AN INTRODUCTION TO COARSE HYPERSPACES

NICOLÒ ZAVA

University of Udine, Italy

Introduction

Coarse geometry, also known as large-scale geometry, is the study of large-scale properties of spaces, ignoring their local, small-scale ones. Intuitively, for this theory two spaces are equivalent if they look the same to an observer whose point of view is getting further and further away. For example, coarse geometry identifies the set of integers and the one of the reals, both endowed with the usual euclidean metric, because they have the same "asymptotic" behaviour, even though they are completely different objects from the topological (small-scale) point of view. Coarse geometry found applications in several branches of mathematics, for example in geometric group theory (following the work of Gromov on finitely generated groups endowed with their word metrics), in Novikov conjecture, and in coarse Baum-Connes conjecture. We refer to [16] for a comprehensive introduction to large-scale geometry of metric spaces, and to [12] for applications to geometric group theory.

Large-scale geometry was initially developed for metric spaces, but then several equivalent structures that capture the large-scale properties of spaces appeared, inspired by the theory of uniform spaces ([14]). Roe introduced coarse spaces ([21]), as a counterpart of Weil's definition of uniform spaces via entourages, Dydak and Hoffland with large-scale structures ([9]) and Protasov with asymptotic proximities ([18]) independently developed the approach via coverings, as Tukey did for uniform spaces, and Protasov and Banakh ([18]) defined balleans, generalising the ball structure of metric spaces.

"Dualising" small-scale concepts has been a fruitful way to introduce new notions in coarse geometry. Among those concepts, in this paper we focus on hyperspaces, i.e, structures induced on power sets. Given a metric space (X, d), Hausdorff introduced the Hausdorff metric d_H on the power set $\mathcal{P}(X)$ of X, so that $(\mathcal{P}(X), d_H)$ is a metric space itself, called metric hyperspace ([13]). Later on, his idea was generalised to arbitrary uniform spaces, by introducing uniform hyperspaces (see, for example, [14]), and, recently, to arbitrary coarse spaces ([3], where the equivalent approach via balleans is used).

The aim of this paper is to provide a gentle introduction to coarse hyperspaces, i.e., power sets of coarse spaces endowed with a coarse structure induced by the one of the starting space. This article is based on some material contained in [3] and in [4], with more results and discussions that provide a useful context. We recall the basic notions of uniform spaces and construct the uniform hyperspace, highlighting the similarities and the differences between these classical objects and the corresponding large-scale notions, such as coarse spaces and coarse hyperspaces. We then focus our attention on some specific properties of coarse spaces, such as connectedness, we count the number of connected components of the coarse hyperspace in many cases, and this study justifies the interest in some important coarse subspaces of the whole hyperspace. The leading example is the $\flat$-coarse hyperspace, whose support is the family of all non-empty bounded subsets, which was already introduced in [19] by using balleans. For the last part of this paper we study another, more algebraic, example of subspace of a coarse hyperspace. Every group can be endowed with its group-coarse structure, which is a coarse structure on it that agrees with its algebraic structure. Then we define the subgroup hyperspace as the subspace of the coarse hyperspace of a group whose support is the family of all subgroups. We characterise the connected components of the subgroup hyperspace and we begin to tackle a specific problem, which we call "rigidity". If two groups are isomorphic as algebraic objects, then their subgroup hyperspaces are isomorphic as coarse spaces, but the opposite implication does not hold in general. A "rigidity result" is a set of conditions ensuring that the opposite implication holds.

The paper is organised as follows. In Section 1 we introduce the needed background in both uniform and coarse spaces, presenting the main examples, such as metric uniformities, metric coarse structures and group-coarse structures, and providing the definitions of morphisms and some first properties. In Section

2 we define both the coarse and the uniform hyperspace, discussing also some special coarse subspaces of the coarse hyperspace. Then Section 3 is devoted to calculate the number of connected components of some special coarse hyperspaces. In order to do that, we investigate, in particular, coarse hyperspaces of thin coarse spaces. Subgroup hyperspaces are introduced in Section 4, and finally in Section 5 we provide some rigidity results.

1. Uniform spaces and coarse spaces

First of all, let us recall some basic topological notion about uniform spaces (see, for example, [14] for a complete introduction of the topic).

An *entourage* of a set X is a subset of the square $X \times X$. For every pair of entourages U, V, every $x \in X$ and $A \subseteq X$,

$$U \circ V := \{(x,z) \mid \exists y \in X:\ (x,y) \in U,\ (y,z) \in V\}, \quad U^{-1} := \{(y,x) \mid (x,y) \in U\},$$
$$U[x] := \{y \in X \mid (x,y) \in U\}, \quad \text{and} \quad U[A] := \bigcup_{x \in A} U[x].$$

In the sequel, the subset $U[x]$ just defined will be often called the *ball centred in x with radius U*. Moreover, a map $f\colon X \to Y$ can be naturally extend to a map between the squares $X \times X$ and $Y \times Y$, which we denote by $f \times f$.

Given a set X, a *uniformity of* X is a family $\mathcal{U}$ of entourages of X such that:

(i) for every $U \in \mathcal{U}$, $\Delta_X := \{(x,x) \mid x \in X\} \subseteq U$;
(ii) $\mathcal{U}$ is a *filter* (i.e., it is closed under taking supersets and finite intersections);
(iii) for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;
(iv) for every $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$.

In this case, the pair $(X, \mathcal{U})$ is called a *uniform space*.

A family of entourages $\mathcal{B}$ of X is a *base of a uniformity* if the closure $\mathcal{U}_{\mathcal{B}}$ of $\mathcal{B}$ under taking supersets is a uniformity.

Let us present an important class of examples of uniformities. Let X be a set and d be an *extended-pseudo-metric* on X, i.e., $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfies the following properties:

(i) $d(x,x) = 0$, for every $x \in X$;
(ii) $d(x,y) = d(y,x)$, for every $x, y \in X$ (*symmetry*);
(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for every $x, y, z \in X$ (*triangle inequality*).

For the sake of simplicity, we refer to d as a *metric* and to (X, d) as a *metric space*. For every $R > 0$, we define a particular entourage, called the *strip of radius* R, as follows:

$$S_R := \bigcup_{x \in X} \Big(\{x\} \times B(x,R)\Big), \tag{1}$$

where $B(x,R)$ denotes the open ball centred in x with radius R. Then the family $\mathcal{B}_d := \{S_R \mid R > 0\}$ is a base of the so-called *metric uniformity* $\mathcal{U}_d := \mathcal{U}_{\mathcal{B}_d}$.

Every uniform space $(X, \mathcal{U})$ carries a topology of X, namely the *uniform topology* $\tau_{\mathcal{U}}$, defined as follows: the filter of neighbourhoods of a point $x \in X$ is given by the family $\{U[x] \mid U \in \mathcal{U}\}$.

A uniform space $(X, \mathcal{U})$ is *Hausdorff* (or *separated*) if one of the following equivalent conditions holds:

(i) $\bigcap \mathcal{U} = \Delta_X$;
(ii) $\tau_{\mathcal{U}}$ is T_0;
(iii) $\tau_{\mathcal{U}}$ is $T_{3,5}$.

If $(X, \mathcal{U})$ is a uniform space, we can endow a subset Y of X with the *subspace uniformity* $\mathcal{U}|_Y := \{U \cap (Y \times Y) \mid U \in \mathcal{U}\}$. The pair $(Y, \mathcal{U}|_Y)$ is a *uniform subspace*.

A map $f\colon (X, \mathcal{U}_X) \to (Y, \mathcal{U}_Y)$ between uniform spaces is:

(i) *uniformly continuous* if, for every $U \in \mathcal{U}_Y$, there exists $V \in \mathcal{U}_X$, such that $(f \times f)(V) \subseteq U$;
(ii) a *uniform isomorphism* if it is bijective and both f and f^{-1} are uniformly continuous;
(iii) a *uniform embedding* if the corestriction of f to its image endowed with the subspace uniformity is a uniform isomorphism.

The large-scale concept associated to uniform spaces is the notion of coarse spaces (see [21] for a comprehensive introduction).

Definition 1.1. Given a set X, a *coarse structure of* X is a family $\mathcal{E}$ of entourages of X such that:

(i) $\Delta_X \in \mathcal{E}$;
(ii) $\mathcal{E}$ is an *ideal* (i.e., it is closed under taking subsets and finite unions);
(iii) for every $E \in \mathcal{E}$, $E \circ E \in \mathcal{E}$;
(iv) for every $E \in \mathcal{E}$, $E^{-1} \in \mathcal{E}$.

In this case, the pair $(X, \mathcal{E})$ is called a *coarse space.*

A family of entourage $\mathcal{B}$ of X is a *base of a coarse structure* if the closure $\mathcal{E}_{\mathcal{B}}$ of $\mathcal{B}$ under taking subsets is a coarse structure.

For every coarse space $(X, \mathcal{E})$ and every point $x \in X$, we define its *connected component* as the subset $\mathcal{Q}_X(x) := \bigcup_{E\in\mathcal{E}} E[x]$. In particular, we say that X is *connected* if one of the following equivalent conditions holds:

(i) $\bigcup \mathcal{E} = X \times X$;
(ii) there exists $x \in X$ such that $\mathcal{Q}_X(x) = X$;
(iii) for every $x \in X$, $\mathcal{Q}_X(x) = X$.

If we compare item (i) with item (i) of the definition of the Hausdorff property of a uniform space, we are justified to claim that connectedness is the large-scale counterpart of Hausdorff property.

It is trivial that, if X is a coarse space, then X is connected if and only if every coarse subspace of X is connected.

Note that $\mathcal{Q}_X(x)$ is connected, for every $x \in X$. The family of connected components of a coarse space is a partition of the set and the cardinality of that family is denoted by $\operatorname{dsc}(X, \mathcal{E})$, which is a measure of how much a space is not connected.

