of a periodic coefficients ODE system $\Phi^1 = \Phi$ is the Poincaré map associated with the system.

We enunciate some concepts already given in Definition 3.1.1 in the case of processes and add some more. Refer to [Hal77] and [Fer90] for these definitions.

Definition 3.3.1. Let Y be a subset of X.

- Y is *forward invariant* with respect to u if and only if

$$u(\sigma, Y, t) \subseteq Y \quad \forall\, \sigma \in \mathbb{R},\, \forall\, t \geq 0;$$

- Y *attracts points* of X if

$$\forall\, x \in X,\ \forall\, \sigma \in \mathbb{R} \quad \exists\, t_x \geq 0 : \quad u(\sigma, x, t) \in Y \quad \forall\, t \geq t_x\,;$$

- Y *attracts compact sets* of X if for each compact subset H of X there exists a neighbourhood $\mathcal{N}(H)$ of H such that

$$\exists\, t_H = t(\mathcal{N}(H)) \geq 0 : \quad u(0, \mathcal{N}(H), t) \subseteq Y \quad \forall t \geq t_H.$$

- Y is a *uniform repeller* with respect to u if it exists $\eta > 0$ such that

$$\liminf_{t\to+\infty} d(Y, u(\sigma, x, t)) \geq \eta \qquad \forall\, \sigma \in \mathbb{R},\, \forall\, x \in X \setminus Y\,.$$

The process u is called

- *dissipative* if for all $x \in X$ $\omega(x) \neq \emptyset$ and the set $\bigcup_{x\in X}\omega(x)$ has compact closure with respect to X;
- *point* [*compact*] *dissipative* if it exists a proper subset $Y \subset X$ which attracts points [compact sets] of X;

- *uniformly persistent* if ∂X is a uniform repeller;
- *permanent* if u is dissipative and uniformly persistent.

From now on let u be a ω-periodic process defined on X. We formulate a result that combines Theorem 3.1.2 and Theorem 3.1.3 in the case of processes.

Proposition 3.3.1. *Let Σ be a compact subset of X such that $X \setminus \Sigma$ is forward invariant. Suppose that Σ can be written as*

$$\Sigma = \Sigma_1 \cup \Sigma_2$$

with Σ_1, Σ_2 closed subsets of Σ such that for Σ_1 it holds

$$\forall x \in \Sigma_1 \quad \exists t_x > 0 \quad \textit{such that} \quad u(0, x, t_x) \in X \setminus \Sigma_1 \tag{3.3.2}$$

while Σ_2 is forward invariant and there exists a continuous function $\Lambda : X \to \mathbb{R}_+$ such that

i) $\Lambda(x) = 0$ *if and only if* $x \in \Sigma_2$ *;*

ii) there exists a lower semicontinuous and bounded below function $\psi : X \to \mathbb{R}$ and $\alpha \in [0, 1]$ such that

$$a) \quad \dot{\Lambda}(x) \geq \Lambda^{\alpha}(x)\psi(x) \qquad \forall x \in X \setminus \Sigma_2$$

$$b) \quad \sup_{t>0} \int_0^t \psi(u(\sigma, x, s))ds > 0 \qquad \forall \sigma \in [0, \omega),\ \forall x \in \overline{\omega(\Sigma_2)}.$$

Then Σ is a uniform repeller.

Remark 3.3.1. The Proposition can be easily generalised to the case

$$\Sigma = \bigcup_{i=1}^{m} \Sigma_i$$

where, say, $\Sigma_1, \ldots, \Sigma_k$ satisfy the first hypothesis while $\Sigma_{k+1}, \ldots, \Sigma_m$ are forward invariant and satisfy condition *i)* and *ii)*. □

We now move to a less general framework, asking $\mathcal{X}$ to be a *compact* metric space. This assumption is quite natural in ecological models and in light of the dissipativity requested before. The set $\bigcup_{x \in X} \omega(x)$ eventually contains the dynamics so nothing is lost of the asymptotic behaviour if the analysis is focused on such set. We can now state the key instrument for the theorem to come, contained in [Hut84].

Theorem 3.3.1. *Let Σ be a compact subset of X such that Σ and $X \setminus \Sigma$ are forward invariant with respect to u. If Σ is a uniform repeller then it exists $K' \subset X \setminus \Sigma$ compact, forward invariant which attracts points of $X \setminus \Sigma$.*

The result of Hutson relates repulsiveness and point-dissipativity. We can show that in our case we gain a stronger property, the compact-dissipativity, via a Lemma by Hale (*ibidem*, Lemma 3.3). Note that in the following we will deal with the discrete dynamical system $\{\Phi^k\}_k$ associated with the ω-periodic process u as defined in (3.3.1): however, at least the following Lemma can be stated for the continuous case, see [Hal77, Section 4.6].

Lemma 3.3.1. *In the hypotheses of Theorem 3.3.1 there exists $K \subset X \setminus \Sigma$ compact and forward invariant which attracts via $\{\Phi^k\}_k$ compact sets of $X \setminus \Sigma$.*

Proof. Let $K' \subset X \setminus \Sigma$ be the compact, forward invariant attractor given by Theorem 3.3.1. Of course, K' is a (point) attractor also for $\{\Phi^k\}_k$. Because K' is compact in $X \setminus \Sigma$, which is open,

$$\exists \bar{\varepsilon} > 0 \quad \text{such that} \quad \mathcal{B}(K', \bar{\varepsilon}) \subset \overline{\mathcal{B}(K', \bar{\varepsilon})} \subset X \setminus \Sigma$$

where $\mathcal{B}(K', \bar{\varepsilon}) = \{x \in X : \ d(x, K') < \bar{\varepsilon}\}$.

