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DE FINETTI AND MARKOWITZ MEAN VARIANCE APPROACH TO  
REINSURANCE AND PORTFOLIO SELECTION PROBLEMS:  
A COMPARISON

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**Abstract**

Based on a critical analysis of de Finetti's paper, where the mean variance approach in finance was early introduced to deal with a reinsurance problem, we offer an alternative interpretative key of such an approach to the standard portfolio selection one. We discuss analogies and differences between de Finetti's and Markowitz's geometrical approaches.

**JEL CLASSIFICATION:** B23; C61; D81; G11; G22;

**KEYWORDS:** FINANCE; MEAN-VARIANCE EFFICIENCY;  
CONSTRAINED QUADRATIC OPTIMIZATION; PORTFOLIO  
SELECTION; PROPORTIONAL REINSURANCE.

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## **1. Introduction**

H. Markowitz was awarded the Nobel Prize in Economics thanks to his three papers (Markowitz 1952, 1956, 1959) in the 1950's in which he introduced the mean-variance approach in quantitative finance in order to solve a problem of a portfolio of risky assets' selection (henceforth investment problem).

Only recently it has been discovered (see Rubinstein, 2006a) that the primacy of the mean-variance approach in finance should be credited to B. de Finetti, who introduced it in a paper (de Finetti, 1940) concerning the selection of individual proportional reinsurance of a set of risks in an insurance company portfolio (henceforth reinsurance problem).

While obviously Markowitz's papers are very well known in financial world, the one of de Finetti, written in Italian language in an actuarial journal at the beginning of the Second World War, went unnoticed to researchers in financial economics and its knowledge remained restricted for a long time to European actuarial circles; this explains its late rediscovery<sup>1</sup>.

At first sight, the investment and the reinsurance problems seem to be quite different; on the contrary, they reveal to have much in common once they are expressed in a formalized version. Hence, one of the goals of this paper is an analysis of analogies and differences between the authors' approaches. A lot of analogies may be found mainly in the first paper (Markowitz, 1952), which is largely based on geometric intuitions, a method often privileged by de Finetti too, at least in his papers devoted to economic and financial applications.

It is convenient to recall here that while Markowitz did not stop investigating the topic, thus being able to apply also newly developed optimization techniques (Karush, 1939; Kuhn-Tucker, 1951), de Finetti completely forgot his paper<sup>2</sup>, so that he did not even consider either to purge

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<sup>1</sup> An early discussion of the connections between de Finetti's work and portfolio theory can be found in Pressacco (1986) as quoted in Rubinstein (2006a). An English translation of the first part of de Finetti's paper has been provided in Barone (2006).

<sup>2</sup> He himself (de Finetti, 1969) did not include it in a list of his own papers bearing some, even minor, connection with economic topics.

it from shortcomings or to extend results to other fields of financial economics.

With this respect, another goal of the paper is to discuss how these shortcomings can be fixed and which enlightening interpretations can be also obtained in the investment problem through further investigation (but remaining faithful to de Finetti's logic).

The plan of the paper is as follows: Section 2 offers a preliminary largely informal glance to the analogies and differences in the authors approach to the problem. Section 3 is devoted to a quick recall of de Finetti's reward risk approach to the reinsurance problem, followed in section 4 by an informal resume of his friendly procedure to find the whole efficient set in the asset space as well as of some shortcomings, signalled by Markowitz (2006) about its applicability to a general correlation structure in the absence of a regularity condition. Section 5 shows the closed form formulae of the reinsurance problem for the case of no correlation and group correlation respectively. Section 6 shows how to adjust de Finetti's rules when the regularity hypothesis does not hold. In the next sections 7 (tools), 8 (regular case), 9 (non regular case), the extension of de Finetti's approach to the standard investment problem is presented in a quite informal way, while a more technical presentation is given in the Appendix. Finally, in section 10 a discussion of some examples offers an enlightening comparison of Markowitz and de Finetti's geometrical intuitions behind the critical line algorithm in the investment problem. Conclusions follow in section 11.

## 2. A preliminary glance at the problem

Let us start from a preliminary statement concerning the strict connection between the (re)insurance and the investment problem.

An insured risk with liability  $X_i$  in a given single period time horizon may be dealt with as a risky asset with random gain  $G_i$ , if we consider the difference  $G_i = P_i - X_i$  where  $P_i$  is the premium received to insure the risk (net of loading expenses but gross of positive safety loadings, so that  $P_i > E(X_i)$ ). A proportional reinsurance on original terms of a quota  $1 - x_i$  of the risk, i.e. retention of a quota  $x_i$ , originates a random post-reinsurance gain  $x_i G_i$ , and an overall post-reinsurance gain  $\sum_i x_i G_i$ .

On the other side, a portfolio of risky assets can be defined from a set of quotas  $x_i$  of the disposable wealth of an investor. If the assets have a single

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period random rate of return  $G_i$ , the random rate of return on the portfolio is  $\sum_i x_i G_i$ . Standard constraints are  $0 \leq x_i \leq 1$  for the reinsurance case and  $x_i \geq 0$ ,  $\sum_i x_i = 1$  for the investment one. These constraints were applied by de Finetti (de Finetti, 1940) to the reinsurance problem faced by an insurance company and in the Markowitz (1952) to the investment one, while in subsequent papers he considered also other constraints, such as industrial ones and so on (for a general treatment of affine constraints see Markowitz, 1987, chapter 6).

In this framework, a first strict analogy between the authors is the choice of an integrated reward-risk approach. Indeed both were unsatisfied with the one-sided approach to decisions, largely prevailing at that time, which was risk driven in the (re)insurance field (where decisions aimed to keep conveniently bounded the ruin probability of the insurance company), and reward driven in the investment sector (a consequence of the wrong idea that naive diversification could fully get rid of any risk<sup>3</sup>).

The second strong analogy concerns the reward and risk measures to be used in the analysis. Both authors reflected carefully on the problem and their common final choice was expectation and variance of the random gain (or rate of return) in a single period horizon, even if de Finetti came to the variance only as a (second best) computationally more convenient, equivalent substitute of the (first best) ruin probability risk measure, and Markowitz dedicated an entire chapter (chap. 9) of his book (Markowitz, 1959) to deal with a mean-semideviation approach. Since that time the mean-variance approach was extraordinarily successful, even if from time to time other reward-risk combinations may be found in theoretical as well in financial applications literature. A few citations may include Benati-Rizzi, 2007; Huang 2008a, 2008b; and Miller-Ruszczyński, 2008.

A third strong analogy is the clear awareness of the difference between efficient (in Paretian sense) solutions and optimal solutions and the idea to concentrate at first on the search for the efficient set of the problem as a preliminary step to any further second stage optimality approach, which in any case should be restricted to the set of efficient solutions. In addition, both authors had quite clear the, now obvious but at that time not so widely known, meaning of efficiency in Paretian sense<sup>4</sup>.

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<sup>3</sup> On the point see the historical survey provided in Rubinstein, 2006b, p. 102-104

<sup>4</sup> On this topic de Finetti had already written in the 1930s a couple of ground breaking papers, see de Finetti, 1937a, 1937b.

On the basis of these pillars, surprisingly similar despite the certain reciprocal ignorance, both authors framed their problems as a constrained quadratic optimization problem (minimization of the variance of the random gain or rate of return), parameterized to a lower bound constraint on the expected gain  $E^5$ , and with the addition of the respective standard constraints.

With regard to the time horizon, both authors distinguished between single period and multiperiod problems. In this paper we neglect the multiperiod question<sup>6</sup> and focus on the analysis of the single period problem, and specifically of the procedure offered by the authors to find the efficient set. Here also the analogies are quite strong and largely prevailing on differences at least as long as we limit our comparison to Markowitz, 1952.

Both authors embedded their feasible sets in the asset space, the  $n$  dimensional unitary cube in the reinsurance case and the convex hull of the feasible vertices with respect to the budget constraint in the investment problem. Both analyzed and exploited the geometric character of isomean (parallel planes) and isovariance (concentric ellipsoids) in that space. Both looked at the efficient set as a feasible path in that space, connecting the MfE point of largest (Maximal) feasible Expectation (mean) with the mfV of smallest (minimal) feasible Variance, but to be run in opposite directions (see fig. 1, sect. 10). Both started reasoning on examples in 3-dimensions and both found that the path was a continuous broken line whose (initial, intermediate and final) corner points played a key role. But the ways they arrived to this conclusion were rather different.

In Markowitz the segments of the broken line were part of (half)lines he named "critical lines". Given a subspace of  $s \leq n$  assets identified by the

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<sup>5</sup> In the reinsurance problem this is equivalent to a lower bound constraint on  $W+E$  with  $W$  a given constant value of the initial guarantee fund of the insurance company.

<sup>6</sup> It was the subject of chapter 11 of Markowitz, 1959 and of the second part of de Finetti, 1940. A comparison between the authors' approach to the multiperiod problem would require another paper. We wish to underline here that Markowitz had a clear idea of the importance of myopia, that is treating any decision as if it were the last. In turn de Finetti found it unconsciously as a byproduct of his strategy to fix retention quotas consistent with an acceptable level of asymptotic ruin probability. The interested reader may find details in Pressacco, 2009, sect. 19.6, p. 527.