Again an important example of coarse structure is the metric coarse structure, whose definition is similar to the one of metric uniformity. In Example 1.2 we take the opportunity to introduce also another big and important class of coarse spaces, namely group coarse structures.

Example 1.2. (i) If (X, d) is a metric space, the family $\mathcal{B}_d$ of all the strips (1) is a base of the so-called *metric coarse structure* $\mathcal{E}_d := \mathcal{E}_{\mathcal{B}_d}$. The metric coarse structure is connected if and only if d doesn't assume the value ∞.
(ii) We can endow a group G with its *group-coarse structure* $\mathcal{E}_G$ defined as follows. For every subset $K \in [G]^{<\omega} := \{F \subseteq G \mid |F| < \omega\}$, define the entourage
$$E_K := \bigcup_{x\in G} (\{x\} \times xK)$$
of G. Then the family $\mathcal{B}_G := \{E_K \mid K \in [G]^{<\omega}\}$ is a base of a coarse structure, named $\mathcal{E}_G$. The pair $(G, \mathcal{E}_G)$ is a *coarse group.* Note that the coarse group $(G, \mathcal{E}_G)$ is connected.

Geometric group theory also studies the large-scale properties of finitely generated groups endowed with their word metrics ([12]). If G is a group and d is a word metric on it, we have $\mathcal{E}_d = \mathcal{E}_G$.

While metric uniformities captures the small-scale properties of metric spaces, metric coarse structures encode their large-scale properties. Let us clarify this notion with an example.

Example 1.3. Let (X, d) be a metric space. Define two more metrics: for every $x, y \in X$,
$$d_1(x, y) := \min\{d(x, y), 1\}, \quad d_2(x, y) := \begin{cases} 0 & \text{if } x = y, \\ \max\{d(x, y), 1\} & \text{otherwise.} \end{cases}$$
Note that d_1 forgets about the large-scale behaviour of d, while d_2 forgets about the small-scale behaviour of d, and we have
$$\mathcal{U}_d = \mathcal{U}_{d_1} \neq \mathcal{U}_{d_2} \quad \text{and} \quad \mathcal{E}_d = \mathcal{E}_{d_2} \neq \mathcal{E}_{d_1}.$$

If $(X, \mathcal{E})$ is a coarse space, we can endow a subset Y of X with the *subspace coarse structure* $\mathcal{E}|_Y := \{E \cap (Y \times Y) \mid E \in \mathcal{E}\}$. The pair $(Y, \mathcal{E}|_Y)$ is said to be a *coarse subspace of* $(X, \mathcal{E})$.

Definition 1.4 ([20, 6])**.** Let $(X, \mathcal{E})$ be a coarse space. A subset A of X is called:

(i) *bounded* if there exists $E \in \mathcal{E}$ such that $A \subseteq E[x]$, for every $x \in A$;
(ii) *large in* X if there exists $E \in \mathcal{E}$ such that $E[A] := \bigcup_{x\in A} E[x] = X$;
(iii) *small in* X if, for every $E \in \mathcal{E}$, $X \setminus E[A]$ is large in X;
(iv) *slim in* X if it is not large in X;
(v) *piecewise large in* X if it is not small in X;
(vi) *meshy in* X if there exists $E \in \mathcal{E}$ such that, for every $x \in X$, $E[x] \setminus A \neq \emptyset$ (equivalently, $X \setminus A$ is large in X).

Let $\flat(X)$, $\mathcal{LA}(X)$, $\mathcal{SM}(X)$, $\mathcal{SL}(X)$, and $\mathcal{PL}(X)$ be the families of all non-empty bounded, large, small, slim, and piecewise large subsets of X, respectively.

Let us introduce the morphisms between coarse spaces. Two maps $f, g\colon S \to (X, \mathcal{E})$ from a set to a coarse space are *close* (and we denote it by $f \sim g$) if $\{(f(x), g(x)) \mid x \in X\} \in \mathcal{E}$. A map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between coarse spaces is:

(i) *bornologous* if, for every $E \in \mathcal{E}_X$, $(f \times f)(E) \in \mathcal{E}_Y$;
(ii) *effectively proper* if, for every $E \in \mathcal{E}_Y$, $(f \times f)^{-1}(E) \in \mathcal{E}_X$;
(iii) an *asymorphism* if one of the following equivalent properties is satisfied:
 (iii$_1$) f is bijective and both f and f^{-1} are bornologous,
 (iii$_2$) f is bijective and both bornologous and effectively proper;
(iv) an *asymorphic embedding* if one of the following equivalent properties is satisfied:
 (iv$_1$) the corestriction of f to its image endowed with the subspace coarse structure is an asymorphism,
 (iv$_2$) f is injective and both bornologous and effectively proper;
(v) a *coarse equivalence* if one of the following equivalent properties is satisfied:
 (v$_1$) f is bornologous and there exists another bornologous map $g\colon Y \to X$ (called *coarse inverse*) such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$,
 (v$_2$) f is bornologous and effectively proper and $f(X)$ is large in Y.

If $f\colon G \to H$ is a homomorphism of groups, then $f\colon (G, \mathcal{E}_G) \to (H, \mathcal{E}_H)$ becomes automatically bornologous. In particular, every isomorphism of groups induces an asymorphism of the corresponding coarse groups. This "functorial property" is widely discussed and studied in [8].

Now that we have introduced morphisms, we can define the category **Coarse** of coarse spaces and bornologous maps between them. In [7, 23] this category and its quotient **Coarse**$/_\sim$ under closeness relation are widely investigated. Moreover, if we put, for every subset A of a coarse space X, $\mathcal{Q}_X(A) := \bigcup_{x\in A} \mathcal{Q}_X(x)$, we have a closure operator of the category **Coarse** (see [5] for the definition). It is actually the only non-trivial closure operator of **Coarse** (see [23]).

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces. We define the *coproduct coarse structure* $\mathcal{E}$ on the disjoint union $X \sqcup Y$ as $\mathcal{E} := \{(i_1 \times i_1)(E) \cup (i_2 \times i_2)(F) \mid E \in \mathcal{E}_X, F \in \mathcal{E}_Y\}$, where $i_1\colon X \to X \sqcup Y$ and $i_2\colon Y \to X \sqcup Y$ are the canonical inclusions. Of course, this definition can be easily extended to the coproduct of a finite number of coarse spaces. As for the infinite case, we refer, for example, to [22]. The coproduct just defined is actually the categorical coproduct of the category **Coarse**.

If $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ are two maps, we define a map $f_1 \sqcup f_2\colon X_1 \sqcup X_2 \to Y_1 \sqcup Y_2$ between the disjoint unions as follows: for every $i_k(x) \in X_1 \sqcup X_2$, $(f_1 \sqcup f_2)(i_k(x)) := i_k(f_k(x))$.

Let us present some further easy facts.

Remark 1.5. (i) Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between coarse spaces. If $\mathcal{B}$ is a base of $\mathcal{E}_X$, then f is bornologous if and only if $f(B) \in \mathcal{E}_Y$, for every $B \in \mathcal{B}$.
(ii) Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be an asymorphism between coarse spaces. If $X = \bigsqcup_{i\in I} X_i$ and $Y = \bigsqcup_{i\in I} Y_i$, where $|I| = \operatorname{dsc} X = \operatorname{dsc} Y$, are the corresponding decompositions in connected components, for every $i \in I$, $f|_{X_i}\colon (X_i, \mathcal{E}_X|_{X_i}) \to (Y_i, \mathcal{E}_Y|_{Y_i})$ is an asymorphism. Conversely, if $\operatorname{dsc} X$ is finite and, for every $i \in I$, $f_i\colon X_i \to Y_i$ is an asymorphism, then the map $f = \bigsqcup_{i\in I} f_i$ between the coproducts of the families of spaces is an asymorphism.
(iii) If $f\colon X \to Y$ is a coarse equivalence between coarse spaces, then $\operatorname{dsc} X = \operatorname{dsc} Y$ and there is a one-to-one correspondence between the connected components of X and Y with the following further property: if X_i is a connected component of X, then X_i is bounded if and only if $\mathcal{Q}_Y(f(X_i))$ is bounded.

All the five families of subsets of a coarse spaces defined in Definition 1.4 satisfies the following property:

Theorem 1.6 ([6])**.** *Let X and Y be two coarse spaces. Let $\mathcal{A}(X)$ ($\mathcal{A}(Y)$) be one of the families of subsets defined in Definition 1.4. If $f\colon X \to Y$ is a coarse equivalence, then, for every $A \in \mathcal{A}(X)$, $f(A) \in \mathcal{A}(Y)$.*

Remark 1.7. In the literature there are other structures, which are equivalent to coarse spaces, to describe large-scale properties of spaces. Let us mention *large-scale structures* ([9]) and *balleans* ([18]). In this remark we want to introduce the latter.

A *ball structure* is given by a triple $\mathfrak{B} = (X, P, B)$, where X and P are sets (P is non-empty) and $B\colon X \times P \to \mathcal{P}(X)$ is a map that associates every pair $(x, r) \in X \times P$ with the *ball $B(x, r)$ centred in x with radius r*, that contains the centre itself. A ball structure $\mathfrak{B} = (X, P, B)$ is a *ballean* if it satisfies the following two properties:

- for every $r \in P$ and $x, y \in X$, $y \in B(x, r)$ if and only if $x \in B(y, r)$;
- for every $r, s \in P$, there exists $t \in P$ such that, for every $x \in X$, $B(B(x, r), s) \subseteq B(x, t)$, where, for every $A \subseteq X$ and $u \in P$, $B(A, u) := \bigcup_{x\in A} B(x, u)$.