From now on let then $\varepsilon \in (0, \bar{\varepsilon})$ be fixed. Call

$$K^* := \Phi(\overline{\mathcal{B}(K', \bar{\varepsilon})}) \subseteq X \setminus \Sigma :$$

this set is of course compact being Φ continuous. If K' attracts points of $X \setminus \Sigma$ then

$$\forall x_0 \in X \setminus \Sigma \quad \exists n_0 = n(x_0, \varepsilon) \quad \text{such that} \quad \Phi^n(x_0) \in \mathcal{B}(K', \varepsilon) \quad \forall n \geq n_0 :$$

then by continuity of Φ^n there is a neighbourhood $\mathcal{N}(x_0)$ of x_0 such that

$$\Phi^{n_0}(\mathcal{N}(x_0)) \subseteq \mathcal{B}(K', \varepsilon) \subseteq K^* .$$

Let now $H \subset X \setminus S$ be an arbitrary non-empty compact set. Then $\{\mathcal{N}(x) : x \in H\}$ is an open cover of H, from which we can extract a finite subcover

$$\{\mathcal{N}(x_1), \mathcal{N}(x_2), \dots, \mathcal{N}(x_l)\}.$$

Call

$$H_0 = \bigcup_{i=1}^{l} \mathcal{N}(x_i)\,, \qquad n_H = \max\{n_1, n_2, \dots, n_l\} + 1$$

where n_i is the index related to $\mathcal{N}(x_i)$: it holds

$$\Phi^{n_H}(H_0) \subseteq K^*.$$

We noted before that K^* is compact as well and so it can be assigned a similar index n_{K^*}. We can therefore define

$$K := K^* \cup \Phi(K^*) \cup \Phi^2(K^*) \cup \ldots \cup \Phi^{n_{K^*}}(K^*)$$

which is compact. By construction it is

$$\Phi^n(K^*) \subseteq K \quad \forall n \in \mathbb{N}.$$

Thus we have that for the neighbourhood H_0 of H it holds

$$\Phi^n(H_0) \subseteq K \quad \forall n \geq n_H$$

which shows that K attracts compact sets of $X \setminus \Sigma$. □

Remark 3.3.2. The framework of [Hal77] in which the original is carried on is quite different as $\mathcal{X}$ is a Banach space. The proof does not rely on this assumption and thus the result holds true also in our context. □

For the last step we require a normed space structure in order to give sense to the concept of convexity. Let then X, Σ be compact subsets of $\mathbb{R}^N$: although strong this assumption is needed but, being the arguments of topological nature, the results extend to any set homeomorphic to $\mathbb{R}^N$. We can now state our main result of existence.

Theorem 3.3.2. *Let $\Sigma \subset X \subset \mathbb{R}^N$ be compact sets such that $X \setminus \Sigma$ is open and convex. If Σ is a uniform repeller and $X \setminus \Sigma$ is forward invariant with respect to $\{\Phi^k\}_k$ then there exists a fixed point of Φ inside $X \setminus \Sigma$.*

For the proof of the theorem we use the previous results plus one coming from the degree theory and known as mod p Theorem, stated independently in [ZK71] and [Ste72] (for good surveys on these topics see [Nus85] and [Ste15]).

Theorem 3.3.3. *In the framework of Theorem 3.3.2 let $Y \subset X \setminus \Sigma$ be an open set and call*

$$F := \{x \in Y : \ \Phi^p(x) = x\}$$

for some prime $p \in \mathbb{N}$. If p is such that F is compact (the case $F = \emptyset$ being included), that $\Phi(F) \subseteq F$ and that Φ is compact on some neighbourhood of F, then

$$i_{X\setminus\Sigma}(\Phi^p, Y) \equiv i_{X\setminus\Sigma}(\Phi, Y) \mod p$$

where $i_{X\setminus\Sigma}$ denotes the fixed point index on $X \setminus \Sigma$.

The original theorem is far more general and contemplates the case when the ambient space (in our case, $X \setminus \Sigma$) is an absolute neighbourhood retract (ANR) and F is the set of fixed points of Φ^m, m being a power of a prime.

Let us now prove our main theorem.

Proof of Theorem 3.3.2. For our goal let us consider instead of K its closed convex hull $\hat{K} := \overline{\mathrm{co}}(K)$, for which it holds $K \subseteq \hat{K} \subset X \setminus \Sigma$ being $X \setminus \Sigma$ convex and open. $\hat{K}$ is compact, hence there exists an open neighbourhood $\mathcal{N}(\hat{K})$ and $\hat{n} = n_{\hat{K}}$ such that

$$\Phi^n(\mathcal{N}(\hat{K})) \subseteq K \quad \forall n \geq \hat{n}.$$

Let p be the first prime number greater or equal than $\hat{n}$. The restriction $\Phi^p : \hat{K} \to \hat{K}$ is a continuous map defined on a convex compact set of $\mathbb{R}^N$: Brouwer's fixed point Theorem holds and thus we gain a fixed point for Φ^p, which of course belongs to the attractor K.

At this point we deploy Steinlein's result. We choose $Y = \mathcal{N}(\hat{K})$ and note that F is not empty thanks to the previous step. The second condition required by the theorem is easily verified:

$$\Phi^p(\Phi(x)) = \Phi(\Phi^p(x)) = \Phi(x) \qquad \forall x \in F$$

so that $\Phi(F) \subseteq F$. The third condition is trivial as Φ is a continuous map, thus necessarily compact everywhere. It remains to show that F itself is compact. Being a subset of X it is bounded: let us show it is also closed. By contradiction suppose that exists $x^* \in \partial F \setminus F$. Then we can find $\{x_k\}_k \subset F$ such that $x_k \to x^*$ for $k \to \infty$ in the topology of Y. By continuity of Φ^p we have

$$x_k \to x^*, \quad x_k = \Phi^p(x_k) \to \Phi^p(x^*) \qquad \text{for } k \to \infty$$

and thus by uniqueness of the limit

$$x^* = \Phi^p(x^*)$$

so that $x^* \in \Sigma$, reaching the contradiction. Thus Σ is compact.

Theorem 3.3.3 applies and so we obtain

$$1 = i_{X\setminus\Sigma}(\Phi^p, Y) \equiv i_{X\setminus\Sigma}(\Phi, Y) \mod p$$

which tells us Φ admits at least one fixed point in $Y = \mathcal{N}(\hat{K}) \supseteq \hat{K}$. In particular the fixed point shall belong to the attractor K. □

An immediate remark for applications is that if u is the flow generated by one or more differential equations the theorem returns the existence of a ω-periodic solution of the equation(s), i.e. a fixed point of the Poincaré map Φ.

Remark 3.3.3. Theorem 3.3.2 can be compared with [Fer90, Theorem 3.4]. The key Lemma for the proof of that theorem was not explicitly proven: we used Theorem 3.3.1, Lemma 3.3.1 and Theorem 3.3.3 in order to rigorously justify that result. We point out that a similar approach has been illustrated in [Zha95, see Theorem 2.3], in the setting of semidynamical systems on Banach spaces.