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budget constraint restricted to the assets of the subspace, and free of individual non-negativity constraints, the critical line of the subspace is the loci of all points (indeed the straight line of tangency between isovariance ellipsoids and isomean planes) with minimum variance for any given level of expectation. He suggested to take the point of minimum feasible variance (mfV) as starting point of the efficient path; this point belongs to a subspace and is obviously a member of the critical line of this subspace. Then, the first segment of the efficient path moves along this critical line in the direction of increasing expectation. Corner points of the efficient path, dictating a change of direction forced by a change of the subspace, may be found for two possible reasons. Either because on the way there is an intersection with a critical line of a larger subspace, which in turn is to be run in the proper direction of increasing mean, or because on the contrary you cannot find any intersection of this type before reaching a stopping point, that is a point where you cannot proceed without breaking one of the individual constraints. Also in this case there is a change of subspace and you move along the critical line of the new smaller subspace (provided that this is at least a two assets subspace, whereas in case of a single asset subspace a "special" rule to choose a new two assets subspace is needed) in the proper direction of increasing mean. The path ends in a final corner where the maximal (largest) feasible expectation (MfE) is reached (which in case of no ties between the assets expectations is a vertex of the convex hull of feasible points; see fig. 2, sect. 10). We suggest to call match (respectively break) corners those of the first (second) type. As we shall see later (sections 6 and 10), there are also "special" corners of a mixed type, coupling at first a break event and then a match event. Variables associated to a match corner (enlarging the space) or to a break corner (lowering the space, or breaking the constraint) will be called match and respectively break variables.

de Finetti took as a starting point the one of largest expectation (MfE), that is full retention of all risks. The efficient path connects through a sequence of segments the MfE point with the endpoint of null variance (zero retention for all risks). The segments are pieces of straight lines (counterpart of critical lines) in a sequence of subspaces (of increasing dimension) corresponding to the subset of risks currently reinsured, while the other are kept fixed at full retention (for a detailed description of the dynamic logic proposed by de Finetti see sect. 4). The risks currently reinsured in a subspace are those sharing the largest advantage from marginal additional reinsurance, measured by the ratio decrease of variance over decrease of

expectation and the direction of movement in the subspace is the one on the straight line preserving the equality of the individual advantages (obviously towards decreasing expectation). All intermediate corner points are matching points found where the advantage of beginning to reinsure another risk, previously fully retained, matches from below the decreasing common advantage shared by all risks currently reinsured. Then, according to de Finetti, the efficient path reveals to be a continuous broken line made by  $n$  segments. In particular, the last segment implies a movement on the  $n$  dimensional space, with joint additional reinsurance for all risks, and ends in the vertex of null retention. In the Markowitz language, it is clear that each segment of de Finetti's path lies on a critical line of the corresponding subset of assets currently reinsured (but with the other kept at the fixed level  $x_i = 1$  of full retention rather than at the level  $x_i = 0$  as in the investment problem).

For a better perception of the symmetry between the two approaches, it is convenient to take note of a shortcoming (only recently discovered by Markowitz, 2006) in de Finetti's procedure: he gave for granted that a matching event may be found in any half-line of the efficient path, thus excluding that along some half-line a matching does not happen before a variable reaches its boundary level<sup>7</sup>. In this case, there a break corner would be found, as proceeding in that direction would break an individual constraint. We discuss the point in the final part of sect. 4.

The gap between the two approaches became quite large if we consider the subsequent papers by Markowitz (1956, 1959). Indeed, in the next few years of the 1950s he realized the big jump from the largely informal and intuitive geometric approach previously discussed to the advanced one exploiting the Kuhn-Tucker, 1951, Dantzig *et al*, 1955 and Frank-Wolfe, 1956 results in optimization<sup>8</sup>. This way he was able to generate the formalized sequential procedure known as critical line algorithm (CLA). The same step was never done by de Finetti; only recently and under the decisive stimulus of the Markowitz remark, de Finetti's tools and rules were embedded in the modern environment of mathematical programming to obtain at first a formalized version of the CLA for the reinsurance case (see Pressacco-Serafini, 2007) and later also what could be named a version *a la* de Finetti of the CLA for the standard investment problem (see Pressacco-Serafini, 2009).

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<sup>7</sup> Quite likely at 0, but 1 cannot be in our opinion logically excluded as there is no necessary monotonic character in the path of individual retentions, see Pressacco-Serafini, 2007, fig. 1b, p. 39.

<sup>8</sup> Quoted by Markowitz, 2006.



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It is interesting to note that Markowitz mimicked de Finetti's choice of the MfE point as starting point of the CLA only in his advanced approach, whereas in the previous intuitive one he had chosen the inverted path direction (starting point mfV).

In addition, it is fair to recognize that de Finetti's version of the CLA requires some additional hypothesis and is then of lesser general applicability than classical Markowitz CLA to cases where non standard constraints hold or there is equality or negativity of assets expectations.

### 3. de Finetti risk reward approach

de Finetti treated a problem of optimal variable quota share proportional reinsurance faced by an insurance company with an initial guarantee fund (free capital)  $W$  and a given portfolio, formally a vector  $X = X_1, X_2, \dots, X_n$ , of insured risks, with  $X_i$  the random liability of the  $i$ -th risk. Let  $P_i = P(X_i)$  be the net premium (gross premium less the expenses loading for commissions, collection, management, but including the safety loading) charged by the company for the  $i$ -th risk and  $P = \sum_i P_i$  denote the overall net premium. A proportional retention generated by a variable quota share proportional reinsurance treaty is a feasible vector  $\mathbf{x}$ , i.e.  $0 \leq x_i \leq 1$ . Under reinsurance on original terms conditions, the insurance company retains a quota  $x_i$  and transfers  $(1 - x_i)$  of the net premium as well as of the liability to the reinsurer; this way the premium retention is  $\Pi = \sum_i \Pi_i = \sum_i x_i P_i$  and the retained liability  $Y$  is given by  $\sum_i Y_i = \sum_i x_i X_i$ . The post retention random profit of the insurer is then  $G = \Pi - Y = \sum_i G_i = \sum_i (\Pi_i - Y_i) = \sum_i x_i (P_i - X_i)$ . How to choose  $\mathbf{x}$ ?

At that time, the largely prevailing approach was exclusively driven by the need to keep the default risk (in a single accounting period or over an extended time horizon) conveniently bounded. In a single period setting, the default risk was roughly measured by the probability that  $W + G < 0$ , or  $W < -G$ . Unsatisfied by this exclusively risk driven approach, de Finetti suggested an integrated reward-risk approach, based on the simple consideration that every reinsurance reduces the default risk *but leads to renounce to part of the profit* (de Finetti, 1940, p. 2, 2nd paragraph, in the translation Barone, 2006). Then, he looked for a reinsurance strategy able to maximize the *reduction in the risk of default for a given loss in profit (ibidem)*. This way he explicitly recognized the expected profit (shortly

mean)  $E$ , and the default or ruin probability  $RP$ , as the proper reward and respectively risk measures. After that, de Finetti treated the question as a two-stage problem: in the first stage, the  $(E, RP)$  Pareto efficient set is selected and, in the second stage, a point within this set is chosen. As strange as it may seem, he did not introduce an explicit formal definition of Pareto efficiency in mean-ruin probability, but there is no doubt that he referred to the standard one:  $\mathbf{x}$  is efficient iff there do not exist other feasible retentions  $\mathbf{y}$  such that both  $E(\mathbf{y}) \geq E(\mathbf{x})$  and  $RP(\mathbf{y}) \leq RP(\mathbf{x})$ , with at least one inequality strict<sup>9</sup>.

Under an additional hypothesis of normality of the post retention random gain, another fundamental step in de Finetti's paper was the transition from the single period ruin probability to the single period variance  $V$  (or standard deviation  $\sigma$ ) of the random gain as the proper risk measure to be used in the first stage of the reinsurance problem. Indeed with  $E(\mathbf{x}) = \sum_i x_i m_i = \sum_i x_i (P_i - E(X_i))$  and  $V(\mathbf{x}) = \sum_i x_i^2 V(X_i) + 2 \sum_i \sum_{j>i} x_i x_j Cov(X_i, X_j)$ , the mean and variance of  $G$ , the ruin probability is given by  $Prob[(G - E)/\sigma] < [-(W + E)/\sigma]$ , or with  $t = (W + E)/\sigma$ , by  $p = Prob\left(\frac{(G - E)}{\sigma} < -t\right)$ . Note that  $t = (W + E)/\sigma$  is surely positive under the hypothesis  $m_i > 0$  for any  $i$ . Under normality of  $G$  (for any  $\mathbf{x}$ ) and with  $N$  the cumulate of the standard normal (of course a decreasing function of  $t$ ), it is  $p = N(-t)$ , a monotone decreasing function of  $t$ . Hence, de Finetti argued that minimizing  $p$  for any given level of  $E$  is equivalent to minimizing  $\sigma$  (see de Finetti, 1940, p. 9, 4<sup>th</sup> paragraph, *as long as we stay on an iso level  $W+E$  constant, determining the maximum of  $t$  - i.e. the minimum ruin probability - is equivalent to determining the minimum of  $\sigma$* ).

