A general existence result for the Toda system on compact surfaces

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A GENERAL EXISTENCE RESULT FOR THE TODA SYSTEM ON COMPACT SURFACES

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ABSTRACT. In this paper we consider the following *Toda system* of equations on a compact surface:

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{1,j} (\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{2,j} (\delta_{p_j} - 1), \end{cases}$$

which is motivated by the study of models in non-abelian Chern-Simons theory. Here h_1, h_2 are smooth positive functions, ρ_1, ρ_2 two positive parameters, p_i points of the surface and $\alpha_{1,i}, \alpha_{2,j}$ non-negative numbers. We prove a general existence result using variational methods.

The same analysis applies to the following mean field equation

$$-\Delta u = \rho_1 \left(\frac{h e^u}{\int_\Sigma h e^u dV_g} - 1 \right) - \rho_2 \left(\frac{h e^{-u}}{\int_\Sigma h e^{-u} dV_g} - 1 \right),$$

which arises in fluid dynamics.

With an appendix by Sadok Kallel (University of Lille 1)

1. Introduction

The Toda system

$$(1) \qquad -\Delta u_i(x) = \sum_{j=1}^N a_{ij} e^{u_j(x)}, \qquad x \in \Sigma, \; i = 1, \dots, N,$$

where Δ is the Laplace operator and $A = (a_{ij})_{ij}$ the *Cartan matrix* of $SU(N+1)$,

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

plays an important role in geometry and mathematical physics. In geometry it appears in the description of holomorphic curves in $\mathbb{CP}^n$, see [7], [11], [15], [29]. In mathematical physics, it is a model for non-abelian Chern-Simons vortices, which might have applications in high-temperature superconductivity and which appear in a much wider variety compared to the Yang-Mills framework, see e.g. [58], [59] and [64] for further details and an up-to-date set of references.

The existence of abelian Chern-Simons vortices has been quite deeply investigated in the literature, see e.g. [10], [13], [50], [55], [57]. The study of the non-abelian case is more recent, and we refer for example to [21], [34], [35], [41], [51], [61].

We will be interested in the following problem on a compact surface Σ. For the sake of simplicity, we will assume that $Vol_g(\Sigma) = 1$.

$$(2) \qquad \begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{1,j} (\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1 \right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1 \right) - 4\pi \sum_{j=1}^m \alpha_{2,j} (\delta_{p_j} - 1). \end{cases}$$

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Here h_1, h_2 are smooth positive functions, and $\alpha_{i,j} \geq 0$. The above system arises specifically from gauged self-dual Schrödinger equations, see e.g. Chapter 6 in [64]: the Dirac deltas represent *vortices* of the wave function, namely points where the latter vanishes.

To describe the history and the main features of the problem, we first desingularize the equation using a simple change of variables. Consider indeed the fundamental solution $G_p(x)$ of the Laplace equation on Σ with pole at p, i.e. the unique solution to

$$-\Delta G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \qquad \text{with} \quad \int_\Sigma G_p(x)\, dV_g(x) = 0. \tag{3}$$

By the substitution

$$u_i(x) \mapsto u_i(x) + 4\pi \sum_{j=1}^{m} \alpha_{i,j} G_{p_j}(x), \qquad h_i(x) \mapsto \widetilde{h}_i(x) = h_i(x) e^{-4\pi \sum_{j=1}^m \alpha_{i,j} G_{p_j}(x)} \tag{4}$$

problem (2) transforms into an equation of the type

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1}\, dV_g} - 1 \right) - \rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2}\, dV_g} - 1 \right), \\ -\Delta u_2 = 2\rho_2 \left(\dfrac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2}\, dV_g} - 1 \right) - \rho_1 \left(\dfrac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1}\, dV_g} - 1 \right), \end{cases} \tag{5}$$

where the functions $\widetilde{h}_j$ satisfy

$$\widetilde{h}_i > 0 \quad \text{on } \Sigma \setminus \{p_1, \dots, p_m\}; \qquad \widetilde{h}_i(x) \simeq d(x, p_j)^{2\alpha_{i,j}}, \text{ near } p_j, \ i = 1,\ 2. \tag{6}$$

Problem (5) is variational, and solutions can be found as critical points of the Euler-Lagrange functional $J_\rho : H^1(\Sigma) \times H^1(\Sigma) \to \mathbb{R}$ ($\rho = (\rho_1, \rho_2)$) given by

$$J_\rho(u_1, u_2) = \int_\Sigma Q(u_1, u_2)\, dV_g + \sum_{i=1}^{2} \rho_i \left(\int_\Sigma u_i dV_g - \log \int_\Sigma \widetilde{h}_i e^{u_i} dV_g \right), \tag{7}$$

where $Q(u_1, u_2)$ is defined as:

$$Q(u_1, u_2) = \frac{1}{3} \left(|\nabla u_1|^2 + |\nabla u_2|^2 + \nabla u_1 \cdot \nabla u_2 \right). \tag{8}$$

A basic tool for studying functionals like J_ρ is the Moser-Trudinger inequality, see (15). Its analogue for the Toda system has been obtained in [29] and reads as

$$4\pi \sum_{i=1}^{2} \left(\log \int_\Sigma h_i e^{u_i} dV_g - \int_\Sigma u_i dV_g \right) \leq \int_\Sigma Q(u_1, u_2)\, dV_g + C \qquad \forall\, u_1, u_2 \in H^1(\Sigma), \tag{9}$$

for some $C = C(\Sigma)$. This inequality immediately allows to find a global minimum of J_ρ provided both ρ_1 and ρ_2 are less than 4π. For larger values of the parameters ρ_i J_ρ is unbounded from below and the problem becomes more challenging. In this paper we use min-max theory to find a critical point of J_ρ in a general non-coercive regime. Our main result is the following:

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^2$ be as in Definition 2.4. Let Σ be a compact surface neither homeomorphic to $\mathbb{S}^2$ nor to $\mathbb{RP}^2$, and assume that $(\rho_1, \rho_2) \notin \Lambda$. Then* (2) *is solvable.*

Let us point out that $\Lambda \subseteq \mathbb{R}^2$ is an explicit set formed by an union of straight lines and discrete points, see Remark 2.6. In particular it is a closed set with zero Lebesgue measure.

Up to our knowledge, *there is no previous existence result in the literature* for the singular Toda system. Our result is hence the first one in this direction, and is generic in the choice of parameters ρ_1 and ρ_2. In the regular case there are some previous existence results, see [28, 41, 44, 46], some of which have a counterpart in [18] and [19] for the scalar case (10) (see also [20] for a higher order problem and [2, 5, 12, 45] for the singular case). However, these require an upper bound either on one of the ρ_i's or both: hence our result covers most of the unknown cases also for the regular problem.

The main difficulties in attacking (5) are mainly of two kinds: compactness issues and the Morse-structure of the functional, which we are going to describe below.

As many geometric problems, also (5) presents loss of compactness phenomena, as its solutions might blow-up. To describe the general phenomenon it is first convenient to discuss the case of the scalar counterpart of (5), namely is a Liouville equation the form:

$$-\Delta u = 2\rho \left(\frac{h\, e^{u}}{\int_{\Sigma} h\, e^{u} dV_g} - 1 \right), \tag{10}$$

where $\rho \in \mathbb{R}$ and where $h(x)$ behaves as in (6) near the singularities. Equation (10) rules the change of Gaussian curvature under conformal deformation of the metric, and that it also describes the abelian counterpart of (2) from the physical point of view. This equation has been very much studied in the literature; there are by now many results regarding existence, compactness of solutions, bubbling behavior, etc. We refer the interested reader to the reviews [43, 59].

Concerning (10) it was proved in [9], [36] and [37] that for the regular case a blow-up point $\overline{x}_R$ for a sequence $(u_n)_n$ of solutions satisfies the following quantization property

$$\lim_{r \to 0} \lim_{n \to +\infty} \rho \int_{B_r(\overline{x}_R)} h\, e^{u_n} dV_g = 4\pi, \tag{11}$$

and that the limit profile of solutions is that of a *bubble*, namely the logarithm of the conformal factor of the stereographic projection from S^2 onto $\mathbb{R}^2$, composed with a dilation.

For the singular case instead, it was proven in [2] and [6] that if blow-up occurs at a singular point $\overline{x}_S$ with weight $-4\pi\alpha$ then one has

$$\lim_{r \to 0} \lim_{n \to +\infty} \rho \int_{B_r(\overline{x}_S)} h\, e^{u_n} dV_g = 4\pi(1+\alpha), \tag{12}$$

whereas (11) still holds true if blow-up occurs at a regular point.

This behaviour helps to explain the blow-up feature for system (5), which inherits some character from the scalar case. Consider first the regular case, that is, (2) with $\alpha_{i,j} = 0$. Here a sequence of solutions can blow-up in three different ways: one component blows-up and the other does not; one component blows-up faster than the other; both components blow-up at the same rate.

It was proved in [28, 30] that the quantization values for the two components are respectively $(4\pi, 0)$ or $(0, 4\pi)$ in the first case, $(8\pi, 4\pi)$ or $(4\pi, 8\pi)$ in the second case and $(8\pi, 8\pi)$ in the third one. Notice that, by the results in [17], [22] and [48], all the five alternatives may indeed happen.

When singular sources are present a similar phenomenon happens, which has been investigated in the recent paper [38]. If blow-up occurs at a point p with values α_1, α_2 (we may allow them to vanish), the corresponding blow-up values would be

$$(4\pi(1+\alpha_1, 0)); \qquad (0, 4\pi(1+\alpha_2)); \qquad (4\pi(1+\alpha_1), 4\pi(2+\alpha_1+\alpha_2));$$

$$(4\pi(2+\alpha_1+\alpha_2), 4\pi(1+\alpha_2)); \qquad (4\pi(2+\alpha_1+\alpha_2), 4\pi(2+\alpha_1+\alpha_2)).$$

Other (finitely-many) blow-up values are indeed allowed, see Theorem 2.5 for details, as more involved situations are not yet excluded (or known to exist). As a consequence, the set of solutions to (5) is compact whenever $(\rho_1, \rho_2) \notin \Lambda$: this is the main reason for our assumption in Theorem 1.1.

Let us now show how we can study the sub-levels of the functional and conclude existence of solutions via min-max methods. The main tool in the variational study of this kind of problems is the so-called Chen-Li inequality, see [14]. In the scalar case, it implies that a suitable *spreading* of the term e^u yields a better constant in the Moser-Trudinger inequality, which in turn might imply a lower bound on the Euler functional I_ρ of (10)

$$I_\rho(u) = \frac{1}{2} \int_{\Sigma} |\nabla_g u|^2 dV_g + 2\rho \left(\int_{\Sigma} u\, dV_g - \log \int_{\Sigma} h\, e^{u} dV_g \right), \qquad u \in H^1(\Sigma). \tag{13}$$

The consequence of this fact is that if $\rho < 4(k+1)\pi$, $k \in \mathbb{N}$, and if $I_\rho(u)$ is large negative (i.e. when lower bounds fail) e^u accumulates near at most k points of Σ, see e.g. [19]. This suggests to introduce the family of unit measures Σ_k which are supported in at most k points of Σ, known as *formal barycenters* of Σ

$$\Sigma_k = \left\{ \sum_{j=1}^{k} t_j \delta_{x_j} : \sum_{j=1}^{k} t_j = 1, \; x_j \in \Sigma \right\}. \tag{14}$$

One can show that, for any integer k, Σ_k is not contractible and that its homology is mapped injectively into that of the low sub-levels of I_ρ. This allows to prove existence of solutions via suitable min-max schemes.

When both ρ_1 and ρ_2 are larger than 4π the description of the sub-levels becomes more involved, since the two components u_1 and u_2 interact in a non-trivial way. See [46] on this respect. In this paper we obtain a *partial* topological characterization of the low energy levels of J_ρ, which is however sufficient for our purposes. This strategy has been used in [4] and in [3] for the singular scalar equation and for a model in electroweak theory respectively, while in this paper the general non-abelian case is treated for the first time.

First, we construct two disjoint simple non-contractible curves γ_1, γ_2 which do not intersect singular points, and define global retractions Π_1, Π_2 of Σ onto these two curves. Such curves do not exist for $\Sigma = \mathbb{S}^2$ or $\mathbb{RP}^2$, and hence our arguments do not work in those cases.

Combining arguments from [14], [44] and [46] we prove that if $\rho_1 < 4(k+1)\pi$ and $\rho_2 < 4(l+1)\pi$, $k, l \in \mathbb{N}$, then either $\widetilde{h}_1 e^{u_1}$ is close to Σ_k or $\widetilde{h}_2 e^{u_2}$ is close to Σ_l in the distributional sense. Then we can map continuously (and naturally) $\widetilde{h}_1 e^{u_1}$ to Σ_k or $\widetilde{h}_2 e^{u_2}$ to Σ_l; using then the retractions Π_i one can restrict himself to targets in $(\gamma_1)_k$ or $(\gamma_2)_l$ only. This alternative can be expressed naturally in terms of the *topological join* $(\gamma_1)_k * (\gamma_2)_l$. Roughly speaking, given two topological spaces A and B, the join $A * B$ is the formal set of segments joining elements of A with elements of B, see Section 2 for details. In this way, we are able to define a global projection Ψ from low sub-levels of J_ρ onto $(\gamma_1)_k * (\gamma_2)_l$.

We can also construct a reverse map Φ_λ (where λ is a large parameter) from $(\gamma_1)_k * (\gamma_2)_l$ into arbitrarily low sub-levels of J_ρ using suitable test functions. Moreover, we show that the composition of both maps is homotopic to the identity map. Finally, $(\gamma_1)_k * (\gamma_2)_l$ is homeomorphic to a sphere of dimension $2k+2l-1$ see Remark 3.2: in particular it is not contractible, and this allows us to apply a min-max argument.

In this step a compactness property is needed, like the Palais-Smale's. The latter is indeed not known for this problem, but there is a way around it using a monotonicity method from [56]. For that, compactness of solutions comes to rescue, and here we use the results of [28] and [38]. This is the reason why we assume $(\rho_1, \rho_2) \notin \Lambda$.

In this paper we also give a general result for a mean field equation, Theorem 6.2, arising from models in fluid dynamics and in the description of constant mean curvature surfaces: to keep the introduction short we discuss its motivation and how our result compares to the existing literature in Section 6.

The plan of the paper is the following: in Section 2 we recall some preliminary results on Moser-Trudinger inequalities, the notion of topological join and a compactness theorem. In Section 3 we construct a family of test functions with low energy modelled on the topological join of $(\gamma_1)_k$ and $(\gamma_2)_l$. In Section 4 we derive suitable improved Moser-Trudinger inequalities to construct projections from low sub-levels of J_ρ into $(\gamma_1)_k * (\gamma_2)_l$. In Section 5 we prove our existence theorem using the min-max argument and finally in Section 6 we discuss the mean field equation.

Acknowledgment: This paper includes an appendix of Sadok Kallel which establishes that Σ_k is a CW-complex. This allows us to give a self-consistent and short proof of Proposition 2.2. The authors are deeply grateful to him for this contribution.

2. Notation and preliminaries

In this section we collect some useful notation and preliminary material. The Appendix at the end of the paper use independent notation, which will be established there.

Given points $x, y \in \Sigma$, $d(x,y)$ will stand for the metric distance between x and y on Σ. Similarly, for any $p \in \Sigma$, $\Omega, \Omega' \subseteq \Sigma$, we set:

$$d(p,\Omega) = \inf \{ d(p,x) \; : x \in \Omega \}, \qquad d(\Omega,\Omega') = \inf \{ d(x,y) \; : \; x \in \Omega, \; y \in \Omega' \}.$$

The symbol $B_s(p)$ stands for the open metric ball of radius s and centre p, and the complement of a set Ω in Σ will be denoted by Ω^c.