Remark 3.3.4. We stress the importance of the convexity of $X \setminus \Sigma$, which can however be weakened to topological convexity. Without it Brouwer's fixed point Theorem fails. It is easy to build a counterexample when such hypothesis is removed. Take for instance

$$X := \overline{\mathcal{B}(0,2)} \subset \mathbb{R}^2, \qquad \Sigma := \partial X \cup \{(0,0)\}.$$

$X \setminus \Sigma$ is open but not convex. Let the flow be of the form

$$\Phi^k(z) = \left(1 + \frac{\rho - 1}{k}\right) e^{ik\theta} \qquad \text{for } z = \rho e^{i\theta} \in X\,.$$

The attractor is $K = \partial\mathcal{B}(0,1)$ but no fixed point exists being K rotated by the flow. We point out that in [BFW86] the authors do not ask for convexity: their result holds true if such condition is added to the forward invariant set $X \setminus \Sigma$. □

Remark 3.3.5. From [BS02, Lemma 3.7] it easily follows that for autonomous systems Theorem 3.3.2 reduces to the existence of an equilibrium point.

In the following we illustrate three examples of biological relevance on which the theoretical approach of this Section applies. The first application is a one-dimensional example based on the so called Kolmogorov equation. It is a good preamble that deploys Proposition 3.3.1 in its simplest form. The second application is the well-known SIR model for infectious diseases as described in [GP97] and with a bit less generality in [Kat14]: the assumption of no loss of biomass reduces the system's degrees of freedom and it becomes planar. Here we prove the existence of a periodic solution as in [Kat14] but we reach also permanence. The third and last application is a model of seasonal phytoplankton blooms we encountered in [HBOS05]: a bidimensional example of competing species. Though similar in settings (3.3.4) and (3.3.7) need different choices of the ambient space $\mathcal{X}$ which raise the need of a further sophistication for the latter case.

3.3.2 Kolmogorov equation

We are interested in studying the T-periodic solutions of

$$\dot{x}(t) = x(t)h(t, x(t)) \qquad t \geq 0 \tag{3.3.3}$$

with $h : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$. Such sort of equations are known as Kolmogorov-type equations and are defined on the positive cone (in our case $x(t) \geq 0$). The derivative is expressed as a product of the function itself and a vector field which is T-periodic in the first variable. This kind of equations is frequently used to describe models of competitive species and has been extensively studied in light of persistence (also known as coexistence of the species): for some references see [Zan92, BZ98, ZC17].

To fit this example into our framework we call $u(0, x_0, t)$ the solution at time t of the initial value problem

$$\begin{cases} x'(t) = x(t)h(t, x(t)) & \text{on } [0, T] \\ x(0) = x_0 \end{cases}$$

with $h(t, s)$ a T-periodic function with respect to the first variable, which returns the periodicity of u as a process as well. Recall that the associated discrete dynamical system is defined as

$$\Phi^k(x) = u(0, x, kT).$$

The result we are going to prove is the following (note that it can also be deduced straightforwardly from [RMB12, Theorem 2]).

Theorem 3.3.4. *Let $h : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be lower semicontinuous, bounded below, T-periodic with respect to the first variable and such that*

i) there exists $M > 0$ such that

$$\forall\, y \geq M \quad h(t, y) \leq \beta(t)$$

where $\beta \in L^1(0, T)$ is such that $\int_0^T \beta(t)dt < 0$;

ii) $\displaystyle\int_0^T h(s, 0)ds > 0$.

Then $\{\Phi^k\}_k$ is uniformly persistent on the set $X := [0, M']$ with

$$M' > Me^{\|\beta\|_{L^1(0,T)}}.$$

Moreover, (3.3.3) *admits a T-periodic solution inside X.*

Proof. We first prove that $\{\Phi^k\}_k$ is dissipative. Let M' be a fixed positive constant satisfying the hypothesis of the theorem and let $x(t)$ be a generic solution of (3.3.3) such that $x(0) = M'$. We claim that $x(T) < M'$. By contradiction suppose this is not the case. Two situations could happen:

a) $x(t) \geq M \quad \forall t \in [0, T]$;

b) $\exists\ t^* \in]0, T[$ such that $x(t^*) < M$.

If we fall under a) then by hypothesis 2) on h it holds

$$\frac{\dot{x}(t)}{x(t)} \leq \beta(t) \quad \forall\, t \in [0, T]$$

from which it easily follows

$$\ln\left(\frac{x(T)}{x(0)}\right) \leq \int_0^T \beta(t) dt < 0$$

and a contradiction is reached as $x(T) < x(0) = M'$. In case b), by continuity of the solution we can find $[t_1, t_2] \subset (t^*, T)$ such that

$$x(t_1) = M, \quad x(t_2) = M', \quad M \leq x(t) \leq M' \text{ for } t \in [t_1, t_2]$$

and $[t_1, t_2]$ is maximal. Retracing step by step the calculations in the previous case, evaluating this time the integral over $[t_1, t_2]$, we reach

$$\ln\left(\frac{x(t_2)}{x(t_1)}\right) \leq \int_{t_1}^{t_2} \beta(t) dt < \|\beta\|_{L^1(0,T)}$$

and taking the exponential on both sides we come to

$$M' \leq M e^{\|\beta\|_{L^1(0,T)}} \quad \text{on } [t_1, t_2]$$

which is a contradiction thanks to the cunning choice of M'. This proves the dissipativity of Φ^k on $\{M'\}$.

To prove the global existence of solutions of (3.3.3) let $t' \in (0, T)$ be such that $x(t') = M'$. There is $t'' > t'$ such that $[t', t'']$ is the maximal interval of existence of the solution x. If it holds $x(t) < M'$ for all $t \geq t'$ we are done, so let $\tilde{t} \in (t', t'')$ be such that

$$x(\tilde{t}) = M', \quad x(t) > M' \; \forall \, t > \tilde{t}.$$

On $[\tilde{t}, t]$ the above steps lead to

$$\ln\left(\frac{x(t)}{x(\tilde{t})}\right) \leq \int_{\tilde{t}}^{t} \beta(s)ds < \|\beta\|_{L^1(0,T)}$$

so that

$$x(t) \leq M' e^{\|\beta\|_{L^1(0,T)}}$$

which is a bound independent of t. We conclude that $t'' = T$.

