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Coarse geometry: a foundational and categorical approach with applications to groups and hyperspaces

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Abstract

The topic of the manuscript is coarse geometry, also known as large-scale geometry, which is the study of large-scale properties of spaces. It found applications in geometric group theory after the work of Gromov, and in Novikov and coarse Baum-Connes conjectures.

The thesis is divided into three distinct parts. In the first one, we provide a foundational and categorical approach to coarse geometry. Large-scale geometry was originally developed for metric spaces and then Roe introduced coarse structures as a large-scale counterpart of uniformities. However, coarse spaces are innerly symmetric objects, and thus are not suitable to parametrise asymmetric objects such as monoids and quasi-metric spaces. In order to fill the gap, we introduce quasi-coarse spaces. Moreover, we consider also semi-coarse spaces and entourage spaces. The latter notion generalises both quasi-coarse spaces and semi-coarse spaces. These objects induce para-bornologies, quasi-bornologies, semi-bornologies, pre-bornologies (also known as bounded structures) and bornologies, and this process is similar to the definition of uniform topology from a (quasi-)uniform space. We study all the notions introduced and recalled to find extensions of classical results proved for metric or coarse spaces, and similarities with notions and properties for general topology. Furthermore, we study the categories of those objects and the relations among them. In particular, since all of them are topological categories, we have a complete understanding of their epimorphisms and monomorphisms, and the description of many categorical constructions. Among them, of particular interest are quotients. We then focus our attention on **Coarse**, the category of coarse spaces and bornologous maps, discussing its closure operators and the cowellpoweredness of its epireflective subcategories, and its quotient category **Coarse**$/_\sim$, which turns out to be balanced and cowellpowered.

The second part is dedicated to study the large-scale geometry of algebraic objects, such as unitary magmas, monoids, loops and groups. In particular, we focus on coarse groups (groups endowed with suitable coarse structures) and we investigate their category. We study different choices, underlining advantages and drawbacks. With some restrictions on the coarse groups that we are considering, if we enlarge the class of morphisms to contain bornologous quasi-homomorphisms (and not just bornologous homomorphisms), every coarse inverse of a homomorphism which is a coarse equivalence is a quasi-homomorphism. This observation is connected to the notion of localisation of a category and could provide a categorical justification to the notion of quasi-homomorphism. Once the categories of coarse groups are fixed, inspired by the notion of functorial topologies, we can introduce functorial coarse structures on

Grp, the category of groups and homomorphisms, and on **TopGrp**, the category of topological groups and continuous homomorphisms. Among them, we pay attention to the ones induced by cardinal invariants, and to those associated to the family of relatively compact subsets. As for the latter functorial coarse structure, we study the transformation of large-scale properties (e.g., metrisability, asymptotic dimension) along Pontryagin and Bohr functors.

Finally, the third part is devoted to coarse hyperspaces, which are suitable coarse structures on power sets of coarse spaces. This construction was introduced following the work of Protasov and Protasova and miming the classical notion of uniform hyperspace. We see how properties of the initial coarse space are reflected on the hyperspace (e.g., cellularity). Since the coarse hyperspace is highly disconnected, it is convenient to consider some special subspaces of it. For example, if the base space is a coarse group, it is natural to consider the subspace structure induced on the lattice of subgroups, called subgroup exponential hyperballean. We show that both the subgroup exponential hyperballean and the subgroup logarithmic hyperballean, another coarse structure on the subgroup lattice, capture many important properties of the group.

Contents

Chapter 1

Introduction

Coarse geometry, also known as large-scale geometry, is the study of large-scale properties of spaces, ignoring their local, small-scale ones. The origin of large-scale geometry goes back to Milnor's problems, Gromov's ideas from geometric group theory, and Mostow's rigidity theorem ([121]).

Intuitively, two spaces are considered equivalent in coarse geometry if they look alike for an observer whose point of view is getting further and further away. For example, every bounded space is indistinguishable from a one-point space. Another example is the pair given by the integer numbers $\mathbb{Z}$ and the real numbers $\mathbb{R}$. From a topological perspective, these equivalences seem to loose too much information of the spaces. In fact, 'small holes' and 'small discontinuities' are ignored, and, for example, we can identify a discrete space, $\mathbb{Z}$, with a connected one, $\mathbb{R}$. However, and somehow unexpectedly, this theory found applications in several branches of mathematics, for example in geometric group theory (following the work of Gromov on finitely generated groups endowed with their word metrics), in Novikov conjecture, and in coarse Baum-Connes conjecture. We refer to [127] for a comprehensive introduction to large-scale geometry of metric spaces, and to [93] for applications to geometric group theory.

This thesis is divided into three parts. In the first one, our aim is to provide the foundations of the coarse geometry of spaces, developing also a categorical approach. In the second part, we apply those techniques to the world of algebraic objects (magmas, loops, monoids and groups). Furthermore, we investigate the large-scale geometry of topological groups associated to the family of all relatively compact subsets. Finally, the third part is devoted to the study of coarse hyperspaces, in particular, of those induced on groups.

1.1 A foundational and categorical approach to coarse geometry

We have said that coarse geometry is the study of those properties of spaces that are preserved for an observer whose point of view is getting further and further away. Let us now more precisely describe the equivalences involved.

A map $f\colon (X,d_X)\to(Y,d_Y)$ between metric spaces is said to be:

- *large-scale Lipschitz* if there exist $L > 0$ and $C \geq 0$ such that $d_Y(f(x), f(y)) \leq L d_X(x, y) + C$, for every $x, y \in X$;
- *bornologous* if, for every $R \geq 0$, there exists $S_R \geq 0$ such that $d_Y(f(x), f(y)) \leq S_R$ if $d_X(x, y) \leq R$, for every $x, y \in X$.

Note that every large-scale Lipschitz map is bornologous.

Two maps $f, g\colon S \to (X, d)$ from a set to a metric space are *close* if there exists $R \geq 0$ such that $d(f(x), g(x)) \leq R$, for every $x \in X$. If f and g are close, we write $f \sim g$.

Definition 1.1.1. Let $f\colon X \to Y$ be a map between two metric spaces. Then f is called a:

- *quasi-isometry* if there exists another map $g\colon Y \to X$ such that both f and g are large-scale Lipschitz maps, and $f \circ g \sim id_Y$ and $g \circ f \sim id_X$;
- *coarse equivalence* if there exists another map $g\colon Y \to X$, called *coarse inverse*, such that both f and g are bornologous, and $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

If f is a quasi-isometry (a coarse equivalence), then the spaces X and Y are called *quasi-isometric* (*coarsely equivalent*, respectively).

A quasi-isometry is, in particular, a coarse equivalence. The converse implication is not true in general. For example, the metric spaces $\{n^2 \mid n \in \mathbb{N}\}$ and $\{n^3 \mid n \in \mathbb{N}\}$, with their metrics inherited by the family of non-negative integers $\mathbb{N}$, are coarsely equivalent, but they are not quasi-isometric. On the other hand, if two metric spaces X and Y are coarsely equivalent to geodesic metric spaces, then X is coarsely equivalent to Y if and only if X is quasi-isometric to Y (see [127]).

Let us consider some easy examples of quasi-isometries. Every bounded metric space (X, d) (i.e., there exists $R \in \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$ such that $X = B_d(x, R) = \{y \in X \mid d(x, y) \leq R\}$, for every $x \in X$) is quasi-isometric to a one-point space $\{*\}$ (just take any inclusion $f\colon \{*\} \to X$ and the constant map $g\colon X \to \{*\}$). The metric spaces $\mathbb{Z}$ and $\mathbb{R}$, endowed with their canonical euclidean metrics are quasi-isometric. In fact, we can take the inclusion map $i\colon \mathbb{Z} \to \mathbb{R}$ and the *floor map* $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$ such that, for every $x \in \mathbb{R}$, $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$. Another choice for a suitable map $g\colon Y \to X$ is the *ceiling map* $g = \lceil\cdot\rceil$, where, for every $x \in \mathbb{R}$, $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$.

Before giving more interesting examples of quasi-isometries, let us introduce two important classes of metric spaces.

Example 1.1.2. Let $\Gamma = (V, E)$ be a non-directed connected graph. Then the set of vertices V can be endowed with the *path metric* d_Γ defined as follows: for every $x, y \in X$,

$$d_\Gamma(x, y) = \min\{n \in \mathbb{N} \mid \exists x_0 = x, x_1, \dots, x_n = y \in V : \\ \forall i = 1, \dots, n, \{x_{i-1}, x_i\} \in E\}.$$

Since Γ is connected, $d_\Gamma\colon V \times V \to \mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$. If we consider also non-connected non-directed graphs, we can extend the path metric by putting $d_\Gamma(x, y) = \infty$ if and only if the vertices x and y are in different connected components.

Example 1.1.3. Let G be a group. We say that G is *finitely generated* if there exists a finite subset Σ of G such that, for every $g \in G$, there exist $n \in \mathbb{N}$ and

$\sigma_1, \dots, \sigma_n \in \Sigma \cup \Sigma^{-1}$ which satisfy $g = \sigma_1 \cdots \sigma_n$ (i.e., $G = \langle \Sigma \rangle$). If a group G is finitely generated by a finite subset Σ, we always assume without loss of generality that $\Sigma = \Sigma^{-1}$ and the identity e_G of G belongs to Σ. In fact, we can replace Σ with $\Sigma \cup \Sigma^{-1} \cup \{e_G\}$.

Let G be a group which is generated by the finite subset $e = e_G \in \Sigma = \Sigma^{-1}$. Let us define the *(left) word metric* d_Σ as follows: for every pair of elements $x, y \in G$,

$$d_\Sigma(x, y) = \min\{n \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : y = x\sigma_1 \cdots \sigma_n\}.$$

Note that d_Σ is *left-invariant*, i.e., for every $x, y, z \in G$, $d_\Sigma(zx, zy) = d_\Sigma(x, y)$. Finally, let us point out that also the *right word metric* d_Σ^ρ can be defined on G. However, it provides no further information since the inverse map $i\colon (G, d_\Sigma) \to (G, d_\Sigma^\rho)$, where, for every $g \in G$, $i(g) = g^{-1}$, is an isometry.

To every finitely generated group G and every finite generating set Σ, we can associate a non-directed graph $\mathrm{Cay}(G, \Sigma) = (G, E)$, called *Cayley graph of G associated to* Σ, where a pair $\{g, h\} \in G \times G$ belongs to E if and only if there exists $\sigma \in \Sigma$ such that $h = g\sigma$. Note that the map $id_G\colon (G, d_\Sigma) \to (G, d_{\mathrm{Cay}(G,\Sigma)})$ is an isometry.

A finitely generated group G can be endowed with several word metrics, in fact, they strongly depend on the finite generating set associated. However, from the large-scale point of view, they coincide as the following result shows.

Proposition 1.1.4. *Let G be a finitely generated group, and Σ and Δ be two symmetric finite generating subsets of G. Then the identity map $id_G\colon (G, d_\Sigma) \to (G, d_\Delta)$ is a quasi-isometry.*

Proposition 1.1.4 can be interpreted as follows: every finitely generated group has precisely one large-scale geometry. Finitely generated groups are a very important object in geometric group theory where the large-scale approach turned out to be fruitful (see, for example, [90] and [93] for a wide discussion of the subject).

An extremely important *coarse invariant* (i.e., a cardinal associated to every metric space in such a way that two coarsely equivalent metric spaces are associated to the same cardinal) is the *asymptotic dimension* which was introduced by Gromov ([90]) as the large-scale counterpart of the classical Čech-Lebesgue covering dimension. Before giving the definition of the asymptotic dimension, let us recall the classical topological notion. We refer to [74] for more information on this topic.

Let $\mathcal{U}$ be a cover of a topological space X. A *refinement* $\mathcal{V}$ of $\mathcal{U}$ is a cover of X such that for every $V \in \mathcal{V}$ there exists an element $U \in \mathcal{U}$ such that $V \subseteq U$. The cover $\mathcal{U}$ is *open* if its elements are open sets. Moreover, we can define the *order of $\mathcal{U}$* $\mathrm{ord}\,\mathcal{U}$ as the value:

$$\mathrm{ord}\,\mathcal{U} = \sup_{x \in X} |\{U \in \mathcal{U} \mid x \in U\}|\,.$$

Definition 1.1.5. Let X be a T_4 topological space. Then we define its *covering dimension*, or *Čech-Lebesgue dimension*, with the following properties:

- $\dim X \leq n$ for some $n \in \mathbb{N}$ if, for every finite open cover $\mathcal{U}$ of X, there exists a finite open cover $\mathscr{V}$ which forms a refinement of $\mathscr{U}$ and satisfies $\mathrm{ord}\,\mathcal{V} \leq n+1$;

- $\dim X = n$ if we have both $\dim X \leq n$ and $\dim X > n-1$;
- $\dim X = \infty$ if there is no $n \in \mathbb{N}$ such that $\dim X \leq n$.

Let us now provide the definition of asymptotic dimension.

Definition 1.1.6. Let X be a metric space and $n \in \mathbb{N}$. Then

- $\operatorname{asdim} X \leq n$ if, for every $R \geq 0$, there exists $S \geq 0$ and a cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ of X such that, for every $U \in \mathcal{U}$ and every $x \in U$, $U \subseteq B_d(x, S)$, and, for every $i = 0, \ldots, n$ and $U, V \in \mathcal{U}_i$, $B_d(U, R) \cap V \neq \emptyset$ if $U \neq V$;
- the *asymptotic dimension of* X is n (and we write $\operatorname{asdim} X = n$) if $\operatorname{asdim} X \leq n$ and $\operatorname{asdim} X > n-1$;
- $\operatorname{asdim} X = \infty$ if, for every $m \in \mathbb{N}$, $\operatorname{asdim} X > m$.

We refer to [18, 19] for a comprehensive introduction of this notion and for other, equivalent, characterisations. Let us just mention for now that, as promised, if X and Y are two coarsely equivalent metric spaces, then $\operatorname{asdim} X = \operatorname{asdim} Y$.

We have already stated that coarse geometry has application in coarse Baum-Connes and Novikov conjecture. In the first results in that direction (due to Yu), the asymptotic dimension played a fundamental role.

Theorem 1.1.7 ([173])**.** *The coarse Baum-Connes conjecture is true for every proper metric space with finite asymptotic dimension.*

Theorem 1.1.8 ([173])**.** *Finitely generated groups of finite homotopy type with finite asymptotic dimension satisfy the Novikov conjecture.*

In [174], Yu noticed that for a proper metric space, the existence of a coarse embedding in a Hilbert space (property which is weaker than having finite asymptotic dimension) still implies that the coarse Baum-Connes conjecture holds. We refer to the monograph [127] and to the paper [169] for a wide bibliography and deep discussion of the topic.

The definition of asymptotic dimension, which takes inspiration from the covering dimension, is a clear example of a paradigm in coarse geometry. Namely, several notions of coarse geometry were created as a large-scale counterpart of small-scale notions (see, for example, the discussion in [28]).

1.1.1 Uniform spaces

A classical generalisation of the notion of metric space is the one of uniform space. Uniform spaces have been widely studied since their introduction by the work of Weil and Tukey in the first half of the last century, and successfully applied in different areas. If X is a set, every subset $U \subseteq X \times X$ is called an *entourage*. For every pair of entourages U, V, we define the *composite of* U *and* V as the entourage

$$U \circ V = \{(x, z) \mid \exists y \in X : (x, y) \in U, (y, z) \in V\},$$

and the *inverse of* U as

$$U^{-1} = \{(y, x) \mid (x, y) \in U\}.$$

Definition 1.1.9 ([104])**.** A *uniform space* is a pair $(X,\mathcal{U})$, where X is a set and $\mathcal{U}$ is a *uniformity* over it, i.e., a family of subsets of $X\times X$ that satisfies the following properties:

(U1) $\mathcal{U}$ is a *filter* (i.e., a family closed under taking finite intersections and supersets);
(U2) for every $U\in\mathcal{U}$, $\Delta_X=\{(x,x)\mid x\in X\}\subseteq U$;
(U3) for every $U\in\mathcal{U}$, $U^{-1}\in\mathcal{U}$;
(U4) for every $U\in\mathcal{U}$, there exists $V\in\mathcal{U}$ such that $V\circ V\subseteq U$.

For instance, if (X,d) is a metric space, then, the family

$$\begin{aligned}\mathcal{U}_d=\{V\supseteq U_R^d\mid R\geq 0\},\ &\text{where, for every } R>0,\\ U_R^d=&\bigcup_{x\in X}(\{x\}\times B_d(x,R)),\end{aligned}\tag{1.1}$$

is a uniformity over X, called *metric uniformity.* If there is no risk of ambiguity and the metric considered is clear, we simply write U_R instead of U_R^d. The metric uniformity captures the small-scale properties (e.g., the topological properties) of metric spaces (see Remark 3.1.4).

A family of entourages $\mathcal{B}$ of X is a *base of a uniformity* if the closure $\mathcal{U}_\mathcal{B}$ of $\mathcal{B}$ under taking supersets is a uniformity.

If $(X,\mathcal{U})$ is a uniform space, we can endow a subset Y of X with the *subspace uniformity*

$$\mathcal{U}|_Y=\{U\cap(Y\times Y)\mid U\in\mathcal{U}\}.$$

The pair $(Y,\mathcal{U}|_Y)$ is a *uniform subspace.*

Let $f\colon X\to Y$ be a map between sets. Denote by $f\times f\colon X\times X\to Y\times Y$ the map defined by the law $(f\times f)(x,y)=(f(x),f(y))$, for every $(x,y)\in X\times X$.

A map $f\colon(X,\mathcal{U}_X)\to(Y,\mathcal{U}_Y)$ between uniform spaces is:

- *uniformly continuous* if, for every $U\in\mathcal{U}_Y$, there exists $V\in\mathcal{U}_X$, such that $(f\times f)(V)\subseteq U$;
- a *uniform isomorphism* if it is bijective and both f and f^{-1} are uniformly continuous;
- a *uniform embedding* if the corestriction of f to its image endowed with the subspace uniformity is a uniform isomorphism.

In order to generalise the large-scale properties of metric spaces, Roe introduced coarse spaces ([157]), as a counterpart of Weil's definition of uniform spaces via entourages, and Protasov and Banakh ([144]) defined balleans, generalising the ball structure of metric spaces. Furthermore, Dydak and Hoffland with large-scale structures ([72]) and Protasov with asymptotic proximities ([140]) independently developed the approach via coverings, as Tukey did for uniform spaces. As for the definition of coarse structures and coarse spaces, we refer to Definition 3.1.1. The notions of bornologous maps, closeness and coarse equivalences that we gave for metric spaces can be extended to the framework of coarse spaces (see §3.1.1 and [157]). Among all the large-scale properties of metric spaces whose definition is generalised to arbitrary coarse spaces, let us also mention the asymptotic dimension for its importance (see [18, 88] and §6 for a brief introduction).

1.1.2 Generalisations of metric spaces and uniformities

In mathematics some generalisations of metric spaces appeared. Let X be a set and $d\colon X\times X\to\mathbb{R}_{\geq 0}\cup\{\infty\}$ be a map such that $d(x,x)=0$, for every $x\in X$. The map d is a *semi-positive-definite map*. Moreover d is a

- *pseudo-semi-metric* if, for every $x,y\in X$, $d(x,y)=d(y,x)$;
- *pseudo-quasi-metric* if, for every $x,y,z\in X$, $d(x,y)\leq d(x,z)+d(z,y)$ (with the convention that $\infty+a=a+\infty=\infty$, for every $a\in\mathbb{R}$).

In particular, a pseudo-metric is both a pseudo-semi-metric and a pseudo-quasi-metric. Note that we allow that the distance between two points is infinite. Usually, the prefix ‘pseudo’ is dropped if, for every $x,y\in X$, $d(x,y)=0$ if and only if $x=y$. However, for the sake of simplicity, we call a pseudo-semi-metric a *semi-metric*, a pseudo-quasi-metric a *quasi-metric*, and a pseudo-metric a *metric*. The pair (X,d) is a *semi-metric space* if d is a semi-metric, a *quasi-metric space* if d is a quasi-metric, and a *metric space* if d is a metric (in this broader meaning).

Let us now give some examples of quasi-metric spaces in order to motivate our interest in those structures. The first example (Example 1.1.10) is due to Hausdorff himself, while Examples 1.1.12 and 1.1.13 are the asymmetric counterparts of Examples 1.1.2 and 1.1.3, respectively.

Example 1.1.10 ([94])**.** Let (X,d) be a metric space. On the power set $\mathcal{P}(X)$ of X we define a map $d_H^q\colon\mathcal{P}(X)\times\mathcal{P}(X)\to\mathbb{R}_{\geq 0}\cup\{\infty\}$ as follows: for every $Y,Z\subseteq X$,

$$d_H^q(Y,Z)=\inf\{R\geq 0\mid Z\subseteq B_d(Y,R)\},$$

where $\inf\emptyset=\infty$. The map d_H^q is actually a quasi-metric, called *Hausdorff quasi-metric*.

Example 1.1.11. Let $(X,\geq)$ be a preordered set. Then the preorder $\geq$ induces a quasi-metric $d_{\geq}$ on X, called *preorder quasi-metric*, defined as follows: for every $x,y\in X$,

$$d_{\geq}(x,y)=\begin{cases}0 & \text{if } x\geq y,\\ \infty & \text{otherwise.}\end{cases}$$

Example 1.1.12. Let $\Gamma=(V,E)$ be a directed graph. Then the set of vertices V can be endowed with the *path quasi-metric* d_Γ defined as follows: for every $x,y\in X$,

$$d_\Gamma(x,y)=\min\{n\in\mathbb{N}\mid \exists x_0=x,x_1,\dots,x_n=y\in V:\\ \forall i=1,\dots,n,(x_{i-1},x_i)\in E\}.$$

Again $\min\emptyset=\infty$, and thus $d_\Gamma(x,y)=\infty$ if and only if there is no directed path from x to y. It is easy to check that d_Γ is actually a quasi-metric.

Before introducing the next example, let us recall some algebraic definitions. A *magma* is a pair $(M,\cdot)$, where M is a set and $\cdot\colon M\times M\to M$ is a map. A magma $(M,\cdot)$ is called *unitary* if there exists a *neutral element* $e\in M$ such that $g\cdot e=e\cdot g=g$, for every $g\in M$. A unitary magma is a *monoid* if $\cdot$ is associative.

Example 1.1.13. Let M be a monoid. We say that M is *finitely generated* if there exists a finite subset Σ of M such that, for every $g \in M$ there exist $n \in \mathbb{N}$ and $\sigma_1, \dots, \sigma_n \in \Sigma$ which satisfy $g = \sigma_1 \cdots \sigma_n$.

Let M be a monoid which is finitely generated by Σ. Let us define the *left word quasi-metric* d_Σ^λ as follows: for every pair of elements $x, y \in M$,

$$d_\Sigma^\lambda(x, y) = \min\{n \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : y = x\sigma_1 \cdots \sigma_n\}.$$

The map $d_\Sigma^\lambda \colon M \times M \to \mathbb{N} \cup \{\infty\}$ is actually a quasi-metric. Similarly, one can define a quasi-metric d_Σ^ρ on M, called *right word quasi-metric*: for every $x, y \in M$,

$$d_\Sigma^\rho(x, y) = \min\{n \mid \exists \sigma_1, \dots, \sigma_n \in \Sigma : y = \sigma_1 \cdots \sigma_n x\}.$$

Moreover, note that d_Σ^λ is *left-non-expanding*, i.e., for every $x, y, z \in M$, $d_\Sigma^\lambda(zx, zy) \leq d_\Sigma^\lambda(x, y)$, and, similarly, d_Σ^ρ is *right-non-expanding*, i.e., for every $x, y, z \in M$, $d_\Sigma^\rho(xz, yz) \leq d_\Sigma^\rho(x, y)$. The left word quasi-metric and the right word quasi-metric are no longer isometric, as in the case of finitely generated groups, and they can be very different.

It is possible to extend the notion of Cayley graph, which is a useful tool to represent a finitely generated group, in the framework of finitely generated monoids. Let M be a monoid and $\Sigma \subseteq M$ a finite subset which generates M. Then the *left Cayley graph of M associated to* Σ is the directed graph $\mathrm{Cay}^\lambda(M, \Sigma) = (M, E)$, where $(x, y) \in E$ if and only if there exists $\sigma \in \Sigma$ such that $y = x\sigma$ or, equivalently, $d_\Sigma^\lambda(x, y) = 1$. Similarly $\mathrm{Cay}^\rho(M, \Sigma)$, the *right Cayley graph*, can be constructed. Also in this case, the maps $id_M \colon (M, d_\Sigma^\lambda) \to (M, d_{\mathrm{Cay}^\lambda(M,\Sigma)})$ and $id_M \colon (M, d_\Sigma^\rho) \to (M, d_{\mathrm{Cay}^\rho(M,\Sigma)})$ are isometries.

We refer to [171] for a general introduction to the subject of quasi-metric spaces. Quasi-metrics are innerly non symmetric, so, if we consider the family $\mathcal{U}_d$ as in (1.1), then (U3) may not be satisfied. In order to fill the gap, quasi-uniform spaces were introduced: a *quasi-uniform space* is a pair $(X, \mathcal{U})$, where $\mathcal{U}$ is a *quasi-uniformity* over the set X, i.e., a family of entourages that satisfies (U1), (U2) and (U4). There is a wide literature investigating those structures and also important applications to computer science were discovered (see the monograph [78] and the survey [112] for a wide-range introduction and a broad bibliography). Similarly, a *semi-uniform space* is a pair $(X, \mathcal{U})$, where $\mathcal{U}$ is a *semi-uniformity* over the set X, i.e., a family of entourages that satisfies (U1)–(U3) (see, for example, [27]).

1.1.3 Weak neighbourhood systems and topologies

Topologies are another, different generalisation of (quasi-)uniformities. In order to recall how a topology is induced by a uniformity, let us introduce a more general structure.

Definition 1.1.14 ([119])**.** Let X be a set. A *weak neighbourhood system* $\vartheta = \{\vartheta(x) \mid x \in X\}$ is a family of non-empty filters of subsets of X such that, for every $x \in X$, $x \in \bigcap \vartheta(x)$. If $x \in X$, then an element V of $\vartheta(x)$ is called a *neighbourhood of x*.

Let ϑ be a weak neighbourhood system on a set X. An *open neighbourhood* is a subset V of X such that, for every $x \in V$, $V \in \vartheta(x)$. A weak neighbourhood system ϑ on a set X is a *neighbourhood system* if it satisfies the following property: for every $x \in X$ and $U \in \vartheta(x)$, there exists $x \in V \subseteq U$ which is an open neighbourhood.

If ϑ is a neighbourhood system of the set X, then the family τ_ϑ of all open neighbourhoods of ϑ is a topology on X. Conversely, if τ is a topology on X, the family $\vartheta_\tau = \{\vartheta_\tau(x) \mid x \in X\}$, where, for every $x \in X$, $\vartheta_\tau(x)$ is the family of the usual neighbourhoods of x, is a neighbourhood system. Moreover, $\tau = \tau_{\vartheta_\tau}$ and $\vartheta = \vartheta_{\tau_\vartheta}$, for every topology τ and neighbourhood system ϑ of X.

Every quasi-uniform space $(X,\mathcal{U})$ carries a topology $\tau_\mathcal{U}$ of X. In order to describe it, we need to introduce some more notation. For every entourage U, every $x \in X$ and $A \subseteq X$, we denote

$$U[x] = \{y \in X \mid (x,y) \in U\}, \quad \text{and} \quad U[A] = \bigcup_{x\in A} U[x].$$

In the sequel, the subsets $U[x]$ and $U[A]$ just defined will be called the *ball centred in x with radius U* and the *ball centred in A with radius U*, respectively.

Let now $(X,\mathcal{U})$ be a semi-uniform or a quasi-uniform space. Then the family $\vartheta_\mathcal{U} = \{\vartheta_\mathcal{U}(x) \mid x \in X\}$, where, for every $x \in X$, $\vartheta_\mathcal{U}(x) = \{U[x] \mid U \in \mathcal{U}\}$, is a weak neighbourhood system. Moreover, if $\mathcal{U}$ is a quasi-uniformity, then $\vartheta_\mathcal{U}$ is a neighbourhood system, and thus we can associate a topology $\tau_\mathcal{U} = \tau_{\vartheta_\mathcal{U}}$, called *quasi-uniform topology.*

It is not true in general that, for every topology τ on a set X, there exists a uniformity $\mathcal{U}$ on X such that $\tau = \tau_\mathcal{U}$. A topology with this property is called *uniformisable*. It is a classical result that a T_0 topology is uniformisable if it satisfies the axiom $T_{3,5}$. On the contrary, for every topology τ, there exists a quasi-uniformity $\mathcal{U}$ such that $\tau = \tau_\mathcal{U}$ (see [75, 112]).

A uniform space $(X,\mathcal{U})$ is *Hausdorff* (or *separated*) if $\bigcap \mathcal{U} = \Delta_X$. Moreover, a uniform space $(X,\mathcal{U})$ is Hausdorff if and only if $\tau_\mathcal{U}$ is T_0 if and only if $\tau_\mathcal{U}$ is $T_{3,5}$.

1.1.4 The large-scale generalisations of metric spaces

We have already mentioned that Roe introduced coarse spaces as the large-scale counterpart of uniform spaces. Moreover, bounded structures (also known as pre-bornologies, which are a generalisation of the classical notion of bornological spaces,also known as bornological sets, see Remark 2.1.3 for their usual definitions) were introduced to relax the definition of coarse spaces. However, both coarse spaces and bounded structures are innerly symmetric objects and they are not suitable to parametrise, for example, quasi-metric spaces. The initial segment of the first part is devoted to the study of the large-scale counterparts of topological spaces, quasi-uniform spaces and semi-uniform spaces.

In order to define a suitable counterpart of the notion of topological spaces, we 'dualise' the notion of weak neighbourhood systems, and introduce para-bornological spaces (Definition 2.1.1). Para-bornological spaces are very weak notions. Hence, sometimes it is convenient to consider some reinforcements of their definition, namely semi-bornological spaces and quasi-bornological spaces.

A space which is both a semi-bornological space and a quasi-bornological space is a pre-bornological space. In order to study para-bornological spaces, we look for similarities with topology. For example, the place of the classical separation axioms is taken by connectedness axioms (Definition 2.2.5), which describe how connected the spaces are, how the points of the spaces can be reached. Pushing forward this approach, we study local simple ends (Definition 2.3.1), the large-scale counterpart of converging nets, and use them to characterise the morphisms of those spaces (Theorem 2.3.5). This characterisation resembles the one of continuous maps via converging nets. Local simple ends are also used to show that the notion of local finiteness is similar to the one of sequential compactedness since, while the latter definition involves converging sequences, we characterise the first one using local simple ends (Theorem 2.4.2).

We introduce large-scale counterparts of quasi-uniform spaces and semi-uniform spaces, respectively, in order to generalise coarse spaces. In particular, we define quasi-coarse spaces and semi-coarse spaces (Definition 3.1.1). Moreover, in order to provide a more comprehensive introduction to these new objects, we consider also entourage spaces, which are structures that generalise both quasi-coarse spaces and semi-coarse spaces. First of all, scratching the surface of this topic, we focus on adapting basic notions of coarse geometry (e.g., morphisms, as bornologous maps, connectedness, boundedness) to this more general setting. Moreover, we present a different characterisation of those structures by using ball structures ([144]). We motivate our interest in quasi-coarse spaces and semi-coarse spaces by providing a wide list of examples in which those structures naturally appear (see also Examples 1.1.10–1.1.13). Most of them are extensions of some classical examples of coarse spaces. In particular, we prove that also every finitely generated monoid can be endowed with precisely just two word quasi-metrics up to asymorphism (Proposition 3.3.6), which coincide if the monoid is abelian. This result is a generalisation of the classical situation involving finitely generated groups endowed with word metrics (Proposition 1.1.4).

Furthermore, we provide a generalisation of the notion of coarse equivalence between spaces in the realm of asymmetric objects, namely, the Sym-coarse equivalence. Using this equivalence, we were able to provide important characterisations of some classes of quasi-coarse spaces: metric entourage spaces induced by extended-quasi-metrics and graphic quasi-coarse spaces, giving an answer to a problem posed by Protasov and Banakh ([144, Problem 9.4]).

Let us also mention that in [168] the notion of asymptotic dimension is extended to quasi-coarse spaces, with a particular focus on its applications to preordered sets.

1.1.5 A categorical approach

In order to study coarse spaces, (pre-)bornological spaces and their generalisations, it is useful to consider their categories. Let us start with considering the category of coarse spaces. At the level of morphisms of the category of coarse spaces, several possible choices have been used (see for example [157], [116], [24] and [88]), but often the choice is so restrictive, that even products are not available (as the natural projections are not morphisms). For the same reason, pullbacks are not available either, while only very special maps (e.g., those with uniformly bounded fibers, see [151, 9]) admit quotients. This rules also out some standard constructions, as adjunction spaces. Recently coarse quotient mappings between metric spaces were studied by Sheng Zhang ([178]).

In [65], a more relaxed condition on the morphisms compared to [157] and [151] has been adopted (asking the maps to be only bornologous and not necessarily proper). This choice turns out to be quite fruitful, since the category **Coarse** (with objects the coarse spaces and morphisms the bornologous maps) proves to be topological (i.e., admits initial and final structures), so arbitrary products, coproducts, as well as pullbacks and pushouts exist. Once the category of coarse spaces was fixed, the categories **Entou** (of entourage spaces), **SCoarse** (of semi-coarse spaces), and **QCoarse** (of quasi-coarse spaces) were easily introduced in [176]. Also these categories turned out to be topological (Theorem 4.2.1), and so a complete characterisation of their epimorphisms and monomorphisms could be provided.

The categories **PrBorn** and **Born** of pre-bornological spaces and bornological spaces were already defined and studied. For example, see [103] and [11], respectively. We extend here this approach by defining the categories **PaBorn**, **SBorn** and **QBorn**, of para-bornological spaces, semi-bornological spaces and quasi-bornological spaces, respectively. Also all these categories are topological, as shown in Theorem 4.2.1.

We then look for connections between **Entou** and its three subcategories, and, separately, **PaBorn** and its three subcategories. The existence of several functors between the four categories of entourage spaces turned out to be very useful. First of all, they have been used to prove that **QCoarse** is a reflective subcategory in **Entou** (but it is not co-reflective), **SCoarse** is a reflective and co-reflective subcategory in **Entou**, **Coarse** is a reflective subcategory in **SCoarse** (but it is not co-reflective), and **Coarse** is a reflective and co-reflective subcategory in **QCoarse** (see Theorem 4.2.4 and §4.3.3). A similar situation can be provided for the subcategories of **PaBorn**. Using different functors, we can show that **QBorn** is a reflective subcategory in **PaBorn** (but it is not co-reflective), **SBorn** is a reflective and co-reflective subcategory in **PaBorn**, **PrBorn** is a reflective subcategory in **SBorn** (but it is not co-reflective), and **PrBorn** is a reflective and co-reflective subcategory in **QBorn** (see Theorem 4.1.5 and §4.3.2). These results just described help in defining some limits (e.g., products, pullbacks and equalisers) and colimits (e.g., coproducts and quotients) in the various categories, which always exist since all of them are topological.

A particular emphasis is given to one of the basic construction, namely quotients. The difficulties with quotients of uniform spaces are well known, we shall cite Plaut ([137]): 'The notion that quotients of uniform spaces always have a uniform structure compatible with the quotient topology has been described not only as being false, but "horribly false" [104] and leading to "unavoidable difficulties" [107].' In §4.3.3 we study the counterpart of this problem in the realm of coarse spaces. More precisely, for a coarse space $(X, \mathcal{E})$ and a surjective map $q\colon X \to Y$, similarly to the case of uniformities, the 'image' $q(\mathcal{E})$ of the coarse structure $\mathcal{E}$ under the map q needs not be a coarse structure on Y. In case q satisfies the quite restrictive condition of uniform boundedness of the fibers ([151, 9]), $q(\mathcal{E})$ turns out to be a coarse structure (necessarily, the quotient coarse structure of Y). As mentioned above, the properness of q, usually imposed so far, was giving as a consequence the uniform boundedness of the fibers of q 'for free', so the issue of when $q(\mathcal{E})$ is a coarse structure never appeared before explicitly, as the maps were necessarily 'too good'. We characterise the maps q such that $q(\mathcal{E})$ is a coarse structure are characterised. We notice that a similar approach can be used to describe the quotients of **QCoarse** (**Coarse** is coreflective in **QCoarse**), while the situation for the categories **SCoarse** and **Entou** is way less complex. Furthermore, we find a similar situation in the categories of para-bornological spaces. We describe in §4.3 these constructions with all the needed details.

The notion of closeness between morphisms of **Coarse** is a congruence, and thus its quotient category **Coarse**$/_\sim$ can be defined and studied. For example, in §5.3.2, its epimorphisms and monomorphisms are characterised, and it is proved that it is a balanced category, i.e., if a map is both a monomorphisms and an epimorphism, then it is an isomorphism (i.e., a coarse equivalence). Furthermore, the functors between the categories of entourage spaces are a fundamental tool to transport the notion of closeness from coarse spaces, to the other, weaker structures, inducing congruences, and thus quotient categories. Among them, the notion of Sym-closeness and Sym-coarse equivalence, already presented at the end of §1.1.4, in the realm of quasi-coarse spaces is particularly relevant.

The characterisation of the epimorphisms of a category opens the problem of discussing its cowellpoweredness. In a category $\mathcal{X}$, two epimorphisms $e_1\colon X \to Y_1$ and $e_2\colon X \to Y_2$ are equivalent if there exists an isomorphism $h\colon Y_1 \to Y_2$ such that $e_2 = h \circ e_1$. The category $\mathcal{X}$ is *cowellpowered* if, for every object X of $\mathcal{X}$, the class of all epimorphisms with domain X (which can be even a proper class) may be labelled up to equivalence by a set. A classic example of a cowellpowered category is the category **Haus** of Hausdorff spaces, where the epimorphisms are the continuous maps with dense image. The epimorphisms of $\mathbf{Top_0}$, the category of T_0 topological spaces, were discussed in [16] and [123].

In 1971, Herrlich asked if there exists a non-cowellpowered subcategory of **Top**. Herrlich himself found the first example of such a subcategory ([97]) and, later, an easier example was provided by Schröder ([160]): the *Urysohn spaces* **Ury** is not cowellpowered. In Schröder's example, the *θ-closure* plays a fundamental role, a fact that motivated Dikranjan's and Giuli's study of *closure operators* as means to describe epimorphisms in subcateogires of **Top** and decide about their cowellpoweredness ([46]).

Closure operators, whose origin seems to date back to foundational work in analysis by Moore and Riesz, have been used in different branches of mathematics. For examples they appear in algebra, in topology, where the leading example is the *(Kuratowski) closure* of a subspace of a topological space, in logic and in lattice theory. In the monograph [60] a detailed bibliography is provided. In [48] the authors described how to introduce closure operators in a categorical setting. The interest to this topic is still alive: recently, Dikranjan and Tholen introduced the notion of *dual closure operator* ([61]).

In [35], a rather general description of the epimorphisms of subcategories of topological categories was given. Closure operators, and in particular those which are *regular* ([159], [105]), were fundamental tools to test cowellpoweredness of some subcategories of **Top**. For example, they were used in [160], [49], and [63] to prove cowellpoweredness of subcategories defined by separation axioms which are stronger than the Hausdorff property, while in [84] and in [165] cowellpoweredness was proved for subcategories containing **Haus**. In [84] the authors extended the subcategory defined in [86]. Another example of a non-cowellpowered subcategory of **Top** which contains **Haus** can be found in [47]. Moreover, we refer to the following papers: [51], where closure operators and epimorphisms of quasi-uniform spaces were studied; [106], where epimorphisms and cowellpoweredness of universal algebras were considered; and [31], which focusses on separated metrically generated theories. For more examples, see [60, Chapter 8] and, in particular, [60, Section 8.9] for more algebraic applications.

If a category is topological, then its epimorphisms are surjective morphisms, and thus it is trivially cowellpowered. This observation rules out all the eight categories, **Entou**, **SCoarse**, **QCoarse**, **Coarse**, **PaBorn**, **SBorn**, **QBorn**, and **PrBorn**, that we consider, which are trivially cowellpowered. Inspired by Herrlich's question, we address the problem of the existence of a non-cowellpowered subcategories of **Coarse**. The negative answer was provided by mean of a complete characterisation of the closure operators of **Coarse**. In the same paper, the cowellpoweredness of **Coarse**$/_\sim$ (as well as its wellpoweredness, the dual notion of cowellpoweredness) was proved. We report these results in Chapter 5, where, for convenience, the category **Ballean** of balleans, which is isomorphic to **Coarse**, is widely used.

We have already mentioned that it is not true in general that every topology τ on a set X is induced by a uniformity on X, while, there always exists a quasi-uniformity on X inducing τ. We discuss in §4.4 the large-scale counterpart of this problem. In particular, we prove that every para-bornology is induced by an entourage structure, every quasi-bornology is induced by a quasi-coarse structure (Theorem 4.4.1), and every pre-bornology is induced by a coarse structure (Theorem 4.4.5).

1.2 Coarse geometry of algebraic objects

Proposition 1.1.4 states that finitely generated groups can be seen as metric spaces in an essentially unique way. This approach in their study has been extremely fruitful in geometric group theory. However, a problem immediately emerged. In fact, it is not true in general that a subgroup of a finitely generated group is finitely generated. In order to overcome this problem, in [70], a solution is described. Recall that a metric d on a set X is *proper* if every closed ball is compact. In the cited paper, it is noticed that every countable group endowed with the discrete topology can be endowed with a left-invariant proper (i.e., the balls are finite) metric. Let us explicitly construct this metric, following [70]. Let G be a countable group and Σ be a symmetric set of generators (not necessarily finite). A *weight function* is a map $w\colon \Sigma \to \mathbb{R}_{\geq 0}$ which satisfies the following properties:
- w is proper, i.e., $|w^{-1}([0,R])| < \infty$, for every $R \geq 0$ (equivalently, $\lim w = \infty$);
- for every $\sigma \in \Sigma$, $w(\sigma) = w(\sigma^{-1})$.

Then w induces a left-invariant proper metric d_w on G defined as follows: for every $x, y \in G$,

$$d_w(x,y) = \min\left\{ \sum_{i=1}^n w(\sigma_i) \mid n \in \mathbb{N}, \sigma_1, \dots, \sigma_n \in \Sigma : y = x\sigma_1 \cdots \sigma_n \right\}. \quad (1.2)$$

Proposition 1.2.1 ([70]). *Let d and d' be two proper left-invariant metrics on a group G inducing the same topology. Then the identity map $id_G\colon (G,d)\to(G,d')$ is a coarse equivalence.*

In particular, Proposition 1.2.1 can be applied to countable discrete groups, showing that every countable group can be endowed with just one left-invariant proper metric, up to coarse equivalence. Note that this class of groups is trivially closed under taking subgroups.

Among the studies of countable groups, let us just cite the paper [13], where the authors provide a complete classification of countable abelian groups up to coarse equivalence. Recall that, for an abelian group G, its *free-rank*, denoted by $r_0(G)$, is the cardinality of the maximal independent subset of G (a subset X of G is *independent* if, for every $n\in\mathbb{N}$, $x_1,\dots,x_n\in X$, and $m_1,\dots,m_n\in\mathbb{Z}$, if $\sum_{i=1}^n m_ix_i=0$, then $m_i=0$, for all $i=1,\dots,n$, where we denote by 0 the neutral element of an abelian group). Equivalently, $r_0(G)=\dim_{\mathbb{Q}} G/\operatorname{Tor}(G)\otimes\mathbb{Q}$, where $\operatorname{Tor}(G)$ is the *torsion subgroup of* G, i.e., the subgroup of G consisting of all its finite-order elements.

Theorem 1.2.2. ([13, Theorem 1]) *For two discrete countable abelian groups G and H endowed with proper left-invariant metrics, the following three statements are equivalent:*

(a) G and H are coarsely equivalent;
(b) $\operatorname{asdim} G=\operatorname{asdim} H$ and G and H are both finitely generated or both infinitely generated;
(c) $r_0(G)=r_0(H)$ and G and H are either both finitely generated or both infinitely generated.

A further step into generalisation was provided by Cornulier and de la Harpe (see their monograph [32] for a comprehensive discussion). They noticed that in the realm of σ-compact locally compact groups the metric approach can be generalised. In fact, every σ-compact locally compact group has a left-invariant proper pseudometric that is locally bounded (i.e., every point has a neighbourhood of finite diameter), and every pair of such pseudometrics are coarsely equivalent.

In order to go beyond those cases, the metric approach is not enough and it is necessary to introduce some special coarse structures on groups and topological groups. As uniform structures are a useful tool to parametrise topological groups (recall that a *topological group* is a group G endowed with a topology τ that makes both the inverse map $i\colon g\mapsto g^{-1}$ and the multiplication $\cdot\colon (g,h)\mapsto gh$ continuous), one can use some special classes of coarse structures to study the large-scale properties of groups. We require that those coarse structures agree with the algebraic structures of the supporting group and this idea leads to the definition of left (right) group coarse structures (and thus to left and right coarse groups). If a coarse structure on a group is both a left and right group coarse structure, we say that it is a uniformly invariant group coarse structure and the coarse group is called a bilateral coarse group. If there is no risk of ambiguity, for the sake of simplicity, we will refer to left group coarse structures as group coarse structures. The study of coarse groups was started by Protasov and Protasova in [145], where this notion was introduced by using balleans. In the same paper they highlighted the fact that coarse groups are uniquely determined by particular ideals of subsets of the group, called group ideals (Definition

7.1.1). The idea is similar to the fact that every group topology is uniquely determined by the filter of neighbourhoods of the identity. More recently, Nicas and Rosenthal ([125]) developed the same approach via entourages.

1.2.1 Large-scale geometry of generalisations of groups

We have already seen that some classical constructions in the realm of groups (see Example 1.1.3 and the definition of Cayley graph of a finitely generated group) can be extended to some weaker structures (Example 1.1.13 and the left and right Cayley graphs of finitely generated monoids). In this thesis we push forward this idea, aiming for a more comprehensive approach to large-scale geometry of algebraic objects.

We have already defined unitary magmas and monoids. Let us present another generalisation of groups. A unitary magma $(M,\cdot)$ is a *loop* if, for every $a,b\in M$ there exist a unique $x\in M$ and a unique $y\in M$ such that

$$a\cdot x=b \quad\text{and}\quad y\cdot a=b. \tag{1.3}$$

Since $e\cdot e=e$, (1.3) implies that e is the only neutral element. By (1.3), for every $g\in M$, there exist two elements $g^\rho,g^\lambda\in M$ such that $g\cdot g^\rho=e$ and $g^\lambda\cdot g=e$. Note that $(g^\rho)^\lambda=(g^\lambda)^\rho=g$, for every $g\in M$ (in fact, $(g^\rho)^\lambda\cdot g^\rho=e$, $g^\lambda(g^\lambda)^\rho=e$, $g\cdot g^\rho=e$, and $g^\lambda\cdot g=e$ and the conclusions follow by uniqueness of the solution of (1.3)). A loop $(M,\cdot)$ has *right inverse property* if, for every $g,h\in M$, $(g\cdot h)\cdot h^\rho=g$. Similarly, a loop $(M,\cdot)$ has *left inverse property* if, for every $g,h\in M$, $g^\lambda\cdot(g\cdot h)=h$. A loop has *inverse property* if it has both right and left inverse property. A loop M is said to have *two-side inverses* if $g^\lambda=g^\rho$, for every $g\in M$, and, in this case, we denote the inverse of g by g^{-1}.

Let $f\colon M\to N$ be a map between two unitary magmas. Then f is called a *homomorphism* if, for every $g,h\in M$, $f(gh)=f(g)f(h)$ and $f(e_M)=e_N$. Moreover, f is an *isomorphism* if it is bijective and both f and f^{-1} are homomorphisms. If that is the case, we say that M and N are *isomorphic* and we write $M\simeq N$.

Similarly to how groups represent a good environment for defining coarse structures, monoids, loops and unitary magmas, are suitable objects for defining some 'compatible' quasi-coarse structures, semi-coarse structures and entourage structures respectively. The beginning of Section 7.1 is devoted to the definition of these structures. Moreover, the particular algebraic structure of the underlining space implies that many large-scale properties can be described by looking at some specific families of subsets of the algebraic object.

1.2.2 Coarse groups and categories of coarse groups

We aim to define categories of coarse groups. The first choice is **l-CGrp**, whose objects are (left) coarse groups and whose morphisms are bornologous homomorphisms. Taking the quotient category **l-CGrp**$/_\sim$ of **l-CGrp** under the closeness relation would be the next step. However, we face some undesired consequences even dealing with basic examples. Before describing the critical point, let us recall what happens in small-scale. Let us define the following categories:

- **Grp** is the category of groups and homomorphisms between them;
- **Top** is the category of topological spaces and continuous maps between them;
- **TopGrp** is the category of topological groups and continuous homomorphisms between them.

Denote by $\mathrm{U}\colon \mathbf{TopGrp} \to \mathbf{Top}$ the forgetful functor that forgets about the group structure. Note that, if f is a morphism of **TopGrp**, f is an isomorphism (i.e., it is a topological isomorphism) provided that $\mathrm{U}\, f$ is an isomorphism (i.e., a homeomorphism). Similarly to the small-scale situation, there exists a forgetful functor $\mathrm{U}\colon \mathbf{l\text{-}CGrp}/_\sim \to \mathbf{Coarse}/_\sim$. However, there exist morphisms $[f]_\sim$ of $\mathbf{l\text{-}CGrp}/_\sim$ that are not isomorphisms, but such that $\mathrm{U}[f]_\sim$ is an isomorphism of $\mathbf{Coarse}/_\sim$. For example, the inclusion homomorphism $i\colon \mathbb{Z} \to \mathbb{R}$, where both groups are endowed with the usual euclidean metric, is one of the first examples of coarse equivalences (it is, in particular, a quasi-isometry). However, there is no coarse inverse of i which is a homomorphism. Hence $[i]_\sim$ is not an isomorphism of $\mathbf{l\text{-}CGrp}/_\sim$. In order to overcome this problem we need the notion of quasi-homomorphism.

A *quasi-homomorphism* (also called *quasi-morphism*) is a map $f\colon G \to \mathbb{R}$ from a group into the real line which is somehow 'close' to be a homomorphism, i.e., there exists a constant $K \geq 0$ such that $|f(x)+f(y)-f(xy)| < K$, for every $x, y \in G$. The notion of quasi-homomorphism dates back to some questions posed by Ulam ([166]) in the realm of linear functional equations. We refer to [166], [109] and [110] for an introduction to this classical subject.

Rosendal ([158]) noticed that the classical notion of quasi-homomorphism can be described and extended to other settings using the large-scale notion of closeness (see Definition 7.3.1 for a rigorous definition). Also in Fujiwara and Kapovich's paper [81], where the authors followed some older sources, there is a generalisation of the classical notion of quasi-homomorphism.

Following [66], we study quasi-homomorphisms in Rosendal's definition in order to refute Kotschick's point of view: 'the notion of a quasi-morphism does not have much to do with category theory' ([109]). We prove that, in the class of bilateral coarse groups (that properly contains all abelian coarse groups), maps close to quasi-homomorphisms are quasi-homomorphisms (Proposition 7.3.4), and composites of bornologous quasi-homomorphisms are bornologous quasi-homomorphisms (Proposition 7.3.6). Finally, in the same class of coarse groups, we show that coarse inverses of quasi-homomorphisms that are coarse equivalences are quasi-homomorphisms. In particular, every coarse inverse of the inclusion map $i\colon \mathbb{Z} \to \mathbb{R}$ is a quasi-homomorphism (for example, the floor map $\lfloor\cdot\rfloor$ and the ceiling map $\lceil\cdot\rceil$).

We then define the quotient category $\mathbf{CGrpQ}/_\sim$ of bilateral coarse groups and equivalence classes of bornologous quasi-homomorphisms between them. In this category, the equivalence class of a homomorphism which is a coarse equivalence is an isomorphism. We study the localisation $\mathbf{CGrp}/_\sim[\mathcal{W}^{-1}]$ of the quotient category $\mathbf{CGrp}/_\sim$, of bilateral coarse groups and equivalence classes of bornologous homomorphisms, by the family $\mathcal{W}$ of equivalence classes of homomorphisms which are coarse equivalences. The category $\mathbf{CGrp}/_\sim[\mathcal{W}^{-1}]$, provided that it exists, is the 'smallest' category containing $\mathbf{CGrp}/_\sim$ for which all morphisms of $\mathcal{W}$ are isomorphisms. We then ask whether it exists and if it coincides with $\mathbf{CGrpQ}/_\sim$. As for the existence, we provide in Corollary 8.4.5 a positive answer in the case of κ-group coarse structures (in particular, of the one associated to the ideal of finite subsets, called finitary-group coarse structure), for which a nice characterisation of morphisms is provided.

1.2.3 Functorial coarse structures

Once that categories of coarse groups are fixed, we can address the notion of functorial coarse structures. This should be compared with the notion of functoriality, which appeared in the category of topological groups as follows. A *functorial topology* is a functor $\mathrm{F}\colon \mathbf{Grp} \to \mathbf{TopGrp}$ that assigns to every *abstract* group G a group topology T_G so that $\mathrm{F}\colon G \mapsto (G, T_G)$ is a functor $\mathrm{F}\colon \mathbf{Grp} \to \mathbf{TopGrp}$, i.e., every group homomorphism $f\colon G \to H$ in $\mathbf{Grp}$ gives rise to a continuous group homomorphism $f\colon (G, T_G) \to (H, T_H)$ in $\mathbf{TopGrp}$ ([41, 117]).

Following the topological notion, we introduce the concepts of functorial coarse structures on **Grp** and on **TopGrp**. Informally, a functorial coarse structure on **Grp** (on **TopGrp**) associates to every group (to every topological group) a coarse structure making every homomorphism (every continuous homomorphism, respectively) bornologous. Moreover, we show that a functorial coarse structure on **Grp** can be seen as a functorial coarse structure on **TopGrp** in a canonical way, endowing every group with its discrete topology.

Among the functorial coarse structures on **Grp**, particularly relevant is the finitary-group coarse structure. Moreover, in §9.2 we generalise the construction and introduce coarse structures induced by cardinal invariants using ideals generated by subgroups (linear coarse structures), that are often functorial on **Grp** (see §9.2.1).

In §9.3 we scrutinise abelian groups under the looking glass of the functorial coarse structure induced by the free-rank, establishing a kind of 'rigidity' of the class of divisible groups with respect to homomorphisms that are coarse equivalences. Let us recall the definition and the basic facts concerning divisible groups.

An abelian group G is *divisible* if, for every $y \in G$ and every $n \in \mathbb{N} \setminus \{0\}$, there exists $x \in G$ such that $nx = y$. Every abelian group G has a largest divisible subgroup $d(G)$. Examples of divisible groups are the additive group of the rational numbers $\mathbb{Q}$ and, for every prime p, the *Prüfer p-group* $\mathbb{Z}_{p^\infty}$, i.e., the subgroup $\mathbb{Z}_{p^\infty} = \langle \{1/p^n \mid n \in \mathbb{N}\} \rangle \leq \mathbb{T}$, where $\mathbb{T}$ denotes the one-dimensional torus. A group G is called *reduced* if $d(G) = \{0\}$. Finite groups are reduced and so, in particular, they are not divisible provided that they are non-trivial. Let us recall, that a divisible subgroup H of an abelian group G always splits, i.e., there exists another subgroup K of G such that $G \simeq H \oplus K$. The class of divisible groups is stable under taking quotients, products and direct sums.

For a group G and $n \in \mathbb{N}$, let $G[n] = \{x \in G \mid x^n = 0\}$. Note that if G is abelian, then $G[n]$ is a subgroup of G. For a prime p let $r_p(G)$ be the *p-rank of* G (defined as $\dim_{\mathbb{Z}/p\mathbb{Z}} G[p]$), and $r(G) = r_0(G) + \sum_p r_p(G)$. Finally, we denote by $\pi(G)$ the set of primes $\{p \mid r_p(G) > 0\}$. Finally, divisible abelian groups are completely characterised by their ranks, as the following folklore fact shows.

Fact 1.2.3 ([79])**.** *If G is a divisible abelian group, then*

$$G \simeq \mathbb{Q}^{r_0(G)} \oplus \Big(\bigoplus_{p \in \pi(G)} (\mathbb{Z}_{p^\infty})^{r_p(G)} \Big).$$

As an example of the above mentioned 'rigidity' of divisible groups, in Theorem 9.3.3 we prove that if a fully decomposable torsion-free abelian group G is coarsely equivalent (i.e., 'as close as possible' from the large-scale point of view) to a divisible group, then G is also 'as close as possible' to a divisible group from algebraic point of view (i.e., $r_0(G/d(G)) < \omega$), in case G is either uncountable or homogeneous. These results go close, more or less, to the spirit of Theorem 1.2.2.

A very important functorial coarse structure on **TopGrp** is the one associated to the family of all relatively compact subsets, called compact-group coarse structure. We say that a subset Y of a topological space X is *relatively compact* if its closure $\overline{Y}$ in X is compact. This particular group coarse structure, to whose investigation Chapter 10 is devoted, has been studied, for example, in [125] and in [32]. Of a particular interest is studying this coarse structure in the realm of locally compact abelian groups, considering its behaviour in connection with the Pontryagin duality.

1.2.4 Pontryagin and Bohr functors

First of all, let us recall the definition of the Pontryagin functor. Let G be a topological abelian group. Denote by $\widehat{G}$ the family of all *continuous characters* $\chi\colon G \to \mathbb{T}$. With the pointwise operation, $\widehat{G}$ is actually an abelian group. We can endow $\widehat{G}$ with the *compact-open topology* $\widehat{\tau}$, defined by the base

$$\{W_{\widehat{G}}(K,U) \mid K \subseteq G \text{ compact, } U \in \vartheta(0_{\mathbb{T}})\}, \text{ where,}$$
$$\text{for every compact subset } K \subseteq G \text{ and every } U \subseteq \vartheta(0_{\mathbb{T}}),$$
$$W_{\widehat{G}}(K,U) = \{\chi \in \widehat{G} \mid \chi(K) \subseteq U\}.$$

With this topology, $\widehat{G}$ is a topological abelian group, called *dual group of G*. For every continuous homomorphism $f\colon G \to H$ between topological abelian groups, there exists a continuous homomorphism $\widehat{f}\colon \widehat{H} \to \widehat{G}$ defined by the law $\widehat{f}(\chi) = \chi \circ f$, for every $\chi \in \widehat{H}$. The Pontryagin-van Kampen duality theorem states that the functor $\widehat{\cdot}\colon$ **LCA** $\to$ **LCA**, called *Pontryagin functor*, induces a duality, where **LCA** is the category of locally compact abelian groups and continuous homomorphisms between them (see [138] for details).

By using the Pontryagin duality, the following fact can be deduced (see [53]).

Theorem 1.2.4. *Let G be a locally compact abelian group. Then G is of the form $G = \mathbb{R}^n \times G_0$, where G_0 has an open compact subgroup K. If G is connected, then $G_0 = K$ is connected as well.*

The inspiration of considering the compact-group coarse structure in relation with the Pontryagin duality comes from a beautiful result due to Nicas and Rosenthal ([126]), where that functor represents as a bridge between small-scale and large-scale dimensions of locally compact abelian groups. More precisely, their result states that the covering dimension of a locally compact abelian group (see [4, 37, 161] for a surveys on dimension theory of topological groups) coincides with the asymptotic dimension of its dual. Let us recall that Pasynkov ([130]) proved that in the realm of locally compact groups the covering dimension coincides with both the small inductive dimension and the large inductive dimension.

After briefly investigating the asymptotic dimension of generic topological groups (in particular, in Theorem 10.1.8 we provide a characterisation of locally compact groups with asymptotic dimension 0), in §10.2 we continue the study of the asymptotic dimension of locally compact abelian groups, providing also an alternative computation making no recourse to the dual group $\widehat{G}$ (see (10.3)). This allows us to obtain, among others, also a new self-contained proof of Nicas and Rosenthal's result. As another consequence of Theorem 10.2.1, we improve the additivity result for the asymptotic dimension of discrete abelian groups due to Dranishnikov and Smith ([70]) by extending it to the realm of locally compact abelian groups endowed with their compact-group coarse structure (Corollary 10.2.3).

According to results of Grave ([89]), there is no relation between $\operatorname{asdim}(X,\mathcal{E}_X)$ and $\operatorname{asdim}(Y,\mathcal{E}_Y)$ of coarse spaces provided with a bornologous injective map $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$. In contrast with this situation in the general case of coarse spaces, we show in §10.2.2 that when X and Y are locally compact abelian groups equipped with their compact-group coarse structure and f is continuous homomorphism with dense image (i.e., an epimorphism in the category **LCA**), then

$$\operatorname{asdim} X \geq \operatorname{asdim} Y. \tag{1.4}$$

In particular, if X and Y have the same support X and $f=id_X$ (i.e., the topology of X is finer than the one of Y), then (1.4) holds. We give numerous examples witnessing the failure of this property in the class of precompact abelian groups. We provide also the "dual result', stating that, for every monomorphism $f\colon G\to H$ of locally compact abelian groups, $\dim G\leq\dim H$ (Theorem 10.2.10).

The Pontryagin duality functor provides a bridge also between other pairs of small-scale and large-scale properties. For example, we prove that a locally compact abelian group is metrisable if and only if the compact-group coarse structure of its dual is metrisable (Theorem 10.2.16), and a locally compact abelian group is compact if and only if the compact-group coarse structure of its dual is locally finite (Proposition 10.2.17). Moreover, in Appendix B the same functor is used in order to connect the coarse entropy of surjective endomorphisms of discrete abelian groups with the topological entropy of the dual morphisms.

The Bohr functor is another endofunctor of the category **TopAbGrp** of topological abelian groups and continuous homomorphisms between them in which we are interested. A subset B of an abelian group G is *big* if there exists a finite subset F of G such that $G=FB$. A topological abelian group is *totally bounded* if every open non-empty subset of G is big.

Every topological group (G,τ) admits a finest totally bounded group topology τ^+, called the *Bohr topology* of (G,τ), such that $\tau^+\subseteq\tau$. Clearly, (G,τ), $id_G\colon(G,\tau)\to(G,\tau^+)$ is continuous. The *Bohr functor* $\cdot^+\colon$ **TopAbGrp** $\to$

TopAbGrp associates to every topological group the same group endowed with its Bohr topology. It is well-defined since, for every continuous homomorphism $f\colon (G,\tau)\to(H,\delta)$ between topological groups, the map $f^+=f\colon(G,\tau^+)\to(H,\delta^+)$ is continuous homomorphism. For the sake of simplicity, in the sequel we denote by G^+ the topological group (G,τ^+), where τ is a group topology on G, if there is no risk of ambiguity.

In §10.3 we study the impact of the Bohr functor on the compact-group coarse structure with particular emphasis on the preservation of the asymptotic dimension. To this end we introduce the class $\mathscr{B}$ of topological abelian groups G such that G and G^+ have the same compact-group coarse structure. The class $\mathscr{B}$ is obviously contained in the larger class $\mathscr{H}$ of topological abelian groups G such that $\operatorname{asdim} G=\operatorname{asdim} G^+$. The choice to introduce this class was inspired by a theorem of Hernández ([95]) regarding the equality $\dim G=\dim G^+$ for locally compact abelian groups G. In order to obtain a better description of the smaller class $\mathscr{B}$, we connect it to the class $\mathscr{G}$ of Glicksberg groups (see Definition 10.3.2, Proposition 10.3.11 and Theorem 10.3.12). Let us briefly mention here that these groups are usually considered within the class MAP of maximally almost periodic groups. Here we propose generalizations that go beyond maximal almost periodicity (they appear here at two distinct levels, see Definition 10.3.2).

1.3 Coarse hyperspaces and related structures

Let (X,d) be a metric space. In Example 1.1.10 we have seen how one can induce a quasi-metric on the power set $\mathcal{P}(X)$ of X. Alternatively, we can say that, for every $Y,Z\subseteq X$ and $R\geq 0$,

$$d_H^q(Y,Z)\leq R \text{ if and only if } Z\subseteq B_d(Y,R).$$

Furthermore, Hausdorff provided a metric on the power set $\mathcal{P}(X)$ of X as follows: for any two subsets $Y,Z\subseteq X$, the *Hausdorff distance* between them is the value

$$d_H(Y,Z)=\inf\{R\geq 0\mid Y\subseteq B_d(Z,R),\ Z\subseteq B_d(Y,R)\}.$$

The pair $(\mathcal{P}(X),d_H)$ is called *metric hyperspace*. We can characterise the Hausdorff metric in a more convenient way for our purpose of extending the notion of hyperspace to both uniform and coarse spaces. Let Y and Z be two subsets of X and $R\geq 0$. Then, according to (1.1), the following properties are equivalent:

(a) $d_H(Y,Z)\leq R$;
(b) $(Y,Z)\in U_R^{d_H}$;
(c) $Z\subseteq B_d(Y,R)$ and $Y\subseteq B_d(Z,R)$;
(d) $Z\subseteq U_R^d[Y]$ and $Y\subseteq U_R^d[Z]$;
(e) $d_H^q(Y,Z)\leq R$ and $d_H^q(Z,Y)\leq R$.

Suppose now that we have a uniform space $(X,\mathcal{U})$. Inspired by the equivalence (b)↔(d) in the above list, for an entourage $U\in\mathcal{U}$, we can define a new entourage $\exp U$ satisfying the following property: for every $Y,Z\subseteq X$, $(Y,Z)\in\exp U$ if and only if $Z\subseteq U[Y]$ and $Y\subseteq U[Z]$. Then the family $\exp B_{\mathcal{U}}=\{\exp U\mid U\in\mathcal{U}\}$ is a base of a uniformity $\exp\mathcal{U}$. The pair $(\mathcal{P}(X),\exp\mathcal{U})$ is called *uniform hyperspace* (also known as *Hausdorff-Bourbaki hyperspace*).

Since we have not used the fact that U is an entourage of an uniformity, the definition of $\exp U$ can be carried out in a wider setting, which will be useful when we will discuss the coarse hyperspace (§11.1). Let X be a set and W be an entourage of X. Then

$$\exp W = \{(Y,Z) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid Y \subseteq W[Z],\, Z \subseteq W[Y]\}. \qquad (1.5)$$

Note that, $(\exp W)^{-1} = \exp W$.

It is easy to check that the definition of uniform hyperspace agrees with the Hausdorff metric. More explicitly, if (X,d) is a metric space, we have $\exp(\mathcal{U}_d) = \mathcal{U}_{d_H}$ (actually, for every $R \in \mathbb{R}_{\geq 0}$, $\exp U_R^d = U_R^{d_H}$).

Let us now present some properties of the uniform hyperspace. Denote by $\imath\colon X \to \mathcal{P}(X)$ the map that associates to every point $x \in X$ the singleton $\{x\}$. The following fact, concerning the map just defined is straightforward.

Fact 1.3.1. *If $(X,\mathcal{U})$ is a uniform space, then $\imath\colon X \to \mathcal{P}(X)$ is a uniform embedding.*

Denote by $\mathcal{S}(X)$ the family of all singletons of a set X. Then, for every uniform space X and every family $\mathcal{S}(X) \subseteq \mathcal{A}(X) \subseteq \mathcal{P}(X)$, Fact 1.3.1 implies that the corestriction $\imath\colon X \to \mathcal{A}(X)$, where $\mathcal{A}(X)$ is endowed with the subspace uniformity induced by the uniform hyperspace, is a uniform embedding.

The uniform hyperspace is not Hausdorff in general, even if the initial uniform space is Hausdorff. In the following proposition we discuss conditions implying the preservation of that property. Denote by $\mathcal{F}(X)$ the family of all closed subsets of a topological space X. If X is a uniform space, $\mathcal{F}(X)$ denotes the subsets that are closed with respect to the uniform topology.

Proposition 1.3.2. *Let $(X,\mathcal{U})$ be a Hausdorff uniform space and $\mathcal{A}(X) \subseteq \mathcal{P}(X)$ be a family closed under finite unions and such that $\mathcal{S}(X) \subseteq \mathcal{A}(X)$. Then the following properties are equivalent:*

(a) $(\mathcal{A}(X), \exp\mathcal{U}|_{\mathcal{A}(X)})$ is Hausdorff;

(b) $\mathcal{A}(X) \subseteq \mathcal{F}(X)$.

Proof. The implication (b)→(a) is a classical result (see, for example, [104]). Conversely, suppose that $A \in \mathcal{A}(X)$ is not closed. Hence, there exists $x \notin A$ such that, for every $U \in \mathcal{U}$, $U[x] \cap A \neq \emptyset$. Then, for every $U \in \mathcal{U}$,

$$A \subseteq U[A \cup \{x\}] \quad \text{and} \quad A \cup \{x\} \subseteq U[A],$$

and so $(\mathcal{A}(X), \exp\mathcal{U}|_{\mathcal{A}(X)})$ is not Hausdorff. □

In the statement of Proposition 1.3.2, the request that $\mathcal{S}(X) \subseteq \mathcal{A}(X)$ is to ensure that the corestriction $\imath\colon X \to \mathcal{A}(X)$ is defined and thus it is still a uniform embedding.

Proposition 1.3.2 is the reason why many authors consider $(\mathcal{F}(X), \exp\mathcal{U}|_{\mathcal{F}(X)})$ as the hyperspace of a uniform space $(X,\mathcal{U})$.

1.3.1 Coarse hyperspaces

Inspired by the uniform hyperspace and the paper [146], we introduce in §11.1 the coarse hyperspace $\exp X$ of a coarse space X, a coarse structure on the power set of X that generalises the large-scale geometry of the metric hyperspace.

As connectedness for coarse spaces is the large-scale counterpart of the Hausdorff property for uniform spaces, it is natural to study connectedness in the coarse hyperspace. As one may have expected, the coarse hyperspace is highly disconnected even in simple cases. For example, we compute the number of connected components for some coarse hyperspaces. In particular, we consider the hyperspace of an ideal coarse structure (Corollary 11.2.2) and of a group endowed with the finitary-group coarse structure (Proposition 11.3.9). In the latter case, if the group is infinite, the number of connected components coincides with the cardinality of the power set of the group.

In order to work with more manageable objects, as we have done for the uniform hyperspace, we can consider some subspaces of the whole coarse hyperspace. For a coarse space X and a family of subsets $\mathcal{A}(X) \subseteq \mathcal{P}(X)$, we denote by $\mathcal{A}\text{-}\exp X$ the subspace of the coarse hyperspace induced on the subspace $\mathcal{A}(X)$, and we call it the $\mathcal{A}$-coarse hyperspace. We discuss how the properties of coarse spaces can be described by means of the coarse hyperspaces or their subspaces. In particular, we are interested in $\mathcal{A}$-coarse hyperspaces induced by notion of sizes.

Combinatorial size of subsets of a group or semigroup has long been studied in combinatorial group theory and harmonic analysis. The fundamental paper [20] of Bella and Malykhin introduced and studied largeness, smallness and extra-largeness in groups as well as their relation in a systematic way. This paper, as well as the consequent ones [21, 5, 54, 91, 118], proposed various challenging problems, many of them arising in the framework of topological groups. Furthermore, Protasov and Banakh showed that balleans provide a nice and unifying way to describe size in a sufficiently general setting. The sizes and various cardinal invariants of balleans related to size have been intensively studied by the Ukrainian school ([114, 115, 135, 139, 141, 148, 149, 150]). The survey [142] is very helpful to get a better idea on the topic. Moreover, we also cite [64], where a comprehensive discussion about sizes in balleans and their preservations along morphisms is provided.

Among the notion of sizes, the family $\flat(X)$ of bounded subsets of a coarse space X is particularly useful in this context. In fact, we prove in Proposition 11.1.10 that the family $\flat(X)$ plays the role of the family $\mathcal{F}(X)$ for the uniform hyperspace in Proposition 1.3.2. More explicitly, the $\flat$-coarse hyperspace is connected provided that the starting coarse space X is connected and $\flat(X)$ is the largest family preserving that property. Let us also point out that the $\flat$-coarse hyperspace was already introduced in [146] in terms of balleans.

Finally, let us add some remarks on the asymptotic dimension of the coarse hyperspace. Since a coarse space can be seen as a subspace of its coarse hyperspace, the asymptotic dimension of the coarse hyperspace is bounded from below by the asymptotic dimension of the original space. Moreover, we show that, if a coarse space X has asymptotic dimension 0, then also its coarse hyperspace has asymptotic dimension 0 (Proposition 11.2.7). In [169], it is shown that the situation is completely different if we assume that a metric space has a certain subspace with positive asymptotic dimension. In particular, if a connected geodesic metric space (e.g., a non-directed connected graph endowed with its path metric) has positive asymptotic dimension, then its $\flat$-coarse hyperspace is not coarsely embeddable into a Hilbert space, and this property implies having infinite asymptotic dimension. However, in the same paper is proved that, if we consider the subspace of the metric hyperspace $(\mathcal{P}(X), d_H)$ induced on the family $[X]^{\leq n} = \{A \subseteq X \mid |A| \leq n\}$, then many infinite-dimensional properties are preserved (as for having finite asymptotic dimension, we refer to [154, 111]).

1.3.2 Coarse structures on the subgroup lattice of a group

Previously, we have suggested to consider some subspaces of a coarse hyperspace based on sizes. For example, the restriction of it to the family of all bounded subsets is particularly interesting. However, if we are dealing with a coarse group, another canonical choice can be made. For a group G, define $L(G)$ as the subgroup lattice of G, i.e., the family of all subgroups of G. In Chapter 12 we investigate the restriction of the coarse hyperspace of a coarse group G to the family $L(G)$. In order to study this coarse space, it is useful to work in the realm of balleans, that provide an equivalent description of the large scale geometry.

Let $\mathfrak{B}$ be a ballean. We denote by $\exp\mathfrak{B}$ the ballean, called *hyperballean*, associated to the coarse hyperspace of the coarse space induced by $\mathfrak{B}$. If $\mathfrak{B}_G$ is the ballean associated to the finitary-group coarse structure on G, we can consider the subballean $\mathcal{L}(G) = \exp\mathcal{B}_G|_{L(G)}$, called the *subgroup exponential hyperballean.* The second ballean structure on $L(G)$, denoted by ℓ-$\mathcal{L}(G)$, is called the *subgroup logarithmic hyperballean.* The latter can be characterised as follows: it is the ballean structure on $L(G)$ induced by the metric

$$d(H,K) = \log(\max\{|H : H \cap K|, |K : H \cap K|\}),$$

where H and K are two subgroups of G. Actually, we provide a ballean structure, ℓ-$\exp\mathcal{B}_G$, on the entire power set of G such that ℓ-$\mathcal{L}(G)$ is the restriction of ℓ-$\exp\mathcal{B}_G$ to $L(G)$. The balleans $\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$ have also the same connected components determined by the property that two subgroups are in the same connected component if and only if they are commensurable. In particular, the existence of isolated points (i.e., points in $L(G)$ whose connected components is just a singleton) is closely related to divisibility. In fact, we show that a subgroup H of G is isolated if and only if it is divisible and has a torsion-free direct summand.

Moreover, while all examples of subgroup exponential hyperballeans we considered have asymptotic dimension 0, the computation of asymptotic dimension of subgroup logarithmic hyperballeans is much more interesting. In particular, we compute it for some well known groups, such as $\mathbb{Z}$ ($\operatorname{asdim}\ell$-$\mathcal{L}(\mathbb{Z}) = \infty$) or the Prüfer p-group $\mathbb{Z}_{p^\infty}$ ($\operatorname{asdim}\mathbb{Z}_{p^\infty} = 1$), where p is a prime, we find necessary conditions on an abelian groups G that imply $\operatorname{asdim}\ell$-$\mathcal{L}(G) < \infty$ (G has to be torsion and layerly finite), and we characterise those groups G with $\operatorname{asdim}\ell$-$\mathcal{L}(G) = 0$ (G has to be torsion, reduced and with all p-ranks finite).

The last part of Part III is focused on answering the following natural question. If G and H are two isomorphic groups, then $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are asymorphic (i.e., isomorphic in the category of balleans and coarse maps) and we write $\mathcal{L}(G) \approx \mathcal{L}(H)$. Moreover, the isomorphism between G and H yields also ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$. However, the converse is not true in general. As a 'rigidity result' we mean a set of conditions that imply that these converse implications holds. In other words, it is a collection of properties that implies that G is isomorphic to H whenever $\mathcal{L}(G) \approx \mathcal{L}(H)$ or ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$. Note that this is not the usual notion of rigidity in large-scale geometry (see, for example, [157]). In particular, we focus on some special cases, namely, for a group G, we investigate the hypothesis $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z})$, $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z}_{p^\infty})$, ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Z})$, and ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Z}_{p^\infty})$, for some p prime, and we obtain the following results.

Theorem 1.3.3. *Let G be a group.*

(a) Suppose that G has an element of infinite order. Then $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z})$ if and only if $G \simeq \mathbb{Z}$.

(b) Suppose that G is abelian, then $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z})$ if and only if either $G \simeq \mathbb{Z}$ or $G \simeq \mathbb{Z}_{p^\infty}$, for some prime p.

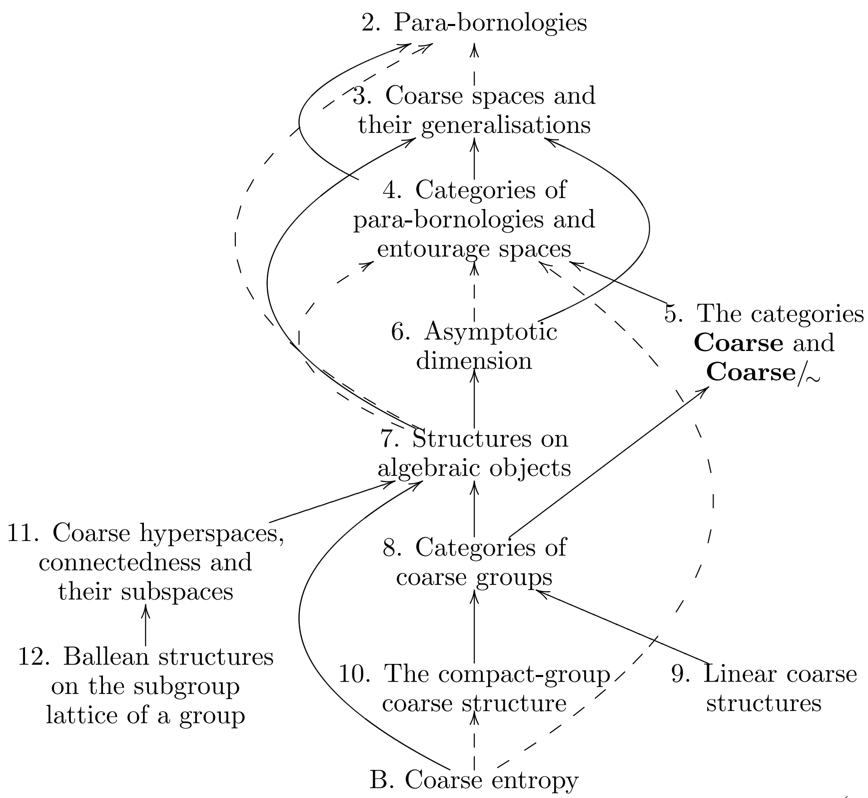
Theorem 1.3.4. *Let G be a group and p be a prime.*

(a) ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Z})$ if and only if $G \simeq \mathbb{Z}$;

(b) ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Z}_{p^\infty})$ if and only if $G \simeq \mathbb{Z}_{q^\infty}$, for some prime q.

1.4 Structure of the thesis

We now want to briefly discuss the structure of the thesis. In diagram (1.6), we summarise the relationships among the content of the various chapters. An arrow from chapter X to chapter Y means that the content of X strongly depends on the notions and results discussed in Y, while, if the arrow is dashed, then the dependency from the material of chapter Y is only partial.



(1.6)

Part of this thesis content is collected in some papers. The results of Chapter 3, and partially of Chapters 4 and 7 are contained in [176]. The content of §5 is divided between [65] and [175]. The paper [66] collects the results of Chapter 9 and, partially, of Chapter 8. The remaining part of the latter chapter is contained in [67], together with the content of §10. Some results of Chapter 11 are a translation in terms of coarse spaces of those contained in [55] and [56]. The latter paper collects also the results of Chapter 12. The notion of the coarse entropy, given in Appendix B, is described in [177]. Finally, for the reader convenience, all the necessary categorical background, mainly taken from [1], is provided in Appendix A.

Part I

A foundational and categorical approach to coarse geometry

Chapter 2

Para-bornologies

2.1 Para-bornologies: objects and morphisms

An *ideal* $\mathcal{I}$ on a set X is a family of subsets of X which is closed under taking subsets and finite unions. For example, for every set X and every cardinal κ, $[X]^{<\kappa} = \{F \subseteq X \mid |F| < \kappa\}$ is an ideal. We call $[X]^{<\omega}$ the *finitary ideal on* X. Note that, an ideal $\mathcal{I}$ on X is a *cover of* X (i.e., $\bigcup \mathcal{I} = X$) if and only if it contains the finitary ideal on X.

Definition 2.1.1. Let X be a set. A *para-bornology* is a collection $\beta = \{\beta(x) \mid x \in X\}$ of ideals on X such that $\{x\} \in \beta(x)$, for every $x \in X$. The pair (X, β) is a *para-bornological space*. An element $A \in \beta(x)$ is said to be *bounded from* x. Conversely, an element $A \notin \beta(x)$ is said to be *unbounded from* x. Moreover, a subset B of X is *bounded* if it is bounded from each of its points.

When the para-bornology is clear, we denote a para-bornological space (X, β) by its support X.

If $\mathcal{F}$ is a family of subsets of a set X, denote by $\mathfrak{cl}(\mathcal{F})$ its *completion*, i.e., $\mathfrak{cl}(\mathcal{F}) = \{A \subseteq X \mid \exists F \in \mathcal{F} : A \subseteq F\}$. If X is a set, a family of families of subsets $\mathcal{B} = \{\mathcal{B}(x) \mid x \in X\}$ of X is a *base for a para-bornology* if $\mathfrak{cl}(\mathcal{B}) = \{\mathfrak{cl}(\mathcal{B}(x)) \mid x \in X\}$ is a para-bornology.

If (X, β) is a para-bornological space, then we can consider the following *global properties* that link the ideals $\beta(x)$, where $x \in X$, with each other:

(G_1) for every $x, y \in X$, if $\{y\} \in \beta(x)$, then $\{x\} \in \beta(y)$;
(G_2) for every $x, y \in X$ and every $A \in \beta(y)$, if $\{y\} \in \beta(x)$, then $A \in \beta(x)$.

Definition 2.1.2. If a para-bornological space (X, β) satisfies (G_1), then β is called a *semi-bornology* and (X, β) a *semi-bornological space*, while, if (X, β) satisfies (G_2), then β is called a *quasi-bornology* and (X, β) a *quasi-bornological space*. Finally, if (X, β) satisfies both (G_1) and (G_2), then β is a *pre-bornology* and (X, β) is a *pre-bornological space*.

It is important to notice that a non-empty subset of a pre-bornological space is bounded from a point if and only if it is bounded.

Remark 2.1.3. Since the notion of pre-bornology can be found in the literature (see, for example, [103]), we need to clarify the terminology introduced in

Definition 2.1.2. Note, in fact, that the notion of pre-bornology can be found in literature. According to [103], a *pre-bornology* (while it can be found also in [10] under the name *bounded structure*) $\mathcal{B}$ *on a set* X is a family of subsets of X satisfying the following properties:
(a) $\mathcal{B}$ is a cover of X;
(b) $\mathcal{B}$ is closed under taking subsets;
(c) for every $A, B \in \mathcal{B}$, $A \cup B \in \mathcal{B}$ provided that $A \cap B \neq \emptyset$.
If $\mathcal{B}$ is a family of subsets satisfying (a)–(c), then the family $\beta_{\mathcal{B}} = \{\beta_{\mathcal{B}}(x) \mid x \in X\}$, where, for every $x \in X$, $\beta_{\mathcal{B}}(x) = \mathfrak{cl}(\{A \in \mathcal{B} \mid x \in A\})$, is a pre-bornology according to Definition 2.1.2. Conversely, if (X, β) is a pre-bornological space, then the family $\mathcal{B}_\beta = \bigcup_{x\in X} \beta(x)$ satisfies (a)–(c). Since this correspondence is one-to-one, without loss of generality, we can use the term pre-bornology for both notions interchangeably.

Pre-bornologies are a generalisation of the notion of bornology (see, for example, [11]). Let us recall the classical definition of bornology. A *bornology* on a set X is a family $\mathcal{B}$ of subsets of X which forms both a cover and an ideal. Hence the only difference with the definition of pre-bornology is the fact that a bornology is closed under taking arbitrary finite unions. We will discuss a bit more this notion in our setting in §2.2.

Let (X, β) be a para-bornological space and Y be a subset of X. Then Y can be endowed with the *subspace para-bornology* $\beta|_Y$ defined as follows: for every $y \in Y$, $\beta|_Y(y) = \{B \cap Y \mid B \in \beta(y)\}$. In this case, the pair $(Y, \beta|_Y)$ is called *para-bornological subspace of* X. Moreover, it is easy to check the following implications:
- $\beta|_Y$ is a semi-bornology if β is;
- $\beta|_Y$ is a quasi-bornology if β is;
- $\beta|_Y$ is a bornology if β is.

Example 2.1.4. (a) Let X be a set. Then there are always two para-bornologies (actually, two pre-bornologies) on X, namely the *discrete pre-bornology* β_{dis}, where, for every $x \in X$, $\beta_{dis}(x) = \{\{x\}, \emptyset\}$, and the *trivial* (or *indiscrete*) *pre-bornology* β_{triv}, where, for every $x \in X$, $\beta_{triv}(x) = \mathfrak{cl}(\{X\})$. Moreover, a singleton $\{*\}$ can be endowed just one para-bornology: $\beta_{dis}(*) = \beta_{triv}(*) = \mathfrak{cl}(\{\{*\}\})$.

(b) Similarly to what happens for topological spaces, metrics and their generalisations define para-bornologies. Namely, if X is a set endowed with a semi-positive-definite map d, then family $\beta_d = \{\beta_d(x) \mid x \in X\}$, where, for every $x \in X$,
$$\beta_d(x) = \mathfrak{cl}(\{B(x, R) \mid R > 0\}),$$
is a para-bornology, called *metric para-bornology.*
Furthermore, if d is a semi-metric, then β_d is a semi-bornology, while, if d is a quasi-metric, then β_d is a quasi-bornology. In particular, note that, if d is a metric, then β_d is a pre-bornology. The previous implications cannot be reverted. For example, consider the quasi-metric d_1 and the semi-metric d_2 on $\mathbb{N}$ defined as follows: for every two points $m, n \in \mathbb{N}$,
$$d_1(m,n) = \max\{|m-n|-1, 0\}, \quad \text{and} \quad d_2(m,n) = \begin{cases} n-m & \text{if } m \leq n, \\ 2(m-n) & \text{otherwise.} \end{cases} \tag{2.1}$$

Although d_1 does not satisfy the triangular inequality, and d_2 is not symmetric, both β_{d_1} and β_{d_2} coincide with the metric para-bornology induced by the usual metric, and so they are pre-bornologies.
Finally, note that this definition is precisely the 'dualisation'. In fact, if d is a quasi-metric, then the metric topology is induced by the neighbourhood system $\vartheta_d = \{\vartheta_d(x) \mid x \in X\}$, where, for every $x \in X$, $\vartheta_d(x) = \{V \subseteq X \mid \exists R > 0 : x \in V \subseteq B(x, R)\}$.
As in topology, actually metric para-bornologies lose information about their 'uniform structure'. This problem will be considered in §3, where we introduce the large-scale counterparts of uniform spaces and their generalisations.

Question 2.1.5. What conditions on a para-bornology β ensure the existence of a semi-positive-definite map d such that $\beta = \beta_d$? In particular, if β is a pre-bornology, what conditions imply that exists a metric d such that $\beta = \beta_d$?

2.1.1 Morphisms between para-bornological spaces

If $f\colon X \to Y$ is a map between sets, and $\mathcal{A}$ and $\mathcal{B}$ are two families of subsets of X and Y, respectively, we denote by $f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\}$ and $f^{-1}(\mathcal{B}) = \{f^{-1}(B) \mid B \in \mathcal{B}\}$.

Definition 2.1.6. Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a map between para-bornological spaces. The map f is:

- *boundedness preserving in x*, where $x \in X$, if images of subsets that are bounded from x are bounded from $f(x)$, i.e., $f(\beta_X(x)) \subseteq \beta_Y(f(x))$;
- *boundedness preserving* if f is boundedness preserving in x, for every $x \in X$;
- *weakly boundedness copreserving* if, for every $x \in X$ and $A \in \beta_Y(f(x))$, there exists $A'_z \in \beta_X(z)$, for every $z \in f^{-1}(f(x))$, such that $f(\bigcup\{A_z \mid z \in f^{-1}(f(x))\}) = A \cap f(X)$ (or, equivalently, $f(\bigcup\{A_z \mid z \in f^{-1}(f(x))\}) \supseteq A \cap f(X)$);
- *boundedness copreserving* if, for every $x \in X$ and $A \in \beta_Y(f(x))$, there exists $A' \in \beta_X(x)$ such that $f(A') = A \cap f(X)$ (or, equivalently, $f(A') \supseteq A \cap f(X)$);
- *proper* if, for every $y \in Y$ and every $A \in \beta_Y(y)$, $f^{-1}(A) \in \beta_X(z)$, for every $z \in f^{-1}(y)$;
- a *large-scale embedding* if it injective, boundedness preserving and proper;
- a *large-scale isomorphism* if it is bijective and both f and f^{-1} are boundedness preserving.

Let us first give some easy examples of the properties just introduced.

Example 2.1.7. Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a map between two para-bornological spaces. Then the following properties trivially hold:

(a) if β_X is the discrete coarse structure, then f is boundedness preserving;
(b) if β_Y is the discrete coarse structure, then f is boundedness copreserving;
(c) if β_X is the trivial coarse structure, then f is proper;
(d) if β_Y is the trivial coarse structure, then f is boundedness preserving.

Let us add some remarks on Definition 2.1.6.

Remark 2.1.8. (a) Composites of boundedness preserving maps are boundedness preserving.

(b) Let $f\colon (X,\beta_X)\to(Y,\beta_Y)$ be a map between para-bornological spaces. Suppose that both β_X and β_Y are pre-bornologies. Then f is boundedness preserving if and only if $f(\bigcup\beta_X)\subseteq\bigcup\beta_Y$.

(c) A trivial, although useful, example of a boundedness preserving map is the following: if $f\colon X\to Y$ is a map between para-bornological spaces and $x\in X$, then f is boundedness preserving in x if Y is bounded from $f(x)$. If X is bounded, then f is trivially boundedness copreserving. Another trivial property is that a bijective map f between para-bornological spaces is boundedness copreserving if and only if f^{-1} is boundedness preserving.

(d) For a map between para-bornological spaces, the following chain of implications is trivial:

$$\text{proper}\longrightarrow\text{boundedness copreserving}\longrightarrow\text{weakly boundedness copreserving.} \tag{2.2}$$

Those implications cannot be reverted in general (Example 2.1.9). However this three concepts coincide if the map is injective.

(e) Let X be a para-bornological space. Then X is bounded if and only if every *constant map* $f\colon X\to Y$ (i.e., $|f(X)|=1$) is proper.

(f) Thanks to item (d), we can give different characterisations to large-scale isomorphisms. For a bijective map $f\colon X\to Y$ between para-bornological spaces, it is easy to check that the following properties are equivalent: f is boundedness preserving if and only if f^{-1} is proper. Hence, f is a large-scale isomorphism if and only if f is bijective, bornologous and f is either proper or boundedness copreserving or weakly boundedness copreserving.

Let us show that the implications in (2.2) cannot be reverted in general. In particular, Example 2.1.9(a) shows that the first arrow cannot be reverted, while Example 2.1.9(b) proves that the second one cannot be reverted.

Example 2.1.9. (a) Let X be a two point space and Y just a singleton. Endow X with the discrete para-bornology. Then the map $f\colon X\to Y$ is boundedness copreserving, but it is not proper.

(b) Let $X=\{a,b,b',c\}$ and $Y=\{a,b,c\}$. Let us endow X and Y with the para-bornologies β_X and β_Y, defined as follows:

$$\beta_X(a)=\beta_X(b)=\mathfrak{cl}(\{\{a,b\}\}),\quad \beta_X(b')=\beta_X(c)=\mathfrak{cl}(\{\{b',c\}\}),$$
$$\beta_Y(a)=\{\{a\},\emptyset\},\quad \beta_Y(c)=\{\{c\},\emptyset\},\quad\text{and}\quad \beta_Y(b)=\mathfrak{cl}(\{Y\}),$$

Consider the map $f\colon X\to Y$ such that $f(a)=a$, $f(b)=f(b')=b$, and $f(c)=c$. Then f is weakly boundedness copreserving, but it is not boundedness copreserving.

Let us now introduce another notion which will be studied in §2.4.

Definition 2.1.10. A para-bornological space (X,β) is *locally finite at x*, where $x\in X$, if $\beta(x)\subseteq[X]^{<\omega}$. Moreover, (X,β) is *locally finite* if it is locally finite at every point.

Proposition 2.1.11. *Let $f\colon (X,\beta_X)\to(Y,\beta_Y)$ be a map between para-bornological spaces.*

(a) If f is proper, then f is boundedness copreserving and every fibre of f is bounded.

(b) *Suppose that X has the property (G_2) and is locally finite. If f is weakly boundedness copreserving and every fibre of f is bounded, then f is proper.*

Proof. (a) The first part of the statement follows from (2.2). Moreover, for every $x \in X$, $f^{-1}(f(x)) \in \beta_X(x)$.

(b) Let $x \in X$ and $A \in \beta_Y(f(x))$, with $A \subseteq f(X)$. Since $f^{-1}(f(x))$ is bounded, then it is finite because of the assumption. Let us denote $f^{-1}(f(x)) = \{z_1 = x, z_2, \dots, z_n\}$. Moreover, since f is weakly boundedness copreserving, for every $i = 1, \dots, n$, there exist $A_i \in \beta_X(z_i)$ (and thus $A_i \in \beta_X(x)$ since $\{z_i\} \in \beta_X(x)$) such that $A = f(B)$, where $B = \bigcup_{i=1}^n A_i$. Finally, since the union is finite and X is a quasi-bornological space, $B \in \beta_X(x)$. □

In Example 2.1.12, we show that the further hypothesis of Proposition 2.1.11(b) cannot be relaxed.

Example 2.1.12. (a) Let $X = \{0, 1, 2, 3\}$ and

$$\beta(0) = \mathfrak{cl}(\{\{3,0,1\}\}), \quad \beta(1) = \mathfrak{cl}(\{\{0,1,2\}\}), \quad \beta(2) = \mathfrak{cl}(\{\{1,2,3\}\}), \quad \text{and} \quad \beta(3) = \mathfrak{cl}(\{\{2,3,0\}\}).$$

Note that X is trivially locally finite and it satisfies (G_1). Consider the map $f\colon (X, \beta) \to (\{0,1\}, \beta_{triv})$ such that $f(0) = f(1) = 0$ and $f(2) = f(3) = 1$. Then f has bounded fibres and it is boundedness copreserving, although $f^{-1}(\{0,1\}) \notin \beta(x)$, for every $x \in X$.

(b) Let $Y = \mathbb{Z}$ and $X = \mathbb{Z} \times \mathbb{Z}$. Let f be the first canonical projection $p_1\colon X \to \mathbb{Z}$, i.e., $f(x, y) = x$, for every $(x, y) \in X$. Endow Y with the trivial pre-bornology $\beta_Y = \beta_{triv}$. Moreover, for every $(x, y) \in X$, define $\beta_X((x, y)) = \{A \subseteq X \mid |f(A)| < \infty\}$, and, with this choice, β_X is a pre-bornology (actually a bornology, see §2.2). Then f is weakly boundedness copreserving and has bounded fibres. However, f is not even boundedness copreserving (and so, in particular, it is not proper).

(c) As in the previous item, let $X = \mathbb{Z}$, $Y = \mathbb{Z} \times \mathbb{Z}$, and $f = p_1$. Again endow Y with the trivial pre-bornology $\beta_Y = \beta_{triv}$. For every $I, J \in [\mathbb{Z}]^{<\omega}$, define the following subset:

$$A_I^J = \Big(\bigcup_{j \in J}(\{j\} \times \mathbb{Z})\Big) \cup \Big(\bigcup_{i \in I}(\mathbb{Z} \times \{i\})\Big).$$

Then take $\beta_X = \mathfrak{cl}(\{A_I^J \mid I, J \in [\mathbb{Z}]^{<\omega}\})$, which is a pre-bornology (actually a bornology, see §2.2) on X. With these choices, f has bounded fibres and it is bounded copreserving, but it is not proper since $f^{-1}(Y) = X$, which is not bounded from any point.

As expected, properties (G_1) and (G_2) are invariant under large-scale isomorphisms.

Proposition 2.1.13. *Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a large-scale isomorphism. Then X satisfies (G_1) ((G_2)) if and only if Y satisfies (G_1) ((G_2), respectively).*

Proof. Suppose that X satisfies (G_1) and let $f(x)$ and $f(y)$ be two arbitrary points of Y. Suppose that $\{f(y)\} \in \beta_Y(f(y))$. Since f^{-1} is boundedness preserving, it implies that $\{y\} = f^{-1}(\{f(y)\}) \in \beta_X(f^{-1}(f(x))) = \beta_X(x)$ and thus, since X has (G$_1$), $\{x\} \in \beta_X(y)$. Finally, an application of the boundedness preserving map f implies the claim.

Now suppose that X satisfies (G$_2$), let $f(x)$ and $f(y)$ be two arbitrary points of Y and $f(A) \in \beta_Y(f(y))$. Since f^{-1} is boundedness preserving, we have that $\{y\} \in \beta_X(x)$ and $A \in \beta_X(y)$, which imply that $A \in \beta_X(x)$. Once again, the application of the boundedness preserving map f implies that $f(A) \in \beta_Y(f(x))$. □

Remark 2.1.14. Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a map between para-bornological spaces, and Z be a subset of X. Then it is easy to check the following implications:

(a) if f is boundedness preserving, then so is the restriction $f|_Z\colon (Z, \beta_X|Z) \to (Y, \beta_Y)$;

(b) if f is proper, then so is $f|_Z$.

However, if f is boundedness copreserving, $f|_Z$ is not even necessarily weakly boundedness copreserving. In Example 4.3.7 an explicit counter-example will be provided.

Let β and β' be two para-bornologies on a set X. Then β is *finer than* β' (and β' is *coarser than* β) if $id_X\colon (X, \beta) \to (X, \beta')$ is boundedness preserving. Moreover, if we fix a set X, then discrete pre-bornology on X, is the finest para-bornology, while the trivial pre-bornology on X, is the coarsest one. Denote by $\mathfrak{PB}(X)$ the family of all para-bornologies on X. Note that $\mathfrak{PB}(X)$ is actually a set and we can endow it with the preorder $\leq$ defined as follows: $\beta \leq \beta'$ if and only if β is finer than β'. Hence, β_{triv} is the top element, while β_{dis} is the bottom element.

For a set X, if $\beta, \beta' \in \mathfrak{PB}(X)$, we define their supremum $\beta \vee \beta' = \{\beta \vee \beta'(x) \mid x \in X\}$ and their infimum $\beta \wedge \beta' = \{\beta \wedge \beta'(x) \mid x \in X\}$ as follows: for every $x \in X$

$$\beta \vee \beta'(x) = \{A \cup B \mid A \in \beta(x),\, B \in \beta'(x)\} \quad \text{and} \quad \beta \wedge \beta'(x) = \beta(x) \cap \beta'(x).$$

It is not hard to check that the families just defined are actually para-bornologies.

Proposition 2.1.15. *Let X be a set and $\beta, \beta' \in \mathfrak{PB}(X)$.*

(a) Suppose that β and β' satisfy (G$_1$). Then both $\beta \vee \beta'$ and $\beta \wedge \beta'$ satisfy (G$_1$).

(b) Suppose that β and β' satisfy (G$_2$). Then $\beta \wedge \beta'$ satisfies (G$_2$).

Proof. Item (a) is trivial. Let us focus on item (b). Let x and y be two points of X, $\{y\} \in \beta \wedge \beta'(x)$, and $A \in \beta \wedge \beta'(y)$. Since $A \in \beta(y)$, $\{y\} \in \beta(x)$, $A \in \beta'(y)$, and $\{y\} \in \beta'(x)$, we can apply the property (G$_2$) to both β and β', and thus $A \in \beta(x)$ and $A \in \beta'(x)$, which yields to the desired claim. □

It is useful to consider also the infinite meet of a family of para-bornologies $\{\beta_i\}_{i \in I}$ on the same set X: it is the para-bornology $\bigwedge_i \beta_i = \{\bigwedge_i \beta_i(x) \mid x \in X\}$, where $\bigwedge_i \beta_i(x) = \bigcap_i \beta_i(x)$, for every $x \in X$. Proposition 2.1.15 can be extended also for arbitrary meets: if β_i has property (G$_1$) ((G$_2$)), for every $i \in I$, then $\bigwedge_i \beta_i$ has property (G$_1$) ((G$_2$), respectively).

Example 2.1.16. Proposition 2.1.15(b) cannot be extended for quasi-bornological spaces. In fact, consider $X = \{0, 1, 2\}$ and two quasi-bornologies β and β', such that:

$$\beta(0) = \mathfrak{cl}(\{\{0,1\}\}), \quad \beta(1) = \{\{1\}, \emptyset\}, \quad \beta(2) = \{\{2\}, \emptyset\},$$
$$\beta'(0) = \{\{0\}, \emptyset\}, \quad \beta'(1) = \mathfrak{cl}(\{\{1,2\}\}), \quad \beta'(2) = \{\{2\}, \emptyset\}.$$

Note that, $\{1\} \in \beta \vee \beta'(0)$ and $\{2\} \in \beta \vee \beta'(1)$, although $\{2\} \notin \beta \vee \beta'(0)$.

2.2 Connectedness axioms

Let (X, β) be a para-bornological space and let x and y be two points of X. If $\{y\} \in \beta(x)$, then we write $x \downarrow y$. Then $\downarrow$ is a reflexive relation. Denote by $\uparrow = \downarrow^{-1}$ its inverse relation, i.e., for every $x, y \in X$; $x \uparrow y$ if and only if $y \downarrow x$. Moreover, let $\searrow$ ($\nearrow$) be the transitive closure of $\downarrow$ (of $\uparrow$, respectively), and $\leftrightsquigarrow$ be the smallest equivalence relation containing $\downarrow$ (equivalently, $\uparrow$).

If X is a quasi-bornological space, we call $\downarrow = \searrow$ the *large-scale specialisation preorder of* X (which can be seen as the large-scale counterpart of the classical specialisation preorder in topological spaces).

Let (X, β) be a para-bornological space. For every subset A of X, we denote:

$$\begin{gathered}\mathcal{Q}_X^{\downarrow}(A) = \bigcup \big\{ \bigcup \beta(x) \mid x \in A \big\} = \{y \in X \mid \exists x \in A,\, x \downarrow y\},\\ \mathcal{Q}_X^{\uparrow}(A) = \{y \in X \mid \exists x \in A,\, y \downarrow x\}, \quad \mathcal{Q}_X^{\searrow}(A) = \{y \in X \mid \exists x \in A,\, x \searrow y\},\\ \mathcal{Q}_X^{\nearrow}(A) = \{y \in X \mid \exists x \in A,\, y \searrow x\}, \quad \text{and}\\ \mathcal{Q}_X(A) = \mathcal{Q}_X^{\leftrightsquigarrow}(A) = \{y \in X \mid \exists x \in A,\, y \leftrightsquigarrow x\}.\end{gathered} \tag{2.3}$$

When A is just a singleton, we usually omit the brackets, for example, $\mathcal{Q}_X(x) = \mathcal{Q}_X(\{x\})$, for a point $x \in X$.

Let (X, β) be a para-bornological space. Since the relation $\leftrightsquigarrow$ is an equivalence relation, X can be partitioned in its equivalence classes $\{\mathcal{Q}_X(x) \mid x \in X\}$, called *connected components*. Moreover, denote by $\operatorname{dsc}(X, \beta)$ the number of connected components of X. Let us note the following trivial facts.

Fact 2.2.1. *If X and Y are two large-scale isomorphic para-bornologies, then* $\operatorname{dsc} X = \operatorname{dsc} Y$.

Fact 2.2.2. *Let β and β' be two para-bornologies on a set X. If $\beta \leq \beta'$, then* $\operatorname{dsc}(X, \beta) \geq \operatorname{dsc}(X, \beta')$.

Fact 2.2.3. *Let X be a para-bornological space.*
(a) If X satisfies (G_1), then $\downarrow = \uparrow$, and $\searrow = \nearrow = \leftrightsquigarrow$.
(b) If X satisfies (G_2), then $\downarrow = \searrow$, and $\uparrow = \nearrow$.
(c) If X satisfies both (G_1) and (G_2), then $\downarrow = \uparrow = \searrow = \nearrow = \leftrightsquigarrow$.

Remark 2.2.4. Let $f \colon X \to Y$ be a boundedness preserving map between two para-bornological spaces. Fix two points $x, y \in X$. Then it is trivial to check that the following implications hold:
(a) $f(x) \downarrow f(y)$ if $x \downarrow y$, and $f(x) \uparrow f(y)$ if $x \uparrow y$;
(b) $f(x) \searrow f(y)$ if $x \searrow y$, and $f(x) \nearrow f(y)$ if $x \nearrow y$;

(c) $f(x) \leftrightsquigarrow f(y)$ if $x \leftrightsquigarrow y$.
Hence, in particular, we have that, for every $A \subseteq X$,

$$\begin{gathered}
f(\mathcal{Q}_X^{\downarrow}(A)) \subseteq \mathcal{Q}_Y^{\downarrow}(f(A)), \quad f(\mathcal{Q}_X^{\uparrow}(A)) \subseteq \mathcal{Q}_Y^{\uparrow}(f(A)), \\
f(\mathcal{Q}_X^{\searrow}(A)) \subseteq \mathcal{Q}_Y^{\searrow}(f(A)), \quad f(\mathcal{Q}_X^{\nearrow}(A)) \subseteq \mathcal{Q}_Y^{\nearrow}(f(A)), \\
\text{and} \quad f(\mathcal{Q}_X^{\leftrightsquigarrow}(A)) \subseteq \mathcal{Q}_Y^{\leftrightsquigarrow}(f(A)).
\end{gathered} \tag{2.4}$$

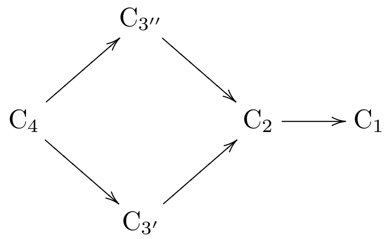
This property of those operators is called *continuity property*, which is fundamental for the definition of closure operators and will be formally discussed in §5.2.