Given a function $u \in L^1(\Sigma)$ and $\Omega \subset \Sigma$, the average of u on Ω is denoted by the symbol

$$\fint_{\Omega} u \, dV_g = \frac{1}{|\Omega|} \int_{\Omega} u \, dV_g.$$

We denote by $\overline{u}$ the average of u in Σ: since we are assuming $|\Sigma| = 1$, we have

$$\overline{u} = \int_{\Sigma} u \, dV_g = \fint_{\Sigma} u \, dV_g.$$

The sub-levels of the functional J_ρ will be indicated as

$$J_\rho^a := \big\{ u = (u_1, u_2) \in H^1(\Sigma) \times H^1(\Sigma) : \; J(u_1, u_2) \leq a \big\}$$

Throughout the paper the letter C will stand for large constants which are allowed to vary among different formulas or even within the same lines. When we want to stress the dependence of the constants on some parameter (or parameters), we add subscripts to C, as C_δ, etc. We will write $o_\alpha(1)$ to denote quantities that tend to 0 as $\alpha \to 0$ or $\alpha \to +\infty$; we will similarly use the symbol $O_\alpha(1)$ for bounded quantities.

We recall next the classical Moser-Trudinger inequality, in its weak form

$$\log \int_{\Sigma} e^{u - \overline{u}} \, dV_g \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla_g u|^2 \, dV_g + C; \qquad u \in H^1(\Sigma), \tag{15}$$

where C is a constant depending only on Σ and the metric g. For the Toda system, a similar sharp inequality was derived in [29] concerning the regular case: indeed, since the weights α_{ij} are positive, that inequality applies to the singular case as well, as the functions $\widetilde{h}_i$ are uniformly bounded.

Theorem 2.1. *([29]) The functional J_ρ is bounded from below if and only if $\rho_i \leq 4\pi$, $i = 1, 2$.*

As it is mentioned in the introduction, some useful information arising from Moser-Trudinger type inequalities and their improvements are the concentration of e^{u_i} when $u = (u_1, u_2)$ belongs to a low sub-level. To express this rigorously, we denote $\mathcal{M}(\Sigma)$ the set of all Radon measures on Σ, and introduce a norm by using duality versus Lipschitz functions, that is, we set:

$$\|\mu\|_{Lip'(\Sigma)} = \sup_{\|f\|_{Lip(\Sigma)} \leq 1} \left| \int_{\Sigma} f \, d\mu \right|; \qquad \mu, \nu \in \mathcal{M}(\Sigma). \tag{16}$$

We denote by $\mathbf{d}$ the corresponding distance, which receives the name of Kantorovich-Rubinstein distance.

When a measure is close in the Lip' sense to an element in Σ_k, see (14), it is then possible to map it continuously to a nearby element in this set. The following result has been proved in [20], but we give here a much shorter and self-consistent proof.

Proposition 2.2. *Given $k \in \mathbb{N}$, for ε_0 sufficiently small there exists a continuous retraction:*

$$\psi_k : \{\sigma \in \mathcal{M}(\Sigma), \; \mathbf{d}(\sigma, \Sigma_k) < \varepsilon_0\} \to \Sigma_k.$$

Here continuity is referred to the distance $\mathbf{d}$. In particular, if $\sigma_n \rightharpoonup \sigma$ in the sense of measures, with $\sigma \in \Sigma_k$, then $\psi_k(\sigma_n) \to \sigma$.

Proof. Observe that the inclusion $Lip(\Sigma) \subset C(\Sigma)$ is compact: therefore, $\mathcal{M}(\Sigma) = C(\Sigma)' \subset Lip(\Sigma)'$ is also compact. Of course, the set $\Sigma_k \subset \mathcal{M}(\Sigma)$, and then it is inside $Lip(\Sigma)'$. Since Σ_k is a Euclidean Neighbourhood Retract (ENR) (see Appendix E of [8]), there exists a neighbourhood $V \supset \Sigma_k$ in the Lip' topology, and a continuous retraction $\psi_k : V \to \Sigma_k$.

Now, if $\sigma_n \rightharpoonup \sigma \in \Sigma_k$ in the sense of measures, by compactness, $f_n \to \sigma$ in Lip', and by continuity, $\psi_k(f_n) \to \psi_k(\sigma)$. But, since ψ_k is a retraction, $\psi_k(\sigma) = \sigma$. ■

Remark 2.1. *In the Appendix to this paper Sadok Kallel proves that Σ_k is a CW-complex. As a consequence it is an Euclidean Neighborhood Retract, see for instance Appendix E of [8]. And this is the key point of the proof of Proposition 2.2.*

At some point of our proof we will be under the assumptions of Proposition 2.2 for either $f = \widetilde{h}_1 e^{u_1}$ or for $f = \widetilde{h}_2 e^{u_2}$. To deal with this alternative it will then be convenient to use the notion of *topological join*, which we recall here. The topological join of two sets A, B is defined as the family of elements of the form

$$\frac{\{(a,b,r): \ a \in A, \ b \in B, \ r \in [0,1]\}}{R},$$

where R is an equivalence relation such that

$$(a_1, b, 1) \overset{R}{\sim} (a_2, b, 1) \quad \forall a_1, a_2 \in A, b \in B \qquad \text{and} \qquad (a, b_1, 0) \overset{R}{\sim} (a, b_2, 0) \quad \forall a \in A, b_1, b_2 \in B.$$

The elements of the join are usually written as formal sums $(1-r)a + rb$.

The next tool we will need is a compactness result from [38]: before stating it it is convenient to introduce a finite set of couples of numbers, which represent possible quantization values for the concentration of the exponential functions. Consider a point p at which (2) has singular weights $\alpha_1 = \alpha_1(p), \alpha_2 = \alpha_2(p)$ in the first and the second component of the equation. We give then the following two definitions.

Definition 2.3. *Given a couple of non-negative numbers (α_1, α_2) we let $\Gamma_{\alpha_1,\alpha_2}$ be the subset of an ellipse in $\mathbb{R}^2$ defined by the equation*

$$\Gamma_{\alpha_1,\alpha_2} := \left\{(\sigma_1, \sigma_2) \ : \ \sigma_1, \sigma_2 \geq 0, \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2(1+\alpha_1)\sigma_1 + 2(1+\alpha_2)\sigma_2\right\}.$$

We then let $\Lambda_{\alpha_1,\alpha_2} \subseteq \Gamma_{\alpha_1,\alpha_2}$ be the set constructed via the following rules:

1. *the points* $(0,0)$, $(2(1+\alpha_1, 0)), (0, 2(1+\alpha_2)), (2(1+\alpha_1), 2(2+\alpha_1+\alpha_2))$, $(2(2+\alpha_1+\alpha_2), 2(1+\alpha_2))$, $(2(2+\alpha_1+\alpha_2), 2(2+\alpha_1+\alpha_2))$ *belong to* $\Lambda_{\alpha_1,\alpha_2}$*;*

2. *if $(a,b) \in \Lambda_{\alpha_1,\alpha_2}$ then also any $(c,d) \in \Gamma_{\alpha_1,\alpha_2}$ with $c = a + 2m$, $m \in \mathbb{N} \cup \{0\}$, $d \geq b$ belongs to $\Lambda_{\alpha_1,\alpha_2}$;*

3. *if $(a,b) \in \Lambda_{a_1,\alpha_2}$ then also any $(c,d) \in \Gamma_{\alpha_1,\alpha_2}$ with $d = b + 2n$, $n \in \mathbb{N} \cup \{0\}$, $c \geq a$ belongs to $\Lambda_{\alpha_1,\alpha_2}$.*

Definition 2.4. *Given $\Lambda_{\alpha_1,\alpha_2}$ as in Definition 2.3, we set*

$$\Lambda_0 = 2\pi \left\{ (2p, 2q) + \sum_{j=1}^{m} n_j (a_j, b_j) \ : \ p, q \in \mathbb{N} \cup \{0\}, n_j \in \{0,1\}, (a_j, b_j) \in \Lambda_{\alpha_{1,j},\alpha_{2,j}} \right\};$$

$$\Lambda_i = 4\pi \left\{ n + \sum_{j=1}^{m} (1 + \alpha_{i,j}) n_j, \ n \in \mathbb{N} \cup \{0\}, \ n_j \in \{0,1\} \right\}, \qquad i = 1, 2.$$

We finally set

$$\Lambda = \Lambda_0 \cup (\Lambda_1 \times \mathbb{R}) \cup (\mathbb{R} \times \Lambda_2) \subseteq \mathbb{R}^2.$$

From the local quantization results in [38], and some standard analysis (see in particular Section 1 in [9]) one finds the following global compactness result.

Theorem 2.5. *([38]) For (ρ_1, ρ_2) in a fixed compact set of $\mathbb{R}^2 \setminus \Lambda$ the family of solutions to* (5) *is uniformly bounded in $C^{2,\beta}$ for some $\beta > 0$.*

Remark 2.6. *The set of lines $\Lambda_1 \times \mathbb{R}$, $\mathbb{R} \times \Lambda_2$ refer to the case of blowing-up solutions in which one component remains bounded, so it is not quantized. The quantization of the blowing up component was obtained in [1] for the singular scalar case. See also [28] for the regular Toda system.*

Instead, the set Λ_0 refers to couples (u_1, u_2) for which both components blow-up. Observe that $\Lambda_{\alpha_1,\alpha_2}$ is finite, and it coincides with the five elements $(4\pi, 0)$, $(0, 4\pi)$, $(8\pi, 4\pi)$, $(4\pi, 8\pi)$, $(8\pi, 8\pi)$ when both α_1 and α_2 vanish. Then, Λ_0 is a discrete set.

In particular, Λ is a closed set in $\mathbb{R}^2$ with zero Lebesgue measure.

3. The topological set and test functions

We begin this section with an easy topological result, which will be essential in our analysis:

Lemma 3.1. *Let Σ be a compact surface not homeomorphic to $\mathbb{S}^2$ nor $\mathbb{RP}^2$. Then, there exist two simple closed curves $\gamma_1, \gamma_2 \subseteq \Sigma$ satisfying (see Figure 1)*

(1) *γ_1, γ_2 do not intersect each other nor any of the singular points p_j, $j = 1 \dots m$;*
(2) *there exist global retractions $\Pi_i : \Sigma \to \gamma_i$, $i = 1, 2$.*

Proof. The result is quite evident for the torus. For the Klein bottle, consider its fundamental square $ABAB^{-1}$. We can take γ_1 as the segment B, and γ_2 a segment parallel to B and passing by the center of the square. The retractions are given by just freezing one cartesian component of the point in the square.

Observe that we can assume that p_i do not intersect those curves.

For any other Σ under the conditions of the lemma, Dyck's Theorem implies that it is the connected sum of a torus and another compact surface, $\Sigma = \mathbb{T}^2 \# M$. Then, one can modify the retractions of the torus so that they are constant on M. □

Remark 3.1. *Observe that each curve γ_i generates a free subgroup in the first co-homology group of Σ. Then, Lemma 3.1 cannot hold for $\mathbb{S}^2$ or $\mathbb{RP}^2$.*

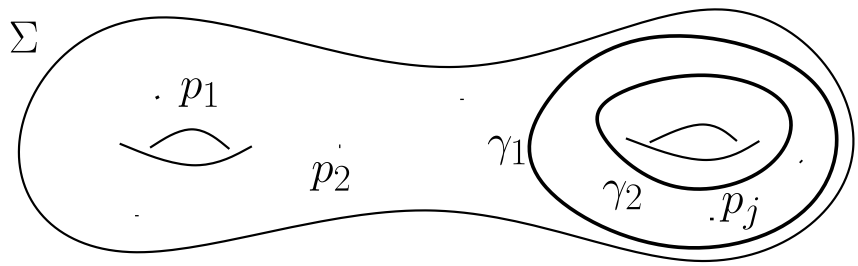

FIGURE 1. The curves γ_i

For $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$ we would like to build a family of test functions modelled on the topological join $(\gamma_1)_k * (\gamma_2)_l$, involving the formal barycenters of the curves γ_1, γ_2, see (14).

Remark 3.2. *Since each γ_i is homeomorphic to S^1, it follows from Proposition 3.2 in [5] that $(\gamma_1)_k$ is homeomorphic to S^{2k-1} and $(\gamma_2)_l$ to S^{2l-1} (the homotopy equivalence was found before in [33]). As it is well-known, the join $S^m * S^n$ is homeomorphic to S^{m+n+1} (see for example [25]), and therefore $(\gamma_1)_k * (\gamma_2)_l$ is homeomorphic to the sphere $S^{2k+2l-1}$.*

Let $\zeta = (1-r)\sigma_2 + r\sigma_1 \in (\gamma_1)_k * (\gamma_2)_l$, where:

$$\sigma_1 := \sum_{i=1}^{k} t_i \delta_{x_i} \in (\gamma_1)_k \qquad \text{and} \qquad \sigma_2 := \sum_{j=1}^{l} s_j \delta_{y_j} \in (\gamma_2)_l.$$

Our goal is to define a test function modelled on any $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, depending on a positive parameter λ and belonging to low sub-levels of J for large λ, that is a map

$$\Phi_\lambda : (\gamma_1)_k * (\gamma_2)_l \to J_\rho^{-L}; \qquad L \gg 0.$$

For any $\lambda > 0$, we define the parameters

$$\lambda_{1,r} = (1-r)\lambda; \qquad \lambda_{2,r} = r\lambda.$$

We introduce $\Phi_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ whose components are defined by

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2 - \frac{1}{2} \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2 \\ -\frac{1}{2} \log \sum_{i=1}^{k} t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2 + \log \sum_{j=1}^{l} s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2 \end{pmatrix}. \tag{17}$$

Notice that when $r = 0$ we have that $\lambda_{2,r} = 0$, and therefore, as $\sum_{j=1}^{l} s_j = 1$, the second terms in both rows are constant, independent of σ_2; a similar consideration holds when $r = 1$. These arguments imply that the function Φ_λ is indeed well defined on $(\gamma_1)_k * (\gamma_2)_l$.

We have then the following result.

Proposition 3.3. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$ and $\rho_2 \in (4l\pi, 4(l+1)\pi)$. Then one has*

$$J_\rho(\varphi_{\lambda,\zeta}) \to -\infty \qquad \text{as } \lambda \to +\infty \qquad\qquad \text{uniformly in } \zeta \in (\gamma_1)_k * (\gamma_2)_l.$$

PROOF. We define $v_1, v_2 : \Sigma \to \mathbb{R}$ as follows;

$$v_1(x) = \log \sum_{i=1}^{k} t_i \left(\frac{1}{1 + \lambda_{1,r}^2 d(x, x_i)^2} \right)^2, \qquad v_2(x) = \log \sum_{j=1}^{l} s_j \left(\frac{1}{1 + \lambda_{2,r}^2 d(x, y_j)^2} \right)^2.$$

With this notation the components of $\varphi(x)$ are given by

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} v_1(x) - \frac{1}{2} v_2(x) \\ -\frac{1}{2} v_1(x) + v_2(x) \end{pmatrix}.$$

We first prove two estimates on the gradients of v_1 and v_2.

$$|\nabla v_i(x)| \le C\lambda_{i,r}, \qquad \text{for every } x \in \Sigma \text{ and } r \in [0,1], \quad i = 1, 2, \tag{18}$$

where C is a constant independent of λ, $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, and

$$|\nabla v_i(x)| \le \frac{4}{d_{\,i,min}(x)}, \qquad \text{for every } x \in \Sigma, \quad i = 1, 2, \tag{19}$$

where $d_{1,min}(x) = \min_{i=1,\dots,k} d(x, x_i)$ and $d_{2,min}(x) = \min_{j=1,\dots,l} d(x, y_j)$.