Now we deploy Proposition 3.3.1 on the boundary of the compact set X, $\Sigma = \partial X = \{0, M'\}$. First we note that $\{0\}$ is forward invariant: we then choose

$$\Lambda(x) := x$$

and see that

$$\dot{x}(t) = x(t)h(t, x(t)) = \Lambda(x(t))h(t, x(t))$$

with Λ satisfying the hypotheses $i)$ and $ii)$ of Proposition 3.3.1, with h playing the role of ψ and satisfying the hypotheses as well thanks to the choices of Theorem 3.3.4. For the above calculations, $\{M'\}$ satisfies the first hypothesis of the Proposition with respect to the discrete dynamical system Φ^k, but a similar result can be obtained for the continuous process u provided that a second constant $M'' > M'$ is chosen (see the global existence calculations). Proposition 3.3.1 holds, granting the uniform repulsiveness of persistence of ∂X, hence the uniform persistence of equation (3.3.3).

As a last step we apply Theorem 3.3.2, being $X \setminus \Sigma =]0, M'[$ convex, open and forward invariant with respect to Φ^k: we get a T-periodic solution of (3.3.3) inside X. $\square$

Remark 3.3.6. It may be possible to build higher dimensional examples by "gluing" together some Kolmogorov models as the one we illustrated here, requiring forward invariant faces and dissipativy on the positive cone. In [Zan92] a discussion of this type is unfolded.

3.3.3 SIR epidemic model

We consider the model

$$\begin{cases} \dot{S}(t) = \mu(1 - S(t)) - \beta(t)S(t)I(t) \\ \dot{I}(t) = \beta(t)S(t)I(t) - (\gamma + \mu)I(t) \\ \dot{R}(t) = \gamma I(t) - \mu R(t). \end{cases} \tag{3.3.4}$$

$S(t), I(t), R(t)$ are non negative functions of time and denote the normalised populations of susceptible, infected and recovered individuals. It holds

$$S(t) + I(t) + R(t) \equiv 1 \qquad \forall\, t \in [0, +\infty) \tag{3.3.5}$$

as no loss of biomass is allowed. $\mu > 0$ is the birth and death rate and is considered here to be constant; $\gamma > 0$ is the recovering rate; $\beta(t)$ is a T-periodic coefficient which usually identifies the transmission rate of the disease. The SIR model hereby displayed is studied in [Kat14] where the existence of a periodic solution is proven via Leray-Schauder degree theory. A similar model that considers a dependence $\beta = \beta(t, I(t))$ is given in [GP97] in order to study complex dynamics and bifurcations.

We are interested in periodic solutions of (3.3.4) without admitting the extinction of the disease, not even asymptotically. As we will see this is linked to the well-known basic reproduction number

$$R_0 = \frac{\bar{\beta}}{\gamma + \mu}$$

where $\bar{\beta} := \int_0^T \beta(t)dt/T$ denotes the average of β over one period.

Let us fit (3.3.4) into our framework. First of all, thanks to (3.3.5) we can reduce the degrees of freedom of the system by analysing only the pairs $(S(t), I(t))$: thus our ambient space shall be $\mathcal{X} = \mathbb{R}^2$. Looking again at (3.3.5) we can also set

$$X = \{(x, y) \in \mathbb{R}^2_+ : \, x + y \leq 1\}$$

and as in the previous Section $\Sigma = \partial X$. For convenience sake we split Σ into

$$\Sigma_x = \{(x, y) \in \Sigma : \, x = 0\}, \qquad \Sigma_y = \{(x, y) \in \Sigma : \, y = 0\}$$

and show that the closed sets Σ_y and Σ_x satisfy respectively the first and the second hypotheses of Proposition 3.3.1. The result is the following.

Theorem 3.3.5. *Let system* (3.3.4) *be defined on*

$$X = \{(x, y) \in \mathbb{R}^2_+ : x + y \leq 1\}.$$

If it holds

$$\sup_{t>0} \int_0^t (\beta(t) - (\gamma + \mu))ds > 0 \tag{3.3.6}$$

then the flow of (3.3.4) *is uniformly persistent and it admits a non trivial T-periodic solution inside X.*

Proof. Let us first prove that (3.3.4) is point dissipative on X. If we evaluate the gradient of $(S(t), I(t))$ against the outward normal vector to $\Sigma_{xy} = \{(x, y) \in \mathbb{R}^2_+ : x + y = 1\}$ we obtain

$$(\dot{S}, \dot{I}) \cdot (1, 1) = \mu(1 - S) - (\gamma + \mu)I = -\gamma I < 0$$

as $S(t) + I(t) \equiv 1$ on Σ_{xy}. The direction of the flow is thus opposite to the outward normal vector, i.e. points of Σ_{xy} are sent by the flow into $X \setminus \Sigma_{xy}$, with the exception of the trivial susceptible-only equilibrium $(1, 0)$.

Now, let us take into account Σ_x, on which (3.3.4) reduces to

$$\begin{cases} \dot{S} = \mu \\ \dot{I} = -(\gamma + \mu)I\,; \end{cases}$$

being $\dot{S} > 0$, the points of Σ_x are sent into $X \setminus \Sigma_x$, fitting hypothesis (3.3.2) of Proposition 3.3.1.

We focus on Σ_y: on it (3.3.4) reads

$$\begin{cases} \dot{S} = \mu(1 - S) \\ \dot{I} = 0 \end{cases}$$

so Σ_y is forward invariant and exponentially attracted to $(1, 0)$, which means $\omega(\Sigma_y) = \{(1, 0)\}$. Now choose

$$\Lambda(x, y) := y.$$

$\Lambda : X \to \mathbb{R}_+$ is non negative, continuous, $\Lambda(x, y) = 0$ if and only if $(x, y) \in \Sigma_y$ and

$$\dot{\Lambda} = \dot{I} = I(\beta S - (\gamma + \mu)) = \Lambda\psi$$

with $\psi(x, y) := \beta x - (\gamma + \mu)$. To show that *ii*) of Proposition 3.3.1 holds we need to evaluate the integral of this function on $\overline{\omega(\Sigma_y)}$:

$$\sup_{t>0} \int_0^t \psi(1, 0) ds = \sup_{t>0} \int_0^t (\beta(s) - (\gamma + \mu)) ds$$

and as this quantity needs to be positive we ask for (3.3.6). Under this hypothesis Proposition 3.3.1 returns the uniform persistence of the flow on X.