Definition 2.2.5. Let (X, β) be a para-bornological space. Then X satisfies:
- C_1 if, for every $x, y \in X$, $x \leftrightsquigarrow y$ (equivalently, for some $x \in X$ (equivalently, for every $x \in X$), $\mathcal{Q}_X(x) = X$);
- C_2 if, for every $x, y \in X$, $x \searrow y$ or $y \searrow x$ (equivalently, for every $x \in X$, $\mathcal{Q}_X^{\searrow}(x) \cup \mathcal{Q}_X^{\nearrow}(x) = X$);
- $C_{3'}$ if, for every $x, y \in X$, $x \searrow y$ and $y \searrow x$ (equivalently, for every $x \in X$, $\mathcal{Q}_X^{\searrow}(x) \cap \mathcal{Q}_X^{\nearrow}(x) = X$);
- $C_{3''}$ if, for every $x, y \in X$, $x \downarrow y$ or $y \downarrow x$ (equivalently, for every $x \in X$, $\mathcal{Q}_X^{\downarrow}(x) \cup \mathcal{Q}_X^{\uparrow}(x) = X$);
- C_4 if, for every $x, y \in X$, $x \downarrow y$ and $y \downarrow x$ (equivalently, for every $x \in X$, $\mathcal{Q}_X^{\downarrow}(x) \cap \mathcal{Q}_X^{\uparrow}(x) = X$).

We call C_1–C_4 *connectedness axioms.*

The connectedness axioms play the role of the large-scale counterpart of separation axioms in topology. We will discuss a bit more this parallelism in Remarks 2.4.4 and 3.1.7.

Note that the following implications trivially hold:



(2.5)

Those implications cannot be reverted in general and this can be shown with very basic examples (Example 2.2.6). However, as consequence of Fact 2.2.3, if a para-bornological space X satisfies (G_1), then (2.5) becomes

$$C_4 \longleftrightarrow C_{3''} \longrightarrow C_{3'} \longleftrightarrow C_2 \longleftrightarrow C_1, \tag{2.6}$$

while, if X satisfies (G_2),

$$C_4 \longleftrightarrow C_{3'} \longrightarrow C_{3''} \longleftrightarrow C_2 \longrightarrow C_1. \tag{2.7}$$

Finally, if X is a pre-bornological space, then all the operators defined in (2.3) and the connectedness axioms C_1–C_4 coincide. For example, if (X, d) is a metric space, then β_d is a pre-bornology, and, furthermore, β_d satisfies C_1 if and only if it satisfies C_4 if and only if d does not assume the value ∞. Moreover, if (X,β) is a para-bornological space that satisfies C_4, then it trivially satisfies (G_1).

Let (X,β) be a pre-bornological space. If X satisfies C_1 (equivalently, C_4), then we say that β is a *bornology* and (X,β) is a *bornological space.* It is easy to check that, if (X,β) is a bornological space, then, for every $x,y\in X$, $[X]^{<\omega}\subseteq\beta(x)=\beta(y)$. In particular, if a subset A of X is bounded from a point, it is bounded from every point of the space. Finally, by following the same reasoning as in Remark 2.1.3, the notion of bornology given here is equivalent to the classical one.

Example 2.2.6. Let us show that the implications of (2.5), (2.6) and (2.7) cannot be reverted in general.

($C_1\not\to C_2$) Consider the following para-bornological space (X,β), where $X=\{0,1,2\}$ and $\beta(0)=\{\{0\},\emptyset\}$, $\beta(1)=\mathfrak{cl}(\{\{1,0\}\})$, and $\beta(2)=\mathfrak{cl}(\{\{2,0\}\})$. Then (X,β) satisfies C_1, but not C_2. Note that (X,β) satisfies also (G_2).

($C_2\not\to C_{3'}$) and ($C_{3''}\not\to C_4$) Let $X=\{0,1\}$, $\beta(0)=\{\{0\},\emptyset\}$ and $\beta(1)=\mathfrak{cl}(\{\{0,1\}\})$. Then, the para-bornological space (X,β) satisfies $C_{3''}$ (and thus C_2), while it does not satisfy $C_{3'}$ (and thus it does not satisfy C_4 either). Moreover, (X,β) has the property (G_2).

($C_2\not\to C_{3''}$) and ($C_{3'}\not\to C_4$) Let $X=\{0,1,2\}$, $\beta(0)=\mathfrak{cl}(\{\{0,1\}\})$, $\beta(1)=\mathfrak{cl}(\{X\})$ and $\beta(2)=\mathfrak{cl}(\{\{1,2\}\})$. Then (X,β) is a counter-example to the two implications we want to refute. Moreover, (X,β) satisfies (G_1).

Proposition 2.2.7. *Let (X,β) be a para-bornological space. Then:*

(a) X satisfies C_1 if and only if every boundedness preserving map $f\colon(X,\beta)\to(\{0,1\},\beta_{dis})$ is constant;

(b) X satisfies $C_{3'}$ if and only if every boundedness preserving map $f\colon(X,\beta)\to(\{0,1\},\beta')$, where $\beta'(0)=\{\{0\},\emptyset\}$ and $\beta'(1)=\mathfrak{cl}(\{\{0,1\}\})$, is constant.

Proof. (a) If X does not satisfy C_1, then there are two points $x,y\in X$ such that $\mathcal{Q}_X(y)\cap\mathcal{Q}_X(x)\neq\emptyset$. Then the map $f\colon(X,\beta)\to(\{0,1\},\beta_{dis})$ defined be the law

$$f(z)=\begin{cases}x & \text{if } z\in\mathcal{Q}_X(x),\\ y & \text{otherwise,}\end{cases}$$

for every $z\in X$, is boundedness preserving. Conversely, (2.4) implies that the image of a para-bornological space satisfying C_1 through a boundedness preserving map satisfies C_1. Since $(\{0,1\},\beta_{dis})$ has not this property, then the claim follows.

(b) If X satisfies $C_{3'}$, then f is trivially constant because of (2.4) as in item (a). Conversely, if X has not the property $C_{3'}$, then there exist two points $x,y\in X$ such that $x\not\searrow y$ or $y\not\searrow x$. Without loss of generality, assume that $x\not\searrow y$. Then consider the map $g\colon X\to\{0,1\}$ defined as follows: for every $z\in X$,

$$g(z)=\begin{cases}0 & \text{if } z\in\mathcal{Q}_X^{\searrow}(x),\\ 1 & \text{otherwise.}\end{cases}$$

Since $y\notin\mathcal{Q}_X^{\searrow}(x)$, then g is not constant. Moreover, g is boundedness preserving. In fact, let $z\in X$ and $A\in\beta(z)$. If $z\in\mathcal{Q}_X^{\searrow}(x)$ then $A\subseteq\mathcal{Q}_X^{\searrow}(x)$ and thus

$g(A) = 0$. Otherwise, there is nothing to prove, since $\{0,1\}$ is bounded from 1. □

Let us isolate a property that has been used in the proof of Proposition 2.2.7(a).

Proposition 2.2.8. *Let $f\colon (X,\beta_X) \to (Y,\beta_Y)$ be a boundedness preserving map between para-bornological spaces.*
(a) If $A \subseteq X$ satisfies C_1, then $f(A) \subseteq Y$ satisfies C_1.
(b) If $B \subseteq Y$ is a connected component of Y, then $f^{-1}(B)$ is a union of connected components of X.

Proof. Item (a) follows from (2.4). Moreover, because of item (a), if a point belongs to $f^{-1}(B)$, then its entire connected component in X is contained in $f^{-1}(B)$, since B is a connected component. □

2.3 Local simple ends and boundedness preserving maps

Let us first recall a definition due to Dydak ([71]). A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a pre-bornological space X is a *simple end* if, for every bounded subset A of X, A contains at most a finite number of elements of $\{x_n\}_n$. Note that simple ends can exist only in infinite spaces. Here we propose a similar, although slightly different, notion in the realm of para-bornological spaces. Let us recall that a *directed set* is a preordered set with the further property that every two elements have an upper bound. More explicitly, a pair $(I,\geq)$ of a set and a preorder on it is a *directed set* if, for every $x,y \in I$, there exists $z \in I$ such that $z \geq x$ and $z \geq y$. A *net* in a set X is a family of points of X indexed by a directed set.

Definition 2.3.1. A net $\{x_i\}_{i\in I}$ in a para-bornological space X is a *local simple end from x* if, for every subset B of X which is bounded from x, there exists $i_B \in I$ such that, for every $k \geq i_B$, $x_k \notin B$.

In other words, a local simple end from a point x is a net that eventually escapes from every subset which is bounded from x. For example, if a space is bounded from a point, then it cannot contain any local simple end from that specified point. Note the similarity with the notion of converging nets in classical topology. A net in a topological space converges to a point if it eventually enters in every neighbourhood of that point.

Remark 2.3.2. Let (X,β) be a quasi-bornological space. Let $x,y \in X$ such that $y \downarrow x$. Then a net $\{x_i\}_{i\in I}$ is a local simple end from x if it is a local simple end from y. In fact, suppose that $\{x_i\}_i$ is not a local simple end from x and let $A \in \beta(x)$ such that $\{x_i\}_i$ cannot eventually escape from A. Then the property (G_2) implies that $A \in \beta(y)$ and thus the claim follows. In particular, if X satisfies C_4 (and thus X is a bornology), then a net is a local simple end from a point if it is a local simple end from any other one.

The next remark relates the notion of local simple end to the one of simple end.

Remark 2.3.3. Let $\{x_n\}_n$ be a sequence of points in a pre-bornological space X.

(a) The sequence $\{x_n\}_n$ is a simple end if and only if $\{x_n\}_n$ is a local simple end from every point $x \in X$.

(b) Let $x \in X$. The following properties are equivalent:

(b$_1$) $\{x_n\}_n$ is a simple end if and only if $\{x_n\}_n$ is a local simple end from x;

(b$_2$) X satisfies C$_1$ (equivalently, C$_4$).

In fact, if X does not satisfy C$_1$, then there exists a point $y \in X$ such that $y \notin \mathcal{Q}_X(x)$. Hence, the constant sequence $\{x_n\}_n$, where, for every $n \in \mathbb{N}$, $x_n = y$ is a local simple end from x, although it is not a simple end. The opposite implication follows from Remark 2.3.2 and item (a).

We now want to prove a characterisation of boundedness preserving maps using local simple ends, which is similar to the one of continuous maps using converging nets. In order to do that, we need a preliminary result.

Lemma 2.3.4. *Let (X, β) be a para-bornological space, $x \in X$ be a point, and A be a subset of X. Then the following properties are equivalent:*

(a) A is unbounded from x;

(b) there exists a local simple end $\{x_i\}_{i\in I}$ from x such that $x_i \in A$, for every $i \in I$.

Proof. The implication (b)→(a) can be trivially shown by considering its contrapositive. Let us now prove the opposite implication. The set $\beta(x)$ endowed with the partial order defined by the inclusion is a directed set. In fact, for every pair of elements $A, B \in \beta(x)$, $A, B \subseteq A \cup B \in \beta(x)$. First of all, suppose that $\beta(x)$ has a biggest element M. Then there exists $\overline{x} \in A \setminus M$, since A is unbounded. Thus the constant sequence $\{x_n\}_n$ such that, for every $n \in \mathbb{N}$, $x_n = \overline{x}$ satisfies the desired properties. Otherwise, for every $B \in \beta(x)$, let x_B be a point in $A \setminus B$, which exists, since $A \notin \beta(x)$. Hence, $\{x_B\}_{B\in\beta(x)}$ is a local simple end from x. □

Theorem 2.3.5. *Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a map between para-bornological spaces.*

(a) Let $x \in X$ be a point. Then f is boundedness preserving in x if and only if, for every net $\{x_i\}_{i\in I}$ of X, $\{x_i\}_i$ is a local simple end from x provided that $\{f(x_i)\}_i$ is a local simple end from $f(x)$.

(b) The map f is boundedness preserving if and only if, for every point $x \in X$, and every net $\{x_i\}_i$ in X, $\{x_i\}_i$ is a local simple end from x provided that $\{f(x_i)\}_i$ is a local simple end from $f(x)$.

Proof. Item (b) trivially descends from item (a). Let us now prove (a). The 'only if' direction is trivial. In fact, if we suppose that f is boundedness preserving and $\{x_i\}_i$ is not a local simple end from x, then it cannot eventually escapes from a subset $A \in \beta(x)$ and thus $\{f(x_i)\}_i$ cannot escapes from $f(A) \in \beta(f(x))$. As for the opposite implication, suppose that f is not boundedness preserving from x. Hence, there exists $B \in \beta(x)$ such that $f(B) \notin \beta(f(x))$. Lemma 2.3.4 implies that exists a local simple end $\{y_i\}_{i\in I}$ from $f(x)$ that lays in $f(B)$. For every $i \in I$, let $x_i \in B$ be an element such that $f(x_i) = y_i$. Then $\{x_i\}_i$ is not a local simple end from x, since it is contained in a subset which is bounded from x, while $\{f(x_i)\}_i$ is a local simple end from $f(x)$. □

2.4 Local finiteness

Let us now focus on the notion of local finiteness, introduced in Definition 2.1.10. We start our study enlisting in the following remark some observations.

Remark 2.4.1. (a) Finite para-bornological spaces are trivially locally finite. More precisely, if X is a para-bornological space such that, for come $x \in X$, $\mathcal{Q}_X^{\downarrow}(x)$ is finite, then X is locally finite at x.

(b) Let $f\colon (X, \beta_X) \to (Y, \beta_Y)$ be a boundedness copreserving surjective map between para-bornological spaces. Then Y is locally finite, whenever X has the same property. In fact, let $x \in X$ and $A \in \beta_Y(f(x))$. Then there exists $B \in \beta_X(x)$ such that $f(B) = A$. Hence, since B is finite, also A has this property. In particular, if f is a large-scale isomorphism, then X is locally finite if and only if Y is locally finite.
Furthermore, note that, if f is a weakly boundedness copreserving map, which is not boundedness copreserving, then the thesis does not hold. In fact, let us define $X = \bigcup_{n\in\mathbb{N}}\{0, 1, \dots, n\} \times \{n\}$ and endow it with the pre-bornology β_X defined as follows: for every $(m, n) \in X$, $\beta_X(m, n) = \mathfrak{cl}(\{\{0, \dots, n\} \times \{n\}\})$. Let $Y = \mathbb{N}$ with the trivial pre-bornology and $f\colon (m, n) \mapsto m$. Then f has the desired properties, while X is locally finite, but Y is not locally finite.

(c) Let $f\colon X \to Y$ be a map between para-bornological spaces. Suppose that Y satisfies C_4, and thus $[Y]^{<\omega} \subseteq \beta_Y(y)$, for every $y \in Y$. Then, for every $x \in X$ for which X is locally finite at x, f is boundedness preserving in x. In particular, two locally finite para-bornological spaces that satisfy C_4 are large-scale isomorphic if and only if the have the same cardinality. For example, if d is the usual euclidean metric, $(\mathbb{Z}, \beta_d)$ and $(\mathbb{Z}^n, \beta_d)$ are large-scale isomorphic, for every $n \in \mathbb{N}$.

Theorem 2.4.2 shows a similarity between of local finiteness in para-bornologies and sequential compactedness in topologies. This connection is furthermore discussed in Remark 2.4.4 and in Proposition 10.2.17.

Theorem 2.4.2. *Let X be an infinite para-bornological space and let x be a point of X. Then X is locally finite at x if and only if, for every sequence $\{x_n\}_n$ of points of X forming an infinite subset of X, there exists a subsequence $\{x_{k_n}\}_n$ of $\{x_n\}_n$ which is a local simple end from x.*

Proof. If X is not locally finite at x, then there exists a infinite subset B of X, which is bounded from x. If $\{x_n\}_n$ is a sequence of distinct infinite points of B, then no subsequence can be a local simple end from x.

Conversely, suppose that X is locally finite at x. Define a map $\varphi\colon \mathbb{N} \to \mathbb{N}$ by the following law: for every $n \in \mathbb{N}$,

$$\varphi(n) = \min\{m \in \mathbb{N} \mid x_m \notin \{x_{\varphi(i)} \mid i = 0, \dots, n-1\}\}.$$

Then φ is a strictly increasing function with the property that $\{x_{\varphi(n)}\}_n$ visits every point of $\{x_n\}_n$ exactly once. Then $\{x_{\varphi(n)}\}_n$ is a subsequence of $\{x_n\}_n$ which is also a local simple end from x since X is locally finite at x. □

Corollary 2.4.3. *Let X be an infinite para-bornological space. Then the following properties are equivalent:*

(a) X is locally finite;

(b) for every $x \in X$, every sequence of points in X with an infinite support has a subsequence which is a local simple end from x;

(c) every sequence of points in X with an infinite support has a subsequence which is a local simple end from every point of X.

Proof. Implication (c)→(b) is trivial, while implication (b)→(a) follows from Theorem 2.4.2. As for the implication (a)→(c), it is enough to notice that, in the notation of the proof of Theorem 2.4.2, the subsequence $\{x_{\varphi(n)}\}_n$ is a local simple end from every point at which X is locally finite. □

Let us conclude this chapter with a remark that relates properties of para-bornologies to their counterparts in classical topology.

Remark 2.4.4. (a) Let (X, β) be a para-bornological space, and $\overline{x} \in X$ be a base point. Then we define a neighbourhood system $\vartheta_{\overline{x}}$ on the one point completion $\overline{X} = X \cup \{\infty\}$ of X as follows: $\vartheta_{\overline{x}}(x) = \{A \subseteq \overline{X} \mid x \in A\}$, for every $x \in X$, while $\vartheta_{\overline{x}}(\infty) = \{(X \setminus A) \cup \{\infty\} \mid A \in \beta(\overline{x})\}$. The fact that $\vartheta_{\overline{x}}$ is a neighbourhood system can be easily shown. Moreover, every neighbourhood of the point ∞ is trivially an open neighbourhood, and note that $(X, \vartheta_{\overline{x}})$ is always T_1, since every point of X can be separated from ∞. The following equivalences are easy to check:

(i) $\mathcal{Q}_X^{\downarrow}(x) = \bigcup \beta(x) = X$ if and only if $(\overline{X}, \vartheta_{\overline{x}})$ is Hausdorff;

(ii) a net $\{x_i\}_i \subseteq X$ is a local simple end from $\overline{x}$ if and only if $\{x_i\}_i$ converges to ∞ in $\overline{X}$;

(iii) X is locally finite in $\overline{x}$ if and only if $\overline{X}$ is compact.

(b) In the notation of the previous item, we can define another neighbourhood system ϑ on $\overline{X}$ as follows: for every $z \in \overline{X}$, $\vartheta(z) = \bigcap\{\vartheta_x(z) \mid x \in X\}$. In particular, $\vartheta(\infty) = \{(X \setminus A) \cup \{\infty\} \mid A \in \beta(x), \forall x \in X\}$. Every neighbourhood of the point ∞ is trivially an open neighbourhood. Also, $(\overline{X}, \vartheta)$ is always T_1. In the sequel of this example, $\overline{X}$ will always be endowed with ϑ.

(i) The space X is C_4 if and only if, for every $x \in X$, $\{x\} \in \beta(y)$, for every $y \in X$. Equivalently, we have that $(X \setminus \{x\}) \cup \{\infty\} \in \vartheta(\infty)$ which is equivalent to the fact that $\overline{X}$ is Hausdorff.

(ii) If X locally finite, then $\overline{X}$ is trivially compact. Conversely, suppose that X is a bornological space. We claim that X is locally finite if $\overline{X}$ is compact. Suppose that X is not locally finite. Then there exists an infinite $A \in \beta(x)$, for some $x \in X$. However, since X is a bornology, A is bounded from every point and thus $(X \setminus A) \cup \{\infty\} \in \vartheta(\infty)$. Then the open cover $\{(X \setminus A) \cup \{\infty\}\} \cup \{\{a\} \mid a \in A\}$ has no finite subcover.

(iii) Every local simple end in X from a point x converges to ∞ in $\overline{X}$. Conversely, suppose that X is a bornological space. Then we claim that a net $\{x_i\}_i \subseteq X$ is a local simple end from any point of x if $\{x_i\}_i$ converges to ∞ in $\overline{X}$. If $\{x_i\}_i$ converges to ∞, it eventually enter every neighbourhood of ∞, which means that it eventually gets out from every subset $A \in \bigcap_{x \in X} \beta(x)$. However, since X is a bornology, $\beta(x) = \beta(y)$, for every $x, y \in X$, and thus $\{x_i\}_i$ is a local simple end from every point.

Chapter 3

Coarse spaces and their generalisations

3.1 Coarse spaces and their generalisations

Definition 3.1.1. Let X be a set. A family $\mathcal{E} \subseteq \mathcal{P}(X \times X)$ is an *entourage structure over* X if it is an ideal on $X \times X$ that contains the diagonal Δ_X. Moreover, an entourage structure $\mathcal{E}$ over X is
- a *semi-coarse structure* if $E^{-1} \in \mathcal{E}$, for every $E \in \mathcal{E}$;
- a *quasi-coarse structure* if $E \circ F \in \mathcal{E}$, for every $E, F \in \mathcal{E}$;
- a *coarse structure* if it is both a semi-coarse and a quasi-coarse structure.

The pair $(X, \mathcal{E})$ is an *entourage space* (a *semi-coarse space*, a *quasi-coarse space*, a *coarse space*) if $\mathcal{E}$ is an entourage structure (a semi-coarse structure, a quasi-coarse structure, a coarse structure, respectively) over X.

If $\mathcal{E}$ is an entourage structure on a set X, then also $\mathcal{E}^{-1} = \{E^{-1} \mid E \in \mathcal{E}\}$ is an entourage structure. Of course, $\mathcal{E} = \mathcal{E}^{-1}$ if and only if $\mathcal{E}$ is a semi-coarse structure. Moreover, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathcal{E}^{-1}$ is a quasi-coarse structure.

Let $(X, \mathcal{E})$ be an entourage space and Y be a subset of X. Then Y can be endowed with the *subspace entourage structure* $\mathcal{E}|_Y = \{E \cap (Y \times Y) \mid E \in \mathcal{E}\}$, and $(Y, \mathcal{E}|_Y)$ is called an *entourage subspace of* $(X, \mathcal{E})$. If $\mathcal{E}$ is a quasi-coarse structure (semi-coarse structure), then $\mathcal{E}|_Y$ is a quasi-coarse structure (semi-coarse structure, respectively).

If X is a set, a family $\mathcal{B}$ of subsets of $X \times X$ such that $\mathcal{E} = \mathfrak{cl}(\mathcal{B})$ is an entourage structure (semi-coarse structure, quasi-coarse structure, coarse structure, respectively) is a *base of the entourage structure* (*base of the semi-coarse structure*, *base of the quasi-coarse structure*, *base of the coarse structure*, respectively) $\mathcal{E}$.

Let us now give some example of these structures.

Example 3.1.2. (a) Every set X can be endowed with two entourage structures which are actually coarse structures: the *discrete coarse structure* $\mathcal{E}_{dis} = \mathfrak{cl}(\{\{\Delta_X\}\})$, and the *trivial* (or *indiscrete*) *coarse structure* $\mathcal{E}_{triv} =$

$\mathcal{P}(X\times X)$. Moreover, the discrete and the trivial coarse structures coincide if the set is a singleton.

(b) A leading example of entourage structures is the metric entourage structure. Let (X,d) be a set endowed with an extended semi-positive-definite map d. We define the following entourage structure:

$$\mathcal{E}_d=\mathfrak{cl}(\{U_R^d\subseteq X\times X\mid R\geq 0\}), \tag{3.1}$$

see (1.1) for the definition of the entourages U_R^d, for every $R\geq 0$. Even though it is not precise, for the sake of simplicity, we call $\mathcal{E}_d$ a *metric entourage structure*. If d is a semi-metric, then $\mathcal{E}_d$ is a semi-coarse structure, while, if d is a quasi-metric, then $\mathcal{E}_d$ is a quasi-coarse structure. There are non-symmetric quasi-metrics and semi-metrics that do not satisfy the triangular inequality which induce coarse structures (the maps defined in (2.1) provide the desired examples). In the sequel, for the sake of simplicity and for consistency with the previous literature, if d is a metric, we call $\mathcal{E}_d$ a *metric coarse structure*.

More examples of entourage spaces will be given in §3.3. However, let us anticipate another classical example of coarse structure.

Example 3.1.3. Let X be a set and $\mathcal{I}$ be an ideal on a set X. We define the *ideal coarse structure* to be the family

$$\mathcal{E}_{\mathcal{I}}=\mathfrak{cl}(\{E_I\mid I\in\mathcal{I}\}),\quad\text{where, for every } I\in\mathcal{I},\quad E_I=\Delta_X\cup(I\times I).$$

Note that, for every $I,J\in\mathcal{I}$, $E_I^{-1}=E_I$, $E_I\cup E_J\subseteq E_{I\cup J}$ and $E_I\circ E_J\subseteq E_{I\cup J}$, which imply that $\mathcal{E}_{\mathcal{I}}$ is actually a coarse structure. The above construction can be carried out in the presence of a filter φ of subsets of the set X. Then the family $\mathcal{I}_\varphi=\{X\setminus V\mid V\in\varphi\}$ is an ideal. The coarse structure induced by a filter φ is also called *filter coarse structure*. These structures have been defined and widely studied in [134]. However, the authors used balleans in order to introduce them. We devote §3.2 to their introduction.

Remark 3.1.4. While uniformities capture the small-scale properties of spaces, coarse structures encode their large-scale behaviour. In order to clarify this idea, let us consider the following constructions. Let (X,d) be a metric space, and let us derived two more metrics from d: for every $x,y\in X$,

$$d_1(x,y)=\min\{d(x,y),1\},\quad\text{and}\quad d_2(x,y)=\begin{cases}0 & \text{if } x=y,\\ \max\{d(x,y),1\} & \text{otherwise.}\end{cases}$$

From the large-scale point of view, d_1 loses a lot of information, in fact $\mathcal{E}_{d_1}=\mathcal{E}_{triv}$, while it keeps all the important features from the small-scale point of view, and, in fact, $\mathcal{U}_d=\mathcal{U}_{d_1}$. Conversely, the metric space (X,d_2) is discrete and $\mathcal{U}_d\neq\mathcal{U}_{d_2}$, but $\mathcal{E}_d=\mathcal{E}_{d_2}$.

Similarly to how uniformities and their generalisations induce (weak) neighbourhood systems, entourage structures induce para-bornologies. Let $(X,\mathcal{E})$ be an entourage space. Then the *uniform para-bornology* $\beta_{\mathcal{E}}$ is defined as follows: for every $x\in X$,

$$\beta_{\mathcal{E}}(x)=\{E[x]\mid E\in\mathcal{E}\}. \tag{3.2}$$

Remark 3.1.5. (a) If $\mathcal{E}$ is a semi-coarse structure (a quasi-coarse structure), then $\beta_{\mathcal{E}}$ is a semi-bornology (a quasi-bornology, respectively).
(b) Let X be a set. Then $\beta_{dis} = \beta_{\mathcal{E}_{dis}}$ and $\beta_{triv} = \beta_{\mathcal{E}_{triv}}$.
(c) Let d be an extended semi-positive-definite map d on a set X. Then $\beta_d = \beta_{\mathcal{E}_d}$.

Let (X, β) be a para-bornological space.

- The para-bornology β is said to be a *uniformisable para-bornology* if there exists an entourage structure $\mathcal{E}$ over X such that $\beta = \beta_{\mathcal{E}}$.
- If β is a semi-bornology, then β is said to be a *uniformisable semi-bornology* provided that there exists a semi-coarse structure $\mathcal{E}$ over X such that $\beta = \beta_{\mathcal{E}}$.
- If β is a quasi-bornology, then β is said to be a *uniformisable quasi-bornology* provided that there exists a quasi-coarse structure $\mathcal{E}$ over X such that $\beta = \beta_{\mathcal{E}}$.
- If β is a pre-bornology, then β is said to be a *uniformisable pre-bornology* provided that there exists a coarse structure $\mathcal{E}$ over X such that $\beta = \beta_{\mathcal{E}}$.

An important notion in coarse geometry is boundedness. A subset A of a coarse space $(X, \mathcal{E})$ is called *bounded* if it satisfies one of the following equivalent properties:

(B_1) there exists $x \in A$ and $E \in \mathcal{E}$ such that $A \subseteq E[x]$ (equivalently, A is bounded from x with respect to the uniform para-bornology $\beta_{\mathcal{E}}$);
(B_2) for every $x \in A$, there exists $E_x \in \mathcal{E}$ such that $A \subseteq E_x[x]$ (equivalently, A is bounded with respect to the uniform para-bornology $\beta_{\mathcal{E}}$);
(B_3) there exists $E \in \mathcal{E}$ such that, for every $x \in A$, $A \subseteq E[x]$ (equivalently, $A \times A \in \mathcal{E}$).

Moreover, the pre-bornology $\beta_{\mathcal{E}}$, seen as family of subsets of X (see Remark 2.1.3) consists of the bounded subsets of X. However, if X is an entourage space, although the implications (B_3)$\to$(B_2)$\to$(B_1) hold, (B_1)–(B_3) are not equivalent any more, as Example 3.1.6 shows.

Example 3.1.6. (a) Let $X = \{0, 1, 2\}$ and consider the semi-coarse structure $\mathcal{E}_1 = \mathfrak{cl}(\{\{(0, 1), (0, 2), (1, 0), (2, 0)\} \cup \Delta_X\})$ and the quasi-coarse structure $\mathcal{E}_2 = \mathfrak{cl}(\{\{(0, 1), (0, 2)\} \cup \Delta_X\})$. Then the whole space X satisfies (B_1) in both $\mathcal{E}_1$ and $\mathcal{E}_2$, but it does not satisfy (B_2).

(b) Let $X = \mathbb{N}$ and d and d' be a semi-metric and a quasi-metric defined as follows: for every $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ \min\{m,n\} & \text{otherwise,} \end{cases} \quad \text{and} \quad d'(m,n) = \begin{cases} 0 & \text{if } n > m, \\ m - n & \text{otherwise.} \end{cases}$$

Then X satisfies (B_2) in both the semi-coarse structure $\mathcal{E}_d$ and the quasi-coarse structure $\mathcal{E}_{d'}$, but it does not satisfy (B_3).

Let $(X, \mathcal{E})$ be an entourage space. Then X is *locally finite* if the uniform para-bornological space $(X, \beta_{\mathcal{E}})$ is locally finite. Let us introduce also the 'uniform' version of this property. The space X has *bounded geometry* if there exists a map $\varphi\colon \mathcal{E} \to \mathbb{N}$ such that, for every $E \in \mathcal{E}$ and $x \in X$, $|E[x]| \leq \varphi(E)$.

Let $(X, \mathcal{E})$ be a locally finite entourage structure. Then a subset A of X satisfies (B_2) if and only if it satisfies (B_3). In fact, if X is locally finite, then every subset A satisfying (B_2) is finite. Hence $E = \bigcup_{x \in A} E_x \in \mathcal{E}$ and this entourage shows that A satisfies (B_3).

A family $\{A_i\}_{i\in I}$ of subsets of an entourage space $(X,\mathcal{E})$ is *uniformly bounded* if there exists $E\in\mathcal{E}$ such that, for every $i\in I$ and every $x\in A_i$, $A_i\subseteq E[x]$. In particular, every element of a uniformly bounded family satisfies (B_3).

Similarly to what we have done for the notion of local finiteness, we can lift all the properties defined for para-bornological spaces to entourage spaces: an entourage space has the property P if the associated uniform para-bornological space has the property P. Hence, for example, an entourage space has the property C_1 if the associated uniform para-bornological space has it. Similarly all the other connectedness axioms can be induced. The same reasoning can be applied to the operators defined in (2.3) that are inherited by entourage spaces, and to the notion of connected component. If $(X,\mathcal{E})$ is an entourage space, then denote $\mathrm{dsc}(X,\beta_{\mathcal{E}})$ by $\mathrm{dsc}(X,\mathcal{E})$.

Remark 3.1.7. Let $(X,\mathcal{E})$ be a coarse space. Then $\beta_{\mathcal{E}}$ satisfies C_1 if and only if $\beta_{\mathcal{E}}$ satisfies C_4 if and only if $\bigcup\mathcal{E}=X\times X$. In that case, the coarse space is called *connected* also in [157]. This remark shows another similarity between connectedness axioms and separation axioms in toplogy. In fact, if $(X,\mathcal{U})$ is a uniform space, the induced topology $\tau_{\mathcal{U}}$ on X satisfies (T_1) if and only if it satisfies $(T_{3,5})$ if and only if $\bigcap\mathcal{U}=\Delta_X$.

Example 3.1.8. One may ask whether there are quasi-coarse spaces that satisfy C_4, but they are not semi-coarse spaces.

Let (X,d) be a metric space and let $h\colon X\to\mathbb{R}$ be an arbitrary function. Then the function $d_h\colon X\to\mathbb{R}_{\geq 0}$, defined by the law

$$d_h(x,y)=\begin{cases} d(x,y)+h(y)-h(x) & \text{if } h(y)-h(x)\geq 0,\\ d(x,y) & \text{otherwise,}\end{cases}$$

for every $x,y\in X$, is a quasi-metric.

Let now $X=\mathbb{Z}$, d be the usual euclidean metric, and $h(x)=x^3$. Then $(\mathbb{Z},\mathcal{E}_{d_h})$ is a quasi-coarse space, since d_h is a quasi-metric, and it is C_4. However, it is not a coarse space. In fact, for every $R\geq 0$ and every $z\in\mathbb{R}$, $d_h(z+R,z)=R$, while $d_h(z,z+R)=R(1+3z^2+3zR+R^2)$, and the latter strongly depends on the point z. Hence, even though $\{(z+R,z)\mid z\in\mathbb{Z}\}\subseteq E_R\in\mathcal{E}_d$, there exists no $S\geq 0$ such that $\{(z,z+R)\mid z\in\mathbb{R}\}\subseteq E_S$.

In Example 3.1.2 we introduced metric entourage structures. We now want to characterise those structures.

Let $(X,\mathcal{E})$ be an entourage structure. Define its *cofinality* as follows: $\mathrm{cf}\,\mathcal{E}=\inf\{|\mathcal{B}|\mid \mathfrak{cl}(\mathcal{B})=\mathcal{E}\}$.

Proposition 3.1.9. *Let $(X,\mathcal{E})$ be an entourage space.*

(a) There exists an extended semi-positive-definite map d on X such that $\mathcal{E}=\mathcal{E}_d$ if and only if $\mathrm{cf}\,\mathcal{E}\leq\omega$.

(b) Suppose that $\mathcal{E}$ is a semi-coarse structure. Then there exists a semi-metric d on X such that $\mathcal{E}=\mathcal{E}_d$ if and only if $\mathrm{cf}\,\mathcal{E}\leq\omega$.

Proof. First of all, the 'only if' implications in both items (a) and (b) are trivial since he family $\{U_n^d\mid n\in\mathbb{N}\}$, in the notation of (3.1), is a base of $\mathcal{E}_d$.

(a,$\leftarrow$) Let $\{F_n\mid n\in\mathbb{N}\}$ be a countable base of $\mathcal{E}$, and, without loss of generality, we can ask that $F_0=\Delta_X$ and $F_n\subseteq F_{n+1}$, for every $n\in\mathbb{N}$. Then define a map $d\colon X\times X\to\mathbb{N}$ as follows: for every $x,y\in X$,

$$d(x,y)=\begin{cases}\min\{n\mid y\in F_n[x]\} & \text{if it exists,}\\ \infty & \text{otherwise.}\end{cases} \tag{3.3}$$

It is easy to check that d satisfies the required properties.

(b,←) Suppose that $\mathcal{E}$ is a semi-coarse structure with $\operatorname{cf}\mathcal{E}\leq\omega$. Then we can choose a base $\{F_n\mid n\in\mathbb{N}\}$ as in item (a) with the further property that $F_n=F_n^{-1}$, for every $n\in\mathbb{N}$. Then the map d defined as in (3.3) satisfies the desired properties. □

Note that the maps d in Proposition 3.1.9 do not assume value ∞ if and only if $(X,\mathcal{E})$ is C_4.

The case where the entourage space is a quasi-coarse space (or a coarse space, in particular, which is a classical result) will be discussed in §3.4.1.

3.1.1 Morphisms between entourage spaces

Let us now introduce the morphisms between those spaces, which are the 'uniform' versions of the ones in Definition 2.1.6.

Definition 3.1.10. A map $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between entourage spaces is said to be

- *bornologous* (or *uniformly boundedness preserving*, *coarsely uniform*, *coarse*) if $(f\times f)(E)\in\mathcal{E}_Y$, for every $E\in\mathcal{E}_X$;
- *uniformly weakly boundedness copreserving* if, for every $E\in\mathcal{E}_Y$, there exists $F\in\mathcal{E}_X$ such that $(f\times f)(F)=E\cap(f(X)\times f(X))$;
- *uniformly boundedness copreserving* if, for every $E\in\mathcal{E}_Y$, there exists $F\in\mathcal{E}_X$ such that, for every $x\in X$, $E[f(x)]\cap f(X)\subseteq f(F[x])$;
- *effectively proper* (or *uniformly proper*) if, for every $E\in\mathcal{E}_Y$, $(f\times f)^{-1}(E)\in\mathcal{E}_X$;
- a *coarse embedding* if it is both bornologous and effectively proper;
- an *asymorphism* if it is bijective and both f and f^{-1} are bornologous.

Let us underline that the term coarse usually has a meaning which is different from the one given in Definition 3.1.10 (and used in some papers, like, for example, [55]). For example in [157], a map between coarse spaces is *coarse* if it is both bornologous and *proper* (i.e., the induced map between the corresponding pre-bornological space is proper).

Note that all the properties introduced in Definition 3.1.10 can be checked just for all the entourages that belong to some base of the entourage structures.

Remark 3.1.11. Let $f\colon(X,d_X)\to(Y,d_Y)$ be a map between metric spaces. Then the following properties hold:

(a) $f\colon(X,d_X)\to(Y,d_Y)$ is bornologous (as a map between metric spaces, as defined in §1.1) if and only if $f\colon(X,\mathcal{E}_{d_X})\to(Y,\mathcal{E}_{d_Y})$ is bornologous (as a map between coarse spaces, Definition 3.1.10);
(b) $f\colon(X,d_X)\to(Y,d_Y)$ is a coarse equivalence (as a map between metric spaces, as defined in §1.1) if and only if $f\colon(X,\mathcal{E}_{d_X})\to(Y,\mathcal{E}_{d_Y})$ is a coarse equivalence (as a map between coarse spaces, Definition 3.1.10).

Similarly to Example 2.1.7, we can provide first trivial examples of the properties enlisted in Definition 3.1.10.

Example 3.1.12. Let $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between two entourage spaces. Then the following properties trivially hold:

(a) if $\mathcal{E}_X$ is the discrete coarse structure, then f is bornologous;
(b) if $\mathcal{E}_Y$ is the discrete coarse structure, then f is uniformly boundedness copreserving;
(c) if $\mathcal{E}_X$ is the trivial coarse structure, then f is effectively proper;
(d) if $\mathcal{E}_Y$ is the trivial coarse structure, then f is bornologous;
(e) if $\mathcal{E}_X$ is discrete and f is effectively proper, then f is an injective coarse embedding;
(f) if $\mathcal{E}_X$ and $\mathcal{E}_Y$ are both discrete or both trivial, then f is an asymorphism if and only if it is bijective.

Let us show that the notions in Definition 2.1.6 are the uniform version of Definition 3.1.10. Let us start with a preliminary result.

Lemma 3.1.13. *Let $f\colon X\to Y$ be a map between sets, and $E\subseteq X\times X$. Then, for every $x\in X$, $(f\times f)(E)[f(x)]=\bigcup_{z\in f^{-1}(f(x))}f(E[z])$.*

Proof. Let us prove the inclusion $(\supseteq)$. In order to do that, it is enough to show that $(f\times f)(E)[f(x)]\supseteq f(E[x])$. Consider an arbitrary $y\in f(E[x])$ and take $z\in f^{-1}(y)$ such that $(x,z)\in E$. Then $(f(x),y)=(f(x),f(z))\in(f\times f)(E)$, which shows that $y\in((f\times f)(E))[f(x)]$.

Conversely, let $f(y)\in(f\times f)(E)[f(x)]$. Since $(f(x),f(y))\in(f\times f)(E)$, there exists $z,w\in X$ such that $(z,w)\in E$, $f(x)=f(z)$, and $f(y)=f(w)$. Hence $f(y)=f(w)\in f(E[z])$, and $z\in f^{-1}(f(x))$. □

Proposition 3.1.14. *Let $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between entourage spaces.*

(a) If f is bornologous, then $f\colon (X,\beta_{\mathcal{E}_X})\to(Y,\beta_{\mathcal{E}_Y})$ is boundedness preserving.
(b) If f is uniformly weakly boundedness copreserving, then $f\colon (X,\beta_{\mathcal{E}_X})\to(Y,\beta_{\mathcal{E}_Y})$ is weakly boundedness copreserving.
(c) If f is uniformly boundedness copreserving, then $f\colon (X,\beta_{\mathcal{E}_X})\to(Y,\beta_{\mathcal{E}_Y})$ is boundedness copreserving.
(d) If f is effectively proper, then $f\colon (X,\beta_{\mathcal{E}_X})\to(Y,\beta_{\mathcal{E}_Y})$ is proper.

Proof. The desired implications are easy to check. Let us just show for example item (c). Let $x\in X$, and $A\in\beta_{\mathcal{E}_Y}(f(x))$. Then there exists $E\in\mathcal{E}_Y$ such that $A=E[f(x)]$. Since f is uniformly weakly boundedness copreserving, there exists $F\in\mathcal{E}_X$ such that $E\cap(f(X)\times f(X))=(f\times f)(F)$. Then, thanks to Lemma 3.1.13, we have the following chain

$$\begin{aligned}A\cap f(X)&=E[f(x)]\cap f(X)\subseteq(E\cap(f(X)\times f(X))[f(x)]=\\&=(f\times f)(F)[f(x)]=\bigcup_{z\in f^{-1}(f(x))}f(F[z]),\end{aligned}$$

where $F[z]\in\beta_{\mathcal{E}_X}(z)$. □

It is easy to check that composites of bornologous maps are bornologous. Moreover, we have the following result.

Proposition 3.1.15. *Let $f\colon (X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between entourage spaces. Then:*

(a) if f is effectively proper, then f is uniformly boundedness copreserving;
(b) if f is uniformly boundedness copreserving, then f is uniformly weakly boundedness copreserving.

Proof. (a) Suppose that f is effectively proper and let $E\in\mathcal{E}_Y$. Then, for every $x\in X$, $E[f(x)]\cap f(X)\subseteq f((f\times f)^{-1}(E)[x])$. In fact, for every $y\in X$ such that $(f(x),f(y))\in E$, $(x,y)\in(f\times f)^{-1}(E)$ and so $f(y)\in f((f\times f)^{-1}(E)[x])$.

(b) Suppose now that f is uniformly boundedness copreserving and let $E\in\mathcal{E}_Y$. Let $F\in\mathcal{E}_X$ be an entourage such that, for every $x\in X$, $E[f(x)]\cap f(X)\subseteq f(F[x])$. We claim that $E\cap(f(X)\times f(X))\subseteq(f\times f)(F)$. Let $(u,v)\in E\cap(f(X)\times f(X))$. There exists $z\in f^{-1}(u)$, and so $v\in E[f(z)]\cap f(X)$, which implies that there exists $w\in F[z]\cap f^{-1}(v)$. Finally, note that $(z,w)\in F$ and $(u,v)=(f(z),f(w))\in(f\times f)(F)$. □

If f is injective, then both implications of Proposition 3.1.15 can be easily reverted. Proposition 3.1.16 gives another condition that implies their reversibility.

Note that a map $f\colon(X,\mathcal{E}_X)\to Y$ from an entourage space to a set has uniformly bounded fibres if and only if $R_f=\{(x,y)\in X\times X\mid f(x)=f(y)\}\in\mathcal{E}_X$. We call such a map *large-scale injective*.

Proposition 3.1.16. *Let $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ be a map between entourage spaces. If f is effectively proper, then f is large-scale injective. Moreover, if $\mathcal{E}_X$ is a quasi-coarse structure, then the following properties are equivalent:*

(a) f is large-scale injective and it is uniformly weakly boundedness copreserving;
(b) f is large-scale injective and it is uniformly boundedness copreserving;
(c) f is effectively proper.

Proof. The first statement can be easily proved: since $\Delta_Y\in\mathcal{E}_Y$, then $R_f=(f\times f)^{-1}(\Delta_Y)\in\mathcal{E}_X$.

In view of Proposition 3.1.15, we just need to show the implication (a)→(c). Suppose now that f is uniformly weakly boundedness copreserving and $R_f\in\mathcal{E}_X$. Let $E\in\mathcal{E}_Y$ and (x,y) be an arbitrary pair in $(f\times f)^{-1}(E)$. Let $F\in\mathcal{E}_X$ such that $(f\times f)(F)=E\cap(f(X)\times f(X))$. Then there exists $(z,w)\in F$ such that $(f(x),f(y))=(f(z),f(w))$ and thus

$$(x,y)=(x,z)\circ(z,w)\circ(w,y)\in R_f\circ F\circ R_f\in\mathcal{E}_X.$$ □

Trivially, for a bijective map $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between entourage spaces the following properties are equivalent:

- f is an asymorphism;
- f is bornologous and uniformly weakly boundedness copreserving;
- f is bornologous and uniformly boundedness copreserving;
- f is bornologous and effectively proper.

Let $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ be two asymorphic entourage spaces. Then $\mathcal{E}_X$ is a semi-coarse structure (quasi-coarse structure) if and only if $\mathcal{E}_Y$ is a semi-coarse structure (quasi-coarse structure, respectively). For the proof of this fact, we address to [144], where the authors used the equivalent approach through ball structures (see §3.2 for the introduction of these structures).

Furthermore, if X and Y are two asymorphic entourage spaces, then X satisfies C_i, where $i \in \{1, 2, 3', 3'', 4\}$, if and only if Y satisfies C_i.

Some properties of entourage spaces are preserved even if the map is not necessarily an asymorphism, as the following result shows.

Proposition 3.1.17. *Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a uniformly boundedness copreserving surjective map between entourage spaces. Then Y has bounded geometry (Y is locally finite) whenever X has bounded geometry (X is locally finite, respectively).*

Proof. Suppose that $\varphi\colon \mathcal{E}_X \to \mathbb{N}$ is a map that demonstrates that $(X, \mathcal{E}_X)$ has bounded geometry. Let $E \in \mathcal{E}_Y$. Then there exists $F \in \mathcal{E}_X$ such that, for every $x \in X$, $E[f(x)] \subseteq f(F[x])$. Hence, $|E[f(x)]| \leq |F[x]| \leq \varphi(F)$. The other implication can be similarly proved. □

Remark 3.1.18. Let us discuss the uniform counterpart of Remark 2.1.14. Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between entourage spaces, and $Z \subseteq X$. One can easily shows that:

(a) if f is bornologous, then so is $f|_Z$;
(b) if f is effectively proper, then so is $f|_Z$;
(c) if f is large-scale injective, then so is $f|_Z$.

However, in Example 4.3.7 we provide a uniformly boundedness copreserving map between entourage spaces whose restriction is not even uniformly weakly boundedness copreserving.

Let X be a set. Similarly to what we have done for the family $\mathfrak{PB}(X)$ of para-bornologies on X, we can construct the lattice of entourage structures on X. Denote that family by $\mathfrak{E}(X)$. The lattice $\mathfrak{E}(X)$ is ordered by inclusion. More precisely, let X be a set and $\mathcal{E}, \mathcal{E}' \in \mathfrak{E}(X)$ be two entourage structures. Then we say that $\mathcal{E}$ is *finer* than $\mathcal{E}'$ if $\mathcal{E} \subseteq \mathcal{E}'$ (and $\mathcal{E}'$ is *coarser* than $\mathcal{E}$). Equivalently, $\mathcal{E}$ is finer than $\mathcal{E}'$ if and only if the map $id_X\colon (X, \mathcal{E}) \to (X, \mathcal{E}')$ is bornologous. Moreover, $\mathfrak{E}(X)$ has the trivial coarse structure as top element and the discrete coarse structure as minimum element. Finally, $\mathfrak{E}(X)$ is a complete lattice. In fact, for every family $\{\mathcal{E}_i\}_{i \in I}$ of entourage structures, $\bigcap_i \mathcal{E}_i$ is an entourage structure and so their meet $\bigwedge_i \mathcal{E}_i$. Moreover, if $\mathcal{E}_i$ is a semi-coarse structure (quasi-coarse structure), for every $i \in I$, then also $\bigcap_i \mathcal{E}_i$ is a semi-coarse structure (quasi-coarse structure, respectively). Hence, the join of a family of entourage structures (semi-coarse structures, quasi-coarse structures, coarse structures) $\{\mathcal{E}_i\}_{i \in I}$ on a set X can be defined as the entourage structure $\bigvee_i \mathcal{E}_i$ (semi-coarse structure, quasi-coarse structure, coarse structure, respectively) *generated* by $\bigcup_i \mathcal{E}_i$, i.e., the finest structure that contains $\mathcal{E}_i$, for every $i \in I$.

Let X be a set and $\mathcal{E}, \mathcal{E}' \in \mathfrak{E}(X)$. According to Proposition 3.1.14, if $\mathcal{E}$ is finer that $\mathcal{E}'$, then also $\beta_{\mathcal{E}}$ is finer than $\beta_{\mathcal{E}'}$. However the converse is not true. In Example 4.3.7(b) we provide two entourage structures $\mathcal{E}_1$ and $\mathcal{E}_2$ that are not compatible (i.e., $\mathcal{E}_1 \not\subseteq \mathcal{E}_2$ and $\mathcal{E}_2 \not\subseteq \mathcal{E}_1$) although they induce the same para-bornology.

3.2 Approach via ball structures

Let $(X, \mathcal{E})$ be an entourage space. Then we can associate to $\mathcal{E}$ a triple $\mathfrak{B}_{\mathcal{E}} = (X, P_{\mathcal{E}}, B_{\mathcal{E}})$, where $P_{\mathcal{E}} = \{E \in \mathcal{E} \mid \Delta_X \subseteq E\}$ and $B_{\mathcal{E}}(x, E) = E[x]$, for every $x \in X$ and every $E \in P$. It is an example of ball structure.

Definition 3.2.1. ([144, 151]) A *ball structure* is a triple $\mathfrak{B} = (X, P, B)$ where X and P are sets, $P \neq \emptyset$, and $B\colon X \times P \to \mathcal{P}(X)$ is a map, such that $x \in B(x, r)$ for every $x \in X$ and every $r \in P$. The set X is called *support of the ball structure*, P – *set of radii of a ball structure*, and $B(x, r)$ – *ball of center x and radius r*. In case $X = \emptyset$, the map B is the empty map.

The terminology and the intuition come from the metric setting: if (X, d) is a metric space, then $\mathfrak{B}_d = (X, \mathbb{R}_{\geq 0}, B_d)$ is a ball structure, called *metric ballean*.

For a ball structure (X, P, B), $x \in X$, $r \in P$ and a subset A of X, one puts

$$B^*(x, r) = \{y \in X \mid x \in B(y, r)\} \quad \text{and} \quad B(A, r) = \bigcup\{B(x, r) \mid x \in A\}.$$

A ball structure $\mathfrak{B} = (X, P, B)$ is said to be:

- *weakly upper multiplicative* if, for every pair of radii $r, s \in P$ there exists $t \in P$ such that $B(x, r) \cup B(x, s) \subseteq B(x, t)$, for every $x \in X$;
- *upper multiplicative* if, for every pair of radii $r, s \in P$ there exists $t \in P$ such that $B(B(x, r), s) \subseteq B(x, t)$, for every $x \in X$;
- *upper symmetric* if, for every pair of radii $r, s \in P$ there exist $r', s' \in P$ such that $B^*(x, r) \subseteq B(x, r')$ and $B(x, s) \subseteq B^*(x, s')$, for every $x \in X$.

It is trivial that upper multiplicativity implies weak upper multiplicativity since every ball contains its center.

Definition 3.2.2. A ball structure is

- a *semi-ballean* if it is weakly upper multiplicative and upper symmetric;
- a *quasi-ballean* if it is upper multiplicative;
- a *ballean* ([144]) if it is both a semi-ballean and a quasi-ballean.

For every entourage space $(X, \mathcal{E})$, $\mathfrak{B}_{\mathcal{E}}$ is indeed a weakly upper multiplicative ball structure. Moreover, if $\mathcal{E}$ is a semi-coarse structure, then $\mathfrak{B}_{\mathcal{E}}$ is a semi-ballean, while, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathfrak{B}_{\mathcal{E}}$ is a quasi-ballean.

We have seen how we construct ball structures from entourage structures. Let us now discuss the opposite construction. Let $\mathfrak{B} = (X, P, B)$ be a weakly upper multiplicative ball structure. Then we can define an associated entourage structure $\mathcal{E}_{\mathfrak{B}}$ of X as follows: for every $r \in P$,

$$E_r = \bigcup_{x \in X} (\{x\} \times B(x, r)),$$

and the family $\{E_r \mid r \in P\}$ is a base for the entourage structure $\mathcal{E}_{\mathfrak{B}}$. Moreover,

- if $\mathfrak{B}$ is a semi-ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a semi-coarse structure;
- if $\mathfrak{B}$ is a quasi-ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a quasi-coarse structure;
- if $\mathfrak{B}$ is a ballean, then $\mathcal{E}_{\mathfrak{B}}$ is a coarse structure.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two weakly upper multiplicative ball structures on the same support X. Then we identify those two ball structures and we write $\mathfrak{B} = \mathfrak{B}'$, if $\mathcal{E}_{\mathfrak{B}} = \mathcal{E}_{\mathfrak{B}'}$. We soon give a characterization of the equality between ball structures. Hence, for every entourage space $(X, \mathcal{E})$ and every weakly upper multiplicative ball structure $\mathfrak{B}$ on X,

$$\mathcal{E}_{\mathfrak{B}_{\mathcal{E}}} = \mathcal{E} \quad \text{and} \quad \mathfrak{B}_{\mathcal{E}_{\mathfrak{B}}} = \mathfrak{B}.$$

The equivalence between coarse structures and balleans have been widely discussed for example in [151, 65].

In the sequel of this thesis, we are going to use ball structures and entourage structures interchangeably. Hence, it will be convenient to translate some of the notions already defined in terms of balls. More notions will be translated in §5.1.

Let $\mathfrak{B} = (X, P_X, B_X)$ and $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ be two weakly upper multiplicative ball structures and $f\colon X \to Y$ be a map. The map f is *bornologous* if the following equivalent properties are fulfilled:

- $f\colon (X, \mathcal{E}_{\mathfrak{B}_X}) \to (Y, \mathcal{E}_{\mathfrak{B}_Y})$ is bornologous;
- for every radius $r \in P_X$, there exists $s \in P_Y$ such that $f(B_X(x, r)) \subseteq B_Y(f(x), s)$, for every $x \in X$.

Similarly, f is *uniformly boundedness copreserving* if the following equivalent properties are satisfies:

- $f\colon (X, \mathcal{E}_{\mathfrak{B}_X}) \to (Y, \mathcal{E}_{\mathfrak{B}_Y})$ is uniformly boundedness copreserving;
- for every $s \in P_Y$, there exists $r \in P_X$ such that $B_Y(f(x), s) \cap f(X) \subseteq f(B_X(x, r))$, for every $x \in X$.

Thanks to the previous characterisation of being uniformly boundedness copreserving, it is clear that this notion generalises the one of $\succ$-mapping ([151]). A map $f\colon (X, P_X, B_X) \to (Y, P_Y, B_Y)$ between balleans is a *$\succ$-mapping* if, for every $s \in P_Y$, there exists $r \in P_X$ such that $B_Y(f(x), s) \subseteq f(B_X(x, r))$, for every $x \in X$. Of course, a surjective map is uniformly boundedness copreserving if and only if it is a $\succ$-mapping. However, the second definition is very restrictive when the map is not surjective. In fact, if a map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ is a $\succ$-mapping, then, if $(f(x), y) \in E$ for some $E \in \mathcal{E}_Y$, $x \in X$ and $y \in Y$, then $y \in f(X)$ (equivalently, $\mathcal{Q}_Y(f(X)) = f(X)$).

Finally, let us give the promised characterisation of the equality between ball structures on the same support. If $\mathfrak{B}$ and $\mathfrak{B}'$ are two ball structure on a set X, then $\mathfrak{B}$ is *finer than* $\mathfrak{B}'$ and we write $\mathfrak{B} \prec \mathfrak{B}'$ if $id_X\colon \mathfrak{B} \to \mathfrak{B}'$ is bornologous. Moreover,

$$\begin{aligned} \mathfrak{B} = \mathfrak{B}' & \text{ if and only if } \mathfrak{B} \prec \mathfrak{B}' \text{ and } \mathfrak{B}' \prec \mathfrak{B} \\ & \text{ if and only if } id_X\colon \mathfrak{B} \to \mathfrak{B}' \text{ is an } \textit{asymorphism}, \end{aligned} \tag{3.4}$$

i.e., a bijective and bornologous map whose inverse is bornologous.

Remark 3.2.3. Let $(X, \mathcal{E})$ be an entourage space and $\mathcal{B}$ be a base of $\mathcal{E}$. Then we can associate to $\mathcal{B}$ a ball structure $\mathfrak{B}_{\mathcal{B}} = (X, P_{\mathcal{B}}, B_{\mathcal{B}})$ as follows: $P_{\mathcal{B}} = \{E \in \mathcal{B} \mid \Delta_X \subseteq E\}$ and, for every $x \in X$ and $E \in P_{\mathcal{B}}$, $B_{\mathcal{B}}(x, E) = E[x]$. Then, using the fact that $\mathcal{B}$ is a base of $\mathcal{E}$, it is easy to see that actually $\mathfrak{B}_{\mathcal{B}} = \mathfrak{B}_{\mathcal{E}}$.

We have briefly recalled how coarse spaces and balleans are equivalent constructions. In the literature, there is a third way to describe coarse spaces by using coverings: the so-called *large-scale structures* ([72], also know as *asymptotic proximities* in [140]). Those are large-scale counterpart of the classical approach to uniformities via coverings (see [104]). Moreover, in [136], the authors presented a way to use the covering approach to describe quasi-uniformities. Hence the following question naturally arises.

Question 3.2.4. Is it possible to give a characterisation of entourage structures, semi-coarse structures or quasi-coarse structures through coverings?

3.3 Examples of entourage spaces

In this section we enlist some examples of entourage spaces.

3.3.1 Relation entourage structures and para-bornologies

Let $\mathscr{R}$ be a reflexive relation over a set X. In other words, $\mathscr{R} \subseteq X \times X$ is an entourage containing the diagonal Δ_X. Then we can canonically define an entourage structure $\mathcal{E}_{\mathscr{R}} = \mathfrak{cl}(\{\mathscr{R}\})$, which is called *relation entourage structure*. Moreover, $\mathscr{R}$ is symmetric if and only if $\mathcal{E}_{\mathscr{R}}$ is a semi-coarse structure, while $\mathscr{R}$ is transitive if and only if $\mathcal{E}_{\mathscr{R}}$ is a quasi-coarse structure. Furthermore, note that, $(\mathcal{E}_{\mathscr{R}})^{-1} = \mathcal{E}_{\mathscr{R}^{-1}}$, where $\mathscr{R}^{-1}$ denotes the inverse of $\mathscr{R}$ as an entourage.

Remark 3.3.1. Note that, if $(X, \geq)$ is a preordered set and $d_{\geq}$ is defined as in Example 1.1.11, then $\mathcal{E}_{d_{\geq}} = \mathcal{E}_{\geq}$.

Another entourage structure that can be defined from a reflexive relation $\mathscr{R}$ on a set X is the following: $\mathcal{E}^{fin}_{\mathscr{R}} = \mathfrak{cl}([\mathscr{R}]^{<\omega} \cup \{\Delta_X\})$.

The previous entourage structures defined on a set X endowed with a reflexive relation $\mathscr{R}$ induce two different para-bornologies: $\beta_{\mathscr{R}} = \beta_{\mathcal{E}_{\mathscr{R}}}$ and $\beta^{fin}_{\mathscr{R}} = \beta_{\mathcal{E}^{fin}_{\mathscr{R}}}$. More explicitly, for every $x \in X$, $\beta_{\mathscr{R}}(x) = \mathfrak{cl}(\{\mathscr{R}[x]\})$, while $\beta^{fin}_{\mathscr{R}}(x) = [\mathscr{R}[x]]^{<\omega}$.

It is easy to verify the following result.

Proposition 3.3.2. *Let $f\colon (X, \mathscr{R}_X) \to (Y, \mathscr{R}_Y)$ be a map between sets endowed with reflexive relations. Then the following properties are equivalent:*

(a) f preserves the relation *(i.e., for every $x, y \in X$, $f(x)\mathscr{R}_Y f(y)$ provided that $x\mathscr{R}_X y$);*
(b) $f\colon (X, \mathcal{E}_{\mathscr{R}_X}) \to (Y, \mathcal{E}_{\mathscr{R}_Y})$ is bornologous;
(c) $f\colon (X, (\mathcal{E}_{\mathscr{R}_X})^{-1}) \to (Y, (\mathcal{E}_{\mathscr{R}_Y})^{-1})$ is bornologous;
(d) $f\colon (X, \mathcal{E}^{fin}_{\mathscr{R}_X}) \to (Y, \mathcal{E}^{fin}_{\mathscr{R}_Y})$ is bornologous;
(e) $f\colon (X, (\mathcal{E}^{fin}_{\mathscr{R}_X})^{-1}) \to (Y, (\mathcal{E}^{fin}_{\mathscr{R}_Y})^{-1})$ is bornologous;
(f) $f\colon (X, \beta_{\mathscr{R}_X}) \to (Y, \beta_{\mathscr{R}_Y})$ is boundedness preserving;
(g) $f\colon (X, \beta_{\mathscr{R}_X^{-1}}) \to (Y, \beta_{\mathscr{R}_Y^{-1}})$ is boundedness preserving;
(h) $f\colon (X, \beta^{fin}_{\mathscr{R}_X}) \to (Y, \beta^{fin}_{\mathscr{R}_Y})$ is boundedness preserving;
(i) $f\colon (X, \beta^{fin}_{\mathscr{R}_X^{-1}}) \to (Y, \beta^{fin}_{\mathscr{R}_Y^{-1}})$ is boundedness preserving.

An equivalence similar to (b)↔(f) of Proposition 3.3.2 can be stated also for other properties of maps.

Proposition 3.3.3. *Let $f\colon (X,\mathscr{R}_X)\to(Y,\mathscr{R}_Y)$ be a map between sets endowed with reflexive relations. Then the following properties hold:*

(a) $f\colon (X,\mathcal{E}_{\mathscr{R}_X})\to(Y,\mathcal{E}_{\mathscr{R}_Y})$ is uniformly weakly boundedness copreserving if and only if $f\colon (X,\beta_{\mathscr{R}_X})\to(Y,\beta_{\mathscr{R}_Y})$ is weakly boundedness copreserving;

(b) $f\colon (X,\mathcal{E}_{\mathscr{R}_X})\to(Y,\mathcal{E}_{\mathscr{R}_Y})$ is uniformly boundedness copreserving if and only if $f\colon (X,\beta_{\mathscr{R}_X})\to(Y,\beta_{\mathscr{R}_Y})$ is boundedness copreserving;

(c) $f\colon (X,\mathcal{E}_{\mathscr{R}_X})\to(Y,\mathcal{E}_{\mathscr{R}_Y})$ is effectively proper if and only if $f\colon (X,\beta_{\mathscr{R}_X})\to(Y,\beta_{\mathscr{R}_Y})$ is proper.

We have discussed how one can construct entourage structures from reflexive relations. Now, we focus on the opposite process. Let $(X,\mathcal{E})$ be an entourage space. Then we define $\mathscr{R}_{\mathcal{E}}=\bigcup\mathcal{E}$, which is a reflexive relation since $\Delta_X\in\mathcal{E}$. Moreover, if $\mathcal{E}$ is a semi-coarse structure, then $\mathscr{R}_{\mathcal{E}}$ is symmetric, and, if $\mathcal{E}$ is a quasi-coarse structure, then $\mathscr{R}_{\mathcal{E}}$ is transitive.

Note that, if $\mathscr{R}$ is a reflexive relation on X, then

$$\mathscr{R}=\mathscr{R}_{\mathcal{E}_{\mathscr{R}}}=\mathscr{R}_{\mathcal{E}^{fin}_{\mathscr{R}}}.$$

Meanwhile, if $(X,\mathcal{E})$ is an entourage space, then

$$\mathcal{E}^{fin}_{\mathscr{R}_{\mathcal{E}}}\subseteq\mathcal{E}\subseteq\mathcal{E}_{\mathscr{R}_{\mathcal{E}}}. \tag{3.5}$$

The inclusions in (3.5) can be strict. Consider, for example, $\mathbb{R}$ endowed with the usual metric d. Then $\mathcal{E}^{fin}_{\mathscr{R}_{\mathcal{E}_d}}\subsetneq\mathcal{E}_d\subsetneq\mathcal{E}_{\mathscr{R}_{\mathcal{E}_d}}$. Furthermore, note that $\mathcal{E}=\mathcal{E}_{\mathscr{R}_{\mathcal{E}}}$ if and only if $\bigcup\mathcal{E}\in\mathcal{E}$ and, thus, every entourage structure $\mathcal{E}$ on a finite set X is a relation entourage structure. This observation will be used also in Remark 3.4.9.

Let us now consider a para-bornological space (X,β). Then we can associate to it a relation $\mathscr{R}_\beta$, namely the reflexive relation $\downarrow$. Note that, if β is a uniform para-bornology, induced by the entourage structure $\mathcal{E}$, then $\mathscr{R}_\beta=\mathscr{R}_{\mathcal{E}}=\bigcup\mathcal{E}$. The relation $\mathscr{R}_\beta$ symmetric if and only if β satisfies (G_1), and is transitive if β satisfies (G_2). Actually, there exist para-bornologies without (G_2) that induce transitive relation. For example, take $X=\mathbb{N}$, $\beta(0)=[\mathbb{N}]^{<\omega}$ and, for every $n\in\mathbb{N}\setminus\{0\}$, $\beta(n)=\mathfrak{cl}(\{\mathbb{N}\})$.

Also for a para-bornological space (X,β) we can deduce a chain similar to (3.5):

$$\beta^{fin}_{\mathscr{R}_\beta}\leq\beta\leq\beta_{\mathscr{R}_\beta}; \tag{3.6}$$

moreover, $\beta^{fin}_{\mathscr{R}_\beta}=\beta$ if and only if β is locally finite and $\beta=\beta_{\mathscr{R}_\beta}$ if and only if, for every $x\in X$, $\bigcup\beta(x)\in\beta(x)$. Furthermore, also the relations in (3.6) can be strict, and this can be again shown by considering $\mathbb{R}$ endowed with the usual metric.

3.3.2 Graphic quasi-coarse structures

In Example 1.1.12, we described how the family of vertices of a directed graph can be endowed with a quasi-metric, namely, the path quasi-metric. The induced metric entourage structure $\mathcal{E}_d$, which is a quasi-coarse structure, is called *graphic quasi-coarse structure.*

The graphic quasi-coarse space can be extended to the points on the graph edges, by identifying every edge with the interval $[0,1]$ endowed with the relation quasi-coarse structure associated to the usual order $\leq$ on $[0,1]$. More precisely, if $\Gamma = (V,E)$ is a directed graph and $(v,w) \in E$, then we identify 0 with v and 1 with w, respectively. This new quasi-coarse structure is called *extended graphic quasi-coarse structure*.

Let $f\colon \Gamma(V,E) \to \Gamma'(V',E')$ be a map between oriented graphs. Then f is said to be a *graph homomorphism* if, for every $(x,y) \in E$, either $f(x) = f(y)$ or $(f(x), f(y)) \in E'$. If $f\colon \Gamma(V,E) \to \Gamma'(V',E')$ is a graph homomorphism, then f sends directed paths into non-longer directed paths. Hence $f\colon (V,d) \to (V',d)$ is *non-expanding* (i.e., $d(f(x),f(y)) \leq d(x,y)$, for every $x,y \in V$), and thus $f\colon (V,\mathcal{E}_d) \to (V',\mathcal{E}_d)$ is bornologous.

3.3.3 Entourage hyperstructures

Let $(X,\mathcal{E})$ be an entourage structure. We define the following two entourage structures on $\mathcal{P}(X)$:

$$\mathcal{H}(\mathcal{E}) = \mathfrak{cl}(\{\mathcal{H}(E) \mid \Delta_X \subseteq E \in \mathcal{E}\}) \quad \text{and}$$
$$\exp\mathcal{E} = \mathfrak{cl}(\{\exp E \mid \Delta_X \subseteq E \in \mathcal{E}\}) = \mathcal{H}(\mathcal{E}) \cap \mathcal{H}(\mathcal{E})^{-1},$$

where, for every $E \in \mathcal{E}$,

$$\mathcal{H}(E) = \{(A,B) \mid B \subseteq E[A]\} \quad \text{and} \quad \exp(E) = \mathcal{H}(E) \cap \mathcal{H}(E)^{-1},$$

named *entourage hyperstructure* and *semi-coarse hyperstructure*, respectively. The way we obtained semi-coarse hyperstructures from entourage hyperstructures will be generalised in §4.2.

Remark 3.3.4. Let (X,d) be a metric space. In Example 1.1.10, we described the Hausdorff quasi-metric d_H^q on $\mathcal{P}(X)$. It is easy to prove that actually $\mathcal{E}_{d_H^q} = \mathcal{H}(\mathcal{E}_d)$.

First of all, note that, if $\mathcal{E}$ is an entourage structure, then both $\mathcal{H}(\mathcal{E})$ and $\exp\mathcal{E}$ are entourage structures since $\mathcal{H}(E) \cup \mathcal{H}(F) \subseteq \mathcal{H}(E \cup F)$, for every $E,F \in \mathcal{E}$. More precisely, $\exp\mathcal{E}$ is actually a semi-coarse structure. Furthermore, if $\mathcal{E}$ is quasi-coarse structure, then $\mathcal{H}(\mathcal{E})$ is a quasi-coarse structure, while $\exp\mathcal{E}$ is a coarse structure. In fact, for every $E,F \in \mathcal{E}$, if $(A,C) \in \mathcal{H}(E) \circ \mathcal{H}(F)$, there exists $B \subseteq X$ such that $(A,B) \in \mathcal{H}(E)$ and $(B,C) \in \mathcal{H}(F)$. Then $B \subseteq E[A]$ and $C \subseteq F[B]$, which implies that $C \subseteq F[E[A]] = (F \circ E)[A]$ and so $(A,C) \in \mathcal{H}(F \circ E)$. Note that $\mathcal{H}(\mathcal{E})$ is not a semi-coarse structure, unless the support X of $\mathcal{E}$ is empty: in fact, $(X,\emptyset) \in \mathcal{H}(\Delta_X)$, although, for every $E \in \mathcal{E}$, $E[\emptyset] = \emptyset$. Moreover, even if we consider the subspace $(\mathcal{P}(X) \setminus \{\emptyset\}, \mathcal{H}(\mathcal{E})|_{\mathcal{P}(X)\setminus\{\emptyset\}})$, it is a semi-coarse structure if and only if X satisfies (B_3). In fact, $(X,\{x\}) \in \mathcal{H}(\Delta_X)$, for every $x \in X$.

Every map $f\colon X \to Y$ between sets can be extended to a map $\overline{f}\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ such that, for every $A \in \mathcal{P}(X)$, $\overline{f}(A) = f(A) \in \mathcal{P}(Y)$.

Proposition 3.3.5. *Let $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ be a map between entourage spaces. The following properties are equivalent:*

(a) $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is bornologous;

(b) $\overline{f}\colon (\mathcal{P}(X), \mathcal{H}(\mathcal{E}_X)) \to (\mathcal{P}(Y), \mathcal{H}(\mathcal{E}_Y))$ *is bornologous.*

Proof. As for the implication (a)→(b), if f is bornologous, then the inclusion $(\overline{f}\times\overline{f})(\mathcal{H}(E)) \subseteq \mathcal{H}((f\times f)(E))$, for every $E \in \mathcal{E}_X$, holds, and the thesis follows. Conversely, (b)→(a) is a consequence of the fact that, for every entourage space $(Z, \mathcal{E}_Z)$, if $E \in \mathcal{E}_Z$ and $x, y \in Z$, then $(x, y) \in E$ if and only if $(\{x\}, \{y\}) \in \mathcal{H}(E)$. □

3.3.4 Finitely generated monoids

In this subsection we want to briefly discuss the existence of precisely two inner quasi-coarse structures on a finitely generated monoid (see Proposition 3.3.6). The proof we give is similar to the proof of Proposition 1.1.4, which is the case of finitely generated groups (see, for example, [92]).

Let M be a monoid which is finitely generated by Σ. In Example 1.1.13 we defined the left, d^λ_Σ, and the right, d^ρ_Σ, word quasi-metrics. These quasi-metrics induce quasi-coarse structures on the monoid.

Proposition 3.3.6. *Let M be a monoid and Σ and Δ be two finite subsets of M which generate the whole monoid. Then $\mathcal{E}_{d^\lambda_\Sigma} = \mathcal{E}_{d^\lambda_\Delta}$ and $\mathcal{E}_{d^\rho_\Sigma} = \mathcal{E}_{d^\rho_\Delta}$.*

Proof. Define $k = \max\{d^\lambda_\Delta(e, \sigma) \mid \sigma \in \Sigma\}$ and $l = \max\{d^\lambda_\Sigma(e, \delta) \mid \delta \in \Delta\}$. Let $x, y \in M$, suppose that $d^\lambda_\Sigma(x, y) = n$ and let $\sigma_1, \dots, \sigma_n \in \Sigma$ such that $y = x\sigma_1 \cdots \sigma_n$. Suppose that $\sigma_i = \delta_{i,1} \cdots \delta_{i,k_i}$, for every $i = 1, \dots, n$, where $k_i = d^\lambda_\Delta(e, \sigma_i)$ and $\delta_{i,j} \in \Delta$, for every $i = 1, \dots, n$ and $j = 1, \dots, k_i$. Then

$$y = x\sigma_1 \cdots \sigma_n = x\delta_{1,1} \cdots \delta_{1,k_1}\delta_{2,1} \cdots \delta_{n,k_n}$$

and so $d^\lambda_\Delta(x, y) \le \sum_{i=1}^n k_i \le nk = kd^\lambda_\Sigma(x, y)$. Hence, $\mathcal{E}_{d^\lambda_\Sigma} \subseteq \mathcal{E}_{d^\lambda_\Delta}$. Similarly, $d^\lambda_\Sigma(x, y) \le l d^\lambda_\Delta(x, y)$ and then $\mathcal{E}_{d^\lambda_\Delta} \subseteq \mathcal{E}_{d^\lambda_\Sigma}$. A similar proof shows that $\mathcal{E}_{d^\rho_\Sigma} = \mathcal{E}_{d^\rho_\Delta}$. □

3.4 The Sym-coarse equivalence

In this section we focus on quasi-coarse spaces. We want to introduce another equivalence notion, which will be more flexible than the one of asymorphism. In order to do that, we need to fix some terminology and notation. Two maps $f, g\colon S \to (X, \mathcal{E})$ from a set to a quasi-coarse space are Sym-*close*, and we denote it by $f \sim_{\mathrm{Sym}} g$, if $\{(f(x), g(x)), (g(x), f(x)) \mid x \in X\} \in \mathcal{E}$. Note that the Sym-closeness relation just defined is an equivalence relation. In §4.2.1 we provide a categorical justification to this definition.

Remark 3.4.1. Let $f, g\colon S \to (X, \mathcal{E})$ be two maps from a set to a quasi-coarse space. If $f = g$, then $f \sim_{\mathrm{Sym}} g$. The converse implication is not always true. However, if $\mathcal{E}_Y$ is the discrete coarse structure over Y, then $f \sim_{\mathrm{Sym}} g$ if and only if $f = g$.

Remark 3.4.2. It will be useful to check that some large-scale properties of a map are shared by all the maps in its equivalent class under Sym-closeness. Let us fix a pair of Sym-close maps $f, g\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between quasi-coarse spaces. Since they are Sym-close, $M = \{(f(x), g(x)), (g(x), f(x)) \mid x \in X\} \in \mathcal{E}_Y$. This remark will be used in §5.3.

(a) We claim that f is bornologous if and only if g is bornologous. In fact, let us assume that f is bornologous, and let $E \in \mathcal{E}_X$ be an arbitrary entourage. Then, for every $(x, y) \in E$

$$(g\times g)(x,y) = (g(x), f(x)) \circ (f(x), f(y)) \circ (f(y), g(y)) \in M \circ (f \times f)(E) \circ M,$$

which shows that $(g \times g)(E) \subseteq M \circ (f \times f)(E) \circ M \in \mathcal{E}_Y$.

(b) Similarly to what we have done for the item (a), we can prove that f is effectively proper if and only if g is effectively proper. In fact, if f is effectively proper, for every $E \in \mathcal{E}_Y$, $(g \times g)^{-1}(E) \subseteq (f \times f)^{-1}(M \circ E \circ M)$.

(c) The map f is large-scale surjective if and only if g is large-scale surjective. Let $E \in \mathcal{E}_Y$ be an entourage such that $E[f(X)] = Y$. Then

$$(M \circ E)[g(X)] = E[M[g(X)]] \subseteq E[f(X)] = Y.$$

Let $f\colon X \to Y$ be a map between quasi-coarse spaces. Then a map $g\colon Y \to X$ is a Sym-*coarse inverse* of f if $g \circ f \sim_{\mathrm{Sym}} id_X$ and $f \circ g \sim_{\mathrm{Sym}} id_Y$.

Definition 3.4.3. Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then f is a Sym-*coarse equivalence* if it is bornologous and has a Sym-coarse inverse $g\colon Y \to X$ which is bornologous.

Two quasi-coarse spaces are Sym-*coarsely equivalent* if there exists a Sym-coarse equivalence between them.

In Theorem 3.4.6 we give other characterisations of Sym-coarse equivalences.

A subset L of a quasi-coarse space $(X, \mathcal{E})$ is Sym-*large* if there exists a symmetric entourage $E = E^{-1} \in \mathcal{E}$ such that $E[L] = X$. A map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between quasi-coarse spaces is *large-scale surjective* if $f(X)$ is Sym-large in Y. If f is also large-scale injective, then it is *large-scale bijective.* The following result characterises large-scale bijective maps between quasi-coarse spaces.

Proposition 3.4.4. *Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then f is large-scale bijective if and only if it has a* Sym-*coarse inverse. In particular, every* Sym-*coarse inverse is large-scale bijective.*

Proof. ($\to$) Let $M = M^{-1} \in \mathcal{E}_Y$ be an entourage such that $M[f(X)] = Y$. For every $y \in Y$, there exists $x_y \in X$ such that $(y, f(x_y)) \in M$. If $y \in f(X)$, suppose that $x_y \in f^{-1}(y)$. Define $g\colon Y \to X$ with the following law: $g(y) = x_y$, for every $y \in Y$. Then $(f(g(y)), y) \in M$ for every $y \in Y$, which witnesses that $f \circ g \sim_{\mathrm{Sym}} id_Y$. The fact that f is large-scale injective proves that $g \circ f \sim_{\mathrm{Sym}} id_X$.

($\leftarrow$) Let now $g\colon Y \to X$ be a Sym-coarse inverse of f. Let $M = M^{-1} \in \mathcal{E}_X$ and $N = N^{-1} \in \mathcal{E}_Y$ be two entourages showing that $g \circ f \sim_{\mathrm{Sym}} id_X$ and $f \circ g \sim_{\mathrm{Sym}} id_Y$, respectively. Note that, for every $y \in Y$, $f(g(y)) \in f(X)$ and $(y, f(g(y))), (f(g(y)), y) \in N$. Hence f is large-scale surjective. Moreover, since $R_f = \{(x, y) \in X \times X \mid f(x) = f(y)\} \subseteq M \circ M$, f is large-scale injective.

The last assertion is trivial since, if g is a Sym-coarse inverse of f, then f is a Sym-coarse inverse of g. □

Proposition 3.4.5. *Let $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ be a large-scale bijective map between quasi-coarse spaces and let g be a* Sym*-coarse inverse of f. Then, the following properties are equivalent:*
(a) f is bornologous;
(b) g is uniformly weakly boundedness copreserving;
(c) g is uniformly boundedness copreserving;
(d) g is effectively proper.
Moreover, every other Sym*-coarse inverse h of g satisfies $h \sim_{\mathrm{Sym}} f$.*

Proof. Since g is large-scale injective, the equivalences (b)↔(c)↔(d) descend from Proposition 3.1.16. Suppose now that f is bornologous. Let $E \in \mathcal{E}_X$ and consider $(g\times g)^{-1}(E)$. Denote by $M = M^{-1}$ the entourage of $\mathcal{E}_Y$ such that $(f(g(z)),z) \in M$, for every $z \in Y$. Then, for every $(x,y) \in (g\times g)^{-1}(E)$,

$$(x,y) = (x, f(g(x)))\circ(f(g(x)), f(g(y)))\circ(f(g(y)),y) \in M\circ(f\times f)(E)\circ M \in \mathcal{E}_Y.$$

Conversely, suppose that g is effectively proper. Denote by $N = N^{-1} \in \mathcal{E}_X$ the entourage showing that $g\circ f \sim_{\mathrm{Sym}} id_X$. Let $E \in \mathcal{E}_X$ and $(x,y) \in E$. Then

$$(g(f(x)), g(f(y))) = (g(f(x)), x)\circ(x,y)\circ(y, g(f(y))) \in N\circ E\circ N \in \mathcal{E}_X$$

and thus $(f(x), f(y)) \in (g\times g)^{-1}(N\circ E\circ N) \in \mathcal{E}_Y$.

Finally, if h is another Sym-coarse inverse of g, then, for every $x \in X$, $(g(f(x)), g(h(x))) = (g(f(x)),x)\circ(x, g(h(x))) \in N\circ K$, where $K = K^{-1} \in \mathcal{E}_X$ is an entourage that shows that $g\circ h \sim_{\mathrm{Sym}} id_X$. Hence $(f(x), h(x)) \in (g\times g)^{-1}(N\circ K)$ and so $f \sim_{\mathrm{Sym}} h$ since $(g\times g)^{-1}(N\circ K) = ((g\times g)^{-1}(N\circ K))^{-1} \in \mathcal{E}_Y$. □

Note that, with an easy variation of the proof of Proposition 3.4.5, one can prove that every large-scale injective map $f\colon (X,\mathcal{E}_X) \to Y$ from a quasi-coarse space to a set has a *partial* Sym*-coarse inverse*, i.e., a map $g\colon Y' \to (X,\mathcal{E}_X)$, where $Y' \subseteq Y$, such that $g\circ f \sim_{\mathrm{Sym}} id_X$.

Theorem 3.4.6. *Let $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ be a map between quasi-coarse spaces. Then the following are equivalent:*
(a) f is a Sym*-coarse equivalence;*
(b) $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is large-scale bijective, bornologous and uniformly weakly boundedness copreserving;
(c) $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is large-scale bijective, bornologous and uniformly boundedness copreserving;
(d) $f\colon (X,\mathcal{E}_X) \to (Y,\mathcal{E}_Y)$ is large-scale surjective, bornologous and effectively proper.

Proof. The equivalences (b)↔(c)↔(d) follow from Proposition 3.1.16.

(a)→(b) Since f has a Sym-coarse inverse g, it is large-scale bijective, thanks to Proposition 3.4.4. Moreover, g is bornologous and thus Proposition 3.4.5 implies that f is uniformly weakly boundedness copreserving.

(d)→(a) Let us construct a map $g\colon Y \to X$ with the desired properties. Since f is large-scale surjective, there exists $M = M^{-1} \in \mathcal{E}_Y$ such that $Y = M[f(X)]$. Hence, for every point $y \in Y$, we can fix another point $x \in X$ with the property that $(f(x_y), y) \in M$. Define the map g by putting $g(y) = x_y$, for every $y \in Y$. Let now $x \in X$. Then $(f(x), f(g(f(x)))) = (f(x), f(x_{f(x)})) \in M$, and so $(x, g(f(x))) \in (f \times f)^{-1}(M) \in \mathcal{E}_X$ since f is effectively proper. Thus $g \circ f \sim_{\mathrm{Sym}} id_X$. If now $y \in Y$, the pair $(y, f(g(y))) \in M$ because of the definition of g, which implies that $f \circ g \sim_{\mathrm{Sym}} id_Y$, and so g is a Sym-coarse inverse of f. The conclusion then follows from Proposition 3.4.5. □

Theorem 3.4.7. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two quasi-coarse spaces. Then X and Y are* Sym*-coarsely equivalent if and only if there exist two subspaces $X' \subseteq X$ and $Y' \subseteq Y$, which are* Sym*-large in X and in Y, respectively, and an asymorphism $f'\colon X' \to Y'$.*

Proof. $(\to)$ Let assume that there exists a Sym-coarse equivalence $f\colon X \to Y$. According to Theorem 3.4.6, f is large-scale surjective, bornologous and effectively proper. Let $X' \subseteq X$ be a subset with the following property: for every $x \in X$, $|X' \cap f^{-1}(f(x))| = 1$. Then $f' = f|_{X'}\colon X' \to Y'$, where $Y' = f(X) = f(X')$, is bijective. Moreover, $f'\colon (X', \mathcal{E}_X|_{X'}) \to (Y', \mathcal{E}_Y|_{Y'})$ is bornologous and effectively proper, since it is a restriction of f. Finally, since $f\colon X \to Y$ is large-scale injective (Proposition 3.1.16), X' is Sym-large in X.

$(\leftarrow)$ Let $M = M^{-1} \in \mathcal{E}_X$ be an entourage such that $M[X'] = X$. Then define a map $h\colon X \to X'$ as follows: if $x \in X'$, then $h(x) = x$, and, if otherwise $x \in X \setminus X'$, then $h(x)$ is a point such that $(h(x), x) \in M$. Similarly we can define a map $k\colon Y \to Y'$. We claim that h and k are bornologous. Let $E \in \mathcal{E}_X$. Then note that $(h \times h)(E) \subseteq M \circ E \circ M \in \mathcal{E}_X$ and thus h is bornologous. The same property can be similarly proved for k. Then the maps $f = f' \circ h$ and $g = (f')^{-1} \circ k$ are bornologous. We claim that g is a Sym-coarse inverse of f. For every $x \in X$, since $k|_{Y'} = id_{Y'}$,

$$(x, g(f(x))) = (x, (f')^{-1}(k(f'(h(x))))) = (x, (f')^{-1}(f'(h(x)))) = (x, h(x)) \in M,$$

and thus $g \circ f \sim_{\mathrm{Sym}} id_X$. The other request can be similarly proved. □

Proposition 3.4.8. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two* Sym*-coarsely equivalent quasi-coarse spaces. If $(X, \mathcal{E}_X)$ is a coarse space, then so it is $(Y, \mathcal{E}_Y)$.*

Proof. Let $f\colon X \to Y$ be a Sym-coarse equivalence and let $g\colon Y \to X$ be a Sym-coarse inverse of f. Moreover, let $E = E^{-1} \in \mathcal{E}_X$ and $F^{-1} = F \in \mathcal{E}_Y$ be two symmetric entourages which witness that $g \circ f \sim_{\mathrm{Sym}} id_X$ and $f \circ g \sim_{\mathrm{Sym}} id_Y$, respectively. Then, for every $K \in \mathcal{E}_Y$ and $(x, y) \in K$,

$$\begin{aligned}(y, x) &= (y, f(g(y))) \circ (f(g(y)), f(g(x))) \circ (f(g(x)), x) \in \\ &\in F \circ (f \times f)(((g \times g)(K))^{-1}) \circ F \in \mathcal{E}_Y,\end{aligned}$$

and then $K^{-1} \in \mathcal{E}_Y$. □

Remark 3.4.9. Let $(X, \mathcal{E})$ be a finite quasi-coarse space. According to the discussion contained in §3.3.1, there exists a pre-order $\geq$ on X such that $\mathcal{E} = \mathcal{E}_{\geq}$ (actually, $\geq = \bigcup \mathcal{E}$). Moreover, $\geq$ induces an equivalence relation $\cong$ on X in the usual way: for every $x, y \in X$, $x \cong y$ if and only if $x \geq y$ and $y \geq x$. Let $q\colon X \to X/_{\cong}$ be the quotient map. Then $\geq$ induces a partial order $\overline{\geq} = (q \times q)(\geq)$ on $\overline{X} = X/_{\cong}$. Moreover, the map $q\colon (X, \mathcal{E}_{\geq}) \to (\overline{X}, \mathcal{E}_{\overline{\geq}})$ is a Sym-coarse equivalence. Hence, finite quasi-coarse spaces from the large-scale point of view are just partial ordered sets.

3.4.1 Characterisation of some special classes of quasi-coarse spaces

Let $(X, \mathcal{E})$ be a quasi-coarse space. Then $(X, \mathcal{E})$ is *monogenic* if there exists an entourage $E \in \mathcal{E}$ such that the family $\{E^n \mid n \in \mathbb{N}\}$ forms a base of $\mathcal{E}$, where E^n is the composite of n copies of E. In the realm of coarse spaces, monogenicity is a classical notion (see, for example [157]). In particular, every monogenic quasi-coarse space has a countable base. Note that, if $(X, \mathcal{E})$ is an entourage space such that there exists $E \in \mathcal{E}$ with the property that $\mathfrak{cl}(\{E^n \mid n \in \mathbb{N}\}) = \mathcal{E}$, then $\mathcal{E}$ is a quasi-coarse structure. An example of a monogenic quasi-coarse space is a directed graph endowed with its graphic quasi-coarse structure.

Proposition 3.4.10. *Let X and Y be two* Sym*-coarsely equivalent quasi-coarse spaces. Then:*
(a) if $i \in \{1, 2, 3', 3'', 4\}$, X satisfies C_i if and only if Y satisfies C_i;
(b) X is monogenic if and only if Y is monogenic.

Proof. First of all, note that all those properties are invariant under asymorphism. Thanks to Theorem 3.4.7, it is enough to prove the claim when Y is a Sym-large subspace of X, and, in this case, item (a) is not hard to shown. Let us now prove item (b).

Suppose that X is monogenic and $E \in \mathcal{E}_X$ is an entourage such that $\{E^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_X$. Let $F \in \mathcal{E}_X|_Y$. Then there exists $n_F \in \mathbb{N}$ such that $F \subseteq E^{n_F}$. Let $(x, y) \in F$. Thus there exist $z_0 = x, z_1, \dots, z_n = y \in X$ such that $(z_i, z_{i+1}) \in E$, for every $i = 0, \dots, n-1$. Moreover, for every $i = 1, \dots, n-1$, there exists $z_i' \in Y$ such that $(z_i, z_i') \in M$. Then, if we define $z_0' = x$ and $z_n' = y$, for every $i = 0, \dots, n-1$, $(z_i', z_{i+1}') \in (M \circ E \circ M) \cap (Y \times Y)$. Hence $\{((M \circ E \circ M) \cap (Y \times Y))^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_Y$.

Conversely, suppose that Y is monogenic and $\{E^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_Y$, for some $E \in \mathcal{E}_Y$. By using a similar argument, it is easy to show that $\{(M \circ E \circ M)^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}_X$. □

Lemma 3.4.11. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two* Sym*-coarsely equivalent quasi-coarse spaces. Then* $\operatorname{cf} \mathcal{E}_X = \operatorname{cf} \mathcal{E}_Y$*.*

Proof. By applying Theorem 3.4.7, we can assume that Y is an entourage subspace of X and the inclusion map $i\colon Y \to X$ is large-scale surjective. It is trivial that $\operatorname{cf} \mathcal{E}_Y \leq \operatorname{cf} \mathcal{E}_X$. Let $f\colon X \to Y$ be a Sym-coarse inverse of i and $M = M^{-1} \in \mathcal{E}_X$ be an entourage such that $(x, f(x)) \in M$, for every $x \in X$. Then, for every base $\{E_i\}_{i \in I}$ of $\mathcal{E}_Y$, we claim that $\{M \circ E_i \circ M\}_i$ is a base of $\mathcal{E}_X$, and thus $\operatorname{cf} \mathcal{E}_X \leq \operatorname{cf} \mathcal{E}_Y$. In fact, let $F \in \mathcal{E}_X$ and $i \in I$ be an index such that $(M \circ F \circ M)|_{Y \times Y} \subseteq E_i$. Then $F \subseteq M \circ E_i \circ M$. □

We are now ready to prove the generalisations of some classical classification results in the framework of quasi-coarse spaces ([144, Theorems 9.1, 9.2], [151, Theorem 2.11]). The following results, together with Proposition 3.1.9, give a complete characterisation of metric entourage structures.

Theorem 3.4.12. *Let $(X, \mathcal{E})$ be a quasi-coarse space. The following properties are equivalent:*
(a) there exists a quasi-metric $d\colon X \times X \to [0, \infty]$ on X such that $\mathcal{E} = \mathcal{E}_d$;

(b) there exists a quasi-metric space (Y, d) *which is* Sym*-coarsely equivalent to* $(X, \mathcal{E})$*;*
(c) $\operatorname{cf} \mathcal{E} \leq \omega$.

Proof. The implications (a)→(b)→(c) are trivial: in particular, (b)→(c) is implied by Lemma 3.4.11.

(c)→(a) Let $\{F_n\}_n$ be a base of $\mathcal{E}$ as in the proof of Proposition 3.1.9(a) with the following further property: for every $m, n \in \mathbb{N}$, $F_m \circ F_n \subseteq F_{m+n}$. We claim that the map $d\colon X \times X \to [0, \infty]$ defined as in (3.3) is a quasi-metric and, in order to show it, proving that d satisfies the triangle inequality is enough. Let $x, y, z \in X$ be three arbitrary points. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ trivially holds if $d(x, y) = \infty$ or $d(y, z) = \infty$. Suppose now that $d(x, y) \leq m$ and $d(y, z) \leq n$. Then $(x, z) = (x, y) \circ (y, z) \in F_m \circ F_n \subseteq F_{m+n}$ and thus $d(x, z) \leq m + n$. Finally, the equality $\mathcal{E} = \mathcal{E}_d$ can be easily proved. □

A quasi-coarse space satisfying the hypothesis of the previous theorem is called *quasi-metrisable*. Since the extended quasi-metric defined in the proof of Theorem 3.4.12 does not assume the value ∞ if and only if the quasi-coarse space is strongly connected, in view of Proposition 3.4.10, Theorem 3.4.12 can be specialised as follows.

Corollary 3.4.13. *Let* $(X, \mathcal{E})$ *be a quasi-coarse space. The following properties are equivalent:*
(a) there exists a quasi-metric d *on* X *which does not assume the value* ∞ *and satisfies* $\mathcal{E} = \mathcal{E}_d$*;*
(b) there exists a quasi-metric space (Y, d) *which does not assume the value* ∞ *and is* Sym*-coarsely equivalent to* $(X, \mathcal{E})$*;*
(c) $(X, \mathcal{E})$ *satisfies* C_4 *and* $\operatorname{cf} \mathcal{E} \leq \omega$.

[151, Proposition 2.1.1] implies that the quasi-metrics in Theorem 3.4.12 and in Corollary 3.4.13 can be taken as metrics if and only if the initial space is a coarse space.

Finally we can answer to a problem posed by Protasov and Banakh [144, Problem 9.4].

Theorem 3.4.14. *Let* $(X, \mathcal{E})$ *be a connected quasi-coarse space. Then the following properties are equivalent:*
(a) $(X, \mathcal{E})$ *is a graphic quasi-coarse space;*
(b) $(X, \mathcal{E})$ *is* Sym*-coarsely equivalent to a graphic quasi-coarse space;*
(c) $(X, \mathcal{E})$ *is monogenic.*

Proof. The implication (a)→(b) is trivial. As for the implication (b)→(c), since graphic quasi-coarse spaces are monogenic, Proposition 3.4.10(b) implies that also $(X, \mathcal{E})$ has the same property.

(c)→(a) Let $\Delta_X \subseteq E \in \mathcal{E}$ be an entourage such that $\mathfrak{cl}(\{E^n \mid n \in \mathbb{N}\}) = \mathcal{E}$. Consider the directed graph $\Gamma = (X, E)$ whose set of edges is the entourage E (i.e., a pair of points (x, y) of X is an edge of Γ if and only if $(x, y) \in E$). Then the graphic quasi-coarse space associated to the graph Γ is asymorphic to $(X, \mathcal{E})$. □

Chapter 4

Categories of para-bornologies and entourage spaces

Let us now discuss the categories of the objects introduced in the previous chapters. Denote by **PaBorn** be the concrete category of all para-bornological spaces and boundedness preserving maps between them. Let us also consider the following full subcategories of **PaBorn**:

- **SBorn**, whose objects are semi-bornological spaces;
- **QBorn**, whose objects are quasi-bornological spaces;
- **PrBorn**, whose objects are pre-bornological spaces.

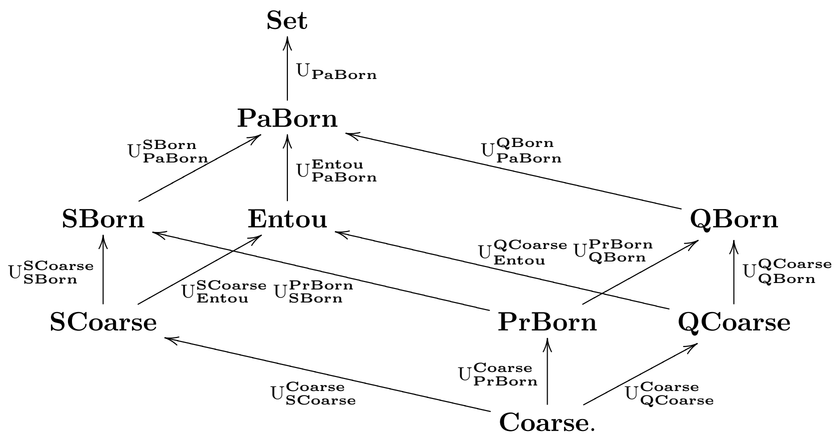
Large-scale isomorphisms are actually the isomorphisms of the categories just defined.

Similarly, let **Entou** be the concrete category of all entourage spaces and bornologous maps,

- **SCoarse** be the full subcategory of **Entou** whose objects are semi-coarse spaces,
- **QCoarse** be the full subcategory of **Entou** whose objects are quasi-coarse spaces, and
- **Coarse** be the full subcategory of **Entou** whose objects are coarse spaces.

The isomorphisms of these categories are the asymorphisms.

As already mentioned, all the previously defined categories are concrete, which means that, if $\mathcal{X}$ is one of them, there is a faithful functor $\mathrm{U}_{\mathcal{X}}\colon \mathcal{X}\to\mathbf{Set}$. Moreover, for every full subcategory $\mathcal{Y}$ of a category $\mathcal{X}$, we have a forgetful functor $\mathrm{U}_{\mathcal{X}}^{\mathcal{Y}}\colon \mathcal{Y}\to\mathcal{X}$. Finally, the definition of the uniform para-bornology induces a functor $\mathrm{U}_{\mathbf{PaBorn}}^{\mathbf{Entou}}\colon \mathbf{Entou}\to\mathbf{PaBorn}$. Moreover, Remark 3.1.5 implies that we have the following restrictions: $\mathrm{U}_{\mathbf{SBorn}}^{\mathbf{SCoarse}}\colon \mathbf{SCoarse}\to\mathbf{SBorn}$, $\mathrm{U}_{\mathbf{QBorn}}^{\mathbf{QCoarse}}\colon \mathbf{QCoarse}\to\mathbf{QBorn}$, and $\mathrm{U}_{\mathbf{PrBorn}}^{\mathbf{Coarse}}\colon \mathbf{Coarse}\to\mathbf{PrBorn}$. These are faithful functors since the triangles created with the forgetful functors to **Set** commute (for instance, $\mathrm{U}_{\mathbf{Entou}}=\mathrm{U}_{\mathbf{PaBorn}}\circ\mathrm{U}_{\mathbf{PaBorn}}^{\mathbf{Entou}}$). In the following diagram the forget functors are represented:



(4.1)

For the sake of simplicity, in (4.1) not all the forgetful functors appear. However, those with **Set** as target that are not inserted in the diagram can be recover as composition of the represented ones. For example,

$$\mathrm{U}_{\mathbf{Coarse}} = \mathrm{U}_{\mathbf{PaBorn}} \circ \mathrm{U}^{\mathbf{Entou}}_{\mathbf{PaBorn}} \circ \mathrm{U}^{\mathbf{SCoarse}}_{\mathbf{Entou}} \circ \mathrm{U}^{\mathbf{Coarse}}_{\mathbf{SCoarse}}\,.$$

When there is no risk of ambiguity, we drop the indices and simply write U for the forgetful functor that we are taking into account.

4.1 Categories of para-bornological spaces

In this section, let us focus our attention on the four categories **PaBorn**, **SBorn**, **QBorn** and **PrBorn**. First of all, we want to prove that those are topological (according to Brümmer's definition, see [23]). In order to prove it, we need to show how we can induce para-bornologies through maps.

Let $f\colon X \to (Y,\beta)$ be a map from a set to a para-bornological space. Define the *initial para-bornology* $f_*(\beta) = \{\beta_f(x) \mid x \in X\}$, where

$$\beta_f(x) = \mathfrak{cl}(\{f^{-1}(A) \mid A \in \beta(f(x))\}),$$

for every $x \in X$. It is easy to check that $\beta_f(x)$ is an ideal, for every $x \in X$. Moreover, $f\colon (X, f_*\beta) \to (Y,\beta)$ is boundedness preserving and proper. Furthermore,

- if X is a semi-bornology, then $f_*(\beta)$ is a semi-bornology, and
- if X is a quasi-bornology, then $f_*(\beta)$ is a quasi-bornology.

Theorem 4.1.1. **PaBorn**, **SBorn**, **QBorn** *and* **PrBorn** *are topological categories.*

Proof. If $\mathcal{X}$ is one of the enlisted categories, it is easy to show that the forgetful functor $\mathrm{U}_{\mathcal{X}}$ is amnestic and transportable, and it has small fibres. Moreover, the constant maps are boundedness preserving maps. The only non trivial request is that every source has an initial lifting. Let $\{f_i\colon X \to (X_i,\beta_i)\}_{i\in I}$ be a family of maps from a fixed set X to a family of para-bornological spaces $\{(X_i, \beta_i)\}_{i\in I}$ that are objects in $\mathcal{X}$. Then, according to the previous observations, the para-bornology $\beta = \bigwedge_i((f_i)_*(\beta_i))$ satisfies the desired properties. □

As a consequence of Theorem 4.1.1, we have a complete characterisation of the monomorphisms and the epimorphisms of **PaBorn**, **SBorn**, **QBorn**, and **PrBorn**. In fact, the monomorphisms and the epimorphisms are the injective and surjective boundedness preserving maps, respectively. Hence a bimorphism is a bijective boundedness preserving map, for example, the map $id_X\colon (X, \beta_{dis}) \to (X, \beta_{triv})$ is a bimorphism. This also shows that those categories are not balanced since the just defined map is not a large-scale isomorphism (i.e., an isomorphism of the category) provided that X has at least two elements. Furthermore, note that those categories are trivially cowellpowered.

Question 4.1.2. Does there exist an (epireflective) subcategory of **PaBorn** which is not cowellpowered?

We want to study the relationships between **PaBorn**, **SBorn**, **QBorn**, and **PrBorn**, and, in order to do that, we define some useful functors between the four categories, which will be summarised in the diagram (4.2). All these functors will only be defined on the objects, since the morphisms are 'fixed' (i.e., if $\mathrm{F}\colon \mathcal{X} \to \mathcal{Y}$ is one of the functors that we are going to define and $f\colon X \to Y$ is a morphism of $\mathcal{X}$, then $\mathrm{U}_{\mathcal{Y}}(\mathrm{F}\, f) = \mathrm{U}_{\mathcal{X}}\, f$). Before defining them, we need to introduce a notion. Let (X, β) be a para-bornological space, and fix a point $x \in X$. Then a subset B of X is called *balloon of X starting in x* if there exist $m \in \mathbb{N}$, and $x_1 = x, x_2, \dots, x_m \in X$ such that, for every $i = 1, \dots, m-1$, $x_{i+1} \in \beta(x_i)$, and $B \in \beta(x_m)$. Let us now introduce the announced functors.

- B-wSym: **PaBorn** → **SBorn** is defined by the law
$$\text{B-wSym}(X, \beta) = (X, \text{B-wSym}(\beta)),$$
for every $(X, \beta) \in$ **PaBorn**, where $\text{B-wSym}(\beta) = \{\text{B-wSym}(\beta)(x) \mid x \in X\}$, and, for every $x \in X$, $\text{B-wSym}(\beta)(x) = \{A \cap \mathcal{Q}_X^{\uparrow}(x) \mid A \in \beta(x)\}$.
- B-USym: **PaBorn** → **SBorn** is defined by the law
$$\text{B-USym}(X, \beta) = (X, \text{B-USym}(\beta)),$$
for every $(X, \beta) \in$ **PaBorn**, where $\text{B-USym}(\beta) = \{\text{B-USym}(\beta)(x) \mid x \in X\}$, and, for every $x \in X$, $\text{B-USym}(\beta)(x) = \{A \cup B \mid A \in \beta(x), B \in [\mathcal{Q}_X^{\uparrow}(x)]^{<\omega}\}$.
- B-W: **PaBorn** → **QBorn** is defined by the law
$$\text{B-W}(X, \beta) = (X, \text{B-W}(\beta)),$$
for every $(X, \beta) \in$ **PaBorn**, where $\text{B-W}(\beta) = \{\text{B-W}(\beta)(x) \mid x \in X\}$, and, for every $x \in X$, a subset of X belongs to $\text{B-W}(\beta)(x)$ if it is a finite union of balloons of X starting in x. Similarly, B-W: **SBorn** → **PrBorn** is defined.
- B-Sym: **QBorn** → **PrBorn** is defined by the law
$$\text{B-Sym}(X, \beta) = (X, \text{B-Sym}(\beta)),$$
for every $(X, \beta) \in$ **QBorn**, where $\text{B-Sym}(\beta) = \{\text{B-Sym}(\beta)(x) \mid x \in X\}$, and, for every $x \in X$, a subset $A \in \beta(x)$ belongs to $\text{B-Sym}(\beta)(x)$ if and only if $A \cup \{x\}$ is bounded.