We show the inequalities just for v_1, as for v_2 the proof is similar. We have that

$$\nabla v_1(x) = -2\lambda_{1,r}^2 \frac{\sum_{i=1}^{k} t_i \big(1 + \lambda_{1,r}^2 d^2(x, x_i)\big)^{-3} \nabla\big(d^2(x, x_i)\big)}{\sum_{j=1}^{k} t_j \big(1 + \lambda_{1,r}^2 d^2(x, x_j)\big)^{-2}}.$$

Using the estimate $\big|\nabla\big(d^2(x, x_i)\big)\big| \le 2d(x, x_i)$ and the following inequality

$$\frac{\lambda_{1,r}^2 d(x, x_i)}{1 + \lambda_{1,r}^2 d^2(x, x_i)} \le C\lambda_{1,r}, \qquad i = 1, \dots, k,$$

with C a fixed constant, we obtain (18). For proving (19) we observe that if $\lambda_{1,r} = 0$ the inequality is trivially satisfied. If instead $\lambda_{1,r} > 0$ we have

$$\begin{aligned} |\nabla v_1(x)| &\le 4\lambda_{1,r}^2 \frac{\sum_{i=1}^{k} t_i \big(1 + \lambda_{1,r}^2 d^2(x, x_i)\big)^{-3} d(x, x_i)}{\sum_{j=1}^{k} t_j \big(1 + \lambda_{1,r}^2 d^2(x, x_j)\big)^{-2}} \le 4\lambda_{1,r}^2 \frac{\sum_{i=1}^{k} t_i \big(1 + \lambda_{1,r}^2 d^2(x, x_i)\big)^{-2} \frac{d(x,x_i)}{\lambda_{1,r}^2 d^2(x,x_i)}}{\sum_{j=1}^{k} t_j \big(1 + \lambda_{1,r}^2 d^2(x, x_y)\big)^{-2}} \\ &\le 4 \frac{\sum_{i=1}^{k} t_i \big(1 + \lambda_{1,r}^2 d^2(x, x_i)\big)^{-2} \frac{1}{d_{\,1,min}(x)}}{\sum_{j=1}^{k} t_j \big(1 + \lambda_{1,r}^2 d^2(x, x_j)\big)^{-2}} = \frac{4}{d_{\,1,min}(x)}, \end{aligned}$$

which proves (19).

We consider now the Dirichlet part of the functional J_ρ. Taking into account the definition of φ_1, φ_2 we have

$$\begin{aligned} \int_\Sigma Q(\varphi_1, \varphi_2)\, dV_g &= \frac{1}{3} \int_\Sigma \big(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2 + \nabla\varphi_1 \cdot \nabla\varphi_2\big)\, dV_g \\ &= \frac{1}{3} \int_\Sigma \left(|\nabla v_1|^2 + \frac{1}{4}|\nabla v_2|^2 - \nabla v_1 \cdot \nabla v_2\right) dV_g + \frac{1}{3} \int_\Sigma \left(|\nabla v_2|^2 + \frac{1}{4}|\nabla v_1|^2 - \nabla v_2 \cdot \nabla v_1\right) dV_g + \\ &+ \frac{1}{3} \int_\Sigma \left(-\frac{1}{2}|\nabla v_1|^2 - \frac{1}{2}|\nabla v_2|^2 + \frac{5}{4}(\nabla v_1 \cdot \nabla v_2)\right) dV_g \\ &= \frac{1}{4} \int_\Sigma |\nabla v_1|^2\, dV_g + \frac{1}{4} \int_\Sigma |\nabla v_2|^2\, dV_g - \frac{1}{4} \int_\Sigma \nabla v_1 \cdot \nabla v_2\, dV_g. \end{aligned}$$

We first observe that the part involving the mixed term $\nabla v_1 \cdot \nabla v_2$ is bounded by a constant depending only on Σ. Indeed, we introduce the sets

$$A_i = \left\{ x \in \Sigma : d(x, x_i) = \min_{j=1}^{k} d(x, x_j) \right\}. \tag{20}$$

Using then (19) we have

$$\begin{aligned} \int_\Sigma \nabla v_1 \cdot \nabla v_2 \, dV_g &\leq \int_\Sigma |\nabla v_1||\nabla v_2| \, dV_g \leq 16 \int_\Sigma \frac{1}{d_{1,min}(x)\, d_{2,min}(x)} \, dV_g(x) \\ &\leq 16 \sum_{i=1}^{k} \int_{A_i} \frac{1}{d(x, x_i)\, d_{2,min}(x)} \, dV_g(x). \end{aligned}$$

We take now $\delta > 0$ such that

$$\delta = \frac{1}{2} \min \left\{ \min_{i \in \{1,\dots k\}, j \in \{1,\dots l\}} d(x_i, y_j), \quad \min_{m,\ n \in \{1,\dots k\}, m \neq n} d(x_m, x_n) \right\}$$

and we split each A_i into $A_i = B_\delta(x_i) \cup (A_i \setminus B_\delta(x_i)), i = 1, \dots k$. By a change of variables and exploiting the fact that $d_{2,min}(x) \geq \frac{1}{C}$ in $B_\delta(x_i)$ we obtain

$$\sum_{i=1}^{k} \int_{B_\delta(x_i)} \frac{1}{d(x, x_i)\, d_{2,min}(x)} \, dV_g(x) \leq C.$$

Using the same argument for the part $A_i \setminus B_\delta(x_i)$ with some modifications and exchanging the role of $d_{1,min}$ and $d_{2,min}$ we finally deduce that

$$\int_\Sigma \nabla v_1 \cdot \nabla v_2 \, dV_g \leq C. \tag{21}$$

We want now to estimate the remaining part of the Dirichlet energy. For convenience we treat the cases $r = 0$ and $r = 1$ separately. Consider first the case $r = 0$: we then have $\nabla v_2(x) = 0$ and we get

$$\int_\Sigma Q(\varphi_1, \varphi_2) \, dV_g = \frac{1}{4} \int_\Sigma |\nabla v_1(x)|^2 \, dV_g(x).$$

We divide now the integral into two parts;

$$\frac{1}{4} \int_\Sigma |\nabla v_1(x)|^2 \, dV_g(x) = \frac{1}{4} \int_{\bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x) + \frac{1}{4} \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x).$$

From (18) we deduce that

$$\int_{\bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2 \, dV_g(x) \leq C.$$

Using then (19) for the second part of the integral, recalling the definition (20) of the sets A_i, one finds that

$$\begin{aligned} \frac{1}{4} \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} |\nabla v_1(x)|^2(x) \, dV_g &\leq 4 \int_{\Sigma \setminus \bigcup_i B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{1,min}^2(x)} \, dV_g(x) + C \\ &\leq 4 \sum_{i=1}^{k} \int_{A_i \setminus B_{\frac{1}{\lambda}}(x_i)} \frac{1}{d_{1,min}^2(x)} \, dV_g(x) + C \\ &\leq 8k\pi \big(1 + o_\lambda(1)\big) \log \lambda + C, \end{aligned}$$

where $o_\lambda(1) \to 0$ as $\lambda \to +\infty$. Therefore we have

$$\int_\Sigma Q(\varphi_1, \varphi_2) \, dV_g \leq 8k\pi \big(1 + o_\lambda(1)\big) \log \lambda + C. \tag{22}$$

Reasoning as in [43], Proposition 4.2 part (ii), it is possible to show that

$$\int_\Sigma v_1 \, dV_g = -4\big(1 + o_\lambda(1)\big) \log \lambda; \qquad \log \int_\Sigma e^{v_1} \, dV_g = -2\big(1 + o_\lambda(1)\big) \log \lambda$$

$$\log \int_\Sigma e^{-\frac{1}{2} v_1} \, dV_g = 2\big(1 + o_\lambda(1)\big) \log \lambda,$$

and clearly

$$\int_\Sigma v_2\, dV_g = O(1); \qquad \log \int_\Sigma e^{v_2}\, dV_g = O(1); \qquad \log \int_\Sigma e^{-\frac12 v_2}\, dV_g = O(1).$$

Therefore we get

$$\int_\Sigma \varphi_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\lambda; \qquad \log \int_\Sigma e^{\varphi_1}\, dV_g = -2\big(1+o_\lambda(1)\big)\log\lambda;$$

$$\int_\Sigma \varphi_2\, dV_g = 2(1+o_\lambda(1))\log\lambda; \qquad \log \int_\Sigma e^{\varphi_2}\, dV_g = -2(1+o_\lambda(1))\log\lambda.$$

Inserting the latter equalities in the expression of the functional J_ρ and using the fact that $\widetilde{h}_i \geq \frac{1}{C}, i = 1,2$ outside a small neighbourhood of the singular points (which are avoided by the curves γ_1, γ_2), we obtain

$$J_\rho(\varphi_1,\varphi_2) \leq \big(8k\pi - 2\rho_1 + o_\lambda(1)\big)\log\lambda + C,$$

where C is independent of λ and σ_1, σ_2.

For the case $r = 1$, by the same argument we have that

$$J_\rho(\varphi_1,\varphi_2) \leq \big(8l\pi - 2\rho_2 + o_\lambda(1)\big)\log\lambda + C.$$

We consider now the case $r \in (0,1)$. By (21) the Dirichlet part can be estimated by

$$\int_\Sigma Q(\varphi_1,\varphi_2)\, dV_g \leq \frac14 \int_\Sigma |\nabla v_1(x)|^2\, dV_g(x) + \frac14 \int_\Sigma |\nabla v_2(x)|^2\, dV_g(x) + C.$$

For general r one can just substitute λ with $\lambda_{1,r}$ in (22) (and similarly for the v_2), to get the following estimate

$$\int_\Sigma Q(\varphi_1,\varphi_2)\, dV_g \leq 8k\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + 8l\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + C, \tag{23}$$

where $\delta_{1,r} > \delta > 0$ as $r \to 1$ and $\delta_{2,r} > \delta > 0$ as $r \to 0$, for some fixed δ. The same argument as for $r = 0,1$ leads to

$$\int_\Sigma v_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + O(1); \qquad \int_\Sigma v_2\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1),$$

therefore we obtain

$$\int_\Sigma \varphi_1\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + 2\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1), \tag{24}$$

$$\int_\Sigma \varphi_2\, dV_g = 2\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) - 4\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1). \tag{25}$$

We consider now the exponential term. We have

$$\int_\Sigma e^{\varphi_1}\, dV_g = \sum_{i=1}^k t_i \int_\Sigma \frac{1}{\big(1+\lambda_{1,r}^2 d(x,x_i)^2\big)^2} \left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,r}^2 d(x,y_j)^2\big)^2} \right)^{-\frac12} dV_g(x).$$

Clearly it is enough to estimate the term

$$\int_\Sigma \frac{1}{\big(1+\lambda_{1,r}^2 d(x,\overline{x})^2\big)^2} \left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,r}^2 d(x,y_j)^2\big)^2} \right)^{-\frac12} dV_g(x)$$

with $\overline{x} \in \{x_1, \dots x_k\}$. Letting $\delta = \dfrac{\min_j\{d(\overline{x},y_j)\}}{2}$ we divide the domain into two regions as follows: $\Sigma = B_\delta(\overline{x}) \cup (\Sigma \setminus B_\delta(\overline{x}))$. When we integrate in $B_\delta(\overline{x})$ we perform a change of variables for the part involving $\lambda_{1,r}$ and observing that $\frac1C \leq d(x,y_j) \leq C, j = 1,\dots,l$, for every $x \in B_\delta(\overline{x})$, we deduce

$$\int_{B_\delta(\overline{x})} \frac{1}{\big(1+\lambda_{1,r}^2 d(x,\overline{x})^2\big)^2} \left(\sum_{j=1}^l s_j \frac{1}{\big(1+\lambda_{2,r}^2 d(x,y_j)^2\big)^2} \right)^{-\frac12} dV_g(x) = \frac{\big(\lambda_{2,r}+\delta_{2,r}\big)^2}{\big(\lambda_{1,r}+\delta_{1,r}\big)^2}\big(1+O(1)\big).$$

On the other hand for the integral over $\Sigma \setminus B_\delta(\overline{x})$ we use that $\frac1C \leq d(x,\overline{x}) \leq C$ to get that this part is a higher-order term and can be absorbed by the latter estimate. Recall now that $\widetilde{h}_1$ stays bounded away from zero in a neighbourhood of the curve γ_1 (see the beginning of the section). Therefore, since the contribution of the integral outside a neighbourhood of γ_1 is negligible, we can conclude that

$$\log \int_\Sigma \widetilde{h}_1 e^{\varphi_1}\, dV_g = 2\log\big(\lambda_{2,r}+\delta_{2,r}\big) - 2\log\big(\lambda_{1,r}+\delta_{1,r}\big) + O(1). \tag{26}$$

Similarly we have that

$$\log \int_\Sigma \widetilde{h}_2 e^{\varphi_2}\, dV_g = 2\log\big(\lambda_{1,r}+\delta_{1,r}\big) - 2\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1). \tag{27}$$

Using the estimates (23), (24), (25), (26) and (27) we finally obtain

$$J_\rho(\varphi_1,\varphi_2) \leq \big(8k\pi - 2\rho_1 + o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + \big(8l\pi - 2\rho_2 + o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1).$$

Recalling that $\rho_1 > 4k\pi, \rho_2 > 4l\pi$ and observing that $\max\limits_{r\in[0,1]}\{\lambda_{1,r},\lambda_{2,r}\} \to +\infty$ as $\lambda \to \infty$, we conclude the proof. ■

4. Moser-Trudinger inequalities and topological join

In this section we are going to give an improved version of the Moser-Trudinger inequality (9), where the constant 4π can be replaced by an integer multiple under the assumption that the integral of $\widetilde{h}_i e^{u_i}$ is distributed on different sets with positive mutual distance. The improved inequality implies that if $J_\rho(u_1,u_2)$ attains very low values, then $\widetilde{h}_i e^{u_i}$ has to concentrate near a given number (depending on ρ_i) of points for some $i \in \{1,2\}$. As anticipated in the introduction, we will see that this induces a natural map from low sub-levels of J_ρ to the topological join of some sets of barycenters. This extends some analysis from [28] and [44], where the authors considered the case $\rho_2 < 4\pi$, and from [46], where both parameters belong to the range $(4\pi, 8\pi)$. We start with a covering lemma:

Lemma 4.1. *Let $\delta > 0$, $\theta > 0$, $k,l \in \mathbb{N}$ with $k \geq l$, $f_i \in L^1(\Sigma)$ be non-negative functions with $\|f_i\|_{L^1(\Sigma)} = 1$ for $i = 1,2$ and $\{\Omega_{1,i},\Omega_{2,j}\}_{i\in\{0,\dots,k\},j\in\{0,\dots,l\}} \subset \Sigma$ such that*

$$d(\Omega_{1,i},\Omega_{1,i'}) \geq \delta \qquad \forall\, i,\ i' \in \{0,\dots,k\} \text{ with } i \neq i';$$
$$d(\Omega_{2,j},\Omega_{2,j'}) \geq \delta \qquad \forall\, j,\ j' \in \{0,\dots,l\} \text{ with } j \neq j',$$

and

$$\int_{\Omega_{1,i}} f_1 dV_g \geq \theta \qquad \forall\, i \in \{0,\dots,k\};$$
$$\int_{\Omega_{2,j}} f_2 dV_g \geq \theta \qquad \forall\, j \in \{0,\dots,l\}.$$

