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# CW-complex Nagata idealizations [☆]

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We introduce a construction which allows us to identify the elements of the skeletons of a CW-complex $P(m)$ and the monomials in $m$ variables. From this, we infer that there is a bijection between finite CW-subcomplexes of $P(m)$, which are quotients of finite simplicial complexes, and certain bigraded standard Artinian Gorenstein algebras, generalizing previous constructions of Faridi and ourselves.

We apply this to a generalization of Nagata idealization for level algebras. These algebras are standard graded Artinian algebras whose Macaulay dual generator is given explicitly as a bigraded polynomial of bidegree $(1, d)$. We consider the algebra associated to polynomials of the same bidegree $(d_1, d_2)$.



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## 0. Introduction

Let $X = V(f) \subset \mathbb{P}^N_{\mathbb{K}}$ be a hypersurface, where the underlying field $\mathbb{K}$ has characteristic 0; the *Hessian determinant of* $f$ (which we call *the Hessian of* $f$ or *the Hessian of* $X$) is the *determinant of the Hessian matrix of* $f$.

Hypersurface with vanishing Hessian were studied for the first time in 1851 by O. Hesse; he wrote two papers ([10,11]) according to which these hypersurfaces should be necessarily cones. In 1876 Gordan and Noether ([7]) proved that Hesse's claim is true for $N \leq 3$, and it is false for $N \geq 4$. They and Franchetta classified all the counterexamples to Hesse's claim in $\mathbb{P}^4$ (see [7,3,4,6]). In 1900, Perazzo classified cubic hypersurfaces with vanishing Hessian for $N \leq 6$ ([14]). This work was studied and generalized in [5], and the problem is still open in spaces of higher dimension.

Hessians of higher degree were introduced in [13] and used to control the so called *Strong Lefschetz Properties* (for short, *SLP*). The Lefschetz properties have attracted a great attention in the last years. The basic papers of the algebraic theory of Lefschetz properties were the original ones of Stanley [15–17] and the book of Watanabe and others [8].

An algebraic tool that occurs frequently in these papers is the *Nagata Idealization*: it is a tool to convert any module $M$ over a (commutative) ring (with unit) $R$ to an ideal of another ring $R \ltimes M$. The starting point is the isomorphism between the idealization of an ideal $I = (g_0, \ldots, g_n)$ of $\mathbb{K}[u_1, \ldots, u_m]$ and its level algebra see [8, Definition 2.72]. In this way, the new ring is a *Standard Graded Artinian Gorenstein Algebra* (*SGAG algebra*, for short). An explicit formula for the Macaulay generator $f$ is:

$$f = x_0 g_0 + \cdots + x_n g_n \in \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]_{(1,d)}.$$

A generalization of this construction is to consider polynomials of the form:

$$f = x_0^d g_0 + \cdots + x_n^d g_n \in \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]_{(d,d+1)};$$

these are called *Nagata polynomials of degree d*. The Lefschetz properties for the relevant associated algebras $A$, the geometry of *Nagata hypersurfaces of degree d*, the interaction between the combinatorics of $f$ and the structure of $A$ were studied in [1], where the $g_i$'s are square free monomials, using a simplicial complex associated to $f$.

In this paper we use the *CW-complexes*, to study Nagata polynomials of bidegree $(d_1, d_2)$. We study the Hilbert vector and we give a complete description of the ideal $I$ for every case, also if the $g_i$'s are not square free monomials.

The geometry of the Nagata hypersurface is very similar to the geometry of the hypersurfaces with vanishing Hessian.

More precisely, we introduce a new Construction 3.10 which allows us to identify each (monic) monomial of degree $d$ in $m$ variables with an element of the $(d-1)$-skeleton of a CW-complex that we call $P(m)$. This CW-complex is constructed by generalizing the construction introduced in [2] which associates to a (monic) square-free monomial in $m$ variables of degree $d$ a unique $(d-1)$-cell of the simplex of dimension $m-1$, and vice versa. We consider an $h$-power $u_i^h$ as a product of $h$ linear forms: $\tilde{u}_1 \cdots \tilde{u}_h$; this corresponds to a $(h-1)$-simplex, and we identify all the $\delta$-faces, with $\delta < h-1$, of this simplex to just one $\delta$-face, recursively, starting from $\delta = 0$ to $\delta = h-2$: for $\delta = 0$ we identify all the points to one, then if $\delta = 1$ we obtain a bouquet of $h$-circles, and we identify all these circles, and so on. Generalizing this construction to a general monic monomial and attaching the corresponding CW-complexes along the common skeletons, we obtain $P(m)$.

The paper is organized as follows: in Section 1 we recall some generalities about graded Artinian Gorenstein Algebras and Lefschetz Properties, with their connections with the vanishings of higher order Hessians. In Section 2 we recall what the Nagata idealization is, what we intend for a higher Nagata idealization and we show its connection with the Lefschetz Properties for bihomogeneous polynomials. Section 3 is the core of this article. After recalling generalities about bigraded algebras and the topological definitions that we need, we give the construction of the CW-complex $P(m)$; then, we apply it to the Nagata polynomials (Definition 2.5) in Theorems 3.16 and 3.18, which give Theorem 3.16 a precise description of the Artinian Gorenstein Algebra associated to a Nagata polynomial and Theorem 3.18 the generators of the annihilator of the polynomial. We show that from these theorems a generalization of the principal results of [1] follows: Corollaries 3.17 and 3.19.

We think that the study of the Nagata hypersurfaces can be—among other things—a useful tool for the classification of the hypersurfaces with vanishing Hessian in $\mathbb{P}^n$.

**Notations.** In this the paper we fix the following notations and assumptions:

- $\mathbb{K}$ is a field of characteristic 0.
- $R := \mathbb{K}[x_0, \dots, x_n]$ will always be the ring of polynomials in $n+1$ variables $x_0, \dots, x_n$.
- $Q := \mathbb{K}[X_0, \dots, X_n]$ will be the ring of differential operators of $R$, where $X_i = \dfrac{\partial}{\partial x_i}$.
- The subscript of a graded $\mathbb{K}$-algebra will indicate the part of that degree; $R_d$ is the $\mathbb{K}$-vector space of the homogeneous polynomials of degree $d$, and $Q_\delta$ the $\mathbb{K}$-vector space of the homogeneous differential operators of order $\delta$.

## 1. Graded Artinian Gorenstein algebras and Lefschetz properties

### 1.1. Graded Artinian Gorenstein algebras are Poincaré algebras

**Definition 1.1.** Let $I$ be a homogeneous ideal of $R$ such that $A = R/I = \displaystyle\bigoplus_{i=0}^{d} A_i$ is a graded Artinian $\mathbb{K}$-algebra, where $A_d \neq 0$. The integer $d$ is the *socle degree of* $A$. The algebra $A$ is said *standard* if it is generated in degree 1. Setting $h_i = \dim_{\mathbb{K}} A_i$, the vector

$\mathrm{Hilb}(A) = (1, h_1, \dots, h_d)$ is called *Hilbert vector of* $A$. Since $I_1 = 0$, then $h_1 = n + 1$ is called *codimension of* $A$.