Let us show that those maps just introduced are actually well-defined and functors.

Proposition 4.1.3. B-wSym, B-USym: **PaBorn** $\to$ **SBorn**, B-W: **PaBorn** $\to$ **QBorn**, B-W: **SBorn** $\to$ **PrBorn**, *and* B-Sym: **QBorn** $\to$ **PrBorn** *are functors.*

Proof. Once we prove that those maps are well-defined, it will be easy to check that they are functors. Well-definition is not hard to show either, and, for instance, we prove that, if X is a semi-bornology, B-W(X) is a quasi-bornology, and that, if X is a quasi-bornology, B-Sym(X) is a pre-bornology.

Let then (X, β) be a semi-bornology. Then B-W(X) is trivially a para-bornology. Let us show first that B-W(X) is a semi-bornology. Fix $x, y \in X$ such that $\{y\} \in$ B-W$(\beta)(x)$. Then there exist $m \in \mathbb{N}$, and $x_1 = x, x_2, \dots, x_m = y \in X$ such that, for every $i = 1, \dots, m-1$, $x_{i+1} \in \beta(x_i)$. Since X has the property (G$_1$), for every $i = 1, \dots, m-1$, $x_i \in \beta(x_{i+1})$, which shows that $\{x\} \in$ B-W$(\beta)(y)$, and so B-W(X) is a semi-bornology. Consider now a subset $A \in$ B-W$(\beta)(y)$. Since B-W$(\beta)(y)$ is closed under finite unions, we can assume without loss of generality that A is a balloon starting in y. Then there exists $n \in \mathbb{N}$, and $y_1 = y, y_2, \dots, y_n \in X$ such that, for every $i = 1, \dots, n-1$, $y_{i+1} \in \beta(y_i)$, and $A \in \beta(y_n)$. If we define

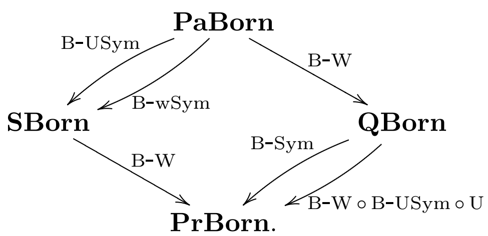
$$z_1 = x_1 = x, \dots, z_m = x_m = y_1 = y, z_{m+1} = y_2, \dots, z_{m+n-1} = y_n,$$

then A is a balloon starting in x, and this observation implies the desired property.

Consider now a quasi-bornology (X, β). Fix a point $x \in X$. The family B-Sym$(\beta)(x)$ trivially contains the singleton $\{x\}$ and it is closed under taking subsets. Pick now two subsets $A, B \subseteq X$ such that $A \cup \{x\}$ and $B \cup \{x\}$ are bounded. If $a \in A$, then $\{x\} \in \beta(a)$, and, since $B \in \beta(x)$ and β is a quasi-bornology, $A \cup B \in \beta(a)$. Similarly $A \cup B$ is bounded from every point of B. Since $A \cup B$ is also bounded from x, $A \cup B \in$ B-Sym$(\beta)(x)$. Moreover, B-Sym(β) has trivially the property (G$_1$). Consider now another point $y \in X$ and $A \subseteq X$ such that $\{y\} \in$ B-Sym$(\beta)(x)$ and $A \in$ B-Sym$(\beta)(y)$. Then A is bounded from x because β has the property (G$_2$). Furthermore, for every $a \in A$, $A \in \beta(a)$. Finally, since $\{y\} \in \beta(a)$ and $\{y\} \in \beta(x)$, $\{x\} \in \beta(a)$ by applying both properties (G$_1$) and (G$_2$). $\square$

Note that, in the proof of Proposition 4.1.3, in order to prove that, if X is a quasi-bornology, B-Sym(X) is a para-bornology, the property (G$_2$) of X is applied.

If we also consider the composite functor B-W $\circ$ B-USym $\circ$ $U^{\mathbf{QBorn}}_{\mathbf{PaBorn}}$: **QBorn** $\to$ **PrBorn**, the situation can be represented by the following diagram:



(4.2)

Moreover, note that, if $\mathrm{F}\colon \mathcal{X} \to \mathcal{Y}$ is one of the functors represented in (4.2), then $\mathrm{F} \circ \mathrm{U}_{\mathcal{X}}^{\mathcal{Y}} = 1_{\mathcal{Y}}$.

In the following example we provide a semi-bornology that admits two maximal pre-bornologies that are finer than the original one.

Example 4.1.4. Consider the following semi-metric d on $\mathbb{Z}^2$: for every pair of points $(x,y),(z,w) \in \mathbb{Z}^2$,

$$d((x,y),(z,w)) = \begin{cases} |x-z| & \text{if } y = w, \\ |y-w| & \text{if } x = z, \\ \infty & \text{otherwise.} \end{cases}$$

Then $(\mathbb{Z}^2, \beta_d)$ is a semi-bornological space. We claim that there are two different maximal pre-bornologies β_1 and β_2 on $\mathbb{Z}^2$ which are finer then β_d. Define the following metrics d_1 and d_2 on $\mathbb{Z}^2$ as follows: for every $(x,y),(z,w) \in \mathbb{Z}^2$,

$$d_1((x,y),(z,w)) = \begin{cases} |x-z| & \text{if } y = w, \\ \infty & \text{otherwise,} \end{cases} \qquad \text{and}$$

$$d_2((x,y),(z,w)) = \begin{cases} |y-w| & \text{if } x = z, \\ \infty & \text{otherwise.} \end{cases}$$

Then $\beta_1 = \beta_{d_1}$ and $\beta_2 = \beta_{d_2}$ satisfy the desired properties.

The functors previously defined can be used to prove the following theorem.

Theorem 4.1.5. *(a)* **QBorn** *is a reflective subcategory in* **PaBorn***;*
(b) **SBorn** *is a reflective and co-reflective subcategory in* **PaBorn***;*
(c) **PrBorn** *is a reflective subcategory in* **SBorn***;*
(d) **PrBorn** *is a reflective and co-reflective subcategory in* **QBorn**.

Proof. The thesis follows by proving the following assertions:
- B-W is a reflector of $\mathrm{U}_{\mathbf{PaBorn}}^{\mathbf{QBorn}}$;
- B-USym is a reflector and B-wSym is a co-reflector of $\mathrm{U}_{\mathbf{PaBorn}}^{\mathbf{SBorn}}$;
- B-W is a reflector of $\mathrm{U}_{\mathbf{SBorn}}^{\mathbf{PrBorn}}$;
- B-W $\circ$ B-USym $\circ\, \mathrm{U}_{\mathbf{PaBorn}}^{\mathbf{QBorn}}$ is a reflector and B-Sym is a co-reflector of $\mathrm{U}_{\mathbf{QBorn}}^{\mathbf{PrBorn}}$;

which are easy checks of the definitions. □

4.2 Categories of entourage spaces

Similarly to what we have done for the categories of para-bornological spaces, first of all we want to prove that the categories **Entou**, **SCoarse**, **QCoarse** and **Coarse** are topological. Also in this case we need to introduce the notion of initial entourage structure.

Let $f\colon X \to (Y,\mathcal{E})$ be a map between a set and an entourage space. We define the *initial entourage structure* $f_*\mathcal{E}$ as the entourage structure over X generated by the base $\{(f \times f)^{-1}(E) \mid E \in \mathcal{E}\}$. If $\mathcal{E}$ is a semi-coarse structure (quasi-coarse structure), then $f_*\mathcal{E}$ is a semi-coarse structure (quasi-coarse structure, respectively). Moreover, $f\colon (X, f_*\mathcal{E}) \to (Y,\mathcal{E})$ is bornologous and effectively proper.

Theorem 4.2.1. *The categories* **Entou***,* **SCoarse***,* **QCoarse** *and* **Coarse** *are topological.*

Proof. Similarly to the situation of Theorem 4.1.1, we only need to show that all the sources in these categories admit initial liftings. Let $\{f_i\colon X \to (Y_i, \mathcal{E}_i)\}_{i\in I}$ be a source of maps from a set to a family of entourage spaces. Define the entourage structure $\mathcal{E}$ over X as $\mathcal{E} = \bigcap_{i\in I}(f_i)_*\mathcal{E}_i$. If $\mathcal{E}_i$ is a semi-coarse structure (a quasi-coarse structure), for every $i \in I$, then $\mathcal{E}$ is a semi-coarse structure (a quasi-coarse structure, respectively). □

Thanks to Theorem 4.2.1, in those categories the monomorphisms are injective morphisms (i.e., injective bornologous maps), and the epimorphisms are surjective morphisms (i.e., surjective bornologous maps). In particular those four categories are not balanced since the bimorphisms are bijective bornologous maps, the isomorphisms are asymorphisms, and the map $id_X\colon (X, \mathcal{E}_{dis}) \to (X, \mathcal{E}_{triv})$ is not an asymorphism provided that X has at least two elements.

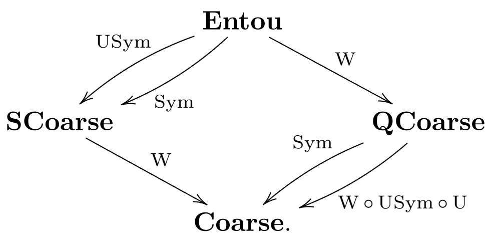
Since the epimorphisms of **Entou**, **SCoarse**, **QCoarse**, and **Coarse** are surjective morphisms, those categories are cowellpowered. Moreover, in the sequel (namely in Section 5.2) we prove that every epireflective subcategory of **Coarse** is cowellpowered. Hence the following question naturally arises.

Question 4.2.2. Does there exist a subcategory of **Entou** containing **Coarse** which is not cowellpowered?

We want to study the relationships between **Entou**, **SCoarse**, **QCoarse**, and **Coarse**, and, in order to do that, we define some useful functors between the four categories, which will be summarised in the diagram (4.3). All these functors will only be defined on the objects, since the morphisms are fixed in the same sense described for the functors between the categories of para-bornological spaces that were introduced in §4.1.

- $\mathrm{Sym}\colon$ **Entou** $\to$ **SCoarse** is defined by the law $\mathrm{Sym}(X, \mathcal{E}) = (X, \mathrm{Sym}(\mathcal{E}))$, where $\mathrm{Sym}(\mathcal{E}) = \mathfrak{cl}(\{E \cap E^{-1} \mid E \in \mathcal{E}\}) = \mathcal{E} \cap \mathcal{E}^{-1}$, for every $(X, \mathcal{E}) \in$ **Entou**. In a similar way, $\mathrm{Sym}\colon$ **QCoarse** $\to$ **Coarse** is defined.
- $\mathrm{USym}\colon$ **Entou** $\to$ **SCoarse** is defined by the law $\mathrm{USym}(X, \mathcal{E}) = (X, \mathrm{USym}(\mathcal{E}))$, where $\mathrm{USym}(\mathcal{E}) = \mathfrak{cl}(\{E \cup E^{-1} \mid E \in \mathcal{E}\})$, for every $(X, \mathcal{E}) \in$ **Entou**.
- $\mathrm{W}\colon$ **Entou** $\to$ **QCoarse** is defined by the law $\mathrm{W}(X, \mathcal{E}) = (X, \mathrm{W}(\mathcal{E}))$, for every $(X, \mathcal{E}) \in$ **Entou**, where $\mathrm{W}(\mathcal{E}) = \mathfrak{cl}(\{E^n \mid n \in \mathbb{N},\, E \in \mathcal{E}\})$. Similarly, $\mathrm{W}\colon$ **SCoarse** $\to$ **Coarse** is defined.

If we also consider the composite functor $\mathrm{W} \circ \mathrm{USym} \circ \mathrm{U}^{\mathbf{QCoarse}}_{\mathbf{Entou}}\colon$ **QCoarse** $\to$ **Coarse**, the situation can be represented by the following diagram:



(4.3)

Similarly to what we noticed for the functors represented in (4.2), if $\mathrm{F}\colon \mathcal{X} \to \mathcal{Y}$ is one of the functors that can be found in (4.3), then $\mathrm{F} \circ \mathrm{U}^{\mathcal{Y}}_{\mathcal{X}} = 1_{\mathcal{Y}}$.

There is another endofunctor J of **Entou** that it is worth mentioning. Every entourage space $(X,\mathcal{E})$ is associated to $\mathrm{J}(X,\mathcal{E})=(X,\mathcal{E}^{-1})$ and every morphism $f\in\mathrm{Mor}_{\mathbf{Entou}}(X,Y)$ is fixed, i.e., $\mathrm{U}\,f=\mathrm{U}(\mathrm{J}\,f)$. Since, for every entourage E of X, $(f\times f)(E^{-1})=((f\times f)(E))^{-1}$, $\mathrm{J}\,f$ is bornologous whenever f is bornologous, and so J is a functor. Note that $\mathrm{J}\,|_{\mathbf{SCoarse}}$ is the identity functor of **SCoarse**.

Remark 4.2.3. (a) Note that the functor Sym generalises the definition of the semi-coarse hyperstructure from the entourage hyperstructure (see §3.3.3 for the definitions). More precisely, if $(X,\mathcal{E})$ is an entourage space, then $\mathrm{Sym}(\mathcal{P}(X),\mathcal{H}(\mathcal{E}))=(\mathcal{P}(X),\exp\mathcal{E})$.

(b) It is not true in general that, if X is a quasi-coarse space, $\exp X=\exp(\mathrm{Sym}\,X)$. In fact, let $X=\mathbb{Z}$ with the relation entourage structure induced by the usual order relation, which is a quasi-coarse structure. Then $\mathrm{Sym}\,X$ is X endowed with the discrete coarse structure since the original relation entourage structure is induced by an antisymmetric relation (actually a total order). However, $2\mathbb{Z}$ and $2\mathbb{Z}+1$ belong to the same connected component of $\exp X$.

Example 4.1.4 provides also a semi-coarse structure that admits two maximal coarse structures that are finer than the original one.

The functors previously defined can be used to prove the following theorem.

Theorem 4.2.4. *(a)* **QCoarse** *is a reflective subcategory in* **Entou***;*
(b) **SCoarse** *is a reflective and co-reflective subcategory in* **Entou***;*
(c) **Coarse** *is a reflective subcategory in* **SCoarse***;*
(d) **Coarse** *is a reflective and co-reflective subcategory in* **QCoarse***.*

Proof. The thesis follows by proving the following assertions:

- W is a reflector of $\mathrm{U}_{\mathbf{Entou}}^{\mathbf{QCoarse}}$;
- USym is a reflector and Sym is a co-reflector of $\mathrm{U}_{\mathbf{Entou}}^{\mathbf{SCoarse}}$;
- W is a reflector of $\mathrm{U}_{\mathbf{SCoarse}}^{\mathbf{Coarse}}$;
- $\mathrm{W}\circ\mathrm{USym}\circ\mathrm{U}_{\mathbf{Entou}}^{\mathbf{QCoarse}}$ is a reflector and Sym is a co-reflector of $\mathrm{U}_{\mathbf{QCoarse}}^{\mathbf{Coarse}}$;

which are easy checks of the definitions. □

Embeddings of reflective subcategories preserve limits (e.g., products), while embeddings of co-reflective subcategories preserve colimits (e.g., coproducts and quotients). See [1] for details. In §4.3, we prove that **QCoarse** (**QBorn**) is not a co-reflective subcategory in **Entou** (**PaBorn**, respectively), and that **Coarse** (**PrBorn**) is not a co-reflective subcategory in **SCoarse** (**SBorn**, respectively) since they do not preserve some colimits.

4.2.1 F-coarse equivalences and quotient categories

A very important notion in coarse geometry is the one of coarse equivalence ([157]). Let $f,g\colon X\to(Y,\mathcal{E})$ be two maps from a set to a coarse space. Then f and g are *close*, and we denote this fact by $f\sim g$, if $\{(f(x),g(x))\mid x\in X\}\in\mathcal{E}$. Since $\mathcal{E}$ is a coarse space, then $\sim$ is an equivalence relation. A map $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between coarse spaces is a *coarse equivalence* if it is bornologous and there exists another bornologous map $g\colon Y\to X$, called *coarse inverse* such that $g\circ f\sim id_X$ and $f\circ g\sim id_Y$.

Let us show that $\sim$ is actually a congruence in the category **Coarse**, so that the quotient category $\mathbf{Coarse}/_\sim$ can be defined.

Lemma 4.2.5. *If $(X,\mathcal{E}_X),(Y,\mathcal{E}_Y)$ and $(Z,\mathcal{E}_Z)$ are quasi-coarse spaces, and the pairs $f,f'\in \mathrm{Mor}_{\mathbf{Coarse}}(X,Y)$, $g,g'\in \mathrm{Mor}_{\mathbf{Coarse}}(Y,Z)$ satisfy $f\sim f\prime$ and $g\sim g\prime$, then $g\circ f\sim g'\circ f'$.*

Proof. Since $f\sim f'$, $\{(f(x),f'(x))\mid x\in X\}\in\mathcal{E}_Y$ and then

$$M=\{(g(f(x)),g(f'(x)))\mid x\in X\}\in\mathcal{E}_Z,$$

because g is bornologous. Moreover, $g\sim g'$ and then $N=\{(g(f'(x)),g'(f'(x)))\mid x\in X\}\in\mathcal{E}_Z$. Finally we have

$$\{(g(f(x)),g'(f'(x)))\mid x\in S\}\subseteq M\circ N\in\mathcal{E}_Z.$$

□

Suppose that F is a functor from a category $\mathcal{X}$ to **Coarse**. Then a notion of closeness can be inherited by $\mathcal{X}$ from **Coarse**. Let $f,g\colon X\to Y$ be two morphisms of $\mathcal{X}$. We say that f is F-*close* to g (and we write $f\sim_{\mathrm{F}} g$) if $\mathrm{F}\,f$ is close to $\mathrm{F}\,g$ in **Coarse**. These new relations are equivalences. Moreover, a morphism $k\colon W\to Z$ of $\mathcal{X}$ is a F-*coarse inverse* of a morphism $h\colon Z\to W$ if $k\circ h\sim_{\mathrm{F}} id_Z$ and $h\circ k\sim_{\mathrm{F}} id_W$. Thanks to this notion, we can define equivalences between the objects of **Entou**, **SCoarse** and **QCoarse**.

Definition 4.2.6. Suppose that $\mathcal{X}$ is a subcategory of **Entou** and F is a functor from $\mathcal{X}$ to **Coarse**. A map $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ between two objects of $\mathcal{X}$ is a F-*coarse equivalence* if $f\colon(X,\mathcal{E}_X)\to(Y,\mathcal{E}_Y)$ is bornologous and it has a bornologous F-coarse inverse $g\colon(Y,\mathcal{E}_Y)\to(X,\mathcal{E}_X)$. In this case, $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ are called F-*coarsely equivalent*.

The concept just introduced induces an equivalence relation between objects of $\mathcal{X}$. Note that the Sym-coarse equivalence introduced in Definition 3.4.3 coincides with the one induced by the functor Sym: **QCoarse** $\to$ **Coarse** according to Definition 4.2.6. This observation explains the name choice for that equivalence relation.

Furthermore, note that, if $\mathcal{X}$ is a category containing **Coarse** as a full subcategory and $\mathrm{F}\colon\mathcal{X}\to$ **Coarse** is the identity functor on **Coarse**, then the induced notion of F-coarse equivalence extends the usual coarse equivalence. More precisely, if, for every coarse space X, $\mathrm{F}\,X=X$, then a map f between coarse spaces is a F-coarse equivalence if and only if it is a coarse equivalence. In particular, this observation can be applied to the functor Sym: **QCoarse** $\to$ **Coarse**, and thus the Sym-coarse equivalence coincides with the coarse equivalence if we restrict to coarse spaces.

Another notion that we have introduced was actually borrowed from the framework of coarse spaces using the functor Sym: **QCoarse** $\to$ **Coarse**. A subset Y of a coarse space $(X,\mathcal{E})$ is *large in* X ([157]) if there exists $E\in\mathcal{E}$ such that $E[Y]=\bigcup_{y\in Y}E[y]=X$. Then a subset Y of a quasi-coarse space X is Sym-large in X if and only if it is large in $\mathrm{Sym}(X)$.

Let $\mathcal{X}$ be one of the categories **Entou**, **SCoarse** or **QCoarse** and let $\mathrm{F}\colon\mathcal{X}\to$ **Coarse** be a functor. As it is shown in the first part of this section, an equivalence relation $\sim_{\mathrm{F}}$ on $\mathcal{X}$ can be induced, which is actually a congruence (it easily follows, since F is a functor and $\sim$ is a congruence, Lemma 4.2.5). Hence it is natural to produce the quotient category $\mathcal{X}/_{\sim_{\mathrm{F}}}$. The quotient category **Coarse**$/_{\sim}$ will be studied in §5.3. Moreover, among the others, also **QCoarse**$/_{\sim_{\mathrm{Sym}}}$ is worth being investigated.

4.3 Categorical constructions

4.3.1 Products, coproducts and pullbacks

Let $\{X_i\}_{i\in I}$ be a family of sets. Denote by $p_j\colon \Pi_i X_i \to X_j$, where $j\in I$, the canonical projection into the j-th component, while denote by $i_k\colon X_k \to \bigsqcup_i X_i$ the canonical inclusion of X_k into the disjoint union. For the sake of simplicity, if $\{X_i\}_i$ is a family of sets and, for every $i\in I$, $A_i\subseteq X_i$, we denote by $\Pi_i A_i$ the subset $\bigcap_i p_i^{-1}(A_i)$.

Let us first introduce products in the eight categories we consider.

Let $\{(X_i,\beta_i)\}_{i\in I}$ be a family of para-bornological spaces. The *product* $\Pi_i(X_i,\beta_i)$ *of the family* $\{(X_i,\beta_i)\}_{i\in I}$ *of para-bornological spaces* is the para-bornological space (X,β), with $X=\Pi_{i\in I}X_i$ and $\beta=\Pi_i\beta_i=\{\beta(x)\mid x\in X\}$ (called *product para-bornology*), where, for every $x\in X$, $\beta(x)=\mathfrak{cl}(\{\Pi_i A_i\mid A_i\in\beta_i(x),\,\forall i\in I\})$. We can check that (X,β) is the product in **PaBorn**. As we have already pointed out, since **PrBorn** is reflective in both **SBorn** and **QBorn**, which are reflective in **PaBorn**, these categories are stable under taking products. Hence, the same construction leads to the product in **SBorn**, **QBorn** and **PrBorn**.

Let $\{(X_i,\mathcal{E}_i)\}_{i\in I}$ be a family of entourage spaces. Let $X=\Pi_i X_i$. Then the *product entourage structure* $\mathcal{E}=\Pi_i\mathcal{E}_i$ is defined as

$$\mathcal{E}=\mathfrak{cl}\bigg(\bigg\{\bigcap_{i\in I}(p_i\times p_i)^{-1}(E_i)\mid E_i\in\mathcal{E}_i,\,\forall i\in I\bigg\}\bigg).$$

Then the pair $(X,\mathcal{E})=\Pi_i(X_i,\mathcal{E}_i)$, called *product entourage space* is the product in **Entou**. Similarly to the previous situation, since **Coarse** is reflective in both **SCoarse** and **QCoarse**, which are reflective in **Entou**, the product entourage structure provide also the product in **SCoarse**, **QCoarse** and **Coarse**. The products in **Coarse** are well-known objects (see, for instance, [151, 65]).

Note that, if $\{(X_i,\mathcal{E}_i)\}_{i\in I}$ is a family of entourage spaces, then $\beta_{\mathcal{E}}=\Pi_i\beta_{\mathcal{E}_i}$. Moreover, the product entourage structure is trivial (discrete) if all the coarse structures composing it are trivial (discrete, respectively).

Remark 4.3.1. Let $\{X_i\}_{i\in I}$ be a family of sets, and, for every $i\in I$, β_i and $\mathcal{E}_i$ be a para-bornology and an entourage structure of X_i, respectively. Denote by β and $\mathcal{E}$ the product para-bornology of the family $\{\beta_i\}_i$ and the product entourage structure of the family $\{\mathcal{E}_i\}_i$, respectively. Then, for every $j\in I$, $p_j\colon(\Pi_i X_i,\beta)\to(X_j,\beta_j)$ is boundedness preserving and copreserving, while $p_j\colon(\Pi_i X_i,\mathcal{E})\to(X_j,\mathcal{E}_j)$ is bornologous and uniformly boundedness copreserving.

The projection maps are rarely proper maps. More precisely, in the previous notations, p_j is proper if and only if, for every $i\in I\setminus\{j\}$, (X_i,β_i) is bounded.

This observation implies that the category $\mathcal{R}$ of coarse spaces and bornologous proper maps between them (see [157, 151]) does not have products.

Let $f\colon X \to Y$ be an application between sets. Then we can define the *graph of* f to be the subset $\mathrm{Graph}(f) = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$.

Proposition 4.3.2. *Let $f\colon X \to Y$ be a map between sets.*
(a) Let β_X and β_Y be para-bornologies of X and Y, respectively. Then $i\colon (X, \beta_X) \to (\mathrm{Graph}(f), \beta_X \times \beta_Y|_{\mathrm{Graph}(f)})$ is a proper bijection. Moreover, i is a large-scale isomorphism if and only if f is boundedness preserving.
(b) Let $\mathcal{E}_X$ and $\mathcal{E}_Y$ be entourage structures of X and Y, respectively. Then $i\colon (X, \mathcal{E}_X) \to (\mathrm{Graph}(f), \mathcal{E}_X \times \mathcal{E}_Y|_{\mathrm{Graph}(f)})$ is an effectively proper bijection. Moreover, i is an asymorphism if and only if f is bornologous.

Proof. Let us show item (a), and item (b) can be proved similarly. Since i^{-1} is the restriction of the projection $p_1\colon X \times Y \to X$ to $\mathrm{Graph}(f)$, it is boundedness preserving and thus i is proper. As for the second assertion, suppose that f is boundedness preserving. Then, for every $x \in X$ and every $A \in \beta(x)$, $i(A) \subseteq A \times f(A) \in \beta((x, f(x)))$, which shows that i is boundedness preserving, and thus a large-scale isomorphism. Conversely, if i is a large scale isomorphism, then $f = p_2 \circ i$ is boundedness preserving since p_2 is boundedness preserving. □

Proposition 4.3.3. *Let $\mathbb{B}$ and $\mathbb{D}$ denote the set $\{0,1\}$ endowed with the trivial and the discrete coarse structure, respectively.*
(a) Every coarse space is trivial if and only if it is isomorphic to some subspace of a product of copies of $\mathbb{B}$.
(b) Every coarse space is discrete if and only if it is isomorphic to some subspace of a product of copies of $\mathbb{D}$.

Proof. Let X be a coarse space. Let us define a map $i\colon X \to \{0,1\}^X$ which associates to every point $x \in X$ its characteristic function $\chi_x \in \{0,1\}^X$. Recall that $\mathbb{B}^X$ is trivial and $\mathbb{D}^X$ is discrete. Example 3.1.12 implies that, if X is trivial (discrete), $i\colon X \to i(X) \subseteq \mathbb{B}^X$ ($i\colon X \to i(X) \subseteq \mathbb{D}^X$, respectively) is an asymorphism. The converse implications are trivial. □

Let us now introduce the coproducts. The *coproduct* $\bigoplus_i (X_i, \beta_i)$ *of the family* $\{(X_i, \beta_i)\}_{i\in I}$ of para-bornological spaces is the para-bornological space (X, β), with $X = \bigsqcup_i X_i$ and $\beta = \bigoplus_i \beta_i = \{\beta(i_k(x)) \mid k \in I,\, x \in X_k\}$ (called *coproduct para-bornology*), where, for every $k \in I$ and $x \in X_k$, $\beta(i_k(x)) = i_k(\beta_k(x))$. It is easy to check that the pair (X, β) is the categorical coproduct in the categories **PaBorn**, **SBorn**, **QBorn**, and **PrBorn**.

Let $\{(X_k, \mathcal{E}_k)\}_{k\in I}$ be a family of entourage spaces. On the disjoint union $X = \bigsqcup_k X_k$ of the supports, we define the *coproduct entourage structure* $\mathcal{E} = \bigoplus_k \mathcal{E}_k$ as follows:

$$\mathcal{E} = \mathfrak{cl}\big(\{E_{J,\varphi} \mid J \in [I]^{<\infty},\, \varphi\colon J \to \bigcup_{k\in I} \mathcal{E}_k,\, \varphi(k) \in \mathcal{E}_k,\, \forall k \in I\}\big)$$
$$\text{and, for every such a } J \text{ and } \varphi, \quad E_{J,\varphi} = \Delta_X \cup \Bigg(\bigcup_{j\in J} (i_j \times i_j)(\varphi(j))\Bigg). \tag{4.4}$$

It is not hard to check that $(X,\mathcal{E})=\bigoplus_k(X_k,\mathcal{E}_k)$, called *coproduct entourage space*, is actually the coproduct in **Entou**, **SCoarse**, **QCoarse** and **Coarse**. Furthermore, note that, in this notation, $\beta_{\mathcal{E}}=\bigoplus_i\beta_{\mathcal{E}_i}$.

Let us underline that we could not have concluded as in the product case that, once we provide the coproduct of **PaBorn** (**Entou**), then it would automatically be the coproduct of the other categories. In fact, **QBorn** (**QCoarse**) is not co-reflective in **QBorn** (**Entou**, respectively) and **PrBorn** (**Coarse**) is not co-reflective in **SBorn** (**SCoarse**, respectively).

Proposition 4.3.4. *Let $X=\bigoplus_{k\in I}X_k$ be a coproduct entourage space and let Y be another entourage space. Denote by $i_k\colon X_k\to X$ the canonical inclusions, for every $k\in I$. A map $f\colon X\to Y$ is bornologous if and only if $f|_{i_k(X_k)}$ is bornologous for every $k\in I$.*

In Proposition 4.3.4, the fact that X is a coproduct is necessary. For example the identity $id_{\{0,1\}}\colon\mathbb{B}\to\mathbb{D}$, as defined in Proposition 4.3.3, is not bornologous, although both $id_{\{0,1\}}|_{\{0\}}$ and $id_{\{0,1\}}|_{\{1\}}$ are bornologous.

The following result can be shown by checking the definitions.

Proposition 4.3.5. *Let $\{f_i\colon X_i\to Y_i\}_{i\in I}$ be a family of maps between entourage spaces. Denote by $\Pi_if_i\colon\Pi_iX_i\to\Pi_iY_i$ and by $\bigoplus_kf_k\colon\bigoplus_kX_k\to\bigoplus_kX_k$ the maps defined as follows: for every $(x_i)_i\in\Pi_iX_i$, $(\Pi_if_i)(x_i)_i=(f_i(x_i))_i$, and, for every $i_j(x)\in\bigoplus_kX_k$, $\bigoplus_kf_k(i_j(x))=i_j(f_j(x))$, respectively. Then the following properties are equivalent:*

(a) Π_if_i is bornologous;
(b) $\bigoplus_if_i$ is bornologous;
(c) for every $i\in I$, f_i is bornologous.

Remark 4.3.6. (a) Let $\{X_i\}_{i\in I}$ be a family of sets, and, for every $i\in I$, β_i and $\mathcal{E}_i$ be a para-bornology and an entourage structure of X, respectively. Denote by $X=\sqcup_iX_i$, by $\beta=\bigoplus_i\beta_i$, and by $\mathcal{E}=\bigoplus_i\mathcal{E}_i$. Then, for every $k\in I$, the map $i_k\colon(X_k,\beta_k)\to(X,\beta)$ is a large-scale embedding, while the map $i_k\colon(X_k,\mathcal{E}_k)\to(X,\mathcal{E})$ is an injective coarse embedding.

(b) Let (X,β) be a para-bornological space. Denote by $\{X_i\}_{i\in I}$ its connected components. Then $(X,\beta)=\bigoplus_{i\in I}(X_i,\beta|_{X_i})$. However, the situation is different if we consider an entourage space. Let $(X,\mathcal{E})$ be an entourage space, and $\{X_i\}_{i\in I}$ be its connected components. While it is true that $\bigoplus_i\mathcal{E}|_{X_i}$ is finer then $\mathcal{E}$, and if I is finite, then $\bigoplus_i\mathcal{E}|_{X_i}=\mathcal{E}$, they do not coincide in general.
Endow $\mathbb{N}$ with the discrete coarse structure and the pair $\{0,1\}$ with the trivial one. Let then X be the product entourage space $\mathbb{N}\times\{0,1\}$, which is a coarse space. The connected components of X are the subsets $\{n\}\times\{0,1\}$, for every $n\in\mathbb{N}$. However, the coarse structure of X is strictly coarser than the one of $\bigoplus_n(\{n\}\times\{0,1\})$.

(c) If $\varphi\colon X\to Y$ is an asymorphism of entourage structures, then φ determines a bijection between the family of non-trivial connected components of X and its counterpart in Y, so one can index both families with the same index set $I=\operatorname{dsc}X=\operatorname{dsc}Y$ and write $X=\bigcup_{i\in I}\mathcal{Q}_X(x_i)$ and $Y=\bigcup_{i\in I}\mathcal{Q}_Y(y_i)$, assuming without loss of generality that $\varphi(x_i)=y_i$ and the restriction of φ determines asymorphisms between $\mathcal{Q}_X(x_i)$ and $\mathcal{Q}_Y(y_i)$. All these are only *necessary conditions* for the existence of an asymorphism between X

and Y (i.e., the bare fact that $\operatorname{dsc} X = \operatorname{dsc} Y$ and $\mathcal{Q}_X(x_i)$ is asymorphic to $\mathcal{Q}_Y(y_i)$, for all $i \in I$, need not imply that X and Y are asymorphic, see the previous item). Indeed, the asymorphism between X and Y imposes a 'uniform coarseness' of the asymorphisms $\mathcal{Q}_X(x_i) \approx \mathcal{Q}_Y(y_i)$.
Moreover, if $\operatorname{dsc} X = \operatorname{dsc} Y$ is finite, then they are indeed sufficient conditions, since, in that case, both X and Y coincide with the coproducts of their connected components using Proposition 4.3.5.

(d) In the case of quasi-coarse spaces, the previous item can be generalised as follows. Let $\varphi\colon X \to Y$ be a Sym-coarse equivalence between two quasi-coarse spaces. Then $\operatorname{dsc} X = \operatorname{dsc} Y$. Moreover, φ induces Sym-coarse equivalences between the connected components of X and Y, respectively. Hence, in particular, for every $x \in X$, $\mathcal{Q}_X(x)$ is bounded if and only if $\mathcal{Q}_Y(\varphi(x))$ is bounded.
Similarly to the previous item, the fact that $\operatorname{dsc} X = \operatorname{dsc} Y$ and the existence of Sym-coarse equivalences between the connected components of X and Y, respectively, are only necessary conditions for the existence of a Sym-coarse equivalence between X and Y. However, those conditions are also sufficient if $\operatorname{dsc} X = \operatorname{dsc} Y < \infty$.

Example 4.3.7. In this example we want to provide the counter-examples promised in Remarks 2.1.14 and 3.1.18.

(a) Consider the following pre-bornological spaces: $X = \{0, 1\}$ endowed with the discrete pre-bornology β_X and $Y = \{a, b\}$ endowed with the trivial pre-bornology β_Y. Let $f = p_2\colon (X, \beta_X) \times (Y, \beta_Y) \to (Y, \beta_Y)$. Then f is boundedness copreserving. However, if we consider $Z = \{(0, b), (1, a)\} \subseteq X \times Y$, then $\{a, b\}$, which is bounded from b, cannot by covered by a subset of $A \subseteq Z$ which is bounded from $(0, b)$. Hence f is not weakly boundedness copreserving.
Since the spaces involved are finite, according to §3.3.1, each pre-bornology is induced by only one coarse structure, namely, the associated entourage structure. Then Proposition 3.3.3 implies that the previous example is a uniformly boundedness copreserving map whose restriction is not uniformly weakly boundedness copreserving.

(b) We now can easily define two coarse structures on a set X that are not comparable, even if they induce the same pre-bornology. In fact, if we take two different coarse structures $\mathcal{E}$ and $\mathcal{E}'$ on a set X such that $\beta_{\mathcal{E}} = \beta_{\mathcal{E}'}$, then $\mathcal{E} \oplus \mathcal{E}'$ and $\mathcal{E}' \oplus \mathcal{E}$ are not comparable, although $\beta_{\mathcal{E}\oplus\mathcal{E}'} = \beta_{\mathcal{E}'\oplus\mathcal{E}}$ (e.g., let $X = \mathbb{R}$, $\mathcal{E}$ be the euclidean metric coarse structure and $\mathcal{E}'$ be the ideal coarse structure associated to the bornology $\beta_{\mathcal{E}}$).

Since all the categories in which we are interested (**PaBorn**,**SBorn**, **QBorn**, **PrBorn**, **Entou**, **SCoarse**, **QCoarse**, and **Coarse**) are topological categories, they have *equalisers of pairs of morphisms* $f, g\colon X \to Y$ defined by $\operatorname{eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$ (more precisely, by the inclusion map $\operatorname{eq}(f, g) \hookrightarrow X$). Since they have also products, this yields the existence of *pullbacks of pairs of morphisms* $f\colon X \to Y$, $e\colon Z \to Y$ defined by the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle e} \\ P & \xrightarrow{\;v\;} & Z. \end{array} \tag{4.5}$$

The pullback can be built as follows, using the product $X \times Z$ and the equalizer $P = \mathrm{eq}(f \circ p_1, e \circ p_2) \to X \times Z$ of the pair of morphisms $f \circ p_1, e \circ p_2$, where $p_1 \colon X \times Z \to X$ and $p_2 \colon X \times Z \to Z$ are the projections of the product. The morphisms u, v of the pullback are obtained by $u = p_1|_P$ and $v = p_2|_P$.

4.3.2 Quotients of para-bornological spaces

Let $q \colon (X, \beta) \to Y$ be a surjective map from a para-bornological space to a set. We want to define the *quotient para-bornology* $\overline{\beta}^q$ on Y, which exists since **PaBorn** is topological. More explicitly, $\overline{\beta}^q$ is the finest para-bornology on Y making the map $q \colon (X, \beta) \to (Y, \overline{\beta}^q)$ boundedness preserving. First of all, consider the family

$$q(\beta) = \{q(\beta)(y) \mid y \in Y\}, \quad \text{where, for every } y \in Y,$$
$$q(\beta)(y) = \{q(A) \mid A \in \beta(x),\, x \in q^{-1}(y)\}.$$

Of course, for every $y \in Y$, $q(\beta)(y) \subseteq \overline{\beta}^q(y)$, however, $q(\beta)$ is not a para-bornology in general since, even though, for each $y \in Y$, $q(\beta)(y)$ contains $\{y\}$ and is closed under taking subsets, it is not necessarily closed under finite unions. In Example 4.3.8 we explicitly provide a surjective map $q \colon (X, \beta) \to Y$ from a pre-bornology to a set such that $q(\beta)$ is not a para-bornology. In the previous notation, $\overline{\beta}^q$ is the finest para-bornology generated by $q(\beta)$. Note that, for every $y \in Y$, $\overline{\beta}^q(y)$ is the closure under finite unions of $q(\beta)(y)$ since $q(\beta)(y)$ is closed under taking subsets. Moreover, since **SBorn** is co-reflective in **PaBorn**, the same construction leads to the quotient of the category **SBorn**.

Example 4.3.8. Let $Z = \{0, 1\}$, $(X, \beta) = (Z, \beta_{triv}) \oplus (Z, \beta_{triv})$, and $Y = \{0, 1, 2\}$. Consider then the map $q \colon X \to Y$, where $q(i_1(0)) = 0$, $q(i_1(1)) = q(i_2(0)) = 1$ and $q(i_2(1)) = 2$. Then $q(\beta)$ is not a para-bornology since $\{0, 1\}, \{1, 2\} \in q(\beta)(1)$, but $Y \notin q(\beta)(1)$. We can easily compute $\overline{\beta}^q = \{\overline{\beta}^q(y) \mid y \in Y\}$:

$$\overline{\beta}^q(0) = \mathfrak{cl}(\{\{0,1\}\}) \quad \overline{\beta}^q(2) = \mathfrak{cl}(\{Y\}), \quad \text{and} \quad \overline{\beta}^q(2) = \mathfrak{cl}(\{\{1,2\}\}).$$

Note that $\overline{\beta}^q$ does not have the property (G_2), although β is a pre-bornology.

Example 4.3.8 has another important consequence. Since a colimit (actually, constructing a quotient) is not preserved if we move from **PaBorn** to **QBorn** or from **SBorn** to **PrBorn**, neither **QBorn** is co-reflective in **PaBorn** nor **PrBorn** is co-reflective in **SBorn**. In the previous notation, the quotient in **QBorn** and in **PrBorn** is B-W($\overline{\beta}^q$), which is called the *quotient quasi-bornology.*

We have noticed that, if $q \colon (X, \beta) \to Y$ is a surjective map from a para-bornological space to a set, the family $q(\beta)$ is not a para-bornology in general. We now want to discuss some properties ensuring that $q(\beta)$ is actually a para-bornology.

A surjective map $q \colon (X, \beta) \to Y$ from a para-bornological space to a set is *soft* if, for every $x \in X$, $y \in q^{-1}(q(x))$, and $A \in \beta(x)$, there exists $B \in \beta(y)$ such that $q(B) = q(A)$.

Proposition 4.3.9. *Let $q \colon (X, \beta) \to Y$ be a surjective map from a para-bornological space to a set. Then $q(\beta)$ is a para-bornology provided that one of the following properties holds:*

(a) q is soft;
(b) β is a quasi-bornology and, for every $y \in Y$, the quasi-bornological subspace $F_y = \{x \in q^{-1}(y) \mid \mathcal{Q}_X^{\downarrow}(x) \neq \{x\}\}$ of X has the property $C_{3''}$.

Proof. Suppose that the hypothesis (a) holds, and let $A, B \in q(\beta)(q(x))$, for some $x \in X$. Since q is soft, there exist $A', B' \in \beta(x)$ such that $q(A') = A$ and $q(B') = B$. Then $A' \cup B' \in \beta(x)$ and thus $A \cup B = q(A' \cup B') \in q(\beta)(q(x))$.

Assume now item (b). Let $y \in Y$ and $A, B \in q(\beta)(y)$. If $A = \{y\}$ or $B = \{y\}$, there is nothing to prove. Otherwise, there exist $x, x' \in F_y$ and $A' \in \beta(x)$ and $B' \in \beta(x')$ such that $q(A') = A$ and $q(B') = B$. Since F_y satisfies $C_{3''}$, we can assume without loss of generality that $\{x'\} \in \beta(x)$. Thus, the property (G_2) implies that $B' \in \beta(x)$ and so $A' \cup B' \in \beta(x)$. Hence, $A \cup B = q(A' \cup B') \in q(\beta)(y)$. □

Remark 4.3.10. (a) Let $q\colon (X, \beta) \to Y$ be a surjective map from a para-bornological space to a set. Note that, if $q(\beta)$ is a para-bornology (e.g., if one of the assumptions of Proposition 4.3.9 is fulfilled), then $q\colon (X, \beta) \to (Y, q(\beta))$ is trivially boundedness copreserving. This is not true in general for the quotient para-bornology. In fact, the map $q\colon (X, \beta) \to (Y, \overline{\beta}^q)$ defined in Example 4.3.8 is not boundedness copreserving.

(b) Using an argument similar to the one followed in the proof of Proposition 4.3.9, we can prove that, if $q\colon (X, \beta) \to Y$ is a surjective soft map from a quasi-bornological space to a set, then also $q(\beta)$ is a quasi-bornology.

(c) In the notation of Proposition 4.3.9, it is not true in general that $q(\beta)$ satisfies (G_2) even if the assumptions (b) are fulfilled. In fact, let $X = \{0, 1, 2, 3\}$ and $\beta = \beta_{\mathscr{R}}$, where $\mathscr{R} = \Delta_X \cup \{(1, 0), (1, 2), (3, 2)\}$. Since $\mathscr{R}$ is a partial order, then β is a quasi-bornology. Define q as the map $q\colon X \to \{0, 1, 2\}$ such that $q(0) = 0$, $q(1) = q(2) = 1$ and $q(3) = 2$. Then, although this choice satisfies all the requests of Proposition 4.3.9(b), $q(\beta)$ is not a quasi-bornology, which is easy to show.

4.3.3 Quotients of entourage spaces

We now want to construct quotients in **Entou** and its three subcategories that we are considering. Let $q\colon (X, \mathcal{E}) \to Y$ be a surjective map from an entourage space to a set. Then the *quotient entourage structure* on Y is $q(\mathcal{E}) = \{(q \times q)(E) \mid E \in \mathcal{E}\}$. Note that the quotient entourage structure is actually a an entourage structure since, in the previous notation, $\Delta_Y \in q(\mathcal{E})$ because q is surjective, $q(\mathcal{E})$ is trivially closed under inclusions, and $(q \times q)(E) \cup (q \times q)(F) = (q \times q)(E \cup F)$, for every $E, F \in \mathcal{E}$. Moreover, the map $q\colon (X, \mathcal{E}) \to (Y, q(\mathcal{E}))$ is automatically uniformly weakly boundedness copreserving.

Furthermore, in the previous notation, if $\mathcal{E}$ is a semi-coarse structure, then $(Y, q(\mathcal{E}))$, called *quotient entourage space*, is a semi-coarse space and thus it is also the quotient structure in **SCoarse**. However, even if $\mathcal{E}$ is a coarse structure, then $q(\mathcal{E})$ is not a quasi-coarse structure in general (see Example 4.3.8). Then the quotient structure in **QCoarse** (in **Coarse**) is $\overline{\mathcal{E}}^q$, where $\overline{\mathcal{E}}^q$ is the finest quasi-coarse structure (coarse structure, respectively) which contains $q(\mathcal{E})$, namely $\mathrm{W}((Y, q(\mathcal{E})))$. See also [88] for the quotient of coarse structures. Hence, in particular, **QCoarse** is not co-reflective in **Entou** and **Coarse** is not co-reflective in **SCoarse**.

Remark 4.3.11. Let $q\colon (X,\mathcal{E}) \to Y$ be a surjective map from an entourage space to a set. Note that, while $q(\mathcal{E})$ is an entourage structure, and thus $\beta_{q(\mathcal{E})}$ is a para-bornology on Y, $q(\beta_\mathcal{E})$ is not necessarily a para-bornology (see Example 4.3.8). Moreover, since the map $q\colon (X,\mathcal{E}) \to (Y, q(\mathcal{E}))$ is bornologous, according to Proposition 3.1.14, $q\colon (X,\beta_\mathcal{E}) \to (Y,\beta_{q(\mathcal{E})})$ is boundedness preserving, and thus $\overline{\beta_\mathcal{E}}^q \leq \beta_{q(\mathcal{E})}$.

The next task is to study sufficient conditions implying that the quotient entourage structure is actually a quasi-coarse structure or a coarse structure.

The following relations between entourages and the equivalence relation R_q will be needed in the sequel.

Proposition 4.3.12. *If R_q is the equivalence relation associated to a surjective map $q\colon X \to Y$ and $E \subseteq X \times X$, then*

$$(q\times q)(E) = (q\times q)(R_q \circ E) = (q\times q)(E\circ R_q) = (q\times q)(R_q\circ E\circ R_q), \quad (4.6)$$

$$(q\times q)(E)\circ(q\times q)(E) = (q\times q)(E\circ R_q\circ E). \quad (4.7)$$

and

$$(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E) = (q\times q)(E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E). \quad (4.8)$$

Moreover, if F is another subset of $X\times X$ such that $(q\times q)(F) \subseteq (q\times q)(E)$, then $F \subseteq R_q\circ E\circ R_q$. Consequently, $(q\times q)^{-1}((q\times q)[E]) = R_q\circ E\circ R_q$.

Proof. To prove (4.6) note first that $E \subseteq R_q\circ E \subseteq R_q\circ E\circ R_q$ and $E \subseteq E\circ R_q \subseteq R_q\circ E\circ R_q$, since $R_q \supseteq \Delta_X$. Therefore, it suffices to check the inclusion $(q\times q)(E) \supseteq (q\times q)(R_q\circ E\circ R_q)$. Pick $(x,y)\in R_q\circ E\circ R_q$. Then there exists $(z,u)\in E$ such that $q(x)=q(z)$ and $q(u)=q(y)$. Then, $(q(x),q(y)) = (q(z),q(u)) \in (q\times q)(E)$.

To prove (4.7), assume $(y,y') \in (q\times q)(E)\circ(q\times q)(E)$. Then there exist $x,x',z,z' \in X$ such that

$$y=q(x),\ y'=q(x'),\ (x,z),(z',x')\in E \ \text{ and } \ q(z)=q(z'),$$

consequently, $(z,z')\in R_q$. This yields $x' \in E\circ R_q\circ E[x]$, i.e., $(x,x')\in E\circ R_q\circ E$. Therefore, $(y,y') = (q(x),q(x')) \in (q\times q)(E\circ R_q\circ E)$. This proves the inclusion $\subseteq$ in (4.7).

Now assume that $(y,y')\in(q\times q)(E\circ R_q\circ E)$. Then $y=q(x)$ and $y'=q(x')$ for $(x,x')\in E\circ R_q\circ E$. So there exist $z,u\in X$ such that $(x,z),(u,x')\in E$ and $(z,u)\in R_q$, i.e., $q(z)=q(u)$. Then the pair $(q(x),q(x'))$ belongs to $(q\times q)(E)\circ(q\times q)(E)$, as $(q(x),q(z)) = (q(x),q(u)) \in (q\times q)(E)$, and $(q(z),q(x')) = (q(u),q(x')) \in (q\times q)(E)$. Therefore, $(y,y') = (q(x),q(x')) \in (q\times q)(E)\circ(q\times q)(E)$. This proves (4.7).

We deduce (4.8) from (4.7) as follows. Let $E_1 = (q\times q)(E)$. Then $E_1\circ E_1 = (q\times q)(R_q\circ E\circ R_q)$ by (4.7). Applying once again (4.7) to

$$E_2 = E_1\circ E_1 = (q\times q)(E\circ R_q\circ E)\circ(q\times q)(E\circ R_q\circ E)$$

we deduce that

$$E_1\circ E_1\circ E_1\circ E_1 = E_2\circ E_2 = (q\times q)(E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E).$$

This proves (4.8).

To prove $F \subseteq R_q \circ E \circ R_q$, under the assumption of $(q \times q)(F) \subseteq (q \times q)(E)$, pick $(x, y) \in F$. Then $(q(x), q(y)) \in (q \times q)(E)$, by our hypothesis. Thus, there exists $(u, v) \in E$, such that $(q(x), q(y)) = (q(u), q(v))$. Then $(x, u) \in R_q$ and $(v, y) \in R_q$ and this yields $(x, y) \in R_q \circ E \circ R_q$, as required.

The last assertion follows from the last proven inclusion and (4.6). □

Corollary 4.3.13. *Let $q\colon (X, \mathcal{E}) \to Y$ be a surjective map from an entourage space to a set. The quotient entourage structure $q(\mathcal{E})$ on Y is bounded (i.e., it coincides with the trivial coarse structure on Y) if and only if there exists $E \in \mathcal{E}_X$ such that $X \times X = R_q \circ E \circ R_q$. In such a case, $q(\mathcal{E})$ is a coarse structure.*

Proof. Clearly, $q(\mathcal{E})$ on Y is bounded if and only if there exists $E \in \mathcal{E}_X$ such that $(q \times q)(E) = Y \times Y = (q \times q)(X \times X)$. According to the last assertion of Proposition 4.3.12, this occurs precisely when $X \times X = R_q \circ E \circ R_q$. □

We propose now two natural sufficient conditions ensuring that the quotient entourage structure of a quasi-coarse space is a quasi-coarse structure.

Definition 4.3.14. Let $(X, \mathcal{E})$ be a quasi-coarse space and $q\colon X \to Y$ be a surjective map. We say that q is:

(a) *uniformly soft* (simply *soft* in [65]) if for all $E \in \mathcal{E}$ there exists a $F \in \mathcal{E}$ such that $R_q \circ E \subseteq F \circ R_q$;

(b) *weakly uniformly soft* (*weakly soft* in [65]) if for all $E \in \mathcal{E}$ there exists a $F \in \mathcal{E}$ such that $E \circ R_q \circ E \subseteq R_q \circ F \circ R_q$;

(c) *2-uniformly soft* (*2-soft* in [65]) if for all $E \in \mathcal{E}$ there exists a $F \in \mathcal{E}$ such that
$$E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E \subseteq R_q \circ F \circ R_q \circ F \circ R_q.$$

Remark 4.3.15. Let $q\colon (X, \mathcal{E}) \to Y$ be a surjective map from a quasi-coarse space to a set.

(a) The property $R_q \circ E \subseteq F \circ R_q$ in the definition of uniform softness reminds a (very) weak form of *commutativity* between $\mathcal{E}$ and R_q in the monoid of all entourages of $X \times X$ with respect to the composition law $\circ$, taken into account the fact that F can be chosen with $E \subseteq F$.

(b) Obviously, $R_q \circ E \subseteq F \circ R_q$ implies
$$R_q \circ E \circ R_q \subseteq F \circ R_q \circ R_q = F \circ R_q, \tag{4.9}$$
as $R_q \circ R_q = R_q$. On the other hand, (4.9) implies $R_q \circ E \subseteq F \circ R_q$ as $R_q \circ E \subseteq R_q \circ E \circ R_q$. Hence, q is uniformly soft if and only if for every $E \in \mathcal{E}_X$ there exists a $F \in E_X$ such that (4.9) holds.
Similarly, one can show that q is weakly uniformly soft (respectively, 2-uniformly soft) if and only if for every $E \in \mathcal{E}_X$ there exists a $F \in \mathcal{E}_X$ such that
$$E \circ R_q \circ E \circ R_q \subseteq R_q \circ F \circ R_q$$
$$(\text{respectively, } E \circ R_q \circ E \circ R_q \circ E \circ R_q \circ E \circ R_q \subseteq R_q \circ F \circ R_q \circ F \circ R_q). \tag{4.10}$$

(c) As one may expect, if the map q is uniformly soft, then $q\colon (X, \beta_{\mathcal{E}}) \to Y$ is soft. In fact, fix two points $x, y \in X$ with $(x, y) \in R_q$ and a subset A which is bounded from x. Then there exists $E \in \mathcal{E}$ such that $A = E[x]$. Let $F \in \mathcal{E}$ be an entourage such that $R_q \circ E \subseteq F \circ R_q$. Finally, since, for every $C \subseteq X$, $q(R_q[C]) = q(C)$,

$$q(A) = q(E[x]) \subseteq q(E[R_q[y]]) = q((R_q \circ E)[y]) \subseteq q(R_q[F[y]]) = q(F[y]),$$

and this shows the claim since $F[y] \in \beta_{\mathcal{E}}(y)$.

Proposition 4.3.16. *Let $q\colon X \to Y$ a surjective map from a quasi-coarse space $(X, \mathcal{E})$. Then the following implications hold:*

(a) if q is large-scale injective (i.e., $R_q \in \mathcal{E}$), then it is uniformly soft;

(b) if q is uniformly soft, then it is weakly uniformly soft;

(c) if q is weakly uniformly soft, then it is 2-uniformly soft.

Proof. (a) As $R_q \circ E \subseteq R_q \circ E \circ R_q$ for every $E \in \mathcal{E}$, our claim follows from $R_q \circ E \in \mathcal{E}$.

(b) If $E \in \mathcal{E}_X$ and $F \in E_X$ satisfies $R_q \circ E \subseteq F \circ R_q$, then

$$E \circ R_q \circ E \subseteq E \circ F \circ R_q \subseteq R_q \circ (E \circ F) \circ R_q.$$

(c) It is an easy application of the definition of weak uniform softness and of the fact that $R_q \circ R_q = R_q$. □

The above lemma gives the following implications between the above four properties of a map:

$$\text{uniformly bounded fibres} \longrightarrow \text{uniformly soft} \longrightarrow \\ \longrightarrow \text{weakly uniformly soft} \longrightarrow \text{2-uniformly soft.} \quad (4.11)$$

Counter-examples witnessing that none of these implications is reversible are given in Example 5.3.5.

The next results justify our interest in the notions introduced in Definition 4.3.14.

Theorem 4.3.17. *Let $q\colon (X, \mathcal{E}) \to Y$ be a surjective map from a quasi-coarse space to a set. Denote by $\overline{\mathcal{E}}^q$ the quotient quasi-coarse structure on Y. Then the following properties are equivalent:*

(a) $\overline{\mathcal{E}}^q = q(\mathcal{E})$;

(b) $q(\mathcal{E})$ is a quasi-coarse structure;

(c) q is weakly uniformly soft;

(d) $q\colon (X, \mathcal{E}) \to (Y, \overline{\mathcal{E}}^q)$ is uniformly weakly boundedness copreserving.

Proof. The equivalence (a)↔(b) is trivial, while (a)↔(d) is easy to check. In fact, as for (d)→(a), if q is uniformly weakly bounded copreserving, then, applying the definition, we can show that $\overline{\mathcal{E}}^q \subseteq q(\mathcal{E})$.

(c)→(b) Suppose that q is weakly uniformly soft. It is enough to show that the family $q(\mathcal{E})$ is a quasi-coarse structure. In order to do that, we need to only check the the property $(q\times q)(E)\circ(q\times q)(E)\in q(\mathcal{E})$ whenever $E\in\mathcal{E}$. Pick $F\in\mathcal{E}$ such that $E\circ R_q\circ E\subseteq R_q\circ F\circ R_q$. Therefore, (4.7) and (4.6) imply

$$(q\times q)(E)\circ(q\times q)(E)=(q\times q)(E\circ R_q\circ E)\subseteq(q\times q)(R_q\circ F\circ R_q)=(q\times q)(F), \tag{4.12}$$

i.e., $(q\times q)(E)\circ(q\times q)(E)\in q(\mathcal{E})$.

(b)→(c) Assume that $q(\mathcal{E})$ on Y is a quasi-coarse structure. Then $(q\times q)(E)\circ(q\times q)(E)\in q(\mathcal{E})$ for every $E\in\mathcal{E}$. By (4.7), $(q\times q)(E\circ R_q\circ E)\in\overline{\mathcal{E}}^q$, so there exists $F\in\mathcal{E}$ such that

$$(q\times q)(E\circ R_q\circ E)\subseteq(q\times q)(F). \tag{4.13}$$

In view of Proposition 4.3.12, (4.13) implies

$$E\circ R_q\circ E\subseteq R_q\circ F\circ R_q.$$

This proves that q is weakly uniformly soft. □

Proposition 4.3.18. *Let $q\colon(X,\mathcal{E})\to Y$ be a surjective map from a quasi-coarse space to a set. Denote by $\overline{\mathcal{E}}^q$ the quotient quasi-coarse structure on Y. Then q is soft if and only if the map $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is uniformly boundedness copreserving.*

Proof. If q is soft, then it is in particular weakly uniformly soft, and so $\overline{\mathcal{E}}^q=q(\mathcal{E})$, according to Theorem 4.3.17. Take an arbitrary entourage $(q\times q)(E)\in q(\mathcal{E})=\overline{\mathcal{E}}^q$ and let $F\in\mathcal{E}$ be an entourage satisfying $R_q\circ E\subseteq F\circ R_q$. Fix now $x\in X$ and an arbitrary element $q(w)\in(q\times q)(E)[q(x)]$. Since $(q(x),q(w))\in(q\times q)(E)$, $(x,w)\in R_q\circ E\circ R_q\subseteq F\circ R_q$, and thus $q(w)\in q((F\circ R_q)[x])=q(R_q[F[x]])=q(F[x])$, which implies that $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is uniformly boundedness copreserving.

Conversely, assume that $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is uniformly boundedness copreserving. Since, in particular, q is uniformly weakly boundedness copreserving, Theorem 4.3.17 implies that $\overline{\mathcal{E}}^q=q(\mathcal{E})$. Consider an entourage $E\in\mathcal{E}$ and pick an $F\in\mathcal{E}$ such that, for every $x\in X$, $((q\times q)(E))[q(x)]\subseteq q(F[x])$, which exists since q is uniformly boundedness copreserving. If a pair $(y,z)\in R_q\circ E$, then there exists $w\in X$ such that $(y,w)\in R_q$ and $(w,z)\in E$. In particular, $(q(y),q(z))=(q(w),q(z))\in(q\times q)(E)$, and $q(z)\in((q\times q)(E))[q(y)]\subseteq q(F[y])$. Thus, there exists $t\in F[y]$ such that $q(z)=q(t)$, and so, finally, $(y,z)=(y,t)\circ(t,z)\in F\circ R_q$. □

Corollary 4.3.19. *Let $\{(X_i,\mathcal{E}_i)\}_{i\in I}$ be a family of quasi-coarse spaces. Denote by $(X,\mathcal{E})$ the product quasi-coarse space. Then, for every $i\in I$, the canonical projection $p_i\colon(X,\mathcal{E})\to X_i$ is soft.*

Proof. Thanks to Proposition 4.3.18 and Theorem 4.3.17, since the projection map $p_i\colon(X,\mathcal{E})\to(X_i,\mathcal{E}_i)$ is uniformly boundedness copreserving, it is enough to prove that $\overline{\mathcal{E}}^q=q(\mathcal{E})$. However, this is easy to show. In fact, for every $E\in\mathcal{E}_i$, the entourage $F=E\times(\Pi_{j\in I\setminus\{i\}}\Delta_{X_j})$ satisfies $(q\times q)(F)=E$. □

Theorem 4.3.20. *Let $q\colon(X,\mathcal{E})\to Y$ be a surjective map from a quasi-coarse space to a set. Denote by $\overline{\mathcal{E}}^q$ the quotient quasi-coarse structure on Y. Then the family of entourages $\mathcal{E}_Y^*=\{(q\times q)(E)\circ(q\times q)(E)\mid E\in\mathcal{E}_X\}$ is a quasi-coarse structure if and only if q is 2-uniformly soft.*

Proof. The 'only if' implication can be shown by applying Proposition 4.3.12. Conversely, suppose that q is 2-uniformly soft. We have to check that the family of entourages $\mathcal{E}_Y^*$ is a quasi-coarse structure. The argument is similar to that of the one of the proof of Theorem 4.3.17. Indeed, suppose that q is 2-uniformly soft. Let us check that $(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)\in\mathcal{E}_Y^*$ whenever $E\in\mathcal{E}_X$. Pick $F\in\mathcal{E}$ such that $E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E\subseteq R_q\circ F\circ R_q\circ F\circ R_q$. Therefore, (4.6), (4.7) and (4.8) imply

$$\begin{aligned}(q\times q)(E)\circ&(q\times q)(E)\circ(q\times q)(E)\circ(q\times q)(E)=\\&=(q\times q)(E\circ R_q\circ E\circ R_q\circ E\circ R_q\circ E)\subseteq\\&\subseteq(q\times q)(R_q\circ F\circ R_q\circ F\circ R_q)=(q\times q)(F\circ R_q\circ F)=\\&=(q\times q)(F)\circ(q\times q)(F)\in\mathcal{E}_Y^*.\end{aligned}$$

□

Let us add one more similar result.

Proposition 4.3.21. *Let $q\colon(X,\mathcal{E})\to Y$ be a surjective map from a quasi-coarse space to a set. Then the following properties are equivalent:*
(a) q is large-scale injective;
(b) the map $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is effectively proper;
(c) the map $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is a Sym*-coarse equivalence.*

Proof. The implications (b)→(c)→(a) follow from Theorem 3.4.6 since $q\colon(X,\mathcal{E})\to(Y,\overline{\mathcal{E}}^q)$ is bornologous and surjective. Let us now show the last implication, (a)→(b). If $R_q\in\mathcal{E}$, then q is weakly uniformly soft and $\overline{\mathcal{E}}^q=q(\mathcal{E})$, according to Theorem 4.3.17. Moreover, for every $E\in\mathcal{E}$, $(q\times q)^{-1}((q\times q)(E))=R_q\circ E\circ R_q\in\mathcal{E}$ since $\mathcal{E}$ is a quasi-coarse structure. The opposite implication is trivial. □

Proposition 4.3.21 implies that for a quotient map q with uniformly bounded fibres $\overline{\mathcal{E}}^q$ is bounded if and only if $(X,\mathcal{E})$ is bounded. Moreover, note that, if X is bounded, then trivially $R_q\in\mathcal{E}$. This witnesses how restrictive is the hypothesis, usually imposed in the literature (see [151, 9]), of uniformly bounded fibres, to define quotients.

We give now an explicit construction of quotients of quasi-coarse spaces in the general case.

Proposition 4.3.22. *Let $(X,\mathcal{E})$ be a quasi-coarse space and let $q\colon X\to Y$ be a surjective map. Denote by $\mathscr{R}_q$ the relation entourage structure on X, actually a coarse structure, induced by the entourage R_q, and let $\mathcal{E}^\#=\mathcal{E}\vee\mathscr{R}_q$. Then $q(\mathcal{E}^\#)=\overline{\mathcal{E}}^q$.*

Proof. Applying Propositions 4.3.16 and 4.3.21 to $q\colon(X,\mathcal{E}^\#)\to Y$, we deduce that the quotient entourage structure $q(\mathcal{E}^\#)$ is a quasi-coarse structure, as q has uniformly bounded fibres, in view of $R_q\in\mathcal{E}^\#$.

It is easy to see that $\mathcal{E}^\#$ is generated by the entourages of the form $W_n=E\circ R_q\circ E\circ\cdots\circ E\circ R_q\circ E$, where $E\in\mathcal{E}$ participates n-times, E runs over $\mathcal{E}$ and $n\in\mathbb{N}$. According to an obvious counterpart of (4.8) from Proposition 4.3.12, $(q\times q)(W_n)=(q\times q)(E)\circ\ldots\circ(q\times q)(E)$, where the composition on the right-hand side has n components. Since this is a typical entourage of $\overline{\mathcal{E}}^q$, the quasi-coarse structure generated by $q(\mathcal{E})$ coincides with the one having the family of all $(q \times q)(W_n)$ as a base, which in turn coincides with $q(\mathcal{E}^{\#})$. □

The previous results (Theorems 4.3.17 and 4.3.20, and Propositions 4.3.18, 4.3.21 and 4.3.22) hold also for quotients in **Coarse** since **Coarse** is co-reflective in **QCoarse**.

Remark 4.3.23. The result of Theorem 4.3.17 is closely related to a similar fact about uniformities (defined by means of a family of entourages) established in [99]: *if $(X,\mathcal{U})$ is a uniform space and $q\colon X \to Y$ is a surjective map, then the family of entourages $q(\mathcal{U}) = \{(q\times q)(U) \mid U \in \mathcal{U}\}$ is a uniformity precisely when for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $U \circ R_q \circ U \subseteq (q\times q)^{-1}((q\times q)(V))$* (taking into account that $(q \times q)^{-1}((q \times q)(V)) = R_q \circ V \circ R_q$, according to Proposition 4.3.12).

4.4 Functors from PaBorn to Entou

In Sections 4.1 and 4.2, we have studied functors that connect separately categories of para-bornological spaces and categories of entourage spaces, respectively. The goal of this section is to further discuss connections between para-bornological spaces and entourage spaces.

Following this spirit, first of all, we want to provide adjoints to the functors $\mathrm{U}^{\mathbf{Entou}}_{\mathbf{PaBorn}}$, $\mathrm{U}^{\mathbf{QCoarse}}_{\mathbf{QBorn}}$, and $\mathrm{U}^{\mathbf{Coarse}}_{\mathbf{PrBorn}}$. While providing the desired adjoints, we also prove that all para-bornologies, quasi-bornologies and pre-bornologies are uniform.

Let X be a set and $\mathcal{U} = \{\mathcal{U}_x \mid x \in X\}$ be a family of ideals on X. Then we can define the *ideal entourage structure* $\mathcal{E}_{\mathcal{U}}$ as follows:

$$\mathcal{E}_{\mathcal{U}} = \mathfrak{cl}(\{E_{I,\psi} \mid I \in [X]^{<\omega},\ \psi\colon I \to \bigcup \mathcal{U} :\ \psi(x) \in \mathcal{U}_x,\ \forall x \in I\}),$$

$$\text{where, for every such } I \text{ and } \psi, \quad E_{I,\psi} = \Delta_X \cup \left(\bigcup_{x\in I} (\{x\} \times (\{x\} \cup \psi(x))) \right). \tag{4.14}$$

Of course the construction described in (4.14) can be carried out when $\mathcal{U}$ is a para-bornology on X. If β is a para-bornology on X, we can assume that we consider only those maps $\psi\colon I \to \bigcup \beta$, where I is a finite subset of X, such that $\{x\} \subseteq \psi(x)$, for every $x \in I$. In that case we can slightly simplify the definition of $E_{I,\psi}$.

For the sake of simplicity, we prove the fact that the ideal entourage structure is indeed an entourage structure only for para-bornologies, even though it could be easily generalised to arbitrary families of ideals.

Theorem 4.4.1. *Let β be a para-bornology on a set X.*
(a) $\mathcal{E}_\beta$ is an entourage structure such that $\beta_{\mathcal{E}_\beta} = \beta$.
(b) If $\mathcal{E}$ is an entourage structure on X such that $\beta_{\mathcal{E}} = \beta$, then $\mathcal{E}_\beta \subseteq \mathcal{E}$.
(c) If β is a quasi-bornology, then $\mathcal{E}_\beta$ is a quasi-coarse structure.

Proof. (a) Let I and J be two finite subsets of X and let $\psi_I, \psi_J\colon I \to \bigcup \beta$ such that $\{x\} \subseteq \psi_I(x) \in \beta(x)$ and $\{y\} \subseteq \psi_J(y) \in \beta(y)$, for every $x \in I$ and $y \in J$.

We want to show that $E_{I,\psi_I} \cup E_{J,\psi_J} \in \mathcal{E}_\beta$. Define $K = I \cup J$ and $\psi_K \colon K \to \bigcup \beta$ as follows: for every $z \in K$,

$$\psi_K(z) = \begin{cases} \psi_I(z) & \text{if } z \in I \setminus J, \\ \psi_J(z) & \text{if } z \in J \setminus I, \\ \psi_I(z) \cup \psi_J(z) & \text{otherwise.} \end{cases}$$

One can easily verify that ψ_K satisfies the desired properties and that $E_{I,\psi_I} \cup E_{J,\psi_J} \subseteq E_{K,\psi_K}$. Finally, the equality $\beta = \beta_{\mathcal{E}_\beta}$ is trivial.

(b) Let E an arbitrary element of $\mathcal{E}_\beta$. Without loss of generality, we can assume that $E = E_{I,\psi}$ for some $I \in [X]^{<\omega}$ and $\psi \colon I \to \bigcup \beta$ with the desired properties. Since $\beta = \beta_\mathcal{E}$, for every $x \in I$, there exists $E_x \in \mathcal{E}$ such that $\psi(x) = E_x[x]$. Then $E_{I,\psi} \subseteq \left(\bigcup_{x \in I} E_x\right) \cup \Delta_X \in \mathcal{E}$.

(c) Similarly to item (a), let I and J be two finite subsets of X and let $\psi_I, \psi_J \colon I \to \bigcup \beta$ such that $\{x\} \subseteq \psi_I(x) \in \beta(x)$ and $\{y\} \subseteq \psi_J(y) \in \beta(y)$, for every $x \in I$ and $y \in J$. We want to show that $E_{I,\psi_I} \circ E_{J,\psi_J} \in \mathcal{E}_\beta$. Define $K = I \cup J$ and $\psi_K \colon K \to \bigcup \beta$ as follows: for every $z \in K$,

$$\psi_K(z) = \begin{cases} \psi_I(z) \cup \left(\displaystyle\bigcup_{y \in \psi_I(z) \cap J} \psi_J(y) \right) & \text{if } z \in I, \\ \psi_J(z) & \text{otherwise.} \end{cases}$$

Since, for every $x \in I$, $\psi_J(x) \cap J$ is finite and β satisfies (G_2) the union still belongs to $\beta(x)$ and so ψ_K is actually a map that satisfies the requested properties. Finally, let us show that $E_{I,\psi_I} \circ E_{J,\psi_J} \subseteq E_{K,\psi_K}$. Suppose that $(x, y) \in E_{I,\psi_I}$ and $(y, z) \in E_{J,\psi_J}$. If $x = y$ or $y = z$, there is nothing to prove, because of the inclusion $E_{I,\psi_I} \cup E_{J,\psi_J} \subseteq E_{K,\psi_K}$. Otherwise, we have $x \in I$, $y \in \psi_I(x) \cap J$ and $z \in \psi_J(y)$, which implies that $z \in \psi_K(x)$. □

Proposition 4.4.2. *Let $f \colon (X, \beta_X) \to (Y, \beta_Y)$ be a map between para-bornological spaces. Then $f \colon (X, \beta_X) \to (Y, \beta_Y)$ is boundedness preserving if and only if $f \colon (X, \mathcal{E}_{\beta_X}) \to (Y, \mathcal{E}_{\beta_Y})$ is bornologous.*

Proof. If f is bornologous, then it is boundedness preserving because of Proposition 3.1.14, and $\beta_{\mathcal{E}_{\beta_X}} = \beta_X$ and $\beta_{\mathcal{E}_{\beta_Y}} = \beta_Y$. Conversely, it is trivial that, for every $I \in [X]^{<\omega}$ and $\psi \colon I \to \bigcup \beta_X$,

$$(f \times f)(E_{I,\psi}) \subseteq E_{f(I),\psi^*}, \quad \text{where, for every } x \in I,$$
$$\psi^*(f(x)) = f\left(\bigcup_{y \in R_f[x] \cap I} \psi(y) \right).$$

□

Theorem 4.4.1 proves, in particular, that every para-bornology is uniform (item (a)) and also every quasi-bornology is uniform (item (c)). Moreover, thanks to both Theorem 4.4.1 and Proposition 4.4.2, we can define a functor $\mathrm{F} \colon$ **PaBorn** $\to$ **Entou** that associates to every para-bornological space (X, β) the entourage structure $\mathrm{F}(X, \beta) = (X, \mathcal{E}_\beta)$. Moreover, we can also consider its restriction $\mathrm{F}_Q = \mathrm{F}|_{\mathbf{QBorn}} \colon$ **QBorn** $\to$ **QCoarse**. Furthermore, note that

$\mathrm{U}^{\mathbf{Entou}}_{\mathbf{PaBorn}} \circ \mathrm{F} = 1_{\mathbf{PaBorn}}$, where $1_{\mathcal{X}}$ is the identity functor of the category $\mathcal{X}$, and thus $\mathrm{U}^{\mathbf{QCoarse}}_{\mathbf{QBorn}} \circ \mathrm{F}_Q = 1_{\mathbf{QBorn}}$.

Unfortunately, it is not true that, if β is a semi-bornology, then $\mathcal{E}_\beta$ is a semi-coarse structure. In fact, consider $(\mathbb{R}, \beta_d)$, where d is the usual metric. Then, $E = \{(0,x) \mid x \in B_d(0,1)\} \in \mathcal{E}_\beta$, although $E^{-1} \notin \mathcal{E}_\beta$, since $B_d(0,1)$ is infinite. As we have seen, infinite balls can create problems. In fact, the following result holds.

Proposition 4.4.3. *Let (X,β) be a semi-bornological space. Then $\mathcal{E}_\beta$ is a semi-coarse structure if and only if X is locally finite. In particular, if (X,β) also satisfies (G_2), then $\mathcal{E}_\beta$ is a coarse structure.*

Proof. Let us assume that $\mathcal{E}_\beta$ is a semi-coarse structure. Then, for every $x \in X$ and every $B \in \beta(x)$, $\Delta_X \cup (\{x\} \times B) \in \mathcal{E}_\beta$, and thus

$$F = (\Delta_X \cup (\{x\} \times B))^{-1} = \Delta_X \cup \{(y,x) \mid y \in B\} \in \mathcal{E}_\beta.$$

Because of the definition of $\mathcal{E}_\beta$, there exists $J \in [X]^{<\omega}$ and $\psi\colon J \to \bigcup \beta$ such that

$$F \subseteq E_{J,\psi} = \Delta_X \cup \left(\bigcup_{y \in I} (\{y\} \times (\{y\} \cup \psi(y))) \right),$$

which implies that $B \setminus \{x\} \subseteq J$ and thus B itself is finite. Since both x and $B \in \beta(x)$ can be arbitrarily taken, we deduce that (X,β) is locally finite.

Conversely, suppose that β is locally finite, and let $E \in \mathcal{E}_\beta$. Without loss of generality, we can assume that $E = E_{I,\psi}$, where $I \in [X]^{<\omega}$ and $\psi\colon I \to \bigcup \beta$ with the desired properties. Define $J = \bigcup_{x \in I} \psi(x)$, which is a finite subset of X, since (X,β) is locally finite, and $\psi\colon J \to \bigcup \beta$ as follows: for every $x \in J$, $\psi(x) = \{y \in I \mid x \in \psi(y)\}$. Note that, for every $x \in J$, $\psi(x) = \bigcup\{\{y\} \mid y \in I,\ \{x\} \in \beta(y)\} \in \beta(x)$ because of the property (G_1). We claim that $E^{-1}_{I,\psi} \subseteq E_{J,\psi}$. Let $(x,y) \in E_{I,\psi}$. If $x = y$, there is nothing to prove. Otherwise, $x \in I$ and $y \in \psi(x)$. Thus $y \in J$ and $x \in \psi(y)$, which implies that $(y,x) \in E_{J,\psi}$.

The last assertion follows from Theorem 4.4.1(c). □

Question 4.4.4. Explicitly describe the finest semi-coarse structure, if it exists, that induces a given semi-bornology.

Proposition 4.4.3 is not fully satisfying also for pre-bornological spaces. Let β be a pre-bornology on a set X. According to Remark 2.1.3, we can assume that β is a cover of X, which is closed under taking both subsets and finite unions provided that the elements have non-empty intersection. We assume this form of pre-bornologies and bornologies for the remaining part of this section. The pre-bornological space (X,β) can be seen as coproduct $\bigoplus_{i \in I}(X_i, \beta_i)$ of its connected components (Remark 4.3.6(b)). For every index $i \in I$, consider the ideal coarse structure $\mathcal{E}_{\beta_i}$ (see Example 3.1.3). Finally, define the coarse structure $\mathcal{E}^C_\beta = \bigoplus_{i \in I} \mathcal{E}_{\beta_i}$.

Theorem 4.4.5. *Let (X,β) be a pre-bornological space. Then β is uniform, and, more precisely, $\beta = \beta_{\mathcal{E}^C_\beta}$. Moreover, for every coarse structure $\mathcal{E}$ of X such that $\beta_{\mathcal{E}} = \beta$, $\mathcal{E}^C_\beta \subseteq \mathcal{E}$.*

Proof. If we prove the statements for a bornology, then the claim follows by coproduct's properties. So, we assume that β is a bornology on X, which implies that the ideal coarse structure $\mathcal{E}_\beta$ is actually a coarse structure. Moreover, for every $x\in X$ and $x\in B\in\beta$, $B=E_B[x]=\Delta_X\cup(B\times B)[x]$ (in the notation of Example 3.1.3). Hence $\beta\subseteq\beta_{\mathcal{E}_\beta^C}$. However, since the family of entourages of the form E_B, where $B\in\beta$, form a base of $\mathcal{E}_\beta^C$, the opposite inequality is also satisfied and $\beta=\beta^C_{\mathcal{E}_\beta}$.

Let now $\mathcal{E}$ be a coarse structure of X such that $\beta_{\mathcal{E}}=\beta$. Let $E\in\mathcal{E}_\beta^C$ and we can assume, without loss of generality, that $E=E_B=\Delta_X\cup(B\times B)$ for some $B\in\beta$. Thus $E\in\mathcal{E}$ since $\beta=\beta_{\mathcal{E}}$ consists of subsets satisfying (B_3). □

Proposition 4.4.6. *Let $f\colon(X,\beta_X)\to(Y,\beta_Y)$ be a map between pre-bornological spaces. Then $f\colon(X,\beta_X)\to(Y,\beta_Y)$ is boundedness preserving if and only if $f\colon(X,\mathcal{E}^C_{\beta_X})\to(Y,\mathcal{E}^C_{\beta_Y})$ is bornologous.*

Proof. If f is bornologous, then f is boundedness preserving by Proposition 3.1.14. Conversely, it is not hard to check that, if $A\in\bigcup\beta_X$, $(f\times f)(E_A)\subseteq E_{f(A)}$ (in the notation of Example 3.1.3) and $f(A)\in\bigcup\beta_Y$ since f is boundedness preserving. □

According to Theorem 4.4.5 and Proposition 4.4.6, we can define a functor $\mathrm{F}_C\colon\mathbf{PrBorn}\to\mathbf{Coarse}$ as follows: for every $(X,\beta)\in\mathbf{PrBorn}$, $\mathrm{F}_C(X,\beta)=(X,\mathcal{E}^C_\beta)$. Also in this case, $\mathrm{U}^{\mathbf{Coarse}}_{\mathbf{PrBorn}}\circ\mathrm{F}_C=1_{\mathbf{PrBorn}}$. Finally, by checking the definitions, the following result can be deduced.

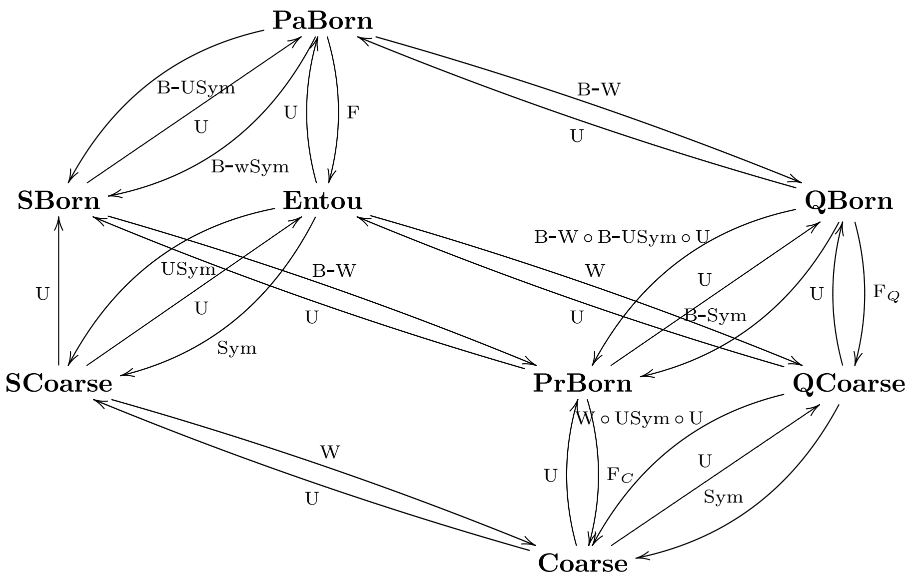
Corollary 4.4.7. F *is an adjoint for* $\mathrm{U}^{\mathbf{Entou}}_{\mathbf{PaBorn}}$*,* F_Q *is an adjoint for* $\mathrm{U}^{\mathbf{QCoarse}}_{\mathbf{QBorn}}$*, and* F_C *is an adjoint for* $\mathrm{U}^{\mathbf{Coarse}}_{\mathbf{PrBorn}}$*.*

In this section we have shown that every para-bornology is uniform. However, many of them can be induced by several entourage structures.

Problem 4.4.8. Characterise the family of all entourage structures (semi-coarse structures, quasi-coarse structures, or coarse structures) that induce the same para-bornology.

Question 4.4.9. Which are the para-bornologies that are induced by just one entourage structure (semi-coarse space, quasi-coarse space, or coarse space)?

In the diagram (4.15) we represent the functors that connect the categories **PaBorn**, **SBorn**, **QBorn**, **PrBorn**, **Entou**, **SCoarse**, **QCoarse** and **Coarse**, merging the diagrams (4.1), (4.2) and (4.3), and the functors defined in this subsection.


PaBorn
B-USym
U
U
F
B-wSym
B-W
U
SBorn
Entou
QBorn
B-W ∘ B-USym ∘ U
USym
B-W
W
U
U
U
U
B-Sym
U
F$_Q$
Sym
SCoarse
PrBorn
QCoarse
W ∘ USym ∘ U
W
U
F$_C$
U
U
Sym
Coarse

(4.15)

Chapter 5

The categories Coarse and Coarse/$\sim$

5.1 The category Ballean

Let us define another category **Ballean**. An object of **Ballean** is an equivalence class of ballean $\{(X, P_i, B_i)\}_{i\in I}$ on the same support, where the equivalence relation that we consider here is the one defined in (3.4). A morphism between two of such equivalence classes, $\{(X, P_i^X, B_i^X)\}_{i\in I}$ and $\{(Y, P_\lambda^Y, B_\lambda^Y)\}_{\lambda\in\Lambda}$, is a map $f\colon X \to Y$ which is bornologous, whenever X and Y are endowed with one, equivalently all, ballean structure from their own equivalence classes. By abuse of notation, in the sequel we consider balleans as objects of **Ballean**: each ballean carries its equivalence class. The connection described in [151] and widely discussed in [65] (see §3.2) shows that **Coarse** and **Ballean** are isomorphic categories. This fact allows us to use the name **Coarse** when we refer to **Ballean** and this choice is done for consistency with Chapter 4, and in order to unify the notation with the papers [65, 175].

Since balleans and coarse spaces are two faces of the same coin, as shown in §3.2, the properties we have defined and studied so far for coarse spaces can be also introduced for balleans in the following canonical way: if P is some property defined for coarse spaces, then a ballean $\mathfrak{B}$ on a set X has the property P if $(X, \mathcal{E}_{\mathfrak{B}})$ has property P. For example, a *discrete ballean* is a ballean $\mathfrak{B} = (X, P, B)$ which is equivalent to $\mathfrak{B}_{\mathcal{E}_{dis}} = (X, P_{\mathcal{E}_{dis}}, B_{\mathcal{E}_{dis}})$, while a *trivial* or *bounded ballean* is a ballean $\mathfrak{B} = (X', P', B')$ which is equivalent to $\mathfrak{B}_{\mathcal{E}_{triv}} = (X', P_{\mathcal{E}_{triv}}, B_{\mathcal{E}_{triv}})$. Let us give a direct characterisation of discrete and trivial balleans: a ballean (X, P, B) is

- discrete if and only if, for every $r \in P$, $B(x, r) = \{x\}$, for every $x \in X$, and
- trivial if and only if there exists $r \in P$ such that $X = B(x, r)$, for every $x \in X$.

Since balleans will be substantially used in the sequel, it is convenient to translate some of the notion introduced using the coarse space terminology. Before starting, let us consider a convenient property of balleans that will be taken as a carpet assumption in the sequel.

Remark 5.1.1. Let $\mathfrak{B} = (X, P, B)$ be a ballean, and $\mathcal{E} = \mathcal{E}_{\mathfrak{B}}$ the coarse structure associated to it. Then it is easy to see that the family $\mathcal{B} = \{E \cup E^{-1} \cup \Delta_X \mid E \in \mathcal{E}\}$ is a base of $\mathcal{E}$. Therefore, according with Remark 3.2.3, $\mathfrak{B}_{\mathcal{B}} = \mathfrak{B}_{\mathcal{E}}$, and, moreover, $\mathfrak{B}_{\mathcal{B}}$ is symmetric, where a ball structure $\mathfrak{B}_1 = (X_1, P_1, B_1)$ is called *symmetric* if, for every $x \in X_1$ and $r \in P_1$, $B_1^*(x, r) = B_1(x, r)$. Of course, symmetry implies upper symmetry. Hence, without loss of generality, every ballean can be assumed to be symmetric.

Let $\mathfrak{B} = (X, P, B)$ be a ballean and $Y \subseteq X$. Then Y inherits the *subballean structure*

$$\mathfrak{B}|_Y = \mathfrak{B}_{\mathcal{E}_{\mathfrak{B}}|_Y} = (Y, P, B|_Y), \text{ where } B|_Y(y, r) = B(y, r) \cap Y,$$

for every $y \in Y$ and $r \in P$.

Let $f, g \colon S \to \mathfrak{B} = (X, P, B)$ be two maps from a set to a ballean. Then $f \colon S \to (X, \mathcal{E}_{\mathfrak{B}})$ and $g \colon S \to (X, \mathcal{E}_{\mathfrak{B}})$ are close if and only if there exists $r \in P$ such that, for every $x \in X$, $g(x) \in B(f(x), r)$. In this case, we will usually write $f \sim_r g$, underlining the role of the radius r.

Let $f \colon \mathfrak{B}_X = (X, P_X, B_X) \to \mathfrak{B}_Y = (Y, P_Y, B_Y)$ be a map between balleans. Then $f \colon (X, \mathcal{E}_{\mathfrak{B}_X}) \to (Y, \mathcal{E}_{\mathfrak{B}_Y})$ is

- large-scale injective if and only if there exists $r \in P_X$ such that, for every $x \in X$, $f^{-1}(f(x)) \subseteq B_X(x, r)$;
- large-scale surjective if and only if $f(X)$ is large in $\mathfrak{B}_Y$ if and only if there exists $r \in P_Y$ such that $B_Y(f(X), r) = Y$.

Remark 5.1.2. Let $\{\mathfrak{B}_i = (X_i, P_i, B_i)\}_{i \in I}$ be a family of balleans. Then we can associate to them a family $\{(X_i, \mathcal{E}_i)\}_{i \in I}$ of coarse spaces as described in §3.2. In §4.3.1 the product $\Pi_i(X_i, \mathcal{E}_i)$ and the coproduct $\bigsqcup_i (X_i, \mathcal{E}_i)$ of that family of coarse spaces. We want to rewrite the balleans associated to the product and the coproduct coarse spaces conveniently.

(a) The *product ballean* is the triple $\mathfrak{B} = (X, P, B) = \Pi_i \mathfrak{B}_i$, where $X = \Pi_i X_i$, $P = \Pi_i P_i$, and, for every $(x_i)_i \in X$ and $(r_i)_i \in P$,

$$B((x_i)_i, (r_i)_i) = \Pi_i B_i(x_i, r_i).$$

(b) The *coproduct ballean* is the triple $\mathfrak{B} = (X, P, B) = \bigoplus_i \mathfrak{B}_i$, where $X = \bigsqcup_i X_i$,

$$P = \{(r_j)_j \in \Pi_{j \in J} P_j \mid J \in [I]^{<\omega}\},$$

and, for every $i_k(x) \in X$ and $(r_j)_{j \in J} \in P$,

$$B(i_k(x), (r_j)_j) = \begin{cases} i_k(B_k(x, r_k)) & \text{if } k \in J, \\ \{i_k(x)\} & \text{otherwise.} \end{cases}$$

Let us end this section with a remark in which we translate some of the notions introduced in §4.3.3 in terms of ball structures. This approach will be widely used in §5.3.1 in order to introduce the adjunction space.

Remark 5.1.3. (a) Let $q \colon X \to Y$ be a surjective map from an entourage space $(X, \mathcal{E})$ to a set. We want to describe the ball structure $\mathfrak{B}_{q(\mathcal{E})}$ associated to the family $q(\mathcal{E})$. Fix a radius $(q \times q)(E)$, where $\Delta_X \subseteq E \in \mathcal{E}$ and hence $\Delta_Y \subseteq (q\times q)(E)$. Then, for every point $x\in X$, one has the following chain of equalities:

$$\begin{aligned} B_{q(\mathcal{E})}(q(x),(q\times q)(E)) &= ((q\times q)(E))[q(x)] = \\ &= \{q(z)\in Y \mid \exists w\in R_q[x] : (w,z)\in E\} = \\ &= q(B_{\mathcal{E}}(R_q[x],E)). \end{aligned} \tag{5.1}$$

The equalities (5.1) suggest the definition of the quotient of a ball structure $\mathfrak{B}_X=(X,P_X,B_X)$ with respect to a surjective map $q\colon X\to Y$. To define the quotient ball structure on Y use the same radii set $P_Y=P_X$ and for every $y\in Y$ and $r\in P_Y$ let

$$B_Y^q(y,r)=q(B_X(q^{-1}(y),r)).$$

In other words, if $y=q(x)$, then $B_Y^q(q(x),r)=q(B_X(R_q[x],r)))$. More generally, one has

$$q(B_X(R_q[A],r))=B_Y^q(q(A),r), \tag{5.2}$$

for arbitrary subsets A of X and not only singletons $\{x\}$ in X. This yields $q^{-1}(B_Y^q(q(A),r))=R_q[B_X(R_q[A],r)]$ for $A\subseteq X$.
This defines a ball structure $q(\mathfrak{B})=(Y,P_Y,B_Y^q)$ on Y, which we call *quotient ball structure*. The chain of equalities (5.1) proves that actually $q(\mathfrak{B})=\mathfrak{B}_{q(\mathcal{E})}$. Obviously, this is the finest ball structure on Y making q bornologous.

(b) Now we reformulate the properties from Definition 4.3.14 in terms of quasi-ballean.
If we use the quasi-ballean form of X, q is uniformly soft if and only if for all $r\in P_X$ there exists $s\in P_X$ such that $B_X(R_q[x],r)\subseteq R_q[B_X(x,s)]$ for every $x\in X$. By applying q to both sides of the previous inclusion, one obtains the inclusion

$$B_Y^q(q(x),r)=q(B_X(R_q[x],r))\subseteq q(B_X(x,s)) \tag{5.3}$$

for every $x\in X$. Conversely, if we take the preimages, (5.3) implies that

$$R_q[B_X(R_q[x],r)]\subseteq R_q[B_X(x,s)],$$

for every $x\in X$, which is equivalent to (4.9).
Focusing on weakly uniformly soft maps, q is weakly uniformly soft if and only if for every $r\in P_X$ there exists $s\in P_X$ such that

$$B_X(R_q[B_X(x,r)],r)\subseteq R_q[B_X(R_q[x],s)]$$

for every $x\in X$. Thus we can apply q and obtain that

$$B_Y^q(B_Y^q(q(x),r),r)\subseteq B_Y^q(q(x),s) \tag{5.4}$$

for every $x\in X$. (5.4) is equivalent to weak uniform softness, since application of the preimage of q leads to

$$R_q[B_X(R_q[B_X(x,r)],r)]\subseteq R_q[B_X(R_q[B_X(R_q[x],r)],r)]\subseteq R_q[B_X(R_q[x],s)]$$

for every $x \in X$, which is equivalent to (4.10).
Similarly, by using (4.10), q is 2-uniformly soft if and only if for every $r \in P_X$ there exists $s \in P_X$ such that

$$B_Y^q(B_Y^q(B_Y^q(B_Y^q(q(x),r),r),r),r) \subseteq B_Y^q(B_Y^q(q(x),s),s)$$

for every $x \in X$.

Let $\mathcal{X}$ be a category and $\mathcal{M}$ be a family (possibly a proper class) of monomorphisms of $\mathcal{X}$. For every object X of $\mathcal{X}$, let $\mathcal{M}/X$ be the family of the monomorphisms of $\mathcal{M}$ whose codomain is X. Define a relation $\leq$ on $\mathcal{M}/X$ as follows: if $f, g \in \mathcal{M}/X$, then

$$f \leq g \quad \text{if and only if} \quad \text{there exists a morphism } h \text{ of } \mathcal{X} \text{ such that } f = g \circ h. \tag{5.5}$$

Since f is a monomorphisms, also h is a monomorphism. Two monomorphisms f and g are equivalent, and we write $f \cong g$, if both $f \leq g$ and $g \leq f$. In this situation, if h is a morphism such that $f = g \circ h$, then h is an isomorphism of $\mathcal{X}$.

A *subobject* of an object X of **Coarse** is the equivalence class of an extremal monomorphism $m\colon M \to X$. Note that, for every $n\colon N \to X$ such that $m \cong n$, $m(M) = n(N)$. Moreover, $m \cong i_{m(M)}$, where $i_{m(M)}$ is the inclusion map of the subballean $m(M)$ into X. Hence, without loss of generality, a subobject $[m]_{\cong}$ of an object X of **Coarse** can be identified with the image of $i_{m(M)}$.

It is useful to introduce some subcategories of **Coarse**. These subcategories $\mathcal{Y}$ are *full* (i.e., for every pair of objects X and Y in $\mathcal{Y}$, $\mathrm{Mor}_{\mathcal{Y}}(X,Y) = \mathrm{Mor}_{\mathbf{Coarse}}(X,Y)$) and then it is enough to characterise their objects.

Definition 5.1.4.
- **Singleton** is the subcategory whose objects are one point balleans.
- **Bounded** is the subcategory whose objects are trivial balleans.
- **Discrete** is the subcategory whose objects are discrete balleans.
- **UBounded** is the subcategory whose objects are those balleans (X, P, B) such that there exists a radius $r \in P$ with the property that $B(x,r) = \mathcal{Q}_X(x)$, for every $x \in X$.
- **Connected** is the subcategory whose objects are connected balleans.

5.2 Closure operators in Coarse and cowellpoweredness

Closure operators can be defined in a wide range of categories ([60]), by using the notion of $\mathcal{M}$-*pullback* (see §5.4.2). However, for the purpose of this chapter, it is enough to specialise this concept in **Coarse**.

Definition 5.2.1. A categorical *c*losure operator $\mathrm{C} = (\mathrm{C}_X)_{X \in \mathbf{Coarse}}$ on **Coarse** assigns to every subobject M of an object X another subobject $\mathrm{C}_X(M)$ with the following properties:

(Extension) $M \subseteq \mathrm{C}_X(M)$;

(Monotonicity) if M is a subobject of N, which is a subobject of X, then $\mathrm{C}_X(M) \subseteq \mathrm{C}_X(N)$;

(Continuity) if $f\colon X \to Y$ is a morphism of **Coarse** and M a subobject of X, then $f(\mathrm{C}_X(M)) \subseteq \mathrm{C}_Y(f(M))$.

As in every concrete category, there exist three closure operators:

- the *discrete operator* D, such that $\mathrm{D}_X(M) = M$ for every object X and subobject M of X;
- the *trivial operator* T, such that $\mathrm{T}_X(M) = X$ for every object X and subobject M of X;
- the *indiscrete operator* G, such that

$$\mathrm{G}_X(M) = \begin{cases} X & \text{if } M \neq \emptyset, \\ \emptyset & \text{otherwise,} \end{cases}$$

for every object X and subobject M of X.

In the conglomerate $\mathcal{CL}(\mathbf{Coarse})$ of all closure operators of **Coarse**, let us consider the preorder relation defined as follows: $\mathrm{C} \leq \mathrm{C}'$ if and only if, for every $X \in \mathbf{Coarse}$ and every subspace M of X, $\mathrm{C}_X(M) \subseteq \mathrm{C}'_X(M)$. Moreover, we write $\mathrm{C} = \mathrm{C}'$ if $\mathrm{C} \leq \mathrm{C}'$ and $\mathrm{C}' \leq \mathrm{C}$, while $\mathrm{C} < \mathrm{C}'$ stands for $\mathrm{C} \leq \mathrm{C}'$ and $\mathrm{C}' \not\leq \mathrm{C}$.

Note that $\mathrm{D} < \mathrm{G} < \mathrm{T}$: D and T are the bottom and the top elements in the conglomerate $\mathcal{CL}(\mathbf{Coarse})$, respectively, while G is the top element with respect to the property of being *grounded*, i.e., $\mathrm{G}(\emptyset) = \emptyset$ (see [60] for the general definition of the groundedness property). Those three closure operators, D, G and T, are called *improper*. Any closure operator which is not improper is called *proper*.

A closure operator C on **Coarse** is called

- *idempotent* if $\mathrm{C}_X(\mathrm{C}_X(M)) = \mathrm{C}_X(M)$ for every object X and every subspace M of X;
- *hereditary* if $\mathrm{C}_Y(M) = \mathrm{C}_X(M) \cap Y$, whenever M is a subobject Y of a subobject of an object X;
- *fully additive* if $\mathrm{C}_X(M) = \bigcup_{x\in M} \mathrm{C}_X(\{x\})$, whenever M is a subobject of an object X;
- *symmetric* if $y \in \mathrm{C}_X(\{x\})$ if and only if $x \in \mathrm{C}_X(\{y\})$, whenever x, y are two points of an object X;
- *productive* if, for every product $\{p_i\colon X \to X_i\}_{i\in I}$ of **Coarse** and every family of subobjects $\{M_i \subseteq X_i\}_{i\in I}$, $\mathrm{C}_X(\Pi_i M_i) = \Pi_i \, \mathrm{C}_{X_i}(M_i)$.

5.2.1 Classification of the closure operators

Let (X, P, B) be a ballean, $x \in X$, and $Y \subseteq X$ be a subspace. It is easy to check that $\mathcal{Q}_X(x) = \mathcal{Q}_{(X,\mathcal{E}_{\mathfrak{B}})} = \bigcup_{r\in P} B(x, r)$, and $\mathcal{Q}_X(Y) = \bigcup_{r\in P} B(Y, r)$. The operator $\mathcal{Q} = (\mathcal{Q}_X)_{X\in\mathbf{Coarse}}$ is a closure operator in **Coarse**.

Theorem 5.2.2. *$\mathcal{Q}$ is a closure operator in* **Coarse**. *Moreover, $\mathcal{Q}$ is hereditary, fully additive, symmetric, idempotent and productive.*

Proof. The only property one needs to check, in order to assure that $\mathcal{Q}$ is a closure operator, is the continuity, which follows from Proposition 2.2.8.

As for the second assertion, by the definition of subballean, hereditariness easily follows. Full additivity and symmetry are trivial and idempotency is a consequence of upper multiplicativity of balleans. Finally, the definition of product ballean implies the productivity of $\mathcal{Q}$. $\square$

Suppose now that there exists a closure operator C on **Coarse** such that $\mathrm{D} \neq \mathrm{C}$ and so $\mathrm{D} < \mathrm{C}$. The aim of this section is to prove that $\mathrm{C} = \mathcal{Q}$, $\mathrm{C} = \mathrm{G}$ or $\mathrm{C} = \mathrm{T}$. Since $\mathrm{D} < \mathrm{C}$, there exists an object X of **Coarse** and a subset M of X such that $\mathrm{D}_X(M) = M \subsetneq \mathrm{C}_X(M)$. Lemma 5.2.4 and Corollary 5.2.5 discuss properties of the pair X and M and they will be used to prove the more general Theorem 5.2.6.

Lemma 5.2.3. *Let (X, P_X, B_X) and (Y, P_Y, B_Y) be two balleans and $f\colon X \to Y$ be a bornologus map. Let $\{a_i \in X \mid i \in I\}$ be a family of distinct points and let $\{y_i \in Y \mid i \in I\}$ be another family such that there exists a radius $u \in P_Y$ with the property that $y_i \in B_Y(f(a_i), u)$ for every $i \in I$. Then the map $\widetilde{f}\colon X \to Y$ defined by the law*

$$\widetilde{f}(x) = \begin{cases} y_i & \text{if } x = a_i, \\ f(x) & \text{otherwise,} \end{cases}$$

where $x \in X$, is bornologous.

Proof. Let $r \in P_X$ be an arbitrary radius, $s \in P_Y$ be a radius such that $f(B_X(x,r)) \subseteq B_Y(f(x), s)$, for every $x \in X$, and $t \in P_Y$ be a radius which satisfies $B_Y(B_Y(B_Y(y,u),s),u) \subseteq B_Y(y,t)$, for every $y \in Y$. Then $\widetilde{f}$ is bornologous, since, for every $x \in X$,

$$\begin{aligned} \widetilde{f}(B_X(x,r)) &\subseteq B_Y(f(B_X(x,r)),u) \subseteq B_Y(B_Y(f(x),s),u) \subseteq \\ &\subseteq B_Y(B_Y(B_Y(\widetilde{f}(x),u),s),u) \subseteq B_Y(\widetilde{f}(x),t). \end{aligned}$$

$\square$

Lemma 5.2.4. *Let C be a closure operator in* **Coarse**. *Suppose that there exists an object (X, P, B) and a subobject $M \subseteq X$ such that $M \subsetneq \mathrm{C}_X(M)$. Then $\mathcal{Q}_X(\mathrm{C}_X(M) \setminus M) \subseteq \mathrm{C}_X(M)$. Moreover:*
(a) if $M \subsetneq \mathcal{Q}_X(M) \cap \mathrm{C}_X(M)$, then $\mathcal{Q}_X(M) \subseteq \mathrm{C}_X(M)$;
(b) if $\mathrm{C}_X(M) \setminus \mathcal{Q}_X(M) \neq \emptyset$, then $\mathrm{C}_X(M) = \mathrm{T}_X(M) = X$.

Proof. Let y be an arbitrary point in $\mathcal{Q}_X(\mathrm{C}_X(M) \setminus M)$ and $z \in \mathrm{C}_X(M) \setminus M$ be a point such that $y \in \mathcal{Q}_X(z)$. We define a map $f\colon X \to X$ such that $f(z) = y$ and $f|_{X\setminus\{z\}} = id_{X\setminus\{z\}}$. Since $z \in B(y,r)$ for some $r \in P$, f is bornologous by Lemma 5.2.3. Hence, $\mathcal{Q}_X(\mathrm{C}_X(M) \setminus M) \subseteq \mathrm{C}_X(M)$, since

$$y = f(z) \in f(\mathrm{C}_X(M)) \subseteq \mathrm{C}_X(f(M)) = \mathrm{C}_X(M).$$

(a) Let $x \in (\mathcal{Q}_X(M) \cap \mathrm{C}_X(M)) \setminus M$ and let us suppose by contradiction that there exists a point $y \in \mathcal{Q}_X(M) \setminus \mathrm{C}_X(M)$. In particular $\mathcal{Q}_X(x) \subseteq \mathrm{C}_X(M)$ by the first statement of this Lemma. If $m \in X$ is an arbitrary point which belongs to the non-empty subset $M \cap \mathcal{Q}_X(y)$, then the following map $f\colon X \to X$ can be defined: for every $z \in X$

$$f(z) = \begin{cases} y & \text{if } z \in \mathcal{Q}_X(x) \setminus M, \\ m & \text{if } z \in \mathcal{Q}_X(x) \cap M, \\ z & \text{otherwise.} \end{cases}$$

This map is bornologous by Proposition 4.3.4 and Remark 3.1.12, in fact the image of $f|_{\mathcal{Q}_X(x)}$ is $\{y, m\}$, which is bounded. One obtains the following chain of inclusions:

$$y \in f(\mathcal{Q}_X(x) \setminus M) \subseteq f(\mathcal{Q}_X(x)) \subseteq f(\mathrm{C}_X(M)) \subseteq \mathrm{C}_X(f(M)) \subseteq \mathrm{C}_X(M),$$

a contradiction, since $y \notin \mathrm{C}_X(M)$.