Then, there exist $\overline{\delta} > 0$, $\overline{\theta} > 0$, independent of f_i, and $\{\Omega_n\}_{n=1}^k \subset \Sigma$ such that

$$d(\Omega_n,\Omega_n') \geq \overline{\delta} \qquad \forall\, n,\ n' \in \{0,\dots,k\} \text{ with } n \neq n'$$

and

$$|\Omega_n| \geq \overline{\theta} \qquad \forall\, n \in \{0,\dots,k\};$$
$$\int_{\Omega_n} f_1 dV_g \geq \overline{\theta} \qquad \forall\, n \in \{0,\dots,k\};$$
$$\int_{\Omega_n} f_2 dV_g \geq \overline{\theta} \qquad \forall\, n \in \{0,\dots,l\}.$$

Proof. We set $\overline{\delta} = \dfrac{\delta}{8}$ and consider the open cover $\big\{B_{\overline{\delta}}(x)\big\}_{x\in\Sigma}$ of Σ; by compactness, $\Sigma \subset \bigcup\limits_{h=1}^{H} B_{\overline{\delta}}(x_h)$ for some $\{x_h\}_{h=1}^H \subset \Sigma$, $H = H\left(\overline{\delta},\Sigma\right)$.
We choose $\{y_{1,i},y_{2,j}\}_{i\in\{0,\dots,k\},j\in\{0,\dots,l\}} \subset \{x_h\}_{h=1}^H$ such that

$$\int_{B_{\overline{\delta}}(y_{1,i})} f_1 dV_g = \max\left\{\int_{B_{\overline{\delta}}(x_h)} f_1 dV_g : \ B_{\overline{\delta}}(x_h)\cap\Omega_{1,i} \neq \emptyset\right\};$$
$$\int_{B_{\overline{\delta}}(y_{2,j})} f_2 dV_g = \max\left\{\int_{B_{\overline{\delta}}(x_h)} f_2 dV_g : \ B_{\overline{\delta}}(x_h)\cap\Omega_{2,j} \neq \emptyset\right\}$$

Since $d(y_{1,i},\Omega_{1,i}) < \overline{\delta}$, we have that $d(y_{1,i},y_{1,i'}) \geq 6\overline{\delta}$ for $i \neq i'$. Analogously, $d(y_{2,j},y_{2,j'}) \geq 6\overline{\delta}$ if $j \neq j'$. In particular, this implies that for any $i \in \{0,\dots,k\}$ there exists at most one $j(i)$ such that $d(y_{2,j(i)},y_{1,i}) < 3\overline{\delta}$. We relabel the index i so that for $i = 1, \dots l$ such $j(i)$ exists, and we relabel the index j so that $j(i) = i$. We now define:

$$\Omega_n := \begin{cases} B_{\overline{\delta}}(y_{1,n}) \cup B_{\overline{\delta}}(y_{2,n}) & \text{if } n \in \{0,\dots,l\} \\ B_{\overline{\delta}}(y_{1,n}) & \text{if } n \in \{l+1,\dots,k\}. \end{cases}$$

In other words, we make unions of balls $B_{\overline{\delta}}(y_{1,n}) \cup B_{\overline{\delta}}(y_{2,n})$ if they are close to each other: for separate balls, we make arbitrary unions. If $k > l$, the remaining balls are considered alone.

It is easy to check that those sets satisfy the theses of Lemma 4.1.

■

To show the improved Moser-Trudinger inequality, we will need a *localized* version of the inequality (9), which was proved in [46].

Lemma 4.2. *([46]) Let $\delta > 0$ and $\Omega \Subset \widetilde{\Omega} \subset \Sigma$ be such that $d\left(\Omega,\partial\widetilde{\Omega}\right) \geq \delta$.*
Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon,\delta)$ such that for any $u = (u_1,u_2) \in H^1(\Sigma)\times H^1(\Sigma)$

$$\log\int_\Omega e^{u_1 - f_{\widetilde{\Omega}} u_1 dV_g} dV_g + \log\int_\Omega e^{u_2 - f_{\widetilde{\Omega}} u_2 dV_g} dV_g \leq \frac{1}{4\pi}\int_{\widetilde{\Omega}} Q(u_1,u_2) dV_g + \varepsilon\int_\Sigma Q(u_1,u_2) dV_g + C.$$

Here comes the improved inequality: basically, if the *mass* of both $\widetilde{h}_1 e^{u_1}$ and $\widetilde{h}_2 e^{u_2}$ is spread respectively on at least $k+1$ and $l+1$ different sets, then the logarithms in (9) can be multiplied by $k+1$ and $l+1$ respectively.

Notice that this result was given in [44] in the case $l = 0$ and in [46] in the case $k = l = 1$.

Lemma 4.3. *Let $\delta > 0$, $\theta > 0$, $k, l \in \mathbb{N}$ and $\{\Omega_{1,i},\Omega_{2,j}\}_{i\in\{0,\dots,k\},j\in\{0,\dots,l\}} \subset \Sigma$ be such that*

$$d(\Omega_{1,i},\Omega_{1,i'}) \geq \delta \qquad \forall\, i,\ i' \in \{0,\dots,k\} \text{ with } i \neq i';$$
$$d(\Omega_{2,j},\Omega_{2,j'}) \geq \delta \qquad \forall\, j,\ j' \in \{0,\dots,l\} \text{ with } j \neq j'.$$

Then, for any $\varepsilon > 0$ there exists $C = C(\varepsilon,\delta,\theta,k,l,\Sigma)$ such that any $u = (u_1,u_2) \in H^1(\Sigma)\times H^1(\Sigma)$ satisfying

$$\int_{\Omega_{1,i}} \widetilde{h}_1 e^{u_1} dV_g \geq \theta\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\, i \in \{0,\dots,k\};$$
$$\int_{\Omega_{2,j}} \widetilde{h}_2 e^{u_2} dV_g \geq \theta\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\, j \in \{0,\dots,l\}$$

verifies

$$(k+1)\log\int_\Sigma \widetilde{h}_1 e^{u_1-\overline{u_1}} dV_g + (l+1)\log\int_\Sigma \widetilde{h}_2 e^{u_2-\overline{u_2}} dV_g \leq \frac{1+\varepsilon}{4\pi}\int_\Sigma Q(u_1,u_2) dV_g + C.$$

Proof. In the proof we assume that $\overline{u_1} = \overline{u_2} = 0$. After relabelling the indexes, we can suppose $k \geq l$ and apply Lemma 4.1 with $f_i = \dfrac{\widetilde{h}_i e^{u_i}}{\int_\Sigma \widetilde{h}_i e^{u_i} dV_g}$ to get $\{\Omega_j\}_{j=0}^k \subset \Sigma$ with

$$d(\Omega_i,\Omega_j) \geq \overline{\delta} \qquad \forall\, i,j \in \{0,\dots,k\} \text{ with } i \neq j$$

and

$$\int_{\Omega_i} \widetilde{h}_1 e^{u_1} dV_g \geq \overline{\theta}\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\, i \in \{0,\dots,k\};$$
$$\int_{\Omega_j} \widetilde{h}_2 e^{u_2} dV_g \geq \overline{\theta}\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\, j \in \{0,\dots,l\}.$$

Notice that:

$$\log\int_\Sigma \widetilde{h}_i e^{u_i} dV_g = \fint_{\widetilde{\Omega}_j} u_i dV_g + \log\int_\Sigma \widetilde{h}_1 e^{u_i - f_{\widetilde{\Omega}_j} u_i dV_g} dV_g,\ i = 1,2.$$

The average on $\widetilde{\Omega}_j$ can be estimated by Poincaré inequality:

$$\fint_{\widetilde{\Omega}_j} u_i dV_g \leq \frac{1}{\left|\widetilde{\Omega}_j\right|}\int_\Sigma |u_i| dV_g \leq C\left(\int_\Sigma |\nabla u_i|^2 dV_g\right)^{1/2} \leq C+\varepsilon\int_\Sigma |\nabla u_i|^2 dV_g,\ i=1,\ 2. \tag{28}$$

We now apply, for any $j\in\{0,\dots,k\}$ Lemma 4.2 with , $\Omega=\Omega_j$ and $\widetilde{\Omega}=\widetilde{\Omega}_j := \left\{x\in\Sigma : d(x,\Omega_j)<\frac{\overline{\delta}}{2}\right\}$: for $j\in\{0,\dots,l\}$ we get

$$\begin{aligned}(29)\qquad & \log\int_\Sigma \widetilde{h}_1 e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log\int_\Sigma \widetilde{h}_2 e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\ \leq\ & 2\log\frac{1}{\theta} + \log\int_{\Omega_j} \widetilde{h}_1 e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log\int_{\Omega_j} \widetilde{h}_2 e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\ \leq\ & C + \log\int_{\Omega_j} e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g + \log\int_{\Omega_j} e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\ \leq\ & C + \frac{1}{4\pi}\int_{\widetilde{\Omega}_j} Q(u_1,u_2) dV_g + \varepsilon\int_\Sigma Q(u_1,u_2) dV_g,\ j=1,\dots l.\end{aligned}$$

For $j\in\{l+1,\dots,k\}$ we have

$$\begin{aligned}(30)\qquad & \log\int_\Sigma \widetilde{h}_1 e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \leq \log\frac{1}{\theta} + \left\|\widetilde{h}_1\right\|_{L^\infty(\Sigma)} + \log\int_{\Omega_j} e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \\ \leq\ & C - \log\int_{\Omega_j} e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g + \frac{1}{4\pi}\int_{\widetilde{\Omega}_j} Q(u_1,u_2) dV_g + \varepsilon\int_\Sigma Q(u_1,u_2) dV_g.\end{aligned}$$

The exponential term on the second component can be estimated by using Jensen's inequality:

$$\begin{aligned}(31)\qquad & \log\int_{\Omega_j} e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g = \log|\Omega_j| + \log\fint_{\Omega_j} e^{u_2-\fint_{\widetilde{\Omega}_j} u_2 dV_g} dV_g \\ \geq\ & \log|\Omega_j| \geq -C.\end{aligned}$$

Putting together (31) and (32), we have:

$$\log\int_\Sigma \widetilde{h}_1 e^{u_1-\fint_{\widetilde{\Omega}_j} u_1 dV_g} dV_g \leq \frac{1}{4\pi}\int_{\widetilde{\Omega}_j} Q(u_1,u_2) dV_g + \varepsilon\int_\Sigma Q(u_1,u_2) dV_g + C,\ j=l+1\dots k. \tag{32}$$

Summing over all $j\in\{0,\dots,k\}$ and taking into account (29), (32), we obtain the result, renaming ε appropriately. ■

We will now use a technical result that gives sufficient conditions to apply Lemma 4.3. Its proof can be found for instance in [19, 44].

Lemma 4.4. *([44, 46]) Let $f\in L^1(\Sigma)$ be a non-negative function with $\|f\|_{L^1(\Sigma)}=1$ and let $m\in\mathbb{N}$ be such that there exist $\varepsilon>0,\ s>0$ with*

$$\int_{\bigcup_{j=0}^m B_s(x_j)} f dV_g < 1-\varepsilon \qquad \forall\,\{x_j\}_{j=0}^m\subset\Sigma.$$

Then there exist $\overline{\varepsilon}>0,\ \overline{s}>0$, not depending on f, and $\{\overline{x}_j\}_{j=1}^m\subset\Sigma$ satisfying

$$\int_{B_{\overline{s}}(\overline{x}_j)} f dV_g > \overline{\varepsilon} \qquad \forall\, j\in\{1,\dots,m\},$$
$$B_{2\overline{s}}(\overline{x}_i)\cap B_{2\overline{s}}(\overline{x}_j)=\emptyset \qquad \forall\, i,j\in\{1,\dots,m\}, i\neq j.$$

Now we have enough tools to obtain information on the structure of very low sub-levels of J_ρ:

Lemma 4.5. *Suppose $\rho_1\in(4k\pi,4(k+1)\pi)$ and $\rho_2\in(4l\pi,4(l+1)\pi)$. Then, for any $\varepsilon>0,\ s>0$, there exists $L=L(\varepsilon,s)>0$ such that for any $u\in J_\rho^{-L}$ there are either some $\{x_i\}_{i=1}^k\subset\Sigma$ verifying*

$$\frac{\int_{\bigcup_{i=1}^k B_s(x_i)} \widetilde{h}_1 e^{u_1} dV_g}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} \geq 1-\varepsilon$$

or some $\{y_j\}_{j=1}^l \subset \Sigma$ *verifying*

$$\frac{\int_{\bigcup_{j=1}^l B_s(y_j)} \widetilde{h}_2 e^{u_2} dV_g}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} \geq 1 - \varepsilon.$$

Proof. Suppose by contradiction that the statement is not true, that is there are $\varepsilon_1, \varepsilon_2 > 0,\ s_1, s_2 > 0$, and $\{u_n = (u_{1,n}, u_{2,n})\}_{n\in\mathbb{N}} \subset H^1(\Sigma) \times H^1(\Sigma)$ such that $J_\rho(u_{1,n}, u_{2,n}) \underset{n\to+\infty}{\longrightarrow} -\infty$ and

$$\frac{\int_{\bigcup_{i=1}^k B_{s_1}(x_i)} \widetilde{h}_1 e^{u_{1,n}} dV_g}{\int_\Sigma \widetilde{h}_1 e^{u_{1,n}} dV_g} < 1 - \varepsilon_1; \qquad \frac{\int_{\bigcup_{j=1}^l B_{s_2}(y_j)} \widetilde{h}_2 e^{u_{2,n}} dV_g}{\int_\Sigma \widetilde{h}_2 e^{u_{2,n}} dV_g} < 1 - \varepsilon_2, \qquad \forall\ \{x_i\}_{i=1}^k, \{y_j\}_{j=1}^l \subset \Sigma.$$

Then, we may apply twice Lemma 4.4 with $f = \dfrac{\widetilde{h}_i e^{u_i}}{\int_\Sigma \widetilde{h}_i e^{u_i} dV_g}$, $\widetilde{\varepsilon} = \varepsilon_i$, $\widetilde{s} = s_i$ and find $\overline{\varepsilon}_1, \overline{\varepsilon}_2 > 0,\ \overline{s}_1, \overline{s}_2 > 0$ and $\{\overline{x}_i\}_{i=0}^k, \{\overline{y}_j\}_{j=0}^l$ with

$$\int_{B_{\overline{s}_1}(\overline{x}_i)} \widetilde{h}_1 e^{u_1} dV_g \geq \overline{\varepsilon}_1 \int_\Sigma \widetilde{h}_1 e^{u_1} dV_g \qquad \forall\ i \in \{0, \dots, k\};$$

$$\int_{B_{\overline{s}_2}(\overline{y}_j)} \widetilde{h}_2 e^{u_2} dV_g \geq \overline{\varepsilon}_2 \int_\Sigma \widetilde{h}_2 e^{u_2} dV_g \qquad \forall\ j \in \{0, \dots, l\},$$

and

$$B_{2\overline{s}_1}(\overline{x}_i) \cap B_{2\overline{s}_1}(\overline{x}_j) = \emptyset \qquad \forall\ i, j \in \{0, \dots, k\} \text{ with } i \neq j;$$

$$B_{2\overline{s}_2}(\overline{y}_j) \cap B_{2\overline{s}_2}(\overline{y}_j) = \emptyset \qquad \forall\ i, j \in \{0, \dots, l\} \text{ with } i \neq j.$$