Although in general, this does not allow us to conclude that the efficient sets  $(E, RP)$  and  $(E, V)$  are coincident, this is fortunately true in the reinsurance problem. Hence, he concluded  $RP$  and  $V$  (or  $RP$  and  $\sigma$ ) were

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<sup>9</sup> It could be unequivocally deduced by his explanation of the formal meaning of his words *maximize the reduction in the risk of default for a given loss in profit* which is given in a footnote (de Finetti, 1940, footnote 2, p. 4, Barone, 2006) as follows: *Precisely, it is an optimum problem, in the sense of my Notes on Problemi di optimum and Problemi di optimum vincolato...*, where the reference is to a couple of enlightening papers (see de Finetti, 1937a, 1937b) concerning the mathematical characterization of the Pareto efficiency set in Economics.

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perfectly equivalent risk measures with respect to the problem of finding the single period mean-risk efficient set<sup>10</sup>.

After this equivalence result, the problem of finding the efficient reward-risk single period retentions could be formally expressed in the following mean-variance setting:

an insurance company is faced with  $n$  risks (policies). The net profit of these risks is represented by a vector of random variables with expected value  $\mathbf{m} := \{m_i > 0: i = 1, \dots, n\}$  and a non-singular covariance matrix  $C := \{\sigma_{ij} > 0: i, j = 1, \dots, n\}$ . The company has to choose a proportional reinsurance or retention strategy specified by a retention vector  $\mathbf{x}$ . The retention strategy is feasible if  $0 \leq x_i \leq 1$  for all  $i$ . A retention  $\mathbf{x}$  induces a random profit with expected value  $E = \mathbf{x}^T \mathbf{m}$  and variance  $V = \mathbf{x}^T C \mathbf{x}$ . A retention  $\mathbf{x}$  is by definition mean-variance efficient or Pareto optimal if, for no feasible retention  $\mathbf{y}$  we have both  $\mathbf{x}^T \mathbf{m} \leq \mathbf{y}^T \mathbf{m}$  and  $\mathbf{x}^T C \mathbf{x} \geq \mathbf{y}^T C \mathbf{y}$ , with at least one inequality strict.

#### **4. de Finetti's procedure to find the mean-variance efficient set**

After this theoretical background, de Finetti looked for a procedure to find the single period  $(E, V)$  efficient retentions set. Working at a time where constrained programming and Kuhn-Tucker conditions were things to come, he applied an approach that is quite close to a dynamic programming framework, conveniently supported by geometric intuitions. Precisely, he suggested to look for such a set  $X^*$  as a path in the  $n$  dimensional unitary cube of feasible retentions. The path starts at the full retention vertex  $\mathbf{x}=\mathbf{1}$ , point of largest expectation (being  $m_i > 0$  for any  $i$ ) and ends at the opposite, full reinsurance or zero retention, vertex  $\mathbf{x}=\mathbf{0}$ , unique point of minimum variance, thanks to the non singularity of  $C$ . At any point  $x^*$  of the

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<sup>10</sup> The event of ruin in a single period could be seen as equivalent to the disaster event which has been the base of the celebrated Roy, 1952 approach to portfolio selection. Inspired by a safety first strategy, Roy looked for the risky portfolio which minimizes the probability of a single period disaster and showed that it is the portfolio that minimizes the ratio  $(E - d)/\sigma$ , where  $(E - d)$  is the excess return with respect to the disaster return  $d$ . This strategy is currently widely applied in a lot of decision problems including financial ones. See for example Huang, 2008a and Dorfleitner-Utz, 2012.

path an efficient movement is made in such a way to get locally the largest benefit measured by the ratio decrease of variance over decrease of expectation. To translate this idea in an operational setting, de Finetti introduced tools and rules. The tools are a set of advantage functions:

$$F_i(\mathbf{x}) = \frac{1}{2} \frac{\frac{\partial V}{\partial x_i}}{\frac{\partial E}{\partial x_i}} = \sum_{j=1}^n \frac{\sigma_{ij}}{m_i} \cdot x_j, \quad i=1, \dots, n \quad (1)$$

designed to capture (half of) the local benefit coming at  $\mathbf{x}$  from an additional or initial reinsurance of the  $i$ -th risk.

The rules give an answer to the following three questions:

1. to which risks should we provide additional reinsurance at  $\mathbf{x}^*$ ?
2. if this set is not a singleton, which feasible direction should we choose? and
3. where and why should we change direction?

The answer to the first question is: at any point  $\mathbf{x}^*$  of  $X^*$  move in such a way to provide (additional or initial) reinsurance only to the set  $I(\mathbf{x}^*)$  of those risks sharing the largest value of  $F_i(\mathbf{x}^*)$  (put  $\max_i F_i(\mathbf{x}^*) = \lambda(\mathbf{x}^*)$ ). If this set is a singleton, the direction of the efficient path is obvious; otherwise, the answer to the second question is: move in the direction preserving the equality of all the  $F_i(\mathbf{x}^*)$  coming from  $I(\mathbf{x}^*)$ . Given the form of the advantage functions, it is easily seen that this implies a movement on a segment of the cube characterized by the set of equations  $F_i(\mathbf{x}) = \lambda(\mathbf{x})$ , for  $i$  belonging to  $I(\mathbf{x}^*)$ . As the direction of movement is driven exclusively by the risks belonging to  $I(\mathbf{x}^*)$  we call them directional risks at  $\mathbf{x}^*$ . The chosen direction is kept up until a change in the  $I$  set happens. This gives, according to de Finetti, the following answer to the third question: change direction at the first point on the segment where a risk  $i$ , previously not belonging to the  $I$  set, matches from below the common value of the advantage functions of the currently directional risks, thus beginning to be reinsured and becoming a new member of the  $I$  set. But this implies the addition of one equation to the system giving the best direction of movement and, as a consequence, a change of direction that corresponds to a corner point in the efficient path. de Finetti was then able to conclude that the efficient  $(E, V)$  path is made by a continuous broken line of  $n$  segments, whose corners (except for the terminal vertex) correspond to points in which a new risk begins to be reinsured.

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Moreover, there is a one to one monotonic correspondence between the efficient set  $X^*$  and the interval of values of the advantage parameter  $0 \leq \lambda \leq F_1(\mathbf{1}) = \max_i F_i(\mathbf{1})$ .

More formally, a sequence of corner values  $\lambda_h$ , corner points  $\mathbf{x}_h$  and sets  $I_{h-1}$  of directional risks in the interval  $\lambda_h < \lambda \leq \lambda_{h-1}$  is described by the equations  $\lambda = F_i(x)$  for  $i \in I_{h-1}$ ,  $x_i = 1$  for  $i \notin I_{h-1}$ , which define  $\mathbf{x}$  parameterized by  $\lambda$ . The corner point  $\mathbf{x}_h$  is still found in this direction, where one of the non-directional risks (previously frozen at the level 1 of full retention) matches from below the common value of the advantage functions of the  $h-1$  directional risks in  $I_{h-1}$ . There, the new matching risk joins the set  $I_{h-1}$  of the previous directional risks, changing state and defining the new set  $I_h$ .

This sequential procedure is to be seen as the informal predecessor of the famous critical line approach applied by Markowitz to find  $(E, V)$  efficient portfolios of financial assets. de Finetti gave for granted that matching points are in a one-to-one correspondence with segments of the efficient path.

In addition, de Finetti found closed-form formulae for the efficient retentions  $\mathbf{x}(\lambda)$  in case of no correlation (see de Finetti, 1940, p. 12). Indeed, he gave an outline of the way to obtain the same result in a particular case of correlation structure (group correlation, see de Finetti, 1940, pp. 28-29). He also underlined that these nice results came from the possibility to define an *a priori* fixed ordering of (entrance in reinsurance) the risks, while, in the general case, the application of the sequential procedure previously described is needed.

A hidden bug in de Finetti's approach went unnoticed until it was recently discovered by Markowitz (2006) in his review of de Finetti's paper. The point is that de Finetti's rules have only an internal coherency but lack an unconditional one. Indeed, it was implicit in de Finetti's reasoning, and a necessary and sufficient condition for the survival of the whole procedure, that a matching event could be found in any segment of the efficient path. Let us shortly denote it by matching hypothesis (MH). But we cannot be sure that MH holds for a general structure of correlation, except for the special cases of no correlation or group correlation. Then, Markowitz, while recognizing de Finetti's primacy in the  $(E, V)$  approach, expressed the opinion that his (de Finetti's) procedure works only in the special case of no

correlation. *For the case of uncorrelated risks, de Finetti solved the problem of computing the set of mean variance efficient portfolios. He explains while the problem with correlated risks is more complicated and solves special cases (the group correlation) of it (see Markowitz, 2006, p.5). He does not solve it in general for correlated risks (see Markowitz, 2006, p.11), while for the general case of correlation it is necessary to keep account of the constraint binding at the break event and to apply advanced mathematical programming techniques.*

From a strict technical point of view this is undisputable, yet we think that a softer sentence would offer a fair historical resume of de Finetti's contribution. Indeed, on one side, de Finetti's procedure may quite likely work in a lot of real world correlation structures of insurance portfolios. On the other side, also in case MH does not hold, de Finetti's logic, properly adjusted to purge it from its incoherency, remains an enlightening way to manage and solve the efficient retention problem. Last but not least, if we approach that problem with the modern tools of constrained quadratic programming, the optimality conditions can be restated in the most natural form through the advantage functions so that the revised version of de Finetti's approach is fully compatible with the modern technology (on the point see Pressacco-Serafini, 2007, pp. 34-36).