Given a coarse space $(X, \mathcal{E})$, we can associate a ballean $\mathfrak{B}_{\mathcal{E}} = (X, P, B_{\mathcal{E}})$ to $(X, \mathcal{E})$ as follows: define $P := \{E \in \mathcal{E} \mid E = E^{-1}, \Delta_X \subseteq E\}$ and, for every $E \in P$ and $x \in X$, $B_{\mathcal{E}}(x, E) := E[x]$. This definition justifies the name ball centred in x with radius E, given to the subset $E[x]$ at the beginning of this section. Conversely, if $\mathfrak{B} = (X, P, B)$ is a ballean, for every $r \in P$, we define, as in the metric case, the *strip with radius r* as the subset

$$S_r := \bigcup_{x\in X} \Big(\{x\} \times B(x, r)\Big).$$

Then the family of entourages $\mathcal{B}_{\mathfrak{B}} = \{S_r \mid r \in P\}$ is a base for a coarse structure $\mathcal{E}_{\mathfrak{B}}$ on X. See [7] for more details about the equivalence between coarse spaces and balleans.

2. Introduction to hyperstructures

Given a metric space (X, d), Hausdorff provided a metric on the power set $\mathcal{P}(X)$ of X as follows: for any two subsets $Y, Z \subseteq X$, the *Hausdorff distance* between them is the value

$$d_H(Y, Z) := \inf\{R > 0 \mid Y \subseteq B(Z, R),\, Z \subseteq B(Y, R)\}.$$

The pair $(\mathcal{P}(X), d_H)$ is called *metric hyperspace*.

In the discussion contained in [13], Hausdorff introduced the notion of *quasi-metrics*, i.e., non symmetric metrics. This fruitful idea paved the way for the introduction of the notions of *quasi-uniform spaces* (see [11, 15] for a wide introduction) and, more recently, of *quasi-coarse spaces* ([22]).

Metric hyperspaces can be generalised by defining uniform hyperspaces and coarse hyperspaces. Let X be a set, W be an entourage of X, and $A \subseteq X$. Then

$$W^* := \{(Y, Z) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid Y \subseteq W[Z],\, Z \subseteq W[Y]\}.$$

Given a uniform space $(X, \mathcal{U})$, the family $\mathcal{B}^*_{\mathcal{U}} = \{U^* \mid U \in \mathcal{U}\}$ is a base of a uniformity $\exp \mathcal{U}$ and $(\mathcal{P}(X), \exp \mathcal{U})$ is the *uniform hyperspace* (also known as *Hausdorff-Bourbaki hyperspace*). Similarly, given a coarse space $(X, \mathcal{E})$, the family $\mathcal{B}^*_{\mathcal{E}} = \{E^* \mid E \in \mathcal{E}\}$ is a base of a coarse structure $\exp \mathcal{E}$ and $(\mathcal{P}(X), \exp \mathcal{E})$ is the *coarse hyperspace*. In the sequel we will denote the pair $(\mathcal{P}(X), \exp \mathcal{E})$ also by $\exp X$.

Proposition 2.1. *If $(X, \mathcal{E})$ is a coarse space, then $\exp \mathcal{E}$ is a coarse structure.*

Proof. Properties (i), (iv) and the closure under taking subsets of item (ii) of Definition 1.1 are trivial. In particular, note that, for every $E \in \mathcal{E}$, $(E^*)^{-1} = E^*$. Fix now two entourages of $\exp \mathcal{E}$ and, without loss of generality, we can assume that those are of the form E^* and F^* for some $E, F \in \mathcal{E}$. As for the second part of item (ii), it is enough to check that $E^* \cup F^* \subseteq (E \cup F)^*$. Finally, if $(Y, Z) \in E^* \circ F^*$, there exists $W \subseteq X$ such that $(Y, W) \in E^*$ and $(W, Z) \in F^*$, which means that, in particular,

$$Y \subseteq E[W] \subseteq E[F[Z]] \quad \text{and} \quad Z \subseteq F[W] \subseteq F[E[Y]]$$

and thus $E^* \circ F^* \subseteq ((E \circ F) \cup (F \circ E))^* \in \mathcal{B}^*_{\mathcal{E}}$. □

Let X be a coarse space and Y be a coarse subspace of X. Then $\exp Y$ can be easily identified with a coarse subspace of $\exp X$, namely $\{Z \subseteq X \mid Z \subseteq Y\}$.

Remark 2.2. The notion coarse hyperspace was introduced in [3]. However, in the cited paper, the authors used the language of balleans (see Remark 1.7) to define the *hyperballean* of a ballean.

First of all note that the definitions just provided agree with the metric hyperspace. In fact, it is easy to check that, if (X,d) is a metric space, we have $\exp(\mathcal{U}_d)=\mathcal{U}_{d_H}$ and $\exp(\mathcal{E}_d)=\mathcal{E}_{d_H}$.

Denote by $\imath\colon X\to\mathcal{P}(X)$ the map that associates to every point $x\in X$ the singleton $\{x\}$. The following fact, concerning the map just defined is straightforward.

Fact 2.3. *If $(X,\mathcal{U})$ is a uniform space ($(X,\mathcal{E})$ is a coarse space), then $\imath\colon X\to\mathcal{P}(X)$ is a uniform embedding (an asymorphic embedding, respectively).*

Before starting the detailed study of coarse hyperspaces, let us state one more result for uniform hyperspaces. Denote by $\mathcal{S}(X)$ the family of all singletons of a set X. Moreover, if X is a topological space, let $\mathcal{F}(X)$ denote the family of all non-empty closed subsets of X.

Proposition 2.4. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space and $\mathcal{A}(X)\subseteq\mathcal{P}(X)$ be a family closed under finite unions and such that $\mathcal{S}(X)\subseteq\mathcal{A}(X)$. Then the following properties are equivalent:*

(i) $(\mathcal{A}(X),\exp\mathcal{U}|_{\mathcal{A}(X)})$ *is Hausdorff;*
(ii) $\mathcal{A}(X)\subseteq\mathcal{F}(X)$.

Proof. The implication (ii)→(i) is a classical result (see, for example, Isbell's book). Conversely, suppose that $A\in\mathcal{A}(X)$ is not closed. Hence, there exists $x\notin A$ such that, for every $U\in\mathcal{U}$, $U[x]\cap A\neq\emptyset$. Then, for every $U\in\mathcal{U}$,

$$A\subseteq U[A\cup\{x\}]\quad\text{and}\quad A\cup\{x\}\subseteq U[A],$$

and so $(\mathcal{A}(X),\exp\mathcal{U}|_{\mathcal{A}(X)})$ is not Hausdorff. □

In the statement of Proposition 2.4, the request that $\mathcal{S}(X)\subseteq\mathcal{A}(X)$ is to ensure that the corestriction $\imath\colon(X,\mathcal{U})\to(\mathcal{A}(X),\exp\mathcal{U}|_{\mathcal{A}(X)})$ is defined and thus it is still a uniform embedding.

Proposition 2.4 is the reason why many authors consider $(\mathcal{F}(X),\exp\mathcal{U}|_{\mathcal{F}(X)})$ as the hyperspace of a uniform space $(X,\mathcal{U})$.

It is useful to consider also some coarse subspaces of coarse hyperspaces. In fact, for example, we will see that the coarse hyperspace is not connected in general (see Proposition 3.1 and Remark 3.2) and, moreover, it could be highly disconnected even in simple cases (Corollaries 3.5 and 3.9).

Definition 2.5. Let $(X,\mathcal{E})$ be a coarse space and $\mathcal{A}(X)$ be a family of subsets of X. Then the $\mathcal{A}$-*coarse hyperspace* is $\mathcal{A}\text{-}\exp X:=(\mathcal{A}(X),\exp\mathcal{E}|_{\mathcal{A}(X)})$.

As we said in Remark 2.2, the notion of hyperballean was introduced in [3]. However, the authors had been inspired from a previous paper ([19]), where the $\flat$-coarse hyperspace was introduced, again in terms of balleans, under the name *hyperballean*.

Remark 2.6. Let $(X,\mathcal{E})$ be a coarse space. Then $\mathcal{LA}\text{-}\exp X$ is connected. In fact $\mathcal{L}A(X)=\mathcal{Q}_{\exp X}(X)$. On the contrary, the $\flat$-coarse hyperspace is not connected in general. Let us focus a bit more on the $\flat$-coarse hyperspace. We claim that $\mathcal{Q}_{\exp X}(\imath(X))=\flat(X)$. In fact, a non-empty subset $A\subseteq X$ belongs to $\flat(X)$ if and only if there exists $E\in\mathcal{E}$ and $x\in A$ such that $A\subseteq E[x]$, which is equivalent to $A\in E^*[\{x\}]$. Since connectedness is preserved under taking asymorphic images and subspaces, $\flat\text{-}\exp X$ is connected if and only if X is connected.

Proposition 2.7. *Let $(X,\mathcal{E})$ be a connected coarse space. Then the followings are equivalent:*

(i) *X is unbounded;*
(ii) *every finite subset of X is small in X;*
(iii) *there is a singleton of X which is small in X;*
(iv) $\mathcal{LA}\text{-}\exp X$ *is unbounded.*

Proof. The equivalences between items (i), (ii) and (iii) is proved in [6, Theorem 2.14].

Assume now (iii). We claim that, for every $E=E^{-1}\in\mathcal{E}$, $E^*[X]\neq\mathcal{L}(X)$ and so $\mathcal{LA}\text{-}\exp X$ is unbounded. Fix an entourage $E=E^{-1}\in\mathcal{E}$ and a point $x\in X$ satisfying the condition. Since $\{x\}$ is small, $X\setminus E[x]\in\mathcal{LA}(X)$. However, $X\setminus E[x]\notin E^*[X]$.