(b) Let $x \in \mathrm{C}_X(M) \setminus \mathcal{Q}_X(M)$. In particular $\mathcal{Q}_X(x) \subseteq \mathrm{C}_X(M)$, since $x \in \mathrm{C}_X(M) \setminus M$. For every point $y \in X$, the map $f_y \colon X \to X$ defined by the laws $f_y|_{X \setminus \mathcal{Q}_X(x)} = id_{X \setminus \mathcal{Q}_X(x)}$ and $f_y(\mathcal{Q}_X(x)) = \{y\}$ is bornologous (by Proposition 4.3.4 and Remark 3.1.12) and then

$$y \in f_y(\mathcal{Q}_X(x)) \subseteq f_y(\mathrm{C}_X(M)) \subseteq \mathrm{C}_X(f_y(M)) = \mathrm{C}_X(M). \qquad \square$$

Corollary 5.2.5. *Let* C *be a closure operator in* **Coarse**. *Suppose that there exist an object X and a subobject $M \subseteq X$ such that $M \subsetneq \mathrm{C}_X(M)$. Then $\mathcal{Q}_X(M) \subseteq \mathrm{C}_X(M)$. Moreover, if $\mathcal{Q}_X(M) \subsetneq \mathrm{C}_X(M)$, then $\mathrm{C}_X(M) = X$.*

Proof. Since $M \subsetneq \mathrm{C}_X(M)$, if $M \subsetneq \mathcal{Q}_X(M) \cap \mathrm{C}(X)$, then $\mathcal{Q}_X(M) \subseteq \mathrm{C}_X(M)$ by applying Lemma 5.2.4(a). Otherwise, if $M = \mathcal{Q}_X(M) \cap \mathrm{C}_X(M)$, then $\mathrm{C}_X(M) \setminus \mathcal{Q}_X(M) \neq \emptyset$ and so $\mathcal{Q}_X(M) \subseteq X = \mathrm{C}_X(M)$ by Lemma 5.2.4(b). The last assertion follows again from Lemma 5.2.4(b). $\square$

Theorem 5.2.6. *The only proper closure operator in* **Coarse** *is $\mathcal{Q}$.*

Proof. If we suppose that $\mathrm{D} < \mathrm{C}$, then there exists a ballean (X, P_X, B_X) and a subobject $M \subseteq X$ such that $M \subsetneq \mathrm{C}_X(M)$. By Corollary 5.2.5, $\mathcal{Q}_X(M) \subseteq \mathrm{C}_X(M)$.

We claim that, for every ballean Y and every subobject $N \subseteq Y$, $\mathcal{Q}_Y(N) \subseteq \mathrm{C}_Y(N)$. For every $n \in N$ and $y \in \mathcal{Q}_Y(n)$, the map $g_{n,y} \colon X \to Y$, defined by the law

$$g_{n,y}(x) = \begin{cases} n & \text{if } x \in M, \\ y & \text{otherwise,} \end{cases}$$

for every $x \in X$, is bornologous (by Remark 3.1.12) and

$$y \in g_{n,y}(\mathrm{C}_X(M) \setminus M) \subseteq g_{n,y}(\mathrm{C}_X(M)) \subseteq \mathrm{C}_Y(g_{n,y}(M)) = \mathrm{C}_Y(\{n\}) \subseteq \mathrm{C}_Y(N).$$

Hence, in particular, $\mathcal{Q} \leq \mathrm{C}$.

Suppose now that $\mathcal{Q} < \mathrm{C}$. Then there exist a ballean X and a subobject M of X such that $\mathcal{Q}_X(M) \subsetneq \mathrm{C}_X(M)$. By Corollary 5.2.5, $\mathrm{C}_X(M) = X$. Let Y be an arbitrary ballean and $N \subseteq Y$ be a subobject. Let $x \in X \setminus \mathcal{Q}_X(M)$, y and n be two arbitrary points of Y and N, respectively, and $h_{n,y} \colon X \to Y$ be a map such that, for every $z \in X$,

$$h_{n,y}(z) = \begin{cases} y & \text{if } z \in \mathcal{Q}_X(x), \\ n & \text{otherwise.} \end{cases}$$

The map $h_{n,y}$ is bornologous by Proposition 4.3.4 and Remark 3.1.12. Thus $Y \subseteq \mathrm{C}_Y(N)$, since

$$y \in h_{n,y}(X) = h_{n,y}(\mathrm{C}_X(M)) \subseteq \mathrm{C}_Y(h_{n,y}(M)) = \mathrm{C}_Y(\{n\}) \subseteq \mathrm{C}_Y(N). \qquad \square$$

Theorem 5.2.6 is inspired by the results in [62].

5.2.2 Epireflective subcategories and delta-subcategories of Coarse

A full subcategory $\mathcal{X}$ of **Coarse** is *epireflective* if it is closed under formation of subobjects and of products. This is not the usual definition of epireflective subcategory, but it is equivalent in this context (see [1]). All the subcategories listed in Definition 5.1.4 are epireflective. Of course, also **Coarse** is an epireflective subcategory of **Coarse** itself.

Epireflective subcategories of **Coarse** are in connection with closure operators of **Coarse** and in this subsection we explore their relationship. This connection still holds in a more general context, see for example [60].

Let C be a closure operator in **Coarse**. A subobject Y of a ballean X is:

(a) C-*closed in* X if $\mathrm{C}_X(Y) = Y$;
(b) C-*dense in* X if $\mathrm{C}_X(Y) = X$.

If C is a closure operator of **Coarse**, denote by $\Delta(\mathrm{C})$ the full subcategory of **Coarse** of all the objects X such that Δ_X is C-closed in $X \times X$. A subcategory $\mathcal{X}$ of **Coarse** is called a *Delta-subcategory* if there exists a closure operator C on **Coarse** such that $\mathcal{X} = \Delta(\mathrm{C})$.

Example 5.2.7. $\Delta(\mathrm{D}) =$ **Coarse**, $\Delta(\mathcal{Q}) =$ **Discrete** and $\Delta(\mathrm{T}) = \Delta(\mathrm{G}) =$ **Singleton**.

The only non-trivial assertion is $\Delta(\mathcal{Q}) =$ **Discrete**. Let X be an object of $\Delta(\mathcal{Q})$. Hence, $\mathcal{Q}_{X\times X}(\Delta_X) = \Delta_X$ and then, since $\mathcal{Q}$ is productive, X is the trivial ballean.

Proposition 5.2.8. *Let $\mathcal{X}$ be an epireflective subcategory of* **Coarse**. *Then:*

(a) if **Singleton** $\subseteq \mathcal{X} \subseteq$ **Discrete***, then* $\mathcal{X} =$ **Singleton** *or* $\mathcal{X} =$ **Discrete***;*
(b) if **Singleton** $\subseteq \mathcal{X} \subseteq$ **Bounded***, then* $\mathcal{X} =$ **Singleton** *or* $\mathcal{X} =$ **Bounded***;*
(c) if $\mathcal{X}$ properly contains **Discrete***, then* **Bounded** $\subseteq \mathcal{X}$*;*
(d) if **Bounded** $\subseteq \mathcal{X}$ *and* **Discrete** $\subseteq \mathcal{X}$*, then* **UBounded** $\subseteq \mathcal{X}$.

Proof. Items (a), (b) and (c) follow from Proposition 4.3.3. If $\mathcal{X}$ contains an object with at least two points, which is discrete, then every discrete object is in $\mathcal{X}$. Similarly, if $\mathcal{X}$ has an object with at least two points, which is bounded, then every bounded object is in $\mathcal{X}$. Finally, if $\mathcal{X}$ has an object with a connected component with at least two points, then $\mathbb{B}$ belongs to $\mathcal{X}$ and so every bounded ballean is an object of $\mathcal{X}$.

(d) For every object X of **UBounded**, consider its representation $X = \bigsqcup_{k\in I} X_k$ as disjoint union of its connected components. We define two balleans: X^b is the bounded ballean on the support X and I^t is the discrete ballean on the support I. The ballean $Y = X^b \times I^t$ belongs to **UBounded**. If $i_k \colon X_k \to X$ are the canonical inclusions, for every $k \in I$, it is easy to check that the map $f \colon X \to Y$ such that, for every $i_k(x) \in X$, $f(i_k(x)) = (i_k(x), k)$, where $i_k(x)$ can obviously be identified with its image in X^t, is an injective coarse embedding and so X is a subobject of Y. □

Item (c) of Proposition 5.2.8 cannot be 'symmetrised', since there exist epireflective subcategories $\mathcal{X}$ of **Coarse** with the property that **Bounded** $\subsetneq \mathcal{X}$, but **Discrete** $\nsubseteq \mathcal{X}$. For example take $\mathcal{X} =$ **Connected**. Moreover, it is remarkable that item (d) uses only the closedness of $\mathcal{X}$ under taking subobjects and finite products.

Denote by $\mathfrak{E}(\{X_i\}_{i\in I})$ the *epireflective hull* of a family of objects $\{X_i\}_{i\in I}$ of **Coarse**, i.e., the smallest epireflective subcategory $\mathcal{X}$ of **Coarse** such that X_i is an object of $\mathcal{X}$, for every $i \in I$.

Corollary 5.2.9. *The following equalities hold:*

$$\mathfrak{E}(\mathbb{B}) = \mathbf{Bounded},\ \mathfrak{E}(\mathbb{D}) = \mathbf{Discrete},$$
$$\textit{and } \mathfrak{E}(\mathbb{B}, \mathbb{D}) = \mathbf{UBounded}.$$

Proof. The equalities $\mathfrak{E}(\mathbb{B}) = \mathbf{Bounded}$ and $\mathfrak{E}(\mathbb{D}) = \mathbf{Discrete}$ follow from Proposition 4.3.3, while Proposition 5.2.8(d) implies the last one. □

5.2.3 Regular closure operators in Coarse

In order to describe how to construct closure operators from epireflective subcategories, we need the concept of equalizer $m = \mathrm{eq}(f, g)$ of a pair of morphisms $f, g\colon X \to Y$. Since **Coarse** is a topological category, for every two morphisms $f, g\colon X \to Y$, their equalizer can be characterized as the subobject $\mathrm{eq}(f, g) = \{x \in X \mid f(x) = g(x)\}$ of X.

Let $\mathcal{X}$ be an epireflective subcategory of **Coarse**. Let us define $\mathrm{reg}^{\mathcal{X}} = (\mathrm{reg}^{\mathcal{X}}_X)_{X\in\mathbf{Coarse}}$, where

$$\mathrm{reg}^{\mathcal{X}}_X(M) = \bigcap\{\mathrm{eq}(f, g) \mid Y \in \mathcal{X},\ f, g \in \mathrm{Mor}_{\mathbf{Coarse}}(X, Y),\ f|_M = g|_M\},$$

for every object X and every subobject M of X. Then $\mathrm{reg}^{\mathcal{X}}$ is a closure operator, called *$\mathcal{X}$-regular closure operator of* **Coarse**. Since $\mathcal{X}$ is epireflective, one can prove that, for every object X of **Coarse** and every subobject M, there exist two morphisms $f, g\colon X \to Y$, such that $f|_M = g|_M$ and Y is an object of $\mathcal{X}$, with the property that $\mathrm{reg}^{\mathcal{X}}_X(M) = \mathrm{eq}(f, g)$.

An immediate fact relates regular closure operators to epimorphisms. In [159] the same result was proved for **Top** and its subcategories.

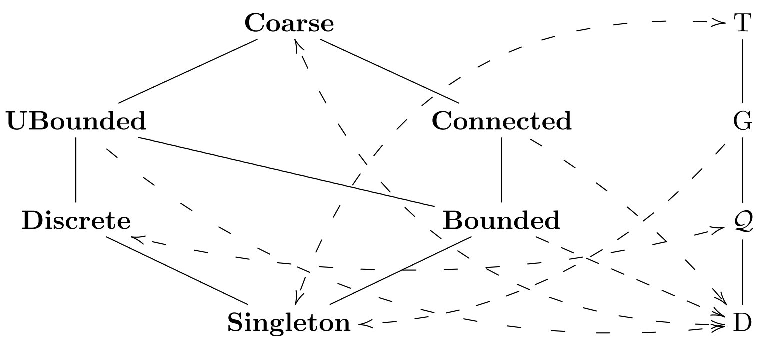
Fact 5.2.10. *Let $\mathcal{X}$ be an epireflective subcategory of* **Coarse** *and $f\colon X \to Y$ be a morphism of $\mathcal{X}$. Then f is an epimorphism of $\mathcal{X}$ if and only if $f(X)$ is* $\mathrm{reg}^{\mathcal{X}}$*-dense in Y.*

Regular closure operators of **Coarse** are idempotent and, moreover, they satisfy the following *monotonicity condition*: if $\mathcal{X} \subseteq \mathcal{Y}$ are two epireflective subcategories of **Coarse**, then $\mathrm{reg}^{\mathcal{Y}} \le \mathrm{reg}^{\mathcal{X}}$.

Theorem 5.2.11. *(a)* $\mathrm{reg}^{\mathbf{Bounded}} = \mathrm{D}$, $\mathrm{reg}^{\mathbf{Discrete}} = \mathcal{Q}$ *and* $\mathrm{reg}^{\mathbf{Singleton}} = \mathrm{T}$.
(b) If $\mathcal{X}$ is an epireflective subcategory of **Coarse** *such that* **Bounded** $\subseteq \mathcal{X}$*, then* $\mathrm{reg}^{\mathcal{X}} = \mathrm{D}$.

Before proving the result, we give a diagram, (5.6), that summarises the situation. On the left hand side there are some relevant epireflective subcategories of **Coarse**, ordered by inclusion, while on the right hand side there are the closure operators in **Coarse**, with their order. The dashed arrows point out the relationships between these two lattices: a dashed arrow from a subcategory

$\mathcal{X}$ to a closure operator C means that $\mathrm{C} = \mathrm{reg}^{\mathcal{X}}$, while a dashed arrow from a closure operator C to a subcategory $\mathcal{X}$ means that $\mathcal{X} = \Delta(\mathrm{C})$.



(5.6)

Proof of Theorem 5.2.11. (a) First of all, the equality $\mathrm{reg}^{\mathbf{Singleton}} = \mathrm{T}$ is trivial.

The next goal is to prove that $\mathrm{reg}^{\mathbf{Bounded}} = \mathrm{D}$. Let X be a ballean and $M \subseteq X$ be an arbitrary subspace of X. We claim that $\mathrm{reg}_X^{\mathbf{Bounded}}(M) = M$, and so $\mathrm{reg}^{\mathbf{Bounded}} = \mathrm{D}$. Suppose, without loss of generality, that $M \neq X$. Let $\mathbb{B}$ the bounded ballean defined in Proposition 4.3.3. We define two maps $f, g\colon X \to \mathbb{B}$ as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in M, \\ 1 & \text{otherwise,} \end{cases}$$

and $g(x) = 0$, for every $x \in X$. Since f and g are bornologous (Remark 3.1.12) and $f|_M = g|_M$, then

$$M \subseteq \mathrm{reg}_X^{\mathbf{Bounded}}(M) \subseteq \mathrm{eq}(f, g) = M.$$

Let X be a ballean and M be a subobject of X such that $M \subsetneq \mathcal{Q}_X(M) \subsetneq X$. It suffices to show that $\mathcal{Q}_X(M) \subseteq \mathrm{reg}_X^{\mathbf{Discrete}}(M) \subsetneq X$ to conclude that $\mathrm{reg}^{\mathbf{Discrete}} = \mathcal{Q}$ by Theorem 5.2.6. Let Y be a discrete ballean and $f, g\colon X \to Y$ be two bornologous maps such that $f|_M = g|_M$. Let us define the bornologous map $f \times g\colon X \to Y \times Y$, where, for every $x \in X$, $(f \times g)(x) = (f(x), g(x))$. Since $Y \times Y$ is discrete, Proposition 2.2.8 implies that $\mathrm{eq}(f, g) = (f \times g)^{-1}(\Delta_Y)$ is union of connected components of X. Moreover, $M \subseteq \mathrm{eq}(f, g)$ and thus $\mathcal{Q}_X(M) \subseteq \mathrm{eq}(f, g)$. Since f and g are arbitrary, $M \subsetneq \mathcal{Q}_X(M) \subseteq \mathrm{reg}_X^{\mathbf{Discrete}}(M)$. Now, let us consider the discrete ballean $\mathbb{D}$ (Proposition 4.3.3) and define a morphism $g\colon X \to \mathbb{D}$ by the law

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathcal{Q}_X(M), \\ 1 & \text{otherwise,} \end{cases}$$

for every $x \in X$. Then both the constant map $f\colon X \to \{0\} \subseteq \mathbb{D}$ and the morphism g are bornologous (Remark 3.1.12 and Proposition 4.3.4), they coincide on M and so they show that $\mathrm{reg}_X^{\mathbf{Discrete}}(M) \subseteq \mathrm{eq}(f, g) = \mathcal{Q}_X(M) \subsetneq X$.

(b) Let $\mathcal{X}$ be an epireflective subcategory of **Coarse** such that **Bounded** $\subseteq \mathcal{X}$. Then, by the monotonicity condition, $\mathrm{D} \leq \mathrm{reg}^{\mathcal{X}} \leq \mathrm{reg}^{\mathbf{Bounded}} = \mathrm{D}$ and so $\mathrm{reg}^{\mathcal{X}} = \mathrm{D}$. □

By combining Example 5.2.7, Proposition 5.2.8, and Theorem 5.2.11 the following result can be proved.

Corollary 5.2.12. *All epireflective subcategories* $\mathcal{X}$ *of* **Coarse** *have* $\mathrm{reg}^{\mathcal{X}} = \mathrm{D}$, *except* $\mathcal{X} =$ **Discrete** *and* $\mathcal{X} =$ **Singleton**, *which satisfy* $\mathrm{reg}^{\mathbf{Discrete}} = \mathcal{Q}$ *and* $\mathrm{reg}^{\mathbf{Singleton}} = \mathrm{T}$.

5.2.4 Consequences for the category Coarse

Let $\mathcal{E}$ be a family of morphisms of a category $\mathcal{X}$ (possibly a proper class), denote by $X/\mathcal{E}$ the family of morphisms of $\mathcal{E}$ whose domain is the object X of $\mathcal{X}$. A category $\mathcal{X}$ is *$\mathcal{E}$-cowellpowered*, where $\mathcal{E}$ is a family of morphisms of $\mathcal{X}$, if, for every object X of $\mathcal{X}$, $X/\mathcal{E}$ has a set of representatives with respect to isomorphisms (i.e., there exists a set $X/\mathcal{E}' \subseteq X/\mathcal{E}$ such that, for every $e \in X/\mathcal{E}$, there exists $e' \in X/\mathcal{E}'$ and an isomorphism f of $\mathcal{X}$ such that $e = f \circ e'$). A category is *cowellpowered* if it is $\mathrm{Epi}_{\mathcal{X}}$-cowellpowered.

Corollary 5.2.13. *For every epireflective subcategory* $\mathcal{X}$ *of* **Coarse**, *the epimorphisms are surjective. Consequently,* $\mathcal{X}$ *is cowellpowered.*

Proof. By Corollary 5.2.12, if $\mathcal{X} \neq$ **Discrete** and $\mathcal{X} \neq$ **Singleton**, $\mathrm{reg}^{\mathcal{X}} = \mathrm{D}$ and so, by Fact 5.2.10, the epimorphisms of $\mathcal{X}$ are the morphisms with D-dense image, i.e., the surjective morphisms. The other two cases similarly follow, once one notices that $\mathrm{reg}^{\mathbf{Singleton}}|_{\mathbf{Singleton}} = \mathrm{T}|_{\mathbf{Singleton}} = \mathrm{D}$ and $\mathrm{reg}^{\mathbf{Discrete}}|_{\mathbf{Discrete}} = \mathcal{Q}|_{\mathbf{Discrete}} = \mathrm{D}$.

The second assertion is a trivial consequence of the first. □

In the category **Coarse** we have been able to classify all closure operators and to prove the cowellpoweredness of all epireflective subcategories. One may ask what happens in related categories.

Question 5.2.14. *(a) Classify all closure operators of* $\mathcal{R}$*, the Roe category of coarse spaces and proper bornologous maps between them.*

(b) Classify all closure operators of **PaBorn**, **SBorn**, **QBorn**, **PrBorn**, **Entou**, **SCoarse**, *and* **QCoarse**.

(c) Is there a non cowellpowered (epireflective) subcategory of $\mathcal{R}$*, of* **PaBorn** *or of* **Entou**?

A subcategory $\mathcal{A}$ of a topological category $\mathcal{X}$ is called *extremely epireflective* if it is epireflective and *closed under monomorphisms*, i.e., if $m\colon M \to A$ is a monomorphism of $\mathcal{X}$ such that A is an object of $\mathcal{A}$, then M belongs to $\mathcal{A}$. In Theorem 5.2.15 we recall a necessary condition for this property. For a generalization of this result, see [60]. A *generator* of a category $\mathcal{X}$ is an object G of $\mathcal{X}$ such that for every pair of distinct morphisms $f, g\colon X \to Y$ of $\mathcal{X}$ there exists a morphism $h\colon G \to X$ such that $f \circ h \neq g \circ h$. An object T of a category $\mathcal{X}$ is *terminal* if, for every other object X of $\mathcal{X}$, $\mathrm{Mor}_{\mathcal{X}}(X, T)$ has exactly one element.

Theorem 5.2.15 (Diagonal Theorem)**.** *Let* $\mathcal{A}$ *be an extremely epireflective subcategory of a topological category* $\mathcal{X}$*. Then* $\mathcal{A}$ *is a* Δ*-subcategory and, in particular,* $\mathcal{A} = \Delta(\mathrm{reg}^{\mathcal{A}})$.

Proof. Since a one point space is both terminal and a generator in a topological category, [85, Corollary 1.2] can be applied to prove the result. □

There are few extremely epireflective proper subcategories of **Coarse**, namely **Discrete** and **Singleton**: in fact, one can easily prove that those two subcategories have that property and Proposition 5.2.8, Corollary 5.2.12, and Theorem 5.2.15 imply that they are the only ones. In Example 5.2.16 two epireflective non-extremely epireflective subcategories are presented.

Example 5.2.16. (a) Because of the previous assertion, the epireflective subcategory **Bounded** is not extremely epireflective and so there exists a monomorphism $m\colon M \to Y$ of **Coarse** such that Y belongs to **Bounded**, but X is not an object of that subcategory. It is enough to put $M = \mathbb{D}$, $Y = \mathbb{B}$ and $m = id_{\{0,1\}}$.

(b) Again because of the previous assertion, **UBounded** is not extremely epireflective. In fact, if $\mathfrak{B} = (X, P, B)$ is an unbounded connected ballean and $\mathfrak{B}' = (X, P \cup \{*\}, B')$, where $B'|_{X\times P} = B$ and $B'(x, *) = X$ for every $x \in X$, then $id_X\colon \mathfrak{B} \to \mathfrak{B}'$ is a monomorphism, $\mathfrak{B}'$ belongs to **Bounded** $\subseteq$ **UBounded**, while the only connected component of $\mathfrak{B}$ is unbounded.

Closure operators induce notions of compactness. Let C be a closure operator in **Coarse**. A map $f\colon X \to Y$ between balleans is C-*closed* if the images of C-closed subobjects of X is C-closed in Y. An object X of **Coarse** is C-*compact* (or *categorically compact with respect to* C) if, for every other object Y of **Coarse**, the canonical projection $p\colon X\times Y \to Y$ is C-closed. Unfortunately, this approach provides no useful notions, as Theorem 5.2.18 shows.

Lemma 5.2.17. *Let $f\colon (X, P_X, B_X) \to (Y, P_Y, B_Y)$ be a surjective uniformly boundedness copreserving map between balleans. Then f is $\mathcal{Q}$-closed.*

Proof. Fix a point $x \in X$. Since f is a uniformly boundedness copreserving map (see §3.2 for the characterisation of this notion in terms of balleans), for every radius $s \in P_Y$, there exists a radius $r_s \in P_X$ such that $B_Y(f(z), s) \subseteq f(B_X(z, r_s))$, for every $z \in X$. Hence,

$$\begin{aligned}\mathcal{Q}_Y(f(x)) &= \bigcup_{\beta\in P_Y} B_Y(f(x), s) \subseteq \bigcup_{\beta\in P_Y} f(B_X(x, r_s)) \subseteq \\ &\subseteq f\Big(\bigcup_{s\in P_Y} B_X(x, r_s)\Big) \subseteq f(\mathcal{Q}_X(x)) \subseteq \mathcal{Q}_Y(f(x))\end{aligned}$$

and so $f(\mathcal{Q}_X(x)) = \mathcal{Q}_Y(f(x))$. Thus, since f is also surjective, images of connected components of X are connected components of Y, and so f is $\mathcal{Q}$-closed. □

Theorem 5.2.18. *In* **Coarse***, every object is categorically compact with respect to any closure operator.*

Proof. Let C be a closure operator in **Coarse**. The cases C = D, C = G and C = T are trivial. As for C = $\mathcal{Q}$, the claim follows from Lemma 5.2.17 and from the fact that every projection map from a product ballean to one of its component is uniformly boundedness copreserving. □

5.3 The quotient category Coarse/$_\sim$

In this section we study the quotient category **Coarse**/$_\sim$ of **Coarse**. In order to characterise the epimorphisms of that category, we first need to introduce the notion of adjunction space.

5.3.1 The adjunction space

Theorem 4.3.17 gives the description of the quotient ballean of a weakly uniformly soft map, namely this is the quotient ball structure (see Example 5.3.5 (c) for an example of a weakly uniformly soft map that is not uniformly soft). We aim to describe the quotient ballean $\overline{\mathfrak{B}}^q$, in a wider range of quotient maps. Here we do it in the case of the quotient map defining the adjunction space $X \sqcup_L X$ which will be substantially used in the sequel. As we show in Theorem 5.3.4 this map is very rarely weakly uniformly soft (the theorem provides a description of the cases when that quotient map can be weakly uniformly soft).

Definition 5.3.1. Let $\mathfrak{B}_X = (X, P_X, B_X)$ be a ballean and L a subset of X. Let $i_1, i_2 \colon X \to X \sqcup X$ be the canonical inclusions of X into the disjoint union $X \sqcup X$. Let $X \sqcup_L X$ be the quotient space $(X \sqcup X)/ \sim_L$ obtained from the equivalence relation

$$x \sim_L y \Leftrightarrow \begin{cases} x = i_1(l), y = i_2(l) \text{ with } l \in L, \\ y = i_1(l), x = i_2(l) \text{ with } l \in L, \\ x = y. \end{cases}$$

If $L = \emptyset$, $X \sqcup_L X$ coincides with $X \sqcup X$, this is why we assume from now on that $L \neq \emptyset$. Our aim is to describe the quotient ballean structure $X \sqcup_L X$ of the quotient of coproduct ballean $\mathfrak{B}_X \oplus \mathfrak{B}_X$ under the canonical map $q \colon X \sqcup X \to X \sqcup_L X$ defined by the equivalence relation $\sim_L$. For every $k = 1, 2$, put $j_k = q \circ i_k$, so that $X \sqcup_L X = j_1(X) \cup j_2(X)$.

Let $p \colon X \sqcup_L X \to X$ be the map defined by $p(j_k(x)) = x$ for all $x \in X$ (this definition is correct as both j_k are injective, $j_1|_L = j_2|_L$ and $X \sqcup_L X = j_1(X) \cup j_2(X)$). Let σ be the obvious involution (symmetry) of the coproduct $X \sqcup X$ and σ' be the involution of $X \sqcup_L X$ induced by σ (so that $\sigma'(j_1(x)) = j_2(x)$ and $\sigma'(j_2(x)) = j_1(x)$ for every $x \in X$). All these maps are conveniently represented in Figure 5.1.

Example 5.3.5(a) shows that the quotient ball structure $q(\mathfrak{B})$ on $X \sqcup_L X$ need not be a ballean in general. This is why we define a new ball structure $\mathfrak{B}^a_{X \sqcup_L X}$, called *adjunction space*, on $X \sqcup_L X$ with radii set P_X and balls defined by

$$B_{X \sqcup_L X}(j_k(x), r) = \begin{cases} j_k(B_X(x, r)) & \text{if } B_X(x, r) \cap L = \emptyset, \\ j_1(B_X(x, r)) \cup j_2(B_X(x, r)) & \text{otherwise}, \end{cases} \tag{5.7}$$

for every $x \in X$, $k = 1, 2$, $r \in P_X$.

Theorem 5.3.2. *$\mathfrak{B}^a_{X \sqcup_L X}$ is the quotient ballean structure on $X \sqcup_L X$.*

Proof. We have to prove that $\mathfrak{B}^a_{X \sqcup_L X}$ is upper multiplicative and upper symmetric, q is bornologous and $\mathfrak{B}^a_{X \sqcup_L X}$ has quotient's universal property.

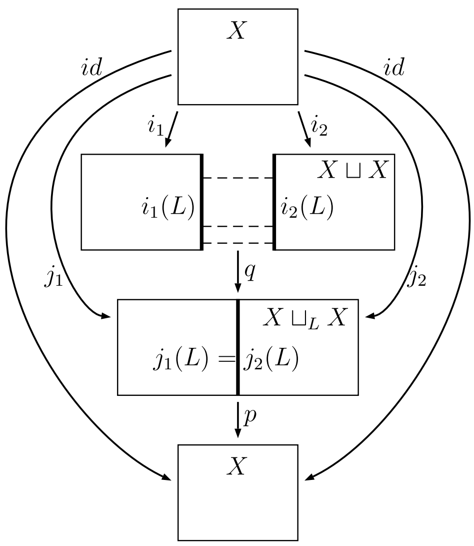


Figure 5.1: A representation of the adjunction space.

First we want to show that it is the upper mulplicative. Fix two radii $r, s \in P_X$ and let $t \in P_X$ be an element such that $B_X(B_X(x,r),s) \subseteq B_X(x,t)$ for every $x \in X$. Then it is easy to check that

$$B_{X\sqcup_L X}(B_{X\sqcup_L X}(j_k(x),r),s) \subseteq B_{X\sqcup_L X}(j_k(x),t), \text{ for } k=1,2 \text{ and } x \in X,$$

since the property $B_X(B_X(x,r),s) \cap L \neq \emptyset$ implies $B_X(x,t) \cap L \neq \emptyset$.

The second thing we want to prove is the upper symmetry. Let us first note that for both embeddings $j_k\colon X \to X \sqcup_L X$ the ball structure induced on $j_k(X)$ coincides with the original ballean structure transported by j_k. Without loss of generality we can assume the ballean $\mathfrak{B}_X$ to be symmetric (Remark 5.1.1). Without loss of generality, fix a point $j_1(x) \in Y$, where $x \in X$, and a radius $r \in P$. Let $j_k(x') \in B_{X\sqcup_L X}(j_1(x),r)$ for some $x' \in X$ and $k=1,2$. If $B_X(x,r) \cap L = \emptyset$, it is trivial to check that $j_1(x) \in B_{X\sqcup_L X}(j_k(x'),r)$, since we have $B_{X\sqcup_L X}(j_1(x),r) = j_1(B_X(x,r))$ and $B_{X\sqcup_L X}(j_k(x'),r) \supseteq j_1(B_X(x',r))$. We consider the case $B_X(x,r) \cap L \neq \emptyset$ in the sequel. Note that one has $\sigma'(B_{X\sqcup_L X}(j_l(z),r)) = B_{X\sqcup_L X}(j_l(z),r)$ when $B_X(z,r) \cap L \neq \emptyset$, for every $l=1,2$, where $z \in X$. Applying p we obtain $x' = p(j_k(x')) \in p(B_Y(j_k(x),r)) = B_X(x,r)$. Hence, so $x \in B_X(x',r)$ by the symmetry of the ball $B_X(x,r)$. Thus, $j_1(x) \in j_1(B_X(x',r)) \subseteq B_{X\sqcup_L X}(j_k(x'),r)$, in case $B_X(x',r) \cap L \neq \emptyset$ or $k=1$. Otherwise, if $B_X(x',r) \cap L = \emptyset$ and $k=2$, we use the fact that $j_1(x) \in B_{X\sqcup_L X}(\sigma'(j_1(x)),r)$ and $\sigma'(j_1(x)) \in j_2(B_X(x',r)) \subseteq B_{X\sqcup_L X}(j_2(x'),r)$. Therefore, $j_1(x) \in B_{X\sqcup_L X}(B_{X\sqcup_L X}(j_2(x'),r),r)$ and we conclude by upper multiplicativity.

So far we have checked that the ball structure $\mathfrak{B}^a_{X\sqcup_L X}$ is a ballean. Since $q(\mathfrak{B}) \prec \mathfrak{B}^a_{X\sqcup_L X}$, in order to conclude we only need to check that $\mathfrak{B}^a_{X\sqcup_L X} \prec \overline{\mathfrak{B}}^q$. As $\overline{\mathfrak{B}}^q$ is the finest coarse structure containing $q(\mathfrak{B})$ (i.e., $q(\mathfrak{B}) \prec \overline{\mathfrak{B}}^q$), this will imply that $\mathfrak{B}^a_{X\sqcup_L X} = \overline{\mathfrak{B}}^q$. In fact, assume that $z \in B_{X\sqcup_L X}(y,r)$ for some $y \in X \sqcup_L X$ and $r \in P$. Assume that $y = q(x)$ and $z = q(x')$ for some

$x, x' \in X \sqcup X$. According to Proposition 4.3.22, it is enough to find a finite chain of points $x_0 = x', x_1, \dots, x_n = x$ in $X \sqcup X$, such that each x_i is either contained in the ball $B_{X\sqcup X}(x_{i+1}, r)$, or $x_i \in R_q[x_{i+1}]$ (i.e., $q(x_i) = q(x_{i+1})$). We can assume without loss of generality that $x = i_1(u) \in i_1(X)$ and $x' = i_k(u') \in i_k(X)$ for $u, u' \in X$ and $k = 1, 2$ (so that $y = j_1(u), z = j_k(u')$). If $k = 1$ we deduce that $u' \in B_X(u, r)$, so $x' \in B_{X\sqcup X}(x, r)$, so we can simply take $n = 1$. If $k = 2$, then $L \cap B_X(x, r) \neq \emptyset$ so there exists $l \in L \cap B_X(u, r)$, consequently, $i_1(l) \in B_{X\sqcup X}(x, r)$. By the symmetry of the balls $u \in B_X(l, r)$. Hence, $\sigma(x) \in B_{X\sqcup X}(i_2(l), r)$. As $x' \in B_{X\sqcup X}(\sigma(x), r)$, and $i_2(l) \in R_q[i_1(l)]$, we can put $n = 4$ and let $x_0 = x'$, $x_1 = \sigma(x)$, $x_2 = i_2(l)$, $x_3 = i_1(l)$, $x_4 = x$ to conclude that

$$x' \in B_{X\sqcup X}(B_{X\sqcup X}(R_q[B_{X\sqcup X}(x, r)], r), r).$$

This concludes the proof of the equality $\mathfrak{B}^a_{X\sqcup_L X} = \overline{\mathfrak{B}}^q$, i.e., $\mathfrak{B}^a_{X\sqcup_L X}$ is the quotient ballean structure on $X \sqcup_L X$. □

Remark 5.3.3. (a) The pair of maps $j_1, j_2 \colon X \to Y = X \sqcup_L X$ associated to the subspace L of X is usually referred to as *cokernel pair* of the inclusion map $m \colon L \to X$ in category theory. In categorical terms, it means that $j_1, j_2 : X \to Y$ is the pushout of the pair $m, m \colon L \to X$ (in other words, it satisfies $j_1 \circ m = j_2 \circ m$ and for every pair of bornologous maps $u_1, u_2 \colon X \to Z$ with $u_1 \circ m = u_2 \circ m$ there exists a unique bornologous map $t : Y \to Z$ such that $u_k = t \circ j_k$ for $k = 1, 2$). Certainly, cokernel pairs exist in **Coarse**, as it is co-complete (being a topological category, by Theorem 4.2.1). The knowledge of its concrete (simple) form described in Theorem 5.3.2, is the relevant issue in this case.

(b) While for a non-empty space X the coproduct $X \sqcup X$ is never connected, the adjunction space $Y = X \sqcup_L X$ is connected precisely when X is connected and $L \neq \emptyset$. This follows from the fact that $X \sqcup_L X = j_1(X) \cup j_2(X)$, both $j_k(X)$ are connected and the union is not disjoint.

The next theorem will provide, among others, more examples showing that the quotient ball structure of a ballean may fail to be a ballean. To this end the quotient map defining the adjunction space, as well as its restrictions, will be used.

Theorem 5.3.4. *For a ballean X and a subballean Y the restriction q_1 of the quotient map $q \colon X \sqcup X \to X \sqcup_Y X$ to $X \sqcup Y$ is weakly uniformly soft. Moreover, the following are equivalent:*

(a) $X = Y \sqcup X \setminus Y$;

(b) the quotient ball structure $q(\mathfrak{B})$ on $X \sqcup_Y X$ is a ballean;

(c) q_1 is uniformly soft.

Proof. It suffices to check that the quotient ball structure of $X \sqcup_Y Y$ coincides with the (ballean) structure of X, then Theorem 4.3.17 will imply that q is weakly uniformly soft. To check this we note that the map $j_1 \colon X \to X \sqcup_Y Y = j_1(X)$ is bijective. Moreover, for every $r \in P$ one has

$$j_1(B_X(x, r)) = B^{q_1}_{j_1(X)}(j_1(x), r) \tag{5.8}$$

This remains true also when $y \in Y$, then $j_1(y) = j_2(y)$, so again (5.8) holds true for $j_1(y) = j_2(y)$. Since these balls define the ball structure of both spaces, our claim is proved.

(a)→(b) To prove that the quotient ball structure on $X \sqcup_Y X$ is a ballean we need to check that it is upper multiplicative. Pick $r, t \in P$ and find a $s \in P$ such that $B_X(B_X(x,r),t) \subseteq B_X(x,s)$ for all $x \in X$. It is enough to show that for every $z \in Z = X \sqcup_Y X$ one has $B^q_Z(B^q_Z(z,r),t) \subseteq B^q_Z(z,s)$. We can assume without loss of generality that $z = j_1(x)$ for some $x \in X$. If $x \in Y$, then $B^q_Z(z,r) = j_1(B_X(x,r)) \cup j_2(B_X(x,r))$. Hence,

$$\begin{aligned} B^q_Z(B^q_Z(z,r),t) &= j_1(B_X(B_X(x,r),t)) \cup j_2(B_X(B_X(x,r),t)) \subseteq \\ &\subseteq j_1(B_X(x,s)) \cup j_2(B_X(x,s)) = B^q_Z(z,s). \end{aligned}$$

In case $x \notin Y$, $B^q_Z(z,r) = j_1(B_X(x,r))$ as $B_X(x,r) \cap Y = \emptyset$. Hence, $B^q_Z(B^q_Z(z,r),t) = B^q_Z(j_1(B_X(x,r)),t)$. Since, our assumption $x \notin Y$ yields $B_X(B_X(x,r),t) \cap Y = \emptyset$, one has

$$B^q_Z(j_1(B_X(x,r)),t) = j_1(B_X(B_X(x,r),t)) \subseteq j_1(B_X(x,s)) \subseteq B^q_Z(z,s).$$

(b)→(a) Assume that there exists $r \in P$ and $y \in Y$, $x \in X \setminus Y$ with $y \in B(x,r)$. Then

$$j_2(x) \in B^q_{X\sqcup_Y X}(B^q_{X\sqcup_Y X}(j_1(x),r),r),$$

but $j_2(x) \notin j_1(X) \supseteq B^q_{X\sqcup_Y X}(j_1(x),s)$ for every $s \in P$, a contradiction.

(a)→(c) To check that q is soft pick an element $z \in Z = X \sqcup Y$. We have to check that

$$R_q[B_Z(R_q[z],r)] \subseteq B_Z(R_q[z],r) \tag{5.9}$$

for every $r \in P$. If $z = (u,k)$ with $u \in X$ and $k = 1,2$, consider two cases. If $u \notin Y$, then necessarily $k = 1$ and $B_X(u,r) \cap Y = \emptyset$. Therefore, $R_q[z] = \{z\}$ and $R_q[B_Z(R_q[z],r)] = i_1(B_X(u,r)) = B_Z(R_q[z],r)$. Hence, (5.9) is proved in this case.

If $u \in Y$, then $R_q[z] = \{i_1(u), i_2(u)\}$, so $B_Z(R_q[z],r) = i_1(B_X(u,r)) \cup i_2(B_X(u,r))$, therefore, $R_q[B_Z(R_q[z],r)] = B_Z(R_q[z],r)$. This proves again (5.9).

(c) → (a) Assume that $y \in Y \cap B_X(x,r)$ for some $r \in P$ and some $x \notin Y$. Then $R_q[x] = \{x\}$. To see that softness at x fails, note that $R_q[i_1(B_X(x,r))] \nsubseteq i_1(B_X(x,s)) \subseteq i_1(X)$, since otherwise for $y \in B_X(x,r)$ one would have

$$i_2(y) \in R_q[i_1(y)] \subseteq R_q[i_1(B_X(x,r))] \subseteq i_1(X),$$

a contradiction. □

The examples provided below show, among others, that none of the implications in (4.11) can be inverted.

Example 5.3.5. (a) Theorem 5.3.4 shows that the quotient ball structure on $X \sqcup_L X$ is not a ballean in general (choose L in such a way that X is not a coproduct of L and $X \setminus L$). Therefore, the map $q\colon X \sqcup X \to X \sqcup_L X$ is not weakly uniformly soft.

(b) Applying Theorem 5.3.4 in the extreme case when $Y = X$ we obtain an example of a uniformly soft map with unbounded fibres. if $(X, \mathcal{E})$ is an unbounded coarse space, then the quotient map $q\colon X \oplus X \to X$ that glues together the two copies of X is soft, but its fibres are not bounded. This example shows also that the first implication in (4.11) cannot be inverted.

(c) Theorem 5.3.4 provides also an example of a weakly uniformly soft map that is not uniformly soft showing that the second implication in (4.11) cannot be inverted (choose L in such a way that X is not a coproduct of L and $X \setminus L$ and consider the weakly uniformly soft map q_1).

(d) Let us see now that the map $q\colon X \sqcup X \to Y = X \sqcup_L X$ is 2-uniformly soft. In conjunction with item (a) this will provide an example witnessing that the last implication in (4.11) cannot be inverted. According to Remark 5.1.3(b), the ball structure $\mathfrak{B}_Y^* = \mathfrak{B}_{\mathcal{E}_Y^*}$ of the quotient Y given by the 'doubled' balls $B_Y^q(B_Y^q(y,r),r)$ $(r \in P)$ is a ballean precisely when the map q is 2-uniformly soft. On the other hand, it is not hard to realize that the ball structure $\mathfrak{B}_Y^*$ is asymorphic to $\mathfrak{B}^q_{X\sqcup_L X}$, shown to be a ball structure in Theorem 5.3.2. Therefore, $\mathfrak{B}_Y^*$ is itself is a ballean, so q is 2-uniformly soft.

5.3.2 Epimorphisms and monomorphisms in the coarse category Coarse/$_\sim$

The morphisms in **Coarse**/$_\sim$ are equivalence classes of morphisms $f\colon X \to Y$ in **Coarse**, nevertheless, we shall often speak of properties of morphisms of **Coarse**/$_\sim$ having in mind some specific representative f in **Coarse** of the equivalence class $[f]$. In some cases, that property is available regardless of the choice of the representative f (see Remark 3.4.2), in other cases this may fail (Remark 5.3.10).

Theorem 5.3.6. *Let $\mathfrak{B}_X = (X, P_X, B_X)$ be a ballean and L be a subset of X. Then the following are equivalent.*
(a) L is large;
(b) every pair of bornologous maps $f, g\colon X \to Y$ with $f|_L \sim g|_L$ are close;
(c) every pair of bornologous maps $f, g\colon X \to Y$ with $f|_L = g|_L$ are close.
Consequently, a morphism $f\colon X \to Y$ in **Coarse** *is an epimorphism in* **Coarse**/$_\sim$ *if and only if $f(X)$ is large in Y.*

Proof. (a)→(b) Assume that L is large in X and let $f, g\colon X \to Y$ be bornologous maps to a ballean $\mathfrak{B}_Y = (Y, P_Y, B_Y)$ with $f|_L \sim g|_L$. Pick $r \in P_X$ such that $B_X(L, r) = X$. Since the maps f, g are bornologous, there exist $s \in P_Y$ be such that

$$f(B_X(y,r)) \subseteq B_Y(f(y),s) \text{ and } g(B_X(y,r)) \subseteq B_Y(g(y),s) \text{ for all } y \in X. \tag{5.10}$$

Since $f|_L \sim g|_L$, there exists $t \in P_Y$ such that $g(l) \in B_Y(f(l),t)$ for every $l \in L$. Let $u \in P_Y$ be a radius such that, for every $y \in Y$, $B_Y(B_Y(B_Y(x,s),t),s) \subseteq B_Y(x,u)$.

Pick arbitrarily $x \in X$. As L is large, one can find $l \in L$ such that $x \in B_X(l,r)$. Applying (5.10) to $y = l$ we deduce that $f(x) \in f(B_X(l,r)) \subseteq B_Y(f(l),s)$ and $g(x) \in f(B_X(l,r)) \subseteq B_Y(f(l),s)$. Hence,

$$\begin{aligned} g(x) &\in B_Y(g(l),s) \subseteq B_Y(B_Y(f(l),t),s) \subseteq \\ &\subseteq B_Y(B_Y(B_Y(f(x),s),t),s) \subseteq B_Y(f(x),u), \end{aligned}$$

and so f and g are close.

(b)→(c) This implication is trivial.

(c)→(a) Consider the canonical maps $j_k\colon X \to X \sqcup_L X$, for $k = 1, 2$, associated to the adjunction space $Y = X \sqcup_L X$. As $j_1|_L = j_2|_L$, our hypothesis implies that j_1 and j_2 are close. Let this be witnessed by $r \in P_X$. Now we show that $X = B_X(L, r)$. Since, for every $x \in X$, $j_2(x) \in B_Y(j_2(x), r)$, (5.7) implies that $B_X(x, r) \cap L \neq \emptyset$, and thus $x \in B_X(L, r)$. □

Theorem 5.3.7. *Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces and $h\colon X \to Y$ a bornologous map between them. Then the following are equivalent:*

(a) h is a coarse embedding, i.e., for every $E \in \mathcal{E}_Y$, $(h \times h)^{-1}(E) \in \mathcal{E}_X$;

(b) for every coarse space $(Z, \mathcal{E}_Z)$ and every pair of bornologous maps $f, g\colon Z \to X$, if $h \circ f \sim h \circ g$, then $f \sim g$.

Consequently, a morphism $h\colon X \to Y$ in **Coarse** *is a monomorphism in* **Coarse**/$_\sim$ *if and only if h is a coarse embedding.*

Proof. (a)→(b) Assume that $f, g\colon Z \to X$ are bornologous maps with $h \circ f \sim h \circ g$. To establish $f \sim g$ we need to check that $M = \{(f(z), g(z)) \mid z \in Z\}$ belongs to $\mathcal{E}_X$. As $h \circ f \sim h \circ g$, one has $(h \times h)(M) = \{(h(f(z)), h(g(z))) \mid z \in Z\} \in \mathcal{E}_Y$. Consequently, $M \subseteq (h \times h)^{-1}((h \times h)(M)) \in \mathcal{E}_X$.

(b)→(a) Suppose for a contradiction that h is not a coarse embedding. This means that there exists an entourage $E \in \mathcal{E}_Y$ such that $E' = (h \times h)^{-1}(E) \notin \mathcal{E}_X$.

Let $Z = E'$ endowed with the discrete coarse structure $\mathcal{E}_Z = \{\Delta_Z\}$. Consider the maps $p_1, p_2\colon Z \to X$ defined by $p_1\colon (x, y) \mapsto x$ and $p_2\colon (x, y) \mapsto y$. These maps are bornologous, because $(Z, \mathcal{E}_Z)$ is discrete (Example 3.1.12). Moreover, $\{(p_1(z), p_2(z)) \mid z \in Z\} = E' \notin \mathcal{E}_X$. This means that p_1 and p_2 are not close.

On the other hand,

$$\begin{aligned}\{((h \circ p_1)(z), (h \circ p_2)(z)) \mid z \in Z\} &= \{(h(p_1(z)), h(p_2(z))) \mid z \in Z\} = \\ &= \{(h \times h)(e) \mid e \in E'\} \subseteq E \in \mathcal{E}_Y,\end{aligned}$$

and so $\{(h \circ p_1)(z)), (h \circ p_2)(z))) \mid z \in Z\} \in \mathcal{E}_Y$. Therefore, $h \circ p_1 \sim h \circ p_2$. This contradicts our hypothesis (b).

As in the previous theorem, the last assertion follows from Remark 3.4.2. □

In particular Theorem 5.3.6 shows that morphisms with large image are epimorphisms in **Coarse**/$_\sim$, while Theorem 5.3.7 implies that the monomorphisms are the coarse embeddings. If we apply Theorem 3.4.6, then we can deduce that the category **Coarse**/$_\sim$ is balanced.

Corollary 5.3.8. *Let $f\colon X \to Y$ a morphism in the category* **Coarse**/$_\sim$*. Then f is a bimorphism if and only if it is an isomorphism. Hence, the category* **Coarse**/$_\sim$ *is balanced.*

Stability of epimorphisms under pullback is an important issue in category theory. This is why we are interested to determine here those morphisms $f\colon X \to Y$ in **Coarse** such that $[f]$ is an epimorphism in **Coarse**/$_\sim$ and for every morphism $e\colon Z \to Y$ in **Coarse** such that $[e]$ is an epimorphisms in **Coarse**/$_\sim$ the class $[u]$ of the pullback $u\colon P \to X$ in (4.5) is an epimorphism in **Coarse**. We shall shortly refer to this property in the sequel by simply saying 'epimorphisms are preserved under taking pullback along f'. As we shall see, this property is not invariant under $\sim$ (see Remark 5.3.10).

A morphism $f\colon X\to Y$ in the category **Coarse** is said to be *$\mathcal{LA}$-reflecting*, if $f^{-1}(L)$ is large in X for every large set L of Y. The properties of maps to preserve or to reflect size properties (for example largeness) is studied in [64] (more definitions from that paper are recalled in §11.1).

A subset A of a ballean X is called *extra-large* if, for every large subset L of X, the intersection $A\cap L$ is still large in X ([144]). The relevance of this notion from categorical point of view is revealed in the following corollary.

Corollary 5.3.9. *Let $f\colon X\to Y$ a representative of an epimorphism in the category* **Coarse**/$_\sim$*. Then the following are equivalent:*

(a) the epimorphisms are preserved under taking pullback along f;

(b) the co-restriction map $f\colon X\to f(X)$ *is $\mathcal{LA}$-reflecting and $f(X)$ is extra-large in Y.*

Proof. We shall simplify the proof by reducing the argument to the case of epimorphisms that are simply inclusions. To this end consider a pullback diagram (4.5), put $Z_1=f^{-1}(e(Z))$ and let $e_1\colon Z_1\hookrightarrow X$ be the inclusion map. Let us see next that

$$Z_1=u(P). \tag{5.11}$$

If $u(p)\in u(P)$ for some $p\in P$, then obviously $f(u(p))=e(v(p))\in e(X)$, so $u(p)\in Z_1$. On the other hand, if $x\in Z_1$, then $f(x)=e(z)$ for some $z\in Z$, hence $(x,z)\in P$ (see the construction of P as an equalizer in §3). Then $x=u(x,z)\in u(P)$. This proves (12.7).

Let $j\colon e(Z)\hookrightarrow Y$ be the inclusion map. Then one can easily see that

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle e_1}\uparrow & & \uparrow{\scriptstyle j} \\ Z_1 & \xrightarrow{\ f|_Z\ } & e(Z). \end{array} \tag{5.12}$$

is a pullback diagram.

It easily follows from Theorem 5.3.6 that

- u is an epimorphisms if and only if $e_1\colon u(P)=Z_1\hookrightarrow X$ is an epimorphism.
- e is an epimorphism precisely when j is an epimorphism.

This makes it clear that the epimorphisms are preserved under taking pullback along f precisely when pullbacks along f of epimorphisms that are inclusions in Y are preserved and the general pullback diagram (4.5) can be replaced by the pullback diagram (5.12), where the vertical arrows are inclusions.

(a)→(b) Assume that epimorphisms are preserved under taking pullback along f. To check that $f(X)$ is extra-large in Y pick a large subset L of Y. Then the inclusion map $j\colon L\hookrightarrow Y$ is an epimorphism in **Coarse**/$_\sim$ by Theorem 5.3.6. Hence, the pullback $j_1\colon f^{-1}(L)\to X$ must be an epimorphism on **Coarse**/$_\sim$. Hence, $f^{-1}(L)$ is large in X by Theorem 5.3.6. It easily follows from the definition of largeness (see [144, Lemma 11.3]), that $f(f^{-1}(L))=f(X)\cap L$ is large in $f(X)$. As $f(X)$ is large in Y (again by Theorem 5.3.6, as f is an epimorphism), we deduce that $f(X)\cap L$ is large in Y. This proves that $f(X)$ is extra-large in Y.

The fact that $f\colon X\to f(X)$ is $\mathcal{LA}$-reflecting follows directly from the definitions.

(b)→(a) Suppose that $f(X)$ is extra-large in Y and let $e\colon Z \to Y$ be an epimorphism. Let us prove that $e_1 = f^{-1}(e)\colon Z_1 \to X$ is an epimorphism in **Coarse**/$_\sim$. By Theorem 5.3.6, $L = e(Z)$ is large in Y. Then $L \cap f(X)$ is large in Y. Consequently, $L \cap f(X)$ is large in $f(X)$. Hence, $f^{-1}(L) = f^{-1}(L \cap f(X))$ is large in X, by hypothesis. As $f^{-1}(L) = e_1(Z_1)$ is large in X, by Theorem 5.3.6 we conclude that e_1 is an epimorphism in **Coarse**/$_\sim$. □

Remark 5.3.10. Unlike Theorems 5.3.6 and 5.3.7, where the characterizing property of the morphism in **Coarse** is available *for all* representatives of the $\sim$-equivalence class (see Remark 3.4.2), the property of item (b) from the above corollary fails to be invariant under closeness. Indeed, the floor map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ does not satisfy (b), as $\mathbb{Z} = f(\mathbb{R})$ is not extra-large in $\mathbb{R}$. Nevertheless, $f \sim id_{\mathbb{R}}$ and $id_{\mathbb{R}}$ obviously satisfies (b). This example shows that the property (a) is also 'fragile' in this sense. This is explained by the fact that while epimorphisms are taken in **Coarse**/$_\sim$, the pullbacks are taken in **Coarse**.

One can prove that a surjective map that is either effectively proper or uniformly soft is $\mathcal{LA}$-reflecting, while a surjective weakly uniformly soft map need not be $\mathcal{LA}$-reflecting ([64]). This gives the following corollary.

Corollary 5.3.11. *Let $f\colon X \to Y$ a morphism in the category* **Coarse** *such that $f(X)$ is extra-large in Y. If the co-restriction map $f\colon X \to f(X)$ is uniformly soft, then the epimorphisms are preserved under taking pullback along f.*

5.4 Cowellpoweredness and wellpoweredness of Coarse/$_\sim$

In Theorem 5.3.6 the epimorphisms of **Coarse**/$_\sim$ have been completely characterised. Hence, the question whether **Coarse**/$_\sim$ is an epireflective category or not naturally arises.

Before discussing the cowellpoweredness of **Coarse**/$_\sim$, let us start this section with an important observation. Since **Coarse**/$_\sim$ is a balanced category (Corollary 5.3.8), all monomorphisms are extremal and so the two, a priori different, classes $\mathrm{Mono}_{\mathbf{Coarse}/_\sim}$ and $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$ coincide.

5.4.1 Products, subobjects, and wellpoweredness of Coarse/$_\sim$

Let $\mathcal{X}$ be a category and $\{f_i\colon X \to X_i\}_{i\in I}$ be a source in $\mathcal{X}$.

(a) $\{f_i\}_i$ is a *mono-source* if, for every pair of morphisms $g, h\colon Y \to X$, $g = h$ whenever $f_i \circ g = f_i \circ h$ for every $i \in I$.
(b) If $\approx$ is a congruence, $\{f_i\}_i$ is a $\approx$*-mono-source* if, for every pair of morphisms $g, h\colon Y \to X$, $g \approx h$ whenever $f_i \circ g \approx f_i \circ h$ for every $i \in I$.

Those two notions relate to each other: $\{f_i\}_{i\in I}$ is a $\approx$-mono-source in $\mathcal{X}$ if and only if $\{[f_i]_\approx\}_{i\in I}$ is a mono-source in $\mathcal{X}/_\sim$.

Example 5.4.1. Consider the category **Coarse** and the closeness relation $\sim$. Then every product $\{p_i\colon X \to X_i\}_{i\in I}$ of **Coarse** is a $\sim$-mono-source. Let $f, g\colon Y \to X$ be two morphisms of **Coarse** such that $p_i \circ f \sim p_i \circ g$ for every $i \in I$ and $r_i \in P_{X_i}$ be a radius such that $p_i \circ f \sim_{r_i} p_i \circ g$, for every index

$i \in I$. Then, by the definition of the product structure, for every $y \in Y$, $g(y) \in B_X(f(y),(r_i)_i)$ and so $f \sim g$.

Proposition 5.4.2. *Let $\mathcal{X}$ be a category and $\approx$ be a congruence. Suppose that $\mathcal{X}$ has (finite) products and they are $\approx$-mono-sources. Then $\mathcal{X}/_\sim$ has (finite) products.*

Proof. Let $\{X_i\}_{i\in I}$ be an indexed family of objects of $\mathcal{X}/_\sim$. Since they are objects of $\mathcal{X}$ too, there exists their product $X = \prod_i X_i$ and the projections $p_i\colon X \to X_i$ in $\mathcal{X}$. We claim that $\{[p_i]_\sim\colon X \to X_i\}_{i\in I}$ is the product of $\{X_i\}_{i\in I}$ in $\mathcal{X}/_\sim$. Suppose that there exists a source of morphisms $[f_i]_\approx\colon Y \to X_i$, where $i \in I$, of $\mathcal{X}/_\sim$. Then by the universal property of the product of $\mathcal{X}$ there exists a unique morphism $f\colon Y \to X$ such that $f_i = p_i \circ f$, for every $i \in I$. Thus, for every $g_i \in [f_i]_\approx$, $g_i \approx p_i \circ f$ and so $[f_i]_\approx = [p_i]_\approx \circ [f]_\approx$. One should prove that $[f]_\approx$ is the unique morphism of $\mathcal{X}/_\sim$ such that $[p_i]_\approx \circ [f]_\approx = [f_i]_\approx$. Let $[g]_\approx$ be a morphism of $\mathcal{X}/_\sim$ such that $[p_i]_\approx \circ [g]_\approx = [f_i]_\approx$, for every $i \in I$. Then $p_i \circ g \approx f_i \approx p_i \circ f$ for every index $i \in I$ and so $g \approx f$, since $\{p_i\colon X \to X_i\}_{i\in I}$ is a $\approx$-mono-source. □

Proposition 5.4.2 and Example 5.4.1 imply the following result. Some applications of this corollary will be shown in the next subsection, once we will have introduced all the needed notions.

Corollary 5.4.3. Coarse/$_\sim$ *has arbitrary products.*

A *skeleton* $\mathcal{M}_0$ of a class of monomorphism $\mathcal{M}$ is a subclass of $\mathcal{M}$ such that, for every object X and every morphism $m \in \mathcal{M}/X$, there exists $m_0 \in \mathcal{M}_0/X$ such that $m \cong m_0$ (see (5.5) for the equivalence relation).

Proposition 5.4.4. *The family $\mathcal{M}_0$ of all the elements $[i]_\sim$, where i varies in* $\mathrm{ExtMono}_{\mathbf{Coarse}}$*, is a skeleton of* $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$.

Proof. Let $[m]_\sim\colon M \to X$ be a monomorphism of **Coarse**/$_\sim$. Since m is a coarse embedding, m has uniformly bounded fibres. By using the axiom of choice, we define a subobject M_0 of M in **Coarse** by choosing just one element for every fibre of m. If $i\colon M_0 \to M$ is the inclusion, $[i]_\sim$ is both a monomorphism and an epimorphism of **Coarse**/$_\sim$, since m has uniformly bounded fibres. Hence $[i]_\sim$ is an isomorphism. Define $m_0 = m \circ i$, which is bornologous and injective, and note that $[m_0] = [m]_\sim \circ [i]_\sim$ and so $[m_0]_\sim \leq [m]_\sim$. Conversely, if j is a coarse inverse of i, then $[j]_\sim$ shows that $[m]_\sim \leq [m_0]_\sim$. □

If X is an object of **Coarse**/$_\sim$, a *subobject of* X is an equivalence class of monomorphism of $(\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim})/X$ under the equivalence relation $\cong$. By Proposition 5.4.4, for every subobject $[[m]_\sim]_\cong$ of an object X of **Coarse**, there exists a representative $i \in (\mathrm{ExtMono}_{\mathbf{Coarse}})/X$ such that $[m]_\sim \cong [i]_\sim$.

Another consequence of Proposition 5.4.4 is the fact that **Coarse**/$_\sim$ is *wellpowered* (Corollary 5.4.5). Wellpoweredness is the dual notion of cowellpoweredness. A category $\mathcal{X}$ is $\mathcal{M}$*-wellpowered*, where $\mathcal{M}$ is a family of morphisms of $\mathcal{X}$, if $\mathcal{M}/X$ has a set of representatives with respect to isomorphisms. $\mathcal{X}$ is *wellpowered*, if it is $\mathrm{Mono}_{\mathcal{X}}$-wellpowered.

Axiom of choice implies that a category $\mathcal{X}$ is wellpowered if and only if, for every sink of monomorphisms $\{m_i\colon M_i \to X\}_{i\in I}$, where I can be even a proper class, there exists a set $J \subseteq I$ with the property that, for every $i \in I$, there exists $j \in J$ such that $m_i \cong m_j$.

Corollary 5.4.5. *The category* **Coarse**$/_\sim$ *is wellpowered.*

Proof. Let $\mathcal{M} = \{[m_i]_\sim \colon M_i \to X\}_{i \in I}$ be a sink of monomorphisms of **Coarse**$/_\sim$. Proposition 5.4.4 states that $\mathcal{M}_0$ is a skeleton of $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$ and then, in particular, $\mathcal{M}_0/X$ is a skeleton of $\mathcal{M}$ (without loss of generality, one can assume that $\mathcal{M}_0/X \subseteq \mathcal{M}$). The conclusion follows from the fact that $\mathcal{M}_0/X$ is a set since every monomorphism $\mathcal{M}_0/X$ has a different injective coarse embedding into X as a representative. □

5.4.2 Coarse$/_\sim$ has neither pullbacks along coarse embeddings nor equalizers

The closure operators of **Coarse** have been widely studied in §5.2. It is desirable to apply those tools also in the quotient category **Coarse**$/_\sim$. In particular those closure operators could be a powerful instrument in order to tackle the problem of cowellpoweredness of **Coarse**$/_\sim$. Unfortunately the category **Coarse**$/_\sim$ is a hostile environment to closure operators, as we will prove in this subsection.

Since **Coarse**$/_\sim$ has products, the existence of all equalizers is equivalent to the existence of all pullbacks ([1, Propositions 11.11, 11.14]). In [73], the authors give an explicit example of a pair of morphisms for which the pullback does not exists, hence, not all equalizers exist either.

If $\mathcal{M}$ is a class of monomorphism of a category $\mathcal{X}$, we say that $\mathcal{X}$ *has* $\mathcal{M}$*-pullbacks* if, for every morphism $f \colon X \to Y$ of $\mathcal{X}$ and every $n \in \mathcal{M}/Y$, a pullback diagram

$$\begin{array}{ccc} M & \xrightarrow{f'} & N \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow{f} & Y \end{array}$$

exists in $\mathcal{X}$ with $m \in \mathcal{M}/X$.

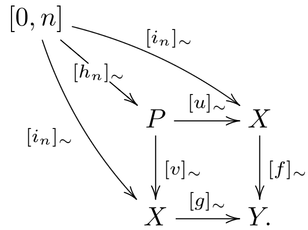
As already mentioned in §5.2, $\mathcal{M}$-pullbacks are a key tool in the standard way of introducing closure operators in a category. In Examples 5.4.6(a) and 5.4.7(a), we provide two pairs of morphisms of **Coarse**$/_\sim$ which do not have $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$-pullbacks. With some effort, we show in Examples 5.4.6(b) and 5.4.7(b) that those pairs do not have equalizers either.

Example 5.4.6. Consider two balleans X and Y defined as follows: X is the discrete ballean over $\mathbb{R}_{\geq 0}$ and Y is the product ballean of X and $\mathbb{R}_{\geq 0}$ with the Euclidean metric ballean structure. Let $\mathbb{N}$ be the radii set of Y. Define two maps $f, g \colon X \to Y$, such that, for every $x \geq 0$, $f(x) = (x, 0)$ and $g(x) = (x, x)$, which are coarse embeddings by Remark 3.1.12. Since $f \not\sim g$, $[f]_\sim \neq [g]_\sim$.

(a) Let us suppose by contradiction that the pair of morphisms $[f]_\sim$ and $[g]_\sim$ has an $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$-pullback: a triple $(P, [u]_\sim, [v]_\sim)$, where $[u]_\sim$ is a monomorphism, such that $[f]_\sim \circ [u]_\sim = [g]_\sim \circ [v]_\sim$. Since also $[f]_\sim$ is a monomorphism, then $[f]_\sim \circ [u]_\sim$ and so $[v]_\sim$ are monomorphisms.

Without loss of generality one can assume that u is an injection. In fact, by Proposition 5.4.4, there exists a monomorphism $u'\colon P' \to X$ of **Coarse** such that $[u']_\sim \cong [u]_\sim$. Let $[h]_\sim\colon P' \to P$ be an isomorphism of **Coarse**/$_\sim$ such that $[u']_\sim = [u]_\sim \circ [h]_\sim$. By defining $v' = v \circ h$, one has a pullback $(P', [u']_\sim, [v']_\sim)$ with the desired properties.

For every $n \in \mathbb{N}$, the inclusion $i_n\colon [0,n] \to X$, where $[0,n]$ is endowed with the discrete ballean structure, is a monomorphism of **Coarse** such that $f \circ i_n \sim g \circ i_n$ and so, by the universal property of the pullback, there exists a unique arrow $[h_n]_\sim\colon [0,n] \to P$ such that both the triangles in the following diagram commute:

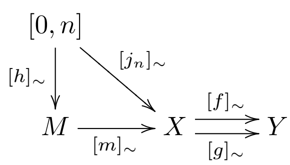


Since $[i_n]_\sim = [u]_\sim \circ [h_n]_\sim$ and $[i_n]_\sim$ is a monomorphism, then also $[h_n]_\sim$ is a monomorphism and so h_n is an injection, by applying Remark 3.1.12. Since also u is an injective coarse embedding, P can be identified with a subspace of X which contains a copy of $[0,n]$, for every $n \in \mathbb{N}$. Hence, $P = X$ and v, which is a coarse embedding, is injective by Remark 3.1.12. Moreover, since X is discrete and $i_n \sim v \circ h_n$ and $i_n \sim u \circ h_n$, Remark 3.4.1 implies that, for every $n \in \mathbb{N}$, $i_n = v \circ h_n$ and $i_n = u \circ h_n$.

Finally we show that $f \circ u \not\sim g \circ v$, which is a contradiction. Let $m \in P_Y$ be an arbitrary radius, one has to prove that $f \circ u \not\sim_m g \circ v$. The point $h_{m+1}(m+1) \in P$ is such that $f \circ u(h_{m+1}(m+1)) = f(m+1) = (m+1, 0)$ and $g \circ v(h_{m+1}(m+1)) = g(m+1) = (m+1, m+1)$ and so $f \circ u \not\sim_m g \circ v$.

(b) Let us suppose that there exists the equalizer of $[f]_\sim$ and $[g]_\sim$ and let $[m]_\sim$ be a representative of the subobject $[\mathrm{eq}([f]_\sim, [g]_\sim)]_\cong$ such that $m\colon M \hookrightarrow X$ is an inclusion (Proposition 5.4.4). In particular the ballean structure on M is the one inherited by X.

For every $n \in \mathbb{N}$, consider the inclusion $j_n\colon [0,n] \hookrightarrow X$, which is a coarse embedding, provided that $[0,n]$ is endowed with the discrete ballean structure. Then $[f]_\sim \circ [j_n]_\sim = [g]_\sim \circ [j_n]_\sim$. Hence, by the universal property of the equalizer, there exists an arrow $[h]_\sim$ such that the diagram



commutes.

We recall that the map $[h]_\sim$ is a monomorphism of **Coarse**/$_\sim$ and so h is a coarse embedding. Since $[0,n]$ is discrete, h is injective by Remark 3.1.12. Hence, for every $n \in \mathbb{N}$, M contains $[0,n]$ as a subspace and M can be assumed contained in X, thus $M = X$ and $m = id_X$.

The fact that $f = f|_M \not\sim g|_M = g$ is a contradiction, since M is taken to be the equalizer of $[f]_\sim$ and $[g]_\sim$.

Now we modify Example 5.4.6 to the Example 5.4.7 having nicer and more natural properties (e.g., X and Y are connected). The proofs will only be sketched.

Example 5.4.7. Let us define two balleans X and Y as follows: X is the *ideal ballean* over $\mathbb{R}_{\geq 0}$ associated to the finitary ideal (i.e., the ballean associated to the ideal coarse structure over $\mathbb{R}_{\geq 0}$ induced by the finitary ideal) and Y is the product ballean between X and $\mathbb{R}_{\geq 0}$ with the Euclidean metric ballean structure. We define two maps $f, g\colon X \to Y$, such that, for every $x \geq 0$, $f(x) = (x, 0)$ and $g(x) = (x, x)$. The fact that both f and g are coarse embeddings can be easily verified. Since $f \not\sim g$, $[f]_\sim \neq [g]_\sim$.

(a) Let us suppose by contradiction the existence of an $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$-pullback $(P, [u]_\sim, [v]_\sim)$ of the pair $[f]_\sim$ and $[g]_\sim$, where $[u]_\sim$ is a monomorphism. Hence, also $[v]_\sim$ is a monomorphism. As in Example 5.4.6(a), u can be considered injective, so that P is a subobject of X and, for every $n \in \mathbb{N}$, there exists monomorphism $[h_n]_\sim\colon [0, n] \to P$, where $[0, n]$ is a subobject of X. Then, for every $n \in \mathbb{N}$ there exists a finite subset F_n of X such that $h_n|_{[0,n]\setminus F_n}$ is injective and let F be another finite subset of P such that $v|_{P\setminus F}$ is injective. Hence, there exists a countable subset N of $\mathbb{R}_{\geq 0}$ such that $\mathbb{R}_{\geq 0} \setminus N \subseteq P$ and $v|_{\mathbb{R}_{\geq 0}\setminus N}$ is injective. Finally, for every radius $(K, m) \in [X]^{<\omega} \times \mathbb{N}$ of Y, there exists a point of $\mathbb{R}_{\geq 0} \setminus N \subseteq P$ which witnesses that $f \not\sim_{(K,m)} g$.

(b) Let, by contradiction, $[m]_\sim\colon M \to X$ be a representative of the equalizer of $[f]_\sim$ and $[g]_\sim$ such that m is an injection. By following the same steps of Example 5.4.6(b), it is possible to prove that, for every $n \in \mathbb{N}$, there exists a coarse embedding $h_n\colon [0, n] \to M$, where $[0, n]$ inherits the ballean structure from X, and so there exists a finite subset F_n of $[0, n]$ such that $h_n|_{[0,n]\setminus F_n}$ is injective. Hence,

$$M \supseteq \bigcup_{n\in\mathbb{N}} \big([0, n] \setminus F_n\big) \supseteq \mathbb{R}_{\geq 0} \setminus N,$$

where N is a countable subset of $\mathbb{R}_{\geq 0}$. Hence, for every $m \in \mathbb{N}$ and every finite subset $F \subseteq X$, there exists a point $x_m \in \mathbb{R}_{\geq 0} \setminus (N \cup F)$ such that $x_m > m$ and so x_m witnesses that $f \not\sim_{(F,m)} g$ and this is a contradiction.

Since every pair of bounded balleans are isomorphic in **Coarse**/$_\sim$, it is not hard to prove that the epireflective subcategory **Bounded**/$_\sim$ of **Coarse**/$_\sim$ has equalizers, pullbacks and $\mathrm{ExtMono}_{\mathbf{Bounded}/_\sim}$-pullbacks.

Question 5.4.8. *Which (epireflective) subcategories $\mathcal{Y}$ of* **Coarse**/$_\sim$ *have equalizers, pullbacks or* $\mathrm{ExtMono}_{\mathcal{Y}}$*-pullbacks?*

The objects of a subcategory $\mathcal{Y}$ of **Coarse**/$_\sim$ which has equalizer, pullbacks or $\mathrm{ExtMono}_{\mathcal{Y}}$-pullbacks have to be connected (Example 5.4.6). However the subcategory of all connected spaces has not this property (Example 5.4.7) and so $\mathcal{Y}$ has to be a proper subcategory of **Connected**/$_\sim$, i.e., the quotient category of **Connected** under the closeness relation, which is a subcategory of **Coarse**/$_\sim$.

5.4.3 Cowellpoweredness of a quotient category

As we have already mentioned in the previous subsection, since **Coarse**/$_\sim$ does not admit all the $\mathrm{ExtMono}_{\mathbf{Coarse}/_\sim}$-pullbacks, we cannot use regular closure operators to investigate cowellpoweredness of **Coarse**/$_\sim$. Hence, we have to follow a different path in order to answer this question.

Lemma 5.4.9. *Let $(\mathcal{X}, U)$ be a topological category. Let us suppose that $\mathcal{F}$ is a class of objects of $\mathcal{X}$ such that UX and UY are isomorphic in* **Set**, *for every $X, Y \in \mathcal{F}$. Then there exists a set $\mathcal{F}'$ of objects of $\mathcal{F}$ such that, for every $X \in \mathcal{F}$ there exists an element $F_X \in \mathcal{F}'$ which is isomorphic to X in $\mathcal{X}$.*

Proof. Let $\mathcal{F}$ be a class of objects of $\mathcal{X}$ which satisfies the hypothesis and X be an object of this class. We denote $A = UX$. Since U has small fibres, the class $\mathcal{F}'$ of all the objects Y of $\mathcal{X}$ such that $UY = A$ is a set. Without loss of generality one can assume $\mathcal{F}' \subseteq \mathcal{F}$. Then $\mathcal{F}'$ has the desired property. In fact, let Z be an arbitrary object of $\mathcal{F}$ and let $h\colon UZ \to A$ be an isomorphism of **Set**. Since U is transportable, h can be lifted to an isomorphism $\overline{h}\colon Z \to W$ of $\mathcal{X}$, where $W \in \mathcal{F}'$. □

Let $\mathcal{X}$ be a category, $\mathcal{M}$ be a subclass of monomorphisms of $\mathcal{X}$ and $\approx$ be a congruence. Then $\approx$ is said to be $\mathcal{M}$-*to*-Mono if, for every $m \in \mathcal{M}$, $[m]_\approx$ is a monomorphism of $\mathcal{X}/_\sim$. The closeness relation $\sim$ is $\mathrm{ExtMono}_{\mathbf{Coarse}}$-to-Mono, while Example 5.4.10 shows that a congruence in a category $\mathcal{X}$ need not be $\mathrm{ExtMono}_{\mathcal{X}}$-to-Mono in general.

Example 5.4.10. There exists a topological category $\mathcal{X}$ and a congruence $\approx$, which is not $\mathrm{ExtMono}_{\mathcal{X}}$-to-Mono.

Let us take $\mathcal{X} = \mathbf{Top}$ and as $\mathcal{X}/_{\sim_h}$ the homotopical category **hTop**, namely the quotient of $\mathcal{X}$ obtained via the homotopy equivalence relation $\sim_h$. Let m be the inclusion of the circle $\mathbb{S}$ into the unit ball B, which is an extremal monomorphism of **Top**. On the other hand, for the maps $f = id_{\mathbb{S}}$ and $g = *$ (any constant self-map of $\mathbb{S}$) one certainly has $f \not\sim_h g$, while $m \circ f \sim_h m \circ g$ (as any pair of maps with target B are homotopic).

Proposition 5.4.11. *Let $(\mathcal{X}, U)$ be a topological category and $\approx$ be a congruence of $\mathcal{X}$. Suppose that $\mathcal{X}/_\sim$ is balanced and $\approx$ is* $\mathrm{ExtMono}_{\mathcal{X}}$*-to-*Mono. *Then $\mathcal{X}/_\sim$ is cowellpowered.*

Proof. Let $\{[e_i]_\approx\colon A \to X_i\}_{i \in I}$ be a source of epimorphisms of $\mathcal{X}/_\sim$ and, for every $i \in I$, let

$$\begin{array}{ccc} A & \xrightarrow{e_i} & X_i \\ {\scriptstyle e'_i}\big\downarrow & \nearrow{\scriptstyle m_i} & \\ M_i & & \end{array}$$

be an $(\mathrm{Epi}_{\mathcal{X}}, \mathrm{ExtMono}_{\mathcal{X}})$-factorization (see Appendix A), which exists since $\mathcal{X}$ is topological. In particular, $[m_i]_\approx$ is a monomorphism of $\mathcal{X}/_\sim$. Moreover, since $[e_i]_\approx = [m_i]_\approx \circ [e'_i]_\approx$ is an epimorphism, $[m_i]_\approx$ is an epimorphism and thus an isomorphism, by the hypothesis.

Let us divide the class $\mathcal{F} = \{M_i\}_{i \in I}$ into equivalence classes of isomorphism of **Set**. Since the epimorphisms of $\mathcal{X}$ are surjective, there is only a set $\{\mathcal{F}_\lambda \mid$

$\lambda \le \kappa\}$ of such classes, where κ is a cardinal. Lemma 5.4.9 implies that, for every $\lambda \le \kappa$, there exists a set $\mathcal{F}'_\lambda \subseteq \mathcal{F}_\lambda$ such that, for every $Z \in \mathcal{F}_\lambda$, there exists an element $F_Z \in \mathcal{F}'_\lambda$ which is isomorphic to Z in $\mathcal{X}$. We define two sets: $\mathcal{F}' = \bigcup_{\lambda \le \kappa} \mathcal{F}'_\lambda$ and $I' = \{i \in I \mid M_i \in \mathcal{F}'\}$.

Finally, for every $i \in I$ there exists $j \in I'$ such that M_i and M_j are isomorphic in $\mathcal{X}$ and then, in particular, M_i and M_j are isomorphic in $\mathcal{X}/_\sim$. Since $[m_i]_\approx$ and $[m_j]_\approx$ are isomorphisms of $\mathcal{X}/_\sim$, X_i and X_j are isomorphic in $\mathcal{X}/_\sim$. □

Corollary 5.4.12. *The category* **Coarse**/$_\sim$ *is cowellpowered.*

Proof. The result follows from Proposition 5.4.11, since the extremal monomorphisms of **Coarse** are injective coarse embeddings and their equivalence classes are monomorphisms of **Coarse**/$_\sim$ (Theorem 5.3.7). □

Remark 5.4.13. By a careful look to the proofs of both Lemma 5.4.9 and Proposition 5.4.11, the reader may notice that the cowellpoweredness of $\mathcal{X}/_\sim$ holds also under weaker hypotheses. Namely, one can suppose that $(\mathcal{X}, U)$ is a concrete category and $\approx$ is a congruence of $\mathcal{X}$ such that U is transportable, the fibres of U are small, $\mathcal{X}/_\sim$ is balanced and every morphism of $\mathcal{X}$ has a $(\mathcal{E}, \mathcal{M})$-factorization, where $\mathcal{E}$ is a class of morphisms with the property that $\mathcal{X}$ is $\mathcal{E}$-cowellpowered and $\mathcal{M}$ is a class of monomorphisms of $\mathcal{X}$ such that $\approx$ is $\mathcal{M}$-to-Mono. Then $\mathcal{X}/_\sim$ is cowellpowered.

Those hypotheses are probably not the weakest sufficient set of assumptions. However, we believe that this approach is worth mentioning.

To the best of our knowledge the following question seems to be open.

Question 5.4.14. *Let* $\sim_h$ *be the homotopy equivalence on* **Top** *and define the* homotopy category **hTop** *to be the quotient category* **Top**/$_{\sim_\mathbf{h}}$. *Is* **hTop** *cowellpowered?*

There are obstacles to apply the same machinery that led us to Corollary 5.4.12: first of all, to the best of our knowledge, it is not known if **hTop** is balanced or not and, moreover, in Example 5.4.10 it is shown that the homotopic equivalence $\sim_h$ is not ExtMono$_{\mathbf{Top}}$-to-Mono.

Chapter 6

Asymptotic dimension

In this chapter we provide the necessary background around the notion of asymptotic dimension of coarse spaces, which is a generalisation of the Definition 1.1.6 given in the framework of metric spaces. This notion is one of the most important invariants in coarse geometry since it has applications both to Novikov and to coarse Baum Connes conjectures (see [19, 18, 127] for a wide discussion of the topic).

6.1 Definition and basic properties

Definition 6.1.1. A coarse space $(X, \mathcal{E})$ has *asymptotic dimension at most* n, where $n \in \mathbb{N}$, and we write $\operatorname{asdim}(X, \mathcal{E}) \leq n$, if, for every $E \in \mathcal{E}$ there exists a uniformly bounded cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ such that, for every $i = 0, \dots, n$ and every pair of distinct elements $U, V \in \mathcal{U}_i$, $E[U] \cap V = \emptyset$ (i.e., $\mathcal{U}_i$ is E-*separated*, for every $i = 0, \dots, n$). We write $\operatorname{asdim}(X, \mathcal{E}) = n$, where $n \in \mathbb{N}$, if $\operatorname{asdim}(X, \mathcal{E}) \leq n$ and $\operatorname{asdim}(X, \mathcal{E}) > n - 1$. Finally, if, for every $n \in \mathbb{N}$, $\operatorname{asdim}(X, \mathcal{E}) > n$, then $\operatorname{asdim}(X, \mathcal{E}) = \infty$.

This notion of asymptotic dimension extends the one provided in Definition 1.1.6 in the following sense. If (X, d) is a metric space, then the value $\operatorname{asdim}(X, d)$ (as in Definition 1.1.6) coincides with $\operatorname{asdim}(X, \mathcal{E}_d)$.

Theorem 6.1.2 ([89])**.** *Let $f\colon X \to Y$ be a coarse embedding between coarse spaces. Then* $\operatorname{asdim} X \leq \operatorname{asdim} Y$.

Theorem 6.1.2 has two important consequences. First of all, the asymptotic dimension is a *coarse invariant*, i.e., if $f\colon X \to Y$ is a coarse equivalence between coarse spaces, then $\operatorname{asdim} X = \operatorname{asdim} Y$. In fact, according to Theorem 3.4.6, f and any Sym-coarse inverse g are coarse embeddings and thus the claim follows from Theorem 6.1.2. Moreover, asymptotic dimension is monotone under taking coarse subspaces since the canonical inclusion map is an asymorphic embedding.

According to results of Grave ([89]), we cannot relax the hypothesis of Theorem 6.1.2 by asking that the map f is just bornologous and injective. In fact, he provided an injective bornologous map $f\colon X \to Y$ between coarse spaces such that $\operatorname{asdim} X > \operatorname{asdim} Y$.

Let us enlist some examples. Every bounded coarse space X satisfies $\operatorname{asdim} X = 0$, while, for every $n \in \mathbb{N}$, $\operatorname{asdim} \mathbb{N}^n = \operatorname{asdim} \mathbb{R}^n = \operatorname{asdim} \mathbb{Z}^n = n$, provided that the spaces are endowed with their usual euclidean metric coarse structures. Moreover, every tree endowed with the path metric has asymptotic dimension equal to one (see [127]). In particular, for every $m \in \mathbb{N} \setminus \{0\}$, the free group F_m with m free generators endowed with its word metrics satisfies $\operatorname{asdim} F_m = 1$ since its Cayley graph associated to the standard generating set, which is asymorphic to the group itself, is a tree.

Let us now present an example of an infinity-dimensional coarse space (see Proposition 6.1.3) which will be useful later in this paper. If X is a set and $f \in \{0,1\}^X$ (i.e., $f\colon X \to \{0,1\}$), the *support of* f is the subset $\operatorname{supt} f = \{x \in X \mid f(x) \neq 0\}$. For an infinite cardinal κ, we consider the *Hamming space*

$$\mathbb{H}_\kappa = \{f \in \{0,1\}^\kappa \mid |\operatorname{supt} f| < \omega\}$$

endowed with the metric $\mathrm{h}(f,g) = |\operatorname{supt} f \triangle \operatorname{supt} g|$, for every $f, g \in \mathbb{H}_\kappa$, and denote by $\mathbb{H}_\kappa^*$ the space $\mathbb{H}_\kappa$ with deleted the zero function. We identify $\mathbb{H}_\kappa$ with the set $[\kappa]^{<\omega}$, where every function $f \in \mathbb{H}_\kappa$ is identified with its support.

Proposition 6.1.3. *For every infinite cardinal κ,* $\operatorname{asdim} \mathbb{H}_\kappa = \infty$*.*

Proof. It is enough to check that $\operatorname{asdim} \mathbb{H}_\omega = \infty$. To see that $\operatorname{asdim} \mathbb{H}_\omega = \infty$, we take an arbitrary $n \in \mathbb{N}$, and we claim that there exists a copy of $\mathbb{N}^n$ in $\mathbb{H}_\kappa$. That property implies the inequality $\operatorname{asdim} \mathbb{H}_\omega \geq n$ according to Theorem 6.1.2. Let $\omega = W^1 \cup \cdots \cup W^n$ be a partition of ω in infinite subsets. Enumerate $W^i = \{a^i_m \mid m \in \mathbb{N}\}$ and define, for every $i = 1, \dots, n$ and every $m \in \mathbb{N}$, $A^i_m = \{a^i_0, \dots, a^i_m\}$.

We define a map $\varphi\colon \mathbb{N}^n \to \mathbb{H}_\kappa$ as follows: for every $(m_1, \dots, m_n) \in \mathbb{N}^n$,

$$\varphi(m_1, \dots, m_n) = \bigcup_{i=1}^n A^i_{m_i}.$$

We claim that φ is an isometry onto its image. In fact, for every

$$(m_1, \dots, m_n), (m'_1, \dots, m'_n) \in \mathbb{N}^n,$$

we have that

$$\begin{aligned}\mathrm{h}(\varphi(m_1, \dots, m_n), \varphi(m'_1, \dots, m'_n)) &= \sum_{i=1}^n |A^i_{m_i} \triangle A^i_{m'_i}| = \sum_{i=1}^n |m_i - m'_i| = \\ &= d((m_1, \dots, m_n), (m'_1, \dots, m'_n)),\end{aligned}$$

which shows the desired property. □

Let us recall a product formula concerning the asymptotic dimension from [88].

Theorem 6.1.4. *Let X and Y be two coarse spaces. Then* $\operatorname{asdim} X \times Y \leq \operatorname{asdim} X + \operatorname{asdim} Y$*.*

The inequality of Theorem 6.1.4 can be also strict (see [25]).

Remark 6.1.5. Let $\mathfrak{B} = (X, P, B)$ be a ballean. Then we define $\operatorname{asdim}\mathfrak{B} = \operatorname{asdim}(X, \mathcal{E}_{\mathfrak{B}})$. It will be convenient to rewrite this definition in terms of radii and balls. Let $n \in \mathbb{N}$. Then $\operatorname{asdim}\mathfrak{B} \le n$ if and only if, for every radius $r \in P$, there exists a uniformly bounded cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ of X (i.e., $\mathcal{U}$ is a cover such that there exists $s \in P$ with the property that, for every $U \in \mathcal{U}$ and $x \in U$, $U \subseteq B(x, s)$) such that, for every $i = 0, \dots, n$ and every pair of distinct elements $U, V \in \mathcal{U}_i$, $B(U, r) \cap V = \emptyset$.

In [168], the authors introduce several notions of asymptotic dimensions in the realm of quasi-coarse spaces. These concepts extend the classical notion and are invariant under Sym-coarse equivalences.

6.2 Some examples of ballean classes: thin and cellular balleans

In this section we focus our attention on some properties of coarse spaces implying zero-dimensionality.

6.2.1 Thin coarse spaces

Let $(X, \mathcal{E})$ be a coarse space. A subset Y of X is *thin* if, for every $E \in \mathcal{E}$, there exists a bounded subset $V \subseteq X$ such that $|E[x] \cap Y| = 1$, for every $x \in Y \setminus V$. A coarse space is *thin* if its whole support is thin.

Among all characterisations of thinness (for example, see [55, 151]), let us remind the following.

Theorem 6.2.1 ([151])**.** *A connected coarse space $(X, \mathcal{E})$ is thin if and only if $\mathcal{E}$ coincides with the ideal coarse structure $\mathcal{E}_{\mathcal{I}}$, where $\mathcal{I}$ is the ideal of bounded subsets of $(X, \mathcal{E})$.*

Thanks to Theorem 6.2.1, it is worth paying a closer attention to ideal coarse spaces. Here we provide some properties that will be important, even though they are easy to check.

Let $\mathcal{I}$ be an ideal on X. The ideal coarse structure $\mathcal{E}_{\mathcal{I}}$ is connected if and only if $\mathcal{I}$ is a cover or, equivalently, $[X]^{<\omega} \subseteq \mathcal{I}$.

Proposition 6.2.2. *Let $f\colon X \to Y$ be a map between two non-empty sets, and $\mathcal{I}$ and $\mathcal{J}$ be two ideals of X and Y, respectively. Then $f\colon (X, \mathcal{E}_{\mathcal{I}}) \to (Y, \mathcal{E}_{\mathcal{J}})$ is bornologous if and only if $f(\mathcal{I}) \subseteq \mathcal{J}$. Hence, f is an asymorphism if and only if f is a bijection such that $f(\mathcal{I}) \subseteq \mathcal{J}$ and $f^{-1}(\mathcal{J}) \subseteq \mathcal{I}$.*

Corollary 6.2.3. *Let X and Y be two sets endowed with the ideal coarse structures induced by their families of finite subsets. Then X and Y are asymorphic if and only if $|X| = |Y|$.*

The notion of thinnes may seem too restrictive. In fact, one can easily see that a non-connected coarse space is thin if and only if all but one connected components are trivial and the non-trivial one is thin. Since some of the coarse spaces we will be considering in this sequel are non-connected, we define the following class. A coarse space is *weakly thin* if every connected component is thin.

6.2.2 Cellular coarse spaces

Let us now recall a characterisation of those coarse spaces that have asymptotic dimension 0. A quasi-coarse space $(X,\mathcal{E})$ is *cellular* if, for every $E\in\mathcal{E}$, $E^{\square}=\bigcup_{n\in\mathbb{N}}E^n\in\mathcal{E}$, where, for every $n\in\mathbb{N}$, E^n is obtained by composing E with itself n times ([147, 153]). Cellular coarse spaces are precisely those with asymptotic dimension 0 ([151]).

Thin coarse spaces and, in particular, bounded coarse spaces are cellular. Moreover, if a weakly thin coarse space has only a finite number of connected components, it is cellular. However, this property does not hold for weakly thin coarse spaces with infinite number of connected components (see Example 6.2.4).

Example 6.2.4. Consider the coarse space $X=\bigsqcup_{n\in\mathbb{N}}[0,n]$ endowed with the following metric: for every $i_k(x),i_{k'}(x')\in\bigsqcup_n[0,n]$,

$$d(i_k(x),i_{k'}(x'))=\begin{cases}|x-x'| & \text{if } k=k',\\ \infty & \text{otherwise.}\end{cases}$$

Then $\operatorname{asdim}X>0$ and thus it is not cellular, although it is weakly thin. Finally, the coproduct coarse space $X=\bigoplus_n[0,n]$, where every interval is endowed with the usual euclidean metric, is cellular and thus X and Y are not coarsely equivalent (and, in particular, not asymorphic).

The family of cellular coarse spaces is stable under taking subspaces (thanks to the characterisation as zero-dimensional spaces and Theorem 6.1.2) and products (Proposition 6.2.5).

Proposition 6.2.5. *Let $\{(X_i,\mathcal{E}_i)\}_{i\in I}$ be a family of cellular coarse spaces. Then the product coarse space $\Pi_i(X_i,\mathcal{E}_i)$ is cellular.*

Proof. Let $E\in\mathcal{E}$. Without loss of generality, we can assume that $E=\Pi_iE_i$, where $E_i\in\mathcal{E}_i$, for every $i\in I$. Then $E^{\square}=\Pi_i(E_i^{\square})\in\mathcal{E}$ since every component is cellular. □

Proposition 6.2.6. *Let $(X,\mathcal{E}_X)$ and $(Z,\mathcal{E}_Z)$ be two coarse spaces such that Z is non-empty and cellular. Then $\operatorname{asdim}X\times Z=\operatorname{asdim}X$.*

This proposition can be easily derived from Theorems 6.1.2 and 6.1.4. In fact, in the notations of Proposition 6.2.6, since X can be embedded in the product $X\times Z$ (take a point $z\in Z$ and define $i(x)=(x,z)$, for every $x\in X$), $\operatorname{asdim}X\leq\operatorname{asdim}(X\times Z)$ and the opposite inequality holds because of Theorem 6.1.4 and the zero-dimensionality of Z. However, for the sake of completeness, we provide a direct proof of this fact.

Proof of Proposition 6.2.6. Theorem 6.1.2 implies that $\operatorname{asdim}X\leq\operatorname{asdim}X\times Z$. We have to prove that $\operatorname{asdim}X\times Z\leq\operatorname{asdim}X$. If $\operatorname{asdim}X=\infty$, there is nothing to prove. Hence, we assume that $\operatorname{asdim}X=n$, for some $n\in\mathbb{N}$.

Denote by $\mathcal{E}$ the product coarse structure on $X\times Z$. Let us fix and entourage $K\in\mathcal{E}$. Without loss of generality, we can assume that $K=E\times F$, where $E\in\mathcal{E}_X$ and $F=F^{\square}\in\mathcal{E}_Z$ are two symmetric entourages. Since X has asymptotic dimension n, there exists a uniformly bounded cover $\mathcal{U} = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_n$ of X such that, for every $i = 0, \dots, n$, $\mathcal{U}_i$ is E-separated. Let $M \in \mathcal{E}_X$ be a uniform bound to the family $\mathcal{U}$. For every $i = 0, \dots, n$, we define a family of subsets of $X \times Z$ as follows:

$$\mathcal{W}_i = \{U \times F[z] \mid U \in \mathcal{U}_i,\, z \in Z\}.$$

We claim that $\mathcal{W}_i$ is K-separated and that $\mathcal{W} = \mathcal{W}_0 \cup \dots \cup \mathcal{W}_n$ is a uniformly bounded cover of $X \times Z$, which imply that $\operatorname{asdim} X \times Z \leq n$.

Let $U \times F[z], U' \times F[z'] \in \mathcal{W}_i$ such that $K[U \times F[z]] \cap U' \times F[z'] \neq \emptyset$. Since $K = E \times F$, the product coarse structure implies that $E[U] \times F[F[z]] \cap U' \times F[z'] \neq \emptyset$. Hence $E[U] \cap U' \neq \emptyset$, which implies that $U = U'$ since $\mathcal{U}_i$ is E-separated. Moreover, since $F = F^{\square}$, $F[F[z]] = F[z]$ and $F[z] \cap F[z'] \neq \emptyset$ implies that $F[z] = F[z']$.

The family $\mathcal{W}$ is trivially a cover, since $\mathcal{U}$ is a cover and, for every $i = 0, \dots, n$, $\bigcup \mathcal{W}_i = \bigcup \mathcal{U}_i \times Z$. Moreover, the entourage $M \times F$ is a uniform bound to $\mathcal{W}$. In fact, for every $U \times F[z] \in \mathcal{W}$ and every $(x, z') \in U \times F[z]$, $U \times F[z] \subseteq M[x] \times F[z'] = (M \times F)[(x, z')]$ since $F^{\square} = F$. $\square$

Since it is going to be useful in the sequel, let us translate the notion of cellularity in terms of balleans. For a ballean (X, P, B), $n \in \mathbb{N}$, $x \in X$ and $r \in P$, we let

$$B^n(x, r) = \underbrace{B(B(\cdots B(B}_{n \text{ times}}(x, r), r) \cdots, r), r) \quad \text{and} \quad B^{\square}(x, r) = \bigcup_{n=1}^{\infty} B^n(x, r).$$

Definition 6.2.7. The triple $\mathfrak{B}^{\square} = (X, P, B^{\square})$ is a ballean called the *cellularisation of* $\mathfrak{B}$. The ballean $\mathfrak{B}$ is said to be *cellular* if $\mathfrak{B} = \mathfrak{B}^{\square}$.

It is trivial to check that a ballean X is cellular if and only if its corresponding coarse space is cellular.

Part II

Coarse geometry of algebraic objects

Chapter 7

Structures on algebraic objects

In this chapter, we investigate the large-scale geometry of large-scale objects and, in particular, of groups and topological groups.

7.1 Coarse groups, bornological groups and their generalisations

Definition 7.1.1. Let M be a unitary magma and $\mathcal{I}$ be a family of subsets of M.

- $\mathcal{I}$ is a *magmatic ideal* if it is an ideal on M such that $\{e\} \in \mathcal{I}$.
- If M is a monoid, $\mathcal{I}$ is a *monoid ideal* if it is a magmatic ideal and, for every $H, K \in \mathcal{I}$, $H \cdot K = \{h \cdot k \mid h \in H,\, k \in K\} \in \mathcal{I}$.
- If M is a loop, a magmatic ideal $\mathcal{I}$ is a *right loop ideal* if, for every $F \in \mathcal{I}$, $F^\rho = \{g^\rho \mid g \in F\} \in \mathcal{I}$, $\mathcal{I}$ is a *left loop ideal* if, for every $F \in \mathcal{I}$, $F^\lambda = \{g^\lambda \mid g \in F\} \in \mathcal{I}$, and $\mathcal{I}$ is a *loop ideal* if it is both a left loop ideal and a right loop ideal.
- If M is a group, $\mathcal{I}$ is a *group ideal* ([151]) if it is both a monoid ideal and a loop ideal.

A *unitary submagma* N of a unitary magma M is a subset $N \subseteq M$ that contains the identity of M and it is closed under the operation. A unitary submagma N is a *subloop* of a loop M if, for every parameters in N, the solutions of (1.3) belongs to N. A unitary submagma N of a monoid M is called a *submonoid*.

If $\mathcal{I}$ is a monoid ideal on a monoid M, then $\bigcup\mathcal{I}$ is a submonoid. Similarly, $\bigcup\mathcal{I}$ is a subgroup if $\mathcal{I}$ is a group ideal on a group M.

The leading examples of magmatic ideals are the finitary magmatic ideal, the finitary monoid ideal, the finitary (left, right) loop ideal, and the finitary group ideal.

Example 7.1.2. Let M be a unitary magma, then the family $\mathcal{I} = [M]^{<\omega} = \{F \subseteq M \mid F \text{ is finite}\}$ is a magmatic ideal, called *finitary magmatic ideal.* If M

is a monoid, then $\mathcal{I}$ is a monoid ideal, called *finitary monoid ideal.* If M is a loop with (right) inverse property, then $\mathcal{I}$ is a (right) loop ideal, called *finitary (right) loop ideal.* Finally, if M is a group, then $\mathcal{I}$ is a group ideal, called *finitary group ideal.*

Let M be a unitary magma and $\mathcal{I}$ a magmatic ideal on M. If A is a subset of $M\times M$, define $M\cdot A=\{(mx,my)\mid m\in M,\,(x,y)\in A\}$. Then we can define the *left magmatic entourage structure* $\mathcal{E}^\lambda_\mathcal{I}$ on M as follows:

$$\begin{gathered}\mathcal{E}^\lambda_\mathcal{I}=\mathfrak{cl}(\{E^\lambda_K\mid K\in\mathcal{I}\}),\quad\text{where, for every } K\in\mathcal{I},\\ E^\lambda_I=M(\{e\}\times K)=\{(x,xk)\mid x\in M,k\in K\}.\end{gathered}$$

Similarly, we can define the *right magmatic entourage structure* $\mathcal{E}^\rho_\mathcal{I}$, where the action of M is on the right:

$$\mathcal{E}^\rho_\mathcal{I}=\mathfrak{cl}(\{E^\rho_K\mid K\in\mathcal{I}\}),\quad\text{where, for every } K\in\mathcal{I},\quad E^\rho_K=(\{e\}\times K)M.$$

Proposition 7.1.3. *Let M be a unitary magma and $\mathcal{I}$ be a magmatic ideal on M.*

(a) $\mathcal{E}^\lambda_\mathcal{I}$ and $\mathcal{E}^\rho_\mathcal{I}$ are entourage structures.
(b) If M is a loop with the right inverse property and $\mathcal{I}$ is a right loop ideal (with the left inverse property and $\mathcal{I}$ is a left loop ideal), then $\mathcal{E}^\lambda_\mathcal{I}$ is a semi-coarse structure, called left loop semi-coarse structure *($\mathcal{E}^\rho_\mathcal{I}$ is a semi-coarse structure, called* right loop semi-coarse structure*, respectively).*
(c) If M is a monoid and $\mathcal{I}$ is a monoid ideal, then $\mathcal{E}^\lambda_\mathcal{I}$ and $\mathcal{E}^\rho_\mathcal{I}$ are quasi-coarse structures, called left monoid quasi-coarse structure *and* right monoid quasi-coarse structure*, respectively.*
(d) If M is a group, then $\mathcal{E}^\lambda_\mathcal{I}$ and $\mathcal{E}^\rho_\mathcal{I}$ are coarse structures, called left group coarse structure *and* right group coarse structure*, respectively.*

Proof. Item (a) is trivial and item (d) has already been proved ([125]).

(b) Let M be a loop with right inverse property and $\mathcal{I}$ be a right loop. Let $K\in\mathcal{I}$. Then, for every $(x,xk)\in E^\lambda_K$, where $x\in M$ and $k\in K$, $(xk,x)=(xk,(xk)k^\rho)\in E^\lambda_{K^\rho}$. Hence $(E^\lambda_K)^{-1}\subseteq E^\lambda_{K^\rho}\in\mathcal{E}^\lambda_\mathcal{I}$. We can prove similarly the other assertion.

Finally, item (c) follows from the observation that, for every $F,K\in\mathcal{I}$, $E^\lambda_F\circ E^\lambda_K\subseteq E^\lambda_{FK}$. □

Of course, if M is an abelian unitary magma and $\mathcal{I}$ is a magmatic ideal, then $\mathcal{E}^\lambda_\mathcal{I}=\mathcal{E}^\rho_\mathcal{I}$. Note that, if M is an abelian loop with the right inverse property, then it has the inverse property. In the next remark we discuss a situation in which the left and the right magmatic entourage structures are asymorphic, even though the may not be equal.

Remark 7.1.4. Let G be a group and $\mathcal{I}$ be a magmatic ideal. Note that $\mathcal{I}^{-1}=\{K^{-1}\mid K\in\mathcal{I}\}$ is still a magmatic ideal and, more precisely, if $\mathcal{I}$ is a loop ideal or a monoid ideal, then so it is $\mathcal{I}^{-1}$. Furthermore, it is easy to check that $\mathcal{E}^\lambda_\mathcal{I}=(\mathcal{E}^\rho_{\mathcal{I}^{-1}})^{-1}$. Consider the map $i\colon G\to G$ such that $i(g)=g^{-1}$, for every $g\in G$. Then $i\colon(G,\mathcal{E}^\lambda_\mathcal{I})\to(G,\mathcal{E}^\rho_{\mathcal{I}^{-1}})=(G,(\mathcal{E}^\lambda_\mathcal{I})^{-1})$ is an asymorphism. In fact, for every $K \in \mathcal{I}$ and every $(x, xk) \in E_K^\lambda$, $(i \times i)(x, xk) = (x^{-1}, k^{-1}x^{-1}) \in E_{K^{-1}}^\rho$. The same conclusion holds if $\mathcal{I}$ is a loop ideal, a monoid ideal, or a group ideal. In particular, if $\mathcal{I}$ is a loop ideal or a group ideal, then $\mathcal{I} = \mathcal{I}^{-1}$ and thus $(G, \mathcal{E}_\mathcal{I}^\lambda)$ and $(G, \mathcal{E}_\mathcal{I}^\rho)$ are asymorphic. Hence, on a group G, if $\mathcal{I}$ is a loop ideal, we simply write $\mathcal{E}_\mathcal{I}$ instead of both $\mathcal{E}_\mathcal{I}^\lambda$ and $\mathcal{E}_\mathcal{I}^\rho$ when there is no risk of ambiguity.

A family of maps $\mathcal{F}$ between two entourage spaces $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ is *equi-bornologous* if, for every $E \in \mathcal{E}_X$, there exists $F \in \mathcal{E}_Y$ such that $(f \times f)(E) \subseteq F$, for every $f \in \mathcal{F}$.

If M is a unitary magma, we define the following families of maps, which are the *left* and the *right shifts* in M:

$$\begin{gathered}\mathcal{S}_M^\lambda = \{s_x^\lambda \mid x \in M\} \text{ and } \mathcal{S}_M^\rho = \{s_x^\rho \mid x \in M\},\\ \text{where, for every } x, y \in M,\ s_x^\lambda(y) = xy \text{ and } s_x^\rho(y) = yx.\end{gathered} \tag{7.1}$$

In the following remark, we describe situations in which the families of morphisms $\mathcal{S}_M^\lambda$ and $\mathcal{S}_M^\rho$ are equi-bornologous.

Remark 7.1.5. (a) For every finitely generated monoid M, if Σ is a finite generating set, then the quasi-coarse space $(M, \mathcal{E}_{d_\Sigma^\lambda})$ makes the family $\mathcal{S}_M^\lambda = \{s_x^\lambda \mid x \in M\}$ equi-bornologous, since d_Σ^λ is left-non-expanding.