To end the proof we deploy Theorem 3.3.2 on $X \setminus \Sigma$, which is again convex, open and forward invariant thanks to the previous calculations and the global dissipativity. We gain a T-periodic solution which is bounded in X. □

Remark 3.3.7. Our fixed point result is in line with the one of Katriel ([Kat14, Theorem 1]) with respect to the existence of a periodic solution. However, we recover also uniform persistence for (3.3.4) on the positive cone. As for the proof, our approach is substantially different and relies on the degree theory only for the last step (the mod p Theorem). As in Section 3.2 (3.3.6) is implied by the standard assumption

$$R_0 = \frac{\bar{\beta}}{\gamma + \mu} > 1$$

which is required in [Kat14], by prescribing that the average integral is positive over the period T:

$$\frac{1}{T} \int_0^T (\beta(t) - (\gamma + \mu)) dt = \bar{\beta} - (\gamma + \mu).$$

Remark 3.3.8. System (3.3.4) can be defined only on X because of (3.3.5) but it can be extended on the whole positive cone allowing the biomass to change in time. If we restrict to the bidimensional system without recovered individuals it is easy to prove the global dissipativity on the set $X_k = \{(x, y) \in \mathbb{R}^2_+ : x + y \leq k\}$ in the same way as in the previous proof. Thanks to the very same argument uniform persistence can be proven for each X_k with $k \geq 1$, obtaining the uniform persistence on the whole positive cone, pretty much as done with the model (3.2.1) in the previous Section.

We note that other approaches such as [GR16] apply flawlessly to this and the following model, being the functional response among those listed thereby. The procedure of the previous Section, based on [RMB14], works as well.

3.3.4 Seasonal phytoplankton blooms

The proposed model in [HBOS05] to describe the seasonal blooming of phytoplankton is, after some rearrangements,

$$\begin{cases} \dot{N}(t) = I - qN(t) - \beta(t)N(t)P(t) \\ \dot{P}(t) = \beta(t)N(t)P(t) - P(t)\,, \end{cases} \tag{3.3.7}$$

where $N(t)$ denotes the nutrients level and $P(t)$ is the phytoplankton biomass, thus positive quantities. $I, q > 0$ are constants depending on the income of nutrients, the rate of uptaking of the phytoplankton and its mortality. $\beta(t)$ is a periodic coefficient describing seasonality. In [HBOS05] an interesting numerical analysis is carried on, showing complex dynamics and bifurcation cascades arising from the model under suitable choices in the coefficients. It is interesting to compare this work with the previously cited [GP97], in which chaotic dynamics are discovered for a particular SIR model.

System (3.3.7) is close in structure to (3.3.4) but has no fixed total biomass, so in this case we analyse our model on the whole positive cone $\mathbb{R}^2_+$. It is easy to show that $(I/q, 0)$ is an equilibrium point: we choose X so that such equilibrium is contained,

$$X := \{(x, y) \in \mathbb{R}^2_+ : x + y \leq k\}$$

where $k > 0$ is large enough in order to have global dissipativity (cfr. Remark 3.3.8). As before $\Sigma := \partial X$, of which we highlight two closed components:

$$\Sigma_x = \{(x, y) \in \Sigma : x = 0\}, \qquad \Sigma_y = \{(x, y) \in \Sigma : y = 0\}.$$

We enunciate immediately the result.

Theorem 3.3.6. *Under the hypothesis*

$$\sup_{t>0} \int_0^t \left(\frac{I}{q} \beta(s) - 1 \right) ds > 0 \tag{3.3.8}$$

the flow generated by (3.3.7) *is uniformly persistent on the whole positive cone* $\mathbb{R}^2_+$*. For any* $k > 0$ *such that*

$$k > \frac{(q+1)I}{q}$$

there exists a non trivial periodic solution of (3.3.7) *inside* X*.*

Proof. As before, global dissipativity is proven by computing the flow against the normal outward vector to $\Sigma_k = \{(x, y) \in \mathbb{R}^2_+ : x + y = k\}$, for any k large enough:

$$(\dot{N}, \dot{P}) \cdot (1, 1) = I - qN - P.$$

We claim this quantity is always negative thanks to the choice of k. While in the case $N \geq I/q$ this is trivial, if $N < I/q$ we rely on the fact that $N + P \equiv k$ to get

$$I - qN - P < I - qN - \left(k - \frac{I}{q}\right) = \frac{q+1}{q} I - k - qN < -qN$$

which is always non positive. As before these entails that the flow sends points of Σ_k into $X \setminus \Sigma_k$ for all k satisfying the prescribed lower bound.

We now move to the analysis of the flow on the boundary. On Σ_x (3.3.7) becomes

$$\begin{cases} \dot{N} = I \\ \dot{P} = -P \end{cases}$$

which in analogy with the previous model grants that hypothesis (3.3.2) of Proposition 3.3.1 holds. On Σ_y instead the system assumes the form

$$\begin{cases} \dot{N} = I - qN \\ \dot{P} = 0 \end{cases}$$

and as before this implies the forward invariance of Σ_y and the attractivity on the axis of the trivial nutrient-only equilibrium $(I/q, 0)$. As our generalised Lyapunov function we choose

$$\Lambda(x, y) := y$$

which is of course non negative, continuous, identically zero only on Σ_y and it holds

$$\dot{\Lambda} = \dot{P} = P(\beta N - 1) = \Lambda\psi$$

with $\psi(x) = \beta x - 1$, of which we need to evaluate the integral over $\overline{\omega(\Sigma_y)} = \{(I/q, 0)\}$:

$$\sup_{t>0} \int_0^t \psi(I/q, 0) ds = \sup_{t>0} \int_0^t (\beta(s)I/q - 1) ds$$

and hypothesis (3.3.8) ensures this quantity is positive, as requested by ii) of Proposition 3.3.1. Uniform persistence is given on X. Now, the argument can be iterated for any $k > 0$ sufficiently large, so the uniform persistence extends to the whole positive cone $\mathbb{R}^2_+$.