Hence, we obtain an improved Moser-Trudinger inequality for $u_n = (u_{1,n}, u_{2,n})$ applying Lemma 4.3 with $\widetilde{\delta} := 2\min\{\overline{s}_1, \overline{s}_2\}$, $\widetilde{\theta} := \min\{\overline{\varepsilon}_1, \overline{\varepsilon}_2\}$ and $\Omega_{1,i} := B_{\overline{s}_1}(\overline{x}_i)$, $\Omega_{2,j} := B_{\overline{s}_2}(\overline{y}_j)$.
Moreover, Jensen's inequality gives

$$\int_\Sigma \widetilde{h}_i e^{u_{i,n} - \overline{u_{i,n}}} dV_g = \int_\Sigma e^{\log \widetilde{h}_i + u_{i,n} - \overline{u_{i,n}}} dV_g \geq e^{\int_\Sigma \log \widetilde{h}_i dV_g},$$

so, choosing

$$\widetilde{\varepsilon} \in \left(0, \min\left\{\frac{4\pi(k+1)}{\rho_1} - 1, \frac{4\pi(l+1)}{\rho_2} - 1\right\}\right)$$

we get

$$\begin{aligned}
-\infty \underset{n\to+\infty}{\longleftarrow}\ & J_\rho(u_{1,n}, u_{2,n}) \\
\geq\ & \left(\frac{4\pi(k+1)}{1+\widetilde{\varepsilon}} - \rho_1\right) \log \int_\Sigma \widetilde{h}_1 e^{u_{1,n} - \overline{u_{1,n}}} dV_g \\
+\ & \left(\frac{4\pi(l+1)}{1+\widetilde{\varepsilon}} - \rho_2\right) \log \int_\Sigma \widetilde{h}_2 e^{u_{2,n} - \overline{u_{2,n}}} dV_g - C \\
\geq\ & \left(\frac{4\pi(k+1)}{1+\widetilde{\varepsilon}} - \rho_1\right) \int_\Sigma \log \widetilde{h}_1 dV_g + \left(\frac{4\pi(l+1)}{1+\widetilde{\varepsilon}} - \rho_2\right) \int_\Sigma \log \widetilde{h}_2 dV_g - C \\
\geq\ & -C
\end{aligned}$$

that is a contradiction. ■

An immediate consequence of the previous Lemma is that at least one of the two $\widetilde{h}_i e^{u_i}$'s (once normalized in L^1) has to be very close respectively to the sets of k-barycenters or l-barycenters over Σ:

Proposition 4.6. *Suppose* $\rho_1 \in (4k\pi, 4(k+1)\pi)$ *and* $\rho_2 \in (4l\pi, 4(l+1)\pi)$. *Then, for any* $\varepsilon > 0$, *there exists* $L > 0$ *such that any* $u \in J_\rho^{-L}$ *verifies either*

$$\mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) < \varepsilon \qquad \text{or} \qquad \mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right) < \varepsilon.$$

PROOF. We apply Lemma 4.5 with $\widetilde{\varepsilon} = \frac{\varepsilon}{4}$, $\widetilde{s} = \frac{\varepsilon}{2}$; it is not restrictive to suppose that the first alternative occurs and that $\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g = 1$. Hence we get L and $\{x_i\}_{i=1}^k$ and we define, for such an $u = (u_1, u_2) \in J_\rho^{-L}$,

$$\sigma_1(u) = \sum_{i=1}^k t_i(u)\delta_{x_i} \in \Sigma_k \qquad \text{where } t_i(u) = \int_{B_{\widetilde{s}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g + \frac{1}{k}\int_{\Sigma\setminus\bigcup_{j=1}^{k} B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g.$$

Then, for any $\phi \in Lip(\Sigma)$,

$$\begin{aligned}
&\left|\int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| = \\
= &\int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)} \widetilde{h}_1 e^{u_1}\phi dV_g \le \int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)} \widetilde{h}_1 e^{u_1} dV_g \|\phi\|_{L^\infty(\Sigma)} < \widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)} \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| \\
= &\left|\int_{\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)} \widetilde{h}_1 e^{u_1}\phi dV_g - \sum_{i=1}^k\left(\int_{B_{\widetilde{s}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g + \frac{1}{k}\int_{\Sigma\setminus\bigcup_{j=1}^k B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g\right)\phi(x_i)\right| \\
= &\left|\int_{\bigcup_{i=1}^k\left(B_{\widetilde{s}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{s}}(x_j)\right)} \widetilde{h}_1 e^{u_1}(\phi - \phi(x_i)) dV_g - \int_{\Sigma\setminus\bigcup_{j=1}^k B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g \phi(x_i)\right| \\
\le &\ \widetilde{s}\|\nabla\phi\|_{L^\infty(\Sigma)}\int_{\bigcup_{i=1}^k\left(B_{\widetilde{s}}(x_i)\setminus\bigcup_{j=1}^{i-1} B_{\widetilde{s}}(x_j)\right)} \widetilde{h}_1 e^{u_1} dV_g + \|\phi\|_{L^\infty(\Sigma)}\int_{\Sigma\setminus\bigcup_{j=1}^k B_{\widetilde{s}}(x_j)} \widetilde{h}_1 e^{u_1} dV_g \\
< &\ \widetilde{s}\|\nabla\phi\|_{L^\infty(\Sigma)} + \widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)}.
\end{aligned}$$

Hence we can conclude the proof:

$$\begin{aligned}
&\mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) \le \mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \sigma_1(u)\right) = \sup_{\|\phi\|_{Lip(\Sigma)}=1}\left|\int_\Sigma\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| \\
= &\sup_{\|\phi\|_{Lip(\Sigma)}=1}\left|\int_{\Sigma\setminus\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| \\
+ &\sup_{\|\phi\|_{Lip(\Sigma)}=1}\left|\int_{\bigcup_{i=1}^k B_{\widetilde{s}}(x_i)}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - \sigma_1(u)\right)\phi dV_g\right| \\
< &\sup_{\|\phi\|_{Lip(\Sigma)}=1} 2\widetilde{\varepsilon}\|\phi\|_{L^\infty(\Sigma)} + \widetilde{s}\|\nabla\phi\|_{L^\infty(\Sigma)} \le 2\widetilde{\varepsilon} + \widetilde{s} = \varepsilon,
\end{aligned}$$

as desired. ■

With the previous estimates, it is now easy to define a projection map in the following form:

Proposition 4.7. *Suppose $\rho_1 \in (4k\pi, 4(k+1)\pi)$, $\rho_2 \in (4l\pi, 4(l+1)\pi)$ and let Φ_λ be as in (17). Then for L sufficiently large there exists a continuous map*

$$\Psi : J_\rho^{-L} \to (\gamma_1)_k * (\gamma_2)_l$$

such that the composition

$$(\gamma_1)_k * (\gamma_2)_l \quad \xrightarrow{\Phi_\lambda} \quad J_\rho^{-L} \quad \xrightarrow{\Psi} \quad (\gamma_1)_k * (\gamma_2)_l$$

is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$ provided that λ is large enough.

The rest of this section is devoted to the proof of this proposition.

By Propositions 4.6 and 4.7 we know that either $\psi_k\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}\right)$ or $\psi_l\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}\right)$ is well defined (or both), since either $\mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right) < \varepsilon$ or $\mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right) < \varepsilon$ (or both).

We then set

$$d_1 = \mathbf{d}\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}, \Sigma_k\right); \qquad d_2 = \mathbf{d}\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}, \Sigma_l\right),$$

and consider a function $\widetilde{r} = \widetilde{r}(d_1, d_2)$ defined as

$$\widetilde{r}(d_1, d_2) = f\left(\frac{d_1}{d_1 + d_2}\right), \tag{33}$$

where f is such that

$$f(z) = \begin{cases} 0 & \text{if } z \in [0, 1/4], \\ 2z - \frac{1}{2} & \text{if } z \in (1/4, 3/4), \\ 1 & \text{if } z \in [3/4, 1]. \end{cases} \tag{34}$$

Consider the global retractions $\Pi_1 : \Sigma \to \gamma_1$ and $\Pi_2 : \Sigma \to \gamma_2$ given in Lemma 3.1, and define:

$$\Psi(u_1, u_2) = (1 - \widetilde{r})(\Pi_1)_* \psi_k\left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g}\right) + \widetilde{r}(\Pi_2)_* \psi_l\left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g}\right), \tag{35}$$

where $(\Pi_i)_*$ stands for the push-forward of the map Π_i. Notice that when one of the two ψ's is not defined the other necessarily is, and the map is well defined by the equivalence relation.

In what follows, we are going to need the following auxiliary lemma:

Lemma 4.8. *Given $n \in \mathbb{N}$, define χ_λ as $\chi_\lambda(x) = \sum_{i=1}^n t_i \left(\frac{\lambda}{1 + \lambda^2 d(x, x_i)^2}\right)^2$. Take a L^∞ function $\tau : \Sigma \to \mathbb{R}$ satisfying:*

i) *$\tau(x) > m > 0$ for all $x \in B(x_i, \delta)$.*
ii) *$|\tau(x)| \le M$ for all $x \in \Sigma$.*

Then, there exist constants $c > 0$, $C > 0$ depending only on Σ, m, M, such that for every $\lambda > 0$,

$$c_0 \min\left\{1, \frac{1}{\lambda}\right\} < \mathbf{d}\left(\frac{\tau\,\chi_\lambda}{\int_\Sigma \tau\,\chi_\lambda\, dV_g}, \Sigma_n\right) < \frac{C_0}{\lambda}.$$

PROOF.

We show the proof for $n = 1$; the general case uses the same ideas and will be skipped. We also assume $\lambda > 1$. First of all, observe that

$$C > \int_\Sigma \chi_\lambda(x)\, dV_g(x) > c > 0$$

for some positive constants c, C.

For the upper estimate, it suffices to show that for any f Lipschitz, $\|f\|_{Lip(\Sigma)} \le 1$,

$$\int_\Sigma \tau(x)\left(\frac{\lambda}{1 + \lambda^2 d(x, x_0)^2}\right)^2 (f(x) - f(x_0))\, dV_g(x) \le \frac{C}{\lambda}.$$

Indeed, by ii),

$$\int_{(B_\delta(x_0))^c} \tau(x)\left(\frac{\lambda}{1 + \lambda^2 d(x, x_0)^2}\right)^2 dV_g(x) \le \frac{C}{\lambda^2},$$

and using geodesic coordinates x centered at x_0, we find

$$\left|\int_{B_\delta(x_0)} \tau(x)\left(\frac{\lambda}{1 + \lambda^2 d(x, x_0)^2}\right)^2 (f(x) - f(x_0))\, dV_g(x)\right|$$

$$\le C \int_{B_{\delta\lambda}(0))} \tau\left(x_0 + \frac{y}{\lambda}\right)\left(\frac{1}{1 + |y|^2}\right)^2 \left|f\left(x_0 + \frac{y}{\lambda}\right) - f(x_0)\right| dy$$

$$\leq C \int_{\mathbb{R}^2} \left(\frac{1}{1+y^2} \right)^2 \left| \frac{y}{\lambda} \right| dy \leq \frac{C}{\lambda}.$$

We now prove the estimate from below. Given $p \in \Sigma$, we estimate $\mathbf{d}(\chi_\lambda,\ \delta_p)$. Define the Lipschitz function $f(x) = d(x,p)$. We now show that:

$$\min_{p \in \Sigma} \int_\Sigma \tau(x) \left(\frac{\lambda}{1+\lambda^2 d(x,x_0)^2} \right)^2 d(x,p)\, dV_g(x) \geq \frac{c}{\lambda}.$$

As above, the integral in the exterior of $B_\delta(x_0)$ is negligible. Moreover, in the same coordinates as above, and taking into account i), we obtain:

$$\int_{B_\delta(x_0)} \tau(x) \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2} \right)^2 d(x,p)\, dV_g(x) \sim \int_{B_{\delta\lambda}(0)} \tau(x) \left(x_0 + \frac{y}{\lambda} \right) \left(\frac{1}{1+|y|^2} \right)^2 \left| x_0 - p + \frac{y}{\lambda} \right| dy$$

$$\geq \frac{m}{\lambda} \int_{B_\delta(0)} \left(\frac{1}{1+|y|^2} \right)^2 |y + \lambda(x_0 - p)|\ dy.$$

It suffices to show that we cannot choose p_λ so that

$$\int_{B_\delta(0)} \left(\frac{1}{1+|y|^2} \right)^2 |y + \lambda(x_0 - p_\lambda)|\ dx \to 0 \text{ as } \lambda \to +\infty. \tag{36}$$

Indeed, if $\lambda|x_0 - p_\lambda| \to +\infty$, the expression (36) diverges. If not, we can assume that $\lambda(x_0 - p_\lambda) \to z \in \mathbb{R}^2$. Then, (36) converges to

$$\int_{B_\delta(0)} \left(\frac{1}{1+|y|^2} \right)^2 |y + z|\ dx > 0.$$

which concludes the proof. ∎

From the previous lemma we deduce the following

Proposition 4.9. *Let φ_i be defined by (17). Then there exist constants $c > 0$, $C > 0$ such that for every $\lambda > 1$ and every $r \in (0,1)$ one has*

$$c_0 \min\left\{1, \frac{1}{\lambda_{1,r}}\right\} \leq \mathbf{d}\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1}\, dV_g}, \Sigma_k \right) \leq \frac{C_0}{\lambda_{1,r}}; \qquad c_0 \min\left\{1, \frac{1}{\lambda_{2,r}}\right\} \leq \mathbf{d}\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2}\, dV_g}, \Sigma_l \right) \leq \frac{C_0}{\lambda_{2,r}}.$$

PROOF. Clearly, it suffices to prove the estimates for φ_1 in the case $\lambda_{1,r} > 1$. By the normalization, it suffices to prove it to the function $\varsigma = \varphi_1 - 2\log(\lambda_{1,r} \max\{1,\ \lambda_{2,r}\})$.

Observe now that we can write $e^\varsigma = \chi_{\lambda_{1,r}}(x)\, \tau(x)$, with:

$$\tau(x) = \widetilde{h}_1(x) \left[\sum_{j=1}^{l} s_j \left(\frac{\max\{1,\ \lambda_{2,r}\}^2}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2 \right]^{-1/2}.$$

It suffices to show that τ satisfy the conditions of Lemma 4.8 to conclude.

∎

We are now in position to prove that the composition $\Psi \circ \Phi_\lambda$ is homotopic to the identity, where Ψ is as in (35) and $\Phi_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ is as in (17). Take $\zeta = (1-r)\sigma_1 + r\sigma_2 \in (\gamma_1)_k * (\gamma_2)_l$, with

$$\sigma_1 = \sum_{i=1}^{k} t_i \delta_{x_i}, \qquad \sigma_2 = \sum_{j=1}^{l} s_j \delta_{y_j}.$$

Set $d_1 = \mathbf{d}\left(\frac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}, \Sigma_k \right)$, $d_2 = \mathbf{d}\left(\frac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}, \Sigma_l \right)$. By the previous proposition and the definition of $\lambda_{1,r}, \lambda_{2,r}$, there exist constants $0 < c_0 < C_0$ such that

$$c_0 \min\left\{1, \frac{1}{\lambda(1-r)}\right\} \leq d_1 \leq \frac{C_0}{\lambda(1-r)}, \quad c_0 \min\left\{1, \frac{1}{\lambda r}\right\} \leq d_2 \leq \frac{C_0}{\lambda r}.$$

Observe then that at least one between d_1 and d_2 must be smaller than $\frac{2C_0}{\lambda}$. Given $\delta > 0$ sufficiently small, we have:

$$r < \delta \Rightarrow \begin{cases} \frac{d_1}{d_1+d_2} \leq \dfrac{\frac{C_0}{\lambda(1-r)}}{\frac{c_0}{\lambda(1-r)} + \frac{c_0}{\lambda r}} = \dfrac{C_0}{c_0} r & \text{if } \lambda r \geq 1; \\ \frac{d_1}{d_1+d_2} \leq \dfrac{\frac{C_0}{\lambda(1-r)}}{c_0 + \frac{c_0}{\lambda(1-r)}} \leq \dfrac{C_0}{c_0}\dfrac{1}{\lambda} & \text{if } \lambda r \leq 1. \end{cases}$$

In any case, by choosing λ, δ adequately, we obtain that $\widetilde{r} = 0$. This fact is important, since the projection $\psi_l\left(\dfrac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right)$ could not be well defined.