Conversely, if X is bounded, then $\mathcal{LA}(X)=\mathcal{P}(X)\setminus\{\emptyset\}$ and it is easy to check that every singleton is large in $\mathcal{LA}\text{-}\exp X$ and so, $\mathcal{LA}\text{-}\exp X$ is bounded. □

Let $f\colon X\to Y$ be a map between sets. Then there is a natural extension $\overline{f}\colon\mathcal{P}(X)\to\mathcal{P}(Y)$, defined as $\overline{f}(A):=f(A)$, for every $A\subseteq X$. If both X and Y are coarse spaces, then we denote by $\exp f$ the map between the hyperspaces. The following result can be easily verified.

Proposition 2.8. *Let $f\colon X\to Y$ be a map between coarse spaces. Then*

(i) f is bornologous if and only if $\exp f$ is bornologous;
(ii) f is effectively proper if and only if $\exp f$ is effectively proper;
(iii) f is an asymorphism if and only if $\exp f$ is an asymorphism;
(iv) f is a coarse equivalence if and only if $\exp f$ is a coarse equivalence.

Proposition 2.8 implies that we have a functor $\exp\colon$ **Coarse** $\to$ **Coarse** that associates to every coarse space its coarse hyperspace.

Remark 2.9. Let us add some remarks on Proposition 2.8. Let $f\colon X\to Y$ be a map between coarse spaces and $\mathcal{A}(X)$ and $\mathcal{B}(Y)$ be two family of subsets of X and Y, respectively.

(i) If f is bornologous and $\exp f(\mathcal{A}(X))\subseteq\mathcal{B}(Y)$, then $\exp f|_{\mathcal{A}(X)}\colon\mathcal{A}\text{-}\exp X\to\mathcal{B}\text{-}\exp Y$ is defined and bornologous. In particular this implication holds for the families $\flat(X)$ and $\flat(Y)$. In fact, if $A\in\flat(X)$ and f is bornologous, then $f(A)\in\flat(Y)$. Hence, we have another functor $\flat\text{-}\exp\colon$ **Coarse** $\to$ **Coarse**.
(ii) If $\mathcal{S}(X)\subseteq\mathcal{A}(X)$, $\mathcal{S}(Y)\subseteq\mathcal{B}(X)$, and $\exp f|_{mathcalA(X)}\colon\mathcal{A}\text{-}\exp X\to\mathcal{B}\text{-}\exp Y$ is bornologous, then $f\colon X\to Y$ is defined and then bornologous since it is the restriction of $\exp f$ to $\mathcal{S}(X)$.
(iii) Suppose that f is a coarse equivalence and let $g\colon Y\to X$ be a coarse inverse of f. Then Theorem 1.6 applied to both f and g implies that the restrictions

$$\exp f|_{\flat(X)}\colon \flat\text{-}\exp X\to\flat\text{-}\exp Y,\quad \exp f|_{\mathcal{LA}(X)}\colon \mathcal{LA}\text{-}\exp X\to\mathcal{LA}\text{-}\exp Y,$$
$$\exp f|_{\mathcal{SM}(X)}\colon \mathcal{SM}\text{-}\exp X\to\mathcal{SM}\text{-}\exp Y,\quad \exp f|_{\mathcal{SL}(X)}\colon \mathcal{SL}\text{-}\exp X\to\mathcal{SL}\text{-}\exp Y,$$
$$\exp f|_{\mathcal{PL}(X)}\colon \mathcal{PL}\text{-}\exp X\to\mathcal{PL}\text{-}\exp Y,\quad \exp g|_{\flat(Y)}\colon \flat\text{-}\exp Y\to\flat\text{-}\exp X,$$
$$\exp g|_{\mathcal{LA}(Y)}\colon \mathcal{LA}\text{-}\exp Y\to\mathcal{LA}\text{-}\exp X,\quad \exp g|_{\mathcal{SM}(Y)}\colon \mathcal{SM}\text{-}\exp Y\to\mathcal{SM}\text{-}\exp X,$$
$$\exp g|_{\mathcal{SL}(Y)}\colon \mathcal{SL}\text{-}\exp Y\to\mathcal{SL}\text{-}\exp X,\quad\text{and}\quad \exp g|_{\mathcal{PL}(Y)}\colon \mathcal{PL}\text{-}\exp Y\to\mathcal{PL}\text{-}\exp X$$

are defined and thus coarse equivalences in view of Proposition 2.8(iv).

3. Connectedness and number of connected components of some hyperspaces

Let us begin this section with the large-scale counterpart of Proposition 2.4.

Proposition 3.1. *Let $(X,\mathcal{E})$ be a coarse space and $\mathcal{A}(X)\subseteq\mathcal{P}(X)$ be a family such that $\mathcal{S}(X)\subseteq\mathcal{A}(X)$. Then the following properties are equivalent:*

(a) $\mathcal{A}\text{-}\exp X$ is connected;
(b) $\mathcal{A}(X)\subseteq\flat(X)$.

Proof. Let $Y\in\mathcal{A}(X)$ and suppose that $Y\notin\flat(X)$. If $Y=\emptyset$, then $\mathcal{A}\text{-}\exp X$ is not connected, in fact $\mathcal{Q}_{\exp X}(\emptyset)=\{\emptyset\}$ (see also Remark 3.2). If Y is non-empty and then unbounded, it cannot be contained in a ball centred in a singleton. Conversely, for every pair of non-empty bounded subsets A and B of X, there exists an entourage E such that $A\subseteq E[B]$ and $B\subseteq E[A]$. In fact, pick two points $x\in A$ and $y\in B$, and, since A and B belong to $\flat(X)$, there exist $E_x\in\mathcal{E}$ and $E_y\in\mathcal{E}$ such that $A\subseteq E_x[x]$ and $B\subseteq E_y[y]$. Moreover, since X is connected, $F:=\{(x,y),(y,x)\}\in\mathcal{E}$. Hence it is enough to define $E:=F\circ(E_x\cup E_y)$. □

Again, as in Proposition 2.4, the request that $\mathcal{S}(X)\subseteq\mathcal{A}(X)$ is justified in order to have the corestriction $\imath\colon(X,\mathcal{E})\to\mathcal{A}\text{-}\exp X$ defined and then an asymorphic embedding.

Remark 3.2. Let us note some basic results concerning the number of connected components of the coarse hyperspace.

(i) Since, for every coarse space $(X,\mathcal{E})$ and every $E\in\mathcal{E}$, $E[\emptyset]=\emptyset$, $\mathcal{Q}_{\exp X}(\emptyset)=\{\emptyset\}$ and thus $\operatorname{dsc}\exp X\geq 2$ provided that X is non-empty. Moreover, it is trivial that $\operatorname{dsc}\exp X\leq|\exp X|=2^{|X|}$.
(ii) If $(X,\mathcal{E})$ is a coarse space such that $\operatorname{dsc}\exp X=2$, then X is non-empty and bounded. In fact, item (i) implies that X has to be non-empty. Moreover, for every $x\in X$, $\{x\}$ and X have to be in the same connected component of $\exp X$, which means that there exists $E\in\mathcal{E}$ such that $X\subseteq E[x]$ and thus the claim follows.

(iii) For every coarse space X, if Y is a coarse subspace of X, then $\operatorname{dsc} X \geq \operatorname{dsc} Y$. In particular, it is true that $\operatorname{dsc}\exp X \geq \operatorname{dsc}\exp Y$.

We want to compute the number of connected components of the coarse hyperspace for particular classes of coarse spaces. Before that, we need to introduce and study another useful class of spaces.

Given an ideal $\mathcal{I}$ on a set X we can define the finest coarse structure $\mathcal{E}_{\mathcal{I}}$ on X such that $\flat(\mathcal{E}_{\mathcal{I}}) = \mathcal{I}$ (i.e., for every other coarse structure $\mathcal{E}$ of X such that $\flat(\mathcal{E}) = \mathcal{I}$, $id_x\colon (X,\mathcal{E}_{\mathcal{I}}) \to (X,\mathcal{E})$ is bornologous), which is called *ideal coarse structure*. Let $K \in \mathcal{I}$ and consider the entourage $E_K^{\mathcal{I}} := \Delta_X \cup (K\times K)$. Then the family $\mathcal{B}_{\mathcal{I}} := \{E_K^{\mathcal{I}} \mid K \in \mathcal{I}\}$ is a base of the coarse structure $\mathcal{E}_{\mathcal{I}}$.

Ideal coarse structures have another remarkable property.

Proposition 3.3 ([4])**.** *Let $f\colon X \to Y$ be a map between sets, and $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X and Y, respectively. Then $f\colon (X,\mathcal{E}_{\mathcal{I}}) \to (Y,\mathcal{E}_{\mathcal{J}})$ is bornologous if and only if $f(\mathcal{I}) \subseteq \mathcal{J}$.*

In particular, if $\mathcal{I} = [X]^{<\omega}$ and $\mathcal{J} = [Y]^{<\omega}$, f is an asymorphism if and only if f is bijective.