To conclude, as before we apply Theorem 3.3.2 to $X \setminus \Sigma$, which is open, convex and forward invariant. We gain a T-periodic solution of (3.3.7) inside X, for any admissible k. □

Remark 3.3.9. The basic reproduction number in this case reads

$$R_0 := \frac{I\bar{\beta}}{q}, \qquad \bar{\beta} = \int_0^T \beta(t) dt,$$

and the condition $R_0 > 1$ is equivalent to hypothesis (3.3.8) as in Theorem 3.3.5.

3.3.5 Further perspectives

To end this Section we point out some further extensions to our approach.

As a pure theoretical exercise Theorem 3.3.2 can be generalised to its most weak hypotheses possible for this framework, i.e. the set $X \setminus \Sigma$ is an absolute neighbourhood retract (ANR) which is open and precompact in a compact normed space $\mathcal{X}$.

On the application side, Proposition 3.3.1 can be applied to many models coming from literature. In a future survey work we plan to incorporate some of them, for example the already cited [BZ98] in which only existence of periodic solutions is proven but not permanence. Many models can be found in the works of Colucci [CCH16, Col13, CNn13], where Kolmogorov-like structures are investigated, and Silva [Sil17, MS17], where more applied eco-epidemiological models are displayed. Especially these last works have interesting analogies with the models we studied in Section 3.2, and this lead us to the idea of investigating the periodic coefficients versions of (3.2.1) and (3.2.2), where β is no more a positive constant but a periodic function of time. Since complex dynamics are shown for these models even in the autonomous case [BH13a], an utopian question could be if techniques as Stretching Along the Paths (see Chapter 1) apply in order to prove rigorously chaotic dynamics as we did with persistence. Another model which could contemplate this questions as well is given in [Kir89].

The comparison between our method and the one exposed in the works of Margheri, Rebelo and Garrione [MR03, RMB12, GR16] is another open question. Since the methods share a common core which are the results contained in [Fon88] it may be expected that the outcomes could be similar. The problem is more subtle than expected and we hope to address it in the above mentioned survey paper in preparation.

Appendix A

Analysis of a Kolmogorov predator-prey model

The underlying prey-predator model to systems (3.2.1) and (3.2.2) is (3.2.3), which we copy for the reader's convenience:

$$\begin{cases} \dot{N} = rN(1-N) - \dfrac{NP}{h+N} = N\left(r(1-N) - \dfrac{P}{h+N}\right) \\ \dot{P} = \dfrac{NP}{h+N} - mP = P\left(\dfrac{N}{h+N} - m\right) . \end{cases}$$

We highlight the Kolmogorov structure of this system. Three equilibrium points can be easily found: the origin $(0, 0)$, the prey-only equilibrium $(1, 0)$ and a non trivial one (N^*, P^*) which is given by

$$N^* = \frac{mh}{1-m} \qquad P^* = r(1-N^*)(h+N^*).$$

The Jacobian with respect to (3.2.3) is

$$J(x, y) = \begin{pmatrix} r(1-2x) - \dfrac{hy}{(h+x)^2} & -\dfrac{x}{h+x} \\ \dfrac{hy}{(h+x)^2} & \dfrac{x}{h+x} - m \end{pmatrix}$$

and evaluation in the equilibrium points reads

$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix}, \quad J(1,0) = \begin{pmatrix} -r & -\dfrac{1}{h+1} \\ 0 & \dfrac{1}{h+1} - m \end{pmatrix}$$

and

$$J(N^*, P^*) = \begin{pmatrix} rm\left(1 - \dfrac{1+m}{1-m}h\right) & -m \\ r(1 - m(1+h)) & 0 \end{pmatrix}.$$

The origin is always a saddle. To avoid the stability of the trivial equilibrium it must be

$$m < \frac{1}{1+h}$$

which also ensures

$$\det J(N^*, P^*) = mr(1 - m(1+h)) > 0$$

so that the nontrivial equilibrium is unstable if

$$\operatorname{tr} J(N^*, P^*) = rm\left(1 - \frac{1+m}{1-m}h\right) > 0 \quad \Longleftrightarrow \quad h < \frac{1-m}{1+m}$$

which is equivalent to condition (3.2.5), namely

$$m < \frac{1-h}{1+h}\,.$$

Three cases are hence given:

- $m > 1/(1+h)$: the logistic equilibrium is stable
- $(1-h)/(1+h) < m < 1/(1+h)$: $(1,0)$ is unstable and (N^*, P^*) is stable
- $m < (1-h)/(1+h)$: both equilibrium points are unstable.

We choose the third hypothesis and prove that a stable and unique limit cycle bifurcates from the unstable equilibrium (N^*, P^*).

For the sake of completeness and coherence we now show that the system (3.2.3) is uniformly persistent on the positive cone $\mathbb{R}^2_+$ by means of Theorem 3.1.3, also if we notice that this comes directly from [GR16, Theorem 3.2(b)]. Once persistence is given assumption (3.2.5) leads to the existence of a unique stable limit cycle.

Persistence of (3.2.3) We first prove that the system is dissipative. On the sets $\{(x, y) \in \mathbb{R}^2_+ : x + y = k\}$ for $k > 0$ it holds

$$(\dot{N}, \dot{P}) \cdot (1, 1) = rN(1 - N) - mP = -rN^2 + (r + m)N - mk$$

which is negative if and only if

$$rN^2 - (r + m)N + mk > 0$$

and by choosing

$$k > \frac{(r + m)^2}{4rm}$$

this is always true. As seen in the main dissertation this entails global dissipativity.

Now, fix k large enough and let

$$X := \{(x, y) \in \mathbb{R}^2_+ : x + y \leq k\}.$$

The set

$$\partial_y X = \{(x, y) \in X : y = 0\}$$

is forward invariant and $\omega(\partial_y X) = \{(1, 0)\}$, thus we apply Theorem 3.1.3 setting

$$\Lambda(x, y) = y.$$

Then, if π is the semidynamical system associated with (3.2.3),

$$\dot{\Lambda}(\pi(x, t)) = \dot{P}(t) = P(t) \left[\frac{N(t)}{h + N(t)} - m \right] = \Lambda(\pi(x, t))\psi(\pi(x, t))$$

with $\psi(x, y) = \dfrac{x}{h+x} - m$ and condition b) of the theorem reads

$$\frac{1}{t}\int_0^t \psi(1,0)ds = \frac{1}{h+1} - m$$

which is positive thanks to the choice of m. Thus $\partial_y X$ is a repeller by Theorem 3.1.3. As for

$$\partial_x X = \{(x, y) \in \mathbb{R}^2_+ : \; x = 0\}$$

we reason the same way but now the only attractive point is the origin. Set then

$$\Lambda(x, y) = x$$

and get

$$\dot{\Lambda}(\pi(x,t)) = \dot{N}(t) = N(t)\left[r(1 - N(t)) - \frac{P(t)}{h + N(t)}\right] = \Lambda(\pi(x,t))\psi(\pi(x,t))$$

with $\psi(x, y) = r(1-x) - y/(h+x)$ and because $\psi(0, 0) = r > 0$ we conclude by the same argument that $\partial_x X$ is a repeller as well. By dissipativity we obtain uniform persistence on the whole positive cone.