Analogously, we have that if $r > (1-\delta)$, then the projection $\psi_k\left(\dfrac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right)$ could not be well defined, but $\widetilde{r} = 1$. Moreover, if $\delta \leq r \leq (1-\delta)$, then $d_i \leq \frac{C}{\delta\lambda}$, and hence both projections $\psi_k\left(\dfrac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right)$, $\psi_l\left(\dfrac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right)$ are well defined.

Letting $\widetilde{\zeta}_\lambda = \Psi \circ \Phi_\lambda(\zeta) = (1-\widetilde{r}_\lambda)\widetilde{\sigma}_{1,\lambda} + \widetilde{r}_\lambda \widetilde{\sigma}_{2,\lambda}$, we consider the following homotopy:

$$H_1 : (0,1] \times ((\gamma_1)_k * (\gamma_2)_l) \to ((\gamma_1)_k * (\gamma_2)_l),$$
$$H_1(\mu, (1-r)\sigma_1 + r\sigma_2) = (1-r_{\mu,\lambda})\widetilde{\sigma}_{1,\frac{\lambda}{\mu}} + r_{\mu,\lambda}\widetilde{\sigma}_{2,\frac{\lambda}{\mu}},$$

where $r_{\mu,\lambda} = (1-\mu)f(r) + \mu\widetilde{r}_\lambda$, and f is given by (34). Observe that $H_1(1,\cdot) = \Psi \circ \Phi_\lambda$.

Suppose now μ tends to zero. Then, as λ is fixed, $\frac{\lambda}{\mu} \to +\infty$, and hence $\dfrac{\widetilde{h}_i e^{\varphi_{i,\frac{\lambda}{\mu}}}}{\int_\Sigma \widetilde{h}_i e^{\varphi_{i,\frac{\lambda}{\mu}}} dV_g} \rightharpoonup \sigma_i$. Proposition 2.2 implies that $\psi_k\left(\dfrac{\widetilde{h}_1 e^{\varphi_1}}{\int_\Sigma \widetilde{h}_1 e^{\varphi_1} dV_g}\right) \to \sigma_1$, $\psi_l\left(\dfrac{\widetilde{h}_2 e^{\varphi_2}}{\int_\Sigma \widetilde{h}_2 e^{\varphi_2} dV_g}\right) \to \sigma_2$. Since Π_i are retractions, we conclude that $\widetilde{\sigma}_{i,\frac{\lambda}{\mu}} \to \sigma_i$. In other words,

$$\lim_{\mu\to 0} H_1(\mu, (1-r)\sigma_1 + r\sigma_2) = (1-f(r))\sigma_1 + f(r)\sigma_2.$$

We now define:

$$H_2 : [0,1] \times ((\gamma_1)_k * (\gamma_2)_l) \to ((\gamma_1)_k * (\gamma_2)_l),$$
$$H_2(\mu, (1-r)\sigma_1 + r\sigma_2) = [1 - (\mu f(r) + (1-\mu)r)]\sigma_1 + (\mu f(r) + (1-\mu)r)\sigma_2.$$

The concatenation of H_1 and H_2 gives the desired homotopy.

5. Min-max scheme

We now introduce the variational scheme which yields existence of solutions: this remaining part follows the ideas of [18] (see also [42]).

By Proposition 3.3, given any $L > 0$, there exists λ so large that $J_\rho(\varphi_{\lambda,\zeta}) < -L$ for any $\zeta \in (\gamma_1)_k * (\gamma_2)_l$. We choose L so large that Proposition 4.7 applies: we then have that the following composition

$$(\gamma_1)_k * (\gamma_2)_l \xrightarrow{\Phi_\lambda} J_\rho^{-L} \xrightarrow{\Psi} (\gamma_1)_k * (\gamma_2)_l$$

is homotopic to the identity map. In this situation it is said that the set J_ρ^{-L} *dominates* $(\gamma_1)_k * (\gamma_2)_l$ (see [25], page 528). Since $(\gamma_1)_k * (\gamma_2)_l$ is not contractible, this implies that

$$\Phi_\lambda((\gamma_1)_k * (\gamma_2)_l) \text{ is not contractible in } J_\rho^{-L}.$$

Moreover, we can take λ larger so that $\Phi_\lambda((\gamma_1)_k * (\gamma_2)_l) \subset J_\rho^{-2L}$.

Define the topological cone with basis $(\gamma_1)_k * (\gamma_2)_l$ via the equivalence relation

$$\mathcal{C} = \frac{(\gamma_1)_k * (\gamma_2)_l \times [0,1]}{(\gamma_1)_k * (\gamma_2)_l \times \{0\}}:$$

notice that, since $(\gamma_1)_k * (\gamma_2)_l \simeq S^{2k+2l-1}$, then $\mathcal{C}$ is homeomorphic to a Euclidean ball of dimension $2k+2l$.

We now define the min-max value:

$$m = \inf_{\xi \in \Gamma} \max_{u \in \mathcal{C}} J(\xi(u)),$$

where

$$\Gamma = \{\xi : \mathcal{C} \to H^1(\Sigma) \times H^1(\Sigma) : \ \xi(\zeta) = \varphi_{\lambda,\zeta} \ \forall \ \zeta \in \partial\mathcal{C}\}. \tag{37}$$

Observe that $t\Phi_\lambda : \mathcal{C} \to H^1(\Sigma) \times H^1(\Sigma)$ belongs to Γ, so this is a non-empty set. Moreover,

$$\sup_{\zeta \in \partial\mathcal{C}} J_\rho(\xi(\zeta)) = \sup_{\zeta \in (\gamma_1)_k * (\gamma_2)_l} J_\rho(\varphi_{\lambda,\zeta}) \leq -2L.$$

We now show that $m \geq -L$. Indeed, $\partial\mathcal{C}$ is contractible in $\mathcal{C}$, and hence in $\xi(\mathcal{C})$ for any $\xi \in \Gamma$. Since $\partial\mathcal{C}$ is not contractible in J_ρ^{-L}, we conclude that $\xi(\mathcal{C})$ is not contained in J_ρ^{-L}. Being this valid for any arbitrary $\xi \in \Gamma$, we conclude that $m \geq -L$.

From the above discussion, the functional J_ρ satisfies the geometrical properties required by min-max theory. However, we cannot directly conclude the existence of a critical point, since it is not known whether the Palais-Smale condition holds or not. The conclusion needs a different argument, which has been used intensively (see for instance [18, 20]), so we will be sketchy.

We take $\nu > 0$ such that

$$[\rho_1 - 2\nu, \rho_1 + 2\nu] \times [\rho_2 - 2\nu, \rho_2 + 2\nu] \subset \mathbb{R}^2 \setminus \Lambda,$$

where Λ is the set defined as in Definition 2.4.

Consider now the parameter $\mu \in [-\nu, \nu]$. It is clear that the min-max scheme described above works uniformly for any μ in this range. In other words, for any $L > 0$, there exists λ large enough so that

$$\sup_{\zeta \in \partial\mathcal{C}} J_{\widetilde{\rho}}(\xi(\zeta)) < -2L; \qquad m_\mu := \inf_{\xi \in \Gamma} \sup_{\zeta \in \mathcal{C}} J_{\widetilde{\rho}}(\xi(\zeta)) \geq -L, \quad \widetilde{\rho} = (\rho_1 + \mu, \rho_2 + \mu). \tag{38}$$

In this way, we are led to a problem depending on the parameter μ that satisfies a uniform min-max structure. In this framework, the following Lemma is well-known, usually taking the name *monotonicity trick*. This technique was first used by Struwe in [56]; a first abstract version was made in [26] (see also [18, 40]).

Lemma 5.1. *There exists $\Upsilon \subset [-\nu, \nu]$ satisfying:*

1. $|[-\nu, \nu] \setminus \Upsilon| = 0$.
2. *For any $\mu \in \Upsilon$, the functional $J_{\widetilde{\rho}}$ possesses a bounded Palais-Smale sequence $(u_{1,n}, u_{2,n})_n$ at level m_μ.*

Conclusion. Consider first $\mu \in \Upsilon$. Passing to a subsequence, the bounded Palais-Smale sequence can be assumed to converge weakly. Standard arguments show that the weak limit is indeed strong and that it is a critical point of $J_{\widetilde{\rho}}$.

Consider now $\mu_n \in \Upsilon$, $\mu_n \to 0$, and let $(u_{1,n}, u_{2,n})$ denote the corresponding solutions. It is then sufficient to apply the compactness result in Theorem 2.5, which yields convergence of $(u_{1,n}, u_{2,n})$ to a solution of (5).

6. The mean field equation

In this section we consider the mean field equation, namely the Liouville-type equation

$$-\Delta u = \rho_1 \left(\frac{he^u}{\int_\Sigma he^u dV_g} - 1 \right) - \rho_2 \left(\frac{he^{-u}}{\int_\Sigma he^{-u} dV_g} - 1 \right) \tag{39}$$

where ρ_1, ρ_2 are two non-negative parameters and h is a smooth positive function. Applying the same analysis developed for the Toda system we give a general existence result.

This equation arises in mathematical physics as a mean field equation of the equilibrium turbulence with arbitrarily signed vortices. The mean field limit was first studied by Joyce and Montgomery [32] and by Pointin and Lundgren [54] by means of different statistical arguments. Later, many authors adopted this model, see for example [16], [39], [49] and the references therein. The case $\rho_1 = \rho_2$ plays also an important role in the study of constant mean curvature surfaces, see [62], [63].

Equation (39) has a variational structure with associated functional $\widetilde{I}_\rho : H^1(\Sigma) \to \mathbb{R}$, with $\rho = (\rho_1, \rho_2)$, defined by

$$\widetilde{I}_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 \, dV_g - \rho_1 \left(\log \int_\Sigma h\, e^u \, dV_g - \int_\Sigma u \, dV_g \right) - \rho_2 \left(\log \int_\Sigma h\, e^{-u} \, dV_g + \int_\Sigma u \, dV_g \right). \tag{40}$$

In [53] the authors derived a Moser-Trudinger inequality for e^u and e^{-u} simultaneously, namely

$$\log \int_\Sigma e^{u-\overline{u}} \, dV_g + \log \int_\Sigma e^{-u+\overline{u}} \, dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 \, dV_g + C,$$

with C depending only of Σ. By this result, solutions to (39) can be found immediately as global minima of the functional $\widetilde{I}_\rho$ whenever both ρ_1 and ρ_2 are less than 8π. For $\rho_i \geq 8\pi$ the existence problem becomes subtler and there are very few results.

The blow-up analysis of (39) was carried out in [53], [52] and [31], see in particular Theorem 1.1, Corollary 1.2 and Remark 4.5 in the latter paper. The following quantization property for a blow-up point $\overline{x}$ and a sequence $(u_n)_n$ of solutions relatively to $(\rho_{1,n}, \rho_{2,n})$ was obtained:

$$\lim_{r\to 0} \lim_{n\to+\infty} \rho_{1,n} \int_{B_r(\overline{x})} h\, e^{u_n} dV_g \in 8\pi\mathbb{N}, \qquad \lim_{r\to 0} \lim_{n\to+\infty} \rho_{2,n} \int_{B_r(\overline{x})} h\, e^{-u_n} dV_g \in 8\pi\mathbb{N}. \tag{41}$$

As for the Toda system, the case of multiples of 8π may indeed occur, see [23] and [24].

Let now define the set $\widetilde{\Lambda}$ by

$$\widetilde{\Lambda} = (8\pi\mathbb{N} \times \mathbb{R}) \cup (\mathbb{R} \cup 8\pi\mathbb{N}) \subseteq \mathbb{R}^2.$$

Combining (41) and the argument in Section 1 of [5] one finds the following result.

Theorem 6.1. *Let (ρ_1, ρ_2) be in a fixed compact set of $\mathbb{R}^2 \setminus \widetilde{\Lambda}$. Then the set of solutions to (39) is uniformly bounded in $C^{2,\beta}$ for some $\beta > 0$.*

Before proving our main result we collect here some known existence results. The first one is given in [31] and treats the case $\rho_1 \in (8\pi, 16\pi)$ and $\rho_2 < 8\pi$. Via a blow up analysis the authors proved existence of solutions on a smooth, bounded, non simply-connected domain Σ in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition. Later, this result is generalized in [65] to any compact surface without boundary by using variational methods. The strategy is carried out in the same spirit as in [43] and [44] for the Liouville equation (10) and the Toda system (5), respectively. The proof relies on some improved Moser-Trudinger inequalities obtained in [14]. The idea is that, in a certain sense, one can recover the topology of low sub-levels of the functional $\widetilde{I}_\rho$ just from the behaviour of e^u. Indeed the condition $\rho_2 < 8\pi$ guarantees that e^{-u} does not affect the variational structure of the problem.

The doubly supercritical regime, namely $\rho_i > 8\pi$, has to be attacked with a different strategy. The only existence result concerning this case has been proved in [27] via variational methods where the author adapted the analysis developed to study the Toda system in [46] for this framework. The main tool is an improved Moser-Trudinger inequality under suitable conditions on the centre of mass and the scale of concentration of both e^u and e^{-u}.

We will give here a general existence result.

Theorem 6.2. *Suppose Σ is not homeomorphic to $\mathbb{S}^2$ nor $\mathbb{RP}^2$, and that $\rho_i \notin 8\pi\mathbb{N}$ for $i = 1, 2$. Then (39) has a solution.*

The proof is an adaptation of the argument introduced for the Toda system. Roughly speaking the role of the function u_2 is played by $-u$.

We start by considering the topological set $(\gamma_1)_k * (\gamma_2)_l$, on which we will base the min-max scheme. We take then two curves $\gamma_1, \gamma_2 \in \Sigma$ with the same properties as before. Let $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, $\zeta = (1-r)\sigma_1 + r\sigma_2$, with

$$\sigma_1 := \sum_{i=1}^k t_i \delta_{x_i} \in (\gamma_1)_k \qquad \text{and} \qquad \sigma_2 := \sum_{j=1}^l s_j \delta_{y_j} \in (\gamma_2)_l.$$

We define now a test function labelled by $\zeta \in (\gamma_1)_k * (\gamma_2)_l$, namely for large L we will find a non-trivial map

$$\widetilde{\Phi}_\lambda : (\gamma_1)_k * (\gamma_2)_l \to \widetilde{I}_\rho^{-L}.$$

We set $\widetilde{\Phi}_\lambda(\zeta) = \varphi_{\lambda,\zeta}$ given by

$$\varphi_{\lambda,\zeta}(x) = \log \sum_{i=1}^k t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2 - \log \sum_{j=1}^l s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2,$$

where $\lambda_{1,r} = (1-r)\lambda, \lambda_{2,r} = r\lambda$.

The following result holds true.

Proposition 6.3. *Suppose $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$. Then one has*

$$\widetilde{I}_\rho(\varphi_{\lambda,\zeta}) \to -\infty \qquad \textit{as } \lambda \to +\infty \qquad\qquad \textit{uniformly in } \zeta \in (\gamma_1)_k * (\gamma_2)_l.$$

Proof. The proof is developed exactly as in Proposition 3.3. Here we just sketch the main features.

We define $\tilde{v}_1, \tilde{v}_2 : \Sigma \to \mathbb{R}$ by

$$\tilde{v}_1(x) = \log \sum_{i=1}^k t_i \left(\frac{1}{1+\lambda_{1,r}^2 d(x,x_i)^2} \right)^2,$$
$$\tilde{v}_2(x) = \log \sum_{j=1}^l s_j \left(\frac{1}{1+\lambda_{2,r}^2 d(x,y_j)^2} \right)^2,$$

so that $\varphi = \tilde{v}_1 - \tilde{v}_2$.