We also recall the following definitions.

**Definition 1.2.** A graded Artinian $\mathbb{K}$-algebra $A = \bigoplus_{i=0}^{d} A_i$ is a *Poincaré algebra* if $\cdot\colon A_i \times A_{d-i} \to A_d$ is a *perfect pairing* for $i \in \{0, \dots, d\}$.

**Definition 1.3.** A graded Artinian $\mathbb{K}$-algebra $A$ is *Gorenstein* if (and only if) $\dim_{\mathbb{K}} A_d = 1$ and it is a Poincaré algebra.

**Remark 1.4.** The Hilbert vector of a Poincaré algebra $A$ is *symmetric with respect to* $h_{\lfloor \frac{d}{2} \rfloor}$, that is $\mathrm{Hilb}(A) = (1, h_1, h_2, \dots, h_2, h_1, 1)$. $\diamond$

*1.2. Graded Artinian Gorenstein quotient algebras of Q*

For any $d \geq \delta \geq 0$ there exists a natural $\mathbb{K}$-bilinear map $B\colon R_d \times Q_\delta \to R_{d-\delta}$ defined by differentiation

$$B(f, \alpha) = \alpha(f)$$

**Definition 1.5.** Let $I = \langle f_1, \dots, f_\ell \rangle$—where $f_1, \dots, f_\ell$ are forms in $R$—be a finite dimensional $\mathbb{K}$-vector subspace of $R$. The *annihilator of* $I$ *in* $Q$ is the following homogeneous ideal

$$\mathrm{Ann}(I) := \{\alpha \in Q \mid \forall f \in I, \alpha(f) = 0\}.$$

In particular, if $I$ is generated by a homogeneous element $f$, we write $\mathrm{Ann}(I) = \mathrm{Ann}(f)$.

Let $A = Q/\mathrm{Ann}(f)$, where $f$ is homogeneous. By construction $A$ is a standard graded Artinian $\mathbb{K}$-algebra; moreover $A$ is Gorenstein.

**Theorem 1.6** *([12], §60ff, [13], theorem 2.1 ). Let $I$ be a homogeneous ideal of $Q$ such that $A = Q/I$ is a standard Artinian graded $\mathbb{K}$-algebra. Then $A$ is Gorenstein if and only if there exist $d \geq 1$ and $f \in R_d$ such that $A \cong Q/\mathrm{Ann}(f)$.*

**Remark 1.7.** Using the notation as above, $A$ is called the *SGAG* $\mathbb{K}$*-algebra associated to* $f$. The socle degree $d$ of $A$ is the degree of $f$ and the codimension is $n+1$, since $I_1 = 0$. $\diamond$

*1.3. Lefschetz properties and the Hessian criterion*

Let $A = \displaystyle\bigoplus_{i=0}^{d} A_i$ be a graded Artinian $\mathbb{K}$-algebra.

**Definition 1.8.** If there exists an $L \in A_1$ such that:

(1) The multiplication map $\cdot L\colon A_i \to A_{i+1}$ is of maximal rank for all $i$, then $A$ has the *Weak Lefschetz Property* (*WLP*, for short);
(2) The multiplication map $\cdot L^k\colon A_i \to A_{i+k}$ is of maximal rank for all $i$ and $k$, then $A$ has the *Strong Lefschetz Property* (*SLP*, for short);

**Definition 1.9.** Let $A$ be the SGAG $\mathbb{K}$-algebra associated to an element $f \in R_d$, and let $\mathcal{B}_k = \{\alpha_j \in A_k \mid j \in \{1,\dots,\sigma_k\}\}$ be an ordered $\mathbb{K}$-basis of $A_k$. The *k-th Hessian matrix* of $f$ with respect to $\mathcal{B}_k$ is

$$\mathrm{Hess}_f^k = (\alpha_i\alpha_j(f))_{i,j=1}^{\sigma_k}.$$

The *k-th Hessian* of $f$ with respect to $\mathcal{B}_k$ is

$$\mathrm{hess}_f^k = \det\left(\mathrm{Hess}_f^k\right).$$

**Theorem 1.10** *([18] Theorem 4). An element $L = a_0X_0 + \cdots + a_nX_n \in A_1$ is a* strong Lefschetz element *of $A$ if and only if $\mathrm{hess}_f^k(a_0,\dots,a_n) \neq 0$ for all $k \in \left\{0,\dots,\left\lfloor \frac{d}{2}\right\rfloor\right\}$. In particular, if for some $k$ one has $\mathrm{hess}_f^k = 0$, then $A$ does not have SLP.*

## 2. Higher order Nagata idealization

### 2.1. Nagata idealizations

**Definition 2.1.** Let $A$ be a ring and let $M$ be an $A$-module. The *Nagata idealization* $A \ltimes M$ of $M$ is the ring with underlying set $A \times M$ and operations defined as follows:

$$(r,m) + (s,n) = (r+s, m+n),\ (r,m)\cdot(s,n) = (rs, sm+rn).$$

#### 2.1.1. Bigraded Artinian Gorenstein algebras

Let $A = \bigoplus_{i=0}^{d} A_i$ be a SGAG $\mathbb{K}$-algebra, it is *bigraded* if:

$$A_d = A_{(d_1,d_2)} \cong \mathbb{K},\ A_i = \bigoplus_{h=0}^{i} A_{(i,h-i)} \text{ for } i \in \{0,\dots,d-1\},$$

since $A$ is a Gorenstein ring, and the pair $(d_1,d_2)$ is said the *socle bidegree of* $A$. In this case we call $A$ an *SBAG algebra*.