Let X be a set and $\mathcal{I}$ be an ideal on it. Let $K \in \mathcal{I}$ and $Z \in (E_K^{\mathcal{I}})^*[Y]$ for some $Y \subseteq X$. Then

$$Z \subseteq E_K^{\mathcal{I}}[Y] = \begin{cases} Y\cup K & \text{if } Y\cap K \neq \emptyset,\\ Y & \text{otherwise,}\end{cases} \quad\text{and}\quad Y \subseteq E_K^{\mathcal{I}}[Z] = \begin{cases} Z\cup K & \text{if } Z\cap K \neq \emptyset,\\ Z & \text{otherwise.}\end{cases}$$

Thus, if $Y\cap K = \emptyset$, $Z = Y$, and, otherwise, $Y\setminus K \subsetneq Z \subseteq Y\cup K$. Then we have computed the subsets

$$(E_K^{\mathcal{I}})^*[Y] = \begin{cases} \{Y\} & \text{if } Y\cap K = \emptyset,\\ \{Z\subseteq X \mid Y\setminus K \subsetneq Z \subseteq Y\cup K\} & \text{otherwise.}\end{cases} \tag{2}$$

Proposition 3.4. *Let X be a set and $\mathcal{I}$ be a* proper *ideal (i.e., $X \notin \mathcal{I}$) on it which is also a cover. Then two subsets $Y, Z \subseteq X$ are in the same connected component of* $\exp(X,\mathcal{E}_{\mathcal{I}})$ *if and only if $X\triangle Y \in \mathcal{I}$.*

Proof. First of all note that the hypothesis lead to the fact that $[X]^{<\omega} \subseteq \mathcal{I}$. If there exists $K \in \mathcal{I}$ such that $Z \in (E_K^{\mathcal{I}})^*[Y]$, then, in particular $Z \subseteq Y\cup K$ and $Y \subseteq Z\cup K$, which imply that $Z\setminus Y \subseteq K \supseteq Y\setminus Z$ and thus $Y\triangle Z \subseteq K \in \mathcal{I}$. Conversely, suppose that $Y\triangle Z \in \mathcal{I}$. Then, if $y \in Y$ and $z \in Z$, $K := Y\triangle Z\cup\{y\}\cup\{z\} \in \mathcal{I}$ has non-empty intersection with both Y and Z and (2) implies that $Z \in (E_K^{\mathcal{I}})^*[Y]$. □

Corollary 3.5. *Let X be an infinite set and $\mathcal{I} = [X]^{<\omega}$. Then* $\operatorname{dsc}\exp(X,\mathcal{E}_{\mathcal{I}}) = 2^{|X|}$.

Proof. According to Proposition 3.4, for every $Y \subseteq X$, $|\mathcal{Q}_{\exp(X,\mathcal{E}_{\mathcal{I}})}(Y)| = |[X]^{<\omega}| = |X|$, since X is infinite. However, $|\mathcal{P}(X)| = 2^{|X|}$ and thus $\operatorname{dsc}\exp(X,\mathcal{E}_{\mathcal{I}}) = 2^{|X|}$. □

For a coarse space $(X,\mathcal{E})$, we define a map $C\colon X \to \mathcal{P}(X)$ by putting $C(x) = X\setminus\{x\}$.

Lemma 3.6. *Let $(X,\mathcal{E})$ be a connected unbounded coarse space. If Y is a subset of X, then $C(Y)$ is bounded in $\exp X$ if and only if there exists $E \in \mathcal{E}$ such that $|E[y]| > 1$, for every $y \in Y$.*

Proof. ($\to$) Since $C(Y)$ is bounded in $\exp X$, there exists $E = E^{-1} \in \mathcal{E}$ such that, for every $x, y \in Y$ with $x \neq y$, $C(y) \in E^*[C(x)]$. Hence $y \in X\setminus\{x\} \subseteq E[X\setminus\{y\}]$ and $x \in X\setminus\{y\} \subseteq E[X\setminus\{x\}]$, in particular, $y \in E[Y\setminus\{y\}]$ and $x \in E[Y\setminus\{x\}]$, from which the conclusion descends.

($\leftarrow$) Since, for every $y \in Y$, there exists $z \in Y\setminus\{y\}$ such that $y \in E[z]$, $C(y) \in E^*[X]$. Hence $C(Y) \subseteq E^*[X]$, and the latter is bounded. □

Theorem 3.7. *Let $(X,\mathcal{E})$ be an unbounded connected coarse space. Then the following properties are equivalent and define a* thin coarse space*:*

(i) for every $E \in \mathcal{E}$, there exists a bounded subset V of X such that, for every $x \in X\setminus V$, $|E[x]| = 1$;
(ii) $(X,\mathcal{E}) = (X,\mathcal{E}_{\mathcal{I}})$, where $\mathcal{I} = \flat(X)$;
(iii) if $A \subseteq X$ is meshy in X, then A is bounded;
(iv) $\mathcal{ME}$-$\exp X$ is connected;
(v) the map $C\colon X \to \mathcal{P}(X)$ is an asymorphism between X and $C(X)$.

Proof. The implication (iii)$\to$(iv) is trivial, since item (iii) implies that $\mathcal{ME}$-$\exp X = \flat$-$\exp X$ (note that $\flat(Y) \subseteq \mathcal{ME}(Y)$ fo a generic coarse space Y) and the latter is connected. Furthermore, (i)$\leftrightarrow$(ii) has already been proved in [20].

(iv)→(iii) Assume that $A\subseteq X$ is meshy. Fix arbitrarily a point $x\in X$. The singleton $\{x\}$ is bounded, hence meshy. By our assumption, $\mathcal{ME}\text{-}\exp X$ is connected and both A and $\{x\}$ are meshy, so there must be a ball centred at x and containing A. Therefore, A is bounded.

(v)→(i) If (i) is not satisfied, then there is an unbounded subset Y of X satisfying Lemma 3.6. Since $C(Y)$ is bounded in $\exp X$, we see that C is not an asymorphism.

(ii)→(v) Suppose that $\mathcal{E}=\mathcal{E}_{\mathcal{I}}$. Fix an element $V\in\mathcal{I}$. Since the family of all E_U, where $U\in\mathcal{I}$ such that $|U|>1$, forms a base of $\mathcal{E}_{\mathcal{I}}$, we can assume that V has at least two elements (Remark 1.5(i)). Now, pick an arbitrary point $x\in X$. Since $|V|>1$, for every $A\in C(X)$, $A\cap V\neq\emptyset$. Hence (2) implies that

$$(E_V^{\mathcal{I}})^*[C(x)]\cap C(X)=\{X\setminus\{y\}\mid (X\setminus\{x\})\setminus V\subsetneq X\setminus\{y\}\subseteq (X\setminus\{x\})\cup V\}. \tag{3}$$

Moreover, if $x\in V$, (3) implies

$$(E_V^{\mathcal{I}})^*[C(x)]\cap C(X)=\{X\setminus\{y\}\mid X\setminus V\subsetneq X\setminus\{y\}\}=C(E_V^{\mathcal{I}}[x]).$$

On the other hand, if $x\notin V$, then (3) implies

$$(E_V^{\mathcal{I}})^*[C(x)]\cap C(X)=\{X\setminus\{y\}\mid X\setminus (V\cup\{x\})\subsetneq X\setminus\{y\}\subseteq X\setminus\{x\}\}=C(E_V^{\mathcal{I}}[x]).$$

(i)→(iii) Suppose that item (i) is satisfied and A is an unbounded subset of X. We claim that A is not meshy. Fix an entourage $E\in\mathcal{E}$ and let $V\subseteq X$ be a bounded subset of X such that $E[x]=\{x\}$, for every $x\notin V$. Since A is unbounded, there exists a point $x_E\in A\setminus V$. Hence $E[x_E]=\{x_E\}\subseteq A$, which shows that A is not meshy.

(iii)→(i) Suppose that item (i) is not satisfied. Then, there exists $E\in\mathcal{E}$ such that, for every bounded subset V of X, there exists $x_V\notin V$ which verifies $|E[x_V]|\geq 2$.

We want to construct, by transfinite induction, a subset $A=A_\kappa=\{y_\lambda\mid\lambda<\kappa\}$, for some limit ordinal κ, and a family of symmetric entourages $\{E_\lambda\}_{\lambda<\kappa}$ (an entourage F is *symmetric* if $F^{-1}=F$) with the following properties:

(a) A is unbounded;
(b) for every $\lambda<\kappa$, $A_\lambda=\{y_{\lambda'}\mid\lambda'<\lambda\}$ is bounded;
(c) $E_\lambda\subseteq E_{\lambda'}$, for every $\lambda<\lambda'<\kappa$ such that there exist a limit ordinal ϑ and two natural numbers m,n with the property that $\lambda=\vartheta+m$ and $\lambda'=\vartheta+n$;
(d) $E_\lambda\not\subseteq E_{\lambda'}$, for every $\lambda'<\lambda<\kappa$;
(e) for every $\lambda<\kappa$, $y_\lambda\notin E_\lambda[A_\lambda]$;
(f) $E\subsetneq E_\lambda$, for every $\lambda<\kappa$;
(g) $|E[y_\lambda]|\geq 2$, for every $\lambda<\kappa$.

Indeed, such an A is unbounded (by item (a)) and $X\setminus A$ is large, since, for every $y\in A$, $|E[y]|\geq 2$ (by item (g)) and $E[y]\cap A=\{y\}$ (by items (c)–(f)) and thus there exists a point $z\in E[y]\setminus A$, which shows that $y\in E[z]\subseteq E[X\setminus A]$. Hence A is meshy.

First of all, note that there exists no $E_{max}\in\mathcal{E}$ such that $F\subseteq E_{max}$, for every $F\in\mathcal{E}$, since, otherwise, X is bounded.

Let $E_1\in\mathcal{E}$ be an arbitrary symmetric entourage such that $E\subsetneq E_1$ and fix a point $y_1\in X$ such that $|E[y_1]|\geq 2$.

Let now κ be an ordinal and suppose that y_ν and E_ν are defined, for every $\nu<\kappa$, and satisfy properties (b)–(g).

Suppose that κ is not a limit ordinal and thus let λ be an ordinal such that $\lambda+1=\kappa$. Let E_κ be a radius such that $E_\lambda\subseteq E_\kappa$. Since A_κ is bounded by item (b), there exists a point $y_\kappa\notin E_\kappa[A_\kappa]$ such that $|E[y_\kappa]|\geq 2$.