The Dirichlet part of the functional $\widetilde{I}_\rho$ is given by

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g = \frac{1}{2}\int_\Sigma \big(|\nabla\tilde{v}_1|^2 + |\nabla\tilde{v}_2|^2 - 2\nabla\tilde{v}_1 \cdot \nabla\tilde{v}_2\big)\, dV_g \le \frac{1}{2}\int_\Sigma |\nabla\tilde{v}_1|^2\, dV_g + \frac{1}{2}\int_\Sigma |\nabla\tilde{v}_2|^2\, dV_g + C,$$

where we have used

$$\left| \int_\Sigma \nabla\tilde{v}_1 \cdot \nabla\tilde{v}_2\, dV_g \right| \le C.$$

We first study the cases $r = 0$ and $r = 1$, starting from $r = 0$. The case $r = 1$ can be treated in the same way and will be omitted. Observing that $\nabla\tilde{v}_2 = 0$ and taking into account the estimates (18), (19) on the gradient of $\tilde{v}_1$ we get

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g \le 16k\pi\big(1+o_\lambda(1)\big)\log\lambda + C,$$

where $o_\lambda(1) \to 0$ as $\lambda \to +\infty$.

Reasoning as in Proposition 3.3 we obtain

$$\int_\Sigma \varphi\, dV_g = -4\big(1+o_\lambda(1)\big)\log\lambda;\ \ \log\int_\Sigma e^{\varphi}\, dV_g = -2\big(1+o_\lambda(1)\big)\log\lambda;\ \ \log\int_\Sigma e^{-\varphi}\, dV_g = 4\big(1+o_\lambda(1)\big)\log\lambda.$$

Therefore we get

$$\widetilde{I}_\rho(\varphi_{\lambda,\zeta}) \le \big(16k\pi - 2\rho_1 + o_\lambda(1)\big)\log\lambda + C,$$

where C is independent of λ and σ_1, σ_2.

We consider now the case $r \in (0,1)$. We can reason as before to estimate the Dirichlet part by

$$\frac{1}{2}\int_\Sigma |\nabla\varphi|^2\, dV_g \le 16k\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + 16l\pi\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big) + C,$$

where $\delta_{1,r} > \delta > 0$ as $r \to 1$ and $\delta_{2,r} > \delta > 0$ as $r \to 0$. Following the argument in Proposition 3.3 we obtain

$$\int_\Sigma \varphi\, dV_g = -4\big(1+o_\lambda(1)\big)\log\big(\lambda_{1,r}+\delta_{1,r}\big) + 4\big(1+o_\lambda(1)\big)\log\big(\lambda_{2,r}+\delta_{2,r}\big)\big) + O(1),$$

$$\log\int_\Sigma e^{\varphi}\, dV_g = 4\log\big(\lambda_{2,r}+\delta_{2,r}\big) - 2\log\big(\lambda_{1,r}+\delta_{1,r}\big) + O(1),$$

$$\log\int_\Sigma e^{-\varphi}\, dV_g = 4\log\big(\lambda_{1,r}+\delta_{1,r}\big) - 2\log\big(\lambda_{2,r}+\delta_{2,r}\big) + O(1).$$

Using these estimates we get

$$\widetilde{I}_\rho(\varphi_{\lambda,\zeta}) \leq \big(16k\pi - 2\rho_1 + o_\lambda(1)\big)\log\big(\lambda_{1,r} + \delta_{1,r}\big) + \big(16l\pi - 2\rho_2 + o_\lambda(1)\big)\log\big(\lambda_{2,r} + \delta_{2,r}\big) + O(1).$$

By assumption we have $\rho_1 > 8k\pi, \rho_2 > 8l\pi$ and exploiting the fact that $\max_{r\in[0,1]}\{\lambda_{1,r}, \lambda_{2,r}\} \to +\infty$ as $\lambda \to \infty$, we deduce the thesis. ■

Once we have this result we can proceed exactly as in Section 4. One gets indeed an analogous improved Moser-Trudinger inequality as in Lemma 4.3. We have just to observe that a local Moser-Trudinger inequality still holds in this case, as pointed out in [27].

Lemma 6.4. *Fix $\delta > 0$, and let $\Omega \Subset \widetilde{\Omega} \subset \Sigma$ be such that $d\left(\Omega, \partial\widetilde{\Omega}\right) \geq \delta$. Then, for any $\varepsilon > 0$ there exists a constant $C = C(\varepsilon, \delta)$ such that for all $u \in H^1(\Sigma)$*

$$\log \int_\Omega e^{u - \fint_{\widetilde{\Omega}} u\, dV_g}\, dV_g + \log \int_\Omega e^{-u + \fint_{\widetilde{\Omega}} u\, dV_g}\, dV_g \leq \frac{1+\varepsilon}{16\pi}\int_{\widetilde{\Omega}} |\nabla_g u|^2\, dV_g + C.$$

Therefore, considering $\rho_1 \in (8k\pi, 8(k+1)\pi)$ and $\rho_2 \in (8l\pi, 8(l+1)\pi)$, we deduce that on low sub-levels of the functional $\widetilde{I}_\rho$ at least one of the component of $\left(\frac{he^u}{\int_\Sigma he^u dV_g}, \frac{he^{-u}}{\int_\Sigma he^{-u} dV_g}\right)$ has to be very close to the sets of k- or l- barycenters over Σ, respectively, see Proposition 4.6 for details. It is then possible to construct a continuous map

$$\widetilde{\Psi} : \widetilde{I}_\rho^{-L} \to (\gamma_1)_k * (\gamma_2)_l$$

with L sufficiently large, such that the composition

$$(\gamma_1)_k * (\gamma_2)_l \quad \xrightarrow{\widetilde{\Phi}_\lambda} \quad \widetilde{I}_\rho^{-L} \quad \xrightarrow{\widetilde{\Psi}} \quad (\gamma_1)_k * (\gamma_2)_l$$

is homotopically equivalent to the identity map on $(\gamma_1)_k * (\gamma_2)_l$ provided that λ is large enough. $\widetilde{\Psi}$ is defined as in (35), where basically e^{u_2} is replaced by e^{-u}:

$$\widetilde{\Psi}(u) = (1-\widetilde{r})(\Pi_1)_*\psi_k\left(\frac{he^u}{\int_\Sigma he^u dV_g}\right) + \widetilde{r}(\Pi_2)_*\psi_l\left(\frac{he^{-u}}{\int_\Sigma he^{-u} dV_g}\right).$$

With this at hand we argue as in Section 5 introducing a min-max scheme based on the set $(\gamma_1)_k * (\gamma_2)_l$. Allowing (ρ_1, ρ_2) to vary in a compact set of $(8k\pi, 8(k+1)\pi) \times (8l\pi, 8(l+1)\pi)$ we obtain a sequence of solutions $(u_n)_n$ corresponding to $(\rho_{1,n}, \rho_{2,n}) \to (\rho_1, \rho_2)$. We finally get a solution for (ρ_1, ρ_2) by applying the compactness result in Theorem 6.1.

7. Appendix: On the CW structure of Barycenter Spaces, by Sadok Kallel

In this appendix we show that barycenter spaces of CW-complexes are again CW. The notation of this appendix is independent of the rest of the paper, and the proofs use arguments from algebraic topology.

We adopt the notation $\mathcal{B}_n$ for barycenter and Sym^{*n} for symmetric join, see [33]. We also need the notation

$$\Delta_{k-1} = \{(t_1, \ldots, t_k) \ \in [0,1]^k \mid \sum t_i = 1\}$$

for the $(k-1)$-dimensional complex. This we view as a CW-complex with faces being subcomplexes. For $k < n$, we write as $\Delta_{k-1} \hookrightarrow \Delta_{n-1}$ the standard face inclusion given by adjoining trivial coordinate entries $(t_1, \ldots, t_k) \mapsto (t_1, \ldots, t_k, 0, \ldots, 0)$. Similarly for based X, with basepoint x_0, we embed $X^k \hookrightarrow X^n$ by adjoining basepoints.

Proposition 7.1. *If X is a based connected CW-complex, then $\mathcal{B}_n(X)$ can be equipped with a CW structure so that all vertical projections in the following diagram are cellular maps and all horizontal maps are subcomplex inclusions*

$$\begin{tikzcd} \Delta_{k-1} \times X^k \arrow[r, hook] \arrow[d] & \Delta_{n-1} \times X^n \arrow[d] \\ \mathcal{B}_k(X) \arrow[r, hook] & \mathcal{B}_n(X). \end{tikzcd}$$

The proof uses standard facts about CW complexes which we now review.

(1) If (X, A) is a relative CW complex, then the quotient space X/A is a CW complex with a vertex corresponding to A.
(2) More generally if A is a subcomplex of a CW complex X, Y is a CW complex, and $f : A \longrightarrow Y$ is a cellular map, then the *pushout* $Y \cup_f X$ has an induced CW complex structure that contains Y as a subcomplex and has one cell for each cell of X that is not in A. We represent this construction by a diagram

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"] & X \arrow[d] \\ Y \arrow[r] & X \cup_f Y \end{tikzcd}$$

with the understanding that all maps arriving at $X \cup_f Y$ are cellular with respect to the induced cell structure there.
(3) A finite group, or more generally a discrete group G acts *cellularly* on X means that: (i) if σ is an open cell of X then $g\sigma$ is again an open cell in X for all $g \in G$, and (ii) if $g \in G$ fixes an open cell σ, that is $g\sigma = \sigma$, then it fixes σ pointwise (i.e. $gx = x$ for all $x \in \sigma$). A CW-complex is a *cellular G-space* if G acts cellularly on X. If a finite group G acts cellulary on X, then X/G is a CW-complex. Furthermore, if $f : X \longrightarrow Y$ is a G-equivariant cellular map between cellular G-spaces, then the induced map $X/G \longrightarrow Y/G$ is cellular with respect to the induced CW-structures.

Properties (1) and (2) can be found in ([47], Chapter 10.2). Property (3) follows from Prop. 1.15 and Ex. 1.17 of [60] (Chapter 2). Throughout we endow X with a CW-structure so that the permutation action of $\mathfrak{S}_n$ on X^n is cellular, and so that x_0 is a 0-cell or vertex.

Proof of Proposition 7.1. We recall the definition of the barycenter spaces. Given X a space, then its n-th barycenter space is the quotient space

$$\mathcal{B}_n(X) := \coprod_{k=1}^{n} \Delta_{k-1} \times_{\mathfrak{S}_k} X^k/_\sim$$

where $\Delta_{k-1} \times_{\mathfrak{S}_k} X^k$ is the quotient of $\Delta_{k-1} \times X^k$ by the symmetric group $\mathfrak{S}_k$ acting diagonally, and where $\sim$ is the equivalence relation generated by:

$$\begin{aligned} (i) \qquad & [t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n; x_1, \dots, x_i, \dots x_n] \\ & \sim [t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n; x_1, \dots, \hat{x}_i, \dots x_n] \end{aligned}$$

(here $\hat{x}_i$ means the i-th entry has been suppressed), and by

$$\begin{aligned} (ii) \qquad & [t_1, \dots, t_i, \dots, t_j, \dots, t_n; x_1, \dots, x_i, \dots, x_j, \dots, x_n] \\ & \sim [t_1, \dots, t_{i-1}, t_i + t_j, t_{i+1}, \dots, \hat{t}_j, \dots, t_n; x_1, \dots, x_i, \dots, \hat{x}_j, \dots, x_n] \qquad \text{if} \quad x_i = x_j. \end{aligned}$$

An intermediate construction is to consider the symmetric join $\mathrm{Sym}^{*n}(X)$ which is the quotient of $\coprod_{k=1}^n \Delta_{k-1} \times_{\mathfrak{S}_k} X^k$ by the equivalence relation (i) only. There are quotient projections

$$\Delta_{n-1} \times X^n \longrightarrow \Delta_{n-1} \times_{\mathfrak{S}_n} X^n \longrightarrow \mathrm{Sym}^{*n}(X) \longrightarrow \mathcal{B}_n(X)$$

and it is convenient to write an equivalence class in $\Delta_{n-1} \times_{\mathfrak{S}_n} X^n$ or any of its images in $\mathrm{Sym}^{*n}X$ and $\mathcal{B}_n(X)$ by

$$\sum_{i=1}^{n} t_i x_i := [t_1, \dots, t_n; x_1, \dots, x_n].$$

Addition means the sum is abelian and this reflects the symmetric group action. The relation (i) means the entry $0x_i$ is suppressed, and relation (ii) means that $t_i x + t_j x = (t_i + t_j)x$.

To show that $\mathcal{B}_n(X)$ is CW, we proceed by induction. When $n = 1$, $\mathcal{B}_1 X = X$ so there is nothing to prove. For the general case, write

$$\mathcal{B}_n X = \mathcal{B}_{n-1} X \cup (\Delta_{n-1} \times_{\mathfrak{S}_n} X^n)/_\sim$$

and write $X^n_{fat} \subset X^n$ the fat diagonal consisting of all n-tuples $(x_1, \dots, x_n)$ with $x_i = x_j$ for some $i \neq j$. Denote by

$$W_n = (\partial \Delta_{n-1} \times_{\mathfrak{S}_n} X^n) \bigcup (\Delta_{n-1} \times_{\mathfrak{S}_n} X^n_{fat})$$

the subspace of $\Delta_{n-1} \times_{\mathfrak{S}_n} X^n$ consisting of all classes $\sum t_i x_i$ with $t_i = 0$ for some i or $x_i = x_j$ for some $i \neq j$. Then W_n is a CW subcomplex of X^n because the $\mathfrak{S}_n$-equivariant decomposition of X^n can always be arranged so that Δ_{fat} is a subcomplex. There is a well-defined quotient map $f : W_n \longrightarrow \mathcal{B}_{n-1}$ sending

$$\begin{aligned} \sum t_j x_j \quad &\longmapsto \quad \sum_{j\neq i} t_j x_j \quad \text{if } t_i = 0 \\ \sum t_j x_j \quad &\longmapsto \quad t_1 x_1 + \cdots + (t_i + t_j) x_i + \cdots + \widehat{t_j x_j} + \cdots + t_n x_n \quad \text{if } x_i = x_j \end{aligned}$$

and we have the pushout diagram

$$(*) \qquad \begin{array}{ccc} W_n & \hookrightarrow & \Delta_{n-1} \times_{\mathfrak{S}_n} X^n \\ \downarrow f & & \downarrow \\ \mathcal{B}_{n-1} X & \longrightarrow & \mathcal{B}_n(X). \end{array}$$

If we can show that f is cellular, then by property (2) and induction, $\mathcal{B}_n(X)$ will be CW as desired.