**Remark 2.2.** By Definition 1.3, $A_i \cong A_{d-i}^{\vee} = \mathrm{Hom}_{\mathbb{K}}(A_{d-i},\mathbb{K})$ and since the duality commutes with direct sums, one has $A_{(i,j)} \cong A_{(d_1-i,d_2-j)}^{\vee}$ for any pair $(i,j)$. ◇

We fix notation as in Theorem 2.4:

- $S := R \otimes_{\mathbb{K}} \mathbb{K}[u_1, \ldots, u_m] = \mathbb{K}[x_0, \ldots, x_n, u_1, \ldots, u_m]$ is the bigraded ring of polynomials in $m + n + 1$ variables $x_0, \ldots, x_n, u_1, \ldots, u_m$;
- We have chosen the natural bigrading of $S$: $x_i$ has bidegree $(1, 0)$ and $u_j$ has bidegree $(0, 1)$;
- Define $S_{(d_1,d_2)}$ to be the $\mathbb{K}$-vector space of bihomogeneous polynomials $f$ of bidegree $(d_1, d_2)$; that is, $f$ can be written as $\sum_{i=0}^{p} a_i b_i$, where $a_i \in R_{d_1} = \mathbb{K}[x_0, \ldots, x_n]_{d_1}$ and $b_i \in \mathbb{K}[u_1, \ldots, u_m]_{d_2}$.
- $T := Q \otimes_{\mathbb{K}} \mathbb{K}[U_1, \ldots, U_m] = \mathbb{K}[X_0, \ldots, X_n, U_1, \ldots, U_m]$ is the (bigraded) ring of differential operators of $S$, where $X_i = \dfrac{\partial}{\partial x_i}$ and $U_j = \dfrac{\partial}{\partial u_j}$; $X_i$ has bidegree $(1, 0)$ and $U_j$ has bidegree $(0, 1)$.

A homogeneous ideal $I$ of $S$ is a *bihomogeneous ideal* if:

$$I = \bigoplus_{i,j=0}^{\infty} I_{(i,j)}, \text{ where } \forall i, j \in \mathbb{N}_{\geq 0},\ I_{(i,j)} = I \cap S_{(i,j)}.$$

Let $f \in S_{(d_1,d_2)}$, then $I = \mathrm{Ann}(f)$ is a bihomogeneous ideal and using Theorem 1.6, $A = T/(\mathrm{Ann}(f))$ is a SBAG $\mathbb{K}$-algebra of socle bidegree $(d_1, d_2)$ (and codimension $m+n+1$).

**Remark 2.3.** Using the above notations, one has:

$$\forall i > d_1, j > d_2,\ I_{(i,j)} = T_{(i,j)}.$$

Indeed, for all $\alpha \in T_{(i,j)}$ with $i > d_1, j > d_2$, $\alpha(f) = 0$; as a consequence:

$$\forall k \in \{0, \ldots, d_1 + d_2\},\ A_k = \bigoplus_{\substack{0 \leq i \leq d_1 \\ 0 \leq j \leq d_2 \\ i+j=k}} A_{(i,j)}.$$

Moreover, the evaluation map $\alpha \in T_{(i,j)} \mapsto \alpha(f) \in A_{(d_1-i,d_2-j)}$ provides the following short exact sequence:

$$0 \longrightarrow I_{(i,j)} \longrightarrow T_{(i,j)} \longrightarrow A_{(d_1-i,d_2-j)} \longrightarrow 0. \qquad \diamond \tag{1}$$

The following theorem, which links Nagata idealizations with bihomogeneous polynomials, holds.

**Theorem 2.4** *([8], Theorem 2.77). Let $S' := \mathbb{K}[u_1, \ldots, u_m]$ and $S := R \otimes_{\mathbb{K}} S'$ be rings of polynomials, let $T' = \mathbb{K}[U_1, \ldots, U_m]$ and $T := Q \otimes_{\mathbb{K}} T'$ be the associated ring of* *differential operators, where* $X_i = \dfrac{\partial}{\partial x_i}$ *and* $U_j = \dfrac{\partial}{\partial u_j}$. *Let* $g_0, \dots, g_n$ *be homogeneous elements of* $S'$ *of degree* $d$, *let* $I$ *be the* $T'$*-submodule of* $S'$ *generated by* $\{\partial(g_i) \in R \mid \partial \in T, i \in \{0, \dots, n\}\}$ *and let* $A' := T'/\operatorname{Ann}(I)$. *Define* $f = x_0 g_0 + \cdots + x_n g_n \in R$, *it is a bihomogeneous polynomial of bidegree* $(1, d)$, *and let* $A := T/\operatorname{Ann}(f)$. *Considering* $I$ *as an* $A'$*-module,* $A' \ltimes I \cong A$.

### 2.2. Lefschetz properties for higher Nagata idealizations

**Definition 2.5.** A bihomogeneous polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$$

is called a *CW-Nagata polynomial of degree* $d_1 \geq 1$ if $g_i \in \mathbb{K}[u_1, \dots, u_m]$, $i = 0, \dots, n$, are linearly independent monomials of degree $d_2 \geq 2$.

**Remark 2.6.** One needs $n \leq \binom{m+d_2-1}{d_2}$ otherwise the $g_i$ cannot be linearly independent.

From now on, we assume that $n$ satisfies this condition. ◇

We will need the following propositions.

**Proposition 2.7** *([4] Proposition 2.5). Let* $n+1 \geq m \geq 2, d_2 > d_1 \geq 1$ *and* $s > \binom{m+d_1-1}{d_1}$; *for any* $j \in \{1, \dots, s\}$, *let* $f_j \in S_{(d_1,0)}$, $g_j \in S_{(0,d_2)}$. *Then the form* $f = f_1 g_1 + \cdots + f_s g_s$ *of degree* $d_1 + d_2$ *satisfies*

$$\operatorname{hess}_f^{d_1} = 0;$$

*that is,* $A = T/\operatorname{Ann}(f)$ *does not have the SLP condition.*

**Proposition 2.8** *([1] Proposition 2.7). Let* $n+1 \geq m \geq 2, d_1 \geq d_2$. *Then* $L = \sum_{i=0}^{n} X_i$ *is a* Weak Lefschetz Element*; that is,* $A = T/\operatorname{Ann}(f)$ *has the WLP condition.*

## 3. CW-complex Nagata idealization of bidegree $(d_1, d_2)$

Let $S$ and $T$ be as in the previous subsection.

**Definition 3.1.** A bihomogeneous CW-Nagata polynomial

$$f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$$

is called a *simplicial Nagata polynomial of degree* $d_1$ if the monomials $g_i$ are square free.

**Remark 3.2.** One needs $n \leq \binom{m}{d_2}$ otherwise the $g_i$ cannot be square free. ◇

*3.1. CW-complexes and bihomogeneous polynomials*

*3.1.1. Abstract finite simplicial complexes*

**Definition 3.3.** Let $V = \{u_1, \dots, u_m\}$ be a finite set. An *abstract simplicial complex* $\Delta$ *with vertex set* $V$ is a subset of $2^V$ such that

(1) $\forall u \in V \Rightarrow \{u\} \in \Delta$,
(2) $\forall \sigma \in \Delta, \tau \subsetneq \sigma, \tau \neq \emptyset \Rightarrow \tau \in \Delta$.

The elements $\sigma$ of $\Delta$ are called *faces* or *simplices*; a face with $q+1$ vertices is called *q-face* or *face of dimension* $q$ and one writes $\dim \sigma = q$; the maximal faces (with respect to the inclusion) are called *facets*; if all facets have the same dimension $d \geq 1$ then one says that $\Delta$ is *of pure dimension* $d$. The set $\Delta^k$ of faces of dimension at most $k$ is called *k-skeleton of* $\Delta$. $2^V$ is called *simplex* (of dimension $m-1$).