(b) Let M be a monoid and $\mathcal{I}$ be a monoid ideal on M. Then $\mathcal{S}_M^\lambda$ and $\mathcal{S}_M^\rho$ are equi-bornologous if M is endowed with the left monoid quasi-coarse structure $\mathcal{E}_\mathcal{I}^\lambda$ and the right monoid quasi-coarse structure $\mathcal{E}_\mathcal{I}^\rho$, respectively. In fact, let $e \in K \in \mathcal{I}$. Then, for every $x \in M$ and every $(y, yk) \in E_K^\lambda$,

$$(s_x^\lambda \times s_x^\lambda)(y, yk) = (xy, xyk) = xy(e, k) \in E_K^\lambda.$$

(c) Consider the unitary magmas $\overline{\mathbb{Z}} = (\mathbb{Z} \cup \{e\}, -, e)$ and $\overline{\mathbb{Q}^*} = (\mathbb{Q}^* \cup \{e\}, /, e)$, obtained by the magmas $(\mathbb{Z}, -)$ and $(\mathbb{Q}^*, /)$, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, by adding a neutral element e. Endow those unitary magmas with the left entourage structures associated to their finitary magmatic ideals. Then the families $\mathcal{S}_{\overline{Z}}^\lambda$ and $\mathcal{S}_{\overline{\mathbb{Q}^*}}^\lambda$ are equi-bornologous. In fact, for every $K \in [\overline{\mathbb{Z}}]^{<\omega}$ and $x \in \overline{\mathbb{Z}}$,

$$\begin{aligned}(s_x^\lambda \times s_x^\lambda)(E_K) &= (s_x^\lambda \times s_x^\lambda)(\{(y, y-k) \mid y \in \overline{\mathbb{Z}}, k \in K\}) =\\ &= \{(x-y, x-(y-k)) \mid y \in \overline{\mathbb{Z}}, k \in K\} \subseteq E_{K'},\end{aligned}$$

where $K' = \{e, k, -k \mid k \in K \setminus \{e\}\}$. Similarly, for every $F \in [\overline{\mathbb{Q}^*}]^{<\omega}$ and $x \in \overline{\mathbb{Q}^*}$,

$$\begin{aligned}(s_x^\lambda \times s_x^\lambda)(E_F) &= (s_x^\lambda \times s_x^\lambda)(\{(y, y/k) \mid y \in \overline{\mathbb{Q}^*}, k \in F\}) =\\ &= \{(x/y, x/(y/k)) \mid y \in \overline{\mathbb{Q}^*}, k \in F\} \subseteq E_{F'},\end{aligned}$$

where $F' = \{e, k, 1/k \mid k \in F \setminus \{e\}\}$. Those results follow from a more general statement.

(d) Let M be a unitary magma such that there exists a map $r\colon M \to M$ with the property that $a(bc) = (ab)r(c)$, for every $a, b, c \in M$. Hence, we claim that, for every magmatic ideal $\mathcal{I}$ on M such that $r(\mathcal{I}) = \{r(K) \mid K \in \mathcal{I}\} \subseteq \mathcal{I}$, the family of left shifts is equi-bornologous whenever M is endowed with the left entourage structure associated to $\mathcal{I}$. Let $K \in \mathcal{I}$ be a generic element of the magmatic ideal. Then, for every $x \in M$,

$$\begin{aligned}(s_x^\lambda \times s_x^\lambda)(E_K) &= \{(xy, x(yk)) \mid y \in M, k \in K\} = \\ &= \{(xy, (xy)r(k)) \mid y \in M, k \in K\} \subseteq E_{r(K)}.\end{aligned}$$

which concludes the proof, since $r(K) \in \mathcal{I}$.

The next proposition shows the importance of having the families of left (right) shifts, defined in (7.1), equi-bornologous. We state the result just for $\mathcal{S}_M^\lambda$, but similar conclusions hold also for $\mathcal{S}_M^\rho$.

Proposition 7.1.6. *Let M be a unitary magma and $\mathcal{E}$ be an entourage structure over M such that $\mathcal{S}_M^\lambda$ is equi-bornologous. Let $\mathcal{I} = \{E[e] \mid E \in \mathcal{E}\} = \beta_{\mathcal{E}}(e)$. Then:*
(a) $\mathcal{I}$ is a magmatic ideal and $\mathcal{E}_{\mathcal{I}}^\lambda \subseteq \mathcal{E}$;
(b) if M is a loop with the inverse property and two-side inverses, and $\mathcal{E}$ is a semi-coarse structure, then $\mathcal{I}$ is a loop ideal and $\mathcal{E}_{\mathcal{I}}^\lambda = \mathcal{E}$;
(c) if M is a monoid and $\mathcal{E}$ is a quasi-coarse structure, then $\mathcal{I}$ is a monoid ideal and $\mathcal{E}_{\mathcal{I}}^\lambda \subseteq \mathcal{E}$;
(d) if M is a group and $\mathcal{E}$ is a coarse structure, then $\mathcal{I}$ is a group ideal and $\mathcal{E}_{\mathcal{I}}^\lambda = \mathcal{E}$.

Proof. The first assertion of item (a) is trivial since $\{e\} \in \beta_{\mathcal{E}}(e)$ which is an ideal. Let now $F \in \mathcal{E}$ be an arbitrary entourage, and so $F[e]$ be an arbitrary element of $\mathcal{I}$. Since $\mathcal{S}_M^\lambda$ is equi-bornologous, there exists $F' \in \mathcal{E}$ such that, for every $x \in M$, $(s_x^\lambda \times s_x^\lambda)(F) \subseteq F'$. Since, for every $(x, y) \in E_{F[e]}^\lambda$, there exists $k \in F[e]$ such that $y = xk$, we have

$$(x, y) = (x, xk) = (s_x^\lambda \times s_x^\lambda)(e, k) \in (s_x^\lambda \times s_x^\lambda)(F) \subseteq F',$$

and so $E_{F[e]}^\lambda \subseteq F'$. Hence, $\mathcal{E}_{\mathcal{I}}^\lambda \subseteq \mathcal{E}$.

(b) Let $E \in \mathcal{E}$ and x be an arbitrary element of $E[e]$. Then,

$$(e, x^{-1}) = (s_{x^{-1}}^\lambda \times s_{x^{-1}}^\lambda)(x, e) \in (s_{x^{-1}}^\lambda \times s_{x^{-1}}^\lambda)(E^{-1}) \subseteq F,$$

for some $F \in \mathcal{E}$ that can be taken independently from the choice of x in $E[e]$ since $\mathcal{S}_M^\lambda$ is equi-bornologous. Thus $E[e]^{-1} \subseteq F[e] \in \mathcal{I}$.

Consider now an arbitrary entourage $E \in \mathcal{E}$. We want to show that there exists $F \in \mathcal{E}$, such that $E \subseteq E_{F[e]}^\lambda$. Let $(x, y) \in E$ and denote by $F \in \mathcal{E}$ an entourage such that $(s_z^\lambda \times s_z^\lambda)(E) \subseteq F$, for every $z \in M$. Then

$$(x, y) = (s_x^\lambda \times s_x^\lambda)(e, x^{-1}y), \quad \text{where} \quad (e, x^{-1}y) = (s_{x^{-1}}^\lambda \times s_{x^{-1}}^\lambda)(x, y) \subseteq F, \tag{7.2}$$

and thus $(x, y) \in E_{F[e]}$. Note that in (7.2) we used that M has the inverse property and two-sided inverses.

(c) Thanks to item (a), we only need to show that $\mathcal{I}$ is a monoid ideal. Take $E, F \in \mathcal{E}$ and consider $E[e] \cdot F[e]$. Let $x \in E[e]$ and $y \in F[e]$ be two arbitrary elements, which means that $(e, x) \in E$ and $(e, y) \in F$. Denote by $E' \in \mathcal{E}$ an entourage such that $(s_x^\lambda \times s_x^\lambda)(F) \subseteq E'$, for every $x \in M$. Then

$$(e, xy) = (e, x) \circ (x, xy) \in E \circ (s_x^\lambda \times s_x^\lambda)(F) \subseteq E \circ E' \in \mathcal{E},$$

which shows that $xy \in (E \circ E')[e]$, and thus $E[e]F[e] \subseteq (E \circ E')[e] \in \mathcal{I}$.

Finally, item (d) descends from items (b) and (c). $\square$

Remark 7.1.7. Let M be a monoid generated by a finite subset Σ. By Remark 7.1.5 and Proposition 7.1.6, $\mathcal{E}_{\mathcal{I}}^{\lambda} \subseteq \mathcal{E}_{d_{\Sigma}^{\lambda}}$, where $\mathcal{I}$ is the family of all subsets of $(M, d_{\Sigma}^{\lambda})$ bounded from e, (i.e., $\mathcal{I} = \beta_{d_{\Sigma}^{\lambda}}(e)$). More precisely, we can show that $\mathcal{I} = [M]^{<\infty}$. We claim that $\mathcal{E}_{\mathcal{I}}^{\lambda} = \mathcal{E}_{d_{\Sigma}^{\lambda}}$. Let $R \geq 0$ and define $F_R = B_{d_{\Sigma}^{\lambda}}(e, R) \in \mathcal{I}$. If $d_{\Sigma}^{\lambda}(x, y) \leq R$, then $y \in xF_R$. Hence $E_R \subseteq E_{F_R}^{\lambda}$, which shows the desired equality. Similarly, $\mathcal{E}_{\mathcal{I}}^{\rho} = \mathcal{E}_{d_{\Sigma}^{\rho}}$.

7.1.1 Coarse groups

Let us focus on coarse structures on groups. First of all, let us specialise Proposition 7.1.6.

Corollary 7.1.8. *Let G be a group and $\mathcal{E}$ be a coarse structure on it. Then the following properties are equivalent:*
(a) for every $E \in \mathcal{E}$, $GE = \{(gx, gy) \mid g \in G, (x, y) \in E\} \in \mathcal{E}$;
(b) the family $\mathcal{S}_G^{\lambda}$ is equi-bornologous;
(c) there exists a group ideal $\mathcal{I} = \beta_{\mathcal{E}}(e)$ on G such that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$.

Proof. The equivalence (b)↔(c) follows from Proposition 7.1.6(d) and Remark 7.1.5(b). The equivalence (a)↔(b) is easy to check. $\square$

A coarse structure $\mathcal{E}$ on a group G satisfying the equivalent properties enlisted in Corollary 7.1.8 is called a *left group coarse structure* and the pair $(G, \mathcal{E})$ is a *left coarse group. Right group coarse structures* and *right coarse groups* can be defined similarly. According to Remark 7.1.4, left and right group coarse structures are asymorphic. In the sequel we will always refer to left group coarse structures and left coarse groups, if it is not otherwise stated, and thus we call them briefly *group coarse structures* (and *coarse groups*) if there is no risk of ambiguity. Moreover, every group coarse structure $\mathcal{E}$ on a group G is induced by a group ideal $\mathcal{I}$ (i.e., $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$), and thus it is convenient sometimes to call $\mathcal{E}$ the *$\mathcal{I}$-group coarse structure.*

There is another way to describe the group ideal of Corollary 7.1.8(c). If G is a group, the map $\pi_G^{\lambda}\colon G \times G \to G$ such that, for every $(g, h) \in G \times G$, $\pi_G^{\lambda}(g, h) = h^{-1}g$ is called *(left) shear map* ([125]). If $\mathcal{E}$ is a left coarse structure satisfying the properties of Proposition 7.1.6, then $\mathcal{I} = \{\pi_G^{\lambda}(E) \mid E \in \mathcal{E}\}$.

According to Corollary 7.1.8, coarse groups are equivalently described in terms of group ideals. This is why it is necessary to provide examples of group ideals. In Example 7.1.9 we provide examples that are defined on arbitrary groups, while in Example 7.1.10 we assume some further structure on the objects.

Example 7.1.9. Let G be a group.
(a) The sigleton $\{\{e\}\}$ is a group ideal and the $\{\{e\}\}$-group coarse structure is the discrete coarse structure.
(b) On the opposite side we have the group ideal $\mathcal{P}(G)$, that induces the trivial coarse structure.

(c) The finitary group ideal $[G]^{<\omega}$ introduced in Example 7.1.2 induces the *finitary-group coarse structure.*
(d) We want to generalise the finitary group ideal. For a given infinite cardinal κ, the family $[G]^{<\kappa}$ is a group ideal. The $[G]^{<\kappa}$-group coarse structure is called *κ-group coarse structure.* Then the finitary-group coarse structure is the ω-group coarse structure.

Example 7.1.10. Let G be a group.
(a) Let τ be a group topology of G. Define $\mathcal{C}(G)$ as the family of all compact subsets of G. Then $\mathfrak{cl}(\mathcal{C}(G))$ coincides with the family $\mathfrak{rC}(G)$ of all relatively compact subsets of G is a group ideal and the $\mathfrak{rC}(G)$-coarse structure is called *compact-group coarse structure.*
(b) Let d be a left-invariant pseudo-metric on G. Then the family $\mathcal{B}_d = \{A \subseteq G \mid \exists R \geq 0 : A \subseteq B_d(e, R)\}$ is a group ideal and the $\mathcal{B}_d$-group coarse structure is called *metric-group coarse structure.* Note that $\mathcal{E}_{\mathcal{B}_d} = \mathcal{E}_d$.
(c) Let G be a topological group. The group ideal

$$\mathcal{OB} = \{A \subseteq G \mid \forall d \text{ left-invariant continuous pseudo-metric}, \\ A \text{ is } d\text{-bounded from } e\}.$$

was defined in [158], where other characterisations of $\mathcal{OB}$ are provided. Then

$$\mathcal{E}_L = \mathcal{E}_{\mathcal{OB}} = \bigcap\{\mathcal{E}_{\mathcal{B}_d} \mid d \text{ is a left-invariant continuous pseudo-metric}\}$$

is defined in [158] and named *left-coarse structure.* The group ideal $\mathcal{OB}$ contains the family $\mathcal{C}(G)$ (and thus $\mathfrak{rC}(G)$) and it coincides with $\mathfrak{rC}(G)$ if G is locally compact and σ-compact ([158, Corollary 2.8]). However, there exist locally compact groups G with $\mathfrak{rC}(G) \subsetneq \mathcal{OB}$. For example, the group $Sym(\mathbb{N})$ of all permutations of $\mathbb{N}$ endowed with the discrete topology has $\mathfrak{rC}(Sym(\mathbb{N})) = [Sym(\mathbb{N})]^{<\omega}$, while $\mathcal{OB} = \mathcal{P}(Sym(\mathbb{N}))$ (see [158, Example 2.16]).
(d) For an infinite cardinal κ, a topological space is *κ-Lindelöf* if every open cover has a subcover of size strictly less than κ (so ω-Lindelöf coincides with compact, while ω_1-Lindelöf is the standard Lindelöf property). For a topological group G, denote by κ-$\mathcal{L}(G)$ the family of all κ-Lindelöf subsets of G. Then $\mathfrak{cl}(\kappa\text{-}\mathcal{L}(G))$ is a group ideal and the $\mathfrak{cl}(\kappa\text{-}\mathcal{L}(G))$-group coarse structure is called *κ-Lindelöf-group coarse structure.*

Remark 7.1.11. Let G be a discrete group. Then, for every infinite cardinal κ, the κ-Lindelöf-group coarse structure coincides with the κ-group coarse structure. In particular, the compact-group coarse structure coincides with the finitary-group coarse structure.

Suppose, furthermore, that G is countable and d is the left-invariant proper metric described in (1.2). Then the finitary-group coarse structure $\mathcal{E}_{[G]^{<\omega}}$, the compact-group coarse structure $\mathcal{E}_{\mathfrak{rC}(G)}$, the metric-group coarse structure $\mathcal{E}_d$ and the left coarse structure $\mathcal{E}_L$ coincide.

According to Remark 7.1.4, for every group G and group ideal $\mathcal{I}$, $(G, \mathcal{E}_{\mathcal{I}}^{\lambda})$ and $(G, \mathcal{E}_{\mathcal{I}}^{\rho})$ are asymorphic. However, these two group coarse structures need not coincide in general. It will be useful in the sequel to characterise those group ideals for which these two group coarse structures coincide.

Proposition 7.1.12. *Let G be a group and $\mathcal{E}$ a coarse structure on it. If the group operation $\cdot\colon G\times G\to G$ is bornologous, then $\mathcal{E}$ is both a left and a right group coarse structure.*

Proof. For every $E\in\mathcal{E}$, $GE=(\cdot\times\cdot)(\Delta_X\times E)\in\mathcal{E}$ and $EG=(\cdot\times\cdot)(E\times\Delta_X)$, and thus the claim follows from Corollary 7.1.8. □

If K is a subset of a group G, and $g\in G$, we define $K^g=g^{-1}Kg$ and $K^G=\bigcup_{h\in G}K^h$. A group ideal $\mathcal{I}$ is *uniformly bilateral* if $K^G\in\mathcal{I}$ for every $K\in\mathcal{I}$. Note that, for every $K\subseteq G$ and $g\in G$,

$$Kg=gg^{-1}Kg=gK^g\subseteq gK^G. \tag{7.3}$$

Similarly, if $E\subseteq G\times G$, and $g\in G$ be an element, we define $E^g=\{(g^{-1}xg,g^{-1}yg)\mid(x,y)\in E\}$ and $E^G=\bigcup_{h\in G}E^h$. We say that a coarse structure $\mathcal{E}$ on G is *uniformly invariant* if $E^G\in\mathcal{E}$ for every $E\in\mathcal{E}$.

The following proposition is the analogue in realm of coarse groups of [96, Proposition 1.2], to which we refer for the proof.

Proposition 7.1.13. *Let G be a group and $\mathcal{E}$ is a left $\mathcal{I}$-group coarse structure on it, for some group ideal $\mathcal{I}$ on G. Then the following conditions are equivalent:*
(a) the inverse map $i\colon(G,\mathcal{E})\to(G,\mathcal{E})$ is bornologous;
(b) the multiplication map $\cdot\colon(G\times G,\mathcal{E}\times\mathcal{E})\to(G,\mathcal{E})$ is bornologous;
(c) $\mathcal{E}$ is also a right $\mathcal{I}$-group coarse structure;
(d) $\mathcal{E}$ is uniformly invariant;
(e) $\mathcal{I}$ is uniformly bilateral.
In particular, when the above conditions are fulfilled, the subgroup $\bigcup\mathcal{I}$ is normal.

A coarse group with uniformly invariant group coarse structure will be called *bilateral coarse group*.

In particular, for every abelian group and every group ideal on it, the conditions of Proposition 7.1.13 are satisfied. It is natural to expect that this remains true for groups close to be abelian, e.g., for groups G having large centre with respect to the finitary-group coarse structure of G. This means that the *centre* $Z(G)=\{g\in G\mid\forall x\in G,gx=xg\}$ *of* G has finite index in G. Then, by Shur's Theorem [156], the *commutator subgroup* $G'=\langle\{g^{-1}h^{-1}gh\mid g,h\in G\}\rangle$ *of* G is finite. As we shall below, this implies that $\mathcal{E}_\mathcal{I}^\lambda=\mathcal{E}_\mathcal{I}^\rho$ (see Corollary 7.2.12). Since finiteness of G' can still be considered as a rather strong restraint, we consider now a weaker condition (but it ensures uniform invariance only of some group coarse structures). Recall that a group G is called an *FC-group*, if all conjugacy classes $x^G=\{x\}^G$ are finite. Obviously, G is an FC-group, if G' is finite.

The next proposition shows that this commutativity condition is the precise measure ensuring uniform invariance of the finitary-group coarse structure. Its easy proof will be omitted.

Proposition 7.1.14. *For every group G the following conditions are equivalent:*
(a) G is an FC-group;
(b) the finitary-group coarse structure of G is uniformly invariant;
(c) for every infinite cardinal κ the κ-group coarse structure of G is uniformly invariant.

Thanks to Proposition 7.1.13 we can easily find a coarse group for which the multiplication is not bornologous. It is the aim of the next example.

Example 7.1.15. Consider the free group F_2, generated by $\{a, b\}$.

(a) Let $\mathcal{I} = [\langle a \rangle]^{<\omega}$. Then, if we endow F_2 with $\mathcal{E}_{\mathcal{I}}^{\lambda}$, $\cdot$ is not bornologous since $\bigcup \mathcal{I}$ is not normal.

(b) If we endow F_2 with the finitary-group coarse structure, μ is not bornologous by item (i) (or by Proposition 7.1.14, as F_2 is not an FC-group, e.g., a^G is not finite).

The following example shows that the compact-group coarse structure of a locally compact group need not be uniformly invariant.

Example 7.1.16. Fix a prime number p and let $\theta\colon \mathbb{Q}_p \to \mathbb{Q}_p$ by the multiplication by p in the p-adic numbers. Then θ is a topological automorphism of $\mathbb{Q}_p$. Let $G = \mathbb{Q}_p \rtimes \langle \theta \rangle$ by the semidirect product defined by means of action determined by this automorphism. Let $K = \mathbb{Z}_p$ be the compact group of p-adic integers. Then K is a compact subgroup of G, yet K^G coincides with $\mathbb{Q}_p$, so it is not relatively compact. Hence, $\mathfrak{C}(G)$ is not uniformly bilateral.

7.1.2 Pre-bornological groups

Following the definition of topological groups, the definition of its large-scale notion can be easily imagined.

Definition 7.1.17. Let G be a group. A para-bornology β on G is a *group pre-bornology* if both the group operation $\cdot\colon (G, \beta) \times (G, \beta) \to (G, \beta)$ and the inverse map $i\colon (G, \beta) \to (G, \beta)$ are boundedness preserving. In that case, the pair (G, β) is called *pre-bornological group.*

The notion of pre-bornological group is an immediate generalisation of the classical definition of bornological group (see, for example, [11, 143]). A *bornological group* is a group G endowed with a bornology β such that both the group operation $\cdot\colon G \times G \to G$ and the inverse map $i\colon G \to G$ are boundedness preserving.

Let us first prove that, a group pre-bornology is actually a pre-bornology.

Fact 7.1.18. *Let G be a group and β be a group pre-bornology. Then β is a pre-bornology.*

Proof. Let $x, y \in G$, and $\{y\} \in \beta(x)$. Since the inverse map is boundedness preserving, $\{y^{-1}\} \in \beta(x^{-1})$. Then, since the multiplication is boundedness preserving, $\{x\} \in \beta(x)$, and $\{y\} \in \beta(y)$, we have that $xy^{-1} \in \beta(xx^{-1}) = \beta(e)$, and so $x = xy^{-1}y \in \beta(y)$.

Let now $A \subseteq G$ be a subset which is bounded from y. Then $y^{-1}A \in \beta(y^{-1}y) = \beta(e)$, and so $A = yy^{-1}A \in \beta(x)$. □

Facts 7.1.18 and 2.2.3 imply that, for a pre-bornological group all the connectedness axioms coincide. This observation shows a similarity with topological algebra. Namely, for topological groups, all the separation axioms from (T_0) to $(\mathrm{T}_{3,5})$ coincide.

It is natural to ask whether pre-bornologies induced by group coarse structures are actually group pre-bornologies. This is true in some cases, as Proposition 7.1.19 shows. Before proving that implication, let us note that, if G is a group, $\mathcal{I}$ is a group ideal on it, $A\subseteq G$ and $x\in G$, then

$$A\in\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}(x)\ (A\in\beta_{\mathcal{E}_{\mathcal{I}}^{\rho}}(x))\text{ if and only if there exists }K\in\mathcal{I}\text{ such that }A\subseteq xK\ (A\subseteq Kx,\text{ respectively}).\quad(7.4)$$

Proposition 7.1.19. *Let G be a a group and $\mathcal{I}$ be a group ideal. Suppose that, for every $A\in\mathcal{I}$ and every $g\in G$, $A^g\in\mathcal{I}$. Then $\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}$ and $\beta_{\mathcal{E}_{\mathcal{I}}^{\rho}}$ are group pre-bornologies.*

Proof. For the sake of simplicity, denote by $\beta=\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}$, while the other case can be similarly shown. Let $x,y\in G$, $A\in\beta(x)$, and $B\in\beta(y)$. Thus, there exists $F,K\in\mathcal{I}$ such that $A\subseteq xF$ and $B\subseteq yK$. Then, $AB\subseteq xFyK=xyF^yK\in\beta(xy)$, because $F^yK\in\mathcal{I}$ taking into account the assumptions. Thus, the multiplication map is boundedness preserving. Moreover, $A^{-1}\subseteq K^{-1}x^{-1}=x^{-1}(K^{-1})^{x^{-1}}\in\beta(x^{-1})$, which shows that also the inverse map i is boundedness preserving. □

In particular, the hypothesis of Proposition 7.1.19, for a group G and a group ideal $\mathcal{I}$, are fulfilled if, for example, $G=\bigcup\mathcal{I}$ (equivalently, $(G,\mathcal{E}_{\mathcal{I}}^{\lambda})$ and $(G,\mathcal{E}_{\mathcal{I}}^{\rho})$ are connected, see §7.2) or the ideal $\mathcal{I}$ is uniformly bilateral.

Corollary 7.1.20. *Let G be a group, H be a subgroup of G and $\mathcal{I}=[H]^{<\omega}$. Then the following properties are equivalent:*
(a) $\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}$ is a group pre-bornology;
(b) $\beta_{\mathcal{E}_{\mathcal{I}}^{\rho}}$ is a group pre-bornology;
(c) H is a normal subgroup of G.

Proof. Let us prove the equivalence (a)↔(c), while (b)↔(c) can be similarly shown. Suppose that (c) holds. Then, for every $K\in[H]^{<\omega}$ and every $g\in G$, K^g is finite and $K^g\subseteq H$ since H is normal. Hence the conclusion follows from Proposition 7.1.19. Conversely, assume that H is not normal. Hence, there exist $k\in H$ and $g\in G$ such that $k^g\notin H$. If $\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}$ is a group pre-bornology, then

$$g^{-1}kgk^{-1}\in\beta(g^{-1}g)=\beta(e)\text{ if and only if }g^{-1}kgk^{-1}\in H,$$

which is a contradiction since it implies $g^{-1}kg\in Hk^{-1}=H$. □

As a consequence of Corollary 7.1.20, for every group G, $\beta_{\mathcal{E}^{\lambda}_{[G]^{<\omega}}}$ and $\beta_{\mathcal{E}^{\rho}_{[G]^{<\omega}}}$ are group pre-bornologies. Furthermore, Corollary 7.1.20 provides examples of group coarse structures that do not induce group pre-bornologies. For example, if $G=S_3$ and $\mathcal{I}=[\langle(1,2)\rangle]^{<\omega}$ then $\beta_{\mathcal{E}_{\mathcal{I}}^{\lambda}}$ and $\beta_{\mathcal{E}_{\mathcal{I}}^{\rho}}$ are not group pre-bornologies since $\langle(1,2)\rangle$ is not normal in S_3.

Theorem 7.1.21. *Let G be a group and β be a para-bornology of G. Then the following properties are equivalent:*
(a) β is a group pre-bornology;
(b) $\beta(e)$ is a group ideal and, for every $x\in G$, $x\beta(e)=\beta(x)=\beta(e)x$.

Moreover, items (a) and (b) imply the following property:
(c) there exists a left (and right) group coarse structure $\mathcal{E}_\mathcal{I}$ such that $\beta = \beta_{\mathcal{E}_\mathcal{I}}$.
Furthermore, if, for every $K \in \beta(e)$ and $g \in G$, $K^g \in \beta(e)$, then item (c) is equivalent to (a) and (b).

Proof. (a)→(b) Since both the multiplication and the inverse map are boundedness preserving, taking into account Fact 7.1.18, it is easy to check that $\beta(e)$ is a group ideal. Let now x ba a point, $A \in \beta(e)$, and $B \in \beta(x)$. Since the multiplication map is boundedness preserving, $Ax, xA \in \beta(x)$, and thus $x\beta(e) \subseteq \beta(x) \supseteq \beta(e)x$. Furthermore, for the same reason, $x^{-1}B, Bx^{-1} \in \beta(e)$, and thus $B = xx^{-1}B = Bx^{-1}x \in x\beta(e) \cap \beta(e)x$.

(b)→(a) Let $x, y \in G$, $A \in \beta(x)$ and $B \in \beta(y)$. Then

$$AB = (xy)(y^{-1}x^{-1}Ay)(y^{-1}B) \in (xy)\beta(e) = \beta(xy),$$

since $y^{-1}x^{-1}Ay, y^{-1}B \in \beta(e)$. Moreover, since $x^{-1}A \in \beta(e)$ and thus $A^{-1}x \in \beta(e)$, $A^{-1} = A^{-1}xx^{-1} \in \beta(x^{-1})$.

The implication (b)→(c) follows from the equalities $\beta = \beta_{\mathcal{E}^\lambda_{\beta(e)}} = \beta_{\mathcal{E}^\rho_{\beta(e)}}$, which are easy to deduce because of the assumptions. Moreover, assuming the further hypothesis, Proposition 7.1.19 implies the claim. □

In particular, Theorem 7.1.21 implies that, for every pre-bornological group G, both the left and the right shifts are large-scale isomorphisms.

7.2 Description of large-scale properties by ideals

7.2.1 Large-scale properties of homomorphisms

Propositions 7.2.1 and 7.2.5 are relaxed versions of classical results in the framework of coarse structures on groups ([125]).

Proposition 7.2.1. *Let $f\colon M \to N$ be a homomorphism between unitary magmas, and $\mathcal{I}_M$ and $\mathcal{I}_N$ be two magmatic ideals on M and N, respectively. Then the following properties are equivalent:*
(a) $f(\mathcal{I}_M) = \{f(K) \mid K \in \mathcal{I}\} \subseteq \mathcal{I}_N$;
(b) $f\colon (M, \mathcal{E}^\lambda_{\mathcal{I}_M}) \to (N, \mathcal{E}^\lambda_{\mathcal{I}_N})$ is bornologous;
(c) $f\colon (M, \mathcal{E}^\rho_{\mathcal{I}_M}) \to (N, \mathcal{E}^\rho_{\mathcal{I}_N})$ is bornologous.

Proof. The implications (b)→(a) and (c)→(a) are trivial. In fact, for every $K \in \mathcal{I}_M$,

$$f(K) = f(E^\lambda_K[e_M]) \subseteq ((f \times f)(E^\lambda_K))[f(e_M)] = ((f \times f)(E^\lambda_K))[e_N].$$

Let us now prove (a)→(b), and (a)→(c) can be similarly shown. Let $K \in \mathcal{I}$. Then, for every $(x, xk) \in E^\lambda_K$, $(f \times f)(x, xk) = (f(x), f(x)f(k)) \in E^\lambda_{f(K)}$, and thus

$$(f \times f)(E^\lambda_K) \subseteq E^\lambda_{f(K)} \in \mathcal{E}^\lambda_{\mathcal{I}_N}. \tag{7.5}$$

□

Fact 7.2.2. *Let M and N be two loops and $f\colon M \to N$ be a homomorphism between them.*
(a) $f(x)^\lambda = f(x^\lambda)$ and $f(x)^\rho = f(x^\rho)$, for every $x \in M$.
(b) $f(M)$ is a subloop *of N.*
(c) If M has two-sided inverses, then $f(M)$ has also two-sided inverses.

Proof. (a) Let $x \in M$. Then

$$f(x)^\lambda f(x) = e_N = f(e_M) = f(x^\lambda x) = f(x^\lambda)f(x) \quad \text{and}$$
$$f(x)f(x)^\rho = e_N = f(e_M) = f(xx^\rho) = f(x)f(x^\rho),$$

and so the conclusion follows by uniqueness of the solutions of (1.3).

(b) Let $f(a)$ and $f(b)$ be two elements in $f(M)$. Then there exists a unique $x \in M$ such that $ax = b$ and so $f(a)f(x) = f(b)$. Moreover, the solution $f(x)$ is unique since N is a loop.

(c) For every $f(x) \in N$, $f(x)^\lambda = f(x^\lambda) = f(x^\rho) = f(x)^\rho$. □

Proposition 7.2.3. *Let M and N be two loops, $\mathcal{I}_M$ and $\mathcal{I}_N$ be two magmatic ideals on M and N, respectively, and $f\colon M \to N$ be a homomorphism. Suppose that M has the left inverse property. Then the following properties are equivalent:*
(a) $f\colon (M, \mathcal{E}^\lambda_{\mathcal{I}_M}) \to (N, \mathcal{E}^\lambda_{\mathcal{I}_N})$ is uniformly boundedness copreserving;
(b) $f\colon (M, \mathcal{E}^\lambda_{\mathcal{I}_M}) \to (N, \mathcal{E}^\lambda_{\mathcal{I}_N})$ is uniformly weakly boundedness copreserving;
(c) for every $K \in \mathcal{I}_H$, there exists $L \in \mathcal{I}_G$ such that $K \cap f(G) \subseteq f(L)$.

Proof. The implication (a)→(b) follows from Proposition 3.1.15.

(b)→(c) Let $K \in \mathcal{I}_H$ and fix an element $L \in \mathcal{I}_G$ such that $E_K \cap (f(G) \times f(G)) \subseteq (f \times f)(E_L)$. We claim that $K \cap f(G) \subseteq f(L)$. Let $k \in K \cap f(G) = (E_K \cap (f(G) \times f(G)))[e]$. Then $k \in ((f \times f)(E_L))[e]$, which means that there exists $(z, w) \in E_L$ such that $f(z) = e$ and $f(w) = k$. Since $w \in zL$, there exists $l \in L$ such that $w = zl$, and thus $l = z^\lambda w$. This implies the following chain of equalities:

$$f(l) = f(z^\lambda w) = f(z)^\lambda f(w) = k$$

since $f(z) = e$, and so the desired claim descends.

(c)→(a) Let $E_K \in \mathcal{E}_{\mathcal{I}_H}$ be an entourage, where $K \in \mathcal{I}_H$, and $L \in \mathcal{I}_G$ satisfying the hypothesis. We claim that, for every $g \in G$, $E_K[f(g)] \cap f(G) \subseteq f(E_L[g])$. Fix an element $g \in G$ and $k \in K$. Assume that $f(g)k \in E_K[f(g)] \cap f(G)$, which means that there exists $h \in G$ such that $f(g)k = f(h)$. Then $k \in f(G)$ and so there exists $l \in L$ such that $f(l) = k$. Finally, $f(g)k = f(g)f(l) = f(gl) \in f(E_L[g])$. □

Fact 7.2.4. *Let $f\colon G \to H$ be a homomorphism between two groups, $\mathcal{I}_G$ and $\mathcal{I}_H$ be a magmatic ideal on G and a monoid ideal on H, respectively. Then the following equivalences hold:*
(a) $f\colon (G, \mathcal{E}^\lambda_{\mathcal{I}_G}) \to H$ is large-scale injective if and only if $\ker f \in \mathcal{I}_G$;
(b) $f\colon G \to (H, \mathcal{E}^\lambda_{\mathcal{I}_H})$ is large-scale surjective if and only if there exists $K = K^{-1} \in \mathcal{I}_H$ such that $f(G)K = H$.

Proof. Since, for every $g \in G$, $f^{-1}(f(g)) = R_f[g] = g\ker f$, the desired equivalence trivially follows. □

Proposition 7.2.5. *Let $f\colon M \to N$ be a homomorphism between two loops with inverse properties, and $\mathcal{I}_M$ and $\mathcal{I}_N$ be two magmatic ideals on M and N, respectively. Assume that M has two-side inverses. Then the following properties are equivalent:*
(a) $f^{-1}(\mathcal{I}_N)\subseteq\mathcal{I}_M$;
(b) $f\colon(M,\mathcal{E}^\lambda_{\mathcal{I}_M})\to(N,\mathcal{E}^\lambda_{\mathcal{I}_N})$ *is effectively proper.*

Proof. Implication (b)→(a) is trivial. In fact, for every $K\in\mathcal{I}_N$, $f^{-1}(K)=f^{-1}(E^\lambda_K[e])\subseteq((f\times f)^{-1}(E^\lambda_K))[e]$ and $(f\times f)^{-1}(E^\lambda_K)\in\mathcal{I}_M$. Conversely, let $e\in K\in\mathcal{I}_N$ and $(x,y)\in(f\times f)^{-1}(E^\lambda_K)$. Then $f(y)\in f(x)K$, which implies that $f(x^{-1}y)\in K$. Hence, $x^{-1}y\in f^{-1}(K)$ and thus $(x,y)\in E^\lambda_{f^{-1}(K)}$. □

Let $f,g\colon S\to M$ be two maps from a set to a monoid, and $\mathcal{I}$ be a monoid ideal on M. Then f and g are Sym-close if and only if there exists $K=K^{-1}\in\mathcal{I}$ such that, for every $x\in X$, $g(x)\in f(x)K$ (or, equivalently, $(f(x),g(x))\in E_K=E_K^{-1}$). In this case, we say that f and g are K-Sym-*close* and we write $f\sim^K_{\mathrm{Sym}}g$ (or, more simply, if M is a group, K-*close* and we write $f\sim_K g$).

The following corollary trivially follows from Propositions 7.2.1 and 7.2.3, Fact 7.2.4, and Theorem 3.4.6.

Corollary 7.2.6. *Let $f\colon G\to H$ be a homomorphism between two groups, and $\mathcal{I}_G$ and $\mathcal{I}_H$ be two monoid ideals on G and H, respectively. Then $f\colon(G,\mathcal{E}^\lambda_{\mathcal{I}_G})\to(H,\mathcal{E}^\lambda_{\mathcal{I}_H})$ is a* Sym-*coarse equivalence if and only if the following conditions hold:*
(a) $\ker f\in\mathcal{I}_G$;
(b) there exists $K=K^{-1}\in\mathcal{I}$ such that $f(G)K=H$;
(c) for every $K\in\mathcal{I}_G$, $f(K)\in\mathcal{I}_H$;
(d) for every $K\in\mathcal{I}_H$, there exists $L\in\mathcal{I}_G$ such that $K\cap f(G)\subseteq f(L)$.

Note that, even if a homomorphism between two coarse groups satisfies the hypotheses of Corollary 7.2.6, and thus it is a coarse equivalence, it is not true in general that there exists a coarse inverse which is a homomorphism. This problem will be fully discussed in §7.3 and in §8.4.

Large-scale properties of homomorphisms between pre-bornological groups

Let us now consider homomorphisms between pre-bornological groups. Recall that the continuity of homomorphisms between topological groups can be checked just on the filters of neighbourhood of the identities. Similarly, the properties of morphisms between pre-bornological groups can be just checked on the group ideal of the subsets bounded from the identity.

Proposition 7.2.7. *Let $f\colon(G,\beta_G)\to(H,\beta_H)$ be a homomorphism between large-scale topological groups. Then:*
(a) f is boundedness preserving if and only if, for every $A\in\beta_G(e_G)$, $f(A)\in\beta_H(e_H)$;
(b) f is weakly boundedness copreserving if and only if, for every $A\in\beta_H(e_H)$ with $A\subseteq f(G)$, there exists $B_z\in\beta_G(z)$, for every $z\in\ker f$, such that $f(\bigcup_{z\in\ker f}B_z)$;
(c) f is boundedness copreserving if and only if, for every $A\in\beta_H(e_H)$ with $A\subseteq f(G)$, there exists $B\in\beta_G(e_G)$ such that $f(B)=A$;

(d) f is proper if and only if, for every $A\in\beta_H(e_H)$, $f^{-1}(A)\in\beta_G(e_G)$.

Proof. All 'only if' implications are trivial. Let us now focus on the 'if' direction.

(a) Let $g\in G$ and $A\in\beta_G(g)$. Then $g^{-1}A\in\beta(e_G)$ and so $f(g^{-1}A)=f(g)^{-1}f(A)\in\beta_H(e_H)$. Finally, $f(A)=f(g)f(g)^{-1}f(A)\in\beta_H(f(g))$.

(b) Let $g\in G$ and $A\in\beta_H(f(g))$, with $A\subseteq f(G)$. Then $f(g)^{-1}A\in\beta_H(e_H)$ and $f(g)^{-1}A\subseteq f(G)$. Applying the hypothesis, we have a family $\{B_z\}_{z\in\ker f}$ of subsets $B_z\in\beta_G(e_G)$ such that $f(\bigcup_{z\in\ker f}B_z)=f(g)^{-1}A$. Define $B'_{gz}=gB_z$, for every $z\in\ker f$. Note that $f^{-1}(f(g))=g\ker f$. Then

$$f\Big(\bigcup_{z\in\ker f}B'_{gz}\Big)=\bigcup_{z\in\ker f}f(g)f(B_z)=f(g)\bigcup_{z\in\ker f}f(B_z)=f(g)f(g)^{-1}A=A.$$

(c) Let $g\in G$ and $A\in\beta_H(f(g))$, with $A\subseteq f(G)$. Then $f(g)^{-1}A\in\beta_H(e_H)$ and $f(g)^{-1}A\subseteq f(G)$. Applying the hypothesis, there exists $B\in\beta_G(e_H)$ such that $f(B)=f(g)^{-1}A$. Then $gB\in\beta_G(g)$ and $f(gB)=f(g)f(g)^{-1}A=A$.

(d) Let $g\in G$ and $A\in\beta_H(f(g))$. Then $f(g)^{-1}A\in\beta_H(e_H)$ and so $f^{-1}(f(g)^{-1}A)\in\beta_G(e)$. It is not hard to prove that $f^{-1}(f(g)^{-1}A)\supseteq g^{-1}f^{-1}(A)$. Then $f^{-1}(A)=gg^{-1}f^{-1}(A)\in\beta_G(g)$. □

Note that the para-bornological spaces in Example 2.1.12(b) are actually pre-bornological groups and so the notions of weak boundedness copreservation and boundedness copreservation do not coincide even in the realm of pre-bornological groups (even when the fibres are bounded). This is a striking difference with the situation in coarse groups. In fact, in Proposition 7.2.3 it is shown that the notions of uniform weak boundedness copreservation and uniform boundedness copreservation coincide in the realm of coarse groups.

Corollary 7.2.8. *Let $\mathcal{I}_G$ and $\mathcal{I}_H$ be two group ideals on the groups G and H, respectively, and $f\colon G\to H$ be a homomorphism. Suppose that, for every $K\in\mathcal{I}_G$, $F\in\mathcal{I}_H$, $g\in G$, and $h\in H$, $K^g\in\mathcal{I}_G$ and $F^h\in\mathcal{I}_H$. Denote by $\mathcal{E}_G=\mathcal{E}^\lambda_{\mathcal{I}_G}$, $\mathcal{E}_H=\mathcal{E}^\lambda_{\mathcal{I}_H}$, $\beta_G=\beta_{\mathcal{E}_G}$, and $\beta_H=\beta_{\mathcal{E}_H}$. Then*

(a) $f\colon(G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ is bornologous if and only if $f\colon(G,\beta_G)\to(H,\beta_H)$ is boundedness preserving;

(b) $f\colon(G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ is effectively proper if and only if $f\colon(G,\beta_G)\to(H,\beta_H)$ is proper;

(c) $f\colon(G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ is uniformly boundedness copreserving if and only if $f\colon(G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ is uniformly weakly boundedness copreserving if and only if $f\colon(G,\beta_G)\to(H,\beta_H)$ is boundedness copreserving.

Proof. The proof immediately follows by applying Propositions 7.1.18, 7.2.1, 7.2.3, 7.2.5 and 7.2.7. □

7.2.2 Connectedness components

Let M be a magma and $\mathcal{I}$ be a magmatic ideal on M. Then, if we endow M with the entourage structure $\mathcal{E}^\lambda_{\mathcal{I}}$, $\mathcal{Q}^\downarrow_M(e)=\bigcup\mathcal{I}$. Moreover, for every $x\in M$, $\mathcal{Q}^\downarrow_M(x)=x\bigcup\mathcal{I}$.

Suppose now that G is a group and $\mathcal{I}$ is a magmatic ideal on G. We want to characterise all the operators defined in (2.3) in terms of the magmatic ideal. It is not hard to show that, for every $x \in G$,

$$\mathcal{Q}_M^{\uparrow}(x) = x\bigcup(\mathcal{I}^{-1}), \quad \mathcal{Q}_M^{\searrow}(x) = x\langle\bigcup\mathcal{I}\rangle_{mon},$$
$$\mathcal{Q}_M^{\nearrow}(x) = x\langle\bigcup\mathcal{I}\rangle_{mon}, \quad \text{and} \quad \mathcal{Q}_M(x) = x\langle\bigcup\mathcal{I}\rangle_{grp},$$

where $\langle A\rangle_{mon}$ and $\langle A\rangle_{grp}$ denote the submonoid and the subgroup of G, respectively, generated by a subset A of G.

Suppose now that $\mathcal{I}$ is a group ideal. Then, in particular, $\mathcal{Q}_G(e) = \bigcup\mathcal{I}$ and G is connected if and only if $G = \bigcup\mathcal{I}$. Since $\bigcup\mathcal{I}$ is a subgroup, the partition of G in its connected components coincides precisely with its partition in left cosets of $\bigcup\mathcal{I}$. Note that $\bigcup\mathcal{I}$ is not necessarily normal (as Example 7.1.15(a) shows). Actually, if G is a group and H is a subgroup of G, then $\mathcal{I} = \mathfrak{cl}(\{H\})$ is a group ideal on G (similar group coarse structures are examples of linear coarse structures, §9), so providing examples of group coarse structures inducing a non-normal connected component of the identity is very easy. This phenomenon is in contrast with the theory of topological groups. In fact, for every topological group, the connected component of the identity is always a normal subgroup.

7.2.3 Categorical constructions

Group ideals are useful also to describe some categorical constructions, in particular products and quotients of coarse groups.

Let $\{G_i\}_{i\in I}$ be a family of groups, and $\mathcal{E}_i$ be a coarse structure on G_i, for every $i \in I$. For the sake of simplicity, we will denote by $\Pi_i E_i$ the entourage $\bigcap_i (p_i \times p_i)^{-1}(E_i)$, where $E_i \in \mathcal{E}_i$, for every $i \in I$. Note that, for every $(x_i)_i \in \Pi_i G_i$, $(\Pi_i E_i)[(x_i)_i] = \Pi_i(E_i[x_i])$, and thus, in particular, if, for every $i \in I$, $\mathcal{E}_i = \mathcal{E}_{\mathcal{I}_i}$ for some group ideal $\mathcal{I}_i$ and $E_i = E_{K_i}$, where $K_i \in \mathcal{I}_i$,

$$(\Pi_i E_{K_i})[e] = \Pi_i(E_{K_i}[e]) = \Pi_i K_i. \tag{7.6}$$

Proposition 7.2.9. *Let $\{(G_i, \mathcal{E}_{\mathcal{I}_i})\}_{i\in I}$ be a family of coarse groups. Then the product coarse structure $\mathcal{E}$ on the direct product $\Pi_i G_i$ is a group coarse structure and it is generated by the base $\mathcal{I} = \{\Pi_i K_i \mid K_i \in \mathcal{I}_i, \forall i \in I\}$.*

Proof. We want to use Corollary 7.1.8. Fix an element $E \in \mathcal{E}$, and, without loss of generality, suppose that $E = \Pi_i E_{K_i}$, where $K_i \in \mathcal{I}_i$, for every $i \in I$. It is easy to check that $(\Pi_i G_i)(\Pi_i E_{K_i}) = \Pi_i(G_i E_{K_i})$. Since, for every $i \in I$, $\mathcal{E}_{\mathcal{I}_i}$ satisfies Corollary 7.1.8, $G_i E_{K_i} \in \mathcal{E}_{\mathcal{I}_i}$, and thus $(\Pi_i G_i)(\Pi_i E_{K_i}) \in \mathcal{E}$. The fact that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$ follows from (7.6) and again Corollary 7.1.8. □

Let us state a trivial, but useful, property.

Fact 7.2.10. *If $f\colon G \to H$ is a homomorphism and $\mathcal{I}$ is a group ideal on G, then $f(\mathcal{I})$ is a group ideal on H.*

Proposition 7.2.11. *Let $q\colon G \to G/N$ be a quotient homomorphism of groups, and $\mathcal{I}$ be a group ideal on G. Then the map $q\colon (G, \mathcal{E}_{\mathcal{I}}) \to (G/N, \mathcal{E}_{q(\mathcal{I})})$ is bornologous and uniformly bounded copreserving, and thus uniformly soft. Moreover, the map q is a coarse equivalence if and only if $N \in \mathcal{I}$.*

Proof. The first claim is trivial, thanks to Propositions 7.2.1, 7.2.3 and 4.3.18. If q is a coarse equivalence, then Proposition 7.2.5 implies that $N = q^{-1}(e_{G/N}) \in \mathcal{I}$. Let us focus on the opposite implication, which can be found also in [125]. Suppose that $N = \ker q \in \mathcal{I}$. Then, in virtue of Corollary 7.2.6, q is a coarse equivalence since q trivially satisfies the conditions. □

The fact that a quotient of a coarse group is a uniformly soft map was proved in [65] in order to provide an example of the notion of uniformly soft map that was introduced in that paper.

As a corollary of Proposition 7.2.11 we obtain the following result.

Corollary 7.2.12. *If G is a group with finite G', then every group coarse structure $\mathcal{E}$ on G is uniformly invariant.*

Proof. In order to see that $\mathcal{E}_\mathcal{I}^\lambda = \mathcal{E}_\mathcal{I}^\rho$ consider the quotient map $q\colon G \to G/G'$ and equip G/G' with the quotient group coarse structure $\mathcal{E}_{f(\mathcal{I})}^\lambda = \mathcal{E}_{f(\mathcal{I})}^\rho$ (they coincide since the group ideal $f(\mathcal{I})$ is uniformly bilateral in the abelian group G/G'). Since

$$q\colon (G, \mathcal{E}_\mathcal{I}^\lambda) \to (G/G', \mathcal{E}_{f(\mathcal{I})}^\lambda) \quad \text{and} \quad q\colon (G, \mathcal{E}_\mathcal{I}^\rho) \to (G/G', \mathcal{E}_{f(\mathcal{I})}^\rho)$$

are coarse equivalences and $\mathcal{E}_{f(\mathcal{I})}^\lambda = \mathcal{E}_{f(\mathcal{I})}^\rho$, we deduce that the identity $(G, \mathcal{E}_\mathcal{I}^\lambda) \to (G, \mathcal{E}_\mathcal{I}^\rho)$ is a coarse equivalence (actually, an asymorphism). □

Another consequence of Proposition 7.2.11 is the following Corollary 7.2.13. A similar result can be proved for the compact-group coarse structure (see Corollary 10.0.1), and for the left-coarse structure (see [158]).

Corollary 7.2.13. *Let G be a topological group and K be a Lindelöf normal subgroup of G. Suppose that G/K is discrete. Then $q\colon G \to G/K$ is a coarse equivalence provided that both G and G/K are endowed with their ω_1-Lindelöf-group coarse structures.*

Proof. According to Proposition 7.2.11, it is enough to show that

$$q(\mathfrak{cl}(\omega_1\text{-}\mathcal{L}(G))) = \mathfrak{cl}(\omega_1\text{-}\mathcal{L}(G/K)).$$

The inclusion ($\subseteq$) follows from general topology properties. Since G/K is discrete, $\mathfrak{cl}(\omega_1\text{-}\mathcal{L}(G/K)) = [G/K]^{<\omega_1}$. Hence, every subset $L \in \mathfrak{cl}(\omega_1\text{-}\mathcal{L}(G/K))$ is countable and thus $q^{-1}(L)$ is a countable union of cosets of K, which is still Lindelöf. □

7.2.4 Metrisability of coarse groups

A coarse group $(G, \mathcal{E}_\mathcal{I})$ is *metrisable* if there exists a left-invariant metric d such that $\mathcal{E}_\mathcal{I} = \mathcal{E}_d$.

Lemma 7.2.14. *Let $(G, \mathcal{E}_\mathcal{I})$ be a coarse group. Then the following properties are equivalent:*

(a) $(G, \mathcal{E}_\mathcal{I})$ is metrisable as a coarse space (i.e., there exists a metric d such that $\mathcal{E}_\mathcal{I} = \mathcal{E}_d$);

(b) $\mathcal{E}_\mathcal{I}$ *has a countable base;*
(c) $\mathcal{I}$ *has a countable cofinal subset with respect to inclusion, i.e., there exists a countable family* $\mathcal{J} \subseteq \mathcal{I}$ *such that* $\mathcal{I} = \mathfrak{cl}(\mathcal{J})$*;*
(d) $(G, \mathcal{E}_\mathcal{I})$ *is metrisable as a coarse group (i.e., there exists a left-invariant metric* d *such that* $\mathcal{E}_\mathcal{I} = \mathcal{E}_d$*).*

Proof. The equivalence between items (a) and (b) is provided in Theorem 3.4.12. The implication (d)→(a) is straightforward.

(b)→(c) Let $\mathcal{B}$ be a countable base of $\mathcal{E}_\mathcal{I}$. We can assume, without loss of generality, that there exists $\mathcal{L} \subseteq \mathcal{I}$ such that $\mathcal{B} = \{E_K \mid K \in \mathcal{L}\}$. Since, for every $M, N \in \mathcal{I}$, $E_M \subseteq E_N$ if and only if $M \subseteq N$, $\mathfrak{cl}(\mathcal{L}) = \mathcal{I}$.

(c)→(d) Without loss of generality, we can assume that there exists a cofinal family $\{K_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{I}$ which satisfies the following properties:
- $K_0 = \{e\}$;
- for every $n \in \mathbb{N}$, $e \in K_n = K_n^{-1}$;
- for every $m, n \in \mathbb{N}$, $K_n K_m \subseteq K_{n+m}$ (in particular it is an increasing sequence).

Then the map $d\colon G \times G \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ defined by the law

$$d(g,h) = \min\{n \in \mathbb{N} \mid h \in gK_n\},$$

for every $g, h \in G$, satisfies the desired properties. □

A proof of Lemma 7.2.14 for the left-coarse structure $\mathcal{E}_L$ can be found in [158].

7.2.5 Asymptotic dimension of coarse groups

It is easy to see that a family $\mathcal{U}$ of subsets of a coarse group $(G, \mathcal{E}_\mathcal{I})$ is uniformly bounded if and only if there exists $K \in \mathcal{I}$ such that $U \subseteq E_K[x] = xK$, for every $U \in \mathcal{U}$ and $x \in U$. If $K \in \mathcal{I}$, we say that $\mathcal{U}$ is *K-disjoint* if, for every pair of distinct elements $U, V \in \mathcal{U}$, $U \cap E_K[V] = U \cap VK = \emptyset$.

Proposition 7.2.15. *A coarse group* $(G, \mathcal{E}_\mathcal{I})$ *has asymptotic dimension at most* n *(*$\operatorname{asdim}(G, \mathcal{E}_\mathcal{I}) \leq n$*), where* $n \in \mathbb{N}$*, if and only if, for every* $K \in \mathcal{I}$*, there exists a uniformly bounded cover* $\mathcal{U} = \mathcal{U}_0 \cup \dots \cup \mathcal{U}_n$ *such that, for every* $i = 0, \dots, n$*,* $\mathcal{U}_i$ *is* K*-disjoint.*

If we take a coarse group $(G, \mathcal{E}_\mathcal{I})$, then, for every $e \in K = K^{-1} \in \mathcal{I}$, $E_K \circ E_K = E_{K \cdot K}$. Hence, a coarse group is cellular if and only if, for every symmetric element $K \in \mathcal{I}$ containing the identity, $E_K^{\square} = E_{\langle K \rangle} \in \mathcal{E}_\mathcal{I}$, which means that $\langle K \rangle \in \mathcal{I}$. We have then showed that a coarse group $(G, \mathcal{E}_\mathcal{I})$ is cellular if and only if $\mathcal{I}$ has a cofinal family, with respect of inclusion, consisting of subgroups. A group coarse structure satisfying that property is called *linear*. This concept will be investigated in §9. The equivalence between cellular coarse groups and linear coarse groups was already pointed out in [133].

The following result descends as an easy consequence of the above discussion. Recall that a group G is *locally finite* if every finitely generated subgroup of G is finite.

Fact 7.2.16. *Let G be a group endowed with its finitary-group coarse structure. Then* $\operatorname{asdim} G = 0$ *if and only if G is locally finite. Moreover, if G is abelian, the previous conditions are equivalent to G being torsion.*

For some coarse groups we have the following criterion for cellularity, which is also proved in [125], but with a stronger hypothesis.

Proposition 7.2.17. *Let $(G, \mathcal{E}_{\mathcal{I}})$ be a coarse group such that there exists an element $K \in \mathcal{I}$ that algebraically generates the whole group G. Then* $\operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}}) = 0$ *if and only if $G \in \mathcal{I}$.*

Proof. Without loss of generality, we can assume that $K = K^{-1}$. Since $G = \langle K \rangle$, for every $U \subsetneq G$, $U \subsetneq E_K[U] = UK$. Hence, the only possible K-disjoint cover $\mathcal{U}$ is $\mathcal{U} = \{G\}$. Finally, $\mathcal{U}$ is uniformly bounded if and only if $G \in \mathcal{I}$. □

Proposition 7.2.18. *Let $(G, \mathcal{E}_{\mathcal{I}})$ be a coarse group and denote by $H = \bigcup \mathcal{I} = \mathcal{Q}_G(e)$. Then* $\operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}}) = \operatorname{asdim}(H, \mathcal{E}_{\mathcal{I}})$.

Proof. Since H is a subspace of G and $\mathcal{E}_{\mathcal{I}} = \mathcal{E}_{\mathcal{I}}|_H$, the monotonicity of the asymptotic dimension (Theorem 6.1.2) implies that $\operatorname{asdim}(H, \mathcal{E}_{\mathcal{I}}) \leq \operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}})$.

We need to show the opposite inequality. If $\operatorname{asdim}(H, \mathcal{E}_{\mathcal{I}}) = \infty$, the claim follows. Suppose then that there exists $n \in \mathbb{N}$ with $\operatorname{asdim}(H, \mathcal{E}_{\mathcal{I}}) \leq n$. We claim that $\operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}}) \leq n$. Fix an element $S \in \mathcal{I}$ and let $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ be a uniformly bounded cover of H such that, for every $k = 0, \ldots, n$, $\mathcal{U}_k$ is S-disjoint. Let $\{g_i\}_{i \in I}$ be a family of representatives of the cosets of H in G. Then, for every $k = 0, \ldots, n$, define $\mathcal{U}'_k = \bigcup_{i \in I} g_i \mathcal{U}_k$, where, for every $i \in I$, $g_i \mathcal{U}_k = \{g_i U \mid U \in \mathcal{U}_k\}$. Then $\mathcal{U}'_0 \cup \cdots \cup \mathcal{U}'_n$ is trivially a uniformly bounded cover. Suppose that there exists $k \in \{0, \ldots, n\}$ and $V, W \in \mathcal{U}'_k$ such that $VS \cap W \neq \emptyset$. Let $i \in I$ be an index such that $V \in g_i \mathcal{U}_k$. Then $VS \subseteq g_i H$ and thus $W \in g_i \mathcal{U}_k$. Let $V', W' \in \mathcal{U}_k$ such that $V = g_i V'$ and $W = g_i W'$. Because of the assumption, there exist $v \in V'$, $w \in W'$, and $s \in S$ such that $g_i v s = g_i w$, which implies that $vs = w$ and so $V'S \cap W' \neq \emptyset$. Since the family $\mathcal{U}_k$ is S-disjoint, then $V' = W'$ and $V = W$. □

Proposition 7.2.18 is a natural counterpart of the following property of the covering dimension in compact-like topological groups (for locally compact groups see Theorem 10.2.2, for monotonicity with respect to taking quotients see [38]).

Theorem 7.2.19. *Let G be a topological group and denote by $c(G)$ the connected component of the identity endowed with the subspace topology. If G is locally compact or countably compact, then* $\dim G = \dim c(G)$.

The locally compact case is folklore, for the countably compact one see [36]. Weaker levels of compactness cannot guarantee this property (see [36] for totally disconnected pseudocompact groups of arbitrarily high dimensions and additional compactness-like properties).

Corollary 7.2.20. *Let $(G, \mathcal{E}_{\mathcal{I}})$ be a coarse group and denote by $H = \bigcup \mathcal{I} = \mathcal{Q}_G(e)$. Suppose that H is normal. Then, if we denote by q the quotient map $q \colon G \to G/H$,* $\operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}}) = \operatorname{asdim}(H, \mathcal{E}_{\mathcal{I}}) + \operatorname{asdim}(G/H, \mathcal{E}_{q(\mathcal{I})})$.

Proof. The claim trivially follows from Proposition 7.2.18 and the fact that $q(\mathcal{I}) = \{\{e_{G/H}\}\}$, which implies $\operatorname{asdim}(G/H, \mathcal{E}_{q(\mathcal{I})}) = 0$. □

For example, we can apply Corollary 7.2.20 if the coarse group is bilateral.

In the notation of Corollary 7.2.20, the assumption of normality of H can be relaxed in the following way. Even if H is not normal, the right coset space G/H can be endowed with a coarse structure induced by the coarse space $(G, \mathcal{E}_{\mathcal{I}})$ and the action of G on G/H by right shifts (see [133]). By the definition of H, the induced coarse structure is the discrete one, which has asymptotic dimension 0. Hence Corollary 7.2.20 can be extended also to generic group ideals. Since in these chapters we are mostly interested in groups, we do not give details of that construction.

Let us recall this additivity result due to Dranishnikov and Smith ([70, Theorem 2.3, Corollary 3.3], see also [17] for the case of finitely generated groups).

Theorem 7.2.21. *Let G be a group and H be a normal subgroup. Then*

$$\operatorname{asdim} G \leq \operatorname{asdim} G/H + \operatorname{asdim} H, \tag{7.7}$$

where all groups are endowed with their finitary-group coarse structure. Moreover, if G is abelian,

$$\operatorname{asdim} G = \operatorname{asdim} G/H + \operatorname{asdim} H. \tag{7.8}$$

In the literature (for example in [70]), the inequality (7.7) is usually called *Hurewicz type formula*.

Remark 7.2.22. The statement of Theorem 7.2.21 is given in [70] in a slightly different, but equivalent form. In fact, in [70], the authors defined the asymptotic dimension of an arbitrary group G as the supremum of the asymptotic dimensions of its finitely generated groups endowed with their word metrics. As a consequence of [125, Theorem 3.13] we have that

$$\operatorname{asdim}(G, \mathcal{E}_{[G]^{<\omega}}) = \sup\{\operatorname{asdim}(\langle K\rangle, \mathcal{E}_{[\langle K\rangle]^{<\omega}}) \mid K \in [G]^{<\omega}\}.$$

Thus, applying Remark 7.1.11 to the finitely generated subgroups of G, the definition of the asymptotic dimension provided in [70] coincides with the asymptotic dimension of G endowed with its finitary-group coarse structure.

Easy examples show that in the non-abelian case even the inequality $\operatorname{asdim} G \geq \operatorname{asdim} G/H$ (a consequence of (7.8)) may fail.

Example 7.2.23. Let $G = F_m$ the free group with m free generators, where $m > 1$, so that F_m is not abelian. The free abelian group $\mathbb{Z}^m$ of rank m is a quotient of F_m; so there exists a surjective homomorphism $f\colon G \to \mathbb{Z}^m$. Yet, $\operatorname{asdim} G = 1 < m = \operatorname{asdim} \mathbb{Z}^m$, if both groups are endowed with their finitary-group coarse structures.

Another result that Dranishnikov and Smith proved for abelian groups is the following theorem. In particular, it generalises the second assertion of Fact 7.2.16.

Theorem 7.2.24. [70, Theorem 3.2] *Let G be an abelian group. Then* $\operatorname{asdim} G = r_0(G)$*, where G is endowed with the finitary-group coarse structure.*

We refer to [163] for a direct computation of the asymptotic dimension of $\mathbb{Q}$ endowed with the finitary-group coarse structure (equivalently, with a proper left-invariant metric).

7.3 Quasi-homomorphisms

Definition 7.3.1 ([158]). A map $f\colon G \to (H, \mathcal{E}_{\mathcal{I}})$ from a group G to a coarse group $(H, \mathcal{E}_{\mathcal{I}})$ is a *quasi-homomorphism* if the maps $f', f''\colon G \times G \to H$, where $f'\colon (g, h) \mapsto f(gh)$ and $f''\colon (g, h) \mapsto f(g)f(h)$, are close (equivalently, if there exists $K \in \mathcal{I}$ such that, for every $g, h \in G$, $f(gh) \in f(g)f(h)K$).

If $E \in \mathcal{E}_{\mathcal{I}}$ is a symmetric entourage such that $\{(f(gh), f(g)f(h)) \mid g, h \in G\} \subseteq E$, then f is called an *E-quasi-homomorphism*. In case $E = E_M$ for some $M \in \mathcal{I}$, we briefly write *M-quasi-homomorphism* to say that $f(gh) \in f(g)f(h)M$, for every $g, h \in G$.

By taking $(H, \mathcal{E}_{\mathcal{I}}) = (\mathbb{R}, \mathcal{E}_{\mathcal{B}_d})$, where d is the usual euclidean metric on $\mathbb{R}$, we recover the classical notion.

Remark 7.3.2. Let G be a group, $(H, \mathcal{E}_{\mathcal{I}})$ be a coarse group, $M \in \mathcal{I}$, and $f\colon G \to H$ be an M-quasi-homomorphism. We can assume, without loss of generality, that $M = M^{-1}$. Since $f(e_G) = f(e_G \cdot e_G) \in f(e_G)f(e_G)M$, we have that $e_H \in f(e_G)M = f(e_G)M^{-1}$ and $f(e_G) \in e_H M = M$. Moreover, for every $x \in G$, $f(e_G) = f(xx^{-1}) \in f(x)f(x^{-1})M$, and thus, in particular, $f(x)^{-1} \in f(x^{-1})Mf(e_G)^{-1} \subseteq f(x^{-1})MM$. Thanks to this computation, in the sequel when we say that f is an M-quasi-homomorphism, we assume that $M \in \mathcal{I}$ satisfies

$$f(e_G) \in M, \quad f(y)^{-1} \in f(y^{-1})M \quad \text{and} \quad f(y^{-1}) \in f(y)^{-1}M,$$

for every $y \in G$.

Let us start with some very easy examples.

Example 7.3.3. Let $f\colon G \to (H, \mathcal{E}_{\mathcal{I}})$ be a map between a group and a coarse group.

(a) If f is a homomorphism, then f is a quasi-homomorphism.
(b) f is a quasi-homomorphism, if f is *bounded* (i.e., $f(G)$ is bounded in H), or, equivalently, if $f(G) \in \mathcal{I}$ by Remark 7.3.2. In particular, f is a quasi-homomorphism when $\mathcal{I} = \mathcal{P}(H)$. As a consequence, we have that every map $f\colon (G, \mathcal{P}(G)) \to (H, \mathcal{P}(H))$ is both a quasi-homomorphism and a coarse equivalence.
(c) If $\mathcal{I} = \{e_H\}$, then $f\colon G \to H$ is a quasi-homomorphism if and only if it is a homomorphism.
(d) An asymorphism may not be a quasi-homomorphism. In fact, for example, for every group G, endowed with the discrete coarse structure $\mathcal{E}_{\{\{e\}\}}$, every bijective self-map $f\colon G \to G$ is automatically an asymorphism. However, f is a quasi-homomorphism if and only if f is an isomorphism, according to item (c). Hence, a counter-example can be easily produced.

Here come two very important properties of quasi-homomorphisms.

Proposition 7.3.4. *Let $f, g\colon G \to (H, \mathcal{E}_{\mathcal{I}})$ be two maps between a group G and a coarse group $(H, \mathcal{E}_{\mathcal{I}})$. Suppose that $f \sim_M g$ for some $M \in \mathcal{I}$. If $M^H \in \mathcal{I}$, then f is a quasi-homomorphism if and only if g is a quasi-homomorphism.*

Proof. Suppose that $K \in \mathcal{I}$ is an element such that f is a K-quasi-homomorphism. Then, for every $x, y \in G$,

$$g(xy) \in f(xy)M \subseteq f(x)f(y)KM \subseteq g(x)Mg(y)MKM \subseteq g(x)g(y)M^H MKM,$$

according to (7.3). Therefore, g is a $M^H MKM$-quasi-homomorphism.

The opposite implication can be similarly shown. □

Inspired by Proposition 7.3.4, the reader may think that every quasi-homomorphism is close to a homomorphism. However, this is not the case, as Example 7.3.5 shows.

Example 7.3.5. (a) Let us recall the intuitive example that was briefly described in §1.1. Consider the floor map $\lfloor\cdot\rfloor\colon \mathbb{R} \to \mathbb{Z}$, which is a quasi-homomorphism if we endow $\mathbb{Z}$ with the finitary-group coarse structure. However, since $\mathbb{R}$ is a divisible group, the only homomorphism from $\mathbb{R}$ to $\mathbb{Z}$ is the null-homomorphism, which is not close to $\lfloor\cdot\rfloor$.

(b) Let $f\colon \mathbb{Z} \to 2\mathbb{Z}$ be the map that associates to every integer n the largest even number smaller than n. If $2\mathbb{Z}$ is endowed with the finitary-group coarse structure, then f is a quasi-homomorphism. However it is not close to any homomorphism.

Proposition 7.3.6. *Let G be a group, $(H, \mathcal{E}_{\mathcal{I}})$ and $(K, \mathcal{E}_{\mathcal{J}})$ be two coarse groups, $f\colon G \to H$ be a quasi-homomorphism, and $g\colon H \to K$ be a bornologous quasi-homomorphism. Then $g \circ f$ is a quasi-homomorphism.*

Proof. Suppose that f is an M-quasi-homomorphism and g is an N-quasi-homomorphism, for some $M \in \mathcal{I}$ and $N \in \mathcal{J}$. Then, for every $x, y \in G$,

$$g(f(xy)) \in g(f(x)f(y)M) \subseteq g(f(x))g(f(y)M)N \subseteq g(f(x))g(f(y))g(M)NN,$$

where $g(M)NN \in \mathcal{J}$ (according to Proposition 7.3.8), and so $g\circ f$ is a $g(M)NN$-quasi-homomorphism. □

Note that, without the assumption of bornology of g in Proposition 7.3.6, it is not true that composition of quasi-homomorphisms is still a quasi-homomorphism (see Example 7.3.7(a)). As mentioned in the introduction, this fact has prevented any categorical systematization of quasi-homomorphisms up to now.

Example 7.3.7. (a) By using Example 7.3.3(b), we are able to construct two quasi-homomorphisms whose composite is not a quasi-homomorphism. Let G be a group and $\mathcal{I}$ be a group ideal on it which is different from $\mathcal{P}(G)$. If $f\colon G \to (G, \mathcal{E}_{\mathcal{I}})$ is not a quasi-homomorphism, we have the following situation:

$$G \xrightarrow{f} (G, \mathcal{E}_{\mathcal{P}(G)}) \xrightarrow{id_G} (G, \mathcal{E}_{\mathcal{I}}),$$

where both arrows are quasi-homomorphisms (the identity is a homomorphism), but their composite is not a quasi-homomorphism. For example, set $G = \mathbb{Z}$, $\mathcal{I} = [\mathbb{Z}]^{<\omega}$, and $f = |\cdot|$, the absolute value.

(b) The inverse of a bijective homomorphism is a homomorphism. However, it is not true a similar result for quasi-homomorphisms. Let G be a group and $f\colon G\to G$ be a bijective map which is not a homomorphism. Then $f\colon (G,\mathcal{E}_{\{\{e\}\}})\to(G,\mathcal{E}_{\mathcal{P}(G)})$ is a quasi-homomorphism, while its inverse is not a quasi-homomorphism (using Example 7.3.3(c), f^{-1} is a quasi-homomorphism if and only if it is a homomorphism, which is not true). In Corollary 7.3.12 we give a condition that guarantees that we can revert a bijective quasi-homomorphism obtaining a quasi-homomorphism as well.

Also quasi-homomorphisms allow us to prove a result (Proposition 7.3.8) similar to Proposition 7.2.1.

Proposition 7.3.8. *Let $f\colon (G,\mathcal{E}_{\mathcal{I}_G})\to(H,\mathcal{E}_{\mathcal{I}_H})$ be a quasi-homomorphism between two coarse groups. Then*
(a) f is bornologous if and only if $f(I)\in\mathcal{I}_H$, for every $I\in\mathcal{I}_G$;
(b) if $\mathcal{I}_H$ is uniformly bilateral, then f is effectively proper if and only if f is proper.

Proof. Both 'only if' implications are trivial. Suppose that f is an M-quasi-homomorphism for some $M=M^{-1}\in\mathcal{I}_H$.

(a,$\leftarrow$) Let $K\in\mathcal{I}_G$ and take an arbitrary element $(x,xk)\in E_K$, where $x\in G$ and $k\in K$. Then

$$f(xk)\in f(x)f(k)M\subseteq f(x)f(K)M,$$

which implies that $(f(x),f(xk))\in E_{f(K)M}$. Thus $(f\times f)(E_K)\subseteq E_{f(K)M}\in\mathcal{E}_{\mathcal{I}_H}$ and so f is bornologous.

(b,$\leftarrow$) Let $K\in\mathcal{I}_H$. Then, for every $(x,y)\in(f\times f)^{-1}(E_K)$, there exists $k\in K$ such that $f(y)=f(x)k$, and thus $f(x)^{-1}f(y)=k\in K$. We have

$$\begin{aligned}f(x^{-1}y)&\in f(x^{-1})f(y)M\subseteq f(x)^{-1}Mf(y)M\subseteq\\&\subseteq f(x)^{-1}f(y)M^{f(y)}M\subseteq KM^HM\in\mathcal{I}_H.\end{aligned}$$

Finally, $x^{-1}y\in f^{-1}(KM^HM)\in\mathcal{I}_G$ and $(x,y)=(x,xx^{-1}y)\in E_{f^{-1}(KM^HM)}$, which finishes the proof. □

Theorem 7.3.9. *Let $f\colon (G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ be a quasi-homomorphism between coarse groups which is a coarse equivalence with coarse inverse $g\colon H\to G$. If $\mathcal{E}_H$ is uniformly invariant, then g is a quasi-homomorphism.*

Proof. Let $F\in\mathcal{E}_H$ be a symmetric entourage such that f is an F-quasi-homomorphism. We claim that there exists $E\in\mathcal{E}_G$ such that, for every $x,y\in H$, $(g(xy),g(x)g(y))\in E$. Let $x,y\in H$. Then, since $\cdot\colon H\times H\to H$ is bornologous (Proposition 7.1.13),

$$\begin{aligned}(f(g(xy)),f(g(x)g(y)))&=(f(g(xy)),xy)\circ(xy,f(g(x))f(g(y)))\circ\\&\circ(f(g(x))f(g(y)),f(g(x)g(y)))\in S\circ(\cdot\times\cdot)(S,S)\circ F\in\mathcal{E}_H,\end{aligned}$$

where $S=\{(f(g(z)),z),(z,f(g(z)))\mid z\in H\}\in\mathcal{E}_H$, and thus it suffices to define $E=(f\times f)^{-1}(S\circ(\cdot\times\cdot)(S,S)\circ F)$. □

Theorem 7.3.10. *Let* $f\colon (G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ *be a quasi-homomorphism between coarse groups which is a coarse equivalence. Then:*

(a) if $\mathcal{E}_H$ *is uniformly invariant, then* $\mathcal{E}_G$ *is uniformly invariant;*
(b) if a coarse inverse of f *is a quasi-homomorphism, then* $\mathcal{E}_G$ *is uniformly invariant if and only if* $\mathcal{E}_H$ *is uniformly invariant.*

Proof. Item (b) follows from item (a). Assume that $\mathcal{E}_H$ is uniformly invariant and let $F\in\mathcal{E}_H$ be a symmetric entourage such that f is a F-quasi-homomorphism. Let $E,E'\in\mathcal{E}_G$. Then, for every $(x,y)\in E$ and $(z,w)\in E'$,

$$(f(xz),f(yw))=(f(xz),f(x)f(z))\circ(f(x)f(z),f(y)f(w))\circ \\ \circ(f(y)f(w),f(yw))\in\in E''\in\mathcal{E}_H,$$

where $E''=F\circ(\cdot((f\times f)(E),(f\times f)(E')))\circ F\in\mathcal{E}_N$, since f is bornologous. Finally, since f is effectively proper, $(\cdot\times\cdot)(E,E')\subseteq(f\times f)^{-1}(E'')\in\mathcal{E}_G$, and thus Proposition 7.1.13 implies the claim. □

Note that the quasi-homomorphisms defined in Example 7.3.5 are coarse inverses of the inclusions $i\colon\mathbb{Z}\to\mathbb{R}$ and $i\colon 2\mathbb{Z}\to\mathbb{Z}$, which are homomorphisms and coarse equivalences. Thus these two inclusions have no coarse inverses which are homomorphisms.

In Theorem 7.3.9, the request of uniformly invariance of $\mathcal{E}_H$ is quite restrictive. In fact, we cannot apply the result to a coarse group $(H,\mathcal{E}_H)$ whose points have unbounded orbits under conjugacy. There is a trade-off between the uniformly invariance and the surjectivity of the map, as Corollary 7.3.12 shows.

Lemma 7.3.11. *Let* $(G,\mathcal{E}_G)$ *and* $(H,\mathcal{E}_H)$ *be two coarse groups,* $E\in\mathcal{E}_H$ *be a symmetric entourage,* $f\colon G\to H$ *be a surjective* E*-quasi-homomorphism, and* $s\colon H\to G$ *be one of its sections (i.e.,* $f\circ s=id_H$*). Suppose that* $(f\times f)^{-1}(E)\in\mathcal{E}_G$*. Then* s *is a quasi-homomorphism.*

Proof. Let $x,y\in H$. Since $f\circ s=id_H$,

$$(f(s(xy)),f(s(x)s(y)))=(f(s(xy)),f(s(x))f(s(y)))\circ \\ \circ(f(s(x))f(s(y)),f(s(x)s(y)))\in\Delta_H\circ E=E,$$

and thus $(s(xy),s(x)s(y))\in(f\times f)^{-1}(E)\in\mathcal{E}_G$. □

Corollary 7.3.12. *Let* $f\colon(G,\mathcal{E}_G)\to(H,\mathcal{E}_H)$ *be a surjective quasi-homomorphism, which is a coarse equivalence. Then there exists a coarse inverse of* f *which is a quasi-homomorphism. In particular the inverse of a quasi-homomorphism which is an asymorphism, is a quasi-homomorphism.*

Proof. Since f is effectively proper, the conditions of Lemma 7.3.11 are fulfilled and thus every section, which is a coarse inverse of f, is a quasi-homomorphism. The second statement trivially follows. □

Remark 7.3.13. Let $f\colon G\to(H,\mathcal{E}_{\mathcal{I}})$ be a surjective E_M-quasi-homomorphism, for some $M\in\mathcal{I}$, between and abelian group G and a coarse group $(H,\mathcal{I})$. Then, for every $g,h\in G$,

$$f(g)f(h)\in f(g+h)M=f(h+g)M\subseteq f(h)f(g)MM. \tag{7.9}$$

In particular (7.9) shows that, for every $k, l \in H$, $[k,l] = k^{-1}l^{-1}kl \in MM$ and so the derived subgroup $[H,H]$ is contained in the subgroup $\langle M \rangle$ generated by M. If $\langle M \rangle \in \mathcal{I}$, then H is coarsely equivalent to the abelian coarse group $(H/[H,H], \mathcal{E}_{q(\mathcal{I})})$ since $q\colon H \to H/[H,H]$ is a coarse equivalence by Proposition 7.2.11.

Chapter 8

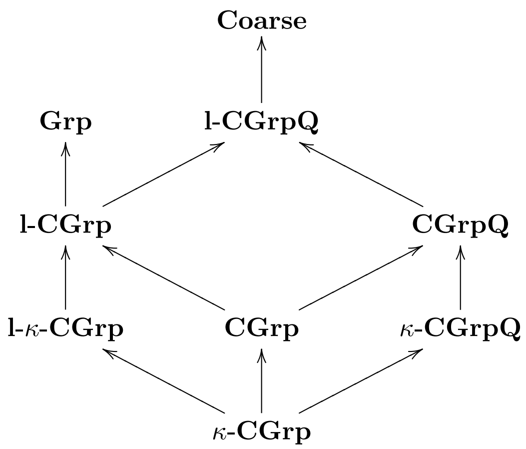
Categories of coarse groups

The aim of this chapter is to provide a categorical treatment of coarse groups. We now introduce a list of categories of coarse groups.

- The category **l-CGrpQ** (**r-CGrpQ**) has left coarse groups as objects (right coarse groups, respectively), and bornologous quasi-homomorphisms as morphisms.
- The category **CGrpQ** is the intersection of **l-CGrpQ** and **r-CGrpQ**, i.e., its objects are coarse groups whose coarse structures are uniformly invariant, and its morphisms are bornologous quasi-homomorphisms (according to Proposition 7.1.13).
- The category **l-CGrp** (**r-CGrp**) has left coarse groups as objects (right coarse groups, respectively), and bornologous homomorphisms as morphisms.
- The category **CGrp** is the intersection of **l-CGrp** and **r-CGrp**, i.e., its objects are coarse groups whose coarse structures are uniformly invariant, and its morphisms are bornologous homomorphisms.
- For any infinite cardinal κ, the subcategory κ-**CGrpQ** (κ-**CGrp**, **l**-κ-**CGrp**, **r**-κ-**CGrp**) of **CGrpQ** (of **CGrp**, **l-CGrp**, **r-CGrp**, respectively) whose objects are groups endowed with κ-group coarse structures.

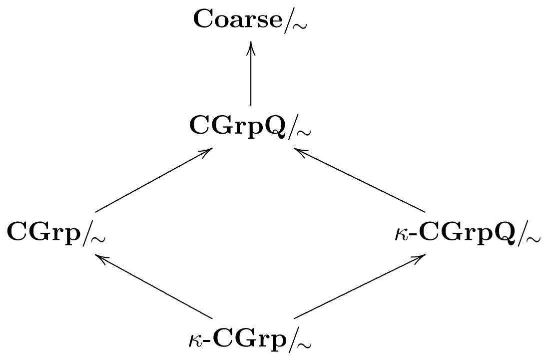
Thanks to Proposition 7.3.6, composites of bornologous quasi-homomorphisms are still quasi-homomorphisms, and thus the categories whose morphisms are bornologous quasi-homomorphisms are indeed categories.

In diagram (8.1), we enlist the categories of coarse groups just defined, where the arrows represent forgetful functors. For the sake of simplicity, we do not include the categories **r-CGrpQ**, **r-CGrp**, and **r-κ-CGrp**.



(8.1)

We have already introduced the quotient category **Coarse**/$_\sim$. In this chapter, we will be also interested in other quotient categories, namely **CGrpQ**/$_\sim$, κ-**CGrpQ**/$_\sim$, **CGrp**/$_\sim$, and κ-**CGrp**/$_\sim$, for every infinite cardinal κ (see diagram (8.2)).



(8.2)

Let us enlist some considerations on the previously defined categories, discussing the consequences of some results we proved in this setting.

Remark 8.0.1. (a) The second assertion of Corollary 7.3.12 implies that, if $X, Y \in$ **l-CGrpQ** ($X, Y \in$ **r-CGrpQ**) and $f\colon X \to Y$ is a morphism in **l-CGrpQ** (**r-CGrpQ**) such that $\mathrm{U} f\colon \mathrm{U} X \to \mathrm{U} Y$ is an isomorphism of **Coarse**, where $\mathrm{U} = \mathrm{U}^{\textbf{l-CGrpQ}}_{\textbf{Coarse}}$, then f is an isomorphism of **l-CGrpQ** (**r-CGrpQ**, respectively).

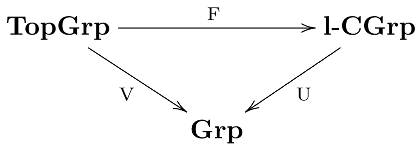
(b) Let $f\colon X \to Y$ be a morphism in **CGrpQ**. Proposition 7.3.4 implies that, for every other morphism $g\colon X \to Y$ in **Coarse**/$_\sim$ such that $g \sim f$, g can be seen as a morphism of **CGrpQ**. Thus the equivalence class of f under closeness relation in **Coarse**/$_\sim$ is equal to the one in **CGrpQ**.

(c) Theorem 7.3.9 implies that, if $X, Y \in$ **CGrpQ**/$_\sim$ and $f\colon X \to Y$ is a morphism in **CGrpQ**/$_\sim$ such that $\mathrm{U} f\colon \mathrm{U} X \to \mathrm{U} Y$ is an isomorphism of **Coarse**/$_\sim$, where $\mathrm{U} = \mathrm{U}^{\mathbf{CGrpQ}/_\sim}_{\mathbf{Coarse}/_\sim}$, then f is an isomorphism of **CGrpQ**/$_\sim$. Note that we cannot replace the category **CGrpQ**/$_\sim$ with **CGrp**/$_\sim$, in fact there are homomorphisms which are coarse equivalences, but they have no coarse inverses which are homomorphisms (Example 7.3.5).

8.1 Functorial coarse structures

Definition 8.1.1. A *functorial coarse structure on* **Grp** is a concrete functor $\mathrm{F}\colon \mathbf{Grp} \to \mathbf{l\text{-}CGrp}$, where concrete means that $\mathrm{U} \circ \mathrm{F}$ is the identity functor, where $\mathrm{U} = \mathrm{U}^{\mathbf{l\text{-}CGrp}}_{\mathbf{Grp}}\colon \mathbf{l\text{-}CGrp} \to \mathbf{Grp}$ is the forgetful functor.

A *functorial coarse structure on* **TopGrp** is a functor $\mathrm{F}\colon \mathbf{TopGrp} \to \mathbf{l\text{-}CGrp}$ such that the following functor diagram

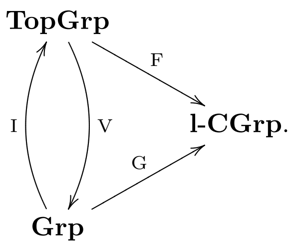


commutes, where $\mathrm{V} = \mathrm{U}^{\mathbf{TopGrp}}_{\mathbf{Grp}}\colon \mathbf{TopGrp} \to \mathbf{Grp}$.

More explicitly, a functorial coarse structure F on **Grp** (on **TopGrp**) associates to a group G (to a topological group G) a coarse structure $\mathcal{E}_{\mathrm{F}(G)}$ on it such that, if $f\colon G \to H$ is a homomorphism (a continuous homomorphism, respectively), then $f\colon (G, \mathcal{E}_{\mathrm{F}(G)}) \to (H, \mathcal{E}_{\mathrm{F}(H)})$ is bornologous.

In Remark 8.1.2 we discuss how functorial coarse structures on **Grp** can be seen also as a particular case of functorial coarse structures on **TopGrp**.

Remark 8.1.2. Let us first fix a functorial coarse structure $\mathrm{F}\colon \mathbf{TopGrp} \to \mathbf{l\text{-}CGrp}$ on **TopGrp**. In order to to associate to F a functorial coarse structure on **Grp** in a canonical way, consider the right adjoint $\mathrm{I}\colon \mathbf{Grp} \to \mathbf{TopGrp}$ of the forgetful functor $\mathrm{V}\colon \mathbf{TopGrp} \to \mathbf{Grp}$ which simply assigns to every group the same group endowed with the discrete topology. Note that, in particular, $\mathrm{V} \circ \mathrm{I}$ is the identity functor of the category **Grp**. Diagram (8.3) represents all functors involved (G is introduced below):



(8.3)

The composition $\mathrm{F} \circ \mathrm{I}$ is a functorial coarse structure on **Grp**. Note that there is a natural transformation from $\mathrm{F} \circ \mathrm{I} \circ \mathrm{V}$ to F whose components have the identity maps as support.

On the other hand, if $\mathrm{G}\colon \mathbf{Grp} \to \mathbf{l\text{-}CGrp}$ is a functorial coarse structures on **Grp**, then there exists a functorial coarse structure H on **TopGrp** with $\mathrm{G} = \mathrm{H} \circ \mathrm{I}$. Indeed, the necessary functorial coarse structure on **TopGrp** can be defined as $\mathrm{H} = \mathrm{G} \circ \mathrm{V}$. In fact, obviously $\mathrm{G} = \mathrm{G} \circ \mathrm{V} \circ \mathrm{I} = \mathrm{H} \circ \mathrm{I}$.

A functorial coarse structure on **Grp** (on **TopGrp**) is called *perfect*, if for every morphism $f\colon G\to H$ in **Grp** (in **TopGrp**, respectively), the morphism $\mathrm{F}(f)$ is uniformly boundedness copreserving.

Remark 8.1.3. Perfect functorial coarse structures $\mathrm{F}\colon$ **TopGrp** $\to$ **l-CGrp** have another remarkable property, namely, for every surjective homomorphism f, $\mathrm{F}(f)$ is a quotient in **Coarse** (and thus in **l-CGrp**). According to Propositions 7.2.1 and 7.2.3, an homomorphism $f\colon (G,\mathcal{E}_{\mathcal{I}_G})\to(H,\mathcal{E}_{\mathcal{I}_H})$ is both bornologous and uniformly bounded copreserving if and only if $f(\mathcal{I}_G)=\mathcal{I}_H\cap\mathcal{P}(f(G))$. Then, if f is surjective, $f\colon (G,\mathcal{E}_{\mathcal{I}_G})\to(H,\mathcal{E}_{\mathcal{I}_H})\cong(G/\ker f,\mathcal{E}_{q(\mathcal{I}_G)})$ is a quotient also in the category **Coarse** (and thus in **l-CGrp**), as it is showed in [65, Proposition 6.5].

One can show that perfect functorial coarse structures on the category **Grp** of abstract groups are completely determined by their 'values' on free groups F_κ of κ generators, where κ is an arbitrary cardinal.

Proposition 8.1.4. *Assume that a group ideal $\mathcal{I}_\kappa$ is assigned to each F_κ in such a way that every homomorphism $f\colon F_\kappa\to F_\mu$ is bornologous and uniformly bounded copreserving when κ and μ vary arbitrarily. Then this assignment can be extended to a perfect functorial coarse structure on the category* **Grp** *assigning to every group G the group ideal $\mathcal{I}_G=q(\mathcal{I}_\kappa)$, provided $q\colon F_\kappa\to G$ is a surjective homomorphism.*

Proof. (a Sketch of a proof) Use the properties of $\mathcal{I}_\kappa$ in the hypotheses to show that:

(a) $\mathcal{I}_G$ is correctly defined (in particular, does not depend on the choice of q);

(b) $G\mapsto(G,\mathcal{I}_G)$ is a perfect functorial coarse structure.

It is enough to prove (a) since (b) will immediately follows. One can use the following two facts. First of all, every group is a quotient of some free group, so that every group can be endowed with a group ideal. Moreover, for every homomorphism $f\colon G\to H$ in **Grp** (including id_G) and for every pair of surjective homomorphisms $q\colon F_\kappa\to G$ and $q'\colon F_\mu\to H$ there is a lifting $\tilde f\colon F_\kappa\to F_\mu$ such that the following diagram commutes

$$\begin{array}{ccc} F_\kappa & \xrightarrow{\tilde f} & F_\mu \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ G & \xrightarrow{f} & H. \end{array}$$

□

A similar result can be shown for the category **AbGrp**, of abelian groups and homomorphisms between them, where the perfect functorial coarse structures are determined by their 'values' on the free abelian groups $A_\kappa=\bigoplus_\kappa\mathbb{Z}$.

One can take as a useful application of Proposition 8.1.4 the case of functorial coarse structures on the class of all groups of size at most κ, where κ is a fixed cardinal. In that case, every group G with $|G|\le\kappa$ is a quotient of the free group F_κ and thus one can define the group ideals of the whole class from its group ideals $\mathcal{I}_\kappa$ that are 'invariant' under endomorphisms of F_κ, i.e., such that, for every endomorphism f, $f\colon(F_\kappa,\mathcal{E}_{\mathcal{I}_\kappa})\to(F_\kappa,\mathcal{E}_{\mathcal{I}_\kappa})$ are bornologous.

Proposition 8.1.5. *All the group-coarse structures defined in Example 7.1.9 and 7.1.10, but the metric-group coarse structure, are functorial on* **Grp** *and on* **TopGrp**, *respectively. Moreover, the discrete, the trivial and the* κ*-group coarse structures are perfect.*

Proof. The proofs are trivial or follow from classical topological results. As for the left-coarse structure, we refer to [158]. □

In Chapter 10 we focus on a particular functorial coarse structure on **TopGrp**, namely the compact-group coarse structure. We will study the preservation of some properties (especially related to dimensions) along the Pontryagin functor and the Bohr functor.

8.2 Preservation of morphisms properties along pullbacks

Several categorical constructions in the category **Coarse** can be carried out in the categories of coarse groups. In particular, we focus here on pullbacks, which will be useful in the sequel. Since **Coarse** is a topological category (see Theorem 4.2.1), we have already discussed that **Coarse** has, in particular, pullbacks (§4.3.1). We can also give a precise description of the pullback of the diagram $Y \xrightarrow{f} Z \xleftarrow{g} X$ in **Coarse** as the triple (P, u, v) in the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{u} & X \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & Z, \end{array} \tag{8.4}$$

where $P = \{(x, y) \in X \times Y \mid g(x) = f(y)\}$ is endowed with the coarse structure inherited by $X \times Y$, and u and v are the restrictions of the canonical projections. Note that, if the diagram $Y \xrightarrow{f} Z \xleftarrow{g} X$ is in **l-CGrp**, in **CGrp**, in **l-**κ**-CGrp**, or in κ**-CGrp**, then (8.4) belongs to **l-CGrp**, to **CGrp**, to **l-**κ**-CGrp**, or to κ**-CGrp**, respectively, and thus it is a pullback also in those categories.

Proposition 8.2.1. *The class* $\mathcal{V}$ *of all coarse embeddings in* **l-CGrp** *is preserved along pullbacks in* **l-CGrp**, *i.e., if the diagram* (8.4) *is a pullback where* $g \in \mathcal{V}$, *then also* $v \in \mathcal{V}$.

Proof. Denote by $\mathcal{I}_Y$, $\mathcal{I}_X$, and $\mathcal{I}_P$ the group ideals associated to Y, X, and P, respectively. Proposition 7.2.9 implies that $\mathcal{I}_P = (\mathcal{I}_X \times \mathcal{I}_Y) \cap \mathcal{P}(P)$. Thanks to Proposition 7.2.5, it is enough to show that, for every $K \in \mathcal{I}_Y$, $v^{-1}(K) \in \mathcal{I}_P$. For every $(x, y) \in v^{-1}(K)$, $g(x) = f(y)$ and $y \in K$. Thus $x \in g^{-1}(f(K))$ and so, since g is a coarse embedding and f is bornologous,

$$v^{-1}(K) \subseteq g^{-1}(f(K)) \times K \in \mathcal{I}_X \times \mathcal{I}_Y,$$

according to Propositions 7.2.1 and 7.2.5. □

Let us now prove a variation of Proposition 8.2.1.

Corollary 8.2.2. *The class $\mathcal{V}'$ of all maps in* **l**-κ-**CGrp** *that are coarse equivalences is preserved along pullbacks in* **l**-κ-**CGrp**.

Proof. According to Proposition 8.2.1, it is enough to show that if g is large-scale surjective, then so it is v. First of all, it is easy to check that $v(P) = f^{-1}(g(X))$. Since g is large-scale surjective, $|Z : g(X)| < \kappa$. Thus

$$|Y : v(P)| = |Y : f^{-1}(g(X))| = |f(Y) : g(X) \cap f(Y)| = |Z : g(X)| < \kappa.$$

□

We could have given a different proof of Corollary 8.2.2 without using Proposition 8.2.1. In fact, since the κ-group coarse structure is functorial and perfect, according to Corollary 7.2.6, it is enough to show that $|\ker v| < \kappa$ and $|Y : v(P)| < \kappa$.

8.3 Categorical properties of cellular coarse groups

In this section we dedicate our attention mainly to cellular coarse groups. This study will be specialise for the compact-group coarse structure in Section 10.1.

Denote by $\mathcal{Z}$ the class of coarse groups G with $\operatorname{asdim} G = 0$. In the sequel we study the stability properties of this class. Let us recall that a full subcategory $\mathfrak{A}$ of the category **l-CGrp** of coarse groups is *epireflective* (*mono-co-reflective*) if every coarse group G admits a surjective (respectively, injective) bornologous homomorphism

$$r_G \colon G \to rG \ \ (\text{resp., } c_G \colon cG \to G) \quad \text{with} \quad rG \in \mathfrak{A} \ \ (\text{resp., } cG \in \mathfrak{A})$$

and the 'arrow' $r_G \colon G \to rG$ (respectively, $c_G \colon cG \to G$) is a (co-)universal arrow to $\mathcal{Z}$, i.e., every bornologous homomorphism $f \colon G \to Z \in \mathcal{Z}$ factorises through r_G (respectively, c_G) via a unique bornologous homomorphism

$$f' \colon rG \to A \ \ (\text{resp.,} \quad f' \colon A \to cG).$$

Let us see that the full subcategory $\mathcal{Z}$ of the category **l-CGrp** of coarse groups is epireflective.

Proposition 8.3.1. *The category $\mathcal{Z}$ is an epireflective subcategory of* **l-CGrp**.

Proof. In order to construct rG, consider the family $\mathcal{F}_G = \{f_i \colon G \to Z_i\}_{i \in I}$ of all surjective bornologous homomorphisms to coarse groups belonging to $\mathcal{Z}$. Note that the family $\mathcal{F}_G$ is non-empty, since it contains $id_G \colon G \to (G, \mathcal{E}_{\mathfrak{cl}(\{G\})})$. Then the product $\Pi_i Z_i$ is a cellular (Proposition 6.2.5) coarse group (Proposition 7.2.9) and then we can define $rG \leq \Pi_i Z_i$ as the image of the diagonal bornologous homomorphism from G to $\Pi_i Z_i$ (i.e., $rG = \{(f_i(g))_i \mid g \in G\}$). □

The above proof is based on the property that the epireflective subcategory of **l-CGrp** are precisely those which are stable under taking subobjects and arbitrary direct products.

In the next proposition we check that $\mathcal{Z}$ is stable also under taking quotients.

Proposition 8.3.2. *Let* $(G, \mathcal{E}_{\mathcal{I}})$ *be a coarse group,* H *be a normal subgroup of* G*. If* $\operatorname{asdim}(G, \mathcal{E}_{\mathcal{I}}) = 0$*, then* $\operatorname{asdim}(G/H, \mathcal{E}_{q(\mathcal{I})}) = 0$.

Proof. We have to prove that G/H is cellular. Fix an element $q(K) \in q(\mathcal{I})$. Then, since G is cellular, there exists a subgroup $K' \in \mathcal{I}$ that contains K. Hence $q(K')$ is a subgroup containing $q(K)$. Hence G/H is cellular. □

This proposition applies especially well for groups equipped with a perfect functorial coarse structure (see [66] for its definition). Then the quotient group G/N automatically carries the quotient coarse structure, so the proposition applies.

Since stability under taking quotients is one of the typical properties of mono-co-reflective subcategories, one may ask whether the category $\mathcal{Z}$ is not also a mono-co-reflective subcategory of **l-CGrp**. The negative answer is provided by the Example 10.1.2(a).

Let us conclude the section stating a cellularity result for the functorial $\mathfrak{cl}(\kappa\text{-}\mathcal{L}(G))$-group coarse structure on **TopGrp**.

Theorem 8.3.3. *If* G *is a locally compact abelian group,* $\operatorname{asdim}(G, \mathcal{E}_{\mathfrak{cl}(\kappa\text{-}\mathcal{L}(G))}) = 0$ *for every* $\kappa > \omega$.

Proof. According to Theorem 1.2.4, $G = \mathbb{R}^n \times G_0$ for some $n \in \mathbb{N}$ and G_0 containing an open compact subgroup K. Then the subgroup $H = \mathbb{R}^n \times K$ of G is Lindelöf (so, κ-Lindelöf as well) and G/H is discrete. Thanks to Corollary 7.2.13, $q\colon G \to G/H$ is a coarse equivalence provided that both groups are endowed with their κ-Lindelöf-group coarse structures, and thus $\operatorname{asdim} G = \operatorname{asdim} G/H$. Since G/H is discrete, its κ-Lindelöf-group coarse structure coincides with the κ-group coarse structure, which is linear ([66]), i.e., $\operatorname{asdim}(G/H, \mathcal{E}_{[G/H]^{<\kappa}}) = 0$, which concludes the proof. □

8.4 Localisation of CoarseGrp/$_\sim$

The reader may be disappointed by Remark 8.0.1(c). In fact, it would be desirable to have a category where all homomorphisms which are coarse equivalences are actually isomorphisms. The category **CGrpQ**/$_\sim$ has that property, but is it the best choice? The aim of this section is to discuss (and give a precise meaning to) this question.

Definition 8.4.1. Let $\mathcal{X}$ be a category and $\mathcal{W}$ be a family of morphisms of $\mathcal{X}$. A *localisation of* $\mathcal{X}$ *by* $\mathcal{W}$ (or *at* $\mathcal{W}$) is given by a category $\mathcal{X}[\mathcal{W}^{-1}]$ and a functor $\mathrm{Q}\colon \mathcal{X} \to \mathcal{X}[\mathcal{W}^{-1}]$ such that:

(a) for every $w \in \mathcal{W}$, $\mathrm{Q}(w)$ is an isomorphism;
(b) for any category $\mathcal{Y}$ and any functor $\mathrm{F}\colon \mathcal{X} \to \mathcal{Y}$ such that $\mathrm{F}(w)$ is an isomorphism, for every $w \in \mathcal{W}$, there exists a functor $\mathrm{F}_{\mathcal{W}}\colon \mathcal{X}[\mathcal{W}^{-1}] \to \mathcal{Y}$ and a natural isomorphism between F and $\mathrm{F}_{\mathcal{W}} \circ \mathrm{Q}$;
(c) for every category $\mathcal{Y}$, the map between functor categories

$$\cdot \circ \mathrm{Q}\colon \operatorname{Funct}(\mathcal{X}[\mathcal{W}^{-1}], \mathcal{Y}) \to \operatorname{Funct}(\mathcal{X}, \mathcal{Y})$$

is full and faithful.

The localisation of a category by a family of morphisms, if it exists, it is unique.

Intuitively, if we localise a category $\mathcal{X}$ by a family of morphisms $\mathcal{W}$, we enrich the family of morphisms of $\mathcal{X}$ by imposing that the elements of $\mathcal{W}$ become isomorphisms. We would like to apply this idea to localise $\mathbf{CGrp}/_\sim$ by the family $\mathcal{W}$ of all equivalence classes of homomorphisms which are coarse equivalences.

Question 8.4.2. *In the previous notations, does the localisation* $\mathbf{CGrp}/_\sim[\mathcal{W}^{-1}]$ *exist? If yes, is it isomorphic to* $\mathbf{CGrpQ}/_\sim$*?*

The functor $\mathrm{U}\colon \mathbf{CGrp}/_\sim \to \mathbf{CGrpQ}/_\sim$ takes every $w \in \mathcal{W}$ to an isomorphism $\mathrm{U}(w)$. Hence, if $\mathbf{CGrp}/_\sim[\mathcal{W}^{-1}]$ exists, and $\mathrm{Q}\colon \mathbf{CGrp}/_\sim \to \mathbf{CGrp}/_\sim[\mathcal{W}^{-1}]$ is the functor guaranteed by the definition, there exists a functor $\mathrm{F}_{\mathcal{W}}\colon \mathbf{CGrp}/_\sim[\mathcal{W}^{-1}] \to \mathbf{CGrpQ}/_\sim$ and a natural transformation between U and $\mathrm{F}_{\mathcal{W}} \circ \mathrm{Q}$.

The final part of this section will be devoted to construct the localisation of κ-$\mathbf{CGrpQ}/_\sim$, for every infinite cardinal κ, by the family $\mathcal{W}$ of all homomorphisms which are coarse equivalences.

The general definition of the localisation of a category is hard to use. However there are some special situations in which constructing it and working with it is easier.

Definition 8.4.3 ([82])**.** A pair $(\mathcal{X},\mathcal{W})$ of a category $\mathcal{X}$ and a class of morphisms $\mathcal{W}$ is said to admit a *calculus of right fractions* if the following conditions holds:

(a) $\mathcal{W}$ contains all identities and it is closed under composition;

(b) (*right Ore condition*) given a morphism $w\colon X \to Z$ in $\mathcal{W}$ and any morphism $f\colon Y \to Z$ in $\mathcal{X}$, there exist a morphism $w'\colon T \to Y$ in $\mathcal{W}$ and a morphism $f'\colon T \to X$ in $\mathcal{X}$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{f'} & X \\ {\scriptstyle w'}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ Y & \xrightarrow{f} & Z \end{array}$$

commutes;

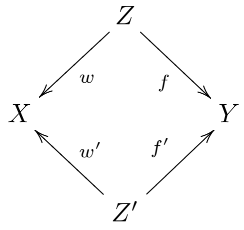
(c) (*right cancellability*) given an arrow $w\colon Y \to Z$ in $\mathcal{W}$ and a pair of morphisms $f, g\colon X \to Y$ such that $w \circ f = w \circ g$, there exists an arrow $w'\colon T \to X$ in $\mathcal{W}$ such that $f \circ w' = g \circ w'$.

The pair $(\mathcal{X},\mathcal{W})$ is a *homotopical category* if, moreover, the following property is fulfilled:

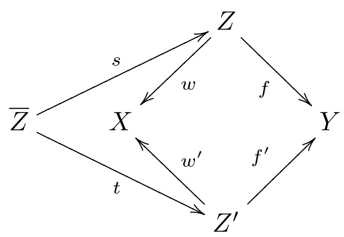
(iv) (2-*out-of*-6-*property*) for every triple of composable morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to T$, if $g \circ f$ and $h \circ g$ are in $\mathcal{W}$, then so are f, g, h (and, necessarily $h \circ g \circ f$).

If $\mathcal{X}$ is a category, a *span* (or *roof*, or *correspondence*) *from an object* X *to an object* Y is a diagram of the form $X \xleftarrow{f} Z \xrightarrow{g} Y$, for some morphisms f and g of $\mathcal{X}$. In this case, f (g) is the *left leg* (*right leg*, respectively) of the span.