Conversely, suppose now that κ is a limit ordinal. If A_κ is unbounded, then we are done. Suppose then that A_κ is bounded. Hence there exists $F\in\mathcal{E}$ such that $A_\kappa\subseteq F[y_1]$. It is not hard to prove that $\mathcal{F}_\kappa=\{E_\lambda\mid\lambda<\kappa\}$ is not a base of $\mathcal{E}$ since, otherwise, A_κ is unbounded by item (e). Thus there exists $E_\kappa=E_\kappa^{-1}\in\mathcal{E}$ such that $E_\kappa\not\subseteq E_\lambda$, for every $\lambda<\kappa$, $F\subsetneq E_\kappa$, and $E\subsetneq E_\kappa$. Since $E_\kappa[A_\kappa]$ is bounded, there exists a point $y_\kappa\notin E_\kappa[A_\kappa]$, such that $|E[y_\kappa]|\geq 2$.

Since $|A_\kappa|=\kappa\leq|X|$ and X is unbounded, $A=A_\kappa$ is unbounded for some limit ordinal $\kappa\leq|X|$. And so A satisfies (a)–(g). □

We refer to [3] for a different proof of Theorem 3.7. In the same paper it is shown that we cannot substitute item (iii) of Theorem 3.7 by asking that all the small subsets are bounded. In fact it is a strictly weaker condition.

Note that condition (i) of Theorem 3.7 is the usual definition of a thin coarse space and it can be applied also to bounded coarse spaces, which are then trivially thin, and to non-connected coarse spaces. The property of being thin is preserved under taking asymorphic images.

Remark 3.8. Let $(X,\mathcal{E})$ be an unbounded connected coarse space. Consider the map $CB\colon \flat\text{-}\exp X \to \exp X$ such that $CB(A)=X\setminus A$, for every bounded A. It is trivial that $C=CB|_X$, where X is identified with the family $\mathcal{S}(X)$ of all its singletons. Hence, if CB is an asymorphic embedding, then C is an asymorphic embedding too, and thus X is thin, according to Theorem 3.7. However, we claim that CB is not an asymorphic embedding if X is thin and then item (v) in Theorem 3.7 cannot be replaced with this stronger property.

Since $(X,\mathcal{E})$ is thin, we can assume that $\mathcal{E}=\mathcal{E}_\mathcal{I}$ (Theorem 3.7) for some ideal $\mathcal{I}$ on X. Fix an element $V\in\mathcal{I}$ of $\exp X_\mathcal{I}$ and suppose, without loss of generality, that V has at least two elements. For every other $W\in\mathcal{I}$, pick an element $A_W\in\mathcal{I}$ such that $A_W\subseteq X\setminus(W\cup V)$. Hence, $CB^{-1}((E_V^\mathcal{I})^*[CB(A_W)])\nsubseteq (E_W^\mathcal{I})^*[A_W]=\{A_W\}$, which implies that CB is not effectively proper. In fact, since $A_W\cup V\in\mathcal{I}$,

$$(E_V^\mathcal{I})^*[CB(A_W)]=\{Z\subseteq X\mid X\setminus(A_W\cup V)\subsetneq Z\subseteq X\setminus A_W\}\subseteq CB(\flat(X)),$$

and thus $|(E_V^\mathcal{I})^*[CB(A_W)]\cap CB(\flat(X))|>1$.

Corollary 3.9. *Let G be a group. If G is finite, then* $\operatorname{dsc}\exp(G,\mathcal{E}_G)=2$. *Otherwise,* $\operatorname{dsc}\exp(G,\mathcal{E}_G)=2^{|G|}$.

Proof. The first claim is trivial since $(G,\mathcal{E}_G)$ is bounded, provided that G is finite. Suppose otherwise that G is infinite. We use [1] to choose a thin subset T of G such that $|T|=|G|$. Since T is a thin subset of G, Theorem 3.7 implies that $\mathcal{E}_G|_T=\mathcal{E}_\mathcal{I}$, where $\mathcal{I}$ is the ideal of all bounded subsets of T (i.e., all finite subsets of T). By Corollary 3.5, $\operatorname{dsc}\exp(T,\mathcal{E}_\mathcal{I})=2^{|T|}$ and thus, because of Remark 3.2(iii),

$$2^{|G|}=|\mathcal{P}(G)|\geq\operatorname{dsc}\exp(G,\mathcal{E}_G)\geq\operatorname{dsc}\exp(T,\mathcal{E}_\mathcal{I})=2^{|T|}=2^{|G|}.$$

□

Let us point out another easy, but useful fact concerning the hyperspace of a coarse group.

Fact 3.10. *Let G be a group and e be its identity. Then every ball of* $\exp(G,\mathcal{E}_G)$ *centred in $\{e\}$ is finite. Hence, for every subset X of G such that $X\in\mathcal{Q}_{\exp G}(\{e\})$ and every finite subset F of G, $(E_F)^*[X]$ is finite.*

4. The subgroup hyperspace of a group

As Corollary 3.9 shows, the coarse hyperspace of a group endowed with its group coarse structure is a very wild object, which is hard to work with. For the sake of simplicity, we can work with $\flat\text{-}\exp(G,\mathcal{E}_G)$, which is connected by Proposition 3.1. However, there is another coarse subspace of $\exp G$ which is worth of interest and relies on the algebraic structure of the group G: the *subgroup hyperspace* $\mathcal{L}\text{-}\exp G$. It is defined as follows: for every group G, denote by $\mathcal{L}(G)$ the family of all subgroups of G and thus $\mathcal{L}\text{-}\exp G$ is defined as in Definition 2.5.

Lemma 4.1. *Let G be a group and let A,B be subgroups of G such that $B\subseteq SA$ for some subset S of G. Then $|B:(A\cap B)|\leq|S|$.*

Proof. We split the proof in three cases.

(Case 1: Assume that $S\subseteq B$) Given any $b\in B$, we pick $s\in S$ such that $b\in sA$. Then $s^{-1}b\in A\cap B$ and $B\subseteq S(A\cap B)$. This proves that $|B:(A\cap B)|\leq|S|$.

(Case 2: Assume that $S\subseteq BA$) Let $S_a:=S\cap Ba$ and note that our assumption provides a partition $S=\bigcup_{a\in A}S_a$. Let $S^*:=\bigcup_{a\in A,S_a\neq\emptyset}S_aa^{-1}$ and note that:
(i) $SA=S^*A$; (ii) $|S^*|\leq|S|$; (iii) $S^*\subseteq B$ (as $S_aa^{-1}\subseteq B$ when $S_a\neq\emptyset$). By (i) and our blanket assumption $B\subseteq SA$, $B\subseteq S^*A$, so by (iii) we can apply Case 1 to A,B and S^* to claim $|B:A\cap B|\leq|S^*|$. Now (ii) allows us to conclude that $|B:(A\cap B)|\leq|S|$.

(Case 3) In the general case let $S_1 := S \cap BA$. Then obviously, $B \subseteq S_1 A$ and $S_1 \subseteq BA$. By case 2, applied to A, B and S_1 we have $|B : A \cap B| \leq |S_1|$. Since obviously $|S_1| \leq |S|$, this yields $|B : A \cap B| \leq |S|$. □

We recall that two subgroups of a group G are *commensurable* if the indices $|A : A \cap B|$ and $|B : A \cap B|$ are finite. Since $E_K[A] = AK$, for every $A \leq G$ and $K \in [G]^{<\omega}$, by Lemma 4.1, the following result immediately follows.

Corollary 4.2. *Let G be a group. Then two subgroups A and B of G are in the same connected component of $\mathcal{L}\text{-}\exp G$ if and only if they are commensurable.*

The previous corollary allow us to easily compute the connected components of some groups.

Remark 4.3. Fix $n \geq 2$. We want to take a closer look at the structure of $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$. First of all note that two commensurable subgroups H and K of $\mathbb{Z}^n$ have same free rank. Moreover, every subgroup H of $\mathbb{Z}^n$ is commensurable with a pure subgroup $\mathrm{sat}(H)$ of $\mathbb{Z}^n$, namely its *saturation* defined by

$$\mathrm{sat}(H) = \{x \in \mathbb{Z}^n \mid mx \in H \text{ for some non-zero } m \in \mathbb{Z}\}$$

(denoted also by H_* by some authors; recall that a subgroup H of an abelian group G is *pure*, whenever $mH = mG \cap H$ for every $m > 0$ [pure subgroups of $\mathbb{Z}^n$ split as direct summands]). For every $H, K \leq \mathbb{Z}^n$, $\mathrm{sat}(H)$ is commensurable with $\mathrm{sat}(K)$ if and only if $\mathrm{sat}(H) = \mathrm{sat}(K)$. Then $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ has a countable number of connected components. Namely, they are:

- $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}^n)}(\{0\}) = \{0\}$,
- $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}^n)}(\mathbb{Z}^n)$,
- for every $0 < k < n$, a countable number of connected components asymorphic to the subballean $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}^n)}(\mathbb{Z}^k)$ of $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ which is asymorphic to the subballean $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}^k)}(\mathbb{Z}^k)$ of $\mathcal{L}\text{-}\exp(\mathbb{Z}^k)$.

In particular, by Remark 1.5(iii), for every $n > 1$, $\mathcal{L}\text{-}\exp(\mathbb{Z})$ is not coarsely equivalent to $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$.

Note that $\mathcal{L}\text{-}\exp(\mathbb{Z})$ has two connected components, while $\mathrm{dsc}(\exp(\mathbb{Z}, \mathcal{E}_{\mathbb{Z}})) = 2^\omega$, as we have proved in Corollary 3.9.

Proposition 4.4. *Let G be one of the groups $\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ for some prime p. Then*

(i) all balls in $\mathcal{L}\text{-}\exp(G)$ are finite;
(ii) $\mathcal{L}\text{-}\exp(G)$ has two connected components, of which one is a singleton (namely, $\{\{0\}\}$, when $G = \mathbb{Z}$, otherwise $\{G\}$);
(iii) $\mathcal{L}\text{-}\exp(G)$ is thin.

Proof. Items (i) and (ii) are trivial.