The map f has two restrictions f_1 and f_2 on the pieces $\partial\Delta_{n-1} \times_{\mathfrak{S}_n} X^n$ and $\Delta_{n-1} \times_{\mathfrak{S}_n} X^n_{fat} \subset W_n$ respectively. To see that f_1 is cellular, write $\partial\Delta_{n-1}$ as a union of faces $F_i = \{(t_1, \ldots, t_n), t_i = 0\}$ each homeomorphic to Δ_{n-2}. Write $X^n_i = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_0\}$ where $x_0 \in X$ is the basepoint. The maps $F_i \times X^n \longrightarrow F_i \times X^n_i$, $(t_1, \ldots, t_n; x_1, \ldots, x_n) \mapsto (t_1, \ldots, t_n; x_1, \ldots, x_0, \ldots, x_n)$; which for a given i replaces x_i by x_0, are cellular and so is their union

$$\bigcup_i F_i \times X^n \longrightarrow \bigcup_i F_i \times X^n_i.$$

This map is $\mathfrak{S}_n$-equivariant and so passes to a cellular map between quotients

$$\begin{array}{ccc} \left(\bigcup_i F_i \times X^n\right)/_{\mathfrak{S}_n} & \longrightarrow & \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n} \\ \| & & \| \\ \partial\Delta_{n-1} \times_{\mathfrak{S}_n} X^n & \xrightarrow{g} & \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1}. \end{array}$$

The restriction f_1 is now the composite of cellular maps

$$\partial\Delta_{n-1} \times_{\mathfrak{S}_n} X^n \xrightarrow{g} \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1} \longrightarrow \mathcal{B}_{n-1}(X)$$

thus it is cellular. We proceed the same way for the restriction f_2. Write $X^n_{fat} = \bigcup_{i<j} X^n_{ij}$ where $X^n_{ij} = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j, i < j\}$. Each X^n_{ij} is identified with X^{n-1}. There are maps $\tau_{ij} : \Delta_{n-1} \times X^n_{ij} \longrightarrow F_i \times X^n_i$ sending

$$\begin{aligned} &(t_1, \ldots, t_n, x_1, \ldots, x_n) \\ &\longmapsto (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{j-1}, t_i + t_j, t_{j+1}, \ldots, t_n; x_1, \ldots, x_{i-1}, x_0, x_{i+1}, \ldots, x_n) \end{aligned}$$

which are cellular being the product of cellular maps (i.e it can be checked that the map $\Delta_{n-1} \longrightarrow \partial\Delta_{n-1}$ sending $(t_1, \ldots, t_n) \longrightarrow (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_{j-1}, t_i + t_j, t_{j+1}, \ldots, t_n)$ sends faces to faces and hence is cellular). The map $\bigcup \tau_{ij}$ is not $\mathfrak{S}_n$-equivariant, but the composite

$$\bigcup_{i<j} \Delta_{n-1} \times X^n_{ij} \longrightarrow \bigcup_i F_i \times X^n_i \longrightarrow \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n}$$

factors through the $\mathfrak{S}_n$-quotient. More precisely, we have the diagram

$$\begin{array}{cccc} \left(\bigcup_{i<j} \Delta_{n-1} \times X^n_{ij}\right)/_{\mathfrak{S}_n} & \longrightarrow & \left(\bigcup_i F_i \times X^n_i\right)/_{\mathfrak{S}_n} & \\ \| & & \| & \\ \Delta_{n-1} \times_{\mathfrak{S}_n} X^n_{fat} & \xrightarrow{\tau} & \Delta_{n-2} \times_{\mathfrak{S}_{n-1}} X^{n-1} & \longrightarrow \mathcal{B}_{n-1}(X) \end{array}$$

with all maps in this diagram cellular. The bottom composite f_2 must therefore be cellular.

In conclusion, the map $f = f_1 \cup f_2$ in the diagram (*) is cellular and this completes the proof. □

Example 7.2. We take a special look at $\mathcal{B}_2(X)$. Consider $\mathrm{Sym}^{*2}X$ which consists of elements of the form $t_1x + t_2y$ with $t_1 + t_2 = 1$ and the identification $0x + 1y = y$. By using the order on the t_i's in $I = [0,1]$, this can be written as

$$\begin{aligned}\mathrm{Sym}^{*2}(X) &= \{(t_1,t_2,x_1,x_2) \mid t_1 \le t_2, t_1+t_2 = 1\}/_\sim \\ &= J \times (X \times X)/_\sim\end{aligned}$$

where $J = \{0 \le t_1 \le t_2 \le 1, t_1 + t_2 = 1\}$ is a copy of the one-simplex, and the identification $\sim$ is such that $(0,1,x,y) \sim (0,1,x',y)$ and $(\frac{1}{2},\frac{1}{2},x,y) \sim (\frac{1}{2},\frac{1}{2},y,x)$. Note that $(0,1)$ and $(\frac{1}{2},\frac{1}{2})$ are precisely the faces or endpoints of J. This is saying that $\mathrm{Sym}^{*2}X$ is precisely the double mapping cylinder

$$\begin{tikzcd} X^2 \times \{(0,1)\} \sqcup X^2 \times \{(\frac{1}{2},\frac{1}{2})\} \arrow[r, hook] \arrow[d, "p_2 \sqcup \pi"] & X^2 \times J \arrow[d] \\ X \sqcup \mathrm{SP}^2 X \arrow[r] & \mathrm{Sym}^{*2}X \end{tikzcd}$$

where p_2 is the projection onto the second factor $X^2 \longrightarrow X$, and π is the $\mathbb{Z}_2$-quotient map $X^2 \longrightarrow \mathrm{SP}^2X$ (see [33]). Both maps p_2 and π are cellular (Property (3)). This gives $\mathrm{Sym}^{*2}(X)$ a CW-structure according to property (3). We can now consider the pushout diagram

$$\begin{tikzcd} J \times X \arrow[r] \arrow[d] & \mathrm{Sym}^{*2}X \arrow[d] \\ X \arrow[r] & \mathcal{B}_2X \end{tikzcd} \tag{42}$$

where the left vertical map $J\times X \longrightarrow X$ is projection hence cellular, while the top map $J\times X \longrightarrow \mathrm{Sym}^{*2}X$, $((t_1,t_2),x) \mapsto t_1x + t_2x$, is a subcomplex inclusion. By property (2), $\mathcal{B}_2(X)$ is CW.

References

[1] D. Bartolucci and G. Tarantello. The Liouville equation with singular data: a concentration-compactness principle via a local representation formula. *J. Differential Equations*, 185(1):161–180, 2002.

[2] D. Bartolucci and G. Tarantello. Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory. *Comm. Math. Phys.*, 229(1):3–47, 2002.

[3] Daniele Bartolucci and Francesca De Marchis. On the Ambjorn-Olesen electroweak condensates. *J. Math. Phys.*, 53(7):073704, 15, 2012.

[4] Daniele Bartolucci, Francesca De Marchis, and Andrea Malchiodi. Supercritical conformal metrics on surfaces with conical singularities. *Int. Math. Res. Not. IMRN*, (24):5625–5643, 2011.

[5] Daniele Bartolucci and Andrea Malchiodi. An improved geometric inequality via vanishing moments, with applications to singular liouville equations. *Comm. Math. Phys., to appear, d.o.i. 10.1007/s00220-013-1731-0.*

[6] Daniele Bartolucci and Eugenio Montefusco. Blow-up analysis, existence and qualitative properties of solutions for the two-dimensional Emden-Fowler equation with singular potential. *Math. Methods Appl. Sci.*, 30(18):2309–2327, 2007.

[7] J. Bolton and L. M. Woodward. Some geometrical aspects of the 2-dimensional Toda equations. In *Geometry, topology and physics (Campinas, 1996)*, pages 69–81. de Gruyter, Berlin, 1997.

[8] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.

[9] Haim Brezis and Frank Merle. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Comm. Partial Differential Equations*, 16(8-9):1223–1253, 1991.

[10] Luis A. Caffarelli and Yi Song Yang. Vortex condensation in the Chern-Simons Higgs model: an existence theorem. *Comm. Math. Phys.*, 168(2):321–336, 1995.

[11] Eugenio Calabi. Isometric imbedding of complex manifolds. *Ann. of Math. (2)*, 58:1–23, 1953.

[12] Alessandro Carlotto and Andrea Malchiodi. Weighted barycentric sets and singular Liouville equations on compact surfaces. *J. Funct. Anal.*, 262(2):409–450, 2012.

[13] Dongho Chae and Oleg Yu. Imanuvilov. The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory. *Comm. Math. Phys.*, 215(1):119–142, 2000.

[14] Wenxiong Chen and Congming Li. Prescribing scalar curvature on S^n. *Pacific J. Math.*, 199(1):61–78, 2001.

[15] Shiing Shen Chern and Jon G. Wolfson. Harmonic maps of the two-sphere into a complex Grassmann manifold. II. *Ann. of Math. (2)*, 125(2):301–335, 1987.

[16] Alexandre J. Chorin. *Vorticity and turbulence*, volume 103 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[17] Manuel del Pino, Michal Kowalczyk, and Monica Musso. Singular limits in Liouville-type equations. *Calc. Var. Partial Differential Equations*, 24(1):47–81, 2005.

[18] Weiyue Ding, Jürgen Jost, Jiayu Li, and Guofang Wang. Existence results for mean field equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(5):653–666, 1999.

[19] Zindine Djadli. Existence result for the mean field problem on Riemann surfaces of all genuses. *Commun. Contemp. Math.*, 10(2):205–220, 2008.

[20] Zindine Djadli and Andrea Malchiodi. Existence of conformal metrics with constant Q-curvature. *Ann. of Math. (2)*, 168(3):813–858, 2008.

[21] G. Dunne. *Self-dual Chern-Simons Theories*. Lecture notes in physics. New series m: Monographs. Springer, 1995.

[22] Pierpaolo Esposito, Massimo Grossi, and Angela Pistoia. On the existence of blowing-up solutions for a mean field equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 22(2):227–257, 2005.

[23] Pierpaolo Esposito and Juncheng Wei. Non-simple blow-up solutions for the Neumann two-dimensional sinh-Gordon equation. *Calc. Var. Partial Differential Equations*, 34(3):341–375, 2009.

[24] Massimo Grossi and Angela Pistoia. Multiple blow-up phenomena for the sinh-poisson equation. *preprint*, 2012.

[25] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[26] Louis Jeanjean. On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 129(4):787–809, 1999.

[27] Aleks Jevnikar. An existence result for the mean field equation on compact surfaces in a doubly supercritical regime. *Proc. Royal Soc. Edinb., to appear.*, 2012.

[28] Jürgen Jost, Changshou Lin, and Guofang Wang. Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. *Comm. Pure Appl. Math.*, 59(4):526–558, 2006.

[29] Jürgen Jost and Guofang Wang. Analytic aspects of the Toda system. I. A Moser-Trudinger inequality. *Comm. Pure Appl. Math.*, 54(11):1289–1319, 2001.

[30] Jürgen Jost and Guofang Wang. Classification of solutions of a Toda system in $\mathbb{R}^2$. *Int. Math. Res. Not.*, (6):277–290, 2002.

[31] Jürgen Jost, Guofang Wang, Dong Ye, and Chunqin Zhou. The blow up analysis of solutions of the elliptic sinh-Gordon equation. *Calc. Var. Partial Differential Equations*, 31(2):263–276, 2008.

[32] Glenn Joyce and David Montgomery. Negative temperature states for the two-dimensional guiding-centre plasma. *Journal of Plasma Physics*, 10:107–121, 7 1973.

[33] Sadok Kallel and Rym Karoui. Symmetric joins and weighted barycenters. *Adv. Nonlinear Stud.*, 11(1):117–143, 2011.

[34] Hsien-Chung Kao and Kimyeong Lee. Self-dual SU(3) Chern-Simons Higgs systems. *Phys. Rev. D (3)*, 50(10):6626–6632, 1994.

[35] Kimyeong Lee. Self-dual nonabelian Chern-Simons solitons. *Phys. Rev. Lett.*, 66(5):553–555, 1991.

[36] Yan Yan Li. Harnack type inequality: the method of moving planes. *Comm. Math. Phys.*, 200(2):421–444, 1999.

[37] Yan Yan Li and Itai Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.*, 43(4):1255–1270, 1994.

[38] Changshou Lin, Juncheng Wei, and Lei Zhang. Classification of blowup limits for su(3) singular toda systems. *preprint*, 2013.

[39] Pierre-Louis Lions. *On Euler equations and statistical physics*. Cattedra Galileiana. [Galileo Chair]. Scuola Normale Superiore, Classe di Scienze, Pisa, 1998.

[40] Marcello Lucia. A deformation lemma with an application to a mean field equation. *Topol. Methods Nonlinear Anal.*, 30(1):113–138, 2007.

[41] Marcello Lucia and Margherita Nolasco. SU(3) Chern-Simons vortex theory and Toda systems. *J. Differential Equations*, 184(2):443–474, 2002.

[42] Andrea Malchiodi. Morse theory and a scalar field equation on compact surfaces. *Adv. Differential Equations*, 13(11-12):1109–1129, 2008.
[43] Andrea Malchiodi. Topological methods for an elliptic equation with exponential nonlinearities. *Discrete Contin. Dyn. Syst.*, 21(1):277–294, 2008.
[44] Andrea Malchiodi and Cheikh Birahim Ndiaye. Some existence results for the Toda system on closed surfaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 18(4):391–412, 2007.
[45] Andrea Malchiodi and David Ruiz. New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces. *Geom. Funct. Anal.*, 21(5):1196–1217, 2011.
[46] Andrea Malchiodi and David Ruiz. A variational analysis of the Toda system on compact surfaces. *Comm. Pure Appl. Math.*, 66(3):332–371, 2013.
[47] J. P. May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.
[48] Monica Musso, Angela Pistoia, and Juncheng Wei. New blow-up phenomena for su$(n+1)$ toda system. *preprint*, 2013.
[49] Paul K. Newton. *The N-vortex problem*, volume 145 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001. Analytical techniques.
[50] Margherita Nolasco and Gabriella Tarantello. Double vortex condensates in the Chern-Simons-Higgs theory. *Calc. Var. Partial Differential Equations*, 9(1):31–94, 1999.
[51] Margherita Nolasco and Gabriella Tarantello. Vortex condensates for the SU(3) Chern-Simons theory. *Comm. Math. Phys.*, 213(3):599–639, 2000.
[52] Hiroshi Ohtsuka and Takashi Suzuki. A blowup analysis of the mean field equation for arbitrarily signed vortices. In *Self-similar solutions of nonlinear PDE*, volume 74 of *Banach Center Publ.*, pages 185–197. Polish Acad. Sci., Warsaw, 2006.
[53] Hiroshi Ohtsuka and Takashi Suzuki. Mean field equation for the equilibrium turbulence and a related functional inequality. *Adv. Differential Equations*, 11(3):281–304, 2006.
[54] Y.B. Pointin and Lundgren T.S. Statistical mechanics of twodimensional vortices in a bounded container. *Physics of Fluids*, 19(10):1459–1470, 1976.
[55] Joel Spruck and Yi Song Yang. Topological solutions in the self-dual Chern-Simons theory: existence and approximation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 12(1):75–97, 1995.
[56] Michael Struwe. On the evolution of harmonic mappings of Riemannian surfaces. *Comment. Math. Helv.*, 60(4):558–581, 1985.
[57] Gabriella Tarantello. Multiple condensate solutions for the Chern-Simons-Higgs theory. *J. Math. Phys.*, 37(8):3769–3796, 1996.
[58] Gabriella Tarantello. *Selfdual gauge field vortices*. Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston Inc., Boston, MA, 2008. An analytical approach.
[59] Gabriella Tarantello. Analytical, geometrical and topological aspects of a class of mean field equations on surfaces. *Discrete Contin. Dyn. Syst.*, 28(3):931–973, 2010.
[60] Tammo tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.
[61] Guofang Wang and Liqun Zhang. Non-topological solutions of the relativistic SU(3) Chern-Simons Higgs model. *Comm. Math. Phys.*, 202(3):501–515, 1999.
[62] Henry C. Wente. Large solutions to the volume constrained Plateau problem. *Arch. Rational Mech. Anal.*, 75(1):59–77, 1980/81.
[63] Henry C. Wente. Counterexample to a conjecture of H. Hopf. *Pacific J. Math.*, 121(1):193–243, 1986.
[64] Yisong Yang. *Solitons in field theory and nonlinear analysis*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.
[65] Chunqin Zhou. Existence result for mean field equation of the equilibrium turbulence in the super critical case. *Commun. Contemp. Math.*, 13(4):659–673, 2011.

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