If $(\mathcal{X},\mathcal{W})$ admits a calculus of right fractions, then we can construct $\mathcal{X}[\mathcal{W}^{-1}]$ as follows. It has the same objects as $\mathcal{X}$, while, as morphisms, we take the spans between objects of $\mathcal{X}$ whose left legs belong to $\mathcal{W}$ under the following equivalence relation: to such spans

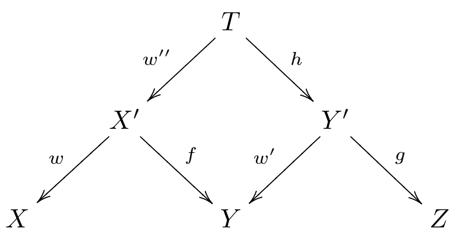


are equivalent if there exist an object $\overline{Z}$ and two morphisms $s\colon \overline{Z} \to Z$ and $t\colon \overline{Z} \to Z'$ such that all the squares in



commute and $w \circ s = w' \circ t \in \mathcal{W}$.

In this category we define the composition of two morphisms as follows: if $X \xleftarrow{w} X' \xrightarrow{f} Y$ and $Y \xleftarrow{w'} Y' \xrightarrow{g} Z$ are two representatives of their equivalence classes, because of Definition 8.4.3(b), there exists another span $X' \xleftarrow{w''} T \xrightarrow{h} Y'$ such that all the squares in



commutes, where $w'' \in \mathcal{W}$ and so does $w \circ w''$ (Definition 8.4.3(a)), and thus we can define the composite as the equivalence class of $X \xleftarrow{w \circ w''} T \xrightarrow{g \circ h} Z$.

The functor $\mathrm{Q}\colon \mathcal{X} \to \mathcal{X}[\mathcal{W}^{-1}]$ fix the objects and sends every morphism $f\colon X \to Y$ of $\mathcal{X}$ in the span $X \xleftarrow{1_X} X \xrightarrow{f} Y$ (note, in fact, that $1_X \in \mathcal{W}$).

If we begin with a homotopical category, the functor Q is exact.

Lemma 8.4.4. *Let $\mathcal{W}$ be the family of all equivalence classes of homomorphisms which are coarse equivalences. Then*

(a) $\mathcal{W}$ contains all the identities and it is closed under composition;
(b) $(\mathbf{CGrp}/_\sim, \mathcal{W})$ has the right cancellability property;
(c) $(\mathbf{CGrp}/_\sim, \mathcal{W})$ has the 2-out-of-6-property;
(d) for every infinite cardinal κ, $(\kappa\text{-}\mathbf{CGrp}/_\sim, \mathcal{W}')$, where $\mathcal{W}' = \mathcal{W} \cap \kappa\text{-}\mathbf{CGrp}/_\sim$, satisfies the right Ore condition.

Proof. Item (a) is trivial.

(b) Let U be the forgetful functor from **CGrp**/$_\sim$ to **Coarse**/$_\sim$. Suppose that $w\colon Y\to Z$ belongs to $\mathcal{W}$ and $f,g\colon X\to Y$ is a pair of morphisms of **CGrp**/$_\sim$ such that $w\circ f=w\circ g$. Since $\mathrm{U}(w)$ is an isomorphism, $f=g$ in **Coarse**/$_\sim$, and thus $f=g$ in **CGrp**/$_\sim$. Hence it is enough to put $w'=1_X$.

Item (c) can be proved similarly to item (b), by using the functor U and the fact that, for every $w\in\mathcal{W}$, $\mathrm{U}(w)$ is an isomorphism of **Coarse**/$_\sim$.

(d) Consider the diagram $Y\xrightarrow{f}Z\xleftarrow{w}X$ in κ-**CGrp**, where $[w]\in\mathcal{W}'$. Take the pullback

$$\begin{array}{ccc} T & \xrightarrow{f'} & X \\ {\scriptstyle w'}\big\downarrow & & \big\downarrow{\scriptstyle w} \\ Y & \xrightarrow{f} & Z \end{array}$$

in the category κ-**CGrp**. Then w' is a coarse equivalence, and $[w']\in\mathcal{W}'$, according to Corollary 8.2.2. □

In Remark 8.4.7 we give a brief comment on the proof of Lemma 8.4.4(d).

From Lemma 8.4.4, the following result immediately descends.

Corollary 8.4.5. *For every infinite cardinal κ, the pair (κ-**CGrp**/$_\sim$, $\mathcal{W}$), where $\mathcal{W}$ is the family of all equivalence classes of homomorphisms which are coarse equivalences, is a homotopical category and thus (κ-**CGrp**/$_\sim[\mathcal{W}^{-1}]$, Q) exists and the functor Q: κ-**CGrp**/$_\sim\to\kappa$-**CGrp**/$_\sim[\mathcal{W}^{-1}]$ is exact.*

Let us specialise Question 8.4.2 in view of Corollary 8.4.5, using the notation of Corollary 8.4.5:

Question 8.4.6. *Is κ-**CGrp**/$_\sim[\mathcal{W}^{-1}]$ isomorphic to κ-**CGrpQ**/$_\sim$?*

Remark 8.4.7. According to Question 8.4.2, we would like to know whether the localisation of the whole category **CGrp**/$_\sim$ by the family $\mathcal{W}$ of all homomorphisms which are coarse equivalences exists or not. One way to provide a positive answer is following the steps that led us to Corollary 8.4.5 and extending them to a more general setting. Then it is worth mentioning that Lemma 8.4.4(a)–(c) holds in general, while the only key point of the proof of Lemma 8.4.4(d) where we actually used the properties of the κ-group coarse structure is when we showed that w' has large image in Y. It is, in fact, the difference between Proposition 8.2.1 and Corollary 8.2.2. If one could extend the proof of just that point, then Corollary 8.4.5 would be immediately generalised, providing a (maybe partial) answer to Question 8.4.2.

Chapter 9

Linear coarse structures

9.1 Cardinal and numerical invariants

A topological abelian group (G,τ), and its topology τ, are called *linear* if τ has a local base at 0_G formed by open subgroups of G. In the non-abelian case some authors impose normality on the open subgroups forming the local base. Motivated by this folklore notion in the area of topological groups, we defined in §7.2.5 the notion of a linear coarse structure. Explicitly, a group coarse structure $\mathcal{E}_\mathcal{I}$ on G is linear if there exists a non-empty family $\mathcal{B}$ of subgroups H_i of G, such that $H_iH_j \in \mathcal{B}$ and $\mathfrak{cl}(\mathcal{B}) = \mathcal{I}$ (note that $H_i \cup H_j \subseteq H_iH_j$).

Note that, if we want $\mathcal{E}_\mathcal{I}$ to be connected, then we have to ask that $\mathcal{I}$ contains all finitely generated subgroups of G.

As far as the group itself is not finitely generated (as a normal subgroup) linear coarse structures do not look trivial. For example, if G is abelian, then for every uncountable cardinal κ the κ-group coarse structure defined in Example 7.1.9(d) is linear. We use this example to introduce a more general construction, namely group coarse structures which come out from *cardinal* and *numerical invariants*.

Definition 9.1.1. A *cardinal invariant* $i(\cdot)$ for abelian groups is an assignment $G \mapsto i(G)$ of a cardinal number $i(G)$ to every abelian group G in such a way that, if $G \cong H$, then $i(G) = i(H)$.

Call a cardinal invariant i

- *subadditive*, if $i(H_1 + H_2) \le i(H_1) + i(H_2)$ whenever H_i $(i = 1, 2)$ are subgroups of some abelian group G;
- *additive*, if $i(G) = i(H) + i(G/H)$ whenever H is a subgroup of G;
- *monotone (with respect to quotients)*, whenever $i(G/H) \le i(G)$ for any subgroup H of G;
- *monotone (with respect to subgroups)*, whenever $i(H) \le i(G)$ for any subgroup H of G;
- *bounded*, whenever $i(G) \le |G|$ for any group G;
- *continuous*, if i is bounded and if $i(G) = \sup_{\lambda\in\Lambda} i(G_\lambda)$, when G is a direct limit of its subgroups $(G_\lambda)_{\lambda\in\Lambda}$;
- *normalised*, if $i(\{0\}) = 0$.

Obviously, additivity implies subadditivity and monotonicity with respect to both quotients and subgroups.

Sometimes it is convenient to consider numerical invariants instead of cardinal invariants. A *numerical invariant for abelian groups* is an assignment $G \mapsto j(G) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $j(G) = j(H)$ provided that $G \cong H$. One can define boundedness, (sub)additivity, continuity, monotonicity, and normalisation also for numerical invariants in the same way. We say that j is a *length function*, if j is continuous and additive. Every cardinal invariant i induces a numerical invariant j_i by 'truncating from above' at ω, i.e., by letting $j_i(G) = \min\{i(G), \infty\}$, for every abelian group G, where, for every $x \in \mathbb{R}_{\geq 0}$, $x < \infty$ and, for every infinite cardinal κ, we assume that $\infty \leq \kappa$.

Example 9.1.2. (a) The *normalised cardinality*, defined by

$$\ell(G) = \begin{cases} |G|, & \text{if } G \text{ is infinite,} \\ \log|G|, & \text{otherwise.} \end{cases}$$

This, maybe somewhat unusual, modification is due to the fact that the size $|G|$ is a cardinal invariant, but it fails to be normalised and subadditive (as far as finite groups are concerned).

(b) The free rank $r_0(G)$ and the p-ranks $r_p(G)$ of an abelian group G are cardinal invariants. Hence also the rank $r(G) = \max\{r_0(G), \sup\{r_p(G) \mid p \in P\}\}$, where P is the set of all prime numbers. In general, $r(G) \leq |G|$, they coincide when $r(G)$ is infinite.

(c) Other invariants can be defined by using functorial subgroups. For example:
- ([40]) the *divisible weight*: $w_d(G) = \inf\{|mG| \mid m > 0\}$,
- ([58]) the *divisible rank*: $r_d(G) = \inf\{r(mG) \mid m > 0\}$.

(d) Using the idea from item (c), for every cardinal invariant i one can define its modification i_d defined similarly to divisible rank: $i_d(G) = \inf\{i(mG) \mid m > 0\}$. It is bounded (normalised), whenever i is, and it has particularly nice properties when i is monotone with respect to taking subgroups and quotients. Then i_d has the same properties and, moreover, i_d is subadditive, whenever i is. This shows that r_d normalise, subadditive, bounded and monotone with respect to taking subgroups and quotients, while w_d has all these properties beyond the first one. To obtain that one too one has to slightly modify its definition as follows

$$\widetilde{w}_d(G) = \inf\{\ell(mG) \mid m > 0\}.$$

It is easy to see that $\widetilde{w}_d(G) = w_d(G)$ is infinite for all unbounded groups, while $\widetilde{w}_d(G) = 0 < 1 = w_d(G)$ for all bounded groups.

All these cardinal invariants are subadditive and bounded, the normalised cardinality $\ell(\cdot)$, the free rank r_0, the divisible weight w_d and the the divisible rank r_k are also monotone with respect to quotients whereas r and r_p are not.

9.2 The linear coarse structures associated to a cardinal invariant

For a cardinal invariant i we define now linear coarse structures depending on a fixed infinite cardinal κ. To this end for any abelian group G denote by

$\mathcal{B}_{i,\kappa}$ the family of all subgroups H of G with $i(H) < \kappa$. If κ is infinite and i bounded, $\mathcal{B}_{i,\kappa}$ is non-empty. Here is a condition ensuring that $\mathcal{B}_{i,\kappa}$ is a base of a group ideal.

Claim 9.2.1. *Let i be a normalised, subadditive cardinal invariant for abelian groups and let κ be an infinite cardinal. For every abelian group G the family $\mathcal{B}_{i,\kappa}$ is a base of a group ideal $\mathcal{I}_{i,\kappa}$ on G.*

Proof. If $H, K \in \mathcal{B}_{i,\kappa}$, then $H \cup K \subseteq H + K \in \mathcal{B}_{i,\kappa}$ since i is subadditive. Moreover, for every subgroup H of G we have $-H = H$ and thus $H \in \mathcal{B}_{i,\kappa}$ if and only if $-H \in \mathcal{B}_{i,\kappa}$. □

The following result is trivial.

Proposition 9.2.2. *Let G be an abelian groups, i be a normalised, subadditive cardinal invariant and κ be an infinite cardinal. Then the trivial homomorphism $(G, \mathcal{E}_{\mathcal{I}_{i,\kappa}}) \to \{0\}$ is a coarse equivalence if and only if $i(G) < \kappa$.*

For a fixed subadditive cardinal invariant i and for any abelian group G denote by $\mathcal{B}_i^0$ the family of all subgroups H of G such that $i(H) = 0$. If the cardinal invariant i is subadditive and normalised, then the family $\mathcal{B}_i^0$ is non-empty and defines a group ideal $\mathcal{I}_i^0$ inducing a cellular coarse structure on abelian groups. This construction can be carried out also in presence of a numerical invariant, and, moreover, for every cardinal invariant i, $\mathcal{B}_i^0 = \mathcal{B}_{j_i}^0$.

Proposition 9.2.3. *Let G be a group and j be a normalised length function. Then group ideal $\mathcal{I}_j^0$ is generated by one element $I \subseteq G$. Moreover, the quotient map $q\colon G \to G/I$ is a coarse equivalence, provided that both groups are endowed with their $\mathcal{I}_j^0$-group coarse structures.*

Proof. The subgroup $I = \sum\{H \mid H \in \mathcal{B}_j^0\}$ satisfies $\mathcal{I}_j^0 = \mathfrak{cl}(\{I\})$. The claim is trivial since j is continuous and then $j(I) = \sup\{j(H) \mid H \in \mathcal{B}_j^0\} = 0$, which prove that $I \in \mathcal{B}_j^0$.

The second statement follows from Proposition 7.2.11 since $j(I) = 0$. □

Let G be an abelian group. Then $\mathcal{I}_{r_0}^0$ is a group ideal on it, since $r_0(\{0\}) = 0$. Moreover, since the numerical invariant induced by r_0 is a length function, we can apply Proposition 9.2.3 to prove that it is generated by the torsion subgroup $\operatorname{Tor}(G)$ of G. Moreover, $q\colon G \to G/\operatorname{Tor}(G)$ is a coarse equivalence. Note that $G/\operatorname{Tor}(G)$ is torsion-free.

The next issue we intend to face is 'how much' the above group coarse structures can 'distinguish' the groups, i.e., is there a great variety of groups that are not coarse equivalent with respect to the linear coarse structures just defined?

Proposition 9.2.4. *Let G and H be two abelian groups, i a cardinal invariant and κ be an infinite cardinal. If there exists an homomorphism which is a coarse equivalence between $(G, \mathcal{E}_{\mathcal{I}_{i,\kappa}})$ and $(H, \mathcal{E}_{\mathcal{I}_{i,\kappa}})$ then either $i(G) < \kappa$ and $i(H) < \kappa$, or $i(G) = i(H)$.*

Example 9.3.4, with $i = r_0$ and $\kappa = \omega$, shows that the implication in the above proposition cannot be reverted.

9.2.1 When linear coarse structures are functorial

Theorem 9.2.5. *Let i be a normalised, subadditive cardinal invariant of abelian groups. Then the following properties are equivalent:*

(a) for every group G and every subgroup $H \leq G$, either $i(G/H) \leq i(G)$, or $i(G/H) < \infty$ whenever $i(G)$ finite;

(b) for every infinite cardinal κ, $\mathcal{E}_{\mathcal{I}_{i,\kappa}}$ defines a cellular functorial coarse structure in the category of abelian group, i.e., every group homomorphism $f\colon G \to H$ is bornologous when G and H carry their linear coarse structures $\mathcal{E}_{\mathcal{I}_{i,\kappa}}$.

Proof. (a)→(b) Let $f\colon G \to H$ be a homomorphism between abelian groups. It is enough to notice that for each $K \subseteq G$, if $K \in \mathcal{B}_{i,\kappa}$, we have $i(f(K)) = i(K/\ker f) \leq i(K) < \kappa$ provided $i(K)$ is infinite. If $i(K)$ is finite, then $i(f(K)) = i(K/\ker f)$ is finite as well, so $i(f(K)) < \kappa$ again. Hence f is bornologous thanks to Proposition 7.2.1.

(b)→(a) Let G be an abelian group and H be a subgroup. Let κ be an infinite cardinal such that $i(G) < \kappa$. Since $f\colon (G, \mathcal{E}_{\mathcal{I}_{i,\kappa}}) \to (G/H, \mathcal{E}_{\mathcal{I}_{i,\kappa}})$ is bornologous and $G \in \mathcal{I}_{i,\kappa}$, then $G/H \in \mathcal{I}_{i,\kappa}$, which means that $i(G/H) < \kappa$. Since the cardinal κ can be taken arbitrarily, then $i(G/H) \leq i(G)$. To check the case when G is finite, just take $\kappa = \omega$. □

The property (a) of Theorem 9.2.5 is obviously implied by the fact that the cardinal invariant i is monotone with respect to quotients. Similarly to the proof of the implication (a)→(b) of Theorem 9.2.5 one obtains the proof of the following result.

Proposition 9.2.6. *Let i be a normalised and subadditive cardinal invariant, which is monotone with respect to taking quotients. Then $\mathcal{E}_{\mathcal{I}_i^0}$ defines a functorial coarse structure.*

If the cardinal invariant is the free rank or the normalised cardinality, then, for every infinite cardinal κ, $\mathcal{E}_{\mathcal{I}_{i,\kappa}}$ defines a perfect functorial coarse structure. In the general case we cannot find the precise conditions on i that ensure this property.

Problem 9.2.7. Determine the properties of the cardinal invariant i such that for every infinite cardinal the functorial coarse structure κ, $\mathcal{E}_{\mathcal{I}_{i,\kappa}}$ is perfect.

9.2.2 Small size vs small asymptotic dimension

For a coarse space $(X, \mathcal{E})$ call a subset A of X *small* if one of the following equivalent properties hold:

(a) for each large set L of X, the set $L \setminus A$ remains large in X;
(b) for every $E \in \mathcal{E}$, $X \setminus E[A]$ is large in X;
(c) $X \setminus A$ is extra-large in X.

This notion, along with other similar notions for size, is due to [144], see also [142] for applications to groups, and [64] for further progress in this direction. Let $\mathcal{SM}(X)$ denote the family of all small subsets of the coarse space X. Furthermore, let $\mathcal{D}_<(X)$ denote the family of all subsets A with $\operatorname{asdim} A < \operatorname{asdim} X$. These two families are ideals in X.

Small sets are considered as the large-scale counterpart of nowhere dense subsets in topology ([15]). It is a classical result that in $\mathbb{R}^n$ the ideal of nowhere dense subsets coincides with the one of those subsets that have covering dimension strictly less than n. Banakh, Chervak and Lyaskovska showed the large-scale counterpart of this classical result, [12, Theorem 1.6], which states that, for every coarse space X, the inclusion $\mathcal{D}_<(X)\subseteq\mathcal{SM}(X)$ holds, while the opposite inclusion holds if X is coarsely equivalent to $\mathbb{R}^n$, endowed with its compact-group coarse structure.

Moreover, for locally compact abelian groups endowed with their compact-group coarse structure, the authors provide the following characterisation.

Theorem 9.2.8. *[12, Theorem 1.7] For a locally compact abelian group the following properties are equivalent:*
(a) $\mathcal{D}_<(G)=\mathcal{SM}(G)$*;*
(b) G *is compactly generated;*
(c) G *is coarsely equivalent to* $\mathbb{R}^n$*, for some* $n\in\mathbb{N}$*.*

They ask a description of the spaces X when the equality $\mathcal{D}_<(X)=\mathcal{SM}(X)$ holds true ([12, Problem 1.3]). Obviously, it holds true when G is compact, since then $\mathcal{D}_<(X)=\mathcal{SM}(X)=\{\emptyset\}$. Here we provide a wealth of counter-examples to this equality which are based on the following trivial observation. If, for a coarse space X, $\operatorname{asdim}X=0$, then $\mathcal{D}_<(X)=\{\emptyset\}$ consists of only the empty subset of X. Therefore, to provide examples where the equality $\mathcal{D}_<(X)=\mathcal{SM}(X)$ does not hold it suffices to find spaces X with $\operatorname{asdim}X=0$ and such that X has a non-empty small set.

Proposition 9.2.9. *Let i be an subadditive, bounded cardinal invariant, κ be an uncountable cardinal and G be an abelian group with $i(G)\geq\kappa$. Then $[G]^{<\kappa}\subseteq\mathcal{SM}(G,\mathcal{E}_{\mathcal{I}_{i,\kappa}})$. In particular,*

$$\mathcal{D}_<(G,\mathcal{E}_{\mathcal{I}_{i,\kappa}})=\{\emptyset\}\subsetneq[G]^{<\kappa}\subseteq\mathcal{SM}(G,\mathcal{E}_{\mathcal{I}_{i,\kappa}}).$$

Proof. Let S be a subset of G with $|S|<\kappa$. To check that $S\in\mathcal{SM}(G,\mathcal{E}_{\mathcal{I}_{i,\kappa}})$ pick a large subset A of G and a subgroup $K\in\mathcal{B}_{i,\kappa}$ such that

$$A+K=G. \tag{9.1}$$

The subadditivity and boundedness of i, combined with (9.1), entail

$$\kappa\leq i(G)\leq i(\langle A\rangle)+i(K).$$

Along with $i(K)<\kappa$, this implies that $\kappa\leq i(\langle A\rangle)$. Therefore, boundedness of i gives

$$|A|=|\langle A\rangle|\geq i(\langle A\rangle)\geq\kappa>|S|.$$

Hence, there exists an element $a\in A\setminus S$. The set $S-a=\{s-a\mid s\in S\}$ belongs to $[G]^{<\kappa}$, so the subgroup $\langle S-a\rangle+K$ belongs to $\mathcal{B}_{i,\kappa}$. On the other hand, it is easy to verify that $(A\setminus S)+(\langle S-a\rangle+K)=G$ (by the choice of a, $(A\setminus S)+(\langle S-a\rangle+K)$ contains S, hence contains A as well, so (9.1) applies). This proves that $A\setminus S$ is large, so S is small.

□

We can refine Proposition 9.2.9 if we consider as cardinal invariant the normalised cardinality. In fact, it is not hard to prove the following statement: *Let G be an infinite group with cardinality κ. Then $[G]^{<\kappa}\subseteq\mathcal{SM}(G,\mathcal{E}_{\mathcal{I}_\kappa})$.*

9.3 Abelian groups with the functorial coarse structure $\mathcal{E}_{r_0,\kappa}$

In Theorem 1.2.2, the free-rank played an important role. So it is reasonable to focus on the linear coarse structures associated to that cardinal invariant. In the sequel of this section we fix the functorial coarse structure $\mathcal{E}_{r_0,\omega}$, so every group we consider is endowed with that functorial coarse structure.

Remark 9.3.1. Let G be a abelian group and let H be a subgroup of G. Then the inclusion $j\colon H \to G$ is an asymorphic embedding.

Since $r_0(\mathrm{Tor}(G)) = 0$, Proposition 7.2.11 implies that every abelian group G is coarsely equivalent, via the quotient homomorphism $q\colon G \to G/\mathrm{Tor}(G)$, to a torsion-free abelian group. That's why we focus on torsion-free abelian groups in the sequel. Due to Remark 9.3.1, the study of the homomorphisms that are coarse equivalences can be reduced to the study of large subgroups. The next proposition provides a necessary condition for that.

Proposition 9.3.2. *If a subgroup H of a torsion-free abelian group G is large, there exists $k \in \mathbb{N}$ such that*

$$r_0(G/H) \leq k \quad \textit{and} \quad r_p(G/H) \leq k \quad \textit{for every prime } p. \tag{9.2}$$

Proof. Suppose that there exists a subgroup S of G with $H + S = G$ of finite free rank $k = r_0(S)$. Then $G/H \cong S/H \cap S$ is a quotient of a torsion-free group. Therefore, $r_0(G/H) \leq k$ and all p-ranks $r_p(G/H) = r_p(S/H \cap S)$ are bounded by k. Indeed, while $r_0(G/H) \leq k$ obviously follows from the monotonicity of r_0, the latter inequality needs more care. As $k = r_0(S)$, we can assume without loss of generality that S is a subgroup of $\mathbb{Q}^k$. Hence, $S/H \cap S$ is a subgroup of $A = \mathbb{Q}^k/H \cap S$. So it suffices to prove that

$$r_p(A) \leq k. \tag{9.3}$$

By the definition of r_p, $r_p(A) = \dim_{\mathbb{Z}/p\mathbb{Z}} A[p]$, where $A[p] = \{a \in A \mid pa = 0\}$. Let $S_1 = \{s \in \mathbb{Q}^k : ps \in H \cap S\}$. Then $A[p] \cong S_1/H \cap S$. To prove that $\dim_{\mathbb{Z}/p\mathbb{Z}} S_1/H \cap S \leq k$ pick a set X with strictly more than k elements of $S_1/H \cap S$. To see that it is linearly dependent, consider a lifting Y of X in $S_1 \leq \mathbb{Q}^k$ along the projection map $q\colon S_1 \to S_1/H \cap S$. As $|Y| > k$, Y satisfies a non-trivial relation $\sum_{y\in Y} k_y y = 0$ in S_1. If not all coefficients are divisible by p, the projection along q immediately gives a linear dependence between the elements of X in $S_1/H \cap S$. If there exists some power p^t dividing all k_y, then we can obtain a new linear combination $\sum_{y\in Y} \frac{k_y}{p^s} y = 0$, as S_1 is torsion-free. By choosing the largest possible t, we obtain a linear combination in which at least one coefficient is coprime with p, se we can argue as before. This proves (9.3). □

In particular, if the inclusion $j\colon H \hookrightarrow G$ is a coarse equivalence, then (9.2) holds for some $k \in \mathbb{N}$. We do not know whether this necessary condition implies that j is a coarse equivalence in the case of arbitrary pairs G, H. Yet, we can say something in case the larger group G is divisible.

Recall that a torsion-free group of the form

$$G = \bigoplus_{i \in I} A_i, \tag{9.4}$$

where all A_i are subgroups of $\mathbb{Q}$, is called *fully decomposable*. Free groups and divisible torsion-free groups are instances of fully decomposable torsion-free groups. A fully decomposable group as in (9.4) is called *homogeneous*, if all groups A_i are pairwise isomorphic. Note that (9.4) is reduced precisely when all A_i are proper subgroups of $\mathbb{Q}$.

Theorem 9.3.3. *Let D be a divisible group and (9.4) be a fully decomposable reduced subgroup of D. Suppose that one of the following conditions holds:*
(i) I is uncountable;
(ii) G is homogeneous.
Then the following properties are equivalent:
(a) the inclusion $j: G \hookrightarrow D$ is a coarse equivalence;
(b) there exists a homomorphism $f: G \to D$ that is a coarse equivalence;
(c) G and D are bounded coarse spaces;
(d) $r_0(G) < \infty$ and $r_0(D) < \infty$.

Proof. Under any of the two assumptions (i) or (ii), the implication (a)$\to$(b) is trivial, as well as the equivalence of (c)$\to$(d), while (c) trivially implies (a) (actually, any homomorphism will do). It only remains to prove (b)$\to$(d) when either (i) or (ii) holds. The initial part of the argument coincides in both cases.

Suppose that $f: G \to D$ is a homomorphism and a coarse equivalence. By Corollary 7.2.6, $r_0(\ker f) < \omega$. Hence, $K = \ker f$ is contained in a finite direct summand $L = \bigoplus_{i \in J} A_i$, with $J \subseteq I$, of G. Moreover, f factorises through an injective homomorphism $f_0: G/K \to D$ and $G/K \cong L/K \oplus G_1$, where $G_1 = \bigoplus_{i \in I \setminus J} A_i$. Since $r_0(L/K) \le r_0(L) < \infty$, the projection $G/K \to G_1$ is a coarse equivalence. Therefore, the restriction $f_1: G_1 \to D$ is still an (injective) coarse equivalence, so $f(G) = f_1(G_1)$ must be a large subgroup of G. We may assume, from now on, that G_1 is simply a subgroup of D, identifying it with $f(G) = f_1(G_1)$. According to Proposition 9.3.2 and (9.2), there exists some $k \in \mathbb{N}$, such that

$$r_0(D/G_1) \le k \ \text{ and } \ r_p(D/G_1) \le k \ \text{ for every prime } p. \tag{9.5}$$

If $r_0(D) < \infty$, this implies $r_0(G_1) < \infty$ and consequently $r_0(G) < \infty$, hence we are done. Assume in the sequel that $r_0(D)$ is infinite. Hence, also $r_0(G_1)$ is infinite by (9.5).

Since D is divisible, the divisible hull $D_i = D(A_i) \cong \mathbb{Q}$ of each A_i, $i \in I \setminus J$, is contained in D along with the direct sum $D' = D(G_1) = \bigoplus_{I \setminus J} D_i$. Hence, the quotient group D/G_1 contains a subgroup isomorphic to $D'/G_1 \cong \bigoplus_{I \setminus J} D_i/A_i$. Since G is reduced, $A_i \neq D_i$ for every $i \in I$, so D_i/A_i is a non-trivial torsion (divisible) group. Therefore, $r_{p_i}(D_i/A_i) > 0$ for some prime p_i. There is some prime q, such that $p_i = q$ for infinitely many indexes $i \in I \setminus J$, so that $r_q(D'/G_1)$ is infinite. In the case (i) this is clear as I is uncountable. In case (ii) this follows from the fact that all groups D_i/A_i are pairwise isomorphic, torsion and non-trivial. This proves, that $r_q(D'/G_1)$ is infinite, hence $r_q(D/G_1)$ is infinite as well. This contradicts (9.5). $\square$

With a slight modification the above proof we can give the following more precise result. Suppose that $f\colon G \to D$ is a homomorphism that is a coarse equivalence and G is fully decomposable, while D is divisible. Then $r_0(G/d(G)) < \omega$ in case G is either uncountable or homogeneous. In other words, if a fully decomposable torsion-free abelian group G is coarsely equivalent (i.e., 'as close as possible' from the large-scale point of view) to a divisible group, then G is also 'as close as possible' to a divisible group from algebraic point of view.

We are not aware if one can replace the group (9.4) in the above theorem by an arbitrary reduced torsion-free group.

As a corollary we prove that there exists no homomorphism which is also a coarse equivalence between a divisible group and a free abelian group, in case at least one of them has infinite free-rank.

Corollary 9.3.4. *Let D be a divisible torsion free abelian group of infinite free rank. Then:*

(a) there is no homomorphism which is also a coarse equivalence from D to any reduced abelian group F;

(b) if F is a free abelian group, then there is no homomorphism from F to D which is also a coarse equivalence

Proof. (a) Assume the existence of a homomorphism $f\colon D \to F$ which is a coarse equivalence. Since D is divisible and F is reduced, f is necessarily the null homomorphism. In particular, the trivial homomorphism $G \to \{0\}$ must be a coarse equivalence. By the above proposition, this yields $r_0(G) < \omega$, a contradiction.

(b) Follows from Theorem 9.3.3. □

Let us note that a much stronger result can be proved than just item (i) in the above theorem: if a homomorphism $f\colon D \to G$ to a torsion-free group G is a coarse equivalence, then $f(D)$ is a finite-co-rank subgroup of G. More precisely, $G = f(D) \oplus G_1$, where $r_0(G_1) < \infty$ and $f(D) \cong D$ is divisible.

Example 9.3.5. It is well-known that surjective homomorphisms preserve various properties of the domain, e.g., having finite rank. Let us see that the counterpart of this property remains true also for quasi-homomorphisms with respect to the group coarse structure $\mathcal{E}_{r_0,\omega}$ in the following weaker form.

Let $f\colon G \to H$ be an H_1-quasi-homomorphism, where, without loss of generality, we can assume that H_1 is a subgroup of H with $r_0(H_1) < \omega$, and suppose that G is finitely generated. Then

$$r_0(\langle f(G)\rangle) < \omega.$$

More precisely, if H_1 contains the images of all (finitely many) generators of G (that can be achieved without loss of generality), then also $f(G)$ is contained in H_1, so has finite free rank.

Let X be the finite set of generators of G and assume that $f(X) \subseteq H_1$. We assume that $e \notin X$. We argue by induction on $n = |X|$. The case $n = 0$, i.e., $G = \{0\}$, is trivial, so we may assume that $n > 0$ and that the assertion is proved for $n - 1$. Then $X \neq \emptyset$ so we can fix an element $x \in X$ and let $Y = X \setminus \{x\}$ and $G_1 = \langle Y\rangle$. Then $f(G_1) \leq H_1$ by our inductive hypothesis. Take any $g \in G =$

$G_1 + \langle x \rangle$, then $g = g_1 + kx$. Therefore, $f(g) \in f(g_1) + f(kx) + H_1 = f(kx) + H_1$, as $f(g_1) \in H_1$. By our assumption, $f(x) \in H_1$. If $k > 0$, then a simple inductive argument shows that $f(kx) \in H_1$ as well. If $k < 0$, then $f(kx) = -f(-kx) + H_1$ (by Remark 7.3.2). Now $-f(-kx) \in H_1$ and we are done.

This example leaves open the question on whether quasi-homomorphisms preserve finiteness of rank.

Question 9.3.6. *If $f\colon G \to H$ is a quasi-homomorphism, and $r_0(G) < \omega$, is it true that $r_0(\langle f(G) \rangle) < \omega$ as well?*

In this section we have provided some results for coarse groups in which the notion of divisibility plays an important role. Let us also mention that divisibility have a great impact in some properties of the coarse structures on the subgroup lattices considered in §12 (following the paper [56]).

Chapter 10

The compact-group coarse structure

In this chapter we focus our attention on a particular functorial coarse structure on **TopGrp**, namely the compact-group coarse structure. We then assume that all the topological groups involved in this chapter are implicitly endowed with that coarse structure. Moreover, in §§10.2 and 10.3, we consider its restrictions to the subcategories **TopAbGrp**, of topological abelian groups, and **LCA**, of locally compact abelian groups, (then the range of the restrictions is in **CAbGrp**, the subcategory of **CGrp** consisting of coarse abelian groups) and the composites with the Pontryagin functor

$$\mathbf{LCA} \to \mathbf{LCA}, \quad G \mapsto \widehat{G}$$

(in §10.2) and (in §10.3) with the Bohr functor

$$\mathbf{TopAbGrp} \to \mathbf{TopAbGrp}, \quad G \mapsto G^+.$$

Let us start with another corollary of Proposition 7.2.11 concerning the compact-group coarse structures (Corollary 10.0.1), which will be useful in the sequel.

Corollary 10.0.1 ([66])**.** *Let G be a topological group, and K be a compact normal subgroup of G. Then the quotient map $q\colon G \to G/K$ is a coarse equivalence provided that both G and G/K are endowed with their compact-group coarse structures.*

It is easy see that the functor $G \mapsto (G, \mathcal{E}_{\mathfrak{r}\mathcal{C}(G)})$ of the compact-group coarse structure preserves embeddings of *closed* subgroups, i.e., if H is a closed topological subgroup of G, then the coarse structure induced from $(G, \mathcal{E}_{\mathfrak{r}\mathcal{C}(G)})$ coincides with $\mathcal{E}_{\mathfrak{r}\mathcal{C}(H)}$. In §10.1 we will show that this property may fail if non-closed subgroups are involved (even if the inclusion $H \hookrightarrow G$ is always bornologous, due to functoriality). To this end we are going to use the criterion from Proposition 7.2.17 for topological groups equipped with their compact-group coarse structure.

10.1 Cellular coarse groups

Following [37], call a topological group G *o-connected*, if the completion of G is connected.

Theorem 10.1.1. *Let G be a metrisable σ-compact group that is either o-connected or precompact. Then* $\operatorname{asdim} G = 0$ *if and only if G is compact. In particular, a countably infinite metrisable group that is either o-connected or precompact has* $\operatorname{asdim} G > 0$.

Proof. We intend to use Proposition 7.2.17, so we need to ensure that the group is compactly generated, whereas we only have at hand the weaker assumption that G is σ-compact. To this end we apply a result, due to Fujita and Shakhmatov [80], which ensures that a σ-compact metrisable group G is compactly generated whenever G is either precompact or o-connected. (Actually, these authors showed that when G is countable, then it is generated by a sequence (possibly eventually constant) converging to its neutral element.) The last assertion follows from the fact that countable groups are σ-compact and non-compact, provided they are infinite. □

Example 10.1.2. Let us highlight two properties of an example given in [125]. Let G be the abelian group $\mathbb{Z}_{2^\infty}$. Then the following properties hold. All the topological groups in this example are endowed with their compact-group coarse structures.

(a) Endow $G \leq \mathbb{T}$ with the topology inherited by the one of the torus. Since the group G is generated by the compact set $\{0\} \cup \{1/2^n \mid n \in \mathbb{N}\}$, but it is not compact, Proposition 7.2.17 implies that $\operatorname{asdim} G > 0$, while $\operatorname{asdim} \mathbb{T} = 0$. Hence the asymptotic dimension is not monotone even under taking dense subgroups. Moreover, every finite subgroup $H_n = \mathbb{Z}(p^n)$ of G has $\operatorname{asdim} H = 0$, while $G = \bigcup H_n$ has $\operatorname{asdim} G > 0$, according to Theorem 10.1.1. In other words, G has no maximum cellular subgroup (that would be the case if $\mathcal{Z}$ were mono-co-reflective).

(b) While $\operatorname{asdim}(G, \tau_{\mathbb{T}}|_G) > 0$, we have that $\operatorname{asdim}(G, \tau_{dis}) = 0$, where τ_{dis} is the discrete topology on G, as it is torsion (Fact 7.2.16). Hence, even though $\tau_{\mathbb{T}}|_G \subseteq \tau_{dis}$, $\operatorname{asdim}(G, \tau_{\mathbb{T}}|_G) > \operatorname{asdim}(G, \tau_{dis})$. Note that $\tau_{\mathbb{T}}|_G$ is not locally compact.

Let α be an irrational number, identified with its image $\alpha + \mathbb{Z}$ in $\mathbb{T}$, and let τ_α be the group topology on $\mathbb{Z}$ induced by $\mathbb{T}$ when we identify $\mathbb{Z}$ with the cyclic subgroup $\langle \alpha \rangle$ of $\mathbb{T}$. Note that the groups $(\mathbb{Z}, \tau_\alpha)$ and $(\mathbb{Z}, \tau_\beta)$ are topologically isomorphic only if $\beta \in \pm\alpha + \mathbb{Z}$. Let $d_\alpha = \operatorname{asdim}(\mathbb{Z}, \tau_\alpha)$. According the Theorem 10.1.1, $d_\alpha > 0$.

Problem 10.1.3. Compute d_α. Does it depend on α? Can d_α be strictly greater than 1 or, even, infinite?

Now we show that the desired counter-examples can be found in every infinite compact metrisable group.

Corollary 10.1.4. *Every infinite compact metrisable group K contains a dense subgroup G such that* $\operatorname{asdim} G > 0$, *hence the inclusion $G \hookrightarrow K$ is not a coarse embedding.*

Proof. In order to build such a subgroup G we build a dense non-compact compactly generated subgroup G of K. Since K is separable, there exists a dense countable subgroup G of K. Now the final assertion of Theorem 10.1.1 applies to give $\operatorname{asdim} G > 0$. The final assertion follows from the fact that $\operatorname{asdim} K = 0$ since K is compact. □

In order to describe cellular coarse groups in that case, we need to introduce functorial subgroups. The assignment of a subgroup $H = \mathbf{r}(G)$ to every topological group is called a *functorial subgroup* (or *preradical*, if for every continuous homomorphism $f\colon G \to G_1$ of topological groups one has $f(\mathbf{r}(G)) \leq \mathbf{r}(G_1)$. Leading examples are the connected component $c(G)$, the quasi-component (of the neutral element) $q(G)$, the subgroup $B(G)$ of compact elements of G (this is the subgroup of G generated by all compact subgroups of G, in case G is abelian this is simply the set of all compact elements of G, i.e., elements $x \in G$ such that $\overline{\langle x \rangle}$ is compact), and the von Neumann kernel $\mathrm{n}(G)$, which will be introduced in the abelian case in §10.3.1. Actually, every mono-co-reflection $c_G\colon cG \to G$ obviously gives rise to a functorial subgroup, namely $\mathbf{r}(G) = cG$. Note that a functorial subgroup is always fully invariant, in particular characteristic (invariant under automorphisms), so normal.

We include the next simple fact only for the sake of references.

Lemma 10.1.5. *Let G be a topological group equipped with their compact-group coarse structure. Then* $\operatorname{asdim} G = 0$ *if and only of $\overline{\langle K \rangle}$ is compact, for every compact subset K of G.*

This proves that every cellular group G satisfies $G = B(G)$, since every singleton must be contained in a compact subgroup of G. In the case of an abelian group, $B(G)$ is simply the union of all compact subgroup of G. One may ask whether an abelian topological group G with $G = B(G)$ must be cellular (this is the case when G is discrete, or LCA, as we shall see below in Corollary 10.2.8). The answer is negative in the general (even abelian) case as the following example shows.

Example 10.1.6. Here we give two examples of non-cellular groups with $B(G) = G$.

(a) An example in the discrete case cannot be abelian. Indeed, even for a non-abelian, but soluble, discrete groups G the subgroup $B(G)$ coincides with the set of all torsion elements of G. Moreover, $B(G)$ is locally finite, hence $\operatorname{asdim} G = 0$ (Fact 7.2.16). To obtain an example in the discrete case one may make recourse to a *Tarskii monster T. A* Tarskii monster of exponent *p, where p is a prime, is an infinite countable group whose proper subgroups are cyclic and have order p. Olshanskii ([128]) built Tarskii monsters for every prime $p > 10^{75}$.*
As T is not locally finite, Fact 7.2.16 implies that $\operatorname{asdim} T > 0$.

(b) The group $G = \mathbb{Z}_{2^\infty} \leq \mathbb{T}$ given in Example 10.1.2 obviously satisfies $G = B(G)$, while $\operatorname{asdim} G > 0$.

Now we characterise the locally compact (not necessarily abelian) groups with asymptotic dimension 0. First of all, let us note that we can restrict ourselves to the case of totally disconnected locally compact groups.

Lemma 10.1.7. *Let G be a locally compact group with* $\operatorname{asdim} G = 0$*. Then $c(G)$ is compact, so $q\colon G \to G/c(G)$ is a coarse equivalence.*

Proof. Since $\operatorname{asdim} G = 0$, G has no subgroups topologically isomorphic to $\mathbb{Z}$. In particular, G has no subgroups topologically isomorphic to $\mathbb{R}$. This implies that $c(G)$ is compact, as a locally compact connected group L without lines (i.e., copies of $\mathbb{R}$) is compact. (This follows from the fact that L is *homeomorphic* to a direct product $\mathbb{R}^n \times K$, where K is a compact (necessarily, connected) subgroup of L [34]. As L has no lines, $n = 0$ and $L = K$ is compact.) The final assertion follows from Corollary 10.0.1. □

Theorem 10.1.8. *Let G be a totally disconnected locally compact group. Then the following properties are equivalent:*

(i) $\operatorname{asdim} G = 0$*;*
(ii) for every open compact subgroup K of G and for every $x \in G$, the subgroup $\langle x, K\rangle$ contains K as a finite index subgroup (and thus $\langle x, K\rangle$ itself is a compact open subgroup).

Moreover, items (i) and (ii) imply the following three equivalent conditions:

(a) for every open compact subgroup K of G and for every $x \in G$, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $x^n \in K$;
(b) there exists an open compact subgroup K of G such that, for every $x \in G$, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $x^n \in K$;
(c) $G = B(G)$;

which are equivalent as well.

Proof. Every totally disconnected locally compact group G has an open compact subgroup K (by Van Dantzig Theorem [33]). It is easy to see that every pair K and K_1 of such subgroups are commensurable, it means that the subgroup $K \cap K_1$ has finite index both in K and K_1.

(i)→(ii) Fix open compact subgroup K of G and $x \in G$. According to Lemma 10.1.5, the compact set $\{x\} \cup K$ must be contained in a compact subgroup K_1 of G which will obviously contain also the subgroup $K_0 = \langle x, K\rangle$ Since K is an open subgroup of G, it is also and open subgroup of K_1. Hence, $[K_1 : K]$ is finite, so $[K_0 : K]$ is finite as well.

(ii)→(ii) It is enough to prove that every compact subset is contained in a compact subgroup of G. Let C be a compact subset and let K be an open compact subgroup of G. Since K is open, C is contained in a finite union $U = \bigcup_{i=0}^n g_i K$ of left cosets of K. We can assume without loss of generality that $g_0 = e_G$. To prove the claim it suffices to show that whenever K is a compact open subgroup of G, then $U = \bigcup_{i=0}^n g_i K$ is contained in a compact subgroup of G.

The case $n = 0$ is trivial, so we assume $n \geq 1$ in the sequel. The base case $n = 1$ follows from (ii), since now $U = K \cup g_1 K \subseteq \langle g_1, K\rangle$ and the subgroup $\langle g_1, K\rangle$ is compact, since it contains the compact subgroup K as a finite index subgroup. Now suppose that $n > 1$ and the assertions is true for $n - 1$. Then there exists a compact subgroup K_1 containing $\bigcup_{i=1}^{n-1} g_i K$. As $K \leq K_1$, the compact subgroup K_1 is also open. Now to K_1 and $U_1 = K_1 \cup g_n K_1$ apply the inductive hypothesis to conclude that $U_1 \subseteq K_2$ for some compact subgroup K_2 of G. Since obviously $U \subseteq U_1$, we are done.

This proves the first part of the theorem. The implication (ii)→(a) is obvious. Since we prove next that (a) and (b) are equivalent, this yields that the first two equivalent conditions imply the second group of three equivalent conditions. Obviously, the implication (a)→(b) holds.

(b)→(c) Let $x \in G$. If the cyclic subgroup C generated by x is finite then obviously, $x \in B(G)$. Assume that C is infinite and pick an open compact subgroup K of G with the property described in (b). By (a), $x^n \in K$ for some $n > 0$. Let L be the compact subgroup of K generated by x^n. Then, if $c_G(x)$ denotes the centraliser of x in G, $L \leq c_G(x)$, as $x^n \in c_G(x)$ and $c_G(x)$ is a closed subgroup. Hence, the subgroup $M = \langle x, L\rangle$ generated by L and x simply coincides with CL and $[M : L] < \infty$. As L is a compact subgroup of M of finite index, M is compact as well. Therefore, $x \in B(G)$.

(c)→(a) Fix an arbitrary open compact subgroup K of G and $x \in G$. Let $C = \langle x\rangle$. If C is finite, then the claim trivially follows. Suppose now that C is infinite. If $C \cap K = \{e\}$, then C is discrete since K is open, which is not allowed because $x \in B(G)$ and $C \cong \mathbb{Z}$. Hence $C \cap K$ is non-trivial and then it contains a power of x. □

The next corollary extends the well known fact that the discrete cellular groups are precisely the locally finite ones (Fact 7.2.16).

Corollary 10.1.9. *In a cellular locally compact group every finite subset is contained in a compact open subgroup.*

Proof. Let G be a locally compact group with $\operatorname{asdim} G = 0$ and let F be a finite subset of G. Then $c(G)$ is compact and $q\colon G \to G/c(G)$ is a coarse equivalence, by Lemma 10.1.7. Then $G_1 = G/c(G)$ is a totally disconnected locally compact group with $\operatorname{asdim} G_1 = 0$. Fix an arbitrary compact open subgroup K of G_1, and applying the equivalence of (i) and (ii) in Theorem 10.1.8 to the finitely many element of $q(F)$ conclude that the subgroup $K_1 = \langle q(F), K\rangle$ contains K as a finite index subgroup. Then K_1 is a compact open subgroup of G_1, and consequently $q^{-1}(K_1)$ is a compact open subgroup of G containing F. □

The second group of equivalent conditions in Theorem 10.1.8 is strictly weaker than the first one, as witnessed by Example 10.1.6(b). It is relevant to notice that while the (weaker) condition $(\forall x \in G)(\exists n > 0) x^n \in K$ is simultaneously valid for all (or just for some) compact open subgroups K (as the equivalence of (a) and (b) says), the condition from (ii) holds for the trivial subgroup $K = \{e\}$ of the Tarski monster T (having $B(T) = T$), but it fails for all non-trivial finite (cyclic) subgroup K of T (in fact, it has $\operatorname{asdim} T > 0$).

Remark 10.1.10. (a) One can deduce from Theorem 10.1.8 also the following property of cellular locally compact groups G in the separable case: for every compact open subgroup K of G there exists an increasing chain

$$K_0 = K \leq K_1 \leq \cdots \leq K_n \leq \cdots, \text{ with } G = \bigcup_{n=1}^{\infty} K_n,$$

of open compact subgroup such that (necessarily) each index $[K_{n+1} : K_n]$, where $n \in \mathbb{N}$, is finite. In fact, it is enough to apply the condition (ii) in Theorem 10.1.8 to a family of representatives of the countable cosets of K in an inductive way.

(b) To conclude, we note that if G is a (necessarily locally compact) group with a compact open normal subgroup K, $\operatorname{asdim} G = 0$ if and only if G/K is locally finite. Indeed, if $\operatorname{asdim} G = 0$, then G/K is locally finite, by Proposition 8.3.2 (or Corollary 10.0.1). Vice versa, if G/K is locally finite, then for every $x \in G$ there exists $n > 0$, such that $x^n \in K$. Since K is normal, this yields that the subgroup $\langle x, K\rangle$ of G contains K as finite-index subgroup. Therefore, $\operatorname{asdim} G = 0$, by Theorem 10.1.8 In particular, if K splits, i.e., $G = K \rtimes D$, where K is compact and $D \cong G/K$ is discrete, then $\operatorname{asdim} G = 0$ if and only if D is locally finite.

10.2 The Pontryagin functor

In this section we focus on investigating how some properties are preserved or transformed under the impact of the Pontryagin functor. Our interest in studying this duality is motivated by a beautiful result due to Nicas and Rosenthal ([126]). In this result (Theorem 10.2.1), they connect the covering dimension of a locally compact abelian group (recall that that Pasynkov in [130] proved that in the realm of locally compact groups the covering dimension coincides with both the small inductive dimension and the large inductive dimension) with the asymptotic dimension of its dual.

Theorem 10.2.1. *Let G be a locally compact abelian group. Then*

$$\operatorname{asdim} G = \dim \widehat{G} \quad \textit{and} \quad \dim G = \operatorname{asdim} \widehat{G}.$$

Before starting our section, let us recall some facts. If G is a locally compact abelian group and H is a subgroup of G, the *annihilator of H in $\widehat{G}$* is the subgroup $(\widehat{G}, H) = \{\chi \in \widehat{G} \mid \chi(H) = 0\}$ of $\widehat{G}$, that we write also briefly $H^\perp$ when no confusion is possible.

If H is a subgroup of a locally compact abelian group G, then

$$\widehat{G/H} \cong (\widehat{G}, H) \quad \text{and} \quad \widehat{H} \cong (G, (\widehat{G}, H)). \tag{10.1}$$

Using the first isomorphism in (10.1) we deduce that a closed subgroup H of G is open if and only if $H^\perp$ is compact. Since

$$c(G) = \bigcap \{H \mid H \text{ open subgroup of } G\}$$

and since $^\perp$ defines an antiisomorphism between the lattices of closed subgroups of G and $\widehat{G}$, we deduce that

$$c(G)^\perp = \sum \{K \mid K \text{ compact subgroup of } \widehat{G}\} = B(\widehat{G}). \tag{10.2}$$

Applying the equality (10.2) to $\widehat{G}$ (instead of G), one obtains $c(\widehat{G})^\perp = B(\widehat{\widehat{G}}) = B(G)$. Hence, $B(G)^\perp = c(\widehat{G})^{\perp\perp} = c(\widehat{G})$.

Let us recall that Theorem 1.2.4 states that every locally compact abelian group G is of the form $\mathbb{R}^n \times G_0$, where G_0 has an open compact subgroup K. Since n is uniquely determined by G, it is denoted sometimes by $n_{\mathbb{R}}(G)$. The compact subgroup K is not uniquely determined by the property of being open in G_0. That is why, it is more convenient to use the functorial subgroup $B(K)$ that contains K and trivially intersects $\mathbb{R}^n$, as it is contained in G_0. One has

$$B(G) = B(G_0) \text{ and } G/B(G) \cong \mathbb{R}^n \times G_0/B(G),$$

where the discrete subgroup $G_0/B(G)$ is torsion-free. The invariant

$$\varrho_0(G) = r_0(G_0/B(G)) = r_0(G/\mathbb{R}^n \times B(G))$$

coincides with the the maximum free rank of a discrete quotient group of G (this maximum is attained by the quotient $G/(\mathbb{R}^n \times B(G))$, see [7], for another invariant $\varrho(G)$, closely related to $\varrho_0(G)$, giving the smallest number of compact sets necessary to cover G).

10.2.1 Around Theorem 10.2.1

We need a classical additivity result for the covering dimension of locally compact groups. We refer to [122] for a proof.

Theorem 10.2.2. *Let G be a locally compact group and H be a closed subgroup of G. Then*

$$\dim G = \dim G/H + \dim H.$$

In particular, if K is another locally compact group, then

$$\dim(G \times K) = \dim G + \dim K.$$

Thanks to Theorem 10.2.1, we can now extend Theorem 7.2.21.

Corollary 10.2.3. *Let G be a locally compact abelian group and H be a closed subgroup of G. Then*

$$\operatorname{asdim} G = \operatorname{asdim} G/H + \operatorname{asdim} H.$$

In particular, if K is a locally compact abelian group, then

$$\operatorname{asdim}(G \times K) = \operatorname{asdim} G + \operatorname{asdim} K.$$

Proof. The first equality follows from the following chain

$$\begin{aligned}\operatorname{asdim} G &= \dim \widehat{G} = \dim(\widehat{G}/(\widehat{G}, H)) + \dim(\widehat{G}, H) = \\ &= \dim \widehat{H} + \dim \widehat{G/H} = \operatorname{asdim} H + \operatorname{asdim} G/H,\end{aligned}$$

which holds because of (10.1) and of Theorem 10.2.1. The second statement can be easily deduced. □

Corollary 10.2.3 cannot be extended to arbitrary locally compact groups. In fact the claim may fails also in the discrete case, as Example 7.2.23 shows.

Theorem 10.2.1 connects $\operatorname{asdim} G$ for a locally compact abelian group G with invariants (namely, covering dimension) of the dual group $\widehat{G}$. In spite of its evident elegance, this connection may lead to some practical troubles, due to the fact that one has to compute the group $\widehat{G}$. Now we offer another description of the asymptotic dimension of G in terms of invariants of the group G itself.

Corollary 10.2.4. *For every locally compact abelian group G*

$$\operatorname{asdim} G = n_{\mathbb{R}}(G) + \varrho_0(G). \tag{10.3}$$

Proof. It is enough to prove that

$$\dim \widehat{G} = \mathbb{R}(G) + \varrho_0(G). \tag{10.4}$$

and applying Theorem 10.2.1 to get (10.3).

Because of Theorem 1.2.4, we can assume that G has the form $G = \mathbb{R}^n \times G_0$, where $n \in \mathbb{N}$ and the group G_0 has an open compact subgroup K. In this setting,

$$\varrho_0(G) = r_0(G_0/K). \tag{10.5}$$

Indeed, obviously $K \leq B(G)$. Moreover, $B(G)/K$ is torsion, as for every compact subgroup C of G (so, of $B(G)$), $C \cap K$ is open in C, so $C/C \cap K$ is finite. This means that $mC \leq K$ for some $m > 0$. Therefore, $B(G)/K$ is torsion, so $r_0(G_0/K) = r_0(G/B(G)) = \varrho_0(G)$. This proves (10.5).

Let $K^\perp$ be the annihilator of K in the dual $\widehat{G}_0$, so that (10.1) implies that the subgroup

$$K^\perp \cong \widehat{G_0/K} \tag{10.6}$$

of $\widehat{G}_0$ is compact and open. In fact, we have the following short exact sequence

$$0 \to K \to G_0 \to G_0/K \to 0,$$

where K is compact and open and thus G_0/K is discrete, which leads to the dual short exact sequence

$$0 \leftarrow \widehat{K} \leftarrow \widehat{G_0} \leftarrow \widehat{G_0/K} \cong K^\perp \leftarrow 0,$$

where $\widehat{K}$ is discrete and thus $K^\perp$ is both compact and open. Moreover,

$$\dim K^\perp = \dim \widehat{G_0/K} = r_0(G_0/K) = \varrho_0(G),$$

in view of (10.6). Since $\widehat{G} \cong \mathbb{R}^n \times \widehat{G}_0$ and $\dim \widehat{G}_0 = \dim K^\perp = \varrho_0(G)$, we obtain (10.4) applying Theorem 10.2.2. □

Remark 10.2.5. There is an alternative proof of (10.3) making use of the functorial subgroup $B(G)$ of a locally compact abelian group G. We first prove the equality

$$\operatorname{asdim} G = \operatorname{asdim} G/B(G), \tag{10.7}$$

which is a counterpart of the well known equality $\dim G = \dim c(G)$ (Theorem 7.2.19). Indeed, from Theorem 10.2.1 we have $\operatorname{asdim} G = \dim \widehat{G} = \dim c(\widehat{G})$. Since $c(\widehat{G}) = B(G)^\perp \equiv \widehat{G/B(G)}$, we deduce that

$$\operatorname{asdim} G = \dim c(\widehat{G}) = \dim \widehat{G/B(G)} = \operatorname{asdim} G/B(G).$$

Since $G/B(G) \cong \mathbb{R}^n \times G_0/B(G)$ is coarsely equivalent to $\mathbb{Z}^n \times G_0/B(G_0)$, we obtain again

$$\operatorname{asdim} G = \operatorname{asdim} G/B(G) = n + r_0(G/B(G)) = n + \varrho_0(G).$$

Question 10.2.6. *Is it true that* $\operatorname{asdim} G = \operatorname{asdim} G/B(G)$*, for every (close-to-abelian) topological group* G*?*

According to Remark 10.2.5, the answer is yes for locally compact abelian groups. The same holds for a finitely generated nilpotent group G endowed with one of its word metrics. Indeed, the subset $\operatorname{Tor}(G)$ of all torsion elements of G is a subgroup and obviously, $B(G) = \operatorname{Tor}(G)$. To see that $\operatorname{asdim} G = \operatorname{asdim} G/B(G)$, notice that G (as well as $G/B(G)$) is polycylic and consequently, $\operatorname{asdim} G = h(G)$ (and $\operatorname{asdim} G/B(G) = h(G/B(G))$), where h denotes the Hirsch length, by [70, Theorem 3.5]. As G is polycylic, $B(G)$ is finite. This implies $h(G) = h(G/B(G))$.

For the group G from Example 7.1.16, $B(G) = \mathbb{Q}_p$ and $G/B(G) \cong \mathbb{Z}$, so $\operatorname{asdim} G/B(G) = 1$ and obviously, $\operatorname{asdim} G \geq 1$ (as G contains a copy of $\mathbb{Z}$). We are not aware whether $\operatorname{asdim} G = \operatorname{asdim} G/B(G) = 1$ holds in this case.

Question 10.2.7. *Does the equality* $\operatorname{asdim} G = \operatorname{asdim} G/B(G) = 1$ *hold for the topological group* G *from Example 7.1.16? Does* G *have finite asymptotic dimension?*

As an immediate corollary of Corollary 10.2.4 (and standard properties of locally compact abelian groups) one obtains the following results.

Corollary 10.2.8. *For a locally compact abelian group* G *the following are equivalent:*
(a) $\operatorname{asdim} G = 0$*;*
(b) $n_{\mathbb{R}}(G) = 0$ *and* $\varrho_0(G) = 0$*;*
(c) every compact subset of G *is contained in a compact subgroup of* G*;*
(d) $G = B(G)$ *(i.e.,* G *is covered by compact subgroups);*
(e) G *has no subgroup topologically isomorphic to* $\mathbb{Z}$*.*
In particular, every locally compact abelian group G *with* $r_0(G) = 0$ *has* $\operatorname{asdim} G = 0$*.*

Proof. (a)↔(b) is a trivial consequence of (10.3), while (c)→(d) is trivial.

(b)→(c) The hypothesis $n_{\mathbb{R}}(G) = 0$ implies, according to Theorem 1.2.4, that G contains an open compact subgroup K. Let $q : G \to G/K$ be the quotient map. Take any $C \in \mathcal{C}(G)$. Then $q(C)$ is finite, as a compact subset of the discrete group G/K. Then $q^{-1}(q(C))$ is compact and contains C.

(d)→(e) It follows from the standard fact that every cyclic subgroup of a locally compact group is either relatively compact or discrete.

(e)→(b) Since $\mathbb{R}$ contains a copy of $\mathbb{Z}$, our blanket assumption implies $n_{\mathbb{R}}(G) = 0$. To prove that $\varrho_0(G) = 0$ consider a discrete quotient G/O, where O is an open subgroup of G. Pick an element $\bar{x} = x + O$ of G/O and consider the cyclic subgroup $C = \langle x \rangle$ of G. If C is finite, then x, as well as $\bar{x}$ are torsion. Assume that C is infinite. Since G has no subgroup topologically isomorphic to $\mathbb{Z}$, C cannot be discrete, hence $C \cap O \neq \{0\}$. Therefore $\langle \bar{x} \rangle \cong C/C \cap O$ is finite. This proves that the group G/O is torsion, so $r_0(G/O) = 0$. Hence, $\varrho_0(G) = 0$. □

Remark 10.2.9. Here is another proof of Corollary 10.2.4 making no recourse to Theorem 10.2.1. In the notation of the proof of Corollary 10.2.4, the quotient homomorphism $q\colon G \to G/K$ is a coarse equivalence thanks to Corollary 10.0.1. Therefore, $\operatorname{asdim} G = \operatorname{asdim} G/K$. But $G/K \cong \mathbb{R}^n \times G_0/K$, where the group G_0/K is discrete with $r_0(G_0/K) = \varrho_0(G)$. Hence, $\operatorname{asdim} G_0/K = \varrho_0(G)$, by Theorem 7.2.24. On the other hand, $\mathbb{R}^n$ is coarsely equivalent to $\mathbb{Z}^n$. Therefore, $\mathbb{R}^n \times G_0/K$ is coarsely equivalent to $\mathbb{Z}^n \times G_0/K$, with

$$r_0(\mathbb{Z}^n \times G_0/K) = n + \varrho_0(G).$$

Hence, $\operatorname{asdim}(\mathbb{Z}^n \times G_0/K) = n + \varrho_0(G)$, by Theorem 7.2.21. This gives

$$\operatorname{asdim} G = \operatorname{asdim} G/K = \operatorname{asdim}(\mathbb{R}^n \times G_0/K) = \operatorname{asdim}(\mathbb{Z}^n \times G_0/K) = n + \varrho_0(G). \tag{10.8}$$

As a by-product, we obtain also a new self-contained proof of Theorem 10.2.1 by comparing (10.4) and (10.8).

10.2.2 Monotonicity of dim and asdim under monos and epis of LCA

It is a well known fact that the monomorphisms in the category **LCA** are the injective continuous homomorphisms, while the epimorphisms are the continuous homomorphisms with dense image [53].

The following theorem establishes monotonicity of the dimension function dim along monomorphisms in **LCA**.

Theorem 10.2.10. *If $f\colon H \to G$ is a monomorphism in* **LCA***, then* $\dim H \leq \dim G$.

Proof. Let $f\colon H \to G$ be a monomorphism in **LCA**, i.e., injective continuous homomorphism. Let us denote by $c(G)$ and $c(H)$ the connected components of the identity in the topological groups G and in H, respectively. Since $\dim H = \dim c(H)$, $\dim G = \dim c(G)$ (Theorem 7.2.19) and $f(c(H)) \leq c(G)$, we can assume without loss of generality that $H = c(H)$ and $G = c(G)$ are connected. Therefore, $H = \mathbb{R}^n \times K$, where $n \in \mathbb{N}$ and K is a compact connected group. As f is injective, its restriction to K gives a topological isomorphism $K \cong f(K)$, since K is compact. Hence, $\dim K = \dim f(K)$.

The additivity of the dimension function dim (Theorem 10.2.2) gives

$$\dim H = \dim K + \dim H/K \text{ and } \dim G = \dim f(K) + \dim G/f(K).$$

For the sake of brevity let $G_1 = G/f(K)$. In order to prove that $\dim H \leq \dim G$ it suffices to check that $n = \dim H/K \leq \dim G_1$, where the equality comes from the isomorphism $H/K \cong \mathbb{R}^n$.

The injective continuous homomorphism $f\colon H \to G$ gives rise to the continuous injective homomorphism (i.e., a monomorphism) $f'\colon H/K \to G_1$. Since $H/K \cong \mathbb{R}^n$ we shall simply write $f'\colon \mathbb{R}^n \to G_1$. We have to prove that $n \leq \dim G_1$.

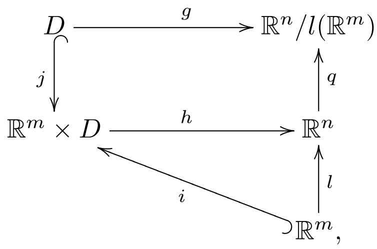
As G_1 is connected, $G_1 = \mathbb{R}^m \times C$, for some $m \in \mathbb{N}$ and a connected compact abelian group C. Let $D = \widehat{C}$ and $s = r_0(D) = \dim C$. Then, by Theorem 10.2.2,

$$\dim G_1 = m + \dim C = m + s. \tag{10.9}$$

Taking the dual of f' we get an epimorphism

$$h\colon \mathbb{R}^m \times D \to \mathbb{R}^n$$

in **LCA**, i.e., $h(\mathbb{R}^m \times D)$ is dense in $\mathbb{R}^n$. Let us consider the following commutative diagram



where j and i are the canonical inclusions, q is the quotient map, $l = h \circ i$, and $g = q \circ h \circ j$. Denote by $p\colon \mathbb{R}^m \times D \to D$ the canonical projection map. Since both h and q are epimorphisms, $g \circ p = q \circ h$ is an epimorphism and thus so it is g. The restriction $l\colon \mathbb{R}^m \to \mathbb{R}^n$ of h is a continuous homomorphism. But since both groups are divisible and torsion-free, it is also $\mathbb{Q}$-linear. By continuity of h, we deduce that l is actually $\mathbb{R}$-linear. So, $l(\mathbb{R}^m)$ is a subspace of $\mathbb{R}^n$ with $\dim_{\mathbb{R}} l(\mathbb{R}^m) \leq m$. Hence, for $U = \mathbb{R}^n/l(\mathbb{R}^m)$ one obviously has $\dim_{\mathbb{R}} U \geq n-m$.

Let $s' = r_0(g(D)) \leq s$. Then

$$s \geq s' \geq \dim_{\mathbb{R}} U \geq n - m. \tag{10.10}$$

We only need to check the middle inequality. To this end assume that that F is a free subgroup of $g(D)$ of rank s'. Then the $\mathbb{R}$-rank of F is at most s', i.e., the $\mathbb{R}$-linear span W of F is isomorphic to $\mathbb{R}^k$, with $k \leq s'$. Since W is the $\mathbb{R}$-linear span of F and since $g(D)$ is contained in the $\mathbb{Q}$-linear span of F (as F is a maximal rank free subgroup of $g(D)$), we deduce that W contains $g(D)$, as $\mathbb{R}^n/l(\mathbb{R}^m)$ is torsion-free.

By the density of $g(D)$ in U, W is dense in U, so $W = U$, so $k = \dim U$. Thus,

$$s \geq s' \geq k = \dim_{\mathbb{R}} U \geq n - m,$$

which proves (10.10). Now (10.9) gives the desired inequality $\dim G_1 \geq n$. □

Remark 10.2.11. Theorem 10.2.10 shows the impact of monomorphisms on the dimensions of their domain and codomain (namely, the dimension cannot properly decrease). A careful analysis of the proof shows that in the case $s > 0$ the density of $g(D)$ in U implies actually a *strict* inequality $s' > k$ that leads to a strict inequality $\dim G > \dim H$. This suggest the question when $\dim G = \dim H$. Following the proof, one can deduce that $\dim G = \dim H$ implies $s = 0$, so both connected components (after taking the quotient with respect to a pair of isomorphic compact connected subgroups) become affine groups $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. As $m = n$, this is possible only if the initial monomorphism $f\colon H \to G$ indices a topological isomorphism between $c(H)$ and $c(G)$. Resuming, *if* $f\colon H \to G$ *is monomorphism between locally compact abelian groups, then* $\dim H = \dim G$ *if and only if* $f|_{c(H)}\colon c(H) \to c(G)$ *is topological isomorphism.*

Taking the duals we obtain mononicity of asdim along epis in **LCA**.

Corollary 10.2.12. *If $f\colon H \to G$ is an epimorphism in* **LCA**, *then* $\operatorname{asdim} H \geq \operatorname{asdim} G$. *Moreover,* $r_0(H) \geq \operatorname{asdim} H$.

Proof. Since the Pontryagin duality functor takes epimorphism to monomorphism, $\widehat{f}\colon \widehat{G} \to \widehat{H}$ is a monomorphism in **LCA**. By the previous theorem, $\dim \widehat{G} \leq \dim \widehat{H}$. Now Theorem 10.2.1 applies.

As for the second inequality, note that $id_H\colon (H, \tau_{dis}) \to H$ is an epimorphism and thus $\operatorname{asdim} H \leq \operatorname{asdim}(H, \tau_{dis}) = r_0(H)$, where the last equality is provided by Theorem 7.2.24.

The same conclusion can be deduced without any recourse to [70], by using only standard facts from the structure theorem of locally compact abelian groups. Indeed, if H is discrete, then $\operatorname{asdim} H = \dim \widehat{H} = r_0(H)$, since $\widehat{H}$ is compact. We can then assume that H is not discrete and thus one has either $r_0(H) = 0$ or $r_0(H) \geq \mathfrak{c}$ ([53]), where $\mathfrak{c}$ denotes the cardinality of the continuum. Hence, if $r_0(H) > 0$, then $r_0(H) \geq \mathfrak{c} \geq \operatorname{asdim} H$. On the other hand, Corollary 10.2.8 implies that $\operatorname{asdim} H = 0$ whenever $r_0(H) = 0$. □

In the non-abelian case the above corollary strongly fails even for discrete groups as Example 7.2.23 shows.

The inequality from Corollary 10.2.12 can be strict even when H and G have the same underlying group and f is the identity of that group. In fact the group $\mathbb{R}$ of real numbers has $\operatorname{asdim} \mathbb{R} = 1$ with the usual topology, but $\operatorname{asdim}(\mathbb{R}, \tau_{dis}) = \infty$.

Corollary 10.2.13. *Let G be an abelian group and δ and τ be two locally compact topologies on it. If $\delta \subseteq \tau$, then*

$$\dim(G, \tau) \leq \dim(G, \delta) \quad \textit{and} \quad \operatorname{asdim}(G, \tau) \geq \operatorname{asdim}(G, \delta).$$

Proof. It is a straightforward application of Theorem 10.2.10 and Corollary 10.2.12 since $id_G\colon (G, \tau) \to (G, \delta)$ is a bimorphism. □

Question 10.2.14. *Let G be an abelian group and $\tau \subseteq \delta$ two group topologies on it. Provide properties of (G, τ) and (G, δ) necessary to conclude* $\operatorname{asdim}(G, \tau) \leq \operatorname{asdim}(G, \delta)$. *What about the case when δ is discrete?*

Because of the Example 10.1.2(b), we cannot have a positive answer even for pairs of locally precompact group topologies (i.e., group topologies having locally compact completion) $\tau \subseteq \delta$ with δ discrete. Note that the specific properties of the group $G = \mathbb{Z}_{2^\infty}$ are not fully exploited in the mentioned example. Indeed, in both items of Example 10.1.2 one only needs a dense non-compact subgroup G of a compact group K that is compactly generated; for item (b) one needs additionally G to be torsion. It is easy to see that every infinite topological subgroup G of the n-dimensional torus $\mathbb{T}^n$ contained in the rational torus $(\mathbb{Q}/\mathbb{Z})^n$ has both these properties (with respect to the topology τ induced by $\mathbb{T}^n$), so fits both (a) and (b).

The next more generic example provides again a counter-example to Corollary 10.2.13 and shows more cases when the asymptotic dimension is not monotone under taking dense subgroups (i.e., can decrease under bimorphisms) in the category of all precompact abelian groups.

Example 10.2.15. We already saw in Corollary 10.1.4 that every infinite compact metrisable group K contains a dense countable subgroup G such that $\operatorname{asdim} G > 0 = \operatorname{asdim} K$ (so the asymptotic dimension decreases under the bimorphism $G \hookrightarrow K$).

In case the torsion subgroup $\mathrm{Tor}(K)$ of K is dense, one can choose G such that $\operatorname{asdim}(G, \tau_{dis}) = 0 < \operatorname{asdim} G$, where G carries the topology induced by K (so the bimorphism $id_G\colon (G, \tau_{dis}) \to G$ strictly increases the asymptotic dimension). Indeed, in this case G in the proof of Corollary 10.1.4 can be chosen to be also torsion, since K is hereditarily separable and $\mathrm{Tor}(K)$ is dense. This ensures $\operatorname{asdim}(G, \tau_{dis}) = 0$, in view of Fact 7.2.16.

Note that if $\mathrm{Tor}(K)$ is simply *infinite* (not necessarily dense in K), then G can be built as above, but it will only be dense in the compact subgroup $\overline{\mathrm{Tor}(K)}$ of K (not necessarily in K).

The example provided by Corollary 10.1.4 does not allow us to achieve arbitrary gaps between the decreasing asymptotic dimensions. In order to do that one can fix a non-trivial metrisable connected locally compact abelian group H. Clearly $0 \leq \operatorname{asdim} H = n_{\mathbb{R}}(H) < \infty$, thanks to Corollary 10.2.4. According to Theorem 1.2.4, $H = \mathbb{R}^n \times K$, for a compact connected group K and $n = n_{\mathbb{R}}(H)$. Then

$$H/\mathbb{Z}^n \cong \mathbb{R}^n/\mathbb{Z}^n \times K \cong \mathbb{T}^n \times K$$

is a compact connected metrisable abelian group, hence it is monothetic, i.e., $H/\mathbb{Z}^n$ has a dense cyclic subgroup $C = \langle c \rangle$ ([98, 53]). Let $q\colon H \to H/\mathbb{Z}^n$ be the quotient map and let $D = q^{-1}(C)$, i.e., $D = \mathbb{Z}^n + \langle h \rangle$, where $h \in H$ is chosen with $q(h) = c$. Then D is torsion-free, as $C \cong \mathbb{Z}$ (algebraically) is torsion-free and $\mathbb{Z}^n$ is torsion-free. Since D is finitely generated, this yields $D \cong \mathbb{Z}^{n+1}$. Moreover, D is a dense subgroup of H. Indeed, if $K = \overline{D}$, then K is a closed subgroup of H containing $\ker q$, hence $q(K)$ is closed in $H/\mathbb{Z}^n$ [53]. Since $q(K)$ contains the dense subgroup C, this yields $q(K) = H/\mathbb{Z}^n$. As $K \geq \ker q$, this yields $K = H$. As $r_0(D_0) = n + 1$, while $r_0(H) = \mathfrak{c}$, one can find a free abelian subgroup F of H with $F \cap D_0 = \{0\}$ and with arbitrary free rank $d = r_0(F) \leq \mathfrak{c}$. Now the subgroup $G = F \oplus D_0$ of H is dense and $r_0(G) = d + n + 1$. Therefore, $\operatorname{asdim}(G, \tau_{dis}) = d + n + 1$, if d is finite; otherwise, $\operatorname{asdim}(G, \tau_{dis}) = \infty$. By varying $d \in \mathbb{N} \cup \{\infty\}$ one can obtain all possible gaps between $\operatorname{asdim}(G, \tau_{dis})$ and $n = \operatorname{asdim} H$.

10.2.3 Metrisability results

Now we see that other properties are also nicely preserved under the action of Pontryagin functor, such as metrisability.

Theorem 10.2.16. *Let (G, τ) be a locally compact abelian group. Then the following properties are equivalent:*
(a) (G, τ) is metrisable;
(b) $(\widehat{G}, \mathcal{E}_{\mathfrak{r}\mathcal{C}(\widehat{G})})$ is metrisable.

If the previous conditions hold, then $(\widehat{G}, \mathcal{E}_L)$ is metrisable.

Proof. It is well-known that G is metrisable if and only if $\widehat{G}$ is σ-compact [98]. Since $\widehat{G}$ is locally compact, $\widehat{G}$ is σ-compact if and only if $\widehat{G}$ is *hemicompact*, i.e.,

$\mathcal{C}(\widehat{G})$ has a countable cofinal subset with respect to inclusion. Finally, Lemma 7.2.14 guarantees that this property is equivalent to $(\widehat{G}, \mathcal{E}_{\mathfrak{r}\mathcal{C}(\widehat{G})})$ being metrisable.

As for the second statement, it is enough to remember that $\mathcal{OB} = \mathfrak{r}\mathcal{C}(\widehat{G})$ if $\widehat{G}$ is locally compact and σ-compact (see Example 7.1.9). □

We want to discuss whether it is possible to extend Theorem 10.2.1 to pairs of infinite-dimensional properties of locally compact abelian groups. A metric space X has *property C* if, for every sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers, there exists an open cover $\mathcal{U}$ of X such that $\mathcal{U} = \bigcup_{n\in\mathbb{N}}\mathcal{U}_n$, where $\mathcal{U}_n$ is a pairwise disjoint family with $\operatorname{diam} U < \varepsilon_n$, for every $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$ (see [76]). Dranishnikov, in [69], introduced the large-scale counterpart of property C. A metric space X has *asymptotic property C* if, for any increasing sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers, there exists a cover $\mathcal{U}$ of X such that $\mathcal{U} = \bigcup_{n\in\mathbb{N}}\mathcal{U}_n$, where $\mathcal{U}_n$ is uniformly bounded and ε_n*-separated* (i.e., E_{ε_n}-separated), for every $n \in \mathbb{N}$.

Let G be a locally compact abelian group. Theorem 10.2.16 implies that G is small-scale metrisable if and only if $\widehat{G}$ is large-scale metrisable. Then it is natural to ask if it is true that such a group G has property C if and only if $\widehat{G}$ has asymptotic property C. However, this is not true, as the following example, kindly provided by Takamitsu Yamauchi, shows. In fact, let $G = \Pi_{\mathbb{N}}\mathbb{T}$, and thus $\widehat{G} = \bigoplus_{\mathbb{N}}\mathbb{Z}$. Then $\widehat{G}$ has asymptotic property C ([172]), while G has not property C (it is actually strongly infinite-dimensional).

Following this idea, let us end this section by proposing another parallelism between small-scale and large-scale properties of groups. Before stating the result, let us just note that the following properties are equivalent for an arbitrary coarse group $(G, \mathcal{E}_{\mathcal{I}})$:

(a) G is locally finite a coarse group;
(b) G has bounded geometry;
(c) $\mathcal{I} \subseteq [G]^{<\omega}$.

Note that a similar equivalence can be stated also for arbitrary unitary magmas endowed with left (right) magmatic entourage structures.

Proposition 10.2.17. *Let G be a locally compact abelian group. Then the following properties are equivalent:*

(a) G is compact;
(b) $\widehat{G}$ is locally finite (as a coarse space);
(c) $\widehat{G}$ has bounded geometry.

Proof. Item (a) holds if and only if $\widehat{G}$ is discrete ([138]). If $\widehat{G}$ is discrete, then every compact subset is finite. Conversely, if every compact subset is finite, then $\widehat{G}$ is discrete, since $\widehat{G}$ is locally compact, and thus G is compact. Finally, the equivalence (b)↔(c) holds in general. □

The previous proposition reinforces the idea that local finiteness is the large-scale counterpart of sequential connectedness, which was already suggested in Theorem 2.4.2.

10.3 The Bohr functor and Hernández Paradigm

All groups in this section are abelian.

10.3.1 Glicksberg groups

In this section we study the preservation (and transformation) of the asymptotic dimension along the Bohr functor.

Let (G, τ) be a topological abelian group. We define its *von Neumann kernel* to be

$$n(G) = \overline{\{1\}}^{\tau^+} = \bigcap\{\ker \chi \mid \chi \in \widehat{G}\}.$$

The von Neumann kernel is a closed subgroup of G which is compact in G^+ (since it is actually indiscrete in G^+) and it is another example of a functorial subgroup (see §10.2). A group G is called *maximally almost periodic*, briefly *MAP*, (respectively, *minimally almost periodic*, briefly *MinAP*) if $n(G) = \{1\}$ (respectively, $n(G) = G$). It is easy to see that a group G is MAP (respectively, MinAP) if and only if G^+ is Hausdorff (respectively, indiscrete).

Following the current terminology in duality theory of abelian groups, we say that a topological group G *respects compactness*, if G satisfies the equality $\mathcal{C}(G) = \mathcal{C}(G^+)$ (this term is used for non-abelian groups as well, [155]). This phenomenon was first revealed in the following well-known theorem of Glicksberg.

Theorem 10.3.1 ([87])**.** *Every locally compact abelian group respects compactness.*

The following notion, inspired by Glicksberg, is given below in three options, of which the second one is the usually known one. We propose the other two since they are closely related to the topic of this section.

Definition 10.3.2. An abelian topological group G is said to be
(a) a *Glicksberg (group)*, if G is MAP and respects compactness.
(b) a *weakly Glicksberg (group)*, if G respects compactness;
(c) a *generalised Glicksberg (group)*, if the group $G/n(G)$ is Glicksberg.

Let $\mathscr{G}$ ($w\mathscr{G}$, $\mathscr{G}^*$, respectively) denote the class of Glicksberg groups (weakly Glicksberg groups, generalised Glicksberg groups, respectively). Clearly

$$\mathscr{G} = w\mathscr{G} \cap MAP = \mathscr{G}^* \cap MAP \subseteq w\mathscr{G} \subseteq \mathscr{G}^*. \tag{10.11}$$

We show in Corollary 10.3.14, that for topological abelian group with infinite compact sets, the notions 'weakly Glicksberg' and 'Glicksberg' are equivalent.

There are many examples of Glicksberg groups beyond the class of locally compact abelian groups, for instance all nuclear groups are Glicksberg (see [14]). The class of nuclear groups, introduced by Banaszczyk, is stable under arbitrary direct products and contains all locally compact abelian groups. Therefore, $\mathscr{G}$ contains also many non-locally compact groups (e.g., $\mathbb{R}^{\mathbb{N}}$). A large class of Glicksberg groups was singled out in [6] (namely, the *Schwartz groups*, introduced in [8]).

10.3.2 Groups respecting relative compactness

In [95], Hernández proved that, for a locally compact abelian group G, $\dim G = \dim G^+$. In the sequel we discuss whether a similar equality holds for the asymptotic dimension for locally compact abelian groups and eventually larger classes of groups (we briefly refer to this property as *Hernández Paradigm*). First, we introduce a class of topological abelian groups G for which the equality $\operatorname{asdim} G = \operatorname{asdim} G^+$ trivially holds.

Proposition 10.3.3. *For a topological abelian group G the following equivalent conditions hold:*
(a) $id_G \colon G \to G^+$ is an asymorphism;
(b) $\mathcal{E}_{\mathfrak{r}\mathcal{C}(G))} = \mathcal{E}_{\mathfrak{r}\mathcal{C}(G^+)}$;
(c) $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$.
If G is weakly Glicksberg, then it satisfies all three properties.

Proof. The implications (c)→(b)→(a) are trivial. The implication (a)→(c) follows from Propositions 7.2.1 and 7.2.5. The last assertion follows from the fact that $\mathcal{C}(G) = \mathcal{C}(G^+)$ when G is weakly Glicksberg. This equality obviously implies (c). □

Definition 10.3.4. We say that a topological abelian group G belongs to the class $\mathscr{B}$ if it satisfies the equivalent conditions of Proposition 10.3.3.

By the conclusion of the above proposition, $\mathscr{B}$ contains $w\mathscr{G}$.

Let $f \colon G \to H$ a continuous homomorphism between two topological abelian groups $G, H \in \mathscr{B}$. Then $f \colon (G, \mathcal{E}_{\mathfrak{r}\mathcal{C}(G)}) \to (H, \mathcal{E}_{\mathfrak{r}\mathcal{C}(H)})$ and $f^+ \colon (G^+, \mathcal{E}_{\mathfrak{r}\mathcal{C}(G^+)}) \to (H^+, \mathcal{E}_{\mathfrak{r}\mathcal{C}(H^+)})$ are bornologous. Moreover, Proposition 10.3.3(a) implies that f is effectively proper/an asymorphism/a coarse equivalence if and only if f^+ has the same property.

$$\begin{array}{ccc} G & \xrightarrow{id_G} & G^+ \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^+} \\ H & \xrightarrow{id_H} & H^+ \end{array}$$

The following result is an immediate corollary of Proposition 10.3.3.

Corollary 10.3.5 (Hernández Paradigm)**.** *Every topological abelian group G from the class $\mathscr{B}$ satisfies* $\operatorname{asdim} G = \operatorname{asdim} G^+$*. In particular, this holds true if G is Glicksberg.*

Remark 10.3.6. A careful analysis of this corollary and Definition 10.3.4 above suggests that the class $\mathscr{B}$ is maybe somewhat narrow compared with the class $\mathscr{H}$ of all topological abelian groups G with the relevant property $\operatorname{asdim} G = \operatorname{asdim} G^+$. Indeed, the above corollary can be resumed in these terms in the concise formula $\mathscr{H} \supseteq \mathscr{B} \supseteq \mathscr{G}$ that immediately follows from Definitions 10.3.2, 10.3.4 and Proposition 10.3.3. The reason we downplay the use of the precise class $\mathscr{H}$ is that while we have a quite precise description of the smaller class $\mathscr{B}$, we are not aware of any reasonable description of the larger class $\mathscr{H}$.

Let us state the special case of Question 10.2.14, with $\tau = \delta^+ \subseteq \delta$.

Problem 10.3.7. Study the class $\mathscr{H}_+$ of topological abelian groups G with $\operatorname{asdim} G \geq \operatorname{asdim} G^+$. How large is $\mathscr{H}_+$? Does it coincide with the class of all topological abelian groups?

The class $\mathscr{H}_+$ contains all MinAP abelian groups G, as $\operatorname{asdim} G^+ = 0$.

Since locally compact abelian groups are Glicksberg, from Corollaries 10.2.12 and 10.3.5 and Theorem 10.2.1, we obtain the following result.

Corollary 10.3.8. *Let G be a locally compact abelian group. Then we have* $\dim \widehat{G} = \operatorname{asdim} G = \operatorname{asdim} G^+ \leq r_0(G)$.

Since the Bohr compactification commutes with arbitrary products [102], $\mathscr{B}$ is stable for them.

The next example illustrates how the class $\mathscr{B}$ is placed in the category all abelian topological groups; more precisely, item (a) shows how big is $\mathscr{B}$, while item (b) and Theorem 10.3.10 show how big is the complement of $\mathscr{B}$.

Example 10.3.9. (a) Obviously, $\mathscr{B}$ contains all precompact groups. Since every abelian group admits a precompact group topology ([53]), this shows that every abelian groups admits a group topology τ such that $(G,\tau) \in \mathscr{B}$.

(b) Let G be a finitely generated group endowed with a non-compact minimally almost periodic (MinAP) topology. Then $G \notin \mathscr{B}$. Indeed, G^+ is compact (actually, indiscrete), so $\operatorname{asdim} G^+ = 0$. On the other hand, G is non-compact, so $\operatorname{asdim} G > 0$, by Proposition 7.2.17. So $G \notin \mathscr{B}$, by Corollary 10.3.5.

The same conclusion holds if G is only compactly generated, instead of being finitely generated.

Now we show that every infinite abelian group, when equipped with appropriate Hausdorff group topology, does not belong to $\mathscr{B}$.

Theorem 10.3.10. *An abelian group G admits a Hausdorff group topology τ such that $(G,\tau) \notin \mathscr{B}$ if and only if G is infinite.*

Proof. Let G be an infinite group. The proof will make use of of the Zariski topology $\mathfrak{Z}_G$ of G (see [57] for its definition and properties, although its roots go back to Bryant and as far back as Markov). Although $(G,\mathfrak{Z}_G)$ is not a topological group, $\mathfrak{Z}_G$ is T_1-topology that makes left and right translates, as well as the inversion, continuous. If $(G,\mathfrak{Z}_G)$ is connected, then G admits a MinAP group topology [59]. Since compact groups are MinAP only when they are trivial, we deduce that (G,τ) is not compact. On the other hand, (G,τ^+) is indiscrete, hence compact. Therefore, by Proposition 10.3.3(c), $(G,\tau) \notin \mathscr{B}$.

In the general case, the connected component $K = c(G)$ of G equipped with the Zariski topology is a finite index subgroup of finite exponent [57] of G and K admits a MinAP group topology τ, as $(K,\mathfrak{Z}_K)$ is connected. Equip G with the group topology τ^* having (K,τ) as an open topological subgroup. Then $(\tau^*)^+$ has K as an open indiscrete (so compact) subgroup, while K is not compact in τ^*, as the induced topology coincides with the MinAP topology τ. Hence, $(G,\tau) \notin \mathscr{B}$.

If G is finite, then every Hausdorff topology τ on G is discrete and $\tau^+ = \tau$, so $(G,\tau) \in \mathscr{B}$. □

10.3.3 Connections between $\mathscr{B}$ and Glicksberg's properties

Proposition 10.3.3(c) implies that the class $\mathscr{G}$ is contained in $\mathscr{B}$. According to the following proposition, these two classes coincide within the realm of MAP groups.

Proposition 10.3.11. *Let G be a topological abelian group which is MAP. Then $G \in \mathscr{B}$ if and only if $\mathcal{C}(G) = \mathcal{C}(G^+)$ (which is equivalent to $G \in \mathscr{G}$).*

Proof. Assume that $G \in \mathscr{B}$ and thus $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$. We want to show that $\mathcal{C}(G) = \mathcal{C}(G^+)$. Since $\tau^+ \subseteq \tau$, it is trivial that $\mathcal{C}(G) \subseteq \mathcal{C}(G^+)$. If $K \in \mathcal{C}(G^+)$, then, in particular, it is closed in τ^+ since G^+ is Hausdorff, and thus in τ. Moreover, the assumption implies that K is relatively compact in G and thus compact since it is closed. The opposite implication has already been discussed. □

In order to give a characterisation of the class $\mathscr{B}$ beyond the class of MAP groups, we make use of the classes $\mathscr{G}^*$ and $w\mathscr{G}$.

Theorem 10.3.12. *Let (G, τ) be a Hausdorff abelian group. Then the following conditions are equivalent:*
(a) $G \in \mathscr{B}$;
(b) G is a generalised Glicksberg group and $\mathrm{n}(G)$ is compact in G.

Proof. Let us consider the following diagram,

$$\begin{array}{ccc} G & \xrightarrow{\quad id_G \quad} & G^+ \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle q^+} \\ G/\,\mathrm{n}(G) & \xrightarrow{\quad id_{G/\,\mathrm{n}(G)} \quad} & (G/\,\mathrm{n}(G))^+ = G^+/\,\mathrm{n}(G), \end{array} \tag{10.12}$$

where $q\colon G \to G/\,\mathrm{n}(G)$ and $q^+\colon G^+ \to (G/\,\mathrm{n}(G))^+ = G^+/\,\mathrm{n}(G)$ are the quotient maps. Note that, $\mathrm{n}(G)$ is always compact in G^+. Moreover, if we assume either (a) or (b), we claim that $\mathrm{n}(G)$ is compact also in G. In item (b) it is actually required. If we assume (a), $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$ and then $\mathrm{n}(G)$ is relatively compact in G. Moreover, since $\mathrm{n}(G)$ is closed in G^+ and $\tau^+ \subseteq \tau$, $\mathrm{n}(G)$ is closed also in G and thus it is compact.

Thanks to Corollary 10.0.1, if we assume (a) or (b) and thus $\mathrm{n}(G)$ is compact both in G and in G^+, we can apply Proposition 7.2.11 to both q and q^+, and claim that they are coarse equivalences. Hence, since the diagram (10.12) commutes, id_G is a coarse equivalence (equivalently, since it is bijective, an asymorphism) if and only if $id_{G/\,\mathrm{n}(G)}$ is a coarse equivalence (equivalently, an asymorphism). Then, Proposition 10.3.3 implies that $G \in \mathscr{B}$ if and only if $G/\,\mathrm{n}(G) \in \mathscr{B}$. Since the group $G/\,\mathrm{n}(G)$ is MAP, $G/\,\mathrm{n}(G) \in \mathscr{B}$ if and only if $G/\,\mathrm{n}(G) \in \mathscr{G}$ by Proposition 10.3.11. □

Corollary 10.3.13. *Let G be a Hausdorff abelian group. If G is weakly Glicksberg, then $G \in \mathscr{B}$ and $\mathrm{n}(G)$ is finite.*

Proof. According to Proposition 10.3.3, weakly Glicksberg groups G belong to $\mathscr{B}$, i.e., satisfy $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$ and $\mathrm{n}(G)$ is compact, by Theorem 10.3.12. Suppose that G is a weakly Glicksberg group with infinite von Neumann kernel. Since $\mathrm{n}(G)$ is infinite by assumption, there exists a subset C of $\mathrm{n}(G)$ which is not closed in G, and thus $C \in \mathfrak{r}\mathcal{C}(G) \setminus \mathcal{C}(G)$. However, $\mathrm{n}(G)$ is actually indiscrete in G^+, hence $C \in \mathcal{C}(G^+)$. Therefore, $\mathcal{C}(G^+) \neq \mathcal{C}(G)$, a contradiction. This proves that $n(G)$ is finite. □

Corollary 10.3.14. *For G a topological abelian group with infinite compact sets, the following are equivalent:*
(a) G is a weakly Glicksberg group;
(b) G is a Glicksberg group.

Proof. The implication (b)→(a) is clear, as Glicksberg groups are weakly Glicksberg group (see (10.11)). The same equation will ensure the implication (a)→(b), if we check that a weakly Glicksberg G is MAP. So we have to prove that $\mathrm{n}(G)$ is trivial.

Assume that G is not MAP, so $\mathrm{n}(G) \neq \{0\}$ and $\mathrm{n}(G)$ is finite, by Theorem 10.3.12. Fix an infinite compact subset K of G. As $F = \mathrm{n}(G) \setminus \{0\}$ is finite and $0 \notin F$, we can choose a symmetric closed neighbourhood U of 0, such that $(U - U) \cap F = \emptyset$. Pick an accumulation point $x \in K$. Replacing K by $K - x$, we can assume without loss of generality that 0 is an accumulation point of the compact set K. Then 0 is an accumulation point also of the compact set $K_1 = K \cap U$. Moreover, the choice of U yields

$$(K_1 - K_1) \cap F = \emptyset, \quad \text{so} \quad K_1 \cap (K_1 + F) = \emptyset.$$

Now let

$$K_0 = (K_1 \setminus \{0\}) \cup F.$$

As $0 \in \overline{K_1 \setminus \{0\}}$, we deduce that $0 \in \overline{K_0}$. On the other hand, obviously $0 \notin K_0$. Hence, K_0 is not closed. Therefore, K_0 is not compact in G.

On other hand, the canonical map $q\colon G \to G/\mathrm{n}(G)$ sends K_0 onto $q(K_1)$, i.e., $q(K_0) = q(K_1)$ is compact both in $G/\mathrm{n}(G)$ and $(G/\mathrm{n}(G))^+$. Therefore, $q^{-1}(q(K_0))$ is compact both in G and G^+. Since G^+ carries the initial topology with respect to $q\colon G \to (G/\mathrm{n}(G))^+$, the compactness of $q(K_0)$ in $(G/\mathrm{n}(G))^+$ yields the compactness of K_0 in G^+. Therefore, G does not respect compactness. This contradicts our assumption that the group G is weakly Glicksberg. □

Theorem 10.3.12 and its corollaries witness the importance of the following classes of topological abelian groups.

Definition 10.3.15. Let G be a topological abelian group. We say that G is a *CMAP* (respectively, *AMAP*) if $\mathrm{n}(G)$ is compact (respectively, finite).

To be precise, the term AMAP, proposed in [113], was more restrictive, imposing by definition additionally also the non-MAP property. This slightly modified version, including in particular the class MAP, is adapted for our exposition.

In these terms Theorem 10.3.12 and Corollary 10.3.13 simply say that

$$\mathscr{B} = \mathscr{G}^* \cap CMAP \quad \text{and} \quad w\mathscr{G} \subseteq \mathscr{B} \cap AMAP.$$

These equality and inclusion can be inserted in the following longer chain of inclusions and equalities extending (10.11):

$$\mathscr{G} = w\mathscr{G}\cap MAP = \mathscr{G}^*\cap MAP \overset{(1)}{\subseteq} w\mathscr{G} \overset{(2)}{\subseteq} \mathscr{B}\cap AMAP \overset{(3)}{\subseteq} \mathscr{B} = \mathscr{G}^*\cap CMAP \subseteq \mathscr{G}^*. \tag{10.13}$$

The last inclusion is proper (witnessed by any MinAP group). The properness of the inclusions (1)–(3) will be discussed in detail §10.3.4. Here we only mention that, due to Corollary 10.3.14, the inclusion (1) can be proper only due to the existence of an AMAP group G that is not MAP and has no infinite compact sets. We are not aware if such groups exist (see the discussion in the next section), anyway for these groups one has:

Corollary 10.3.16. *For G a topological abelian group without infinite compact sets, the following properties are equivalent:*
(a) G is a weakly Glicksberg group;
(b) $G \in \mathscr{B}$ and $\mathrm{n}(G)$ is finite;
(c) $G \in \mathscr{B}$.

Proof. (a)→(b), by Theorem 10.3.12; the implication (b)→(c) is trivial.

Our blanket assumption gives $\mathfrak{r}\mathcal{C}(G) = \mathcal{C}(G) = [G]^{<\omega}$. Since $\mathfrak{r}\mathcal{C}(G^+) = \mathfrak{r}\mathcal{C}(G) = [G]^{<\omega}$, we deduce that $\mathcal{C}(G) = \mathcal{C}(G^+) = [G]^{<\omega}$ as well. Hence, $G \in w\mathscr{G}$. This proves the remaining implication (c)→(a). □

The above corollary shows that within the class of groups without infinite compact sets, both (2) and (3) become equalities.

Remark 10.3.17. In Corollary 10.3.14 we showed that the Glicksberg properties $\mathscr{G}$ and $w\mathscr{G}$ coincide within the class of groups with infinite compact sets. There is a variant of Glicksberg property defined by means of convergent sequences (in place of compact sets): a MAP topological group G is said to have the *Schur property*, if G and G^+ have the same convergent sequences; more precisely, if $x_n \to x$ in G^+, then also $x_n \to x$ in G.

Unlike the case of the Glicksberg properties $\mathscr{G}$ and $w\mathscr{G}$, it is well known that imposing the Schur property on a not necessarily MAP Hausdorff topological group G yields that G is MAP (see Propositions 19.2 and 19.3 of Xabier Dominguez's PhD thesis). A short proof of this fact was also kindly provided to the authors by Vaja Tarieladze. (If $a \in \mathrm{n}(G)$, then the constant null sequence (x_n) converges to a in G^+, so it must converge to a in G as well; hence $a = 0$, as G is Hausdorff)

10.3.4 Questions

In this section we collect some open questions and problems.

According to Proposition 10.3.11, for a MAP group G, the equality $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$ is equivalent to $\mathcal{C}(G) = \mathcal{C}(G^+)$. We are not aware whether this equivalence holds in general, i.e., whether $\mathscr{B} = w\mathscr{G}$ (according to Corollary 10.3.16, this is true for groups G without infinite compact sets).

Question 10.3.18. *Is the equality $\mathscr{B} = w\mathscr{G}$ true? Equivalently, does $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$ imply $\mathcal{C}(G) = \mathcal{C}(G^+)$, for every topological abelian group G?*

Clearly, $w\mathscr{G} = \mathscr{B}$ precisely when both inclusions (2) and (3) in (10.13) are equalities. Hence, a positive answer to this question is equivalent to the conjunction of positive answers to Questions 10.3.19 and 10.3.20 given below for the sake of completeness.

Question 10.3.19. *Is the inclusion (2) in (10.13) an equality?*

One can also ask about the inclusion (3) in (10.13), i.e.

Question 10.3.20. *Does a Hausdorff abelian group $G \in \mathscr{B}$ with infinite von Neumann kernel exist?*

As already mentioned, the class $\mathscr{B}$ contains $\mathscr{G}$ which, in turn, properly contains the class **LCA**. Indeed, all nuclear groups are Glicksberg ([14]), witnessing the huge gap between $\mathscr{B}$ and **LCA**. The chain (10.13) measures the gap between the class $\mathscr{B}$ and the class $\mathscr{G}$ of Glicksberg groups. However, so far even the 'simpler' question, namely whether such a gap exists at all (i.e., the problem to distinguish the classes $\mathscr{B}$ and $\mathscr{G}$) seems to be open. It is equivalent to ask (in a counter-positive form) whether all three inclusions (1)–(3) in (10.13) are equalities simultaneously:

Question 10.3.21. *Does there exist a non-Glicksberg group $G \in \mathscr{B}$?*

Equivalently (according to Theorem 10.3.12), one may ask how far is the class of generalised Glicksberg groups G with $\mathrm{n}(G)$ compact from the class $\mathscr{G}$ of Glicksberg groups. In these terms, we do not know even if there exists a generalised Glicksberg groups G with compact and non-trivial $\mathrm{n}(G)$ (so that $G \notin \mathscr{G}$) at all. Let us try to explain the difficulty of this, apparently 'simplest', question.

It is not hard to realise that the class MinAP shares similar properties with the class of connected groups (so MAP is associated with total disconnectedness, while $\mathrm{n}(G)$ with is associated with $c(G)$). This is why one may expect that the subgroup $\mathrm{n}(G)$ is MinAP (suggested by the fact $c(G)$ is connected). To the best of our knowledge this questions was raised by Lukásc in October 2004. A natural way to find a counter-example was to build an AMAP non-MAP group. The existence of such a topological abelian group (as well as its support, the Prüfer group) was briefly conjectured in [52, Corollary 4.9], although no explicit construction was given there, neither the specific term AMAP was used at that time. The first examples were built by Lukásc [113] and Nguyen [124]. All known constructions of these groups are quite entangled, using the powerful (yet rather sophisticated) technique of T-sequence, created by Protasov and Zelenyuk [152]. Nevertheless, it turned out that AMAP non-MAP groups are quite profuse in some sense. Namely, Gabriyelyan ([83]) proved that every abelian group with non-trivial torsion subgroup $\mathrm{Tor}(G)$ admits an AMAP non-MAP group topology (the restraint on $\mathrm{Tor}(G)$ is obviously necessary, as all finite subgroup of G are contained in $\mathrm{Tor}(G)$). This explains why the mere existence of an abelian group G with non-trivial compact $\mathrm{n}(G)$ (regardless of the fact whether $G/\,\mathrm{n}(G) \in \mathscr{G}$ or not) is highly non-trivial. We are not aware whether for the existing examples of groups G with non-trivial finite $\mathrm{n}(G)$ satisfy $G/\,\mathrm{n}(G) \in \mathscr{G}$.

Obviously, the compact-group coarse structure is too coarse to be useful in some situations (e.g., when the group G itself is compact), this is why it makes sense to consider group ideals $\mathcal{I}_G$ properly contained in $\mathfrak{r}\mathcal{C}(G)$, for a topological group G. A good candidate can be the group ideal $\mathcal{S}_G$ generated by the converging sequences. Taking only finite unions of converging sequences, one obtains an ideal, but in order to make it a group ideal, one has to add also finite products of such unions. Then the assignment $G \mapsto (G, \mathcal{E}_{\mathcal{S}_G})$ will be a functorial structure, as continuous homomorphisms $f\colon G \to H$ preserve this kind of group ideals, i.e., $f\colon (G, \mathcal{E}_{\mathcal{S}_G}) \to (H, \mathcal{E}_{\mathcal{S}_H})$ is bornologous. Call $\mathscr{B}_S$ the class of groups for which the Bohr functor induces an asymorphism $id_G\colon (G, \mathcal{E}_{\mathcal{S}_G}) \to (G, \mathcal{E}_{\mathcal{S}_{G^+}})$. It appears to be of some interest to investigate the connection of $\mathscr{B}_S$ to the Shur classes considered in Remark 10.3.17.

Part III

Coarse hyperspaces and related structures

Chapter 11

Coarse hyperspace, connectedness and their subspaces

11.1 Coarse hyperspaces

Metric hyperspaces can be generalised by defining both uniform and coarse hyperspaces in a similar, but opposite way. Let $(X,\mathcal{E})$ be a coarse space. Then the family

$$\exp\mathcal{E} = \mathfrak{cl}(\{\exp E \mid E \in \mathcal{E}\})$$

(see (1.5) for the definition of the entourages $\exp E$), of entourages of $\mathcal{P}(X)$ is a coarse structure called *coarse hyperstructure* and the pair $(\mathcal{P}(X), \exp\mathcal{E})$ is called *coarse hyperspace*. Thanks to the characterisation given in §3.3.3 and in Remark 4.2.3, we can deduce that the coarse hyperstructure is actually a coarse structure. However, for the sake of completeness, we prefer to give a direct proof of this fact.

Proposition 11.1.1. *If $(X,\mathcal{E})$ is a coarse space, then $\exp\mathcal{E}$ is a coarse structure.*

Proof. It is trivial to check that $\Delta_{\mathcal{P}(X)} = \exp\Delta_X \in \exp\mathcal{E}$, and that $\exp\mathcal{E}$ is closed under taking subsets and inverses (note that, for every $E \in \mathcal{E}$, $\exp E$ is symmetric). Fix now two entourages of $\exp\mathcal{E}$ and, without loss of generality, we can assume that those are of the form $\exp E$ and $\exp F$, for some $E, F \in \mathcal{E}$. As for the closure under taking finite unions, it is enough to check that $\exp E \cup \exp F \subseteq \exp(E \cup F)$. Finally, if $(Y,Z) \in \exp E \circ \exp F$, there exists $W \subseteq X$ such that $(Y,W) \in \exp E$ and $(W,Z) \in \exp F$, which implies that, in particular,

$$Y \subseteq E[W] \subseteq E[F[Z]] \quad \text{and} \quad Z \subseteq F[W] \subseteq F[E[Y]].$$

Thus $\exp E \circ \exp F \subseteq \exp((E \circ F) \cup (F \circ E)) \in \exp\mathcal{E}$. □

Let us note that the definition of coarse hyperspace agrees with the metric hyperspace. In fact, it is easy to check that, if (X, d) is a metric space, we have $\exp(\mathcal{E}_d) = \mathcal{E}_{d_H}$, where d_H is the Hausdorff metric induced on $\mathcal{P}(X)$ by d.