## 5. Cases of no correlation or group correlation: some results

In the case of no correlation (see de Finetti, 1940, p. 12)  $F_i(\mathbf{x})$  reduces to  $F_i(\mathbf{x}) = (x_i V_i / m_i) = x_i v_i$ . There is a natural ordering of risks induced by  $v_1 > v_2 > \dots > v_n$  and the efficient set may be expressed in closed form formulae. For any value  $0 = \lambda(\mathbf{0}) \leq \lambda \leq \lambda(\mathbf{1}) = v_1$ , an efficient retention is given by  $x_i(\lambda) = \min(\lambda/v_i; 1)$ ,  $i=1, \dots, n$ ; conversely  $\lambda(\mathbf{x}) = \max_i x_i v_i$  so that a one-to-one monotone correspondence between  $\lambda$  and  $\mathbf{x}$  is established.

Putting  $\lambda$  values on the abscissa, the graph of the functions  $x_i(\lambda)$  are broken lines made by two segments, the first one going out from the origin with slope  $1/v_i$  which at  $\lambda = v_i$  becomes the horizontal line at level 1.

After this analysis de Finetti was able (see de Finetti, 1940, p. 12) to express both  $E$  and  $V$  of any efficient retention as a function of  $\lambda$ . Precisely, in the interval  $\lambda_h < \lambda \leq \lambda_{h-1}$  ( $h-1$  risks already reinsured), it is:

$$E(\lambda) = \sum_{i=1}^{h-1} x_i m_i + \sum_{i=h}^n m_i = \lambda \sum_{i=1}^{h-1} (m_i/V_i) m_i + \sum_{i=h}^n m_i \quad (2)$$

and putting  $A_{h-1} = \sum_{i=1}^{h-1} m_i^2/V_i$  e  $B_{h-1} = \sum_{i=h}^n m_i$ , we get:

$$E(\lambda) = \lambda A_{h-1} + B_{h-1} \quad (3)$$

$$\begin{aligned} V(\lambda) &= \sum_{i=1}^{h-1} x_i^2 V_i + \sum_{i=h}^n V_i = \lambda^2 \sum_{i=1}^{h-1} (m_i^2/V_i^2) V_i + \sum_{i=h}^n V_i = \\ &= \lambda^2 \sum_{i=1}^{h-1} m_i^2/V_i + \sum_{i=h}^n V_i \end{aligned} \quad (4)$$

and with  $C_{h-1} = \sum_{i=h}^n V_i$ , it is:

$$V(\lambda) = \lambda^2 A_{h-1} + C_{h-1} \quad (5)$$

It turns out that, in the no correlation case,  $E$  and  $V$  are piecewise linear and respectively piecewise quadratic functions of  $\lambda$ . Moreover, it is easy to show that they are continuous but not differentiable at the connection points (corner points of the efficient path). Unfortunately de Finetti neglected to analyze the properties of the efficient set in a  $(E, V)$  reference system; in particular he did not exploit the above relations to obtain  $V$  as the following immediate piecewise quadratic function of  $E$ :

$$V = C_{h-1} + ((E - B_{h-1})^2/A_{h-1}) \quad (6)$$

By doing so he could have had at his disposal an easy road to show that, also at the connection points,  $V$  is a continuous and differentiable function of  $E$  with, not surprisingly,  $\partial V/\partial E = 2\lambda$ . Note that  $\lambda$  could be interpreted as a

shadow price of insurance and  $F_i(\mathbf{x})$  as marginal utility at  $\mathbf{x}$  of further reinsurance of the  $i$ -th risk, so that, given  $\lambda$ , reinsurance is provided up to the point where decreasing marginal utility equals the shadow price<sup>11</sup>.

By group correlation following de Finetti, who at that time considered this structure a good proxy of the real insurance world, we mean that the risks are partitioned into a number  $g$  of groups,  $q=1, \dots, g$  each one characterized by a couple  $(l_q, \rho_q)$  of constants. Denoting by  $m_{iq}$  and  $\sigma_{iq}$  the expectation and the standard deviation of the  $i$ -th risk of the group  $q$ ,  $l_q$  is a group specific loading coefficient used to charge net insurance premiums through a safety loading principle inspired by the standard deviation principle so as  $m_{iq}=l_q\sigma_{iq}$ <sup>12</sup>. The constant  $\rho_q > 0$  is a group specific correlation coefficient and plays a role in specifying the covariance matrix  $C$ , which is a block diagonal matrix  $(C_1, \dots, C_g)$ , with non null elements only on the main diagonal squared blocks given, for  $i_q \neq j_q$ , by  $\rho_q\sigma_{iq}\sigma_{jq}$ .

Under this structure, the advantage functions became, with  $x_q$  the retention vector of the group  $q$ :

$$F_{iq}(\mathbf{x}_q) = l_q^{-1} \left( x_{iq}\sigma_{iq} + \sum_{j_q \neq i_q} x_{jq}\sigma_{jq} \right) \quad (7)$$

de Finetti argued that, under this hypothesis, some of the nice properties of the no correlation case still hold and this would allow a quasi closed-form solution to the problem. He gave only a sketch of how to get this result. It comes from the possibility to define an *a priori* ordering of risks within each group (based on the standard deviation ranking  $\sigma_{1q} > \sigma_{2q} > \dots > \sigma_{nq}$ ), as well as an *a priori* ordering of groups (based on the advantage function ranking of the first risk of each group at full retention  $F_{11}(\mathbf{1}) > F_{12}(\mathbf{1}) > \dots > F_{1g}(\mathbf{1})$ ).

Indeed, applying these suggestions and following de Finetti's procedure, we were able elsewhere (Pressacco-Ziani, 2012, p. 352) to check that the matching condition really holds, and that the value of the advantage

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<sup>11</sup> de Finetti's paper had a large impact on the actuarial sciences, even if the applications remained restricted to the no correlation case. On the point see Bühlmann- Gerber, 1978; Gerber, 1984; Gerber-Shiu, 2003.

<sup>12</sup> In this section, the first index refers to a risk and the second to a group and the symbol must not be confused with notations like  $\sigma_{ij}$  which means covariance between the two risks  $i$  and  $j$ .



*Title*

parameter at which the  $i$ -th risk of the group  $q$  starts to be reinsured, is given by:

$$\lambda_{iq} = l_q^{-1} \left[ \sigma_{iq} (1 + \rho_q (i - 2)) \right] + \sum_{j=2}^{n_q} \sigma_{jq} \quad (8)$$

and that closed-form formulae hold for the efficient retentions both in the retention space (see Pressacco-Ziani, 2012, p. 352) and in the mean-variance one (for details, Pressacco-Serafini-Ziani, 2011, pp. 444-448).

In particular, we found that in the  $(E, V)$  plane the efficient set is still piecewise parabolic continuous and differentiable, also at the connection points. That is enough to show that de Finetti's logic could surely work also under (admittedly very special) conditions of non null correlation. Besides that, we suggest that this model could be considered a counterpart of the simplified models of covariance that (in order to make the problem computationally more manageable) have been developed in the wave of portfolio selection and CAPM models (Markowitz, 1999 quoted Sharpe, 1963, Cohen-Pogue, 1977 and Elton-Gruber, 1973).

## **6. Adjusting de Finetti's procedure for the case of break points**

de Finetti's procedure may be adjusted also when the MH does not hold (for details see Pressacco-Serafini, 2007, discussion in sect. 5 as well as the example in sect. 8). Here we just summarize the results.

It turns out that the corner points may be of three different types: match (in turn distinct between below match and above match), break and mixed corners. At a below match corner the advantage function of a risk previously fully retained matches from below the current value of the advantage parameter, joins the set of directional risks and begins to be reinsured; at an above match corner the advantage function of a risk previously fully reinsured matches from above the current value of the advantage parameter, joins the set of directional risks and ceases to be fully reinsured; at a break corner the retention level of a previously partially reinsured risk reaches a boundary value (quite likely 0, but 1 cannot be excluded), leaves the set of directional risks and is (may be temporarily) frozen at the boundary level; at a mixed corner (surely corresponding to a vertex of the unitary cube) there is at first a break when the retention level of the unique variable previously

partially reinsured reaches the lower boundary 0, and leaves the set of directional risks which as a consequence becomes empty. Then, in order to leave the vertex, we must wait for a subsequent matching between the decreasing value of the advantage parameter and the highest value at the vertex of the advantage functions of the currently fully retained risks. Except in case the efficient set contains one or more vertices (mixed corners) the one to one correspondence between the interval  $0 \leq \lambda \leq F_1(\mathbf{1})$  and the vector of efficient retentions  $\mathbf{x}^*(\lambda)$  is preserved.