(iii, Case $G = \mathbb{Z}$) To show that $\mathcal{L}\text{-}\exp(\mathbb{Z})$ is thin take an arbitrary finite subset F of $\mathbb{Z}$ and choose m so that $F \subseteq [-m, m] \cap \mathbb{Z}$. Pick $n > 3m$. We claim that $(E_F)^* \cap \mathcal{L}\text{-}\exp(\mathbb{Z}) = \{n\mathbb{Z}\}$. We carry out the proof for $F = [-m, m] \cap \mathbb{Z}$, obviously, this implies the general case.

Consider the quotient map $q \colon \mathbb{Z} \to \mathbb{Z}(n) := \mathbb{Z}/n\mathbb{Z}$ and notice that the subset $q(F)$ of $\mathbb{Z}(n)$ contains no non-trivial subgroups, by the assumption $3m < n$. Pick a subgroup $H \in (E_F)^*[\langle n \rangle]$, then $q(H) \subseteq q(F)$, so $q(H) = \{0\}$ in $\mathbb{Z}/n\mathbb{Z}$, hence $H \leq n\mathbb{Z}$. Thus, $H = l\mathbb{Z}$ for some multiple l of n. Since $n\mathbb{Z} \in (E_F)^*[H]$, with $l \geq n \geq 3m$, the previous argument implies $n\mathbb{Z} \leq H$. Therefore, $H = n\mathbb{Z}$.

(iii, Case $G = \mathbb{Z}_{p^\infty}$) We consider now the group $G = \mathbb{Z}_{p^\infty}$, where p is a prime. Denote by H_n the subgroup of $\mathbb{Z}_{p^\infty}$ of order p^n, take an arbitrary finite subset F of $\mathbb{Z}_{p^\infty}$ and choose m so that $F \subseteq H_m$. Then $(E_F)^*[H_n] \cap \mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty}) = \{H_n\}$, for each $n > m$. □

Corollary 4.5. *For every prime p, $\mathcal{L}\text{-}\exp(\mathbb{Z})$ and $\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})$ are asymorphic.*

Proof. By Proposition 4.4(ii), both $\mathcal{L}\text{-}\exp(\mathbb{Z})$ and $\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})$ have two connected components, namely,

$$\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z}),\ \mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\{0\}) = \{0\},\ \mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}_{p^\infty}\},\ \text{ and }\ \mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\}).$$

Moreover, $|\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})| = |\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\})| = \omega$. Since $\mathcal{L}(\mathbb{Z})$ and $\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})$ are thin, in particular, also $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\})$ are thin. Hence, Theorem 3.7 implies that $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\})$ coincide with the ideal coarse spaces associated to the ideals of all their bounded subsets, i.e., finite subsets, namely

$$\mathcal{E}|_{\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})} = \mathcal{E}_{\mathcal{I}} \quad \text{and} \quad \mathcal{E}|_{\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\})} = \mathcal{E}_{\mathcal{J}}, \tag{4}$$

where $\mathcal{I} = [\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})]^{<\infty}$ and $\mathcal{J} = [\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})}(\{0\})]^{<\infty}$.

Fix a bijecton $\varphi\colon \mathcal{L}\text{-}\exp(\mathbb{Z}) \to \mathcal{L}\text{-}\exp(\mathbb{Z}_{p^\infty})$ such that $\varphi(\{0\}) = \mathbb{Z}_{p^\infty}$. We claim that φ is an asymorphism. We can apply Remark 1.5(ii) and the claim follows once we prove that both $\varphi|_{\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\{0\})}$ and $\varphi|_{\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})}$ are asymorphisms. While the first restriction is trivially an asymorphism, Proposition 3.3 and (4) imply that also the second one is an asymorphism. □

In contrast to $\mathcal{L}\text{-}\exp(\mathbb{Z})$, for $n > 1$ $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ has non-thin connected components and thus, in particular, it is not thin. To see that, we put $F = \{(1, 0, \ldots, 0), (0, \ldots, 0)\}$ and note that $2\mathbb{Z} \times S \in (E_F)^*[\mathbb{Z} \times S]$ for each subgroup S of $\mathbb{Z}^{n-1}$.

A coarse space $(X, \mathcal{E})$ is *cellular* if, for every $E \in \mathcal{E}$, $\bigcup_{n\in\mathbb{N}} E^n \in \mathcal{E}$, where E^n is the result of n compositions of E with itself. Thin coarse spaces are, in particular, cellular. A coarse space is cellular if and only if it has asymptotic dimension 0 ([20]).

Question 4.6. *Is $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ cellular for every $n \in \mathbb{N}$?*

For every *locally finite* group G (i.e., every finitely generated subgroup of G is finite), the coarse structure $\mathcal{E}_G$ is cellular, so $\exp(G, \mathcal{E}_G)$ and $\mathcal{L}\text{-}\exp(G)$ are cellular since cellularity is preserved under taking the coarse hyperspace (see [4]).

Question 4.7. *Is the coarse space $\mathcal{L}\text{-}\exp(G)$ cellular for an arbitrary group G?*

Theorem 4.8. *Let $n \in \mathbb{N}$. Then $\mathcal{L}\text{-}\exp(\mathbb{Z}^2)$ is asymorphic to $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ if and only if $n = 2$.*

Proof. Note that that $\mathcal{L}\text{-}\exp(\mathbb{Z})$ is not asymorphic to $\mathcal{L}\text{-}\exp(\mathbb{Z}^2)$ since they have different numbers of connected components as it is shown in Remark 4.3.

Now suppose that $n \geq 3$. Fix, by contradiction, an asymorphism $\varphi\colon \mathcal{L}\text{-}\exp(\mathbb{Z}^2) \to \mathcal{L}\text{-}\exp(\mathbb{Z}^n)$. As recalled in Remark 1.5(ii), φ induces asymorphisms between the connected components of those two coarse spaces. Because of Remark 4.3, one of those restrictions is an asymorphism between $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}\text{-}\exp(\mathbb{Z}^2)}(\mathbb{Z}^2)$. However, this is an absurd, since the first coarse space is thin, while the second one has not that property. □

Question 4.9. *Is it true that $\mathcal{L}\text{-}\exp(\mathbb{Z}^n)$ is asymorphic to $\mathcal{L}\text{-}\exp(\mathbb{Z}^m)$ if and only if $n = m$?*

Remark 4.10. Let G be an arbitrary group. According to Fact 3.10, all balls in $\mathcal{L}\text{-}\exp(G)$ centered at $\{e_G\}$ are finite. Nevertheless, this is not true for all balls of $\mathcal{L}\text{-}\exp(G)$. One can find examples of abelian groups G such that some balls in $\mathcal{L}\text{-}\exp(G)$ centred at G are infinite. For example, let $G = \Pi_{n\in\mathbb{N}} G_n$, where $G_n \simeq \mathbb{Z}/2\mathbb{Z}$, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, denote by a_n the element of G such that $p_n(a_n) = 1$ and, for every $i \neq n$, $p_i(a_n) = 0$. Then, for every $n \in \mathbb{N}$, $\langle \{a_i \mid i \in \mathbb{N}\setminus\{1, n\}\} \cup \{a_n + a_1\} \rangle \in (E_{\langle a_1 \rangle})^*[G] \cap \mathcal{L}\text{-}\exp G$ and thus this ball contains infinitely many elements.

5. Rigidity results

If two groups G and H are isomorphic, then Proposition 2.8 and Remark 2.9 imply that $\mathcal{L}\text{-}\exp(G)$ is asymorphic to $\mathcal{L}\text{-}\exp(H)$. However, the converse is not true in general (for example, $\mathcal{L}\text{-}\exp(\mathbb{Z})$ is asymorphic to $Lexp(\mathbb{Z}_{p^\infty})$ which is asymorphic to $\mathcal{L}\text{-}\exp(\mathbb{Z}_{q^\infty})$, where p and q are primes). In this section we want to determine conditions that ensures that the opposite implication holds. We call such results "rigidity results".

Let us start with some technical results which hold for the subgroup coarse structure $\mathcal{L}\text{-}\exp(G)$.

Lemma 5.1. *Let X be a coarse space.*

(i) If X is asymorphic to $\mathcal{L}\text{-}\exp(\mathbb{Z})$, then X has two connected components. Moreover, one connected component is a singleton, while the other one is infinite and unbounded.
(ii) If X is coarsely equivalent to $\mathcal{L}\text{-}\exp(\mathbb{Z})$, then X has two connected components. Moreover, one connected component is bounded, while the other one is unbounded.

Proof. The proof is an application of Remarks 4.3 and 1.5. □

An infinite group is said to be *quasi-finite* if every proper subgroup is finite. Example of quasi-finite groups are the Prüffer p-groups and the Tarskii monsters ([17]). Moreover, if an abelian group is quasi-finite, then it is isomorphic to Prüffer p-group for some prime p.

Proposition 5.2. *Let G be a group. Suppose that $\mathcal{L}$-$\exp(G)$ has precisely two connected components, one of them is a singleton and the other one is infinite. Then G must be infinite. Moreover:*

(i) if G contains an element of infinite order then $G \simeq \mathbb{Z}$;
(ii) if G is a torsion group then G is quasi-finite.

Proof. The first statement is trivial, since, otherwise, $\mathcal{L}$-$\exp(G)$ would be bounded.

(i) Let g be element of infinite order of G. Then $\langle g\rangle \in \mathcal{L}(G)$ is infinite, $\langle g\rangle \in \mathcal{Q}_{\mathcal{L}\text{-}\exp(G)}(G)$ and thus $\mathcal{Q}_{\mathcal{L}\text{-}\exp(G)}(G)$ is infinite (as it contains the subgroups of the form $\langle g^k\rangle$, where $k \in \mathbb{N}$), while $\mathcal{Q}_{\mathcal{L}\text{-}\exp(G)}(\{e_G\}) = \{e_G\}$. Since each infinite subgroup of G is, in particular, large in G, it has finite index and, by Fedorov's theorem [10], $G \simeq \mathbb{Z}$.