Recall that $\imath\colon X \to \mathcal{P}(X)$ is the map that associates to every point $x \in X$ the singleton $\{x\}$. The following fact, concerning the map just defined is straightforward.

Fact 11.1.2. *If $(X, \mathcal{E})$ is a coarse space, then $\imath\colon X \to \mathcal{P}(X)$ is an asymorphic embedding.*

Let $f\colon X \to Y$ be a map between sets. Then there is a natural extension $\exp f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$, defined as $\exp f(A) = f(A)$, for every $A \subseteq X$. The following result can be easily verified.

Proposition 11.1.3. *Let $f\colon X \to Y$ be a map between coarse spaces. Then*
(a) f is bornologous if and only if $\exp f$ is bornologous;
(b) f is effectively proper if and only if $\exp f$ is effectively proper;
(c) f is an asymorphic embedding if and only if $\exp f$ is an asymorphic embedding;
(d) f is an asymorphism if and only if $\exp f$ is an asymorphism;
(e) f is a coarse equivalence if and only if $\exp f$ is a coarse equivalence.

Proposition 11.1.3 implies that we have a functor $\exp\colon$ **Coarse** $\to$ **Coarse** that associates to every coarse space its coarse hyperspace.

Let us consider another consequence of Proposition 11.1.3. Let X be a coarse space and Y be a subspace of X. Then the inclusion map $j\colon Y \to X$ is an asymorphic embedding. Thanks to Proposition 11.1.3, also $\exp j\colon \exp Y \to \exp X$ is an asymorphic embedding, and thus we can identify the coarse hyperspace of Y as a coarse subspace of the coarse hyperspace of X.

It is useful to study also some coarse subspaces of coarse hyperspaces. In fact, for example, we will see that the coarse hyperspace is not connected in general (see Proposition 11.1.10 and Remark 11.1.11) and, moreover, it could be highly disconnected even in simple cases (Corollary 11.2.2 and Proposition 11.3.9).

Definition 11.1.4. Let $(X, \mathcal{E})$ be a coarse space and $\mathcal{A}(X)$ be a family of subsets of X. Then the *$\mathcal{A}$-coarse hyperspace* $\mathcal{A}$-$\exp X$ is the coarse subspace $\mathcal{A}\text{-}\exp X = (\mathcal{A}(X), \exp\mathcal{E}|_{\mathcal{A}(X)})$.

We will discuss in particular some special subspaces of coarse hyperspaces induced by size notions. We have already introduced the notions of boundedness, largeness and smallness. In the next definition we add more size properties.

Definition 11.1.5 ([151, 64])**.** Let $(X, \mathcal{E})$ be a coarse space. A subset A of X is called:
(a) *slim in X* if it is not large in X;
(b) *piecewise large in X* if it is not small in X;
(c) *meshy in X* if there exists $E \in \mathcal{E}$ such that, for every $x \in X$, $E[x] \setminus A \neq \emptyset$ (equivalently, $X \setminus A$ is large in X).

Let $\flat(X)$, $\mathcal{LA}(X)$, $\mathcal{SM}(X)$, $\mathcal{SL}(X)$, $\mathcal{PL}(X)$, and $\mathcal{ME}(X)$ be the families of all non-empty bounded, large, small, slim, piecewise large, and meshy subsets of X, respectively.

A map between $f\colon X \to Y$ is *$\mathcal{LA}$-preserving* if $f(\mathcal{LA}(X)) \subseteq \mathcal{LA}(Y)$. Similarly, the notions of *$\flat$-preserving*, *$\mathcal{SM}$-preserving*, *$\mathcal{SL}$-preserving*, and *$\mathcal{PL}$-preserving* maps can be introduced (see [64]).

Let us recall the following result concerning some of the previously introduced families defined by sizes.

Theorem 11.1.6. *A coarse equivalence is $\flat$-preserving, $\mathcal{LA}$-preserving, $\mathcal{SM}$-preserving, $\mathcal{SL}$-preserving, and $\mathcal{PL}$-preserving.*

Proof. The first assertion is trivial since every coarse equivalence is, in particular, a boundedness preserving map between the induced pre-bornological spaces. For a proof of the other claims we refer to [64]. □

Remark 11.1.7. Let $(X, \mathcal{E})$ be a coarse space. Then $\mathcal{LA}$-$\exp X$ is connected. In fact, because of the definition of largeness, we have $\mathcal{LA}(X) = \mathcal{Q}_{\exp X}(X)$. On the contrary, the $\flat$-coarse hyperspace is not connected in general. Let us focus a bit more on the $\flat$-coarse hyperspace. We claim that $\mathcal{Q}_{\exp X}(\imath(X)) = \flat(X)$. In fact, a non-empty subset $A \subseteq X$ belongs to $\flat(X)$ if and only if there exists $E \in \mathcal{E}$ and $x \in A$ such that $A \subseteq E[x]$, which is equivalent to $A \in \exp E[\{x\}]$. Since connectedness is preserved under taking asymorphic images and subspaces, $\flat$-$\exp X$ is connected if and only if X is connected.

Proposition 11.1.8. *Let $(X, \mathcal{E})$ be a connected coarse space. Then the following properties are equivalent:*
(a) X is unbounded;
(b) every finite subset of X is small in X;
(c) there is a singleton of X which is small in X;
(d) $\mathcal{LA}$-$\exp X$ is unbounded.

Proof. To prove the implication (a)→(b) pick $x \in X$ and an entourage $E \in \mathcal{E}$, our aim is to prove that $Y = X \setminus E[x]$ is large in X. Since X is unbounded, there exists $y \in Y$. As X is connected, $\{(y,x)\} \in \mathcal{E}$, and note that $E[x] = F[y]$, where $F = \{(y,x)\} \circ E$. Hence $F[Y] = X$. This proves that all singletons of X are small. We are done as finite unions of small sets are small ([144]).

The implication (b)→(c) is trivial.

Assume now item (c) and fix a point x such that $\{x\}$ is small in X. We claim that, for every $E = E^{-1} \in \mathcal{E}$, $\exp E[X] \neq \mathcal{LA}(X)$ and so $\mathcal{LA}$-$\exp X$ is unbounded. Pick an arbitrary entourage $E = E^{-1} \in \mathcal{E}$. Since $\{x\}$ is small, $X \setminus E[x] \in \mathcal{LA}(X)$. However, $X \setminus E[x] \notin \exp E[X]$.

Let us prove now the implication (d)→(a). If X is bounded, then $\mathcal{LA}(X) = \mathcal{P}(X) \setminus \{\emptyset\}$ and it is easy to check that every singleton is large in $\mathcal{LA}$-$\exp X$ and so, $\mathcal{LA}$-$\exp X$ is bounded. □

The equivalence of items (a), (b) and (c) of Proposition 11.1.8 was proved in [64].

Remark 11.1.9. Let us add some remarks on Proposition 11.1.3. Let $f\colon X \to Y$ be a map between coarse spaces and $\mathcal{A}(X)$ and $\mathcal{B}(Y)$ be two family of subsets of X and Y, respectively.

(a) If f is bornologous and $\exp f(\mathcal{A}(X)) \subseteq \mathcal{B}(Y)$, then $\exp f|_{\mathcal{A}(X)}\colon \mathcal{A}\text{-}\exp X \to \mathcal{B}\text{-}\exp Y$ is defined and thus bornologous. In particular, this implication holds for the families $\flat(X)$ and $\flat(Y)$. Hence, we have a functor

$$\flat\text{-}\exp\colon \mathbf{Coarse} \to \mathbf{Coarse}.$$

(b) Recall that $\mathcal{S}(X)$ denotes the family of singletons of the coarse space X. If $\mathcal{S}(X) \subseteq \mathcal{A}(X)$, $\mathcal{S}(Y) \subseteq \mathcal{B}(X)$, and $\exp f|_{\mathcal{A}(X)}\colon \mathcal{A}\text{-}\exp X \to \mathcal{B}\text{-}\exp Y$ is bornologous, then $f\colon X \to Y$ is defined and then bornologous since it is the restriction of $\exp f$ to $\mathcal{S}(X)$.

(c) Suppose that f is a coarse equivalence and let $g\colon Y \to X$ be a coarse inverse of f. Then Theorem 11.1.6, applied to both f and g, implies that the restrictions

$$\begin{aligned}
\exp f|_{\flat(X)}&\colon \flat\text{-}\exp X \to \flat\text{-}\exp Y,\\
\exp g|_{\flat(Y)}&\colon \flat\text{-}\exp Y \to \flat\text{-}\exp X,\\
\exp f|_{\mathcal{LA}(X)}&\colon \mathcal{LA}\text{-}\exp X \to \mathcal{LA}\text{-}\exp Y,\\
\exp g|_{\mathcal{LA}(Y)}&\colon \mathcal{LA}\text{-}\exp Y \to \mathcal{LA}\text{-}\exp X,\\
\exp f|_{\mathcal{SM}(X)}&\colon \mathcal{SM}\text{-}\exp X \to \mathcal{SM}\text{-}\exp Y,\\
\exp g|_{\mathcal{SM}(Y)}&\colon \mathcal{SM}\text{-}\exp Y \to \mathcal{SM}\text{-}\exp X,\\
\exp f|_{\mathcal{SL}(X)}&\colon \mathcal{SL}\text{-}\exp X \to \mathcal{SL}\text{-}\exp Y,\\
\exp g|_{\mathcal{SL}(Y)}&\colon \mathcal{SL}\text{-}\exp Y \to \mathcal{SL}\text{-}\exp X,\\
\exp f|_{\mathcal{PL}(X)}&\colon \mathcal{PL}\text{-}\exp X \to \mathcal{PL}\text{-}\exp Y, \quad \text{and}\\
\exp g|_{\mathcal{PL}(Y)}&\colon \mathcal{PL}\text{-}\exp Y \to \mathcal{PL}\text{-}\exp X
\end{aligned}$$

are defined and thus coarse equivalences in view of Proposition 11.1.3(e).

Proposition 11.1.10. *Let $(X,\mathcal{E})$ be a coarse space and $\mathcal{A}(X) \subseteq \mathcal{P}(X)$ be a family such that $\mathcal{S}(X) \subseteq \mathcal{A}(X)$. Then the following properties are equivalent:*
(a) $\mathcal{A}\text{-}\exp X$ is connected;
(b) $\mathcal{A}(X) \subseteq \flat(X)$.

Proof. Let $Y \in \mathcal{A}(X)$ and suppose that $Y \notin \flat(X)$. If $Y = \emptyset$, then $\mathcal{A}\text{-}\exp X$ is not connected, in fact $\mathcal{Q}_{\exp X}(\emptyset) = \{\emptyset\}$ (see also Remark 11.1.11). If Y is non-empty and then unbounded, it cannot be contained in a ball centred in a singleton. Conversely, for every pair of non-empty bounded subsets A and B of X, there exists an entourage E such that $A \subseteq E[B]$ and $B \subseteq E[A]$. In fact, pick two points $x \in A$ and $y \in B$, and, since A and B belong to $\flat(X)$, there exist $E_x \in \mathcal{E}$ and $E_y \in \mathcal{E}$ such that $A \subseteq E_x[x]$ and $B \subseteq E_y[y]$. Moreover, since X is connected, $F = \{(x,y),(y,x)\} \in \mathcal{E}$. Hence it is enough to define $E = F \circ (E_x \cup E_y)$. □

Note that Proposition 11.1.10 is the large-scale counterpart of Proposition 1.3.2, which holds in the realm of uniform spaces. Also in this case, the request that $\mathcal{S}(X) \subseteq \mathcal{A}(X)$ is justified in order to have the corestriction $\imath\colon (X,\mathcal{E}) \to \mathcal{A}\text{-}\exp X$ defined and so an asymorphic embedding.

Remark 11.1.11. Let us note some basic results concerning the number of connected components of the coarse hyperspace. Recall that $\operatorname{dsc} X$ denote the number of connected components of a coarse space X.

(a) Since, for every coarse space $(X,\mathcal{E})$ and every $E\in\mathcal{E}$, $E[\emptyset]=\emptyset$, $\mathcal{Q}_{\exp X}(\emptyset)=\{\emptyset\}$ and thus $\operatorname{dsc}\exp X\geq 2$ provided that X is non-empty. Moreover, it is trivial that $\operatorname{dsc}\exp X\leq|\exp X|=2^{|X|}$.
(b) A coarse space $(X,\mathcal{E})$ has $\operatorname{dsc}\exp X=2$ if and only if X is non-empty and bounded. Suppose that X is non-empty and bounded. Since X is non-empty, item (a) implies that $\operatorname{dsc}\exp X\geq 2$. Moreover, there exists $E\in\mathcal{E}$ such that, for every $x\in X$, $\exp E[\{x\}]=\mathcal{P}(X)\setminus\{\emptyset\}$. Conversely, item (a) implies that X has to be non-empty. Moreover, for every $x\in X$, $\{x\}$ and X have to be in the same connected component of $\exp X$, which means that there exists $E\in\mathcal{E}$ such that $X\subseteq E[x]$ and thus the claim follows.
(c) For every coarse space X, if Y is a coarse subspace of X, then $\operatorname{dsc}X\geq\operatorname{dsc}Y$. In particular, it is true that $\operatorname{dsc}\exp X\geq\operatorname{dsc}\exp Y$.

Later in this chapter we will compute the number of connected components of the coarse hyperspace for particular classes of coarse spaces. In particular, we show that the upper bound provided in Remark 11.1.11(a) can be achieved.

Remark 11.1.12. Since it is going to be very useful later on, let us characterise the coarse hyperspace in terms of balleans (§3.2). Let $\mathfrak{B}=(X,P,B)$ be a ballean. Then the *hyperballean* $\exp\mathfrak{B}=\mathfrak{B}_{\exp\mathcal{E}_{\mathfrak{B}}}$ can be characterised as follows: $\exp\mathfrak{B}=(\mathcal{P}(X),P,\exp B)$, where, for every $Y\subseteq X$, and $r\in P$,

$$\exp B(Y,r)=\{Z\subseteq X\mid Y\subseteq B(Z,r)\text{ and }Z\subseteq B(Y,r)\}.$$

If $\mathfrak{B}=(X,P,B)$ is a ballean and $\mathcal{A}(X)$ is a family of subsets of X, then the *$\mathcal{A}$-hyperballean* $\mathcal{A}\text{-}\exp\mathfrak{B}$ is the subballean $\mathcal{A}\text{-}\exp\mathfrak{B}=\exp\mathfrak{B}|_{\mathcal{A}(X)}$. In the sequel we will use the characterisation with coarse spaces or balleans interchangeably.

The notion of hyperballean was introduced in [55]. The authors had been inspired from a previous paper ([146]), where only the $\flat$-hyperballean was defined, under the name *hyperballean*.

11.2 Hyperspace of thin and cellular coarse spaces and balleans

11.2.1 Thin coarse spaces and their hyperspaces

We have introduced thin coarse spaces in §6.2.1. Moreover, according to Theorem 6.2.1, every thin coarse space is equivalent to an ideal coarse structure. It is important to focus on the coarse hyperspace of ideal coarse structures.

Let X be a set and $\mathcal{I}$ be an ideal on it. Recall that the ideal coarse structure $\mathcal{E}_{\mathcal{I}}$ is generated by the family of entourages $E_K=\Delta_X\cup(K\times K)$, where $K\in\mathcal{I}$ (see Example 3.1.3). Let $K\in\mathcal{I}$ and $Z\in(\exp E_K)[Y]$ for some $Y\subseteq X$. Then,

$$Z\subseteq E_K[Y]=\begin{cases}Y\cup K & \text{if } Y\cap K\neq\emptyset,\\ Y & \text{otherwise,}\end{cases}\qquad\text{and}$$
$$Y\subseteq E_K[Z]=\begin{cases}Z\cup K & \text{if } Z\cap K\neq\emptyset,\\ Z & \text{otherwise.}\end{cases}$$

Thus, if $Y\cap K=\emptyset$, $Z=Y$, and, otherwise, $Y\setminus K\subsetneq Z\subseteq Y\cup K$. Then we have computed the subsets

$$(\exp E_K)[Y] = \begin{cases} \{Y\} & \text{if } Y \cap K = \emptyset, \\ \{Z \subseteq X \mid Y \setminus K \subsetneq Z \subseteq Y \cup K\} & \text{otherwise.} \end{cases} \tag{11.1}$$

Proposition 11.2.1. *Let X be a set and $\mathcal{I}$ be a* proper *ideal (i.e., $X \notin \mathcal{I}$) on it which is also a cover. Then two subsets $Y, Z \subseteq X$ are in the same connected component of* $\exp(X, \mathcal{E}_\mathcal{I})$ *if and only if $X \triangle Y \in \mathcal{I}$.*

Proof. First of all note that the hypothesis leads to the fact that $[X]^{<\omega} \subseteq \mathcal{I}$. If there exists $K \in \mathcal{I}$ such that $Z \in (\exp E_K)[Y]$, then, in particular $Z \subseteq Y \cup K$ and $Y \subseteq Z \cup K$, which imply that $Z \setminus Y \subseteq K \supseteq Y \setminus Z$ and thus $Y \triangle Z \subseteq K \in \mathcal{I}$. Conversely, suppose that $Y \triangle Z \in \mathcal{I}$. Then, if $y \in Y$ and $z \in Z$, $K = Y \triangle Z \cup \{y\} \cup \{z\} \in \mathcal{I}$ has non-empty intersection with both Y and Z, and (11.1) implies that $Z \in (\exp E_K)[Y]$. □

Corollary 11.2.2. *Let X be an infinite set and $\mathcal{I} = [X]^{<\omega}$. Then*

$$\operatorname{dsc} \exp(X, \mathcal{E}_\mathcal{I}) = 2^{|X|}.$$

Proof. According to Proposition 11.2.1, for every $Y \subseteq X$, $|\mathcal{Q}_{\exp(X,\mathcal{E}_\mathcal{I})}(Y)| = |[X]^{<\omega}| = |X|$, since X is infinite. However, $|\mathcal{P}(X)| = 2^{|X|}$, and thus we have $\operatorname{dsc} \exp(X, \mathcal{E}_\mathcal{I}) = 2^{|X|}$. □

For a coarse space $(X, \mathcal{E})$, we define a map $C \colon X \to \mathcal{P}(X)$ by putting $C(x) = X \setminus \{x\}$.

Lemma 11.2.3. *Let $(X, \mathcal{E})$ be a connected unbounded coarse space. If Y is a subset of X, then $C(Y)$ is bounded in $\exp X$ if and only if there exists $E \in \mathcal{E}$ such that $|E[y]| > 1$, for every $y \in Y$.*

Proof. ($\to$) Since $C(Y)$ is bounded in $\exp X$, there exists $E = E^{-1} \in \mathcal{E}$ such that, for every $x, y \in Y$ with $x \neq y$, $C(y) \in \exp E[C(x)]$. Hence $y \in X \setminus \{x\} \subseteq E[X \setminus \{y\}]$ and $x \in X \setminus \{y\} \subseteq E[X \setminus \{x\}]$, in particular, $y \in E[Y \setminus \{y\}]$ and $x \in E[Y \setminus \{x\}]$, from which the conclusion descends.

($\leftarrow$) Since, for every $y \in Y$, there exists $z \in Y \setminus \{y\}$ such that $y \in E[z]$, $C(y) \in \exp E[X]$. Hence $C(Y) \subseteq \exp E[X]$, and the latter is bounded. □

Theorem 11.2.4. *Let $(X, \mathcal{E})$ be an unbounded connected coarse space. Then the following properties are equivalent and define a* thin coarse space*:*
(a) $(X, \mathcal{E})$ is thin;
(b) $(X, \mathcal{E}) = (X, \mathcal{E}_\mathcal{I})$, where $\mathcal{I} = \flat(X)$;
(c) if $A \subseteq X$ is meshy in X, then A is bounded;
(d) $\mathcal{ME}$-$\exp X$ is connected;
(e) the map $C \colon X \to \mathcal{P}(X)$ is an asymorphism between X and $C(X)$.

Proof. The implication (c)$\to$(d) is trivial, since item (c) implies that $\mathcal{ME}$-$\exp X = \flat$-$\exp X$ (note that $\flat(Y) \subseteq \mathcal{ME}(Y)$ fo a generic coarse space Y) and the latter is connected. Furthermore, (a)$\leftrightarrow$(b) can be found in Theorem 6.2.1.

(d)$\to$(c) Assume that $A \subseteq X$ is meshy. Fix arbitrarily a point $x \in X$. The singleton $\{x\}$ is bounded, hence meshy. By our assumption, $\mathcal{ME}$-$\exp X$ is connected and both A and $\{x\}$ are meshy, so there must be a ball centred at x and containing A. Therefore, A is bounded.

(e)→(a) If (a) is not satisfied, then there is an unbounded subset Y of X satisfying Lemma 11.2.3. Since $C(Y)$ is bounded in $\exp X$, we see that C is not an asymorphism.

(b)→(e) Suppose that $\mathcal{E} = \mathcal{E}_{\mathcal{I}}$. Fix an element $V \in \mathcal{I}$. Since the family of all E_U, where $U \in \mathcal{I}$ such that $|U| > 1$, forms a base of $\mathcal{E}_{\mathcal{I}}$, we can assume that V has at least two elements. Now, pick an arbitrary point $x \in X$. Since $|V| > 1$, for every $A \in C(X)$, $A \cap V \neq \emptyset$. Hence (11.1) implies that

$$\begin{aligned}(\exp E_V^{\mathcal{I}})[C(x)] \cap C(X) &= \{X \setminus \{y\} \mid (X \setminus \{x\}) \setminus V \subsetneq X \setminus \{y\} \subseteq \\ &\subseteq (X \setminus \{x\}) \cup V\}.\end{aligned} \tag{11.2}$$

Moreover, if $x \in V$, (11.2) implies

$$(\exp E_V^{\mathcal{I}})[C(x)] \cap C(X) = \{X \setminus \{y\} \mid X \setminus V \subsetneq X \setminus \{y\}\} = C(E_V^{\mathcal{I}}[x]).$$

On the other hand, if $x \notin V$, then (11.2) implies

$$\begin{aligned}(\exp E_V^{\mathcal{I}})[C(x)] \cap C(X) &= \{X \setminus \{y\} \mid X \setminus (V \cup \{x\}) \subsetneq X \setminus \{y\} \subseteq X \setminus \{x\}\} = \\ &= C(E_V^{\mathcal{I}}[x]).\end{aligned}$$

(a)→(c) Suppose that item (a) is satisfied and A is an unbounded subset of X. We claim that A is not meshy. Fix an entourage $E \in \mathcal{E}$ and let $V \subseteq X$ be a bounded subset of X such that $E[x] = \{x\}$, for every $x \notin V$. Since A is unbounded, there exists a point $x_E \in A \setminus V$. Hence $E[x_E] = \{x_E\} \subseteq A$, which shows that A is not meshy.

(c)→(a) Suppose that item (i) is not satisfied. Then, there exists $E \in \mathcal{E}$ such that, for every bounded subset V of X, there exists $x_V \notin V$ which verifies $|E[x_V]| \geq 2$.

We want to construct, by transfinite induction, a subset $A = A_\kappa = \{y_\lambda \mid \lambda < \kappa\}$, for some limit ordinal κ, and a family of symmetric entourages $\{E_\lambda\}_{\lambda<\kappa}$ (an entourage F is *symmetric* if $F^{-1} = F$) with the following properties:

(i) A is unbounded;
(ii) for every $\lambda < \kappa$, $A_\lambda = \{y_{\lambda'} \mid \lambda' < \lambda\}$ is bounded;
(iii) $E_\lambda \subseteq E_{\lambda'}$, for every $\lambda < \lambda' < \kappa$ such that there exist a limit ordinal ϑ and two natural numbers m, n with the property that $\lambda = \vartheta + m$ and $\lambda' = \vartheta + n$;
(iv) $E_\lambda \nsubseteq E_{\lambda'}$, for every $\lambda' < \lambda < \kappa$;
(v) for every $\lambda < \kappa$, $y_\lambda \notin E_\lambda[A_\lambda]$;
(vi) $E \subsetneq E_\lambda$, for every $\lambda < \kappa$;
(vii) $|E[y_\lambda]| \geq 2$, for every $\lambda < \kappa$.

Indeed, such an A is unbounded (by item (i)) and $X \setminus A$ is large, since, for every $y \in A$, $|E[y]| \geq 2$ (by item (vii)) and $E[y] \cap A = \{y\}$ (by items (iii)–(vi)) and thus there exists a point $z \in E[y] \setminus A$, which shows that $y \in E[z] \subseteq E[X \setminus A]$. Hence A is meshy.

First of all, note that there exists no $E_{max} \in \mathcal{E}$ such that $F \subseteq E_{max}$, for every $F \in \mathcal{E}$, since, otherwise, X is bounded.

Let $E_1 \in \mathcal{E}$ be an arbitrary symmetric entourage such that $E \subsetneq E_1$ and fix a point $y_1 \in X$ such that $|E[y_1]| \geq 2$.

Let now κ be an ordinal and suppose that y_ν and E_ν are defined, for every $\nu < \kappa$, and satisfy properties (ii)–(vii).

Suppose that κ is not a limit ordinal and thus let λ be an ordinal such that $\lambda + 1 = \kappa$. Let E_κ be a radius such that $E_\lambda \subseteq E_\kappa$. Since A_κ is bounded by item (ii), there exists a point $y_\kappa \notin E_\kappa[A_\kappa]$ such that $|E[y_\kappa]| \geq 2$.

Conversely, suppose now that κ is a limit ordinal. If A_κ is unbounded, then we are done. Suppose then that A_κ is bounded. Hence there exists $F \in \mathcal{E}$ such that $A_\kappa \subseteq F[y_1]$. It is not hard to prove that $\mathcal{F}_\kappa = \{E_\lambda \mid \lambda < \kappa\}$ is not a base of $\mathcal{E}$ since, otherwise, A_κ is unbounded by item (v). Thus there exists $E_\kappa = E_\kappa^{-1} \in \mathcal{E}$ such that $E_\kappa \nsubseteq E_\lambda$, for every $\lambda < \kappa$, $F \subsetneq E_\kappa$, and $E \subsetneq E_\kappa$. Since $E_\kappa[A_\kappa]$ is bounded, there exists a point $y_\kappa \notin E_\kappa[A_\kappa]$, such that $|E[y_\kappa]| \geq 2$.

Since $|A_\kappa| = \kappa \leq |X|$ and X is unbounded, $A = A_\kappa$ is unbounded for some limit ordinal $\kappa \leq |X|$. And so A satisfies (i)–(vii). □

We refer to [55] for a different proof of Theorem 11.2.4. In the same paper it is shown that we cannot substitute item (c) of Theorem 11.2.4 by asking that all the small subsets are bounded. In fact it is a strictly weaker condition.

Remark 11.2.5. Let $(X, \mathcal{E})$ be an unbounded connected coarse space. Consider the map $CB\colon \flat\text{-}\exp X \to \exp X$ such that $CB(A) = X \setminus A$, for every bounded A. It is trivial that $C = CB|_X$, where X is identified with the family $\mathcal{S}(X)$ of all its singletons. Hence, if CB is an asymorphic embedding, then C is an asymorphic embedding too, and thus X is thin, according to Theorem 11.2.4. However, we claim that CB is not an asymorphic embedding if X is thin and then item (v) in Theorem 11.2.4 cannot be replaced with this stronger property.

Since $(X, \mathcal{E})$ is thin, we can assume that $\mathcal{E} = \mathcal{E}_\mathcal{I}$ (Theorem 11.2.4) for some ideal $\mathcal{I}$ on X. Fix an element $V \in \mathcal{I}$ of $\exp X_\mathcal{I}$ and suppose, without loss of generality, that V has at least two elements. For every other $W \in \mathcal{I}$, pick an element $A_W \in \mathcal{I}$ such that $A_W \subseteq X \setminus (W \cup V)$. Hence, $CB^{-1}((\exp E_V^\mathcal{I})[CB(A_W)]) \nsubseteq (\exp E_W^\mathcal{I})[A_W] = \{A_W\}$, which implies that CB is not effectively proper. In fact, since $A_W \cup V \in \mathcal{I}$,

$$(\exp E_V^\mathcal{I})[CB(A_W)] = \{Z \subseteq X \mid X \setminus (A_W \cup V) \subsetneq Z \subseteq X \setminus A_W\} \subseteq CB(\flat(X)),$$

and thus $|(\exp E_V^\mathcal{I})[CB(A_W)] \cap CB(\flat(X))| > 1$.

11.2.2 Hyperspaces of cellular balleans

Since the inclusion $\imath\colon X \to \exp X$ is an asymorphic embedding, for every ballean X, Theorem 6.1.2 implies that

$$\operatorname{asdim} X \leq \operatorname{asdim} \exp X. \qquad (11.3)$$

The equality $\operatorname{asdim} X = \operatorname{asdim} \exp X$ is not available in general and may strongly fail (see [154, 169]). However, that equality holds for spaces with asymptotic dimension 0, as we will show in Proposition 11.2.7.

Lemma 11.2.6. *Let (X, P, B) be a ballean. Let $\emptyset \neq Y \subseteq X$ and α be an arbitrary element of $\exp X$ and of P, respectively. Then*

$$(\exp B)^n(Y, \alpha) \subseteq \{Z \in \exp X \mid Z \subseteq B^n(Y, \alpha),\, Y \subseteq B^n(Z, \alpha)\}. \qquad (11.4)$$

Hence, in particular,

$(\exp B)^{\Box}(Y,\alpha) \subseteq \{Z \in \exp X \mid Z \subseteq B^{\Box}(Y,\alpha),\, Y \subseteq B^{\Box}(Z,\alpha)\} = \exp(B^{\Box})(Y,\alpha).$

Proof. We prove (11.4) by induction. As for the base step, suppose that $n = 2$ and then $Z \in \exp B(\exp B(Y,\alpha),\alpha)$. Thus, there exists $W \in \exp B(Y,\alpha)$ such that $Z \in \exp B(W,\alpha)$, i.e.,

$$W \subseteq B(Y,\alpha), \quad Y \subseteq B(W,\alpha), \quad Z \subseteq B(W,\alpha), \quad W \subseteq B(Z,\alpha).$$

Hence, in particular, $Y \subseteq B(B(Z,\alpha),\alpha)$ and $Z \subseteq B(B(Y,\alpha),\alpha)$.

Suppose now that (11.4) holds for n and let $Z \in \exp B^{n+1}(Y,\alpha)$. There exists $W \in \exp B(Y,\alpha)$ such that $Z \in \exp B^n(W,\alpha)$ and so, by using the inductive hypothesis,

$$W \subseteq B(Y,\alpha), \quad Y \subseteq B(W,\alpha), \quad Z \subseteq B^n(W,\alpha), \quad W \subseteq B^n(Z,\alpha),$$

from which the claim follows. □

Proposition 11.2.7. *If a ballean* (X,P,B) *is cellular, then* $\exp X$ *is cellular.*

Proof. Let α be an arbitrary radius and let $\beta \in P$ be another radius such that $B^{\Box}(x,\alpha) \subseteq B(x,\beta)$, for every $x \in X$. Then, by using Lemma 11.2.6, for every $Y \in \exp X$,

$$\begin{aligned}\exp B^{\Box}(Y,\alpha) &\subseteq \{Z \in \exp X \mid Y \subseteq B^{\Box}(Z,\alpha),\, Z \subseteq B^{\Box}(Y,\alpha)\} \subseteq \\ &\subseteq \{Z \in \exp X \mid Y \subseteq B(Z,\beta),\, Z \subseteq B(Y,\beta)\} = \exp B(Y,\beta).\end{aligned}$$

□

11.3 Other ballean structures on the power set of a group

In this section, we study ballean structures on the power set of a group G. Denote by $\mathfrak{B}_G$ the ballean, called *group ballean*, associated to the finitary group coarse structure $\mathcal{E}_G$, i.e., $\mathfrak{B}_G = \mathfrak{B}_{\mathcal{E}_G}$. More explicitly, we can characterise $\mathfrak{B}_G$ as the triple $(G,[G]^{<\omega},B_G)$, where, for every $g \in G$ and $K \in [G]^{<\omega}$, $B_G(g,K) = g(K \cup \{e\})$.

First of all, we can consider the hyperballean $\exp \mathfrak{B}_G$. Concerning this hyperballean, we have the following trivial fact which will be used in the sequel.

Fact 11.3.1. *Let* G *be a group and* e *be its neutral element. Then every ball of* $\exp \mathfrak{B}_G$ *centred in* $\{e\}$ *is finite. Hence, for every subset* X *of* G *such that* $X \in \mathcal{Q}_{\exp \mathcal{B}_G}(\{e\})$*, and every finite subset* F *of* G*, the ball* $\exp B_G(X,F)$ *is finite.*

Proof. It is enough to note that, for every finite subset F of G, $X \subseteq F$ provided that $X \in \exp B_G(\{e\},F)$. The second statement follows, since $\exp \mathcal{B}_G$ is upper multiplicative. □

11.3.1 The logarithmic hyperballean ℓ- $\exp G$

Let us now introduce another ballean structure on $\mathcal{P}(G)$, where G is a group.

Definition 11.3.2. For a group G, we define a function $d\colon \mathcal{P}(G)\times\mathcal{P}(G)\to \mathbb{N}\cup\{\infty\}$ as follows. If $Y,Z\subseteq G$ are two subsets which are in distinct connected components of $\exp\mathcal{B}_G$ then $d(Y,Z)=\infty$. Otherwise, we define

$$\mu(Y,Z)=\min\{\max\{|F|,|S|\}\mid FY\supseteq Z,\ SZ\supseteq Y,\\ F\in[G]^{<\omega},\ S\in[G]^{<\omega},\ e\in F\cap S\}$$

and put

$$d(Y,Z)=\log\mu(Y,Z).$$

The next claim is not hard to check, yet we give an argument for the sake of completeness.

Claim 11.3.3. *The function d is a metric.*

Proof. In fact, the only non-trivial property is the triangular inequality. Fix then three subsets X,Y,Z of G such that $d(X,Y)\leq\log n$ and $d(Y,Z)\leq\log m$. Pick four finite subsets $F_1,F_2,K_1,K_2\subseteq G$ such that
(a) $e\in F_1\cap F_2\cap K_1\cap K_2$,
(b) $|F_1|,|F_2|\leq n$,
(c) $|K_1|,|K_2|\leq m$,
(d) $X\subseteq F_1Y$ and $Y\subseteq F_2X$,
(e) $Y\subseteq K_1Z$ and $Z\subseteq K_2Y$.
In particular $X\subseteq F_1Y\subseteq F_1K_1Z$ and, similarly, $Z\subseteq K_2F_2Y$. Since both $|F_1K_1|\leq mn$ and $|F_2K_2|\leq mn$, this proves the claim. □

Finally we define the *logarithmic hyperballean* as the metric ballean induced by d, namely

$$\ell\text{-}\exp\mathfrak{B}_G=(\mathcal{P}(G),\mathbb{R}_{\geq 0},B_d),\ \text{where, for every } Y\subseteq G \text{ and } R\geq 0,\\ B_d(Y,R)=\{Z\mid d(Y,Z)\leq R\}.$$

The metric d is invariant under left and right actions of G on $\mathcal{P}(G)$, i.e., the maps $F\mapsto gF$ and $F\mapsto Fg$, for every $g\in G$ and every $F\subseteq G$, are isometries.

Furthermore, if G and H are two isomorphic groups, then ℓ- $\exp\mathfrak{B}_G$ and ℓ- $\exp\mathfrak{B}_H$ are asymorphic.

Remark 11.3.4. (a) Clearly, the connected components of the balleans $\exp\mathfrak{B}_G$ and ℓ- $\exp\mathfrak{B}_G$ coincide.
(b) For every group G, $\exp\mathfrak{B}_G$ is finer than ℓ- $\exp\mathfrak{B}_G$. In fact, if two subsets X and Y of G satisfy $X\in B_G(Y,F)$ for some finite subset $F\in[G]^{<\omega}$, then $d(X,Y)\leq\log|F|$.
(c) If G is infinite, then $\exp\mathfrak{B}_G$ is strictly finer than ℓ- $\exp\mathfrak{B}_G$. First of all, note that, for every two distinct singletons $\{x\}$ and $\{y\}$ of G, $d(\{x\},\{y\})=1$ and thus $B_d(\{x\},1)\supseteq\{\{z\}\mid z\in G\}$. However, a subset K of G such that $\{x\}\subseteq\{y\}K$, for every $x,y\in G$, must satisfy $K=G$, which is not a radius of $\exp\mathcal{B}_G$ since G is infinite.

(d) If G is abelian, then the ℓ-$\exp\mathfrak{B}_G$ can also be defined by the metric d', where, for every two subsets $Y, Z \subseteq G$,

$$d'(Y,Z) = \log\min\{|S| \mid S+Y \subseteq Z,\ S+Z \subseteq Y,\ S \in [G]^{<\omega},\ 0 \in S\},$$

if Y, Z are in the same connected component of $\exp\mathfrak{B}_G$, and $d'(Y,Z)=\infty$, otherwise.

Remark 11.3.5. Let G be a group of cardinality κ. We consider two ballean structure on the family $[G]^{<\omega} \setminus \{\emptyset\}$, which is equal to $\flat(G)$ if G is endowed with the group ballean structure $\mathfrak{B}_G$. The first one is the subballean structure $\flat$-ℓ-$\mathfrak{B}_G = \ell$-$\exp\mathfrak{B}_G|_{\flat(G)}$, while the second one is given by the identification of $\flat(G)$ with $\mathbb{H}^*_\kappa$, i.e., the metric ballean induced by h, where, for every $X, Y \in \flat(G)$, $\mathrm{h}(X,Y) = |X \triangle Y|$ (see §6.1).

We claim that $\mathbb{H}^*_\kappa$ is finer than $\flat$-ℓ-$\mathfrak{B}_G$ and, moreover, if G is infinite, it is strictly finer. Let $R \in \mathbb{N}$ and $X, Y \in \flat(G)$ such that $\mathrm{h}(X,Y) \le R$. Fix two elements $\overline{x} \in X$ and $\overline{y} \in Y$ and define

$$F = \{x\overline{y}^{-1} \mid x \in X \setminus Y\} \cup \{e\} \quad \text{and} \quad K = \{y\overline{x}^{-1} \mid y \in Y \setminus X\}.$$

Then $|F|, |K| \le R+1$ and

$$X \subseteq (X\setminus Y) \cup Y = F\overline{y} \cup Y \subseteq FY \quad \text{and}$$
$$Y \subseteq (Y\setminus X) \cup X = K\overline{x} \cup X \subseteq KX,$$

which implies the first part of the statement. Suppose now that G is infinite. Then, for every $n \in \mathbb{N}$, there exists $F \in \flat(G)$ and $g \in G$ such that $F \cap gF = \emptyset$. Then $d(F, gF) = 1$, while $\mathrm{h}(F, gF) = 2n$. Since n can be chosen arbitrarily, $\mathbb{H}^*_\kappa$ is strictly finer than $\flat$-ℓ-$\mathfrak{B}_G$.

Question 11.3.6. (a) For a countable group G, are $\flat$-ℓ-$\mathfrak{B}_G$ and $\mathbb{H}^*_\omega$ asymorphic? Coarsely equivalent?

(b) If the answer to item (a) is affirmative then, if G and H are countable groups, are $\flat$-ℓ-$\mathfrak{B}_G$ and $\flat$-ℓ-$\mathfrak{B}_H$ asymorphic? In particular, what does it happen if $G = \mathbb{Z}$ and H is the countable group of exponent 2?

11.3.2 The G-hyperballean G-$\exp\mathfrak{B}_{\mathcal{I}}$

Definition 11.3.7. Let G be a group, X be a G-space with action $G \times X \longrightarrow X$, $(g,x) \longmapsto gx$, and $\mathcal{I}$ be a group ideal on G. The ballean $\mathfrak{B}(G,X,\mathcal{I})$ is defined as $(X, \mathcal{I}, B)$, where $B(x,A) = Ax \cup \{x\}$ for all $x \in X$, $A \in \mathcal{I}$.

By [133, Theorem 1], every ballean $\mathfrak{B}$ with support X is asymorphic to $\mathfrak{B}(G,X,\mathcal{I})$ under appropriate choice of G as a subgroup of the group S_X of all permutations of X and a group ideal $\mathcal{I}$.

Note that the finitary ballean $\mathfrak{B}_G$ on a group G is precisely $\mathfrak{B}(G,G,[G]^{<\omega})$ with the action of G on G by left shifts.

For $\mathfrak{B} = \mathfrak{B}(G,X,\mathcal{I})$, we introduce a *$G$-hyperballean* G-$\exp\mathfrak{B}$ as

$$G\text{-}\exp\mathfrak{B}(\mathcal{P}(X), \mathcal{I}, G\text{-}\exp B), \text{ where}$$
$$G\text{-}\exp B(Y,A) = \{Y\} \cup \{gY \mid g \in A\},$$

for every $Y \subseteq X$ and every $A \in \mathcal{I}$. Since G acts by bijections, if Y and Z are two subsets of X, then

$$|Y| = |Z| \text{ if there exists } \{g\} \in \mathcal{I} \text{ such that } Z \in G\text{-}\exp B(Y, \{g\}). \qquad (11.5)$$

Proposition 11.3.8. *For every ballean* $\mathfrak{B} = \mathfrak{B}(G, X, \mathcal{I})$*,* $G\text{-}\exp \mathfrak{B} \prec \exp \mathfrak{B}$*. Moreover, the following properties are equivalent:*
(a) $G\text{-}\exp \mathfrak{B} = \exp \mathfrak{B}$*;*
(b) each ball in $\exp \mathfrak{B}$ *around a singleton consists of sigletons;*
(c) $\mathfrak{B}$ *is discrete.*

Proof. Fix a radius $A \in \mathcal{I}$ and assume, without loss of generality, that it satisfies $A = A^{-1}$. Then, for every subset Y of X, if $Z \in G\text{-}\exp B(Y, A)$, then $Z = gY$ for some $g \in A$. Thus

$$Z = gY \subseteq AY \quad \text{and} \quad Y = g^{-1}Z \subseteq AZ,$$

which implies that $Z \in \exp B(Y, A)$.

The implications (b)→(c)→(a) are trivial. Suppose now that $G\text{-}\exp \mathfrak{B} = \exp \mathfrak{B}$. Then, for every $\{x\} \subseteq X$ and every $A \in \mathcal{I}$, $|Y| = 1$, provided that $Y \in \exp B(\{x\}, A)$, because of (11.5). □

Proposition 11.3.9. *For an infinite group* G*,* $\mathrm{dsc}(G\text{-}\exp \mathfrak{B}_G) = \mathrm{dsc}(\exp \mathfrak{B}_G) = 2^{|G|}$.

Proof. We use [30] to choose a thin subset T of G such that $|T| = |G|$.

Since T is a thin subset of G, Proposition 6.2.1 implies that $\mathfrak{B}_G|_T$ coincides with the ideal ballean $\mathfrak{B}_{\mathcal{I}}$, where $\mathcal{I}$ is the ideal of all bounded subsets of T (i.e., all finite subsets of T). By [55, Proposition 4.1] the equivalence relation $A \sim B$, where $A, B \subseteq T$, if and only if $A \triangle B$ is finite, is precisely the equivalence relation of belonging to the same connected component in $\exp(\mathfrak{B}_G|_T)$. Hence, each connected component of $\exp(\mathfrak{B}_G|_T)$ has cardinality precisely $|T| = |G|$, and thus there are $2^{|G|}$ such connected components. Finally, since $G\text{-}\exp \mathfrak{B}_G \prec \exp \mathcal{B}_G$, we can apply Fact 2.2.2 and so

$$2^{|G|} \geq \mathrm{dsc}(G\text{-}\exp \mathfrak{B}_G) \geq \mathrm{dsc}(\exp \mathfrak{B}_G) \geq \mathrm{dsc}(\exp(\mathfrak{B}_G|_T)) = 2^{|G|}.$$

□

Example 11.3.10. Denote by S_ω the group of all permutations of ω. Let us take the ballean $\mathfrak{B} = \mathfrak{B}(S_\omega, \omega, [S_\omega]^{<\omega})$ and show that $\exp \mathfrak{B}$ has only three connected components: the singleton $\{\emptyset\}$, the family of all non-empty finite subsets of ω, and the family of all infinite ones.

For any two non-empty finite subset X_1, X_2 of ω and each $x \in X_1$, $y \in X_2$, let $s_{x,y}$ be the transposition with support $\{x, y\}$, i.e., $s_{x,y}(x) = y$, $s_{x,y}(y) = x$ and $s_{x,y}|_{X \setminus \{x,y\}} = id|_{X \setminus \{x,y\}}$. We take an arbitrary infinite subset Y of ω, partition Y into infinite subsets $Y = Y_1 \cup Y_2$, and partition $\omega = W_1 \cup W_2$ so that $Y_1 \subseteq W_1$, $Y_2 \subseteq W_2$. Then we choose two permutations f_1, f_2 of ω so that $f_1(Y_1) = W_2$, $f_2(Y_2) = W_1$ and put $F = \{id_\omega, f_1, f_2\}$. Then $\omega \in \exp B(Y, F)$. In contrast to $\exp \mathcal{B}$, the ballean $G\text{-}\exp \mathfrak{B}(S_\omega, \omega, \mathcal{F}_{S_\omega})$ has countably many connected components: for every $n \in \mathbb{N}$, the families $\{F \subset \omega \mid |F| = n\}$ and $\{Y \subseteq \omega \mid |\omega \setminus Y| = n\}$, and the family $\{Y \subseteq \omega \mid |Y| = |\omega \setminus Y| = \omega\}$.

In Chapter 12, we study ballean structures on the subgroup lattice $L(G)$ of a group G. In particular, we consider the subballeans induced by the balleans on the power set that have been introduced in this chapter. However, since non-trivial cosets of a subgroup are never subgroups, for every group G, the subballean $G\text{-}\exp \mathfrak{B}_G|_{L(G)}$ is trivial and so not interesting for our purpose in Chapter 12.

Chapter 12

Ballean structures on the subgroup lattice of a group

According to Proposition 11.3.9, the hyperspace of a group is highly disconnected, and thus it is a hard object to study. However, since we are dealing with groups, it is natural to consider the subballeans whose supports are the family of all subgroups. Recall that, for every group G, we denote by $L(G)$ the lattice of all subgroups of G. In this chapter we focus on the following subballeans of $\exp\mathfrak{B}(G)$ and $\ell\text{-}\exp\mathfrak{B}(G)$:

- the *subgroup exponential hyperballean* $\mathcal{L}(G) = L\text{-}\exp\mathfrak{B}_G = \exp\mathfrak{B}_G|_{L(G)}$, and
- the *subgroup logarithmic hyperballean* $\ell\text{-}\mathcal{L}(G) = \ell\text{-}\exp\mathfrak{B}(G)|_{L(G)}$.

For the sake of brevity, in the sequel we write $X \approx Y$ if the two balleans X and Y are asymorphic.

12.1 Connected components of the subgroup hyperballeans $\mathcal{L}(G)$ and $\ell\text{-}\mathcal{L}(G)$

First of all, we want to give a different and useful characterisation of the subgroup logarithmic hyperballean $\ell\text{-}\mathcal{L}(G) = \ell\text{-}\exp\mathfrak{B}_G|_{L(G)}$, where G is a group, and, in order to do that, we need the following result.

Lemma 12.1.1. *Let G be a group and let A, B be subgroups of G such that $B \subseteq SA$ for some subset S of G. Then $|B : (A \cap B)| \leq |S|$.*

Proof. We split the proof in three cases.

Case 1. *Assume that $S \subseteq B$.* Given any $b \in B$, we pick $s \in S$ such that $b \in sA$. Then $s^{-1}b \in A \cap B$ and $B \subseteq S(A \cap B)$. This proves that $|B : (A \cap B)| \leq |S|$.

Case 2. *Assume that $S \subseteq BA$.* Let $S_a = S \cap Ba$ and note that our assumption provides a partition $S = \bigcup_{a\in A} S_a$. Let $S^* = \bigcup_{a\in A, S_a\neq\emptyset} S_a a^{-1}$ and note that:
(a) $SA = S^*A$; (b) $|S^*| \leq |S|$; (c) $S^* \subseteq B$ (as $S_a a^{-1} \subseteq B$ when $S_a \neq \emptyset$).

By (a) and our blanket assumption $B \subseteq SA$, $B \subseteq S^*A$, so by (c) we can apply Case 1 to A, B and S^* to claim $|B : A \cap B| \le |S^*|$. Now (b) allows us to conclude that $|B : (A \cap B)| \le |S|$.

Case 3. In the general case let $S_1 = S \cap BA$. Then obviously, $B \subseteq S_1 A$ and $S_1 \subseteq BA$. By case 2, applied to A, B and S_1 we have $|B : A \cap B| \le |S_1|$. Since obviously $|S_1| \le |S|$, this yields $|B : A \cap B| \le |S|$. □

Definition 12.1.2. We recall that two subgroups of a group G are *commensurable* if the indices $|A : A \cap B|$ and $|B : A \cap B|$ are finite.

By Lemma 12.1.1 and Remark 11.3.4(a), two subgroups A and B of G are in the same connected component of $\mathcal{L}(G)$ (ℓ-$\mathcal{L}(G)$, equivalently) if and only if A and B are commensurable.

Moreover, Lemma 12.1.1 also implies a different characterization of ℓ-$\mathcal{L}(G)$, which is much more manageable. Namely, for every group G and every pair of subgroups A and B of G, define

$$d'(A,B) = \begin{cases} \log \max\{|A : A \cap B|, |B : A \cap B|\} & \text{if } A \text{ and } B \text{ are commensurable,} \\ \infty & \text{otherwise,} \end{cases}$$

which is a metric on $L(G)$. By Lemma 12.1.1, the metric ballean on $L(G)$ induced by d' coincides with ℓ-$\mathcal{L}(G)$.

Thanks to the previous characterisation of the subgroup logarithmic hyperballean, we can easily provide an example of a group G such that ℓ-$\mathcal{L}(G)$ has some infinite ball (see Example 12.2.20).

Remark 12.1.3. Fix $n \ge 2$. We want to take a closer look at the structure of $\mathcal{L}(\mathbb{Z}^n)$. First of all note that two commensurable subgroups H and K of $\mathbb{Z}^n$ have same free rank. Moreover, every subgroup H of $\mathbb{Z}^n$ is commensurable with a pure subgroup $\mathrm{sat}(H)$ of $\mathbb{Z}^n$, namely its *saturation* defined by

$$\mathrm{sat}(H) = \{x \in \mathbb{Z}^n \mid mx \in H \text{ for some non-zero } m \in \mathbb{Z}\}$$

(denoted also by H_* by some authors; recall that a subgroup H of an abelian group G is *pure*, whenever $mH = mG \cap H$ for every $m > 0$ [pure subgroups of $\mathbb{Z}^n$ split as direct summands]). For every $H, K \le \mathbb{Z}^n$, $\mathrm{sat}(H)$ is commensurable with $\mathrm{sat}(K)$ if and only if $\mathrm{sat}(H) = \mathrm{sat}(K)$. Then $\mathcal{L}(\mathbb{Z}^n)$ has a countable number of connected components. Namely, they are:

- $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}^n)}(\{0\}) = \{0\}$,
- $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}^n)}(\mathbb{Z}^n)$,
- for every $0 < k < n$, a countable number of connected components asymorphic to the subballean $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}^n)}(\mathbb{Z}^k)$ of $\mathcal{L}(\mathbb{Z}^n)$ which is asymorphic to the subballean $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}^k)}(\mathbb{Z}^k)$ of $\mathcal{L}(\mathbb{Z}^k)$.

In particular, by Fact 2.2.1 and Proposition 3.1.14, for every $n > 1$, neither $\mathcal{L}(\mathbb{Z}) \approx \mathcal{L}(\mathbb{Z}^n)$, nor ℓ-$\mathcal{L}(\mathbb{Z}) \approx \ell$-$\mathcal{L}(\mathbb{Z}^n)$.

Note that $L(\mathbb{Z})$ has two connected components, while, according to Proposition 11.3.9, $\mathrm{dsc}(\exp \mathcal{B}_{\mathbb{Z}}) = \mathrm{dsc}(\ell\text{-}\mathcal{L}\mathcal{B}_{\mathbb{Z}}) = 2^\omega$.

12.1.1 $\mathcal{I}soL(G)$: the chase for isolated points of $\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$

According to Remark 11.3.4(a), the isolated points of $\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$ coincide. We denote this set of common isolated points by $\mathcal{I}soL(G)$.

In this subsection we show how the existence of isolated points is related to divisibility. The equivalence of (a) and (b) in the following claim is folklore, yet we give a proof for the sake of completeness.

Claim 12.1.4. *For an abelian group G the following are equivalent:*
(a) G is divisible;
(b) G has no proper subgroups of finite index;
(c) $\{G\} \in \mathcal{I}soL(G)$.

Proof. (a)→(b) It suffices to note that if H is a proper subgroup of G of finite index, then the quotient G/H is a non-trivial finite group, so cannot be divisible, while divisibility is preserved under taking quotients.

(b)→(a) If G is not divisible, then $pG \neq G$ for some prime p. Hence, G/pG is a non-trivial abelian group of exponent p, i.e., a vector space over $\mathbb{Z}/p\mathbb{Z}$. Then G/pG admits a non-zero homomorphism $G/pG \to \mathbb{Z}/p\mathbb{Z}$, which is necessarily surjective, so provides a quotient of G isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Finally, (b) and (c) are obviously equivalent. □

Proposition 12.1.5. *For a subgroup A of an abelian group G the following are equivalent:*
(a) $A \in \mathcal{I}soL(G)$;
(b) A is divisible and $G \simeq A \oplus B$, where B is a torsion-free subgroup of G.
(c) $\mathrm{Tor}(G) \leq A \leq d(G)$ and A is divisible;

Proof. (b)→(a)

If $G = A \oplus B$, with A divisible and B torsion-free, then A has no proper subgroup of finite index (Claim 12.1.4). So if C is a subgroup of B commensurable with A, then $C \cap A = A$, i.e., $A \leq C$. Since A is divisible, this gives $C = A \oplus C_1$, for some subgroup C_1 of B. As C and A are commensurable, $C_1 \simeq C/A$ is finite. Since B is torsion-free, this yields $C_1 = \{0\}$, i.e., $C = A$. Thus, $A \in \mathcal{I}soL(G)$.

(a)→(b) Now assume that $A \in \mathcal{I}soL(G)$. Then A has no proper subgroups of finite index, so A is divisible, by Claim 12.1.4. Then there exists a subgroup B of G such that $G = A \oplus B$. If $b \in B$ were a non-zero torsion element of B, then $A \oplus \langle b \rangle$ and A are commensurable, so our hypothesis implies that $\langle b \rangle = \{0\}$. This proves that B is torsion-free.

Finally, the equivalence (b)↔(c) is trivial. □

Corollary 12.1.6. *The following are equivalent for an abelian group G:*
(a) $\mathcal{I}soL(G) \neq \emptyset$;
(b) $G/d(G)$ is torsion-free;
(c) $\mathrm{Tor}(G) \leq d(G)$;
(d) $d(G) \in \mathcal{I}soL(G)$.

Proof. (a)→(b) Assume that $\mathcal{I}soL(G) \neq \emptyset$ and pick $A \in \mathcal{I}soL(G)$. Then $G \simeq A \oplus B$, where B is a torsion-free subgroup of G. As $A \leq d(G)$, one has also $G = d(G) \oplus R(G)$, and one can arrange to have $R(G)$ a subgroup of B, so $R(G) \simeq G/d(G)$ is torsion-free.

The implications (b)→(c)→(d)→(a) are obvious (the second one in view of Proposition 12.1.5). □

In particular, one can easily isolate the following sufficient conditions for the (non-)existence of isolated points.

Corollary 12.1.7. *For an abelian group G one has:*
(a) $G \in \mathcal{I}soL(G)$ (so $\mathcal{I}soL(G) \neq \emptyset$) whenever G is divisible;
(b) if G is reduced, then $\mathcal{I}soL(G) = \emptyset$ if and only if G is not torsion-free; otherwise, if $\mathcal{I}soL(G) \neq \emptyset$, then $\mathcal{I}soL(G) = \{\{0\}\}$ is a singleton.

Proof. (a) follows from Proposition 12.1.5(c).

(b) If G is reduced, then $d(G) = \{0\}$, so $\mathcal{I}soL(G) \neq \emptyset$ precisely when $\mathrm{Tor}(G) = \{0\}$, according to the above corollary. The last assertion again follows from Corollary 12.1.6. □

Now we provide a sharper result that complements the previous corollaries which characterised when $\mathcal{I}soL(G) = \emptyset$. More precisely, we show that the size of $\mathcal{I}soL(G)$ is completely determined by the free-rank $r_0(d(G))$.

Proposition 12.1.8. *Let G be an abelian group with $\mathcal{I}soL(G) \neq \emptyset$ (i.e., $\mathrm{Tor}(G) \leq d(G)$). Then:*
(a) $\mathcal{I}soL(G)$ has size 1 (more precisely, $\mathcal{I}soL(G) = \{d(G)\}$) if and only if $r_0(d(G)) = 0$, i.e., $d(G) = \mathrm{Tor}(d(G))$ is torsion;
(b) $\mathcal{I}soL(G)$ has size 2 (more precisely, $\mathcal{I}soL(G) = \{\mathrm{Tor}(d(G)), d(G)\}$) if and only if $r_0(d(G)) = 1$, i.e., $d(G) = \mathrm{Tor}(d(G)) \oplus D$ with $D \simeq \mathbb{Q}$ torsion-free;
(c) $|\mathcal{I}soL(G)| = \omega$ if and only if $1 < r_0(d(G)) = n < \omega$ (then $d(G) = \mathrm{Tor}(d(G)) \oplus D$ with $D \simeq \mathbb{Q}^n$); and
(d) $\mathcal{I}soL(G)$ is uncountable (more precisely, $|\mathcal{I}soL(G)| = 2^{r_0(d(G))}$) if and only if $r_0(d(G)) \geq \omega$.

Proof. Items (a), (b) and (c) follow from the above corollaries, Fact 1.2.3, and the fact that $\mathbb{Q}^n$ has countably many divisible subgroups when $1 < n < \omega$. Similar arguments work for (d). □

Proposition 12.1.8, along with Fact 2.2.1 and Proposition 3.1.14, provides a large series of non-asymorphic pairs of spaces, like:

$$\mathcal{L}(\mathbb{Z}^n) \not\approx \mathcal{L}(\mathbb{Q}) \not\approx \mathcal{L}(\mathbb{Q}^m) \quad \text{and} \quad \mathcal{L}(\mathbb{Z}^n) \not\approx \mathcal{L}(\mathbb{Q} \oplus \mathbb{Z}) \not\approx \mathcal{L}(\mathbb{Q}^m),$$

for any n and $m > 1$, and the same holds for the corresponding subgroup logarithmic hyperballeans.

In the next remark we discuss further some other immediate consequences of the above results concerning (only) the set $\mathcal{I}soL(G)$ isolated points of an abelian group G on the group structure of G.

Remark 12.1.9. (a) Let G be a virtually divisible abelian group (i.e., $d(G)$ is a finite index subgroup of G) and D be a divisible abelian group. Then G is divisible, provided that either $\mathcal{L}(G) \approx \mathcal{L}(D)$ or ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(D)$. In fact, since $\mathcal{I}soL(D)$ is non-empty, $G/d(G)$ must be torsion-free, by Corollary 12.1.6. On the other hand, $G/d(G)$ must be finite (hence, torsion), since G be a virtually divisible. Thus, $G = d(G)$ is divisible.

(b) Let G be a divisible abelian group and H be a finitely generated abelian group such that $\mathcal{L}(G) \approx \mathcal{L}(H)$ or ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$. Then G is torsion and H is free. In fact, assume first of all that both G and H are non-trivial. By Corollary 12.1.7, $\mathcal{I}soL(G) \neq \emptyset$, so $\mathcal{I}soL(H) \neq \emptyset$ as well. Since H is reduced, this fact implies (by the same corollary) that H is torsion-free. Hence $H \simeq \mathbb{Z}^n$, for some $n \geq 0$, so $|\mathcal{I}soL(H)| = 1$. This yields $|\mathcal{I}soL(G)| = 1$. Since G is divisible, Proposition 12.1.8 applies to entail that G must be torsion.

(c) Let G be a divisible torsion-free abelian group. Then Proposition 12.1.8 implies that $G \simeq \mathbb{Q}$, provided that $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Q})$ or ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Q})$.

(d) Under the assumption of the Generalised Continuum Hypothesis, if G, D are divisible torsion-free abelian group such that at least one of them has infinite rank, then the following statements are equivalent:
(d_1) $\mathcal{L}(G) \approx \mathcal{L}(D)$;
(d_2) ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(D)$;
(d_3) $|\mathcal{I}soL(G)| = |\mathcal{I}soL(D)|$;
(d_4) $r_0(G) = r_0(D)$;
(d_5) $G \simeq D$.
The implications (d_5)$\to$(d_1)$\to$(d_3) and (d_5)$\to$(d_2)$\to$(d_3) are trivial and were already discussed. Implication (d_3)$\to$(d_4) follows from Proposition 12.1.8, in particular the equality $|\mathcal{I}soL(G)| = 2^{r_0(d(G))}$, and GCH. The implication (d_4)$\to$(d_5) follows from the fact that divisible torsion-free abelian groups are determined by their free-rank up to isomorphism (see Fact 1.2.3).

The assertions in Remark 12.1.9(c) and (d) are examples of what we call 'rigidity results', to which Section 12.3 is devoted. It is trivial that, if G and H are two isomorphic groups, then $\mathcal{L}(G) \approx \mathcal{L}(H)$ (more precisely, $\exp\mathfrak{B}_G \approx \exp\mathfrak{B}_H$), and ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$. The converse implication is not true (Corollaries 12.2.2 and 12.2.9). A rigidity result is a list of conditions on balleans $\mathcal{L}(G)$ and $\mathcal{L}(H)$ (ℓ-$\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$), where G and H are two groups, which imply that $G \simeq H$, provided that $\mathcal{L}(G) \approx \mathcal{L}(H)$ (ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$, respectively). We mention here that there is another, more common meaning of rigidity in the coarse context (see [157]).

12.2 Asymptotic dimension of the subgroup hyperballeans $\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$

12.2.1 The subgroup exponential hyperballean $\mathcal{L}(G)$

First of all, we provide some basic, although very important, examples of $\mathcal{L}(G)$. For example, if G is a finite group, then both $\mathfrak{B}_G$ and $\exp\mathfrak{B}_G|_{\mathcal{P}(G)\setminus\{\emptyset\}}$ are bounded. In particular, $\mathcal{L}(G)$ is bounded as well.

Proposition 12.2.1. *Let G be one of the groups $\mathbb{Z}$ and $\mathbb{Z}_{p^\infty}$ for some prime p. Then:*

(a) all balls in $\mathcal{L}(G)$ are finite;

(b) $\mathcal{L}(G)$ has two connected components, of which one is a singleton (namely, $\{\{0\}\}$, when $G = \mathbb{Z}$, otherwise $\{G\}$);

(c) $\mathcal{L}(G)$ is thin and thus $\operatorname{asdim}\mathcal{L}(G) = 0$ since it is cellular.

Proof. Items (a) and (b) are trivial.

(c) **Case $G = \mathbb{Z}$.** To show that $G = \mathcal{L}(\mathbb{Z})$ is thin take an arbitrary finite subset F of $\mathbb{Z}$ and choose m so that $F \subseteq [-m, m] \cap \mathbb{Z}$. Pick $n > 3m$. We claim that $B_{\mathcal{L}(\mathbb{Z})}(n\mathbb{Z}, F) = \{n\mathbb{Z}\}$. We carry out the proof for $F = [-m, m] \cap \mathbb{Z}$, obviously, this implies the general case.

Consider the quotient map $q\colon \mathbb{Z} \to \mathbb{Z}(n) = \mathbb{Z}/n\mathbb{Z}$ and notice that the subset $q(F)$ of $\mathbb{Z}(n)$ contains no non-trivial subgroups, by the assumption $3m < n$. Pick $H \in B(\langle n\rangle, F)$, then $q(H) \subseteq q(F)$, so $q(H) = \{0\}$ in $\mathbb{Z}/n\mathbb{Z}$, hence $H \leq n\mathbb{Z}$. Thus, $H = l\mathbb{Z}$ for some multiple l of n. Since $n\mathbb{Z} \in B(H, F)$, with $l \geq n \geq 3m$, the previous argument implies $n\mathbb{Z} \leq H$. Therefore, $H = n\mathbb{Z}$.

Case $G = \mathbb{Z}_{p^\infty}$. We consider now the group $G = \mathbb{Z}_{p^\infty}$, where p is a prime. Denote by H_n the subgroup of $\mathbb{Z}_{p^\infty}$ of order p^n, take an arbitrary finite subset F of $\mathbb{Z}_{p^\infty}$ and choose m so that $F \subseteq H_m$. Then $B(H_n, F) = \{H_n\}$ for each $n > m$. □

Corollary 12.2.2. *$\mathcal{L}(\mathbb{Z})$ and $\mathcal{L}(\mathbb{Z}_{p^\infty})$ are asymorphic, for every prime p.*

Proof. By Proposition 12.2.1(b), both $\mathcal{L}(\mathbb{Z})$ and $\mathcal{L}(\mathbb{Z}_{p^\infty})$ have two connected components, namely,

$$\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z}),\ \mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\{0\}) = \{0\},\ \mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}_{p^\infty}\},\ \text{ and }\ \mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\}).$$

Moreover, $|\mathcal{Q}_{L(\mathbb{Z})}(\mathbb{Z})| = |\mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\})| = \omega$. Since $\mathcal{L}(\mathbb{Z})$ and $\mathcal{L}(\mathbb{Z}_{p^\infty})$ are thin, in particular, also $\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\})$ are thin. Hence, Proposition 6.2.1 implies that $\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\})$ coincide with the ideal balleans associated to the ideals of all their bounded subsets, i.e., finite subsets, namely

$$\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z}) = \mathfrak{B}_{\mathcal{I}} \quad\text{and}\quad \mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\}) = \mathfrak{B}_{\mathcal{J}}, \tag{12.1}$$

where $\mathcal{I} = [\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z})]^{<\omega}$ and $\mathcal{J} = [\mathcal{Q}_{\mathcal{L}(\mathbb{Z}_{p^\infty})}(\{0\})]^{<\omega}$.

Fix a bijecton $\varphi\colon \mathcal{L}(\mathbb{Z}) \to \mathcal{L}(\mathbb{Z}_{p^\infty})$ such that $\varphi(\{0\}) = \mathbb{Z}_{p^\infty}$. We claim that φ is an asymorphism. We can apply Remark 4.3.6(c) and the claim follows once we prove that both $\varphi|_{\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\{0\})}$ and $\varphi|_{\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z})}$ are asymorphisms. While the first restriction is trivially an asymorphism, Fact 6.2.2 and (12.1) imply that also the second one is an asymorphism. □

In contrast to $\mathcal{L}(\mathbb{Z})$, for $n > 1$ $\mathcal{L}(\mathbb{Z}^n)$ is not weakly thin, and, in particular, it is not thin. To see that $\mathcal{L}(\mathbb{Z}^n)$, $n > 1$, has a non-thin connected component, we put $F = \{(1, 0, \dots, 0), (0, \dots, 0)\}$ and note that $2\mathbb{Z} \times S \in B(\mathbb{Z} \times S, F)$ for each subgroup S of $\mathbb{Z}^{n-1}$.

Question 12.2.3. Is $\mathcal{L}(\mathbb{Z}^n)$ cellular for every $n \in \mathbb{N}$?

For every $n \in \mathbb{N}$, denote by $\mathcal{R}(\mathbb{Z}^n)$ the subballean of $\mathcal{L}(\mathbb{Z}^n)$ whose support is the family $R(\mathbb{Z}^n)$ of rectangular subgroups of $\mathbb{Z}^n$, i.e., $R(\mathbb{Z}^n) = \{k_1\mathbb{Z} \times \cdots \times k_n\mathbb{Z} \mid k_1, \dots, k_n \in \mathbb{Z}\}$. Then, for every $n \in \mathbb{N}$, $\mathcal{R}(\mathbb{Z}^n)$ is cellular. In fact, it is trivial that $\mathcal{R}(\mathbb{Z}^n) \approx \Pi_{i=1}^n \mathcal{L}(\mathbb{Z})$ and products of cellular balleans are cellular.

For every locally finite group G, the ballean $\mathcal{B}_G$ is cellular, equivalently, $\operatorname{asdim} \mathfrak{B}_G = 0$, so $\exp \mathfrak{B}_G$ and $\mathcal{L}(G)$ are cellular (Proposition 11.2.7).

Question 12.2.4. Is the ballean $\mathcal{L}(G)$ cellular for an arbitrary group G?

Theorem 12.2.5. *Let $n \in \mathbb{N}$. Then $\mathcal{L}(\mathbb{Z}^2) \approx \mathcal{L}(\mathbb{Z}^n)$ if and only if $n = 2$.*

Proof. We have already proved that $\mathcal{L}(\mathbb{Z}) \not\approx \mathcal{L}(\mathbb{Z}^2)$.

Now suppose that $n \geq 3$. Fix, by contradiction, an asymorphism $\varphi\colon \mathcal{L}(\mathbb{Z}^2) \to \mathcal{L}(\mathbb{Z}^n)$. As recalled in Remark 4.3.6(c), φ induces asymorphisms between the connected components of those two balleans. Because of Remark 12.1.3, one of those restrictions is an asymorphism between $\mathcal{Q}_{\mathcal{L}(\mathbb{Z})}(\mathbb{Z})$ and $\mathcal{Q}_{\mathcal{L}(\mathbb{Z}^2)}(\mathbb{Z}^2)$. However, this is an absurd, since the first ballean is thin, while the second one has not that property. □

Question 12.2.6. Is it true that $\mathcal{L}(\mathbb{Z}^n) \approx \mathcal{L}(\mathbb{Z}^m)$ if and only if $n = m$?

Remark 12.2.7. Let G be an arbitrary group. According to Fact 11.3.1, all balls in $\mathcal{L}(G)$ centered at $\{e_G\}$ are finite. Nevertheless, this is not true for all balls of $\mathcal{L}(G)$. One can find examples of abelian groups G such that some balls in $\mathcal{L}(G)$ centred at G are infinite. For example, let $G = \Pi_{n\in\mathbb{N}} G_n$, where $G_n \simeq \mathbb{Z}/2\mathbb{Z}$, for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, denote by a_n the element of G such that $p_n(a_n) = 1$ and, for every $i \neq n$, $p_i(a_n) = 0$. Then, for every $n \in \mathbb{N}$, $\langle \{a_i \mid i \in \mathbb{N} \setminus \{1, n\}\} \cup \{a_n + a_1\}\rangle \in B_{\mathcal{L}(G)}(G, \langle a_1\rangle)$ and thus this ball contains infinitely many elements.

12.2.2 The subgroup logarithmic hyperballean ℓ-$\mathcal{L}(G)$

Proposition 12.2.8. *For every prime p, ℓ-$\mathcal{L}(\mathbb{Z}_{p^\infty})$ is asymorphic to the coproduct of $\mathbb{N}$ and a singleton.*

Proof. It is easy to check that the subspace S of all finite subgroups of $\mathbb{Z}_{p^\infty}$ is isometric to $(\log p)\mathbb{N}$ with the metric $d(x, y) = |x - y|$, for every $x, y \in (\log p)\mathbb{N}$. □

Corollary 12.2.9. *For every pair of prime numbers p, q,* $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) \approx \operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{q^\infty})$. *Moreover,*

$$\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) = 1.$$

Theorem 12.2.10. $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}) = \infty$.

Proof. For distinct primes $p_1, \ldots, p_n$, we put $S_n = \{p_1^{m_1} \cdots p_n^{m_n}\mathbb{Z} \mid m_i \in \mathbb{N}, \forall i = 1, \ldots, n\}$, and let $\psi(p_1^{m_1} \cdots p_n^{m_n}\mathbb{Z}) = (m_1, \ldots, m_n) \in \mathbb{N}^n$. In the sequel we denote the n-tuples

$$(m_1, \ldots, m_n), (m_1', \ldots, m_n') \in \mathbb{N}^n$$

by $\bar{m}, \bar{m}'$, and the n-tuple $(\max\{m_1, m_1'\}, \ldots, \max\{m_n, m_n'\})$ by $\max\{\bar{m}, \bar{m}'\}$.

Equip $\mathbb{N}^n$ with the taxi driver metric d_T, defined by

$$d_T(\bar{m}, \bar{m}') = \Sigma_i |m_i - m_i'|,$$

for every pair $(\bar{m}, \bar{m}') \in \mathbb{N}^n$.

We prove below that $\psi\colon S_n \to \mathbb{N}^n$ is an asymorphism. Since $\operatorname{asdim} \mathbb{N}^n = n$, this asymorphism will provide subballeans of arbitrary finite asymptotic dimension of ℓ-$\mathcal{L}(\mathbb{Z})$, hence $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}) = \infty$.

Consider now a second metric on $\mathbb{N}^n$, namely the logarithmic metric $d_{\log}$, induced from S_n through the bijection ψ. More precisely,

$$d_{\log}(\bar{m}, \bar{m}') = d_{\ell\text{-}\mathcal{L}(\mathbb{Z})}(\psi^{-1}(\bar{m}), \psi^{-1}(\bar{m}')),$$

for any pair $\bar{m}, \bar{m}' \in \mathbb{N}^n$. We can assume without loss of generality that $p_1, \dots, p_n$ are greater or equal than the base of the logarithm. We claim that those two metrics induce the same ballean structures on $\mathbb{N}^n$.

First of all, we want to prove that

$$d_{\log}(\bar{m}, \bar{m}') = \begin{cases} \sum_{i=1}^n \log(p_i) \max\{m_i - m_i', 0\} & \text{if } p_1^{m_1} \cdots p_n^{m_n} \geq p_1^{m_1'} \cdots p_n^{m_n'}, \\ \sum_{i=1}^n \log(p_i) \max\{m_i' - m_i, 0\} & \text{otherwise.} \end{cases} \tag{12.2}$$

For $\bar{m} \in \mathbb{N}^n$, let $A_{\bar{m}} = p_1^{m_1} \cdots p_n^{m_n}\mathbb{Z} = \psi^{-1}(\bar{m})$. Then, for $\bar{m}, \bar{m}' \in \mathbb{N}^n$, $A_{\bar{m}} \cap A_{\bar{m}'} = A_{\max\{\bar{m}, \bar{m}'\}}$, hence

$$\begin{aligned} \max\{|A_{\bar{m}} : (A_{\bar{m}} \cap A_{\bar{m}'})|, |A_{\bar{m}'} : (A_{\bar{m}} \cap A_{\bar{m}'})|\} &= |A_{\bar{s}} : A_{\max\{\bar{m}, \bar{m}'\}}| = \\ &= \prod_{i=1}^n p_i^{\max\{m_i, m_i'\} - s_i} = \\ &= \prod_{i=1}^n p_i^{\max\{m_i - s_i, m_i' - s_i\}}, \end{aligned} \tag{12.3}$$

where

$$\bar{s} = \begin{cases} \bar{m}' & \text{if } p_1^{m_1} \cdots p_n^{m_n} \geq p_1^{m_1'} \cdots p_n^{m_n'}, \\ \bar{m} & \text{otherwise.} \end{cases} \tag{12.4}$$

Hence, (12.2) can be obtained by combining (12.3) and (12.4).

We are left with the proof of $\mathfrak{B}_{d_T} = \mathfrak{B}_{d_{\log}}$. Fix $R \geq 0$ and consider a pair $\bar{m}, \bar{m}' \in \mathbb{N}^n$ with $d_T(\bar{m}, \bar{m}') \leq R$. Let $K = \max\{\log(p_i) \mid i = 1, \dots, n\}$. By our assumption, $K \geq 1$. Then

$$d_{\log}(\bar{m}, \bar{m}') \leq \sum_{i=1}^n \log(p_i)|m_i - m_i'| \leq RK,$$

witnessing that $\mathfrak{B}_{d_T} \prec \mathfrak{B}_{d_{\log}}$.

Conversely, let $S \geq 0$ and fix a pair $\bar{m}, \bar{m}' \in \mathbb{N}^n$ with $d_{\log}(\bar{m}, \bar{m}') \leq S$. Split $I = \{1, \dots, n\} = I_+ \cup I_-$, with $I_+ = \{i \in I \mid m_i \geq m_i'\}$ and $I_- = \{i \in I \mid m_i < m_i'\}$ and consider two cases, according to (12.2).

Suppose first that $\Pi_{i \in I} p_i^{m_i} \geq \Pi_{i \in I} p_i^{m_i'}$. Then $I_+ \neq \emptyset$ (if $I_- = \emptyset$ we set $\prod_{i \in I_-} p_i^{m_i} = 1$ below). Hence

$$\Big(\prod_{i \in I_+} p_i^{m_i}\Big)\Big(\prod_{i \in I_-} p_i^{m_i}\Big) \geq \Big(\prod_{i \in I_+} p_i^{m_i'}\Big)\Big(\prod_{i \in I_-} p_i^{m_i'}\Big),$$

equivalently,

$$\prod_{i \in I_+} p_i^{m_i - m_i'} \geq \prod_{i \in I_-} p_i^{m_i' - m_i}. \tag{12.5}$$

In particular, (12.5) implies $p_j^{m'_j - m_j} \leq \Pi_{i\in I_+} p_i^{m_i - m'_i}$, for every $j \in I_-$, so

$$\begin{aligned} |m_j - m'_j| &= m'_j - m_j = \log_{p_j}(p_j^{m'_j - m_j}) \leq \sum_{i\in I_+}(m_i - m'_i)\log_{p_j} p_i \leq \\ &\leq T\sum_{i\in I}(\log p_i)\max\{m_i - m'_i, 0\} \leq TS, \end{aligned} \tag{12.6}$$

where $T = \max\{\log_{p_i}(p_j) \mid i, j \in I\} \geq 1$. Since, for every $k \in I_+$,

$$\begin{aligned} |m_k - m'_k| &= m_k - m'_k \leq (m_k - m'_k)\log p_k \leq \\ &\leq \sum_{i\in I}(\log p_i)\max\{m_i - m'_i, 0\} \leq S, \end{aligned}$$

the inequalities (12.6) imply that $d_T(\bar{m}, \bar{m}') \leq nTS$.

The remaining case $\Pi_{i\in I} p_i^{m_i} < \Pi_{i\in I} p_i^{m'_i}$ is similar. Hence, $\mathfrak{B}_{d_{\log}} \prec \mathfrak{B}_{d_T}$. This proves the equality $\mathfrak{B}_{d_T} = \mathfrak{B}_{d_{\log}}$. □

In particular, Corollary 12.2.9 and Theorem 12.2.10 imply that ℓ-$\mathcal{L}(\mathbb{Z}_{p^\infty})$ is not even coarsely equivalent to ℓ-$\mathcal{L}(\mathbb{Z})$. Note the difference with Corollary 12.2.2.

As we have already noticed, for $n > 1$, ℓ-$\mathcal{L}(\mathbb{Z}^n)$ and ℓ-$\mathcal{L}(\mathbb{Z})$ are not asymorphic because ℓ-$\mathcal{L}(\mathbb{Z})$ has two connected components but ℓ-$\mathcal{L}(\mathbb{Z}^n)$ infinitely (countably) many. It will be nice to answer the following less obvious question:

Question 12.2.11. Are ℓ-$\mathcal{L}(\mathbb{Z}^n)$ and ℓ-$\mathcal{L}(\mathbb{Z}^m)$ asymorphic for all distinct $n, m > 1$?

In order to characterise the abelian groups G with $\operatorname{asdim} \ell$-$\mathcal{L}(G) < \infty$ we need to rule out the groups that are not finitely layered. For a group G and $n \in \mathbb{N}$ let

$$X_n = \{x \in G \mid o(x) = n\} = \bigcup_{d|n} X_d.$$

Note that, even if G is abelian, then X_n is not necessarily a subgroup of G (unlike $G[n]$). Call G *layerly finite*, if the set X_n is finite for every n (or equivalently, when $G[n]$ is finite for each n).

Theorem 12.2.12. *Let G be an abelian group, and p be a prime number. If the subgroup $G[p]$ is infinite then* $\operatorname{asdim} \ell$-$\mathcal{L}(G) = \infty$.

Proof. We take a subgroup $H = \bigoplus_\omega H_n$ of G which is a direct sum of ω copies H_n of $\mathbb{Z}_p$, denote by $\mathcal{S}$ the set of all subgroups of H of the from $H_F = \bigoplus_{n\in F} H_n$, where F is a finite subset of ω, $\bigoplus_{n\in\emptyset} H_n = \{0\}$. Then the correspondence $H_F \mapsto F$ defines an asymorphism between $\mathcal{S}$ and the Hamming space $\mathbb{H}_\omega$. According to Proposition 6.1.3, $\operatorname{asdim} \mathbb{H}_\omega = \infty$. Therefore, $\operatorname{asdim} \mathcal{S} = \infty$. This yields $\operatorname{asdim} \ell$-$\mathcal{L}(G) = \infty$. □

Corollary 12.2.13. *Let G be an abelian group with* $\operatorname{asdim} \ell$-$\mathcal{L}(G) < \infty$. *Then G is torsion and layerly finite.*

Proof. By $\operatorname{asdim} \ell$-$\mathcal{L}(G) < \infty$ and by Theorem 12.2.10, G is a torsion group. By Theorem 12.2.12, G is layerly finite. □

We can characterise now the abelian groups G such that $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = 0$ as the reduced torsion finitely layered abelian groups. For a prime p we denote by S_p the *Sylow p-subgroup* of G, i.e., is the maximal p-subgroup of G.

Theorem 12.2.14. *For an abelian group G, $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = 0$ if and only if G is a torsion group and for every prime p the Sylow p-subgroup S_p of G is finite.*

Proof. By Theorem 12.2.10 and Corollary 12.2.13, G is a torsion layerly finite group. If some S_p is infinite then S_p has a subgroup isomorphic to $\mathbb{Z}_{p^\infty}$ but $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) = 1$.

We assume that each S_p is finite and show that ℓ-$\mathcal{L}(G)$ is cellular. Let $G = \bigoplus_{p\in\pi(G)} S_p$. We take an arbitrary $n \in \mathbb{N}$ and put $G_n = \bigoplus\{S_p \mid p \in \pi(G), \log p > n\}$. If $A, B \in L(G)$ and $d_{\ell\text{-}\mathcal{L}(G)}(A,B) \leq n$ then $A \cap G_n = B \cap G_n$. It follows that $B^{\square}(A,n) \subseteq B(A,m)$, where $m = \sum\{\log|S_p| \mid p \in \pi(G), \log\ p \leq n\}$. □

Now we show that for every $n \in \mathbb{N}$ one can easily build a (divisible) abelian group G with
$$\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = n.$$

Example 12.2.15. (a) For distinct primes $p_1, \dots, p_n$ consider the group $G = \mathbb{Z}_{p_1^\infty} \oplus \dots \oplus \mathbb{Z}_{p_n^\infty}$. Then

$$\ell\text{-}\mathcal{L}(G) \approx \prod_{i=1}^{n} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) = \coprod_{i=0}^{n} \Bigg(\coprod_{j=1}^{\binom{n}{i}} \mathbb{N}^{n-i} \Bigg),$$

so in particular $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = n$. For a proof one has to use the fact that the lattice $L(G)$ is isomorphic to the direct product of the lattices $L(\mathbb{Z}_{p_1^\infty}) \times \dots \times L(\mathbb{Z}_{p_n^\infty})$ since every subgroup H of G has the form

$$H = \bigoplus_{i=1}^{n} H_{p_i}, \text{ where } H_{p_i} \text{ is a subgroup of } \mathbb{Z}_{p_i^\infty}.$$

(b) More generally, for a set π of primes let $G_\pi = \bigoplus_{p\in\pi} \mathbb{Z}_{p^\infty}$. Then $\operatorname{asdim} G_\pi = |\pi|$. Indeed, for finite π this follows from (a). Otherwise, consider subsets $\pi_n \subseteq \pi$ with $|\pi_n| = n$ and apply again (a) to the subgroup G_{π_n} to deduce $\operatorname{asdim} G_{\pi_n} = n$ and conclude $\operatorname{asdim} G_\pi = \infty$.

Remark 12.2.16. Let G be an abelian group, $n \in \mathbb{N}$. If $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = n$ then G is torsion and there exist distinct primes $p_1, \dots, p_m$, $m \leq n$, a layerly finite subgroup G_1 of G which is a direct sum of cyclic subgroups, such that

$$G \simeq \mathbb{Z}_{p_1^\infty} \oplus \dots \oplus \mathbb{Z}_{p_m^\infty} \oplus G_1. \tag{12.7}$$

Indeed, by Corollary 12.2.13, G is torsion and finitely layered. Hence, its maximal divisible subgroup $d(G) = \mathbb{Z}_{p_1^\infty} \oplus \dots \oplus \mathbb{Z}_{p_m^\infty}$ has $r(G) = m \leq n$. So G splits as in (12.7). Furthermore, letting $G_2 = \mathbb{Z}_{p_1^\infty} \oplus \dots \oplus \mathbb{Z}_{p_m^\infty}$, one may have $\pi(G_1) \cap \pi(G_2) \neq \emptyset$, but it is possible to split $G_1 = G_1^* \oplus F$, where F is a finite group, such that, with $G_2^* = G_2 \oplus F$, one has

$$G = G_1^* \oplus G_2^*\ , \quad G_2^* = \mathbb{Z}_{p_1^\infty} \oplus \dots \oplus \mathbb{Z}_{p_m^\infty} \oplus F \quad \text{and} \quad \pi(G_1^*) \cap \pi(G_2^*) = \emptyset.$$

More precise results depend on the following:

Problem 12.2.17. Compute $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty})$.

Note that ℓ-$\mathcal{L}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty})$ contains, as a subspace, the family $\mathcal{S}$ of all proper subgroups of the form $H = H_1 \oplus H_2$, where H_i is a subgroup of $\mathbb{Z}_{p^\infty}$ for $i = 1, 2$. Since $\mathcal{S} \approx \mathbb{N}^2$, $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}) \geq \operatorname{asdim} \mathbb{N}^2 = 2$.

We do not state in Remark 12.2.16 that the converse implication is true. More precisely, if G is as in Remark 12.2.16 with primes $p_1, \dots, p_m$ not necessarily distinct, we cannot claim that $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = n$ is finite with $n \geq m$. In case $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}) = \infty$ occurs for all primes p (see Remark 12.2.19), we can claim that $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = n$ entails that the primes $p_1, \dots, p_m$ are pairwise distinct (and so, $m = n$).

Proposition 12.2.18. *Let G be a p-group. Consider the subballean of ℓ-$\mathcal{L}(G)$ whose support is the family $C(G)$ of all cyclic subgroups of G. Then* $\operatorname{asdim} C(G) \leq 1$. *Moreover,* $\operatorname{asdim} C(G) = 0$ *if and only if G has finite exponent.*

Proof. We claim that $C(G)$ is asymorphic to a tree, which implies $\operatorname{asdim} C(G) \leq 1$ (see [157, Proosition 9.8]). Define a graph T having $C(G)$ as set of vertices and, for $X, Y \in C(G)$, the pair $\{X, Y\}$ is an edge if and only if $X \leq Y$ and $|Y : X| = p$, or $Y \leq X$ and $|X : Y| = p$. Then T is trivially asymorphic to $C(G)$. Obviously, $(T, \leq)$ is also a partially ordered set, where the order is defined by the inclusion of subgroups. We want to show that T is actually a tree. Consider $X, Y, Z \in C(G)$ such that $Y, Z \leq X$. Let $X = \langle x \rangle$. Since $Y, Z \in C(G)$, $Y = \langle x^{p^y} \rangle$, and $Z = \langle x^{p^z} \rangle$, for some $y, z \in \mathbb{N}$. If $z \leq y$, then $Y \leq Z$ since $x^{p^y} = (x^{p^z})^{p^{y-z}}$. Similarly, if $y \leq z$, then $Z \leq Y$. Since the set T_X of vertices below this fixed vertex X is finite, hence it is well-ordered. This shows that the partially ordered set $(T, \leq)$ is a tree with root the trivial subgroup of G and height equal to the (logarithm of the) exponent of G, hence at most ω.

Finally, $\operatorname{asdim} C(G) = 0$ if and only if T is bounded and this is equivalent to G having finite exponent. □

For $G = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ we proved $\operatorname{asdim} C(G) = 1$ in Proposition 12.2.18. However, $\operatorname{asdim} \mathcal{Q}_{\ell\text{-}\mathcal{L}(G)}(\{0\}) \geq 2$.

Let us conclude our discussion about Remark 12.2.16 with a final remark towards an answer to the question whether $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty} \bigoplus \mathbb{Z}_{p^\infty}) = \infty$.

Remark 12.2.19. Unlike the group $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{q^\infty}$, with primes $p \neq q$, the group $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ has $\mathfrak{c}$ many subgroups, actually $\mathfrak{c}$ many divisible subgroups isomorphic to $\mathbb{Z}_{p^\infty}$.

One can easily see that $G = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ has three types of subgroups:

(a) finite subgroups;
(b) infinite proper divisible subgroups, they are all isomorphic to $\mathbb{Z}_{p^\infty}$;
(c) infinite proper non-divisible subgroups, they are all isomorphic to $\mathbb{Z}(p^n) \oplus \mathbb{Z}_{p^\infty}$.

There are countably many finite subgroups and $\mathfrak{c}$ many subgroups of each type (b) and (c).
We conjecture that $\operatorname{asdim} \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}) = \infty$.

Example 12.2.20. Let T be a Tarskii monster of exponent p, where p is a suitable prime. Since every proper subgroup is finite, then $L(T) = \{T\} \sqcup \mathcal{Q}_{L(T)}(\{e\})$, where $L(T)$ can be endowed both with the subgroup exponential hyperballean structure and with the subgroup logarithmic hyperballean structure.

(a) First of all, we focus on the subgroup exponential ballean $\mathcal{L}(T)$. Fact 11.3.1 implies that every ball centred in a proper subgroup of T is finite. We are not aware whether $\mathcal{L}(T)$ is thin. Since $\mathcal{L}(T) = \{T\} \sqcup \mathcal{Q}_{L(T)}(\{e\})$, if $\mathcal{L}(T)$ is thin, then $\mathcal{Q}_{L(T)}(\{e\}) = \mathcal{B}_{\mathcal{I}}$, where $\mathcal{I} = [\mathcal{Q}_{L(T)}(\{e\})]^{<\omega}$.

(b) We now consider $\ell\text{-}\mathcal{L}(T)$. The definition of T implies that the ball centred at the identity of radius $\log p$ contains all proper subgroups of the group, which are infinitely many. Hence, $\ell\text{-}\mathcal{L}(T) = \{T\} \sqcup V$, where V is the family of all proper subgroups of T, and V is bounded. In particular, $\ell\text{-}\mathcal{L}(T)$ is thin and 0-dimensional.

Remark 12.2.21. For every group G, there is a natural map $i\colon G \to L(G)$ that sends every element $g \in G$ in the subgroup $\langle g\rangle$. (One may consider also the co-restriction $G \to C(G)$ of i, where $C(G)$ is the family of all cyclic subgroups of G.) The cardinalities of its fibres have a uniform bound (i.e., i has uniformly bounded fibres) if and only if there is an upper bound for the size of of all finite cyclic subgroups of G (e.g., the groups of finite exponent as well as torsion-free groups have this property). Hence, one might think that i could be a coarse embedding if $L(G)$ is endowed with a subgroup hyperballean structure. For example, if G is finite, then i is trivially a coarse equivalence. However, if G is infinite this may fail even in simple cases. For $G = \mathbb{Z}$ the map $i\colon G \to L(G)$ is surjective, with $\operatorname{asdim} G = 1$, yet $\operatorname{asdim} \mathcal{L}(G) = 0$ and $\operatorname{asdim} \ell\text{-}\mathcal{L}(G) = \infty$; hence i is not a coarse equivalence in both cases.

If G is infinite and has finite exponent n, then $i\colon G \to \ell\text{-}\mathcal{L}(G)$ is not proper. In fact, every cyclic subgroup belongs to the ball in $\ell\text{-}\mathcal{L}(G)$ centred in $\{e\}$ with radius $\log n$ (see also Example 12.2.20) and those subgroups are infinitely many. Hence, $i^{-1}(B_{\ell\text{-}\mathcal{L}}(\{e\}, \log n)$ is unbounded in $\mathfrak{B}_G$, i.e., infinite.

12.3 Rigidity results

As we have already mentioned (see comments on Remark 12.1.9), if two groups G and H are isomorphic, then $\mathcal{L}(G) \approx \mathcal{L}(H)$ and $\ell\text{-}\mathcal{L}(G) \approx \ell\text{-}\mathcal{L}(H)$. However, the converse is not true in general (for example, $\mathcal{L}(\mathbb{Z}) \approx \mathcal{L}(\mathbb{Z}_{p^\infty}) \approx \mathcal{L}(\mathbb{Z}_{q^\infty})$ and $\ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) \approx \ell\text{-}\mathcal{L}(\mathbb{Z}_{q^\infty})$). In this section we want to determine conditions that ensures that the opposite implication holds.

Let us start with some technical results which hold for the subgroup hyperballeans $\mathcal{L}(G)$ and $\ell\text{-}\mathcal{L}(G)$.

Lemma 12.3.1. *Let X be a ballean.*

(a) If X is asymorphic to $\mathcal{L}(\mathbb{Z})$ or to $\ell\text{-}\mathcal{L}(\mathbb{Z})$, then X has two connected components. Moreover, one connected component is a singleton, while the other one is infinite and unbounded.

(b) If X is coarsely equivalent to $\mathcal{L}(\mathbb{Z})$ or to $\ell\text{-}\mathcal{L}(\mathbb{Z})$, then X has two connected components. Moreover, one connected component is bounded, while the other one is unbounded.

Proof. The proof is an application of Remarks 4.3.6(d) and 12.1.3, and Proposition 12.2.1(b). □

An infinite group is said to be *quasi-finite* if every proper subgroup is finite. Example of quasi-finite groups are the Prüfer p-groups and the Tarskii monsters (see Example 12.2.20). Moreover, if an abelian group is quasi-finite, then it is isomorphic to Prüfer p-group for some prime p.

Proposition 12.3.2. *Let G be a group. Suppose that $\mathcal{L}(G)$ (ℓ-$\mathcal{L}(G)$, equivalently) has precisely two connected components, one of them is a singleton and the other one is infinite. Then G must be infinite. Moreover:*
(a) if G contains an element of infinite order, then $G \simeq \mathbb{Z}$;
(b) if G is a torsion group, then G is quasi-finite.

Proof. The first statement is trivial, since, otherwise, $\mathcal{L}(G)$ and ℓ-$\mathcal{L}(G)$ would be bounded.

(a) Let g be element of infinite order of G. Then $\langle g\rangle \in L(G)$ is infinite, $\langle g\rangle \in \mathcal{Q}_{L(G)}(G)$ and thus $\mathcal{Q}_{L(G)}(G)$ is infinite (as it contains the subgroups of the form $\langle g^k\rangle$, where $k \in \mathbb{N}$), while $\mathcal{Q}_{L(G)}(\{e_G\}) = \{e_G\}$. Since each infinite subgroup of G is, in particular, large in G, it has finite index and, by Fedorov's theorem ([77]), $G \simeq \mathbb{Z}$.

(b) Since G is torsion, for every $g \in G$, $\langle g\rangle$ is a finite subgroup and thus belongs to the connected component $\mathcal{Q}_{L(G)}(e_G)$. Hence, the connected component of G is a singleton and every proper subgroup is finite. □

12.3.1 Rigidity results on the subgroup exponential hyperballean $\mathcal{L}(G)$

Corollary 12.3.3. *If a group G contains an element of infinite order, then $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z})$ if and only if $G \simeq \mathbb{Z}$.*

Proof. Lemma 12.3.1(a) implies that $\mathcal{L}(G)$ has two connected components, one is infinite and the other one is just a singleton. Hence the conclusion follows from 12.3.2(a). □

Theorem 12.3.4. *For an abelian group G, $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Z})$ if and only if either $G \simeq \mathbb{Z}$ or $G \simeq \mathbb{Z}_{p^\infty}$, for some p is prime.*

Proof. The 'if part' of the statement is proved in Corollary 12.2.2.

Conversely, let us divide the proof in two cases. If G is torsion, then Lemmas 12.3.1(a) and 12.3.2(b) imply that every proper subgroup of G is finite. Hence, since G is abelian, $G \simeq \mathbb{Z}_{p^\infty}$, for some prime p. Otherwise, there exists and element $g \in G$ of infinite order and then the claim follows from Corollary 12.3.3. □

Can we relax the hypothesis of Theorem 12.3.4? Namely, we wonder whether the request of G being abelian can be relaxed or not. Let us state it as a question.

Question 12.3.5. Let G be a torsion group such that $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are asymorphic. Is $G \simeq \mathbb{Z}_{p^\infty}$ for some prime p?

An affirmative answer to the question of Example 12.2.20, along with a proof similar to that of Corollary 12.2.2, would show that $\ell\text{-}\mathcal{L}(T) \approx \ell\text{-}\mathcal{L}(\mathbb{Z})$, for a Tarskii monster T. This would provide a negative answer to Question 12.3.5.

12.3.2 Rigidity results on the subgroup logarithmic hyperballean $\ell\text{-}\mathcal{L}(G)$

Theorem 12.3.6. *Let G be a group and p be a prime.*
(a) $\ell\text{-}\mathcal{L}(G) \approx \ell\text{-}\mathcal{L}(\mathbb{Z})$ if and only if $G \simeq \mathbb{Z}$;
(b) $\ell\text{-}\mathcal{L}(G) \approx \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty})$ if and only if $G \simeq \mathbb{Z}_{q^\infty}$ for some prime q.

Proof. (a) Assume that $\ell\text{-}\mathcal{L}(G)$ is asymorphic to $\ell\text{-}\mathcal{L}(\mathbb{Z})$. If G has an element of infinite order then $G \simeq \mathbb{Z}$, by Lemma 12.3.1(a) and Proposition 12.3.2(a). Suppose now, by contradiction, that G is a torsion group. By Proposition 12.3.2(b), G is quasi-finite. We show that G is layerly finite. If A, B are subgroup of order n then $A \subseteq AB$, $B \subseteq BA$ so $d_{\ell\text{-}\mathcal{L}(G)}(A, B) \leq \log n$. If some X_n is infinite then $\ell\text{-}\mathcal{L}(G)$ has an infinite ball of radius $\log n$, but each ball in $\ell\text{-}\mathcal{L}(\mathbb{Z})$ is finite. By [29], G either has a subgroup $H, H \simeq \mathbb{Z}_{p^\infty}$ or G is the subdirect product of finite groups. Since this implies the existence of proper (normal) subgroups of finite index and G is quasi-finite, the second case is impossible. So we are left with $H \simeq \mathbb{Z}_{p^\infty}$. Since H is infinite and G is quasi-finite, $H \simeq G$. This contradicts the conjunction of Corollary 12.2.9 & Theorem 12.2.10.

(b) Corollary 12.2.9 implies that $\ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) \approx \ell\text{-}\mathcal{L}(\mathbb{Z}_{q^\infty})$ for every pair of primes p and q. Conversely, suppose that $\ell\text{-}\mathcal{L}(G) \approx \ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty})$. If, by contradiction, G contains an element of infinite order, then $G \simeq \mathbb{Z}$, by Proposition 12.3.2(a). This contradicts $\ell\text{-}\mathcal{L}(\mathbb{Z}_{p^\infty}) \not\approx \ell\text{-}\mathcal{L}(\mathbb{Z})$ established in Corollary 12.2.9 and Theorem 12.2.10. Hence G is torsion. Using Proposition 12.3.2(b) as above, we conclude that G is quasi-finite and layerly finite, and consequently, $H \simeq \mathbb{Z}_{q^\infty}$ for some prime q. □

Note that in Theorem 12.3.6 we do not require that the group is abelian.

12.3.3 Rigidity results and questions on divisible and finitely generated abelian groups

We pointed out in §12.1.1 that divisibility of a group is related to some strong property of its hyperballean. So it is natural to ask if we can find some rigidity result in this setting.

Lemma 12.3.7. *For no cardinal κ, $\ell\text{-}\mathcal{L}(\mathbb{Q}^\kappa)$ has a connected component asymorphic to $\mathbb{N}$.*

Proof. Let H be an arbitrary subgroup of $\mathbb{Q}^\kappa$ and suppose that H is not divisible since, otherwise, $\mathcal{Q}_{L(\mathbb{Q}^\kappa)}(H) = \{H\} \not\approx \mathbb{N}$. Since H is not divisible, there exists $n \in \mathbb{N}$ such that $nH \lneq H$. Note that, this is equivalent to $H \lneq (1/n)H$. Hence, in particular, we can construct a chain of subgroups as follows:

$$\cdots \lneq n^k H \lneq \cdots \lneq n^2 H \lneq nH \lneq H \lneq \frac{1}{n} H \lneq \frac{1}{n^2} H \lneq \cdots \lneq \frac{1}{n^k} H \lneq \cdots .$$

Note that this chain is asymorphic to $\mathbb{Z}$, which is not asymorphic to $\mathbb{N}$ and this observation concludes the proof. $\square$

Proposition 12.3.8. *Let D and D' be two divisible abelian groups. Then D is torsion-free if and only if D' is torsion-free, provided that ℓ-$\mathcal{L}(D) \approx \ell$-$\mathcal{L}(D')$.*

Proof. Suppose that D is torsion-free. Let D' have torsion. Then, by Theorem 1.2.3,

$$D' \simeq \mathbb{Q}^{r_0(D')} \oplus \mathbb{Z}_{p^\infty} \oplus H,$$

where p is a prime and $H \leq t(D')$. Define $K = \mathbb{Q}^{r_0(D')} \oplus \{0\} \oplus H$. Then $\mathcal{Q}_{\ell\text{-}\mathcal{L}(D')}(K) \approx \mathbb{N}$. As D is torsion-free, $D \simeq \mathbb{Q}^{r_0(D)}$ and there is no connected component asymorphic to $\mathbb{N}$, by Lemma 12.3.7. $\square$

Moreover, we can prove a stronger version of Remark 12.1.9(c) and (d).

Corollary 12.3.9. *Let G be a divisible group.*

(a) Then ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Q})$ if and only if $G \simeq \mathbb{Q}$.

(b) Suppose that κ is an infinite cardinal. Then, under the assumption of the Generalised Continuum Hypothesis, ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Q}^\kappa)$ if and only if $G \simeq \mathbb{Q}^\kappa$.

Proof. Proposition 12.3.8 implies that G is torsion-free and thus we can apply Remark 12.1.8 to prove both claims. $\square$

Question 12.3.10. Let G be an abelian group and D be a divisible abelian group. Is it true that G is divisible, provided that ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(D)$?

As an evidence for a possible positive answer to this question, consider G and D as in Question 12.3.10. Then $G \simeq d(G) \oplus H$ for some subgroup H of G. By Corollary 12.1.7, $\mathcal{I}soL(D) \neq \emptyset$. This, along with Corollary 12.1.6, implies $\mathrm{Tor}(H) = \{0\}$. A full positive answer would simply give $H = \{0\}$.

Question 12.3.11. Let G be an abelian group such that $t(d(G)) = \{0\}$. Is it true that $G \simeq \mathbb{Q}$, provided that either $\mathcal{L}(G) \approx \mathcal{L}(\mathbb{Q})$ or ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(\mathbb{Q})$?

Question 12.3.12. Is it true that $\mathcal{L}(\mathbb{Q}) \approx \mathcal{L}(\mathbb{Q} \oplus \mathbb{Z}_{p^\infty})$?

Question 12.3.13. Is it true that $\mathcal{L}(\mathbb{Q} \oplus \mathbb{Z}) \approx \mathcal{L}(\mathbb{Q})$ or ℓ-$\mathcal{L}(\mathbb{Q} \oplus \mathbb{Z}) \approx \ell$-$\mathcal{L}(\mathbb{Q})$?

Question 12.3.14. Let G be an abelian group and H be a finitely generated abelian group. Suppose that ℓ-$\mathcal{L}(G) \approx \ell$-$\mathcal{L}(H)$. Is it true that G is finitely generated?

Note that $\mathcal{L}(\mathbb{Z}_{p^\infty}) \approx \mathcal{L}(\mathbb{Z})$, where $\mathbb{Z}$ is finitely generated and it is not divisible, while $\mathbb{Z}_{p^\infty}$ is not finitely generated, although it is divisible. This is why we formulate Questions 12.3.10 and 12.3.14 only for the subgroup logarithmic hyperballean ℓ-$\mathcal{L}(G)$

12.3.4 Results on coarsely equivalent subgroup exponential hyperballeans

Lemma 12.3.15. *Let G and H be two groups.*

(a) If there exist two homomorphisms $f\colon G \to H$ and $g\colon H \to G$ such that $f \circ g \sim id_H$ and $g \circ f \sim id_G$, then $f\colon \mathfrak{B}_G \to \mathfrak{B}_H$ is a coarse equivalence, with coarse inverse $g\colon \mathfrak{B}_H \to \mathfrak{B}_G$, and $\mathcal{L}(f) = \exp f|_{L(G)}\colon \mathcal{L}(G) \to \mathcal{L}(H)$ is a coarse equivalence, with inverse $\mathcal{L}(g)\colon \mathcal{L}(H) \to \mathcal{L}(G)$.

(b) Let H be a finite normal subgroup of G. Then the quotient map $q\colon \mathcal{L}(G) \to \mathcal{L}(G/H)$ is a coarse equivalence and, moreover, $\mathcal{L}(q)\colon \mathcal{L}(G) \to \mathcal{L}(G/H)$ is a coarse equivalence.

Proof. (a) Note that $f\colon \mathfrak{B}_G \to \mathfrak{B}_H$ is trivially a coarse equivalence. Moreover, it is easy to check that $\exp f\colon \exp \mathfrak{B}_G \to \exp \mathfrak{B}_H$ is a coarse equivalence with coarse inverse $\exp g\colon \exp \mathfrak{B}_H \to \exp \mathfrak{B}_G$ (see also [55]). Since both f and g are homomorphisms, the restrictions $\mathcal{L}(f)$ and $\mathcal{L}(g)$ are well-defined and thus they are coarse equivalences.