In the  $(E, V)$  plane the efficient set is given by a continuous union of parabolic arcs corresponding to the segments of the broken line. The graph is continuous also at the connection points and differentiable everywhere (except if the efficient set includes one or more vertices as there are kinks at the connection points of the graph corresponding to such vertices). A convenient hypothesis of non degeneracy is implicitly assumed in the above description. It may be resumed by the condition that at any transitional value of the advantage parameter  $\lambda$ , only one risk is in transition.

We signal that Markowitz pointed out clearly (Markowitz, 2006) that, if de Finetti would have completed a careful analysis of the problem without relying too much on the wrong intuition of the absence of break corners, he could have made a giant step toward the Kuhn-Tucker conditions (Kuhn-Tucker, 1951).

## 7. Advantage functions in the investment problem

In the next sections, we will see how to extend de Finetti's approach to a standard portfolio selection problem. Let us recall the essential of such a problem.

An investor is faced with  $n$  assets. The net rate of return of these assets is represented by a vector of random variables with expected value  $\mathbf{m} := \{m_i: i = 1, \dots, n\}$  and a non-singular covariance matrix  $C := \{\sigma_{ij}: i, j = 1, \dots, n\}$ . In addition to the usual non degeneracy (singularity) conditions on the covariance matrix, we will assume a labeling of the assets coherent with a strict ordering of expectations, namely  $m_1 > m_2 > \dots > m_n$ . The investor has a budget, conveniently normalized to 1, to invest in the given assets. Let  $x_i$  be the fraction of budget invested in the asset  $i$ . If short positions are not allowed, the portfolio strategy is feasible if  $x_i \geq 0$  for all  $i$  and  $\sum_i x_i = 1$ . A portfolio  $\mathbf{x}$  induces a random rate of return with expected value  $E = \mathbf{x}^T \mathbf{m}$

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and variance  $V = \mathbf{x}^T C \mathbf{x}$ . A feasible portfolio  $\mathbf{x}$  is by definition mean-variance efficient or Pareto optimal if for no feasible portfolio  $\mathbf{y}$  we have both  $\mathbf{x}^T \mathbf{m} \leq \mathbf{y}^T \mathbf{m}$  and  $\mathbf{x}^T C \mathbf{x} \geq \mathbf{y}^T C \mathbf{y}$ , with at least one inequality strict. Let  $X^*$  be the set of optimal portfolios.

de Finetti did not treat at all (neither in his 1940 paper nor even later) the asset portfolio problem; yet we found that de Finetti's procedure, once the behind logic is transferred from the reinsurance to the investment problem, remains a good strategy. To obtain in a natural and straightforward way something analogous to the critical line algorithm, we make recourse to a modified version of the advantage functions tailored to the constraints of the investment problem.

Now, the role of the "elementary" advantage function is played by  $F_{ij}$ , i.e. the advantage associated to the portfolio adjustments coming from a "small" trading between asset  $i$  (decreasing) and asset  $j$  (increasing).

More formally and with reference to a portfolio  $\mathbf{x}$  with positive quotas of both assets  $i$  and  $j$ , let us call "bilateral" trading in the  $i$ - $j$  direction the portfolio adjustment obtained through a "small" exchange between  $i$  decreasing and  $j$  increasing. If the benefit (burden) measure is given by the ratio decrease (increase) of variance over decrease (increase) of expectation, the advantage functions ought to be defined as<sup>13</sup>:

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<sup>13</sup> Note that the sign of the denominator is the same as  $(j-i)$ ; and also that, as both  $i$  and  $j$  are active assets (that is with positive quotas) in the current portfolio  $\mathbf{x}$ , a feasible bilateral trading may happen also in the  $j-i$  direction. Then both  $F_{ij}(\mathbf{x})$  and  $F_{ji}(\mathbf{x})$  describe the results of a feasible bilateral trading at  $\mathbf{x}$ . Moreover it is immediate to check that  $F_{ij}(\mathbf{x}) = -F_{ji}(\mathbf{x})$ . Yet, it is convenient to think that the economic meaning of the two functions is symmetric: precisely, if without loss of generality  $i$  is less than  $j$ ,  $F_{ij}(\mathbf{x})$  describes a benefit in algebraic sense, while  $F_{ji}(\mathbf{x})$  describes a burden. If in the current portfolio  $x_i$  is positive and  $x_j$  is null, then the only feasible bilateral trading may be in the direction  $i-j$ . And  $F_{ij}(\mathbf{x})$  describes a benefit if  $i$  is less than  $j$  or a burden in the opposite case. Obviously, if both  $x_i$  and  $x_j$  are at level 0 no feasible trade between  $i$  and  $j$  may take place.

$$F_{ij}(\mathbf{x}) := \frac{1}{2} \frac{\frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j}}{\frac{\partial E}{\partial x_i} \frac{\partial E}{\partial x_j}} = \sum_{h=1}^n \frac{\sigma_{ih} - \sigma_{jh}}{m_i - m_j} x_h \quad (9)$$

## 8. A procedure for the standard investment problem under regularity

Let us now use the advantage functions to build a friendly procedure to find the optimum mean-variance set for the standard portfolio problem.

The starting point of the mean-variance path is  $\mathbf{x}_2^* = (1, 0, 0, \dots, 0)$ , which is the point with largest expectation due to the ordering convention. The choice of the index 2 may sound strange, but it is justified as it denotes that a second asset starts to be active at this point. Indeed, we leave  $\mathbf{x}_2^*$  in the direction granting the largest benefit, that is the largest value over  $j=2, \dots, n$  of

$$F_{1j}(\mathbf{x}_2^*) := \frac{\sigma_{11} - \sigma_{j1}}{m_1 - m_j} := \lambda(\mathbf{x}_2^*) \quad (10)$$

Let us label asset  $j_2$  this asset. This means that the bilateral trading in the direction  $1 - j_2$  gives the largest benefit  $\lambda(\mathbf{x}_2^*)$  and dictates the efficient path leaving  $\mathbf{x}_2^*$  in the direction  $(-\varepsilon, 0, 0, 0, \varepsilon, \dots, 0)$ . While the trade takes effect, the  $\mathbf{x}$  values change and consequently  $F_{1j}(\mathbf{x}) := \lambda(\mathbf{x})$  decreases. In the regular case, the bilateral trading  $1 - j_2$  remains the most efficient one until we find a point on the above segment. There the benefit granted by this trade is matched (and regularity means just that such a feasible matching point may be found before the end of the segment) by another bilateral trade, that is until the nearest point (labelled  $\mathbf{x}_3^*$ ), where  $F_{1j_2}(\mathbf{x}_3^*) = F_{1j}(\mathbf{x}_3^*) = \lambda(\mathbf{x}_3^*)$  for some  $j \neq j_2$ . Let us label asset  $j_3$  this asset. Some remarks are in order now.

Remark a): at  $\mathbf{x}_3^*$  also the bilateral trade  $j_2 - j_3$  matches the same benefit  $\lambda(\mathbf{x}_3^*)$ . Indeed the following result holds: for any triplet  $i, j, h$  of assets and any portfolio  $\mathbf{x}$  such that at least  $x_i$  and  $x_j$  are strictly positive,

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$$F_{ij}(\mathbf{x}) = F_{ih}(\mathbf{x}) = \lambda \Rightarrow F_{ih}(\mathbf{x}) = \lambda \quad (11)$$

If  $i < j < h$ , the advantage functions value describes a matching of the benefits given by any bilateral trade  $i$ - $j$ ,  $i$ - $h$ ,  $j$ - $h$ . The proof is straightforward.

Remark b): small feasible changes in a portfolio composition may come as well from joint movements of more than two assets, to be seen as a multilateral trade. However, any multilateral feasible trade may be defined as a proper combination of feasible bilateral trades and if all the bilateral trades share the same benefit, then the benefit of the multilateral trade too matches the common benefits. This explains why we may concentrate on the benefits from bilateral trades neglecting the analysis of the consequences of multilateral ones.

Remark c): in some cases, it could be advantageous to split a multilateral trade in bilateral components, implying also one or more trading of the type  $j$ - $i$  with  $j > i$ . Surely in isolation this cannot be a drift of movements along the efficient path. However, as we shall see later, it could add efficiency when inserted in the context of a multilateral trade.

Let us go back now to the point  $\mathbf{x}_3^*$  where all bilateral trades between two of the first three assets and hence any feasible multilateral trade among them shares the same efficiency. At first sight, there is here an embarrassing lot of opportunities to leave  $\mathbf{x}_3^*$  along directions granting the same benefit. However, help is given by the second rule we receive in heritage from the reinsurance problem (see Section 4): move along the path (indeed the segment) which preserves the equality of all implied advantage functions. In our case move along the direction  $(\varepsilon_1, 0, 0, \varepsilon_{j_2}, \dots, \varepsilon_{j_3}, 0, \dots, 0)$  (with  $\varepsilon_1 + \varepsilon_{j_2} + \varepsilon_{j_3} = 0$ ) which preserves the  $F_{1j_2}(\mathbf{x}) = F_{1j_3}(\mathbf{x})$  equality until a new matching point is found.

Under regularity, that is if a feasible matching point may be found at each step, it is easy to prove (see Appendix) that a sequential application of this procedure defines a piecewise linear optimal path with corner points  $\mathbf{x}_h^*$  where the matching asset henceforth labeled  $j_h$  joins the other assets already active in the portfolio.