**Remark 3.4.**

(1) (cfr. [1, Remark 3.4]) There is a natural bijection, introduced in [2], between the square free monomials, of degree $d$, in the variables $u_1, \dots, u_m$ and the $(d-1)$-faces of the simplex $2^V$, with vertex set $V = \{u_1, \dots, u_m\}$. In fact, a square free monomial $g = u_{i_1} \cdots u_{i_d}$ corresponds to the subset $\{u_{i_1}, \dots, u_{i_d}\}$ of $2^V$. Vice versa, to any subset $F$ of $V$ with $d$ elements one associates the free square monomial $m_F = \prod_{u_i \in F} u_i$ of degree $d$.
(2) Let $f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$ be a simplicial Nagata polynomial; by hypothesis there is bijection between the monomials $g_i$ and the indeterminates $x_i$. From this, we can associate to $f$ a simplicial complex $\Delta_f$ with vertices $u_1, \dots, u_m$ where the facet which corresponds to $g_i$ is identified with $x_i^{d_1}$. ◇

*3.1.2. CW-complexes*

For the topological background, we refer to [9]. We start by fixing some notations.

**Definition 3.5.** Let $k \in \mathbb{N}_{\geq 1}$. A topological space $e^k$ homeomorphic to the open (unitary) ball $\{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 < 1\}$ of dimension $k$ (with the natural topology induced by $\mathbb{R}^{k+1}$) is called a *k-cell*. Its boundary, i.e. the $(k-1)$-*dimensional sphere* will be denoted by $\mathbb{S}^{k-1} = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 = 1\}$ and its closure, i.e.

the closed (unitary) $k$-dimensional disk will be denoted by $\mathbb{D}^k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_1^2 + \cdots + x_k^2 \leq 1\}$.

We reall the following

**Definition 3.6.** A *CW-complex* is a topological space $X$ constructed in the following way:

(1) There exists a fixed and discrete set of points $X^0 \subset X$, whose elements are called 0-*cells*;
(2) Inductively, the *k-skeleton* $X^k$ *of* $X$ is constructed from $X^{k-1}$ by attaching $k$-cells $e_\alpha^k$ (with index set $A_k$) via continuous maps $\varphi_\alpha^k \colon \mathbb{S}_\alpha^{k-1} \to X^{k-1}$ (the *attaching maps*). This means that $X^k$ is a quotient of $Y^k = X^{k-1} \bigcup_{\alpha \in A_k} \mathbb{D}_\alpha^k$ under the identification $x \sim \varphi_\alpha(x)$ for $x \in \partial \mathbb{D}_\alpha^k$; the *elements of the k-skeleton* are the (closure of the) attached $k$ cells;
(3) $X = \bigcup_{k \in \mathbb{N}_{\geq 0}} X^k$ and a subset $C$ of $X$ is closed if and only if $C \cap X^k$ is closed for any $k$ (*closed weak topology*).

**Definition 3.7.** A subset $Z$ of a CW-complex $X$ is a *CW-subcomplex* if it is the union of cells of $X$, such that the closure of each cell is in $Z$.

**Definition 3.8.** A CW-complex is *finite* if it consists of a finite number of cells.

We will be interested mainly in finite CW-complexes.

**Example 3.9** *(Geometric realization of an abstract simplicial complex).* It is an obvious fact that to any simplicial complex $\Delta$ one can associate a finite CW-complex $\widetilde{\Delta}$ via the *geometric realization* of $\Delta$ as a simplicial complex (as a topological space) $\widetilde{\Delta}$. △

In what follows we will always identify abstract simplicial complexes with their corresponding simplicial complexes.

**Construction 3.10.** In Remark 3.4, we saw that to any degree $d$ square-free monomial $u_{i_1} \cdots u_{i_d} \in \mathbb{K}[u_1, \ldots, u_m]_d$ one can associate the $(d-1)$-face $\{u_{i_1}, \ldots, u_{i_d}\}$ of the abstract $(m-1)$-dimensional simplex $\Delta(m) := 2^{\{u_1, \ldots, u_m\}}$, and vice versa: if we call

$$\rho_d := \{f \in \mathbb{K}[u_1, \ldots, u_m]_d \mid f \neq 0 \text{ is a square-free monic monomial}\}$$
$$D(m)_d := \Delta(m)^d \setminus \Delta(m)^{d-1},$$

we have a bijection

$$\sigma_d \colon \rho_d \to D(m)_d$$

$$u_{i_1}\cdots u_{i_d} \mapsto \{u_{i_1},\dots,u_{i_d}\}\,.$$

Alternatively, we can associate to $u_{i_1}\cdots u_{i_d}$ the element of the $(d-1)$-skeleton $\overline{\{u_{i_1},\dots,u_{i_d}\}} \in \widetilde{\Delta(m)}^{d-1}$, so we have a bijection

$$\sigma_d\colon \rho_d \to \widetilde{\Delta(m)}^{d-1}$$
$$u_{i_1}\cdots u_{i_d} \mapsto \overline{\{u_{i_1},\dots,u_{i_d}\}}$$

between the square-free monomials and the $(d-1)$-faces of the (topological) simplex $\widetilde{\Delta(m)}$.

Using CW-complexes, we will extend this construction to the *non-square-free monic monomials.* We proceed as follows. Let $g := u_1^{j_1}\cdots u_m^{j_m}$ be a generic degree $d := j_1+\cdots+j_m$ monomial. Consider the following finite set: $W := \left\{u_1^1,\dots,u_1^{j_1},\dots,u_m^1,\dots,u_m^{j_m}\right\}$, and if $\Delta(d) := 2^W$ is the abstract associated (finite) simplex, we consider the corresponding (topological) simplex (which is a CW-complex) $\widetilde{\Delta(d)}$.

If $j_k \le 1$ we do nothing, while if $j_k \ge 2$, we recursively identify, for $\ell$ varying from 0 to $j_k-2$, the $\ell$-faces of the subsimplex $2^{\widetilde{\left\{u_k^1,\dots,u_k^{j_k}\right\}}} \subset \widetilde{\Delta(d)}$: start with $\ell=0$, and we identify all the $j_k$ points to one point—call it $u_k$. Then, for $\ell=1$, we obtain a bouquet of $\binom{j_k+1}{2}$ circles, and we identify them in just one circle $\mathbb{S}^1$ passing through $u_k$, and so on, up to the facets of $2^{\widetilde{\left\{u_k^1,\dots,u_k^{j_k}\right\}}}$, i.e. its $j_k+1$ $(j_k-1)$-faces, which, by the construction, have all their boundary in common, and we identify all of them.