Existence and uniqueness of the limit cycle The existence of the limit cycle comes in a standard way from the Poincaré-Bendixson annular region Theorem: refer to [Lef63] for the following formulation.

Theorem A.0.1. *Suppose R is a bounded region of the plane enclosed by two simple closed curves γ_1 and γ_2, and π is a semidynamical system defined on the plane. If*

i) at each point of γ_1 and γ_2 the flow π points towards the interior of R

ii) R contains no critical points

then the system has a closed trajectory (i.e. a limit cycle) lying inside R.

Proof. In our case let us choose $\gamma_1 = \partial X$ and $\gamma_2 = \partial I^*$ where $I^* \subset X$ is a (sufficiently small) neighbourhood of the unstable nontrivial equilibrium (N^*, P^*). Let then $R = \mathrm{int}\,(X \setminus \overline{I^*})$. Because of the dissipativity and uniform persistence proven before the flow on γ_1 is always pointing inside R and because of the instability of the equilibrium on γ_2 the flow is pointing outside I^*, thus inside R. We then get the existence of a limit cycle. □

To show that the limit cycle is unique we make use of a result proven in [Che81] although already conjectured in [HHW78b]. The system hereby taken into account is

$$\begin{cases} \dot{N} = \Gamma N \left(1 - \dfrac{N}{K}\right) - \dfrac{m}{y}\dfrac{NS}{a+N} \\ \dot{S} = S\left(\dfrac{mN}{a+N} - D_0\right) \end{cases}$$

which is indeed the same system as (3.2.3) with some adjustments in the coefficients. The requested hypotheses therein are

$$\frac{m}{D_0} > 1, \qquad K > a + 2\frac{a}{b-1}$$

which referring to (3.2.3) come to be

$$m < 1, \qquad 1 > h + 2N^* \iff m < \frac{1-h}{1+h},$$

indeed satisfied by our choice of m. The result in [HHW78a] states that under these hypotheses there exists at least one limit cycle and if it is unique then it is stable (see also the more classical [Lef63]). In [Che81] the uniqueness of the limit cycle is proven. For a general result on the limit cycles of Kolmogorov systems like (3.2.3) see [YCDY12].

Appendix B

Stability analysis of predator-prey models

B.1 Diseased predator model

The equilibria on the boundary displayed by system (3.2.1) are the origin, the trivial logistic equilibrium $(1, 0, 0)$ and the disease-free equilibrium $(N^*, P^*, 0)$. If we choose m as in (3.2.5) a limit cycle γ^* appears in the disease-free plane and on that plane all critical points are unstable and the limit cycle is stable. We now want to study the stable manifolds of the equilibrium points.

Recall that $X_1 = \{(x, y, z) \in \mathbb{R}^3_+ : \ x + y + z \leq k_1\}$. Let us write the linearised matrix for the system (3.2.1):

$$J(x, y, z) = \begin{pmatrix} r(1-2x) - \dfrac{h(y+z)}{(h+x)^2} & -\dfrac{x}{h+x} & -\dfrac{x}{h+x} \\ \dfrac{h(y+z)}{(h+x)^2} & \dfrac{x}{h+x} - m - \beta z & \dfrac{x}{h+x} - \beta y \\ 0 & \beta z & \beta y - (m+\mu) \end{pmatrix}.$$

Evaluating in the equilibrium points:

$$J(0,0,0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -(m+\mu) \end{pmatrix}$$

$$J(1,0,0) = \begin{pmatrix} -r & -\dfrac{1}{h+1} & -\dfrac{1}{h+1} \\ 0 & \dfrac{1}{h+1} - m & \dfrac{1}{h+1} \\ 0 & 0 & -(m+\mu) \end{pmatrix}$$

$$J(N^*, P^*, 0) = \begin{pmatrix} rm\left(1 - \dfrac{1+m}{1-m}h\right) & -m & -m \\ r(1 - m(1+h)) & 0 & m - \beta P^* \\ 0 & 0 & \beta P^* - (m+\mu) \end{pmatrix}.$$

It is easy to see that

$$W^s(\{(0,0,0\}) \cap X_1 = \partial_x X_1 = \{(x,y,z) \in X_1 : \; x = 0\}.$$

As for $(1,0,0)$, the tangent plane to its stable manifold is given by

$$k_1(1,0,0) + k_2\left(\frac{\mu}{(1+\mu(h+1))(m+\mu-r)}, -\frac{1}{1+\mu(h+1)}, 1\right), \quad k_1, k_2 \in \mathbb{R}$$

which lies strictly outside the positive cone except for the points in $\{(x,y,z) \in X_1 : \; y = z = 0\}$, all belonging to the stable manifold of $(1,0,0)$ except for the origin. It holds $W^s(\{(1,0,0)\}) \cap \operatorname{int} X_1 = \emptyset$ because no bidimensional manifold could approach $(1,0,0)$ tangentially with respect to the above plane from the inside of X_1 as its boundary is either forward invariant $(\partial_z X_1, \partial_x X_1)$ or repulsive $(\partial_y X_1)$. Thus

$$W^s(\{(1,0,0)\}) \cap X_1 = \{(x,y,z) \in X_1 : \; x > 0, \; y = z = 0\}.$$

The first two eigenvalues λ_1, λ_2 for the non trivial equilibrium $(N^*, P^*, 0)$ are strictly positive as illustrated in Appendix A. being

$$\det J(N^*, P^*, 0) = rm(1 - m(1+h))(\beta P^* - (m+\mu))$$
$$\operatorname{tr} J(N^*, P^*, 0) = rm\left(1 - \frac{1+m}{1-m}h\right) + \beta P^* - (m+\mu).$$