The ending point of the efficient path is the point  $\mathbf{x}_p^*$  of absolute minimum variance with benefit  $\lambda(\mathbf{x}_p^*) = \mathbf{0}$  for all advantage functions.

To summarize, in case of regularity, corner points of the optimum path are always matching points and each matching point  $\mathbf{x}_h^*$  corresponds to a matching of the advantage function  $F_{1j_h}(\mathbf{x}^*)$  of a newcomer asset  $j_h$  with the common value  $\lambda(\mathbf{x}^*)$  of the advantage functions  $F_{1i}(\mathbf{x}^*)$  of the current active assets  $i$ . Yet only in the didactic but fully unrealistic case of no correlation, the labeling induced by the expectation vector is coincident with the entrance (matching) ordering (except of course for the asset of largest expectation, which is still associated with the starting point). In addition, except for the no correlation case, we cannot a priori but only ex post say if there is regularity or not. We remark that under regularity the computational burden is quite lower than what may appear at first glance. Indeed, there is no need to compute values of the advantage functions for all pairs  $i$ - $j$  (at least  $i$  active), but it is enough to evaluate the  $n-1$  values  $F_{1j}(\mathbf{x})$ . Here we may say that the first asset (asset 1) plays overall the optimum path the role of reference variable. More generally, it is required that the current referent variable is active. And the optimality condition to be checked is that there is a non-negative  $\lambda$  such that, for all active assets  $i$ ,  $F_{1i}(\mathbf{x}) = \lambda$ , while, for the other non-active  $j$ ,  $F_{1j}(\mathbf{x}) \leq \lambda$ , with strict equality holding only at corner points just for the newcomer matching asset. As for the other feasible values of  $F_{ij}(\mathbf{x})$  and their superfluous role in checking the optimality conditions the following result holds:

- both  $i$  and  $j$  active then  $F_{ij}(\mathbf{x}) - \lambda = 0$
- only  $i$  active then  $F_{1i}(\mathbf{x}) = \lambda$  and  $F_{1j}(\mathbf{x}) < \lambda \Rightarrow (F_{ij}(\mathbf{x}) - \lambda)(j - i) < 0$

This explains that the optimality conditions at  $\mathbf{x}$  require simply that all basic bilateral tradings between each pair of active assets share the same benefit level  $\lambda$ , while all basic tradings between an active  $i$  and a non-active  $j$  have a lower efficiency level<sup>14</sup>.

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<sup>14</sup> More precisely less efficiency means lower benefit ( $i < j$ ) or greater burden ( $j > i$ ). At matching corners the matching variable becomes efficient and even if it is at the moment still non-active shares the efficiency of all previous active assets.

## **9. The standard investment problem under non-regularity**

Let us now pass to treat the non-regular case. The non-regularity comes from the fact that there is a failure in the matching sequence in the sense that along a segment of the optimum path one of the active assets reaches its boundary value 0 before a matching occurs. This is the counterpart of a break event in the reinsurance case and it should be clear that at a break point the break variable leaves the efficient set and remains frozen, maybe temporarily, at the level 0.

The new set of the other active assets (given that there are at least 2), determines the new direction, either preserving the equality between the advantage functions in case of three or more active assets, or in the unique feasible direction if there are only two assets. Before discussing what happens in case of only one surviving active asset, we underline that the behaviour at a break corner is the only difference between the non-regular and the regular case. As to the computational burden, it is still enough to compute at any point of the optimum path only the values of  $(n-1)$  advantage functions.

Yet an additional effort may be required at those break points, where the break asset has been playing the role of the reference asset. Indeed its exit from the set of active assets requires the choice of a new reference (the reference must be active) and then the need to compute the values of a new set of  $(n-1)$  advantage functions. Hereafter the usual conditions of efficiency still hold for the new set of advantage functions.

Let us finally discuss what happens at a break point where the set of active assets is a singleton, so that the point is a vertex of the  $n$  dimensional simplex of feasible allocations. This being the case, a resetting of the procedure is in order. Denoting by  $k$  the original label of the singleton asset, we look for leaving the vertex in such a way as to maximize the efficiency of a bilateral trade of the  $k$ - $j$  type. This means looking for the largest positive value of  $F_{kj}(\mathbf{x})$  over all  $j > k$ . The corresponding value of the benefit parameter could be seen as a match value. We may say that such vertex corners are mixed corners, that is points where both a break (at first for a larger  $\lambda$ ) and a match event (later for a smaller  $\lambda$ ) happen. At these corners the one-to-one correspondence between efficient portfolio and values of  $\lambda$  is lost: there is an interval of  $\lambda$  values to which the same corner points  $\mathbf{x}$  is

associated. Consequently, the graph of the function  $V(E)$  in the mean-variance space has a kink. For details see the example in sect. 10.

## 10. Comparison between de Finetti and Markowitz

Let us comment on the affinity and differences between Markowitz and de Finetti's oriented procedure as regards the use of geometric intuitions in the solution of the investment problem. They inspired the paper of Pressacco-Serafini, 2009 in which an extension of de Finetti's logic to the investment problem is offered (resumed here in sections from 7 to 9) and were a pillar of the Markowitz earliest paper (Markowitz, 1952) and of chapter 7 (entitled Geometric analysis of efficient sets, pp. 129-153) of his subsequent book (Markowitz, 1959). In the first paper, he analyzed in detail the three asset case in the standard version with non negativity constraints  $x_i \geq 0$ , and one collective budget constraint  $\sum_i x_i$ . Exploiting the budget constraint the feasible set is represented in a bidimensional setting (in the plane  $(x_1, x_2)$ ) by the rectangular triangle whose vertices are the points (0,0), (0,1), (1,0). Then, for a while, he went to what he subsequently (Markowitz, 1987, p. 39) called Black model, whose only constraint is the budget one; then exploiting it in the form  $x_3 = 1 - (x_1 + x_2)$ , transformed the Black model in a free problem in 2-dimensions. He showed that, in the plane  $(x_1, x_2)$ , isomean lines are parallel straight lines whose equation (adding here the additional hypothesis of no ties between assets expectations) is:

$$x_2 = \frac{(m-m_3)}{(m_2-m_3)} - x_1 \frac{(m_1-m_3)}{(m_2-m_3)} \quad (12)$$

while isovariance curves are a set of concentric ellipses of equation:

$$V = x_1^2(V_1 + V_3 - 2\sigma_{13}) + x_2^2(V_2 + V_3 - 2\sigma_{23}) + 2x_1x_2(\sigma_{12} + V_3 - \sigma_{13} - \sigma_{23}) + 2x_1(\sigma_{13} - V_3) + 2x_2(\sigma_{23} - V_3) + V_3 \quad (13)$$

i.e.

$$V = ax_1^2 + bx_2^2 + 2cx_1x_2 + 2dx_1 + 2ex_2 + f \quad (14)$$

whose center is the point of (unconditional) minimum absolute Variance (maV) with coordinates:  $x_1 = (ce - bd)/(ab - c^2)$ ,  $x_2 = (cd - ae)/(ab - c^2)$ . Then, he argued that the set of portfolios of minimum variance for any fixed level of mean (geometrically points of tangency between the



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corresponding isomean and isovariance lines) is a straight line whose equation is given in implicit form by:

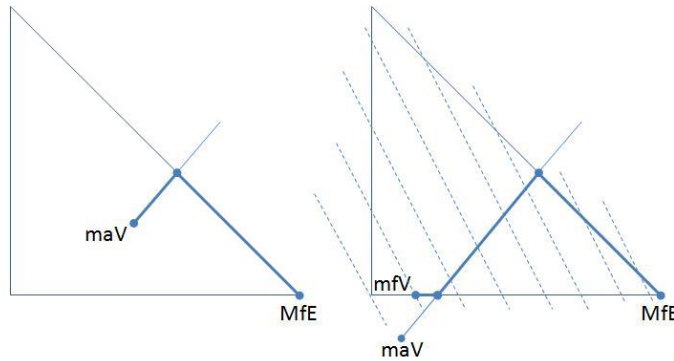
$$(m_2 - m_3)(ax_1 + cx_2 + d) = (m_1 - m_3)(bx_2 + cx_1 + e) \quad (15)$$

He defined critical line such a line, which includes of course the maV point. The half-line starting from maV and going in the direction of increasing mean is the set of efficient mean-variance points (portfolios) of the Black problem in the 2-dimensional picture. This could be considered the main critical half line in the three asset space, but other critical (half)lines should be considered in the two asset (sub)spaces (on the direction of the sides of the triangle) and even in the one asset (sub)spaces (single points). In the intuition of Markowitz, the critical lines played a key role to find the efficient mean variance set. Indeed he suggested (see Markowitz, 1952, p. 87, footnote 10) the following dynamic vision of the efficient path: *the efficient set may be traced out by starting at the point of minimum feasible variance moving continuously along various subspaces according to definite rules...typically we proceed along a given critical line until either this line intersects one critical line of a larger subspace or meets a boundary and simultaneously the critical line of a lower dimensional subspace. In either of these cases the efficient line turns and continues along the new line. The efficient line terminates when a point with maximum feasible mean is reached.*

In other words, the fundamental result is that the efficient path is a continuous broken line made by a sequence of segments lying on critical lines of (sub)spaces so as corner points of the efficient path correspond to an intersection of two adjacent critical (half)lines.