Make all these identifications for all $j_1,\dots,j_m$; in this way, we obtain a finite CW-complex $X = X_g$ of dimension $d-1$, with 0-skeleton $X^0 = \{u_i \mid j_i \neq 0\} \subset \{u_1,\dots,u_m\}$, obtained from the $(d-1)$-dimensional simplex $\widetilde{\Delta(d)}$, with the above identification.

In this way, we obtain a finite CW-complex $X = X_g$ of dimension $d-1$, with 0-skeleton $X^0 = \{u_i \mid j_i \neq 0\} \subset \{u_1,\dots,u_m\}$, obtained from the $(d-1)$-dimensional simplex $\widetilde{\Delta(d)}$, with the above identification. Under this identification each closure of a $(j_k-1)$-cell $\overline{\left\{u_k^1,\dots,u_k^{j_k}\right\}}$ becomes a point if $j_k=1$, a circle $\mathbb{S}^1$ if $j_k=2$, a topological space with fundamental group $\mathbb{Z}_3$ if $j_k=2$ (i.e. it is not a topological surface), etc. We will denote these spaces in what follows by $\epsilon_k^{j_k-1}$, i.e. $\epsilon_k^{j_k-1}$ corresponds to $u_k^{j_k}$, and vice versa:

**Proposition 3.11.** *Every power in $u_1^{j_1}\cdots u_m^{j_m}$ (up to a permutation of the variables) corresponds to a $\epsilon_k^{j_k-1}$, and vice versa.*

We can see $X_g$ as a $(d-1)$-dimensional *join* between these spaces $\epsilon_k^{j_k-1}$ and the span of the 0-skeleton $X^0$ i.e. the simplex $S_X \subset \widetilde{\Delta(m)}$ associated to it; $S_X \cong \widetilde{\Delta(\ell)}$, where $\ell = \#X^0 \le m$.

**Remark 3.12.** This last observation suggests we consider an alternative construction: recall that the cellular decomposition of the real projective space is obtained attaching a single cell at each passage; indeed, $\mathbb{P}^n_{\mathbb{R}}$ is obtained from $\mathbb{P}^{n-1}_{\mathbb{R}}$ by attaching one $n$-cell with the quotient projection $\varphi^{n-1}\colon \mathbb{S}^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$ as the attaching map.

Then, to each power $u_k^{j_k}$ we associate a real projective space of dimension $j_k - 1$ $\mathbb{P}_k^{j_k-1}$ and immersions $i_{k-1}\colon \mathbb{P}_k^{j_k-1} \hookrightarrow \mathbb{P}_k^{j_k}$; so $\mathbb{P}_k^0 = u_k \in \mathbb{P}_k^{j_k-1}$.

Finally, to $g = u_1^{j_1} \cdots u_m^{j_m}$ we associate the join between the $\mathbb{P}_k^{j_k-1}$ and the $S_X$ defined above; if we call this join by $X_g$, we can proceed in an equivalent way, by changing $\epsilon_k^{j_k-1}$ with $\mathbb{P}_k^{j_k-1}$. $\diamond$

It is clear how to glue two of these finite CW-complexes—say $X = X_{u_1^{j_1}\cdots u_m^{j_m}}$ and $Y = Y_{u_1^{k_1}\cdots u_m^{k_m}}$, of degree $d = j_1 + \cdots + j_m$ and $d' = k_1 + \cdots + k_m$—along $\widetilde{\Delta(m)}$: we simply attach $X$ and $Y$ via the inclusion maps $S_X \subset \widetilde{\Delta(m)}$ and $S_Y \subset \widetilde{\Delta(m)}$, where $S_X$ and $S_Y$ are the simplexes associated to, respectively, $X$ and $Y$.

Finally, taking all these finite CW-complexes together, we obtain a CW-complex $P$ in the following way:

$$C := \bigsqcup_{u_1^{j_1}\cdots u_m^{j_m} \in \mathbb{K}[u_1,\dots,u_m]} X_{u_1^{j_1}\cdots u_m^{j_m}} \qquad P(m) := C/{\sim}$$

where $\sim$ is the equivalence relation induced by the above gluing.

**Proposition 3.13.** *There is bijection between the monomials of degree $d$ in $\mathbb{K}[u_1, \dots, u_m]$ and the elements of the $(d-1)$-skeleton of $P(m)$.*

In other words, if we define

$$\rho'_d := \{f \in \mathbb{K}[u_1, \dots, u_m]_d \mid f \neq 0 \text{ is a monic monomial}\}$$

we have a bijection, using the above notation

$$\begin{aligned} \sigma'_d\colon \rho'_d &\to P(m)_{d-1} \\ u_1^{j_1} \cdots u_m^{j_m} &\mapsto X_{u_1^{j_1}\cdots u_m^{j_m}}. \end{aligned}$$

**Proposition 3.14.** $X_{u_1^{j_1}\cdots u_m^{j_m}} \subset X_{u_1^{k_1}\cdots u_m^{k_m}}$ *if and only if* $u_1^{j_1} \cdots u_m^{j_m}$ *divides* $u_1^{k_1} \cdots u_m^{k_m}$.

Let $f = \sum_{i=0}^{n} x_i^{d_1} g_i \in S_{(d_1,d_2)}$ be a CW-Nagata polynomial; by hypothesis there is bijection between the monomials $g_i$ and the indeterminates $x_i$. From this, we can associate to $f$ a finite $(d_2-1)$-dimensional, CW-subcomplex of $P(m)$, $\Delta_f$ where the $(d_2-1)$-skeleton is given by the $X_{g_i}$'s glued together with the above procedure. Each $X_{g_i}$ can be identified with $x_i^{d_1}$ as before.

The previous construction generalizes the analogous one given in [1].

### 3.2. The Hilbert function of SBAG algebras

The first main result of this paper is the following general theorem.

**Remark 3.15.** In order to state it, we observe that the canonical bases of

$$S_{(d_1,d_2)} = \mathbb{K}[x_0,\dots,x_n]_{d_1} \otimes \mathbb{K}[u_1,\dots,u_m]_{d_2}$$

and

$$T_{(d_1,d_2)} = \mathbb{K}[X_0,\dots,X_n]_{d_1} \otimes \mathbb{K}[U_1,\dots,U_m]_{d_2}$$

given by monomials are dual bases each other, i.e.

$$X_0^{k_0}\cdots X_n^{k_n}U_1^{\ell_1}\cdots U_m^{\ell_m}(x_0^{i_0}\cdots x_n^{i_n}u_1^{j_1}\cdots u_m^{j_m}) = \delta^{i_0,\dots,i_n,j_1,\dots,j_m}_{k_0,\dots,k_n,\ell_1,\dots,\ell_m}$$

where $i_0+\cdots+i_n = k_0+\cdots+k_n = d_1$, $j_1+\cdots+j_m = \ell_1+\cdots+\ell_m = d_2$ and $\delta^{i_0,\dots,i_n,j_1,\dots,j_m}_{k_0,\dots,k_n,\ell_1,\dots,\ell_m}$ is the Kronecker delta.