The third eigenvalue, $\lambda_3 = \beta P^* - (m + \mu)$, can in principle possess a one dimensional stable manifold if

$$R_0^* := \frac{\beta P^*}{m+\mu} < 1.$$

The tangent line in $(N^*, P^*, 0)$ to this manifold is

$$(N^*, S^*, 0) + k\left(\frac{\mu m}{f(\lambda_1, \lambda_2, \lambda_3)}, \frac{\mu(\lambda_1 + \lambda_2 - \lambda_3)}{f(\lambda_1, \lambda_2, \lambda_3)}, 1\right), \quad k \in \mathbb{R}$$

where $f(\lambda_1, \lambda_2, \lambda_3) = (\lambda_1 + \lambda_2)\lambda_3 - (\lambda_1\lambda_2 + \lambda_3^2)$. Having non zero z-component the line cuts through the disease-free face, hence the stable manifold passes through the interior of X_1 and can potentially lead to the extinction of the disease. To avoid this chance we ask for

$$R_0^* > 1.$$

B.2 Diseased prey model

The diseased prey model exhibits one more equilibrium point on the prey-only face, given by the positive intersection of the curves

$$I = \left(\frac{\beta}{r} - 1\right) S - \frac{\mu}{r}, \qquad I = \frac{S(1-S)}{\left(\frac{\beta}{r} + 1\right) S - 1}$$

which are obtained after rearrangement of (3.2.6). The intersection occurs only if the first equation describe a line pointing upwards ($\beta > r$) and

$$\beta > \mu + r$$

holds, which is indeed true for the parameters chosen in [BH13b]. Some calculations lead to

$$S^{\#} = \frac{1}{2\beta}\left(C + \sqrt{C^2 - 4\mu r}\right), \qquad C = \frac{r\mu}{\beta} + \mu + r$$

and $I^\#$ can be obtained by substitution (note that for the other intersection point of the curves $I^\#$ is always negative and hence excluded).

The Jacobian $J(x,y,z)$ for the system (3.2.2) is

$$\begin{pmatrix} r(1-y-2x)-\dfrac{z(h+y)}{H^2} & r(1-x)+\dfrac{xz}{H^2} & -\dfrac{x}{H} \\ \left(\beta+\dfrac{z}{H^2}-r\right)y & \beta x-\dfrac{z(h+x)}{H^2}-(\mu+rx+2ry) & -\dfrac{y}{H} \\ \dfrac{hz}{H^2} & \dfrac{hz}{H^2} & \dfrac{x+y}{H}-m \end{pmatrix}$$

where we put $H = h+x+y$ for sake of brevity. Evaluating in the equilibrium points:

$$J(0,0,0) = \begin{pmatrix} r & r & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -m \end{pmatrix},$$

$$J(1,0,0) = \begin{pmatrix} -r & -\beta & -\dfrac{1}{h+1} \\ 0 & \beta-(\mu+r) & 0 \\ 0 & 0 & \dfrac{1}{h+1}-m \end{pmatrix}$$

$$J(N^*,0,P^*) = \begin{pmatrix} rm\left(1-\dfrac{1+m}{1-m}h\right) & r-\dfrac{mh}{1-m}(rm(h+1)-\beta) & -m \\ 0 & \beta N^*-(r+\mu) & 0 \\ r(1-m(h+1)) & r(1-m(h+1)) & 0 \end{pmatrix}$$

$$J(S^\#,I^\#,0) = \begin{pmatrix} r-2rS^\#-(\beta+r)I^\# & r-(\beta+r)S^\# & -\dfrac{S^\#}{H^\#} \\ (\beta-r)I^\# & -rI^\# & -\dfrac{I^\#}{H^\#} \\ 0 & 0 & \dfrac{S^\#+I^\#}{H^\#}-m \end{pmatrix}$$

with $H^\# = h+S^\#+I^\#$.

As in the predator case the origin is always a saddle: its stable manifold is tangent in the origin to the plane

$$k_1\left(-\frac{r}{r+\mu},1,0\right)+k_2(0,0,1), \qquad k_1,k_2\in\mathbb{R}$$

which lays outside the positive cone and intersects it only in the predator-only axis, which is exponentially attracted to the origin. We conclude that

$$W^s(\{(0,0,0)\}) \cap X_1 = \{(x,y,z) \in X_1 : x = y = 0\}.$$

Thanks to (3.2.5) the trivial susceptible prey-only equilibrium $(1,0,0)$ is unstable. Under the hypothesis for the existence of the $(S^\#, I^\#, 0)$ equilibrium, $\beta > \mu + r$, the only negative eigenvalue is the first, which gives rise to a one dimensional stable manifold tangential in $(1,0,0)$ to the susceptible prey-only axis. Since because of the logistic dynamics the whole axis (origin excluded) is attracted to the equilibrium we conclude that

$$W^s(\{(1,0,0)\}) \cap X_1 = \{(x,y,z) \in X_1 : y = z = 0,\ x > 0\}.$$

As for the non trivial disease-free equilibrium we get the same conclusions as before: while two eigenvalues are always positive thanks to (3.2.5) the third one, namely $\beta N^* - (r+\mu)$, could be negative and hence a stable manifold may arise when

$$R_0^* = \frac{\beta N^*}{r+\mu} < 1$$

(note that the parameter m of the predator mortality has been replaced with the logistic parameter r of the prey).

Eventually, for the new equilibrium $(S^\#, I^\#, 0)$ we get that for the Jacobian restricted to the prey-only face $J_2^\#$ it holds

$$\begin{aligned}
\det J_2^\# &= I^\# \left((r^2 + \beta r) I^\# + (r^2 + \beta^2) S^\# - \beta r \right) \\
\mathrm{tr} J_2^\# &= r - 2rS^\# - (2r + \beta) I^\#
\end{aligned}$$

and it can be shown through some calculations that the corresponding eigenvalues have always negative real part. The third eigenvalue is

$$\frac{S^\# + I^\#}{h + S^\# + I^\#} - m$$

and is positive if and only if hypothesis d) of Theorem 3.2.3 holds. The same condition comes from the persistence approach with the generalised Lyapunov function $\Lambda(x,y,z) = z$, refer to the proof of Theorem 3.2.3.

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