To see more in detail how this procedure should work, it is convenient to discuss a couple of examples. In the first one (analyzed in Markowitz, 1952, p. 85) the maV is internal to the triangle (hence feasible), while the MfE is the vertex (1,0,0) of the triangle (see fig. 1 left side).

**Figure 1: Efficient path (bold face segments):** first (left) and second (right) case. Horizontal axis:  $x_1$ . Vertical axis:  $x_2$ . Diagonal side:  $x_1 + x_2 = 1$  i.e.  $x_3 = 0$ . Assets are labelled according to their expectations ( $m_1 > m_2 > m_3$ ), so that isomean are parallel straight lines with negative slope greater, in absolute value, of the one of the diagonal side of the feasible triangle.



The first segment of the efficient path (left side) lies on the critical half-line of the three asset space; the path has a (break) corner at the point where this critical line intersects a side (the diagonal) of the feasible triangle; then the efficient path goes on the critical half-line of the corresponding two asset subspace ending in the MfE vertex.

According to de Finetti's point of view, the path would start at the MfE vertex of largest expectation, would move in the direction of the largest trading advantage, i.e. the diagonal side of the triangle, where trading between asset 1 and asset 2 is the most efficient (so that portfolios made by the first two assets are efficient), up to the corner identified by a matching of the advantage obtained by trading also with the third asset. Here a new direction of movement along the direction preserving the equality of the advantage of all bilateral tradings, implies portfolios with positive quotas of all three assets and the procedure ends in the point where there is no longer any advantage in further bilateral trading, which is the maV point.

It is clear that what is a subspace with  $s$  assets in Markowitz, is equivalent to trading involving  $s$  assets in de Finetti's logic; according to the first point of view, the critical line is the loci of smallest variance for any level of feasible expectation in the subspace, in the other approach it is the direction of largest advantage by trading with those assets. The inversion of the path

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direction implies also an inversion of the interpretation of match and break corners in the two procedures. In this example, there is a unique intermediate corner; in Markowitz it is found where the critical line of the 3 asset space intersects the boundary (diagonal side) of the triangle. In this direction this ought to be interpreted as a break event, because going on in that direction would imply to break one of the constraints:  $x_3 \geq 0$ . Conversely, in de Finetti's logic at that corner there is a match between the advantage of trading between assets number 1 and 2 and the one coming from involving in trade also asset number 3: a match event.

In the other example<sup>15</sup> treated by Markowitz (1952, p. 86), the maV is not feasible, but part of the critical half line of the 3 asset space is feasible (internal to the triangle). Here the mfV is on the basis side of the triangle, so the first segment is on the critical half line of the corresponding 2 assets subspace. A first intermediate corner is found where this line intersects the critical line of the 3 assets space (to be seen as a match event) and a second and last intermediate corner comes when this critical line intersects the diagonal side of the triangle (a break event). Finally, the last part of the efficient path is the segment lying on the critical line of the corresponding two assets subspace to reach once more the MfE vertex.

In de Finetti's logic (starting point at MfE, right side of fig. 1) the first segment corresponds to the last one of Markowitz, the first intermediate corner is driven by a match (like the one of the previous example), the second intermediate corner is found when a break event happens (proceeding in that direction would be breaking the short selling constraint for the asset number 2). Hereafter, the asset number 2 is no more part of efficient portfolios and the last segment directs, through the most efficient constrained trading between asset 1 and asset 3, toward the end point of mfV where further trading is no more advantageous. Summarizing, the ordering of segments and corners in de Finetti's and Markowitz geometric driven procedures are exactly opposite as well as the match and break events.

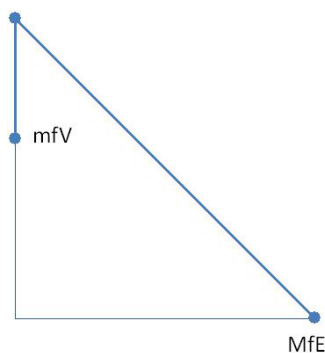
In another example (Markowitz, 1959, p. 143), Markowitz discussed a case in which the main critical half line is fully external to the feasible triangle, so that the efficient path is exclusively made by segments on two sides of the triangle including the vertex (0,1,0), which he defined intermediate corner with interesting features (see fig. 2). *The set of efficient*

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<sup>15</sup> In order to have the same ordering of expectations in both cases, our labelling of assets is different from the one adopted by Markowitz in the second example.

*portfolios seems to reverse direction at the vertex. Nevertheless, it always moves in the direction of increasing return. Another interesting feature will be noted later in connection with the relationship between expected return and variance of return of efficient portfolios. This remark concerns the behaviour of the graph of efficient portfolios in mean-variance space and not in the asset one. Markowitz underlined (p. 153) that to each segment on a critical line of the asset space there corresponds a parabola in the mean variance space and that at the intersection of two successive critical lines the typical relation between the corresponding parabolas is that they are tangent: they touch but do not cross (said another way the graph of  $V(E)$  is continuous and differentiable also at connection points), there are no kinks. It is however possible for the curve to have a kink when the set of efficient portfolios turns a vertex corner.*

**Figure 2: Efficient path: an intermediate vertex corner.**



This shows that Markowitz had already in the 1950s a quite clear idea on kinks of the efficient set in the mean-variance space. On the point he offered sometimes later (see Markowitz, 1987), chapter 11 entitled “Kinks in the set of efficient  $EV$  combinations”) a clear and conclusive proof involving the advantage function as revealed by the following sentence: *there is a kink only if there is a discontinuity of the function  $dV/dE$  at the intersection point of two consecutive critical lines.* And it is clear from the following discussion (see Markowitz, 1987, p. 259) that a kink occurs in the standard case only at a vertex of the feasible polytope or if all the assets currently in portfolio have the same mean.

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At that time, this was not an original result; a few years before a paper by Dybvig (1984) had given the end to the debate on kinks on the  $(E, V)$  space that, notwithstanding the clear ideas of the father of the  $EV$  approach, remained still open 25 years later the cited Markowitz remark (Markowitz, 1959). In the abstract of the paper (Dybvig, 1984, p. 239), after having defined switching points in the mean-variance frontier those corresponding to changes in the set of assets held, Dybvig recalls that *traditional wisdom holds that each switching point corresponds to a kink, while Ross (1977) has claimed that kinks never occur! The paper shows that the truth lies between the two views, since the efficient frontier may or may not be kinked at a switching point. There is some indication that kinks are rare, since a kink corresponds to a portfolio in which all active assets have the same expected return. In particular as an immediate corollary if all securities have different expected returns a kink may occur only if a single security is held there,* (Dybvig, 1984, p. 243). Proofs were based on arguments regarding properties of the mean-variance frontier.

We wish to underline here that de Finetti's approach offers, at least under the assumption of no ties in assets expectations, the most simple way to understand the reasons why kinks may occur only in an intermediate vertex corner of the asset space: only at these mixed corners there is a sudden downward jump in the value of the advantage parameter  $\lambda$  corresponding to (half) the ratio derivative of variance over derivative of expectation on the efficient path, hence only at these points there is a jump in the first derivative of the function  $V(E)$  (see the last paragraph in sect. 9).

This easy deduction is nothing but a byproduct of the new insights about the fundamental properties of the efficient set, both in the asset and in the mean variance space, allowed by (the adjusted version of) de Finetti's approach also in the investment problem.

To conclude it is fair to recall that de Finetti never treated in detail the behaviour of efficient portfolios in a  $EV$  space; on the contrary Markowitz was able to immediately recognize the importance of this reference space, which became of overwhelming importance in the subsequent development of portfolio theory and capital markets equilibria.

## 11. Conclusions

It is today plainly recognized that de Finetti introduced the mean variance approach in financial problems under risk and, dealing with a problem of proportional reinsurance, offered a correct procedure to find the mean variance efficient set in case of no correlation and, under a convenient regularity assumption (no break points), also in case of correlation. In this paper we concentrate on a couple of additional insights provided by de Finetti's approach. As regards the reinsurance problem, we explain how a natural adjustment of his procedure, fully respecting his logic and applying his tools (the advantage functions) and rules, provides correct solutions also in case of non regularity (in case of breaks too). Then, with reference to the standard portfolio selection problem, we show that, exploiting a convenient modification of the advantage functions tool, the extension of this adjusted procedure to the standard portfolio selection problem is able to offer an alternative way to look at the set of efficient portfolios and a new clear characterization of its properties in terms of benefit from asset trading. In particular, as shown in the example discussed in sect. 10, this approach offers a simple, yet insightful, explanation of when and why corner points appear in the path of efficient portfolios in the asset space, and clearly describes the properties of different types of corner points. Moreover, it offers an equally simple explanation of the properties at the corner points of the graph of efficient portfolios in the mean-variance space.