This simple observation allows us to identify—given a CW-Nagata polynomial $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in S_{(d_1,d_2)}$— the *dual differential operator* $G_r$ of the monomial $g_r$—i.e. the monomial $G_r \in \mathbb{K}[U_1,\dots,U_m]_{d_2}$ such that $G_r(g_r) = 1$ and $G_r(g) = 0$ for any other monomial $g \in \mathbb{K}[u_1,\dots,u_m]_{d_2}$—with the same element of the $(d_2-1)$-skeleton of $\Delta_f$ associated to $g_r$. In other words, we associate to $g_r = u_1^{j_1}\cdots u_m^{j_m}$ and to $G_r = U_1^{j_1}\cdots U_m^{j_m}$ the CW-subcomplex of $\Delta_f \subset P(m)$, $X_{u_1^{j_1}\cdots u_m^{j_m}}$.

**Theorem 3.16.** *Let $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, with $g_r = u_1^{j_1}\cdots u_m^{j_m}$, be a CW-Nagata polynomial of (positive) degree $d_1$, where $n \le \binom{m}{d_2}$, let $\Delta_f$ be the CW-complex associated to $f$ and let $A = Q/\operatorname{Ann}(f)$. Then*

$$A = \bigoplus_{h=0}^{d=d_1+d_2} A_h$$

*where*

$$A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)},\ p \le d_1,\ q \le d_2,\ A_d = A_{(d_1,d_2)}$$

*and moreover, $\forall j \in \{0,1,\dots,d_2\}$,*

$$\dim A_{(i,j)} = a_{i,j} = \begin{cases} f_j & i = 0 \\ \sum_{r=0}^{n} f_{j,r} & i \in \{1,\dots,d_1-1\}, \\ f_{d_2-j} & i = d_1 \end{cases}$$

*where:*

- *$f_j$ is the number of the elements of the $(j-1)$-skeleton of the CW-complex $\Delta_f$ (with the convention that $f_0 = 1$);*
- *$f_{j,r}$ is the number of the elements of the $(j-1)$-skeleton of the CW complex $X_{G_r}$ (with the convention that $f_{0,r} = 1$, so that $\dim A_{(i,0)} = n+1$).*

*More precisely, a basis for $A_{(i,j)}$, $\forall j \in \{0,1,\dots,d_2\}$, is given by*

*(1) If $i = 0$, $\{\Omega_1,\dots,\Omega_{f_j}\}$, where any $\Omega_s := U_1^{s_1}\cdots U_m^{s_m}$, with $s_1+\cdots+s_m = j$, is associated to the element $X_{u_1^{s_1}\cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of $\Delta_f$;*
*(2) If $i = 1,\dots,d_1-1$, $\left\{\Omega_s^{i,s_1,\dots,s_m}\right\}_{\substack{s\in\{0,\dots,n\}\\ s_k\leq,r_k,k=1,\dots,m\\ \sum_k s_k=j}}$ where $\Omega_s^{i,s_1,\dots,s_m} := X_s^i \cdot U_1^{s_1}\cdots U_m^{s_m}$ is associated to the element $X_{u_1^{s_1}\cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of $X_{g_s}$;*
*(3) If $i = d_1$, $\left\{X_0^{d_1}\Omega_1(f),\dots X_n^{d_1}\Omega_{f_{d_2-j}}(f)\right\}$, where $\left\{\Omega_1,\dots\Omega_{f_{d_2-j}}\right\}$ is the basis for $A_{(0,d_2-j)}$ of case (1).*

*In the cases (1) and (2) the basis are given by monomials, in the case (3), in general, not.*

**Proof.** We divide the proof into computing the dimension of $A_{(i,j)}$ and find a basis for it, as $i$ varies:

$i=0$: $A_{(0,0)} \cong \mathbb{K}$.
Then, by definition, if $j \in \{1,\dots,d_2\}$, $A_{(0,j)}$ is generated by the (canonical images of the) monomials $\Omega_s \in Q_j = \mathbb{K}[U_1,\dots,U_m]_j \cong Q_{(0,j)}$ that do not annihilate $f$. This means that, if we write

$$\Omega_s = U_1^{s_1}\cdots U_m^{s_m} \qquad s_1+\cdots+s_m = j,$$

there exists an $r_s \in \{0,\dots,n\}$ such that $g_{r_s} = u_1^{s_1}\cdots u_m^{s_m} g'_{r_s}$, where $g'_{r_s} \in R_{d_2-j}$ is a (nonzero) monomial; this means that $X_{u_1^{s_1}\cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of the CW-complex $\Delta_f$ by Proposition 3.14.
We need to prove that these monomials are linearly independent over $\mathbb{K}$: let $\{\Omega_1,\dots,\Omega_{f_j}\}$ be a system of monomials of $Q_{(0,j)}$, where any $\Omega_s = U_1^{s_1}\cdots U_m^{s_m}$ with $s_1+\cdots+s_m = j$, is associated to an element of the $(j-1)$-

skeleton of the CW-complex $\Delta_f$; take a linear combination of them and apply it to $f$:

$$0=\sum_{s=1}^{f_j} c_s\Omega_s(f)=\sum_{s=1}^{f_j} c_s\sum_{r=0}^{n} x_r^{d_1}\Omega_s(g_r)=\sum_{r=0}^{n} x_r^{d_1}\sum_{s=1}^{f_j} c_s\Omega_s(g_r).$$

By the linear independence of the $x_r^{d_1}$'s

$$\sum_{s=1}^{f_j} c_s\Omega_s(g_r)=0, \qquad \forall r\in\{0,\dots,n\}. \tag{2}$$

By hypothesis, for any index $s$ there exists an $r_s\in\{0,\dots,n\}$ such that $\Omega_s(g_{r_s})=g'_{r_s}\in R_{d_2-j}\setminus\{0\}$, then for any index $s$ one has $c_s=0$, since the linear combinations in (2) are formed by linearly independent monomials ($g_r$ is fixed in each linear combination!). In other words, $\dim A_{(0,j)}=f_j$.