(b) Since q is a surjective homomorphism, the finitary group coarse structure is perfect, and $\ker q = H \in [G]^{<\omega}$, $q\colon \mathfrak{B}_G \to \mathfrak{B}_{G/H}$ is a coarse equivalence (see Corollary 7.2.6). In particular,

$$\mathcal{L}(q) = \exp q|_{\mathcal{L}(G)}\colon \mathcal{L}(G) \to \mathcal{L}(G/H),$$

which is well-defined, is bornologous. Moreover, $g\colon \mathcal{L}(G/H) \to \mathcal{L}(G)$ defined by the law $g(K) = q^{-1}(K)$, where $K \leq G/H$, is bornologous and a coarse inverse of $\mathcal{L}(q)$. □

Theorem 12.3.16. *Let a group G contain an element g of infinite order. Then $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent if and only if G has a finite normal subgroup H such that $G/H \simeq \mathbb{Z}$.*

Proof. ($\to$) Assume that $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent. Lemma 12.3.1(b) implies that $\mathcal{L}(G)$ has two connected components: one is unbounded (hence, infinite) and one is bounded. Let us see that the connected component $C = \mathcal{Q}_{\mathcal{L}(G)}(\{e\})$ of $\{e\}$ is the bounded one. To prove that C is bounded it is enough to observe that it does not contain the infinite subgroup $\langle g\rangle$ as well as its infinitely many proper subgroups $\langle g^n\rangle$, where $n \geq 2$. Since this family is certainly unbounded in $\mathcal{L}(G)$, C must be the bounded component. Consequently, C is finite being contained into a ball around $\{e\}$ (see Fact 11.3.1).

Since C contains all finite order elements $h \in G$, this implies that the set H of all the elements of finite order of G is finite. By Ditsman's lemma ([68]), H is a subgroup. Moreover, since conjugacy does not change the order of an element, H is normal in G. Then G/H is torsion free.

Since $\mathcal{L}(G/H)$ is coarsely equivalent to $\mathcal{L}(G)$ (Lemma 12.3.15(b)) and thus to $\mathcal{L}(\mathbb{Z})$, in particular, we can apply again the usual argument and prove that every proper subgroup K of G/H is large in G/H and so $|G/H : K|$ is finite. By Fedorov's theorem, G/H is isomorphic to $\mathbb{Z}$.

($\leftarrow$) On the other hand, if H is finite and $G/H \simeq \mathbb{Z}$ then $G = \langle a\rangle H$, $\langle a\rangle \simeq \mathbb{Z}$ and $\mathcal{L}(\langle a\rangle)$ is large in $\mathcal{L}(G)$, so $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent. □

Lemma 12.3.17. *Let G be a group.*

(a) If H is a subgroup of G of finite index, then G has only finitely many subgroups containing H.

(b) If $\mathcal{H}$ is a family of subgroups of G stable under under finite intersections, and there exists $n \in \mathbb{N}$ such that $|G : H| \leq n$ for every $H \in \mathcal{H}$, then $\mathcal{H}$ is finite.

Proof. (a) Let H_G be the *core of H in G* (i.e., the biggest normal subgroup of G which is contained in H), which has still finite index in G. Consider the map $q\colon G \to G/H_G$. Then q induces a bijection between the family of subgroups of G containing H_G and the one of the subgroups of G/H_G. Since the latter is finite, we are done.

(b) Assume for contradiction that $\mathcal{H}$ has infinitely many pairwise distinct members $\{H_m\}_{m\in\mathbb{N}}$. One can assume, without loss of generality that they form a decreasing chain (indeed, using (a) just replace H_m by the intersection $H_1 \cap \cdots \cap H_m$). As $|G : H_m|$ is bounded, this decreasing chain stabilises. Let us call that common intersection K (obviously, $K \in \mathcal{H}$). Since all H_m contain K, this contradicts Lemma 12.3.17. □

Theorem 12.3.18. *For an abelian group G, $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent if and only if there exists a finite subgroup H of G such that either $G/H \simeq \mathbb{Z}$ or $G/H \simeq \mathbb{Z}_{p^\infty}$, for some prime p.*

Proof. Assume that $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent. If G has an element of infinite order then we apply Theorem 12.3.16. Otherwise, suppose that G is a torsion group. Since $\mathcal{L}(G)$ and $\mathcal{L}(\mathbb{Z})$ are coarsely equivalent, we deduce from Lemma 12.3.1, that $\mathcal{L}(G)$ has two connected components and one of them is bounded, while the other one is unbounded. Since G is torsion, Fact 11.3.1 implies that $\mathcal{Q}_{\mathcal{L}(G)}(\{0\})$ must be unbounded. Hence, the family $\mathcal{H}$ of all finite index subgroups of G satisfies the hypothesis of Lemma 12.3.17(b) and thus $\mathcal{H}$ is finite and, in particular, has a minimum element K. Then G/K is finite and K is quasi-finite. Thus, since G is abelian, $K \simeq \mathbb{Z}_{p^\infty}$, for some prime p. Hence the claim follows. □

We cannot state similar results for the subgroup logarithmic hyperballean, since the balls centred at $\{0\}$ can have infinitely many elements (see Example 12.2.20).

Part IV

Appendix

Appendix A

Basic notions in category theory

In this appendix let us collect and recall some background in category theory that is used in this work. All the definitions and facts here enlisted can be found in [1].

Let $\mathcal{X}$ be a category. For every pair of objects $X, Y \in \mathcal{X}$, denote by $\mathrm{Mor}_{\mathcal{X}}(X, Y)$ the class of morphisms between X, the *domain*, and Y, the *codomain*, in $\mathcal{X}$. A functor $\mathrm{F}\colon \mathcal{X} \to \mathcal{Y}$ between two categories is called

- *faithful* if, for every $X, Y \in \mathcal{X}$ and every $f, g \in \mathrm{Mor}_{\mathcal{X}}(X, Y)$, $\mathrm{F}\,f = \mathrm{F}\,g$ if $f = g$;
- *full* if, for every $X, Y \in \mathcal{X}$ and every $g \in \mathrm{Mor}_{\mathcal{Y}}(\mathrm{F}\,X, \mathrm{F}\,Y)$, there exists $f \in \mathrm{Mor}_{\mathcal{X}}(X, Y)$ such that $\mathrm{F}\,f = g$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two categories. Define the class $\mathrm{Funct}(\mathcal{X}, \mathcal{Y})$ as the family of functors from $\mathcal{X}$ to $\mathcal{Y}$. Moreover, if $\mathcal{Z}$ is another category and $\mathrm{F}\colon \mathcal{X} \to \mathcal{Z}$ is a functor, then we can define a map $\cdot \circ \mathrm{F}\colon \mathrm{Funct}(\mathcal{Z}, \mathcal{Y}) \to \mathrm{Funct}(\mathcal{X}, \mathcal{Y})$ that associates to every $\mathrm{G} \in \mathrm{Funct}(\mathcal{Z}, \mathcal{Y})$ the composite functor $\mathrm{G} \circ \mathrm{F}$.

A subcategory $\mathcal{Y}$ of a category $\mathcal{X}$ is *full* if the inclusion functor $\mathrm{I}\colon \mathcal{Y} \to \mathcal{X}$ is full.

Denote by **Set** the category of sets and maps between them. Following [1], a *forgetful functor* is a faithful functor. Usually we use this term to consider functors from a category to **Set**, but it is useful to consider also a more general setting (see, for example, Chapter 4).

A morphism $\alpha\colon X \to X'$, in a category $\mathcal{X}$, is called:

- an *isomorphism* if there exists another morphism $\beta\colon X' \to X$ of $\mathcal{X}$ such that $\alpha \circ \beta = id_{X'}$ and $\beta \circ \alpha = id_X$;
- an *epimorphism* if every pair of morphisms $\beta, \gamma\colon X' \to X''$ such that $\beta \circ \alpha = \gamma \circ \alpha$ satisfies $\beta = \gamma$;
- a *monomorphism* if every pair of morphisms $\beta, \gamma\colon X'' \to X$ such that $\alpha \circ \beta = \alpha \circ \gamma$ satisfies $\beta = \gamma$;
- a *bimorphism* if it is both epimorphism and monomorphism.

Denote by $\mathrm{Mono}_{\mathcal{X}}$ and $\mathrm{Epi}_{\mathcal{X}}$ the classes of all monomorphisms and all epimorphisms of $\mathcal{X}$, respectively. In any category $\mathcal{X}$, an isomorphism is, in particular, a bimorphism, but the converse implication does not hold in general. The category $\mathcal{X}$ is called *balanced* if bimorphisms are exactly the isomorphisms.

It is easy to check that, if a composite morphism $g \circ f$ is an epimorphism (a monomorphism), then g is an epimorphism (f is a monomorphism, respectively).

A monomorphism m of a category $\mathcal{X}$ is called *extremal*, if, for every factorization $m = f \circ e$ in $\mathcal{X}$, e is an isomorphism, whenever e is an epimorphism. Denote by $\mathrm{ExtMono}_{\mathcal{X}}$ the class of all extremal monomorphisms of $\mathcal{X}$.

A category $\mathcal{X}$ has an $(\mathcal{E}, \mathcal{M})$*-factorisation*, if every morphism f of $\mathcal{X}$ factorises as $f = m \circ e$, for some $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Although the standard definition of $(\mathcal{E}, \mathcal{M})$-factorisation is, in general, stronger (see [60]), this relaxed version is strong enough for the purpose of this work.

A *concrete category* $(\mathcal{X}, U)$ is a pair where $\mathcal{X}$ is a category and $\mathrm{U}\colon \mathcal{X} \to \mathbf{Set}$ is a faithful functor. In that situation, for every set A the *fibre* of A is the family of all X in $\mathcal{X}$ such that $\mathrm{U}\,X = A$. If $(\mathcal{X}, \mathrm{U})$ is a concrete category and X and Y are two objects of $\mathcal{X}$, a morphism $f\colon \mathrm{U}\,X \to \mathrm{U}\,Y$ is a $\mathcal{X}$*-morphism with respect to* X *and* Y whenever there exists $\overline{f} \in \mathrm{Mor}_{\mathcal{X}}(X, Y)$ such that $\mathrm{U}\,\overline{f} = f$.

A *source* in a category $\mathcal{X}$ is a family (possibly a proper class) $\{f_i\colon X \to X_i\}_{i\in I}$ of morphisms of $\mathcal{X}$. Dually, a *sink* in a category $\mathcal{X}$ is a family $\{f_i\colon X_i \to X\}_{i\in I}$ of morphisms, where X and X_i, for every $i \in I$, are objects of $\mathcal{X}$.

Suppose now that $(\mathcal{X}, \mathrm{U})$ is a concrete category. A source $\{f_i\colon X \to X_i\}_{i\in I}$ of $\mathcal{X}$ is *initial* if, for every morphism $f\colon \mathrm{U}\,A \to \mathrm{U}\,X$ of $\mathbf{Set}$, such that $f_i \circ \mathrm{U}\,f\colon \mathrm{U}\,A \to \mathrm{U}\,X_i$ is an $\mathcal{X}$-morphism, then f is an $\mathcal{X}$-morphism. An *initial lifting* of a source $\{f_i\colon A \to \mathrm{U}\,X_i\}$ in $\mathbf{Set}$, where, for every $i \in I$, X_i is an object of $\mathcal{X}$, is an initial source $\{g_i\colon B \to X_i\}_{i\in I}$ of $\mathcal{X}$ such that $UB = A$ and $\mathrm{U}\,g_i = f_i$, for every $i \in I$.

Definition A.0.1 ([23])**.** A concrete category $(\mathcal{X}, \mathrm{U})$ is *topological* if:

(a) U is *amnestic* (i.e. $f = 1_X$, whenever $f\colon X \to X$ is an isomorphism of $\mathcal{X}$ such that $\mathrm{U}\,f = 1_{\mathrm{U}\,X}$);
(b) U is *transportable* (i.e., for every object A of $\mathcal{X}$ and every isomorphism $h\colon \mathrm{U}\,A \to X$ of $\mathbf{Set}$, there exists an object B of $\mathcal{X}$ and an isomorphism $f\colon A \to B$ of $\mathcal{X}$ such that $\mathrm{U}\,f = h$);
(c) constant maps are morphisms of $\mathcal{X}$;
(d) U has *small fibres* (i.e., the fibres are sets);
(e) every source $\{f_i\colon A \to \mathrm{U}\,X_i\}_{i\in I}$ of $\mathbf{Set}$, has an initial lifting.

The set of conditions given in Definition A.0.1 is not optimal and there are some logical dependencies. In fact, for example, item (b) is implied by the remaining properties (see [1]). However, we prefer to enlist it since it is going to be explicitly used in this work.

Let us enlist some consequences of being a topological category (see [1]). Let $\mathcal{X}$ be a topological category. Then:

- the epimorphisms of $\mathcal{X}$ are the surjective morphisms;
- the monomorphisms of $\mathcal{X}$ are the injective morphisms;
- the bimorphisms of $\mathcal{X}$ are the bijective morphisms;
- $\mathcal{X}$ has an $(\mathrm{Epi}_{\mathcal{X}}, \mathrm{ExtMono}_{\mathcal{X}})$-factorisation;
- $\mathcal{X}$ is *complete*, i.e., it has all limits (e.g., products, pullbacks and equalisers, see §A.1 for the definitions), and *co-complete*, i.e., it has all colimits (e.g., coproducts and quotients, §A.1).

A.1 Categorical constructions

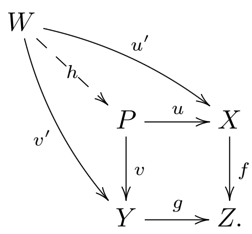
Let $\mathcal{X}$ be a category and $\{X_i\}_{i\in I}$ be a family of objects of $\mathcal{X}$. A source $\{p_i\colon X \to X_i\}_{i\in I}$, where X is an object of $\mathcal{X}$, is the *product of* $\{X_i\}_{i\in I}$ *in* $\mathcal{X}$ if it satisfies the following universal property: for every other source $\{f_i\colon Y \to X_i\}_{i\in I}$, where Y is another object of $\mathcal{X}$, there exists a unique morphism $f\colon Y \to X$ such that $f_i = p_i \circ f$, for every $i \in I$.

Let $\mathcal{X}$ be a category and $\{X_k\}_{k\in I}$ be a family of objects of $\mathcal{X}$. A sink $\{i_k\colon X_k \to X\}_{k\in I}$, where X is an object of $\mathcal{X}$, is the *coproduct of* $\{X_k\}_{k\in I}$ *in* $\mathcal{X}$ if it satisfies the following universal property: for every other sink $\{f_k\colon X_k \to Y\}_{k\in I}$, where Y is another object of $\mathcal{X}$, there exists a unique morphism $f\colon X \to Y$ such that $f_k = f \circ i_k$, for every $k \in I$.

Let $(\mathcal{X}, \mathrm{U})$ be a concrete category. Let X be an object of $\mathcal{X}$, A be a set, and $f\colon \mathrm{U}\,X \to A$ be an epimorphism in **Set** (i.e., a surjective map). Then the *quotient* of f and X is a morphism $\overline{f}\colon X \to Y$ of $\mathcal{X}$ with $\mathrm{U}\,\overline{f} = f$, and that satisfies the following universal property: for every other morphism $g\colon \mathrm{U}\,Y \to \mathrm{U}\,Z$ of **Set**, g is an $\mathcal{X}$-morphism, provided that $g \circ f$ is an $\mathcal{X}$-morphism.

Let $\mathcal{X}$ be a category and $f, g\colon X \to Y$ be two morphisms between two objects of $\mathcal{X}$. A morphism $m\colon M \to X$ is the *equaliser of the pair f and g* and we write $m = \mathrm{eq}(f, g)$ if $f \circ m = g \circ m$ and m satisfies the following universal property: if $n\colon N \to X$ is another morphism such that $f \circ n = g \circ n$, then there exists a unique morphism $h\colon N \to M$ such that the $n = m \circ h$. Every equaliser is a monomorphism.

Let $\mathcal{X}$ be a category, $f\colon X \to Z$ and $g\colon Y \to Z$ be two morphisms. The *pullback* of the pair f and g consists of an object P and two morphisms $u\colon P \to X$ and $v\colon P \to Y$ of $\mathcal{X}$ such that $f \circ u = g \circ v$, and which, moreover, satisfy the following universal property: if W is an object and $u'\colon Z \to X$ and $v'\colon Z \to Y$ are two morphisms of $\mathcal{X}$ such that $f \circ u' = g \circ v'$, then there exists a unique morphism $h\colon W \to P$ such that all the triangles of the following diagram



commute.

If a category $\mathcal{X}$ has (finite) products, then the existence of pullbacks is equivalent to the existence of equalisers. In fact, suppose that a category $\mathcal{X}$ has finite products. If $\mathcal{X}$ has equalisers (e.g., $\mathcal{X}$ is topological), the pullback of the pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ is the equaliser of the pair of morphisms $f \circ p_1\colon X \times Y \to Z$ and $g \circ p_2\colon X \times Y \to Z$. Vice versa, suppose that $\mathcal{X}$ has pullbacks, and let $f, g\colon X \to Y$ be a pair of morphisms. We define the maps $\langle f, g\rangle\colon X \to Y \times Y$ by the law $\langle f, g\rangle(x) = (f(x), g(x)) \in Y \times Y$, and $\langle 1_Y, 1_Y\rangle$ similarly. Then the equaliser of f and g is the map u in the following pullback diagram:

$$\begin{array}{ccc} E & \longrightarrow & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \langle 1_Y, 1_Y\rangle} \\ X & \xrightarrow{\langle f,g\rangle} & Y \times Y. \end{array}$$

A.2 Adjoints and co-adjoints, and reflective and co-reflective subcategories

Let $\mathrm{F}, \mathrm{G}\colon \mathcal{X} \to \mathcal{Y}$ be two functors between two categories. A *natural transformation* η from F to G (in symbols, $F \xrightarrow{\eta} G$) associates to each object X of $\mathcal{X}$, a morphism $\eta_X\colon \mathrm{F}\,X \to \mathrm{G}\,X$ of $\mathcal{Y}$ such that, for every other morphism $f\colon X \to X'$ of $\mathcal{X}$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{F}\,X & \xrightarrow{\eta_X} & \mathrm{G}\,X \\ {\scriptstyle \mathrm{F}\,f}\downarrow & & \downarrow{\scriptstyle \mathrm{G}\,f} \\ \mathrm{F}\,X' & \xrightarrow{\eta_{X'}} & \mathrm{G}\,X'. \end{array}$$

Let now $\mathrm{G}\colon \mathcal{X} \to \mathcal{Y}$ and $\mathrm{F}\colon \mathcal{Y} \to \mathcal{X}$ be two functors. Then F is *co-adjoint* for G and G is *adjoint* for F if there exist two natural transformations $\eta\colon id_{\mathcal{Y}} \to \mathrm{G} \circ \mathrm{F}$ (called *unit*) and $\varepsilon\colon \mathrm{F} \circ \mathrm{G} \to id_{\mathcal{X}}$ (called *co-unit*) such that the following two *triangular identities* hold: for every object $X \in \mathcal{X}$ and every $Y \in \mathcal{Y}$, the following triangles commutes,

$$\begin{array}{ccc} \mathrm{G}\,X & \xrightarrow{\eta_{\mathrm{G}\,X}} & \mathrm{G}\,\mathrm{F}\,\mathrm{G}\,X \\ & {\scriptstyle id_{\mathrm{G}\,X}}\searrow & \downarrow{\scriptstyle \mathrm{G}\,\varepsilon_X} \\ & & \mathrm{G}\,X, \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{F}\,X & \xrightarrow{\mathrm{F}\,\eta_Y} & \mathrm{F}\,\mathrm{G}\,\mathrm{F}\,Y \\ & {\scriptstyle id_{\mathrm{F}\,Y}}\searrow & \downarrow{\scriptstyle \varepsilon_{\mathrm{F}\,Y}} \\ & & \mathrm{F}\,Y. \end{array}$$

Let $\mathcal{Y}$ be a full subcategory of a category $\mathcal{X}$. Then $\mathcal{Y}$ is *reflective* in $\mathcal{X}$ if there exists a functor $\mathrm{G}\colon \mathcal{X} \to \mathcal{Y}$, called *reflector*, which is a co-adjoint for the inclusion functor $\mathrm{I}\colon \mathcal{Y} \to \mathcal{X}$. Dually, $\mathcal{Y}$ is *co-reflective* in $\mathcal{X}$ if there exists a functor $\mathrm{G}\colon \mathcal{X} \to \mathcal{Y}$, called *co-reflector*, which is an adjoint for $\mathrm{I}\colon \mathcal{Y} \to \mathcal{X}$.

A.3 Quotient categories

Let $\mathcal{X}$ be a category and $\sim$ be a *congruence* on $\mathcal{X}$, i.e., for every $X, Y \in \mathcal{X}$, $\sim$ is an equivalence relation in $\mathrm{Mor}_{\mathcal{X}}(X, Y)$ such that, for every $f, g \in \mathrm{Mor}_{\mathcal{X}}(X, Y)$ and $h, k \in \mathrm{Mor}_{\mathcal{X}}(Y, Z)$, $h \circ f \sim k \circ g$, whenever $f \sim g$ and $h \sim k$. Hence the *quotient category* $\mathcal{X}/_{\sim}$ can be defined as the one whose objects are the same of $\mathcal{X}$ and whose morphisms are equivalence classes of morphisms of $\mathcal{X}$, i.e., $\mathrm{Mor}_{\mathcal{X}/_{\sim}}(X, Y) = \{[f]_{\sim} \mid f \in \mathrm{Mor}_{\mathcal{X}}(X, Y)\}$, for every $X, Y \in \mathcal{X}/_{\sim}$. For the sake of simplicity, if $f \in \mathrm{Mor}_{\mathcal{X}}(X, Y)$ is a representative of the equivalence class $[f]_{\sim}$, we often write simply f instead of $[f]_{\sim}$.

Appendix B

Coarse entropy

In 1865 Clausius defined the notion of entropy in physics, but it was only in 1948 that Shannon ([162]) introduced it in mathematics, and, more precisely, in information theory. Inspired by that concept, several other entropies have been introduced in mathematics so far. For example, let us cite Kolmogorov ([108]) and Sinai's ([**?**]) measure theoretic entropy in ergodic theory, and Adler, Konheim and McAndrew's topological entropy ([2]). Other notions of topological entropy were given by Bowen ([22]) and Hood ([101]). In algebraic dynamics, we can cite the work of Adler, Konheim and MacAndrew ([2]), the entropy defined by Weiss in [170], and the one introduced by Peters ([131], and deeply studied in [50]), that was then generalised in [44] for endomorphisms of abelian groups (we refer to [42] for the definition in the non-abelian case, while to [45] for the extension to endomorphisms of semigroups). Later, Peters in [132] gave an extension of the algebraic entropy defined in [131] for topological automorphisms of locally compact abelian groups. This definition was generalised by Virili ([167]) to all endomorphisms of locally compact abelian groups. This definition can be found in [42] also for non-abelian groups. We finally mention the paper [45], where a unifying approach to several notions of entropy is provided by using normed semigroups. More recently the entropy generated by actions of amenable semigroups and groups have been studied. In particular, Ornstein and Weiss introduced topological and measure entropy of amenable group actions ([129]), whose approach was extended to the case of actions of amenable cancellative semigroups by Ceccherini-Silberstein, Coornaert and Krieger ([26]), Hofmann and Stoyanov studied topological entropy of locally compact semigroup actions on metric spaces ([100]), and Dikranjan, Fornasiero and Giordano Bruno in [39] defined and discussed the algebraic entropy of an action of an amenable cancellative semigroup on an abelian group.

In this appendix we introduce coarse entropy $\mathrm{h}_c(f)$ (Definition B.1.1), defined on the class of bornologous self-maps f of locally finite quasi-coarse spaces. Let us also mention that, more recently, Misiurewicz and Geller defined a different notion of coarse entropy inspired by Bowen's entropy ([120]).

The definition of the coarse entropy involves a limit superior, which is not a limit even when we consider the identity map (Example B.1.3), and two supremum operations. The first one is among all entourages, while the second one is among all base points. As for the first one, we show that it is enough to consider just a base of the quasi-coarse structure, while in the second one we just need to evaluate the points that are maximal in the large-scale specialisation preorder (Proposition B.1.2). We then prove basic properties of coarse entropy, such as the weak logarithmic law (Proposition B.2.3), how it behaves while taking products (weak addition theorem) and coproducts (Theorem B.2.5), and the monotonicity under taking invariant coarse subspaces (Corollary B.2.8).

As for conjugation invariance results, that play an important role in the theory of entropy, we have to distinguish two cases. If we consider asymorphisms as isomorphisms, then we obtain a conjugation invariance result for all bornologous self-maps of locally finite quasi-coarse spaces (Corollary B.2.9(a)). If, otherwise, we consider Sym-coarse equivalences as isomorphisms then we prove the desired result for quasi-coarse spaces with bounded geometry and for some particular bornologous self-maps (Corollary B.2.9(b)).

Among the maps for which the conjugation invariance result for Sym-coarse equivalences holds, of a particular interest is the identity map. We can rewrite this specific case as follows: if X and Y are two Sym-coarsely equivalent quasi-coarse spaces with bounded geometry, then $\mathrm{h}_c(id_X) = \mathrm{h}_c(id_Y)$ (Corollary B.3.1). Moreover, for every locally finite quasi-coarse space X, $\mathrm{h}_c(id_X) \in \{0, \infty\}$ (Theorem B.3.4). Those results play a key role in connecting the coarse entropy with the growth of metric spaces ([3]), extending the known results for finitely generated groups ([93, 42]). In particular, we show that, if X is a monogenic metric space and Y is a bounded geometry skeleton of X, then the coarse entropy of the identity id_Y can tell whether X has subexponential or exponential growth type (Theorem B.3.9).

Every monoid endowed with its monoid-quasi-coarse structure has bounded geometry. Hence, it is natural to compare in this setting the coarse entropy with the algebraic entropy (in the definition provided in [44]). It turns out that the algebraic entropy provides an upper bound to the coarse entropy, while they coincide if the endomorphism is surjective (Theorem B.4.2). We also provide examples in which the two notions differ for non-surjective endomorphisms (Examples B.4.3 and B.4.4). Thanks to this result and using the Pontryagin duality, we are able to connect the coarse entropy with the topological entropy and the measure entropy in particular cases (Corollaries B.4.9 and B.4.12).

The appendix is organised as follows. Section B.1 is devoted to introducing the coarse entropy, discussing thoughtfully the definition and providing the first non-trivial examples, i.e., left shifts of monoids and groups (Example B.1.6). In Section B.2 we collect the basic properties of the coarse entropy, such as the logarithmic law, theorems involving products and coproducts, monotonicity under taking invariant subspaces and conjugation invariance results. The focus of Section B.3 is discussing the coarse entropy of the identity, providing also connections with the growth of metric spaces. Finally, Section B.4 is dedicated to describing connections with other known entropies, such as the algebraic (§B.4.1), the topological and the measure entropies (§B.4.2). Moreover, the coarse entropy of some group endomorphisms is computed in §B.4.1.

In this appendix, all monoids and groups will be endowed with the left monoid quasi-coarse structures and with the left group coarse structures induced by the finitary monoid ideals and finitary group ideals, respectively. For the sake of simplicity, we simply call them *monoid-quasi-coarse structure* and *group-coarse structure*, respectively. Moreover, if M is a monoid and G is a group, we denote them by $\mathcal{E}_M$ and $\mathcal{E}_G$, respectively.

B.1 Definition of coarse entropy

Let $(X,\mathcal{E})$ be a quasi-coarse space and $f\colon X\to X$ be a bornologous self map. Then, for every $x\in X$ and $E\in\mathcal{E}$, define the following families of subsets recursively as follows:

$$\begin{cases}\mathrm{T}_1(f,x,E)=E[x],\\ \mathrm{T}_{n+1}(f,x,E)=(f^n\times f^n)(E)[\mathrm{T}_n(f,x,E)], \quad \text{for every } n\in\mathbb{N}.\end{cases} \tag{B.1}$$

More explicitly, for every $n\in\mathbb{N}$, $x\in X$, $E\in\mathcal{E}$,

$$\mathrm{T}_{n+1}(f,x,E)=(E\circ(f\times f)(E)\circ\cdots\circ(f^{n-1}\times f^{n-1})(E)\circ(f^n\times f^n)(E))[x], \tag{B.2}$$

which is called the $n+1$-*coarse trajectory* $\mathrm{T}_{n+1}(f,x,E)$ *with respect to* x *and* E. When there is no risk of ambiguity, we will simply call it $n+1$ trajectory. Note that, if X is locally finite, for every bornologous self-map, every trajectory is a finite subset according to (B.2).

Before defining the coarse entropy, in the notation above, let us focus a bit more on the entourages of the form $(f^n\times f^n)(E)$. For every self-map $g\colon X\to X$ and every subset $A\subseteq X$, we have that

$$(g\times g)(E)[A]=g(E[g^{-1}(A)])\subseteq g(X). \tag{B.3}$$

In particular, if $A\cap g(X)=\emptyset$, then $(g\times g)(E)[A]=\emptyset$. Moreover, even if $\Delta_X\subseteq E$, (B.3) implies that the trajectories can decrease.

Note that, for every $n\in\mathbb{N}\setminus\{0\}$, $x\in x$, and $E\in\mathcal{E}$,

$$\mathrm{T}_n(f,x,E)\subseteq \mathcal{Q}_X^{\downarrow}(x)\cap f^{n-1}(X). \tag{B.4}$$

In fact, if $y\notin f^{n-1}(X)$, then, for every other $z\in X$, $(z,y)\notin(f^{n-1}\times f^{n-1})(E)$.

Let us define the coarse entropy.

Definition B.1.1. Let $(X,\mathcal{E})$ be a locally finite quasi-coarse space and $f\colon X\to X$ be a bornologous self map. If $x\in X$, $E\in\mathcal{E}$, and $n\in\mathbb{N}\setminus\{0\}$, we define

$$d_n=\frac{\log|\mathrm{T}_n(f,x,E)|}{n},\quad \mathrm{H}_c(f,x,E)=\limsup_{n\to\infty}d_n, \tag{B.5}$$

$$\mathrm{H}_c^{loc}(f,x)=\sup_{E\in\mathcal{E}}\mathrm{H}_c(f,x,E),\quad \text{and, finally,}\quad \mathrm{h}_c(f)=\sup_{x\in X}\mathrm{H}_c^{loc}(f,x). \tag{B.6}$$

The value $\mathrm{H}_c^{loc}(f,x)$ and $\mathrm{h}_c(f)$ are called the *local entropy of* f *in* x and the *coarse entropy of* f, respectively.

Proposition B.1.2 discusses more in detail the two supremum operations in (B.6).

Proposition B.1.2. *Let* $(X,\mathcal{E})$ *be a locally finite quasi-coarse space and* $f\colon X\to X$ *be a bornologous self-map.*

(a) If $E, F \in \mathcal{E}$ are two entourages such that $E \subseteq F$, then, for every $x \in X$, $\mathrm{H}_c(f, x, E) \leq \mathrm{H}_c(f, x, F)$. In particular, if $\mathcal{F}$ is a base of $\mathcal{E}$, then $\mathrm{H}_c^{loc}(f, x) = \sup\{\mathrm{H}_c(f, x, F) \mid F \in \mathcal{F}\}$, for every $x \in X$.
(b) If $x \downarrow y$, then $\mathrm{H}_c^{loc}(f, x) \geq \mathrm{H}_c^{loc}(f, y)$. In particular, if X has a maximum x, then $\mathrm{h}_c(f) = \mathrm{H}_c^{loc}(f, x)$.

Proof. (a) Since, if $E \subseteq F \in \mathcal{E}$, then, for every $n \in \mathbb{N}$, $(f^n \times f^n)(E) \subseteq (f^n \times f^n)(F)$, the conclusion is trivial.

(b) Let $E \in \mathcal{E}$. Define $E' = \{(x, y) \circ E\} \cup E$ and then, for every $n \in \mathbb{N} \setminus \{0\}$,

$$\begin{aligned}\mathrm{T}_n(f, x, E') &= (f^{n-1} \times f^{n-1})(E')[\cdots[(f \times f)(E')[E'[x]]]\cdots] \supseteq \\ &\supseteq (f^{n-1} \times f^{n-1})(E)[\cdots[(f \times f)(E)[E[y]]]\cdots] = T_n(f, y, E)\end{aligned}$$

since $E'[x] = (\{(x, y)\} \circ E)[x] = E[y]$, which shows the desired inequality. The second part of the assertion trivially follows. □

The reader may wonder if the limit superior in (B.5) is a limit or not. In Example B.1.3 we provide a locally finite metric space (Example B.1.3(a)) and a metric space with bounded geometry (Example B.1.3(c)) such that, for suitable inputs, the sequence d_n defined in (B.5) has no limit.

Example B.1.3. (a) We want to define a non-directed graph $X = (V, E)$, where $V \subseteq \mathbb{N} \times \mathbb{N}$ and a pair $\{(m, n), (m', n')\} \in E$, where $(m, n), (m', n') \in V$, if and only if $|m - m'| = 1$. In order to define the set V, we need to inductively construct a sequence $\{K_n\}_n$ of natural numbers. Let $K_0 = 1$. Suppose that we have defined $K_0, \dots, K_{m-1}$. Then

$$K_m = \begin{cases} 1 & \text{if } \displaystyle\sum_{i=0}^{m-1} K_i > 2m, \\ 2^m - \displaystyle\sum_{i=0}^{m-1} K_i & \text{otherwise.} \end{cases} \tag{B.7}$$

Finally $V = \bigcup_{m \in \mathbb{N}} (\{m\} \times \{0, \dots, K_m\})$.
Endow X with its path metric and then with the induced metric coarse structure. In particular, X is locally finite, even though it has not bounded geometry. Let us consider the map id_X. Since X is connected, in order to compute its coarse entropy, we can just consider the trajectories centred in $(0, 0)$. Thanks to the definition of X, there exists a strictly increasing sequence $(a_n)_n$ of natural numbers such that

$$\sum_{i=0}^{a_n - 1} K_i \leq 2a_n \quad \text{and} \quad \sum_{i=0}^{a_n} K_i = 2^{a_n}.$$

Then

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log|\mathrm{T}_n(id_X, (0,0), 1)|}{n} &\leq \lim_{n\to\infty} \frac{\log(2n+1)}{n} = 0, \quad \text{and} \\ \limsup_{n\to\infty} \frac{\log|\mathrm{T}_n(id_X, (0,0), 1)|}{n} &\geq \lim_{n\to\infty} \frac{\log(2^n)}{n} = \log 2.\end{aligned}$$

Hence the limit does not exist. Moreover, $\mathrm{h}_c(id_X) \geq \log 2$. Actually, we will show that $\mathrm{h}_c(id_X)$ can only take values in $\{0, \infty\}$ (Theorem B.3.4), and thus $\mathrm{h}_c(id_X) = \infty$.

(b) Let us slightly modify item (a), by changing the sequence $\{K_n\}_n$ in (B.7) to

$$K_m = \begin{cases} 1 & \text{if } \sum_{i=0}^{m-1} K_i > 2m, \\ m^m - \sum_{i=0}^{m-1} K_i & \text{otherwise.} \end{cases} \tag{B.8}$$

The induced coarse space X is still locally finite. However, it can be easily proved that $\mathrm{H}_c(id_X, (0,0), 1) = \infty$.

(c) We want to provide now an example of a metric space with bounded geometry for which the sequence d_n does not have a limit even when we consider the identity map.
We will define a non-directed graph X and endow it with its path metric. For every $n \in \mathbb{N}$, denote by T_n the complete 3-ary tree of height $n+1$, by a_n its root, and by $x_1^n, \dots, x_{3^n}^n$ its leaves. Let i_n and j_n be the two canonical inclusions of T_n in the disjoint union $T_n \sqcup T_n$. Note that $|T_n| = \sum_{i=0}^n 3^i$. Consider the smallest equivalence relation $\sim_n$ on $T_n \sqcup T_n$ satisfying $i_n(x_i^n) \sim_n j_n(x_i^n)$, for every $i = 1, \dots, 3^n$, and define $D_n = (T_n \sqcup T_n)/_{\sim_n}$. Moreover, for every $n \in \mathbb{N}$, denote by L_n the graph consisting of $n+1$ points $y_0^n, \dots, y_n^n$ with the edges $\{y_i^n, y_{i+1}^n\}$, for every $i = 0, \dots, n-1$.
Let us now describe the graph X through a limit process. Define $k_0' = 1$ and $X_0 = L_1 = L_{k_0'}$. Then, for every $n \in \mathbb{N}$, let

$$\begin{aligned} k_{n+1} &= \min\{k \in \mathbb{N} \mid |X_{2n}| + \sum_{i=1}^{k} 3^i \geq 2^{\mathrm{diam}(X_{2n})+k}\}, \text{ and} \\ X_{2n+1} &= (X_{2n} \sqcup D_{k_{n+1}})/_{\approx_n}, \text{ where } \approx_n \text{ is the finest equivalence} \\ &\qquad \text{relation satisfying } y_{k_n'}^{k_n'} \approx_n i_{k_{n+1}}(a_{k_{n+1}}), \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} k_{n+1}' &= \min\{k \in \mathbb{N} \mid |X_{2n+1}| + k \leq 2(\mathrm{diam}(X_{2n+1}) + k)\}, \quad \text{and} \\ X_{2n+2} &= (X_{2n+1} \sqcup L_{k_{n+1}'})/_{\approx_n'}, \text{ where } \approx_n' \text{ is the finest equivalence} \\ &\qquad \text{relation satisfying } j_{k_{n+1}}(a_{k_{n+1}}) \approx_n' y_0^{k_{n+1}'}. \end{aligned} \tag{B.10}$$

For the sake of simplicity, for every $n \in \mathbb{N}$, in (B.9) and (B.10) we have identified the points of $D_{k_{n+1}}$ and $L_{k_{n+1}'}$ with their images in $X_{2n} \sqcup D_{k_{n+1}}$ and in $X_{2n+1} \sqcup L_{k_{n+1}'}$, respectively. Then the graph X is the direct limit of the family $\{X_n\}_{n\in\mathbb{N}}$ of finite graphs (with the family of obvious inclusion maps). In Figure B.1 a representation of this space is provided. The metric space associated with the graph X has bounded geometry since every vertex has degree at most 4.
Similarly to what we have done for item (a), it is not hard to prove that, in the notation of (B.5), if $E = E_1$ and $x = y_0^1$, then

$$\liminf_{n\to\infty} d_n = 0 < \log 2 \leq \limsup_{n\to\infty} d_n.$$

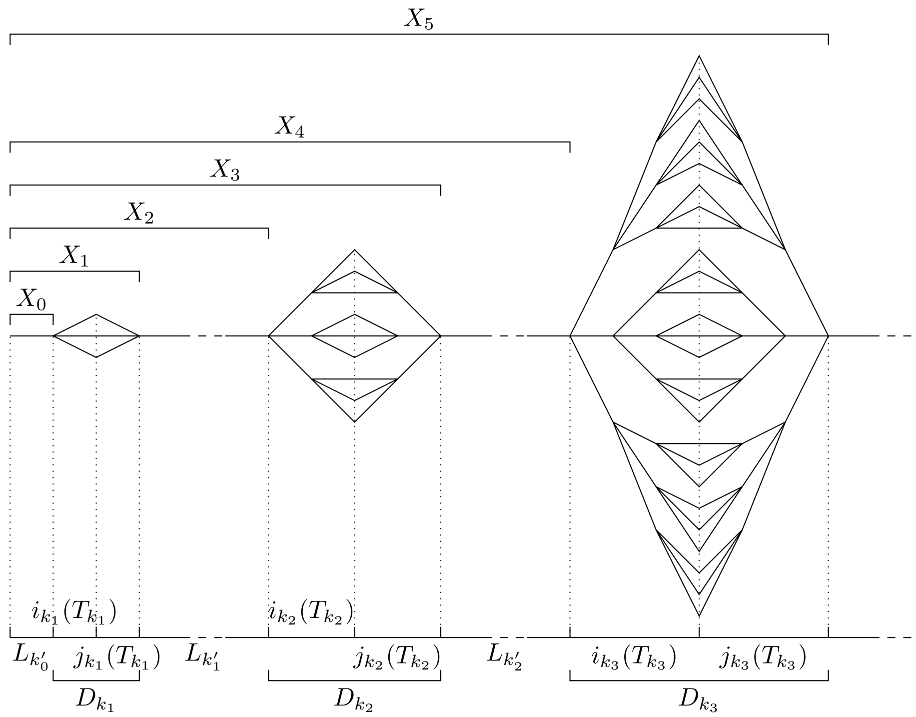


Figure B.1: A representation of the graph X defined in Example B.1.3(c), underlining the different pieces composing the elements of the family of finite graphs $\{X_n\}_n$.

Hence the sequence $\{d_n\}_n$ has no limit.

In Example B.1.3(b) we provided an example of a locally finite coarse space $(X, \mathcal{E})$ for which there exist $x \in X$ and $E \in \mathcal{E}$ with $\mathrm{H}_c(id_X, x, E) = \infty$. Note that, if $(Y, \mathcal{E}')$ is a quasi-coarse space with bounded geometry, then, for every point $y \in Y$ and every $E' \in \mathcal{E}'$, $\mathrm{H}_c(id_Y, y, E') < \infty$. More precisely, if $\delta(E')$ is a uniform bound to the cardinality of the balls with radius E' (i.e., $|E'[x]| \leq \delta(E')$, for every $x \in X$), then

$$\mathrm{H}_c(id_Y, y, E') \leq \limsup_{n\to\infty} \frac{\log(\delta(E')^{n-1})}{n} = \log(\delta(E')).$$

However, the answer to the following question is not known.

Question B.1.4. Let $(X, \mathcal{E})$ be a quasi-coarse space with bounded geometry and $f\colon X \to X$ be a bornologous self-map. Is it true that, for every $x \in X$ and $E \in \mathcal{E}$, $\mathrm{H}_c(f, x, E) < \infty$?

Remark B.1.5. Let $(X, \mathcal{E})$ be a locally finite quasi-coarse space and $f\colon X \to X$ be a bornologous self-map.

(a) Suppose that f is a map such that, for a base $\mathcal{F}$ of $\mathcal{E}$, it satisfies $(f \times f)(F) = F$, for every $F \in \mathcal{F}$. Then, for every $n \in \mathbb{N} \setminus \{0\}$, $x \in X$ and $F \in \mathcal{F}$, $\mathrm{T}_n(f, x, F) = \mathrm{T}_n(id, x, F)$ and thus, according to Proposition B.1.2, $\mathrm{h}_c(f) = \mathrm{h}_c(id)$. If, otherwise, there exists a base $\mathcal{H}$ of $\mathcal{E}$ such that $H \supseteq (f \times f)(H) \in \mathcal{H}$, for every $H \in \mathcal{H}$, then $\mathrm{h}_c(f) \leq \mathrm{h}_c(id)$.

(b) Let $\mathcal{E}'$ be another quasi-coarse structures on X such that $\mathcal{E}$ is finer than $\mathcal{E}'$. Denote $f_{\mathcal{E}}\colon (X, \mathcal{E}) \to (X, \mathcal{E})$ and $f_{\mathcal{E}'}\colon (X, \mathcal{E}') \to (X, \mathcal{E}'))$. If $f_{\mathcal{E}'}$ is bornologous, then $\mathrm{h}_c(f_{\mathcal{E}}) \leq \mathrm{h}_c(f_{\mathcal{E}'})$ since, for every $x \in X$, $\mathrm{H}_c^{loc}(f_{\mathcal{E}}, x) \leq \mathrm{H}_c^{loc}(f_{\mathcal{E}'}, x)$.

(c) If X is cellular, then $\mathrm{h}_c(id) = 0$ since the trajectories stabilise, more precisely, for every choice of $E \in \mathcal{E}$, and $x \in X$, the n-th trajectory is contained in a subset, namely $E^{\square}[x]$, which is bounded from x and thus finite. Note that a monoid, endowed with the monoid-quasi-coarse structure is cellular if and only if it is *locally finite* (i.e., for every $K \in [M]^{<\omega}$, the submonoid generated by K is still finite).

Example B.1.6. Let M be a monoid endowed with the monoid-quasi-coarse structure and $x \in M$. We want to discuss the entropy of the left shift $f = s_x^\lambda\colon y \mapsto xy$. First of all, let us note that, for every $K \in [M]^{<\omega}$,

$$(f \times f)(E_K) \subseteq E_K, \tag{B.11}$$

which shows, in particular, that f is bornologous. Since the neutral element e is a maximum in M, then we just need to consider the trajectories with respect to e (Proposition B.1.2).

(a) Suppose that x is an invertible element of M. Then f is an asymorphism with inverse $s_{x^{-1}}^\lambda$. Moreover, it is easy to see that (B.11) becomes an equality and thus Remark B.1.5(a) implies that $\mathrm{h}_c(f) = \mathrm{h}_c(id_M)$. In particular this equality holds if M is a group.

(b) Let M be left-cancellative (i.e., for every $y \in M$, the left shift s_y^λ is injective) and commutative. Split $K = (K \cap f(M)) \cup (K \setminus f(M))$. Note that $(f \times f)(E_K)[K] = (f \times f)(E_K)[K \cap f(M)]$, according to (B.3). Hence, without loss of generality, we can assume that $K = xF$, for some non-empty $F \in [M]^{<\omega}$. By using induction, (B.3), and the commutativity of M, we can prove that, for every $n \in \mathbb{N}$, $\mathrm{T}_n(f, e, E_K) = nx + nF$, where nF is the family of elements of M that can be written as sum of n elements of F. Moreover, since M is left-cancellative, $|\mathrm{T}_n(f, e, E_K)| = |nF| = |\mathrm{T}_n(id, e, E_F)| = |\mathrm{T}_n(id, e, E_K)|$. Thus $\mathrm{H}_c(f, e, E_K) = \mathrm{H}_c(id, e, E_K) = \mathrm{H}_c(id, e, E_F)$.

(c) Let M be the free monoid generated by two elements a and b, and $x = a$. We claim that $\mathrm{h}_c(f) = \infty$. Because of Proposition B.1.2 we can restrict ourselves to study the base $\{E_{B(e,m)} \mid m \in \mathbb{N}\}$, where $B(e, m) = B_{d^{\wedge}_{\{a,b\}}}(e, m)$, for every $m \in \mathbb{N}$. If $K = B(e, m)$, we claim that

$$\mathrm{T}_{n+1}(f, e, E_K) = a^{n+1}B(e, (n+1)m-(n+1)) + a^n bB(e, (n+1)m-(n+1)). \tag{B.12}$$

If $n = 0$, then (B.12) is trivial. Suppose now that (B.12) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
\mathrm{T}_{n+2}(f, e, E_K) &= ((f^{n+1} \times f^{n+1})(E_K))[\mathrm{T}_{n+1}(f, e, K)] = \\
&= ((f^{n+1} \times f^{n+1})(E_K))[a^{n+1}B(e, (n+1)m - (n+1)) + \\
&\qquad + a^n bB(e, (n+1)m - (n+1))] = \\
&= f^{n+1}(E_K[(f^{n+1})^{-1}(a^{n+1}B(e, (n+1)m - (n+1)) + \\
&\qquad + a^n bB(e, (n+1)m - (n+1)))]) = \\
&= a^{n+1}B(e, (n+1)m - (n+1))B(e, m) = \\
&= a^{n+1}B(e, (n+2)m - (n+1)) = \\
&= a^{n+2}B(e, (n+2)m - (n+2)) + \\
&\qquad + a^{n+1}bB(e, (n+2)m - (n+2)),
\end{aligned}$$

which shows (B.12). Then,

$$\mathrm{H}_c(f, e, E_{B(e,m)}) \geq \limsup_{n\to\infty} \frac{\log(2^{nm-(n-1)})}{n} \geq (m-1)\log 2,$$

and so the claim since m can be arbitrarily taken.

B.2 Basic properties of the coarse entropy

Before computing more, less trivial, examples of the coarse entropy (in Section B.4 many examples appear, in relation with the algebraic entropy), we focus on proving some standard properties of the coarse entropy.

Proposition B.2.1. *Let $(X, \mathcal{E})$ be a locally finite quasi-coarse space, $f\colon X \to X$ be a bornologous self-map and $x \in X$.*

(a) If there exists $F \in \mathcal{E}$ such that $\mathcal{Q}^{\downarrow}_X(x) = F[x]$, then $\mathrm{H}^{loc}_c(f, x) = 0$.

(b) If there exists $E \in \mathcal{E}$ and $n \in \mathbb{N}$ such that $f^n(X) \subseteq E[x]$, then $\mathrm{h}_c(f) = 0$.

(c) If there exists $n \in \mathbb{N}$ such that $x \notin \mathcal{Q}^{\uparrow}_X(f^n(X))$, then $\mathrm{H}^{loc}_c(f, x) = 0$.

(d) If $x \notin \bigcap_n \mathcal{Q}^{\uparrow}_X(f^n(X))$, then $\mathrm{H}^{loc}_c(f, x) = 0$.

Proof. Items (a) and (b) trivially follow from (B.4) and from the fact that X is locally finite and thus $F[x]$ and $E[x]$ are finite. Finally, it is enough to show item

(c) in order to prove item (d). If $x \notin \mathcal{Q}_X^{\uparrow}(f^n(X))$, then $\mathcal{Q}_X^{\downarrow}(x) \cap f^n(X) = \emptyset$. Hence, (B.4) implies that, $\mathrm{T}_{n+1}(f,x,E) \subseteq \mathcal{Q}_X^{\downarrow}(x) \cap f^{n-1}(X) = \emptyset$, for every $E \in \mathcal{E}$. Morevoer, for every $k > n$, $f^k(X) \subseteq f^n(X)$, and thus $\mathrm{T}_k(f,x,E) = \emptyset$. Hence, $\mathrm{H}_c(f,x,E) = 0$ and $\mathrm{H}_c^{loc}(f,x) = 0$. □

Let X be a locally finite quasi-coarse space, and $f\colon X \to X$ be a bornologous self-map. A subset Y of X is called *f-invariant* if $f(Y) \subseteq Y$.

Corollary B.2.2. *Let $(X,\mathcal{E})$ be a locally finite quasi-coarse space and $f\colon X \to X$ be a bornologous self-map. The subspace $Y = \bigcap_n \mathcal{Q}_X^{\uparrow}(f^n(X))$ is f-invariant and $\mathrm{h}_c(f) = \mathrm{h}_c(f|_Y)$.*

Proof. Let $y \in Y$. Then, for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\{(y, f^n(x_n))\} \in \mathcal{E}$. In particular, since f is bornologous, $\{(f(y), f^{n+1}(x_n))\} \in \mathcal{E}$, which implies that $y \in \mathcal{Q}_X^{\uparrow}(f^{n+1}(X))$. Since the chain $\{\mathcal{Q}_X^{\uparrow}(f^n(X)) \mid n \in \mathbb{N}\}$ is decreasing, $f(y) \in \mathcal{Q}_X^{\uparrow}(f(X))$ and thus the first statement is proved. The equality follows from Proposition B.2.1(d). □

Proposition B.2.3 (Weak logarithmic law)**.** *Let $(X,\mathcal{E})$ be a locally finite quasi-coarse space and $f\colon X \to X$ be a bornologous self-map. If f is surjective, for every $k > 0$, $\mathrm{h}_c(f^k) \leq k \cdot \mathrm{h}_c(f)$.*

Proof. Fix a positive integer $k > 0$. Then, for every $n \in \mathbb{N}$, $x \in X$, and $\Delta_X \subseteq E \in \mathcal{E}$, $\mathrm{T}_n(f^k,x,E) \subseteq T_{kn-k+1}(f,x,E)$ since the surjectivity of f implies that $\Delta_X \subseteq (f^s \times f^s)(E)$, for every $s \in \mathbb{N}$, and so

$$\begin{aligned}\mathrm{H}_c(f^k,x,E) &= \limsup_{n\to\infty} \frac{\log|\mathrm{T}_n(f^k,x,E)|}{n} \leq \\ &\leq \limsup_{n\to\infty} \frac{\log|\mathrm{T}_{kn-k+1}(f,x,E)|}{kn-k+1} \cdot \frac{kn-k+1}{n} \leq \\ &\leq k\,\mathrm{H}_c(f,x,E) \leq k\,\mathrm{h}_c(f),\end{aligned}$$

from which the stated inequality follows. □

Question B.2.4. Does the opposite inequality in Proposition B.2.3 hold?

The next result, Theorem B.2.5, states that the coarse entropy behaves as expected in relation with (finite) products and coproducts of quasi-coarse spaces. We consider only finite products since arbitrary products of locally finite quasi-coarse spaces are not necessarily locally finite.

Theorem B.2.5. *(a) Let $(X,\mathcal{E}_X)$ and $(Y,\mathcal{E}_Y)$ be two locally finite quasi-coarse spaces and $f\colon X \to X$ and $g\colon Y \to Y$ be two bornologous self-maps. Then $\mathrm{h}_c(f \times g) = \mathrm{h}_c(f) + \mathrm{h}_c(g)$.*

(b) Let $\{(X_k,\mathcal{E}_k)\}_{k\in I}$ be a family of locally finite quasi-coarse spaces and, for every $k \in I$, $f_k\colon X_k \to X_k$ be a bornologous self-map. Then $\mathrm{h}_c(\bigoplus_k f_k) = \sup_{k\in I} \mathrm{h}_c(f_k)$.

Proof. (a) Because of Proposition B.1.2(a), we can just evaluate the trajectories $\mathrm{T}_n(f \times g, (x,y), E \times F)$, where $n \in \mathbb{N} \setminus \{0\}$, $(x,y) \in X \times Y$, $E \in \mathcal{E}_X$ and

$F \in \mathcal{E}_Y$. Note that $(E \times F)[(x,y)] = E[x] \times F[y]$, $((f \times g) \times (f \times g))(E \times F) = (f \times f)(E) \times (g \times g)(F)$, and $(f \times g)^k = f^k \times g^k$, for every $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathrm{T}_n(f \times g, (x,y), E \times F) = &((f \times g)^{n-1} \times (f \times g)^{n-1})(E \times F)[\cdots \\ &\cdots[((f \times g) \times (f \times g)(E \times F))[E \times F[(x,y)]]]\cdots] = \\ = &((f^{n-1} \times f^{n-1})(E) \times (g^{n-1} \times g^{n-1})(F))[\cdots \\ &\cdots[((f \times f)(E) \times (g \times g)(F))[E \times F[(x,y)]]]\cdots] = \\ = &\mathrm{T}_n(f, x, E) \times \mathrm{T}_n(g, y, F),\end{aligned}$$

and thus

$$|\mathrm{T}_n(f \times g, (x,y), E \times F)| = |\mathrm{T}_n(f, x, E)| \cdot |\mathrm{T}_n(g, y, F)|,$$

from which the equality $\mathrm{H}_c(f \times g, (x,y), E \times F) = \mathrm{H}_c(f, x, E) + \mathrm{H}_c(g, y, F)$ and the desired claim follows.

Item (b) trivially follows from the observation that, for every $n \in \mathbb{N} \setminus \{0\}$, $i_j(x) \in \bigsqcup_k X_k$, and $E_{J,\varphi} \in \bigoplus_k \mathcal{E}_k$, defined as in (4.4), where $J \in [I]^{<\omega}$ and $\varphi\colon J \to \bigcup_k \mathcal{E}_k$ with the desired properties,

$$\mathrm{T}_n\left(\bigoplus_{k \in I} f_k, i_j(x), E_{J,\varphi}\right) = i_j(T_n(f_j, x, F)), \quad \text{where} \quad F = \begin{cases} E_j & \text{if } j \in J, \\ \Delta_{X_j} & \text{otherwise.}\end{cases}$$

□

Conjugation results are particularly important in developing entropies (see also Remark B.2.10). The final part of this section is devoted to prove conjugation results for the coarse entropy.

Lemma B.2.6. *Let X be a set and $h\colon X \to X$ be a self-map. Then:*
(a) if $E, F \subseteq X \times X$, then $(h \times h)(E \circ F) \subseteq (h \times h)(E) \circ (h \times h)(F)$;
(b) if $E \subseteq X \times X$ and $x \in X$, then $h(E[x]) \subseteq ((h \times h)(E))[h(x)]$.

Proof. (a) Let $(h(x), h(z)) \in (h \times h)(E \circ F)$ such that $(x,z) \in E \circ F$. Then there exists $y \in X$ such that $(x,y) \in E$ and $(y,z) \in F$, which implies $(h(x), h(y)) = (h(x), h(y)) \circ (h(y), h(z)) \in (h \times h)(E) \circ (h \times h)(F)$.

(b) Consider an arbitrary $y \in h(E[x])$ and take $z \in h^{-1}(y)$ such that $(x,z) \in E$. Then $(h(x), y) = (h(x), h(z)) \in (h \times h)(E)$, which shows that $y \in ((h \times h)(E))[h(x)]$. □

Theorem B.2.7. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{E}_Y)$ be two locally finite quasi-coarse spaces, $f\colon X \to X$ and $g\colon Y \to Y$ be two bornologous self-maps, and $h\colon X \to Y$ be a bornologous map such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y. \end{array} \tag{B.13}$$

If there exists $K \in \mathbb{N}$ such that $\sup\{|h^{-1}(y)| \mid y \in Y\} \leq K$, *then* $\mathrm{h}_c(f) \leq \mathrm{h}_c(g)$.

Proof. Let $x \in X$, $E \subseteq \mathcal{E}_X$, and $x \in X$. Then, by applying Lemma B.2.6 and the commutativity of (B.13), for every $n \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned}
\mathrm{T}_n(f,x,E) &\subseteq h^{-1}(h(T_n(f,x,E))) = \\
&= h^{-1}(h((E \circ (f \times f)(E) \circ \cdots \circ (f^{n-1} \times f^{n-1})(E))[x])) \subseteq \\
&\subseteq h^{-1}((h \times h)(E \circ (f \times f)(E) \circ \cdots \circ (f^{n-1} \times f^{n-1})(E))[h(x)]) \subseteq \\
&\subseteq h^{-1}((h \times h)(E) \circ (h \times h)((f \times f)(E)) \circ \cdots \\
&\qquad \cdots \circ ((h \times h)(f^{n-1} \times f^{n-1})(E))[h(x)]) = \\
&= h^{-1}((h \times h)(E) \circ (g \times g)((h \times h)(E)) \circ \cdots \\
&\qquad \cdots \circ (g^{n-1} \times g^{n-1})((h \times h)(E))[h(x)]) = \\
&= h^{-1}(\mathrm{T}_n(g, h(x), (h \times h)(E))).
\end{aligned}$$

The computation shows that

$$\begin{aligned}
\mathrm{H}_c(f,x,E) &= \limsup_{n\to\infty} \frac{\log|\mathrm{T}_n(f,x,E)|}{n} \leq \\
&\leq \limsup_{n\to\infty} \frac{\log(K|\mathrm{T}_n(g,h(x),(h\times h)(E))|)}{n} = \\
&= \limsup_{n\to\infty} \frac{\log K + \log|\mathrm{T}_n(g,h(x),(h\times h)(E)|}{n} = \\
&= \mathrm{H}_c(g,h(x),(h\times h)(E))
\end{aligned}$$

because of the assumption on the fibres of h. Then, since both $E \in \mathcal{E}$ and $x \in X$ are arbitrary, $\mathrm{h}_c(f) \leq \mathrm{h}_c(g)$. □

Corollary B.2.8. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{E}_Y)$ be two locally finite quasi-coarse spaces, $f\colon X \to X$ and $g\colon Y \to Y$ be two bornologous self-maps, and $h\colon X \to Y$ be a bornologous map such that the diagram* (B.13) *commutes. Suppose, moreover, that one of the following properties holds:*
(a) h is injective;
(b) $(X, \mathcal{E})$ has bounded geometry and h is large-scale injective.
Then $\mathrm{h}_c(f) \leq \mathrm{h}_c(g)$.

Proof. We want to apply Theorem B.2.7. If h is injective, then we can set $K = 1$. If h is large-scale injective, then $R_h = (h \times h)^{-1}(\Delta_Y) \in \mathcal{E}_X$. Since X has bounded geometry, there exists $K \in \mathbb{N}$ such that $K \geq |R_h[x]| = |h^{-1}(h(x))|$, for every $x \in X$. □

As an immediate consequence of Corollary B.2.8 we have the monotonicity of the coarse entropy under taking invariant subspaces. Let $(X, \mathcal{E})$ be a locally finite quasi-coarse space, $f\colon X \to X$ be a bornologous self-map, and Y be a f-invariant subset of X, then $\mathrm{h}_c(f|_Y) \leq \mathrm{h}_c(f)$. Moreover, the same result implies, in the case of coarse spaces with bounded geometry, the monotonicity of the coarse entropy under taking coarse embeddings. In fact, in the notation of Corollary B.2.8, if h is a coarse embedding, then item (b) is fulfilled.

From Corollary B.2.8 the following important invariance result trivially follows.

Corollary B.2.9 (Invariance under conjugation)**.** *Let* $(X, \mathcal{E})$ *and* $(Y, \mathcal{E}_Y)$ *be two locally finite quasi-coarse spaces,* $f\colon X \to X$ *and* $g\colon Y \to Y$ *be two bornologous self-maps, and* $h\colon X \to Y$ *be a map such that the diagram* (B.13) *commutes. Suppose, moreover, that one of the following properties holds:*

(a) h *is an asymorphism;*

(b) X *and* Y *have bounded geometry,* h *is a* Sym*-coarse equivalence with a* Sym*-coarse inverse* $k\colon Y \to X$ *such that* $f \circ k = k \circ g$*.*

Then $\mathrm{h}_c(f) = \mathrm{h}_c(g)$*.*

Remark B.2.10. Let $\mathcal{X}$ be a category. We define the *category* **Flow**$\mathcal{X}$ *of flows in* $\mathcal{X}$. As objects, it has pairs (X, f), where $X \in \mathcal{X}$ and $f\colon X \to X$ is a morphism of $\mathcal{X}$. Moreover, a morphism between two such pairs (X, f) and (Y, g) is a map $h\colon X \to Y$ in $\mathcal{X}$ such that $h \circ f = g \circ h$. Moreover h is an isomorphism in **Flow**$\mathcal{X}$ is h is an isomorphism in $\mathcal{X}$.

Denote by **LF-QCoarse** the full subcategory of **QCoarse** of locally finite quasi-coarse spaces. Consider then the category **FlowLF-QCoarse**. Thanks to Corollary B.2.9, if (X, f) and (Y, g) are two isomorphic flows, then $\mathrm{h}_c(f) = \mathrm{h}_c(g)$. Hence, h_c associates a value in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ to every isomorphism class of flows in **FlowLF-QCoarse**.

B.3 Coarse entropy of the identity and growth of quasi-coarse spaces

Let us focus on the identity map of a quasi-coarse space. Corollary B.2.8 implies that, the coarse entropy of the identity map is an invariant under Sym-coarse equivalence in the realm of quasi-coarse spaces with bounded geometry.

Corollary B.3.1. *Let* X *and* Y *be two* Sym*-coarsely equivalent quasi-coarse spaces with bounded geometry. Then* $\mathrm{h}_c(id_X) = \mathrm{h}_c(id_Y)$*.*

Proof. Let $f\colon X \to Y$ be a Sym-coarse equivalence. Then we can easily apply Corollary B.2.8, which implies that $\mathrm{h}_c(id_X) \leq \mathrm{h}_c(id_Y)$. The opposite inequality can be similarly proved. □

We want to show that the identity function cannot have arbitrary values. More precisely, if X is a quasi-coarse space, then $\mathrm{h}_c(id_X) \in \{0, \infty\}$.

Lemma B.3.2. *Let* $\{a_n\}_n$ *and* $\{b_n\}_n$ *be two sequences of non-negative real numbers such that the limit of* $\{a_n\}_n$ *exists and it is strictly positive. Then*

$$\limsup_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n). \tag{B.14}$$

Proof. Since, the two sequences have non-negative values, for every $n \in \mathbb{N}$, $\sup_{k\geq n}(a_k b_k) \leq \sup_{k\geq n} a_k \cdot \sup_{k\geq n} b_k$ and thus the inequality $(\leq)$ in (B.14) follows. As for the opposite inequality, fix a value $0 < \varepsilon < l = \lim_{n\to\infty} a_n$. Then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $a_n > l - \varepsilon$. Then, for every $k \geq N$,

$$\sup_{n\geq k} a_n b_n \geq \sup_{n\geq k} (l - \varepsilon) b_n = (l - \varepsilon) \sup_{n\geq k} b_n$$

since $l-\varepsilon>0$, and thus

$$\limsup_{k\to\infty} a_kb_k \geq (l-\varepsilon)\limsup_{k\to\infty} b_k = (\lim_{n\to\infty} a_n - \varepsilon)\limsup_{k\to\infty} b_k. \tag{B.15}$$

Since ε can be arbitrarily taken, (B.15) implies the desired inequality. □

Lemma B.3.3. *Let $\{a_n\}_n$ be an increasing sequence of positive real numbers. If we denote by $l=\limsup_{n\to\infty}\log a_n/n$, then we have $\limsup_{n\to\infty}\log(a_{tn})/n = tl$, for every $t\in\mathbb{N}\setminus\{0\}$.*

Proof. If $t=1$, then there is nothing to prove. For the sake of simplicity, we prove the result for $t=2$, but the argument can be easily generalised.

Fix $n\geq 1$ and define two sequences $\{u_k^n\}_{k\geq n}$ and $\{v_s^n\}_{s\geq 2n-1}$ as follows:

$$u_k^n = \frac{\log(a_{2k})}{k}, \quad \text{and} \quad v_s^n = \frac{\log(a_s)}{\lceil s/2\rceil}, \quad \text{for every } k\geq n \text{ and } s\geq 2n-1.$$

Since, for every $k\geq n$, $u_k^n = v_{2k}^n$, $\sup_{k\geq n} u_k^n \leq \sup_{s\geq 2n-1} v_s^n$. We want to show the opposite inequality. Let then $s\geq 2n-1$. If $s=2k$ for some $k\geq n$, then $v_s^n=u_k^n$. Otherwise, if s is odd, $v_s^n\leq u_{(s+1)/2}^n$ and thus

$$\sup_{k\geq n} u_k^n = \sup_{s\geq 2n-1} v_s^n = \sup_{s\geq 2n-1}\frac{\log(a_s)}{s}\cdot\frac{s}{\lceil s/2\rceil}. \tag{B.16}$$

Then, since (B.16) holds for every $n\geq 1$ and the sequence $\{n/\lceil n/2\rceil\}_n$ converges to 2, by applying Lemma B.3.2, we obtain that

$$\limsup_{n\to\infty}\frac{\log(a_{2n})}{n} = \lim_{n\to\infty}\frac{n}{\lceil n/2\rceil}\cdot\limsup_{n\to\infty}\frac{\log(a_n)}{n} = 2l.$$

□

Theorem B.3.4. *Let $(X,\mathcal{E})$ be a locally finite quasi-coarse space, $f\colon X\to X$ be a bornologous self-map, $\Delta_X\subseteq E\in\mathcal{E}$, and $x\in X$. Then, $\mathrm{H}_c(id,x,E^k) = k\,\mathrm{H}_c(id,x,E)$, for every $k\in\mathbb{N}$. In particular, $\mathrm{h}_c(id)\in\{0,\infty\}$.*

Proof. Fix a point $x\in X$, $\Delta_X\subseteq E\in\mathcal{E}$, and $k\in\mathbb{N}$. Consider the sequence $a_n = |\mathrm{T}_n(id,x,E)| = |E^{n-1}[x]|$, for every $n\in\mathbb{N}$. Since $\Delta_X\subseteq E$, $\{a_n\}_n$ is increasing and non-negative. Hence, Lemma B.3.3 implies the claim since $\mathrm{H}_c(id,x,E)=\limsup_{n\to\infty}\log(a_n)/n$. □

From Proposition B.1.2(a) and Theorem B.3.4 we can deduce an easy corollary.

Corollary B.3.5. *Let $(X,\mathcal{E})$ be a locally finite quasi-coarse space. Then $\mathrm{h}_c(id_X)=\infty$ if and only if there exists $x\in X$ and $F\in\mathcal{E}$ such that $\mathrm{H}_c(id_X,x,F)>0$. Moreover, if X is monogenic, $E\in\mathcal{E}$ is an entourage such that $\{E^n\mid n\in\mathbb{N}\}$ is a base of $\mathcal{E}$, and $x\in X$. Then:*

(a) $\mathrm{H}_c^{loc}(id,x)=0$ if and only if $\mathrm{H}_c(id,x,E)=0$;
(b) $\mathrm{H}_c^{loc}(id,x)=\infty$ if and only if $\mathrm{H}_c(id,x,E)>0$.

Let us now connect the coarse entropy with the growth of a metric space. Here we present a generalisation of the approach for metric spaces that can be found in [3]. A quasi-coarse space $(X,\mathcal{E})$ has a *bounded geometry skeleton* if there exists a subset Y (called *bounded geometry skeleton*) of X such that

- $(Y,\mathcal{E}|_Y)$ has bounded geometry;
- Y is Sym-large in X (i.e., Y is large in $\mathrm{Sym}(X)$).

In [3], this notion is given only for metric spaces under the name *quasi-lattice*, and a metric space is said to have *coarse bounded geometry* if it has a quasi-lattice.

Not every quasi-coarse space has a bounded geometry skeleton, as the following examples show.

Example B.3.6. (a) Let X be an infinite set. Fix a point $\overline{x}\in X$ and define the quasi-coarse structure $\mathcal{E}$ defined by the base $\{\Delta_X\cup(\{\overline{x}\}\times X)\}$. Then Δ_X is the only symmetric entourage in $\mathcal{E}$ and thus X itself is the only Sym-large subset. However X is not (uniformly) locally finite.
(b) Let X be an uncountable group (e.g., take $X=\mathbb{R}$) and consider the group ideal $[X]^{<\omega_1}=\{Y\subseteq X\mid |Y|<\omega_1\}$. For the induced group-coarse structure, if a subset Y of X is Sym-large, then it is uncountable. Then there are countable infinite subsets of Y, which are balls. Hence Y is not (uniformly) locally finite.

Question B.3.7. Does there exist a locally finite quasi-coarse space without a bounded geometry skeleton?

Let X be a quasi-coarse space having a bounded geometry skeleton. We define $\mathrm{h}_c^{bg}(X)=\mathrm{h}_c(id_Y)$, where Y is a bounded geometry skeleton of X. Note that $\mathrm{h}_c^{bg}(X)$ is well-defined. In fact, if Y and Z are two bounded geometry skeletons of X, then Y is Sym-coarsely equivalent to Z, and thus Corollary B.3.1 implies that $\mathrm{h}_c(id_Y)=\mathrm{h}_c(id_Z)$.

Because of the monotonicity of the coarse entropy under taking invariant subspaces, $\mathrm{h}_c^{bg}(X)\leq\mathrm{h}_c(id_X)$. Moreover, there exists a locally finite quasi-coarse space X such that $\mathrm{h}_c^{bg}(X)<\mathrm{h}_c(id_X)$. In order to find an example showing the strict inequality, we have to choose X without bounded geometry. Let X as in Example B.1.3(a). Then Theorem B.3.4 implies that $\mathrm{h}_c(id_X)=\infty$. However, $Y=\mathbb{N}\times\{0\}$ is Sym-large in X and $\mathrm{h}_c(id_Y)=0$. Hence $\mathrm{h}_c^{bg}(X)=0$.

Let us recall the classical notion of growth type of non-decreasing functions from $\mathbb{N}$ to $\mathbb{R}_{\geq 0}$. Let $u,v\colon\mathbb{N}\to\mathbb{R}_{\geq 0}$ be two non-decreasing functions. We say that v *dominates* u, and we write $u\preceq v$, if there are $a,b\geq 1$ and $c>0$ such that, for every $n\geq c$, $u(n)\leq av(bn)$. Then u and v *have the same growth type* ($u\cong v$) if $u\preceq v\preceq u$.

Let $u\colon\mathbb{N}\to\mathbb{R}_{\geq 0}$ be a non-decreasing function. Then the *growth type* of u is:

- *polynomial* if u is dominated by a polynomial function of some exact degree;
- *sub-exponential* if u does not dominate any exponential function $n\mapsto a^n$;
- *exponential* if u dominates an exponential function $n\mapsto a^n$.

We can characterise the previous classes of growth types as follows. If u is a non-decreasing function, then

$$\begin{gathered} u \text{ has polynomial growth type if and only if } \limsup_{n\to\infty} \frac{\log u(n)}{\log(n)} < \infty, \\ u \text{ has sub-exponential growth type if and only if } \limsup_{n\to\infty} \frac{\log u(n)}{n} = 0, \quad \text{and} \\ u \text{ has exponential growth type if and only if } 0 < \liminf_{n\to\infty} \frac{\log u(n)}{n} < \infty. \end{gathered} \tag{B.17}$$

Definition B.3.8 ([3])**.** Let (X, d) be a connected metric space having a bounded geometry skeleton. Define the *growth rate of* X as the growth type of the function

$$\gamma(n) = |B_Y(y, n)|, \quad \text{where } n \in \mathbb{N},$$

for every bounded geometry skeleton Y of X and any point $y \in Y$.

It is proved in [3] that the growth rate of a metric space does not depend on the bounded geometry skeleton and on the point chosen.

We want to estimate the growth rate of a coarse space X having a bounded geometry skeleton Y which is monogenic. However, thanks to Proposition 3.4.10(b), the existence of such a Y ensures that X itself is monogenic. Hence we can consider monogenic quasi-coarse spaces having bounded geometry skeletons.

Theorem B.3.9. *Let (X, d) be a metric space such that $(X, \mathcal{E}_d)$ is monogenic and connected, and has a bounded geometry skeleton Y. Let $E \in \mathcal{E}|_Y$ be an entourage such that $\{E^n \mid n \in \mathbb{N}\}$ is a base of $\mathcal{E}|_Y$, and $y \in Y$.*

(a) If X has polynomial growth type then $\mathrm{H}_c(id_Y, y, E) = 0$ and $\mathrm{h}_c^{bg}(X) = \mathrm{h}_c(id_Y) = 0$.
(b) X has sub-exponential growth type if and only if $\mathrm{H}_c(id_Y, y, E) = 0$ if and only if $\mathrm{h}_c^{bg}(X) = \mathrm{h}_c(id_Y) = 0$.
(c) X has exponential growth type if and only if $\mathrm{H}_c(id_Y, y, E) > 0$ if and only if $\mathrm{h}_c^{bg}(X) = \mathrm{h}_c(id_Y) = \infty$.

Proof. Item (a) and the first equivalences of items (b) and (c) follow from (B.17). Corollary B.3.5 and Proposition B.1.2(b) imply the other two equivalences. □

B.4 Relationships with other entropies

In this section we discuss the relationships of the coarse entropy with other well-known entropy notions in other branches of mathematics. In particular, we consider the coarse entropy of endomorphisms of groups and we connect it with the algebraic entropy (in §B.4.1) and, through the Pontryagin functor, with the topological and the measure entropy (in §B.4.2).

B.4.1 Relationship with the algebraic entropy

Let M be a monoid and $f\colon M \to M$ be an endomorphism of M. Fix a finite subset $K \in [M]^{<\omega}$, and $n \in \mathbb{N} \setminus \{0\}$. Then the *$n$-algebraic trajectory of f with* *respect to* K is the subset

$$\mathrm{T}_n^{alg}(f, K) = K \cdot f(K) \cdot \dots \cdot f^{n-1}(K).$$

Definition B.4.1 ([44])**.** Let M be a monoid, and $f \colon M \to M$ be an endomorphism of M. Then the *algebraic entropy of* f *with respect to* K is defined as

$$\mathrm{H}_{alg}(f, K) = \lim_{n\to\infty} \frac{\log|\mathrm{T}_n^{alg}(f, K)|}{n}. \tag{B.18}$$

Finally, the *algebraic entropy of* f is $\mathrm{h}_{alg}(f) = \sup\{\mathrm{H}_{alg}(f, K) \mid K \in [M]^{<\omega}\}$.

A standard approach to prove that the limit in (B.18) exists is by using Fekete's Lemma (see, for example, [42]). We refer to [42] for a comprehensive survey on the algebraic entropy.

In this subsection, every monoid is endowed with its monoid-quasi-coarse structure. Since every monoid endomorphism is automatically bornologous, it is natural to compare its algebraic entropy with its coarse entropy.

Theorem B.4.2. *If* M *is a monoid and* $f \colon M \to M$ *is an endomorphism, then*

$$\mathrm{h}_c(f) \leq \mathrm{h}_{alg}(f).$$

Moreover, if f *is surjective, then*

$$\mathrm{h}_c(f) = \mathrm{h}_{alg}(f).$$

Proof. Note that $\{E_K \mid K \in [M]^{<\omega}\}$ is cofinal in $\mathcal{E}_M$, and thus Proposition B.1.2(a) implies that we just need to compute $\mathrm{H}_c^{loc}(f, x) = \sup\{\mathrm{H}_c(f, x, E_K) \mid K \in [M]^{<\omega}\}$, for every $x \in M$. Moreover, since $\mathcal{Q}_M^{\downarrow}(e) = M$, Proposition B.1.2(b) implies that $\mathrm{h}_c(f) = \mathrm{H}_c^{loc}(f, e)$. For every $K \in [M]^{<\omega}$ and $n \in \mathbb{N} \setminus \{0\}$, according to (7.5), we have that

$$\begin{aligned}\mathrm{T}_n(f, e, E_K) &= (f^{n-1} \times f^{n-1})(E_K)[\cdots[(f \times f)(E_K)[E_K[x]]]\cdots] \subseteq \\ &\subseteq E_{f^{n-1}(K)}[\cdots[E_{f(K)}[E_K[e]]]\cdots] = \\ &= eKf(K)\cdots f^{n-1}(K) = \mathrm{T}_n^{alg}(f, K).\end{aligned} \tag{B.19}$$

Hence $\mathrm{H}_c(f, e, E_K) \leq \mathrm{H}_{alg}(f, K)$ and thus

$$\begin{aligned}\mathrm{h}_c(f) &= \mathrm{H}_c^{loc}(f, e) = \sup\{\mathrm{H}_c(f, e, E_K) \mid K \in [M]^{<\omega}\} \leq \\ &\leq \sup\{\mathrm{H}_{alg}(f, K) \mid K \in [M]^{<\omega}\} = \mathrm{h}_{alg}(f).\end{aligned}$$

Moreover, if f is surjective, $(f \times f)(E_K) = E_{f(K)}$, for every $K \in [M]^{<\omega}$, and thus the inclusion in (B.19) becomes an equality, which proves the desired equality. □

Thanks to Theorem B.4.2, we can reinterpret what we have obtained so far as generalisations, in the case of a surjective homomorphism of monoids, of classical results in the realm of the algebraic entropy. Results that can be seen from this point of view are Proposition B.1.2(a), Remark B.1.5(c), Proposition

B.2.3, Theorem B.2.5(a), Corollary B.2.9, Theorem B.3.4, and Theorem B.3.9 in the case of finitely generated groups (see [42] for a comprehensive survey on algebraic entropy containing the mentioned results).

Let M be a monoid and $f\colon M \to M$ be an endomorphism. We want to specialise formula (B.1) in this setting. Let K be a finite subset of M. Then

$$(g \times g)(E_K)[A] = g(E_K[g^{-1}(A)]) = g(g^{-1}(A)K) = (A \cap g(M))g(K). \quad \text{(B.20)}$$

The inequality in Theorem B.4.2 may be strict, as Examples B.4.3 and B.4.4 show.

Example B.4.3. Let G be a finitely generated group endowed with one of its word metrics d. Consider the homomorphism $f\colon \bigoplus_{n\in\mathbb{N}} G \to \bigoplus_{n\in\mathbb{N}} G$, called the *right Bernoulli shift*, such that $f(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$, for every $(x_n)_n \in \bigoplus_n G$. We claim that $\mathrm{h}_c(f) = 0$, while, for example, if $G = \mathbb{Z}_m$, the finite cyclic group with m elements, then $\mathrm{h}_{alg}(f) = \log m$.

For every $(i, m, n) \in \mathbb{N}^3$, define the subset

$$A(i, m, n) = \underbrace{\{0\} \oplus \cdots \oplus \{0\}}_{i \text{ times}} \oplus \underbrace{B_d(e, m) \oplus \cdots \oplus B_d(e, m)}_{n \text{ times}} \oplus \{0\} \oplus \cdots .$$

Note that $\{A(0, m, m)\}_{m\in\mathbb{N}}$ forms a base of the group ideal $[\bigoplus_n G]^{<\omega}$ and thus Proposition B.1.2 implies that, in order to compute the coarse entropy, we can restrict ourselves to just consider the values $\mathrm{H}_c(f, 0, E_{A(0,m,m)})$, for every $m \in \mathbb{N}$. Note that, if $m \in \mathbb{N}$, $\mathrm{T}_1(f, 0, E_{A(0,m,m)}) = A(0, m, m)$, and, moreover, we claim that, for every $0 \le n \le m$,

$$\mathrm{T}_{n+1}(f, 0, E_{A(0,m,m)}) = \prod_{i=m-n}^{m} A(n, m, i). \quad \text{(B.21)}$$

Suppose that, for some $0 \le n < m$, (B.21) holds. Then, according to (B.20),

$$\begin{aligned}
\mathrm{T}_{n+2}(f, 0, E_{A(0,m,m)}) &= \Bigg(\mathrm{T}_{n+1}(f, 0, E_{A(0,m,m)}) \cap f^{n+1}\Big(\bigoplus_{n\in\mathbb{N}} G\Big)\Bigg)\cdot \\
&\qquad \cdot f^{n+1}(A(0, m, m)) = \\
&= \Bigg(\Big(\prod_{i=m-n}^{m} A(n, m, i)\Big) \cap f^{n+1}\Big(\bigoplus_{n\in\mathbb{N}} G\Big)\Bigg)\cdot \\
&\qquad \cdot f^{n+1}(A(0, m, m)) = \\
&= \Big(\prod_{i=m-n}^{m} A(n+1, m, i-1)\Big) A(n+1, m, m) = \\
&= \prod_{i=m-(n+1)}^{m} A(n+1, m, i).
\end{aligned}$$

Hence, we have proved (B.21), for every $0 \le n \le m$. By substituting $n = m$, we have

$$\mathrm{T}_{m+1}(f, 0, E_{A(0,m,m)}) = \prod_{i=0}^{m} A(m, m, i).$$

Applying again (B.20), we can show that, for every $k \in \mathbb{N}$,

$$\mathrm{T}_{m+k+1}(f, 0, E_{A(0,m,m)}) = \prod_{i=0}^{m} A(m+k, m, i),$$

and so $|\mathrm{T}_{m+1}(f, 0, E_{A(0,m,m)})| = |\mathrm{T}_{m+k+1}(f, 0, E_{A(0,m,m)})|$. Since the cardinality of the trajectories stabilises, $\mathrm{H}_c(f, 0, E_{A(0,m,m)}) = 0$, for every $m \in \mathbb{N}$, and thus $\mathrm{h}_c(f) = 0$.

Example B.4.4. Let $G = \mathbb{Z}$, and, for $k \in \mathbb{N} \setminus \{0\}$, define the endomorphism $f = f_k \colon x \mapsto kx$ of G. We claim that $\mathrm{h}_c(f) = 0$, while $\mathrm{h}_{alg}(f) = \log k$.

The case $k = 1$ easily follows, since $f_1 = id_{\mathbb{Z}}$ and $\mathrm{h}_c(f_1) = \mathrm{h}_{alg}(f_1) = 0$ according to Theorem B.4.2, so we can assume that $k > 1$. Since G is connected, we can consider the trajectories centred at 0 (Proposition B.1.2(b)). Moreover, the family $\{[-m, m]\}_{m \in \mathbb{N}}$ forms a base of the group ideal $[G]^{<\omega}$ and thus we can restrict ourselves to considering only those induced entourages (Proposition B.1.2(a)). According to (B.19) and the definition of algebraic trajectories, for every $n \in \mathbb{N}$,

$$\mathrm{T}_n(f, 0, E_{[-m,m]}) \subseteq \mathrm{T}_n^{alg}(f, [-m, m]) = \Big[-m\Big(\sum_{i=0}^{n-1} k^i\Big), m\Big(\sum_{i=0}^{n-1} k^i\Big)\Big]. \quad \text{(B.22)}$$

Hence, (B.22) and (B.20) imply that, for every $n \in \mathbb{N}$,

$$\mathrm{T}_{n+1}(f, 0, E_{[-m,m]}) \subseteq (\mathrm{T}_n^{alg}(f, [-m, m]) \cap f^n(G)) + f^n([-m, m]),$$

and thus

$$\begin{aligned} |\mathrm{T}_{n+1}(f, 0, E_{[-m,m]})| &\leq \left| \Big[-m\Big(\sum_{i=0}^{n-1} k^i\Big), m\Big(\sum_{i=0}^{n-1} k^i\Big)\Big] \cap f^n(M) \right| \cdot (2m+1) \leq \\ &\leq \left\lceil 1 + \frac{1}{k^n} 2m \sum_{i=0}^{n-1} k^i \right\rceil (2m+1) \leq (2m+2)^2. \end{aligned}$$

Since the trajectories have bounded cardinality, $\mathrm{H}_c(f, 0, E_{[-m,m]}) = 0$ and thus $\mathrm{h}_c(f) = 0$.

Furthermore, if we consider the group $H = \mathbb{Q}$ and the endomorphism $f = f_k \colon H \to H$, since f is surjective, $\mathrm{h}_c(f) = \mathrm{h}_{alg}(f) = \log k$ (Theorem B.4.2). Hence, there is no Addition Theorem for the coarse entropy. In fact, consider the following diagram

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{i} & \mathbb{Q} & \xrightarrow{q} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle f|_{\mathbb{Z}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ \mathbb{Z} & \xrightarrow{i} & \mathbb{Q} & \xrightarrow{q} & \mathbb{Q}/\mathbb{Z}, \end{array}$$

where i and q are the canonical inclusion and quotient, respectively, and $f|_{\mathbb{Z}}$ is the restriction of f to $\mathbb{Z}$ and $\overline{f}$ is the projection of f. While $\mathrm{h}_{alg}(f) = \log k = \mathrm{h}_{alg}(f|_{\mathbb{Z}}) + \mathrm{h}_{alg}(\overline{f})$ (an instance of the Addition Theorem for the algebraic entropy, see for example [44]), $\mathrm{h}_c(f) = \log k$, while $\mathrm{h}_c(f|_{\mathbb{Z}}) = 0$ and $\mathrm{h}_c(\overline{f}) = \mathrm{h}_{alg}(\overline{f}) = 0$, since $\overline{f}$ is surjective, and thus $\mathrm{h}_c(f) \neq \mathrm{h}_c(f|_{\mathbb{Z}}) + \mathrm{h}_c(\overline{f})$.

Example B.4.5. Let F_2 be the free group freely generated by two letters a and b. Consider the injective endomorphism $f\colon F_2 \to F_2$ such that $f(a) = a^2$ and $f(b) = b^2$. We claim that $\mathrm{h}_c(f) = 0$.

Thanks to Proposition B.1.2, we can just consider the trajectories with respect to e and $E_{B(e,m)}$, where $m \in \mathbb{N}$ and F_2 is endowed with the word metric associated with the standard generating set $\{a, b, a^{-1}, b^{-1}\}$. Let us inductively define a sequence of natural numbers $\{g_n\}_n$ as follows:

$$\begin{cases} g_0 = 0, \\ g_{n+1} = m + \lfloor g_n/2 \rfloor. \end{cases}$$

By induction, it is easy to see that the sequence $\{g_n\}_n$ is bounded by $2m$.

We claim that, for every $n \in \mathbb{N}$,

$$\mathrm{T}_{n+1}(f, e, E_{B(e,m)}) = f^n(B(e, g_{n+1})). \tag{B.23}$$

If $n = 0$, then (B.23) is satisfied. Suppose that for some $n \in \mathbb{N}$, (B.23) holds. Then, applying (7.5),

$$\begin{aligned} \mathrm{T}_{n+2}(f, e, E_{B(e,m)}) &= ((f^{n+1} \times f^{n+1})(E_{B(e,m)}))[\mathrm{T}_{n+1}(f, e, E_{B(e,m)})] = \\ &= f^{n+1}(f^{-n-1}(f^n(B(e, g_{n+1})))B(e, m)) = \\ &= f^{n+1}(B(e, \lfloor g_{n+1}/2 \rfloor)B(e, m)) = f^{n+1}(B(e, g_{n+2})). \end{aligned}$$

Finally, (B.23) implies that

$$\mathrm{H}_c(f, e, E_{B(0,m)}) = \limsup_{n\to\infty} \frac{\log(|B(e, g_{n+1})|)}{n+1} \le \limsup_{n\to\infty} \frac{g_{n+1} \log 5}{n+1} = 0$$

since $g_{n+1} \le 2m$. Hence, $\mathrm{h}_c(f) = 0$.

In Examples B.4.3, B.4.4, and B.4.5 we provided an endomorphism of group $f\colon G \to G$ such that $\bigcap_n f^n(G) = \{e\}$ and $\mathrm{h}_c(f) = 0$. Hence the following question naturally arises.

Question B.4.6. Let $f\colon M \to M$ be a endomorphism of a monoid M such that $\bigcap_n f^n(M)$ is finite. Is it true that $\mathrm{h}_c(f) = 0$?

Note that, in the notation of Example B.1.6(c), even though $\bigcap_n f^n(M) = \emptyset$, $\mathrm{h}_c(f) = \infty$, but f is not an endomorphism.

B.4.2 Relationship with the topological and the measure entropy

In this subsection we want to relate the coarse entropy with the topological entropy and the measure entropy, using the Pontryagin duality. We have already noticed in Section 10.2 that the Pontryagin duality is a powerful tool to translate concepts and results from the small-scale geometry to the large-scale geometry of locally compact abelian groups. Moreover, in the realm of entropies, the Pontryagin duality plays an important role since it connects the algebraic entropy with the topological entropy (see Theorem B.4.8).

Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of a set X. Then we define

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}.$$

Let now X be a compact space. Denote by $\mathfrak{cov}(X)$ the family of all open covers of X. For every cover $\mathcal{U} \in \mathfrak{cov}(X)$, denote by $N(\mathcal{U})$ the minimum of the cardinalities of finite subcovers of $\mathcal{U}$. For every $\mathcal{U} \in \mathfrak{cov}(X)$, define the *entropy of* $\mathcal{U}$ as $\mathrm{H}(\mathcal{U}) = \log N(\mathcal{U})$.

Definition B.4.7 ([2])**.** Let $f\colon X \to X$ be a continuous self-map of a compact space. For $\mathcal{U} \in \mathfrak{cov}(X)$, define the *topological entropy of* f *with respect to* $\mathcal{U}$ as

$$\mathrm{H}_{top}(f,\mathcal{U}) = \lim_{n\to\infty} \frac{\mathrm{H}(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

Then the *topological entropy of* f is

$$\mathrm{h}_{top}(f) = \sup\{\mathrm{H}_{top}(f,\mathcal{U}) \mid \mathcal{U} \in \mathfrak{cov}(X)\}.$$

Theorem B.4.8 (Bridge theorem, [43])**.** *Let* $f\colon G \to G$ *be an endomorphism of a group* G*. Then*

$$\mathrm{h}_{alg}(f) = \mathrm{h}_{top}(\widehat{f}).$$

By combining Theorems B.4.2 and B.4.8 we obtain the following result.

Corollary B.4.9. *Let* $f\colon G \to G$ *be an endomorphism of a group* G*. Then*

$$\mathrm{h}_c(f) \le \mathrm{h}_{alg}(f) = \mathrm{h}_{top}(\widehat{f}).$$

Moreover, if f *is surjective, then*

$$\mathrm{h}_c(f) = \mathrm{h}_{alg}(f) = \mathrm{h}_{top}(\widehat{f}).$$

Before ending the section, let us connect the coarse entropy also with the measure entropy. Let $(X, \mathfrak{B}, \mu)$ be a measure space and $\xi = \{A_i \mid i = 1, \dots, n\}$ be a measurable partition of X. Define the *entropy of* ξ by

$$\mathrm{H}(\xi) = -\sum_{i=1}^{n} \mu(A_i) \log(\mu(A_i)).$$

Definition B.4.10 ([108, **?**])**.** Let X be a measure space and $f\colon X \to X$ be a measure preserving map. If ξ is a measurable partition of X, the *measure entropy of* f *with respect to* ξ is

$$\mathrm{H}_{mes}(f,\xi) = \lim_{n\to\infty} \frac{H(\xi \vee f^{-1}(\xi) \vee \cdots \vee f^{-n+1}(\xi))}{n}.$$

The *measure entropy of* f is

$$\mathrm{h}_{mes}(f) = \sup\{\mathrm{H}_{mes}(f,\xi) \mid \xi \text{ measurable partition of } X\}.$$

In the realm of compact groups, the following result was proved by Stoyanov.

Theorem B.4.11 ([164]). *Let G be a compact metrisable group and $f\colon G \to G$ be a continuous surjective homomorphism. If G is endowed with its Haar measure, then*

$$\mathrm{h}_{top}(f) = \mathrm{h}_{mes}(f).$$

Corollary B.4.12. *Let $f\colon G \to G$ be an automorphism of a countable group G. Then*

$$\mathrm{h}_c(f) = \mathrm{h}_{alg}(f) = \mathrm{h}_{top}(\widehat{f}) = \mathrm{h}_{mes}(\widehat{f}).$$

Proof. Corollary B.4.9 states the first two equalities. Since G is countable and discrete, $(G, \mathcal{E}_{\mathfrak{r}\mathcal{C}(G)}) = (G, \mathcal{E}_G)$ is metrisable ([70]). Hence Theorem 10.2.16 implies that $\widehat{G}$ is metrisable as topological group. Moreover, since f is an isomorphism, also $\widehat{f}$ is an isomorphism and so we can apply Theorem B.4.11 to get the desired claim. □

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Index

List of Symbols

$1_{\mathcal{X}}$ identity functor of the category $\mathcal{X}$, page 79

0 neutral element of an abelian group, page 13

$[X]^{<\kappa}$ family of subsets of X with cardinality strictly less than κ, page 26

$\approx$ asymorphism relation, page 192

$\mathfrak{B} \prec \mathfrak{B}'$ $\mathfrak{B}$ is finer than $\mathfrak{B}'$, page 48

$\bigoplus_i (X_i, \beta_i)$ coproduct para-bornological space, page 67

$\bigoplus_i \mathfrak{B}_i$ coproduct ballean, page 83

$\bigoplus_i \beta_i$ coproduct para-bornology, page 67

$\bigoplus_k (X_k, \mathcal{E}_k)$ coproduct entourage space, page 68

$\bigoplus_k \mathcal{E}_k$ coproduct entourage structure, page 67

$\bigoplus_k f_k$ coproduct map, page 68

$\mathrm{C} \leq \mathrm{C}'$ for every $X \in$ **Coarse** and $M \subseteq X$, $\mathrm{C}_X(M) \subseteq \mathrm{C}'_X(M)$, page 86

$\cdot$ magmatic operation, page 6

$\cdot^+$ Bohr functor, page 19

$\downarrow$ relation induced by a para-bornology, page 32

$\flat(X)$ family of subsets that are bounded in X, page 180

$\lceil \cdot \rceil$ ceiling map, page 2

$\leftrightsquigarrow$ equivalence relation generated by $\downarrow$, page 32

$\lfloor \cdot \rfloor$ floor map, page 2

$\nearrow$ transitive closure of $\uparrow$, page 32

$\overline{\cdot}$ topological closure of a subset, page 17

$\pi(G)$ $= \{p \text{ prime} \mid r_p(G) > 0\}$, page 16

$\Pi_i(X_i, \beta_i)$ product of para-bornological spaces, page 66

$\Pi_i(X_i, \mathcal{E}_i)$ product entourage space, page 66

$\Pi_i \mathfrak{B}_i$ product ballean, page 83

$\Pi_i \beta_i$ product para-bornology, page 66

$\Pi_i \mathcal{E}_i$ product entourage structure, page 66

$\Pi_i A_i$ $= \bigcap_i p_i^{-1}(A_i)$, page 66

$\Pi_i E_i$ $= \bigcap_i (p_i \times p_i)^{-1}(E_i)$, page 127

$\Pi_i f_i$ product map, page 68

$\searrow$ transitive closure of $\downarrow$, page 32

$\sim$ closeness relation, page 2

$\sim$ closeness relation, page 64

$\sim_{\mathrm{F}}$ F-closeness relation, page 65

$\sim_{\mathrm{Sym}}$ Sym-closeness, page 52

$\simeq$ isomorphism relation, page 14

$\uparrow$ inverse relation of $\uparrow$, page 32

$\vee$ join, page 31

$\wedge$ meet, page 31

$\widehat{\cdot}$ Pontryagin functor, page 17

$B^*(x, r)$ $= \{y \in X \mid x \in B(y, r)\}$, page 47

$f \leq g$ there exists h such that $f = g \circ h$, page 85

$\mathcal{A}$-exp $\mathfrak{B}$ $\mathcal{A}$-hyperballean, page 183

AbGrp category of abelian groups and homomorphisms, page 140

$\mathcal{A}$-exp X $\mathcal{A}$-coarse hyperspace of X, page 180

asdim asymptotic dimension of a coarse space, page 108

asdim asymptotic dimension of a metric space, page 4

$\mathbb{B}$ $\{0, 1\}$ with the trivial coarse structure, page 67

$\mathscr{B}$ class of groups G for which $\mathfrak{r}\mathcal{C}(G) = \mathfrak{r}\mathcal{C}(G^+)$, page 171

Ballean category of equivalence classes of balleans and bornologous maps, page 82

$\mathfrak{B}_d$ metric ballean, page 47

$B_d(x, R)$ closed metric ball centred in x with radius R, page 2

$\mathfrak{B}_{\mathcal{E}} = (X, P_{\mathcal{E}}, B_{\mathcal{E}})$ ball structure induced by an entourage structure, page 47

CGrpQ $=$ **l-CGrpQ** $\cap$ **r-CGrpQ**, page 137

$\mathfrak{cl}(\mathcal{B})$ completion of a base of a para-bornology, page 26

$\mathcal{CL}$(**Coarse**) conglomerate of all closure operators of **Coarse**, page 86

$\mathfrak{cl}(\mathcal{F})$ completion of the family $\mathcal{F}$, page 26

Coarse category of coarse spaces and bornologous maps, page 58

Coarse$/_\sim$ category of coarse spaces and equivalence classes of bornologous maps, page 66

Connected category of connected balleans and bornologous maps, page 85

D discrete closure operator, page 86

$\mathbb{D}$ $\{0,1\}$ with the discrete coarse structure, page 67

$\mathcal{D}_<(X)$ family of subsets Y of X such that $\operatorname{asdim} Y < \operatorname{asdim} X$, page 150

Δ(C) Delta-subcategory induced by the closure operator C, page 89

$d(G)$ maximal divisible subgroup of G, page 16

$d_\geq$ preorder quasi-metric, page 6

d_H^q Hausdorff quasi-metric, page 6

dim covering dimension, page 4

Discrete category of discrete balleans and bornologous maps, page 85

dsc number of connected components, page 32

d_Σ word metric, page 3

d_Σ^λ left word quasi-metric, page 7

d_Σ^ρ right word quasi-metric, page 7

Δ_X diagonal of a set X, page 5

$\mathcal{E}^{-1}$ $= \{E^{-1} \mid E \in \mathcal{E}\}$, page 39

$\mathcal{E}_{\mathfrak{B}}$ entourage structure associated to a ball structure, page 47

$\mathcal{E}_\beta^C$ $= \bigoplus_{i\in I} \mathcal{E}_{\beta_i}$, where $I = \operatorname{dsc}(X,\beta)$, page 79

$\mathcal{E}_d$ metric entourage structure, page 40

$\mathcal{E}_{dis}$ discrete coarse structure, page 39

$\mathcal{E}_G$ finitary group coarse structure, page 187

$\mathcal{E}_G$ group-coarse structure, page 216

E^G $= \bigcup_{h\in G} E^h$, page 120

E^g $= \{(g^{-1}xg, g^{-1}yg) \mid (x,y) \in E\}$, page 120

EG	$= \{(xg, yg) \mid (x,y) \in E, g \in G\}$, page 115
$\mathcal{E}_{\mathcal{I}}$	left (right) loop structure on a loop, page 116
$E_{I,\psi}$	$= \Delta_X \cup \left(\bigcup_{x\in I} \left(\{x\} \times (\{x\} \cup \psi(x))\right)\right)$, page 77
$\mathcal{E}_{\mathcal{I}}^{\lambda}$	left magmatic entourage structure, page 115
$\mathcal{E}_{\mathcal{I}}^{\rho}$	right magmatic entourage structure, page 115
$E_{J,\varphi}$	$= \Delta_X \cup \left(\bigcup_{j\in J} (i_j \times i_j)(\varphi(j))\right)$, page 68
E_K^{λ}	$= M(\{e\} \times K)$, page 115
E_K^{ρ}	$= (\{e\} \times K)M$, page 115
$\mathcal{E}_L$	$= \mathcal{E}_{\mathcal{OB}}$, left-coarse structure, page 119
$\mathcal{E}_M$	monoid-quasi-coarse structure, page 216
e_M	identity of the magmatic ideal M, page 3
Entou	category of entourage spaces and bornologous maps, page 58
$\mathrm{Epi}_{\mathcal{X}}$	class of epimorphisms of $\mathcal{X}$, page 210
$\overline{\mathcal{E}}^q$	quotient quasi-coarse structure or quotient coarse structure, page 71
$\mathcal{E}_{\mathscr{R}}$	relation entourage structure, page 49
$E^{\square}$	$= \bigcup_{n\in\mathbb{N}} E^n$, page 111
$\mathcal{E}_{triv}$	trivial or indiscrete coarse structure, page 40
$\mathcal{E}_{\mathcal{U}}$	ideal entourage structure, page 77
$\mathfrak{E}(\{X_i\}_i)$	epireflective hull of the family of objects $\{X_i\}_i$, page 90
$\mathfrak{E}(X)$	family of all entourage structures on X, page 46
$\exp f$	maps between the hyperspaces induced by f, page 180
$\exp \mathfrak{B}$	$= (\mathcal{P}(X), P, \exp B)$, hyperballean, page 183
$\exp E$	$= \mathcal{H}(E) \cap \mathcal{H}(E)^{-1}$, page 51
$\exp \mathcal{E}$	coarse hyperstructure, page 179
$\exp \mathcal{E}$	semi-coarse hyperstructure, page 51
$\exp W$	$= \{(Y,Z) \mid Y \subseteq W[Z],\ Z \subseteq W[Y]\}$, page 20
$\mathrm{ExtMono}_{\mathcal{X}}$	class of extremal monomorphisms of $\mathcal{X}$, page 211
$\overline{f}$	map induced on the power sets, page 51
$\widehat{f}$	dual homomorphism, page 17

Haus category of Hausdorff spaces and contnuous homomorphisms between them, page 11

$\mathrm{h}_c^{bg}(X)$ $=\mathrm{h}_c(id_Y)$, where Y is a bounded geometry skeleton of X, page 227

$\mathrm{h}_c(f)$ coarse entropy of f, page 216

$\mathrm{H}_c(f,x,E)$ $=\limsup_{n\to\infty}(\log|\mathrm{T}_n(f,x,E)|)/n$, page 216

$\mathrm{H}_c^{loc}(f,x)$ local entropy of f in x, page 216

$\mathcal{H}(\mathcal{E})$ entourage hyperstructure, page 51

$\mathcal{H}(E)$ $=\{(A,B)\mid B\subseteq E[A]\}$, page 51

$\mathbb{H}_\kappa$ Hamming space, page 109

$\mathbb{H}_\kappa^*$ Hamming space without the zero-function, page 109

$\mathrm{h}_{mes}(f)$ measure entropy of X, page 233

$\mathrm{h}_{top}(f)$ topological entropy of f, page 233

$\mathcal{I}^{-1}$ $=\{K^{-1}\mid K\in\mathcal{I}\}$, page 115

$\mathcal{I}_i^0$ $=\mathfrak{cl}(\mathcal{B}_i^0)$, page 149

$\mathcal{I}_{i,\kappa}$ $=\mathfrak{cl}(\mathcal{B}_{i,\kappa})$, page 149

i_k canonical inclusion of a set into the disjoint union, page 66

$\imath$ $:X\to\mathcal{P}(X)$ such that, for every $x\in X$, $\imath(x)=\{x\}$, page 20

$\mathcal{I}soL(G)$ isolated points of $L(G)$, page 193

κ-**CGrp** full subcategory of **CGrp** whose objects are groups endowed with κ-group coarse structures, page 137

κ-**CGrpQ** full subcategory of **CGrpQ** whose objects are groups endowed with κ-group coarse structures, page 137

κ-$\mathcal{L}(G)$ family of κ-Lindelöf subsets of G, page 119

K^G $=\bigcup_{h\in G}K^h$, page 120

K^g $=g^{-1}Kg$, page 120

$\mathcal{LA}(X)$ family of subsets that are large in X, page 180

LCA category of locally compact abelian groups and continuous homomorphisms, page 17

l-CGrp category of left coarse groups and bornologous homomorphisms, page 137

l-CGrpQ category of left coarse groups and bornologous quasi-homomorphisms, page 137

$\ell\text{-}\exp\mathfrak{B}_G$ logarithmic hyperballean, page 188

UBounded category of balleans whose connected components are uniformly bounded and bornologous maps, page 85

$U \circ V$ composite of two entourages, page 4

$\mathcal{U}_d$ metric uniformity, page 5

$\mathrm{U}_{\mathcal{X}}$ forgetful functors from $\mathcal{X}$ to **Set**, page 58

$U[x]$ ball centred in x with radius U, page 8

$\mathrm{U}_{\mathcal{X}}^{\mathcal{Y}}$ forgetful functor from $\mathcal{Y}$ to $\mathcal{X}$, page 58

$w\mathscr{G}$ class of weakly Glicksberg groups, page 170

$W_{\widehat{G}}(K,U)$ $=\{\chi \in \widehat{G} \mid \chi(K) \subseteq U\}$, page 17

$X/\mathcal{E}$ family of morphisms of $\mathcal{E}$ whose domain is X, page 92

x^G conjugacy class of x, page 120

X_n $=\{x \in G \mid o(x) = n\}$, page 200

$\mathcal{X}[\mathcal{W}^{-1}]$ localisation of $\mathcal{X}$ by $\mathcal{W}$, page 143

$\mathbb{Z}$ set of integer numbers, page 1

$\mathscr{Z}$ class of coarse cellular groups, page 142

$Z(G)$ centre of G, page 120

$\mathbb{Z}_{p^\infty}$ Prüfer p-group, page 16