(ii) Since G is torsion, for every $g \in G$, $\langle g\rangle$ is a finite subgroup and thus belongs to the connected component $\mathcal{Q}_{\mathcal{L}\text{-}\exp(G)}(e_G)$. Hence, the connected component of G is a singleton and every proper subgroup is finite. □

Corollary 5.3. *If a group G contains an element of infinite order, then $\mathcal{L}$-$\exp(G)$ is asymorphic to $\mathcal{L}$-$\exp(\mathbb{Z})$ if and only if $G \simeq \mathbb{Z}$.*

Proof. Lemma 5.1(i) implies that $\mathcal{L}$-$\exp(G)$ has two connected components, one is infinite and the other one is just a singleton. Hence the conclusion follows from Proposition 5.2(a). □

Theorem 5.4. *For an abelian group G, $\mathcal{L}$-$\exp(G) \approx \mathcal{L}$-$\exp(\mathbb{Z})$ if and only if either $G \simeq \mathbb{Z}$ or $G \simeq \mathbb{Z}_{p^\infty}$, for some p is prime.*

Proof. The "if part" of the statement is proved in Corollary 4.5.

Conversely, let us divide the proof in two cases. If G is torsion, then Lemma 5.1(i) and Proposition 5.2(ii) imply that every proper subgroup of G is finite. Hence, since G is abelian, $G \simeq \mathbb{Z}_{p^\infty}$, for some prime p. Otherwise, there exists and element $g \in G$ of infinite order and then the claim follows from Corollary 5.3. □

Can we relax the hypothesis of Theorem 5.4? Namely, we wonder whether the request of G being abelian can be relaxed or not. Let us state it as a question.

Question 5.5. *Let G be a torsion group such that $\mathcal{L}$-$\exp(G)$ and $\mathcal{L}$-$\exp(\mathbb{Z})$ are asymorphic. Is $G \simeq \mathbb{Z}_{p^\infty}$ for some prime p?*

Lemma 5.6. *Let G and H be two groups.*

(i) If there exist two homomorphisms $f\colon G \to H$ and $g\colon H \to G$ such that $f \circ g \sim id_H$ and $g \circ f \sim id_G$, then $f\colon (G, \mathcal{E}_G) \to (H, \mathcal{E}_H)$ is a coarse equivalence, with coarse inverse $g\colon (H, \mathcal{E}_H) \to (G, \mathcal{E}_G)$, and $\mathcal{L}$-$\exp(f) := \exp f|_{\mathcal{L}\text{-}\exp(G)}\colon \mathcal{L}$-$\exp(G) \to \mathcal{L}$-$\exp(H)$ is a coarse equivalence, with inverse $\mathcal{L}$-$\exp(g)\colon \mathcal{L}$-$\exp(H) \to \mathcal{L}$-$\exp(G)$.
(ii) Let H be a finite normal subgroup of G. Then the quotient map $q\colon \mathcal{L}$-$\exp(G) \to \mathcal{L}$-$\exp(G/H)$ is a coarse equivalence and, moreover, $\mathcal{L}$-$\exp(q)\colon \mathcal{L}$-$\exp(G) \to \mathcal{L}$-$\exp(G/H)$ is a coarse equivalence.

Proof. (a) Note that $f\colon (G, \mathcal{E}_G) \to (H, \mathcal{E}_H)$ is trivially a coarse equivalence. Moreover, Proposition 2.8 implies that $\exp f\colon \exp(G, \mathcal{E}_G) \to \exp(H, \mathcal{E}_H)$ is a coarse equivalence with inverse $\exp g\colon \exp(H, \mathcal{E}_H) \to \exp(G, \mathcal{E}_G)$. Since both f and g are homomorphisms, the restrictions $\mathcal{L}$-$\exp(f)$ and $\mathcal{L}$-$\exp(g)$ are well-defined and thus they are coarse equivalences.

(b) Since q is a homomorphism, $q\colon (G, \mathcal{E}_G) \to (G/H, \mathcal{E}_{G/H})$ is bornologous. In particular, $\mathcal{L}$-$\exp(q) = \exp q|_{\mathcal{L}\text{-}\exp(G)}\colon \mathcal{L}$-$\exp(G) \to \mathcal{L}$-$\exp(G/H)$, which is well-defined, is bornologous as well. Moreover, $g\colon \mathcal{L}$-$\exp(G/H) \to \mathcal{L}$-$\exp(G)$ defined by the law $g(K) = q^{-1}(K)$, where $K \leq G/H$, is bornologous and a coarse inverse of $\mathcal{L}$-$\exp(q)$. □

Theorem 5.7. *Let a group G contain an element g of infinite order. Then $\mathcal{L}$-$\exp(G)$ and $\mathcal{L}$-$\exp(\mathbb{Z})$ are coarsely equivalent if and only if G has a finite normal subgroup H such that $G/H \simeq \mathbb{Z}$.*

Proof. ($\to$) Assume that $\mathcal{L}$-$\exp(G)$ and $\mathcal{L}$-$\exp(\mathbb{Z})$ are coarsely equivalent. Lemma 5.1(ii) implies that $\mathcal{L}$-$\exp(G)$ has two connected components: one is unbounded (hence, infinite) and one is bounded. Let us see that the connected component $C := \mathcal{Q}_{\mathcal{L}\text{-}\exp(G)}(\{e\})$ of $\{e\}$ is the bounded one. To prove that C is bounded it is enough to observe that it does not contain the infinite subgroup $\langle g\rangle$ as well as its infinitely many proper subgroups $\langle g^n \rangle$, where $n \geq 2$. Since this family is certainly unbounded in $\mathcal{L}$-exp(G), C must be the bounded component. Consequently, C is finite being contained into a ball around $\{e\}$ (see Fact 3.10).

Since C contains all finite order elements $h \in G$, we have that the set H of all the elements of finite order of G is finite. By Ditsmans lemma [2], H is a subgroup. Moreover, since conjugacy doesn't change the order of an element, H is normal in G. Then G/H is torsion free.

Since $\mathcal{L}$-exp(G/H) is coarsely equivalent to $\mathcal{L}$-exp(G) (Lemma 5.6(ii)) and thus to $\mathcal{L}$-exp$(\mathbb{Z})$, in particular, we can reapply the argument contained in the proof of Proposition 5.2(i) and prove that every proper subgroup K of G/H is large in G/H and so $|G/H : K|$ is finite. By Federov's theorem, G/H is isomorphic to $\mathbb{Z}$.

($\leftarrow$) On the other hand, if H is finite and $G/H \simeq \mathbb{Z}$ then $G = \langle a \rangle H$, $\langle a \rangle \simeq \mathbb{Z}$ and $\mathcal{L}$-exp$(\langle a \rangle)$ is large in $\mathcal{L}$-exp(G), so $\mathcal{L}$-exp(G) and $\mathcal{L}$-exp$(\mathbb{Z})$ are coarsely equivalent. □

Lemma 5.8. *Let G be a group.*

(i) If H is a subgroup of G of finite index, then G has only finitely many subgroups containing H.
(ii) If $\mathcal{H}$ is a family of subgroups of G stable under under finite intersections, and there exists $n \in \mathbb{N}$ such that $|G : H| \leq n$ for every $H \in \mathcal{H}$, then $\mathcal{H}$ is finite.

Proof. (i) Let H_G be the core of H in G (i.e., the biggest normal subgroup of G which is contained in H), which has still finite index in G. Consider the map $q\colon G \to G/H_G$. Then q induces a bijection between the family of subgroups of G containing H_G and the one of the subgroups of G/H_G. Since the latter is finite, we are done.

(ii) Assume by contradiction that $\mathcal{H}$ has infinitely many pairwise distinct members $\{H_m\}_{m\in\mathbb{N}}$. One can assume, without loss of generality that they form a decreasing chain (indeed, using (a) just replace H_m by the intersection $H_1 \cap \cdots \cap H_m$). As $|G : H_m|$ is bounded, this decreasing chain stabilises. Let us call that common intersection K (obviously, $K \in \mathcal{H}$). Since all H_m contain K, this contradicts Lemma 5.8. □

Theorem 5.9. *For an abelian group G, $\mathcal{L}$-exp(G) and $\mathcal{L}$-exp$(\mathbb{Z})$ are coarsely equivalent if and only if there exists a finite subgroup H of G such that either $G/H \simeq \mathbb{Z}$ or $G/H \simeq \mathbb{Z}_{p^\infty}$, for some prime p.*

Proof. Assume that $\mathcal{L}$-exp(G) and $\mathcal{L}$-exp$(\mathbb{Z})$ are coarsely equivalent. If G has an element of infinite order then we apply Theorem 5.7. Otherwise, suppose that G is a torsion group. Since $\mathcal{L}$-exp(G) and $\mathcal{L}$-exp$(\mathbb{Z})$ are coarsely equivalent, we deduce from Lemma 5.1(ii), that $\mathcal{L}$-exp(G) has two connected components and one of them is bounded, while the other one is unbounded. Since G is torsion, Fact 3.10 implies that $\mathcal{Q}_{\mathcal{L}\text{-exp}(G)}(\{0\})$ must be unbounded. Hence, the family $\mathcal{H}$ of all finite index subgroups of G satisfies the hypothesis of Lemma 5.8(b) and thus $\mathcal{H}$ is finite and, in particular, it has a minimum element K. Then G/K is finite and K is quasi-finite and thus, since G is abelian, $K \simeq \mathbb{Z}_{p^\infty}$, for some prime p. Hence the claim follows. □

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Department of Mathematical, Computer and Physical Sciences, University of Udine, 33100 Udine, Italy
E-mail address: nicolo.zava@gmail.com