$0<i<d_1$: Observe that $X_aX_b(f)=0$ if $a\neq b$. Therefore $A_{(i,j)}$ is generated by the only (canonical images of) the monomials $\Omega_s^{i,s_1,\dots,s_m}:=X_s^iU_1^{s_1}\cdots U_m^{s_m}\in Q_{(i,j)}$, with $s_1+\cdots+s_m=j$, that do not annihilate $f$. In particular, a basis for $A_{(i,0)}$ is given by $X_0^i,\dots,X_n^i$ and we can suppose from now on that $j>0$. Since

$$\Omega_s^{i,s_1,\dots,s_m}(f)=x_s^{d_1-i}\left(U_1^{s_1}\cdots U_m^{s_m}\right)(g_s),$$

in order to obtain that this is not zero, we must have that $g_s=u_1^{s_1}\cdots u_m^{s_m}g'_s$, where $g'_s\in R_{d_2-j}$ is a nonzero monomial. This means $X_{u_1^{s_1}\cdots u_m^{s_m}}\subset X_{g_s}$ by Proposition 3.14.

As above, we can prove that these monomials are linearly independent over $\mathbb{K}$: let

$$\left\{\Omega_s^{i,s_1,\dots,s_m}\right\}_{\substack{s\in\{0,\dots,n\}\\ s_k\leq,r_k,k=1,\dots,m\\ \sum_k s_k=j}}$$

be a system of monomials of $Q_{(i,j)}$, where any $\Omega_s^{i,s_1,\dots,s_m}=X_s^i\cdot U_1^{s_1}\cdots U_m^{s_m}$ is associated to the element $X_{u_1^{s_1}\cdots u_m^{s_m}}$ of the $(j-1)$ skeleton of $X_{g_s}\subset\Delta_f$, i.e. $X_{u_1^{s_1}\cdots u_m^{s_m}}\subset X_{g_s}\subset\Delta_f$ by Proposition 3.14.

Take a linear combination of them and apply it to $f$:

$$\begin{aligned}0&=\sum_{\substack{s\in\{0,\dots,n\}\\ s_k\leq,r_k,k=1,\dots,m\\ \sum_k s_k=j}} c_s^{i,s_1,\dots,s_m}\Omega_s^{i,s_1,\dots,s_m}(f)\\ &=\sum_{s=0}^{n} x^{d_1-i}\sum_{\substack{s_k\leq,r_k,k=1,\dots,m\\ \sum_k s_k=j}} c_s^{i,s_1,\dots,s_m}g_s^{i,s_1,\dots,s_m}\end{aligned} \tag{3}$$

where $g_s^{i,s_1,\dots,s_m} \in R_{d_2-j}$ is the nonzero monomial such that $g_s = u_1^{s_1}\cdots u_m^{s_m} g_s^{i,s_1,\dots,s_m}$. From (3) we deduce, as in the preceding case, that

$$\sum_{\substack{s_k \leq, r_k, k=1,\dots,m \\ \sum_k s_k = j}} c_s^{i,s_1,\dots,s_m} g_s^{i,s_1,\dots,s_m} = 0 \qquad s = 0,\dots,n; \tag{4}$$

as before, given one choice of $s_1,\dots,s_m$ there exists an $s \in \{0,\dots,n\}$ such $g_s^{i,s_1,\dots,s_m}(f)$ is a nonzero monomial, and the (nonzero) $g_s^{i,s_1,\dots,s_m}$'s in (4) are linearly independent since are obtained by a fixed $g_s$.

$i = d_1$: By duality, see Remark 2.2, $A_{(d_1,j)} \cong A^\vee_{(0,d_2-j)}$ so $\dim A_{(d_1,j)} = f_{d_2-j}$. To find a basis for $A_{(d_1,j)}$, we consider the exact sequence (1) given by evaluation at $f$, which in this case reads

$$0 \to I_{(0,d_2-j)} \to Q_{(0,d_2-j)} \to A_{(d_1,j)} \to 0, \tag{5}$$

then a basis for $A_{(d_1,j)}$ is obtained in the following way: if $\{\Omega_1,\dots\Omega_{f_{d_2-j}}\}$ is the basis for $A_{(0,d_2-j)} \cong Q_{(0,d_2-j)}/I_{(0,d_2-j)}$ of the case $i = 0$, then a basis for $A_{(d_1,j)}$ is $\left\{X_0^{d_1}\Omega_1,\dots,X_n^{d_1}\Omega_{f_{d_2-j}}(f)\right\}$. □

As a corollary of Theorem 3.16 we see that we can deduce the general case of the simplicial Nagata polynomial, which is a slight improvement of the first part of [1, Theorem 3.5].

**Corollary 3.17.** *Let $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, with $g_r = x_{r_1}\cdots x_{r_{d_2}}$, be a simplicial Nagata polynomial of (positive) degree $d_1$, where $n \leq \binom{m}{d_2}$, let $\Delta_f$ be the simplicial complex associated to $f$ and let $A = Q/\operatorname{Ann}(f)$. Then*

$$A = \bigoplus_{h=0}^{d=d_1+d_2} A_h$$

*where*

$$A_h = A_{(h,0)} \oplus \cdots \oplus A_{(p,q)} \oplus \cdots \oplus A_{(0,h)},\ p \leq d_1,\ q \leq d_2,\ A_d = A_{(d_1,d_2)}$$

*and moreover, $\forall j \in \{0,1,\dots,d_2\}$,*

$$\dim A_{(i,j)} = a_{i,j} = \begin{cases} f_j & i = 0 \\ \sum_{r=0}^{n} f_{j,r} & i \in \{1,\dots,d_1-1\}, \\ f_{d_2-j} & i = d_1 \end{cases}$$

*where:*

- $f_j$ *is the number of* $(j-1)$*-cells of the* $\Delta_f$ *(with the convention that* $f_0 = 1$*);*
- $f_{j,r}$ *is the number of* $(j-1)$*-subcells of* $\Delta_{g_r}$*, i.e. the* $(d_2-1)$*-cell of the* $\Delta_f$ *associated to* $g_r$ *(with the convention that* $f_{0,r} = 1$*, so that* $\dim A_{(i,0)} = n+1$*).*

*More precisely, a basis for* $A_{(i,j)}$*,* $\forall j \in \{0, 1, \dots, d_2\}$*, is given by*

(1) *If* $i = 0$*,* $\{\Omega_1, \dots, \Omega_{f_j}\}$*, where any* $\Omega_s := U_{s_1} \cdots U_{s_j}$ *is associated to the* $(j-1)$*-subcell* $\{u_{s_1}, \dots, u_{s_j}\}$ *of* $\Delta_f$*;*
(2) *If* $i = 1, \dots, d_1 - 1$*,* $\left\{\Omega_s^{i,s_1,\dots,s_j}\right\}_{\substack{s\in\{0,\dots,n\} \\ s_1,\dots,s_j\in\{r_1,\dots,r_{d_2}\}}}$ *where* $\Omega_s^{i,s_1,\dots,s_j} := X_s^i U_{s_1} \cdots U_{s_j}$ *is associated to the* $(j-1)$*-subcell* $\{u_{s_1}, \dots, u_{s_j}\}$ *of* $\Delta_{g_s} (\subset \Delta_f)$*;*
(3) *If* $i = d_1$*,* $\left\{X_0^{d_1}\Omega_1(f), \dots X_n^{d_1}\Omega_{f_{d_2-j}}(f)\right\}$*, where* $\{\Omega_1, \dots \Omega_{f_{d_2-j}}\}$ *is the basis for* $A_{(0,d_2-j)}$ *of case (1).*