### **Appendix: a mathematical programming formulation for the standard investment problem.**

In the appendix (based on sect. 19.7 Pressacco-Serafini, 2009) we provide a mathematical foundation of the approach through advantage functions illustrated in sections 7-9. The problem we investigate can be restated as the following quadratic problem:

$$\begin{aligned} \min & \frac{1}{2} \mathbf{x}^T C \mathbf{x} \\ & \mathbf{m}^T \mathbf{x} \geq E \\ & \mathbf{1}^T \mathbf{x} = 1 \end{aligned}$$

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$$\mathbf{x} \geq 0 \quad (16)$$

for every attainable  $E$ , i.e.  $\min m_i \leq E \leq \max m_i$ . The strict convexity of the objective function guarantees that there is a one-to-one correspondence between points in  $X$  and optimal solutions of (16) for all attainable  $E$  such that the constraint  $\mathbf{m}^T \mathbf{x} \geq E$  is active. The Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality of (16), since the constraints are regular and the objective function is strictly convex (see Shapiro, 1979; Karush, 1939; Kuhn and Tucker, 1951). The conditions are expressed through the Lagrangean function:

$$L(\mathbf{x}, \lambda, \mu, \mathbf{v}) = 1/2 \mathbf{x}^T C \mathbf{x} + \lambda(E - \mathbf{m}^T \mathbf{x}) + \mu(1 - \mathbf{1}^T \mathbf{x}) - \mathbf{v} \mathbf{x} \quad (17)$$

and state that  $\mathbf{x}^*$  is optimal if and only if there exist Lagrange multipliers  $(\lambda^*, \mu^*, \mathbf{v}^*)$ ,  $\lambda^* \geq 0$ ,  $\mathbf{v}^* \geq 0$ , such that:

- 1)  $\mathbf{x}^*$  minimizes  $L(\mathbf{x}, \lambda^*, \mu^*, \mathbf{v}^*)$
- 2)  $\mathbf{x}^*$  is feasible in formulae (16)
- 3) either  $x_j^* = 0$  or  $v_j^* = 0$  (or both) and either  $\mathbf{m}^T \mathbf{x} = E$  or  $\lambda^* = 0$  (or both) (18)

In order to verify 1) of (18), since  $\mathbf{x}$  is unconstrained (in the Lagrangean minimization), it is enough to compute:

$$\partial L / \partial \mathbf{x} = C \mathbf{x} - \lambda \mathbf{m} - \mu \mathbf{1} - \mathbf{v} = \mathbf{0} \quad (19)$$

i.e., componentwise

$$\sum_j \sigma_{hj} x_j - \lambda m_h - \mu - v_h = 0, \quad h = 1, \dots, n \quad (20)$$

We want to rephrase the optimality condition by showing how the optimal variables depend on  $\lambda$ . They depend also on  $\mu$ , but we prefer to hide this dependence by solving (19) first on  $\mu$ . We assume that the indices are ordered as  $m_1 > m_2 > \dots > m_n$ . Let  $k$  be any index such that  $x_k > 0$ . We denote this variable as the reference variable. We have  $v_k = 0$  by complementarity and, for  $h=k$ , formula (20) for  $h=k$  is:

$$\sum_j \sigma_{kj} x_j - \lambda m_k - \mu = 0 \quad (21)$$

Now we subtract (20) from (21):

$$\sum_j(\sigma_{kj} - \sigma_{hj})x_j - \lambda(m_k - m_h) + v_h = 0, \quad h \neq k \quad (22)$$

or equivalently:

$$\frac{\sum_j(\sigma_{kj}-\sigma_{hj})x_j}{m_k-m_h} + \frac{v_h}{m_k-m_h} = \lambda \quad h \neq k \quad (23)$$

Observe that solving (23) is equivalent to solving (20). Indeed, once (23) is solved,  $\mu$  can be easily computed from all other variables. We have defined the advantage functions as:

$$F_{kh}(\mathbf{x}) = \frac{\sum_j(\sigma_{kj}-\sigma_{hj})x_j}{m_k-m_h} \quad h \neq k \quad (24)$$

(note that  $F_{kh}(\mathbf{x}) = F_{hk}(\mathbf{x})$ ) and by using the advantage functions we may rephrase (23) as:

$$F_{kh}(\mathbf{x}) + \frac{v_h}{m_k-m_h} = \lambda \quad h \neq k \quad (25)$$

Now we partition the variable indices  $\{1, \dots, n\} \setminus k$  into three sets as:

$$I_k^* = \{h \neq k: x_h > 0\}, I_0^* = \{h < k: x_h = 0\}, I_k^0 = \{h > k: x_h = 0\} \quad (26)$$

For the sake of notational simplicity, we omit to denote that these subsets actually depend also on  $\mathbf{x}$ . Moreover, let  $I^* := I_k^* \cup \{k\}$  and  $I^0 := I_0^k \cup I_k^0$  (respectively the sets of positive and null variables independently of the reference variable). Then, taking into account that  $v_h \geq 0$ ,  $m_k > m_h$ , if  $h \in I_k^0$  and  $m_k < m_h$ , if  $h \in I_0^k$ , the complementarity condition can be restated through the advantage functions in the following form:

**Optimality condition.** Let  $k$  such that  $x_k > 0$ . Then  $\mathbf{x} \geq 0$  is optimal if and only if  $\mathbf{1}^T \mathbf{x} = 1$  and there exists  $\lambda \geq 0$  such that:

$$F_{kh}(\mathbf{x}) \geq \lambda, h \in I_0^k, F_{kh}(\mathbf{x}) = \lambda, h \in I_k^*, F_{kh}(\mathbf{x}) \leq \lambda, h \in I_k^0 \quad (27)$$

The following facts can be easily deduced from the optimality condition:

**Corollary.** Let  $i \in I_k^*$ . If  $j \in I_k^*$  then  $F_{ij}(x) = \lambda$ . If  $j \in I^0$  then

$$F_{ij}(\mathbf{x}) \leq \lambda \text{ if } i > j, \quad F_{ij}(\mathbf{x}) \geq \lambda \text{ if } i < j \quad (28)$$

This result implies that the role of reference variable can be subsumed by any variable in  $I_k^*$  without changing the optimality conditions, provided the sets  $I_k^*$ ,  $I_0^k$  and  $I_k^0$  are duly redefined according to the new reference, and,



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more importantly, for the same value of  $\lambda$ . In other words, resetting the reference does not affect the value of  $\lambda$ .

The set  $I_k^*$  can be empty only in the extreme cases  $x_k = 1$  and  $x_h = 0$ ,  $h \neq k$ . In this case (27) becomes:

$$F_{kh}(\mathbf{x}) = \frac{\sigma_{kk} - \sigma_{hk}}{m_k - m_h} \leq \lambda, h > k \text{ and } F_{kh}(\mathbf{x}) = \frac{\sigma_{kk} - \sigma_{hk}}{m_k - m_h} \geq \lambda, h < k \quad (29)$$

Hence, if:

$$\max \left\{ \max_{h>k} \frac{\sigma_{kk} - \sigma_{hk}}{m_k - m_h}; 0 \right\} \leq \min_{h<k} \frac{\sigma_{kk} - \sigma_{hk}}{m_k - m_h} \quad (30)$$

the point  $x_k = 1$  and  $x_h = 0$ ,  $h \neq k$ , is optimal with  $\lambda$  taking any value within the above interval. Note that the point  $(1, 0, \dots, 0)$  is always optimal (since the r.h.s. term is missing) and that  $x_k = 1$  and  $x_h = 0$ ,  $h \neq k$ , can be optimal only if  $\sigma_{kk} < \sigma_{hk}$  for all  $h < k$  (necessary but not sufficient condition). In particular, the point  $(0, 0, \dots, 1)$  of absolute minimum mean can be also mean-variance efficient if and only if  $\sigma_{mm} < \sigma_{hm}$  for all  $h$ . In this case it is the end point of the set  $X^*$ .

If the set  $I_k^*$  is not empty, the optimality condition  $F_{kh}(x) = \lambda$ ,  $h \in I_k^*$ , is a linear system in the variables in  $I^*$ :

$$\sum_{j \in I^*} \frac{\sigma_{kj} - \sigma_{hj}}{m_k - m_h} x_j = \lambda, \quad h \in I_k^* \quad (31)$$

Adding the condition  $\sum_{j \in I^*} x_j = 1$  yields a square linear system whose solution is an affine function of  $\lambda$ :

$$x_h := w_h + \lambda z_h, \quad h \in I^* \quad (32)$$

(with  $w$  solution of the linear system with r.h.s.  $(0, 0, \dots, 0, 1)$  and  $z$  solution with r.h.s.  $(1, 1, \dots, 1, 0)$ ) and clearly  $x_h = 0$ ,  $h \in I^0$ .

As stated in Pressacco-Serafini (2007) the minimum portfolio variance in (16) is a strictly convex monotonically increasing function of the mean  $E$  and the multiplier  $\lambda$  is its derivative (or a subgradient on the points of non-differentiability). Therefore the set  $X^*$  can be parameterized via  $\lambda$  instead of  $E$ , taking into account that some points of  $X^*$ , where the derivative of the function has a discontinuity jump, correspond to an interval of values for  $\lambda$ . It is easy to understand that this discontinuity jump corresponds to the so-called kinks of the parabolic efficient frontier in the mean-variance space.

Basing on the advantage functions we could obtain a computational procedure, analogous to the critical line algorithm by Markowitz, to describe  $X^*$  parameterized via  $\lambda$ .

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