*In the cases (1) and (2) the bases are given by monomials, in the case (3), in general, not.*

**Theorem 3.18.** *Let* $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in S_{(d_1,d_2)}$*, with* $g_r = x_1^{r_1} \cdots x_m^{r_m}$ *such that* $r_1 + \cdots + r_m = d_2$*, be a CW-Nagata polynomial whose associated CW-complex is* $\Delta_f$*, as in the preceding theorem.*

*Then* $I := \operatorname{Ann}(f)$ *is generated by:*

(1) $X_i X_j$ *and* $X_k^{d_1+1}$*, for* $i, j, k \in \{0, \dots, n\}$*,* $i < j$*;*
(2) $\langle U_1, \dots, U_m\rangle^{d_2+1}$*, i.e. all the (monic) monomials of degree* $d_2 + 1$*;*
(3) *The monomials* $U_1^{s_1} \cdots U_m^{s_m}$ *such that* $s_1 + \cdots + s_m = j$*, where* $X_{u_1^{s_1} \cdots u_m^{s_m}}$ *is a (minimal) element of the* $(j-1)$*-skeleton of* $P(m)$ *not contained in* $\Delta_f$ *(for* $j \in \{1, \dots, d_2\}$*);*
(4) *The monomials* $X_r U_i$*, where* $u_i$ *does not divide* $g_r$ *(i.e.* $\{u_i\}$ *is not an element of the* $0$*-skeleton of* $X_{g_r}$*);*
(5) *The monomials* $X_s U_1^{r_1} \cdots U_m^{r_m}$ *such that* $r_1 + \cdots + r_m = j$*, where* $u_1^{r_1} \cdots u_m^{r_m}$ *is minimal among those that do not divide* $g_s$ *(i.e. the (minimal) element of the* $(j-1)$*-skeleton of* $P(m)$*,* $X_{u_1^{r_1} \cdots u_m^{r_m}}$*, is not contained in* $X_{g_s}$*), for* $j \in \{1, \dots, d_2\}$*;*
(6) *The binomials* $X_r^{d_1} U_1^{\rho_1} \cdots U_m^{\rho_m} - X_s^{d_1} U_1^{\sigma_1} \cdots U_m^{\sigma_m}$ *with* $\rho_1 + \cdots + \rho_m = \sigma_1 + \cdots + \sigma_m = j$ *such that* $g_{r,s} = \operatorname{GCD}(g_r, g_s)$ *and* $g_r = u_1^{\rho_1} \cdots u_m^{\rho_m} g_{r,s}$*,* $g_s = u_1^{\sigma_1} \cdots u_m^{\sigma_m} g_{r,s}$ *(i.e.* $X_{g_{r,s}}$ *is the element of the* $(d_2 - j - 1)$*-skeleton of* $\Delta_f$ *which represents the intersection of* $X_{g_r}$ *and* $X_{g_s}$*:* $X_{g_{r,s}} = X_{g_r} \cap X_{g_s}$*).*

**Proof.** Let $A := T/I$, where $T = \mathbb{K}[X_0, \dots, X_n, U_1, \dots, U_m]$.

By Theorem 3.16, (1) a basis for $A_{(0,j)}$, $\forall j \in \{1,\ldots,d_2\}$, is $\{\Omega_1,\ldots,\Omega_{f_j}\}$, where $\Omega_s := U_1^{s_1}\cdots U_m^{s_m}$, with $s_1+\cdots+s_m=j$, is associated to the element $X_{u_1^{s_1}\cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of $\Delta_f$. Therefore, using the identification introduced in Remark 3.15, a basis for $I_{(0,j)}$ is given by the monomials $U_1^{s_1}\cdots U_m^{s_m}$ such that $s_1+\cdots+s_m=j$, where $X_{u_1^{s_1}\cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of $P(m)$ not contained in $\Delta_f$ (for $j\in\{1,\ldots,d_2\}$);

Observe that $X_iX_j(f)=0$ if $i\neq j$ and $X_k^{d_1+1}(f)=0=U_1^{i_1}\cdots U_m^{i_m}(f)$ with $\sum_{j=1}^m i_j = d_2+1$, for degree reasons. Set

$$\beta := (X_0X_1,\ldots,X_{n-1}X_n, X_0^{d_1+1},\ldots,X_n^{d_1+1},\langle U_1,\ldots,U_m\rangle^{d_2+1});$$

this is a homogeneous ideal such that $\beta\subset I$ and $A\cong \dfrac{T}{\beta}/\dfrac{I}{\beta}$.

By Theorem 3.16, (2), if $i=1,\ldots,d_1-1$, a basis for $A_{(i,j)}$ $\forall j\in\{1,\ldots,d_2\}$, is given by

$$\left\{\Omega_s^{i,s_1,\ldots,s_m}\right\}_{\substack{s\in\{0,\ldots,n\}\\ s_k\leq, r_k, k=1,\ldots,m\\ \sum_k s_k=j}}$$

where $\Omega_s^{i,s_1,\ldots,s_m} := X_s^i\cdot U_1^{s_1}\cdots U_m^{s_m}$ is associated to the element $X_{u_1^{s_1}\cdots u_m^{s_m}}$ of the $(j-1)$-skeleton of $X_{g_s}$.

Again using the identification introduced in Remark 3.15, a basis for $\left(\dfrac{I}{\beta}\right)_{(i,j)}$ is given by

- The monomials $X_r^iU_1^{s_1}\cdots U_m^{s_m}$ such that $s_1+\cdots+s_m=j$, with $r\neq s$, where $u_1^{s_1}\cdots u_m^{s_m}$ divides $g_s$ (i.e. $X_{u_1^{s_1}\cdots u_m^{s_m}}$ is an element of the $(j-1)$-skeleton of $X_{g_s}$), for $i=1,\ldots,d_1-1$, and
- The monomials $X_s^iU_1^{r_1}\cdots U_m^{r_m}$ such that $r_1+\cdots+r_m=j$, where $u_1^{r_1}\cdots u_m^{r_m}$ does not divide $g_s$ (i.e. the element of the $(j-1)$-skeleton of $P(m)$, $X_{u_1^{r_1}\cdots u_m^{r_m}}$, is not contained in $X_{g_s}$),

for $j\in\{1,\ldots,d_2\}$.

It remains to find the generators of $I$ of bidegree $(d_1,j)$, with $j\in\{1,\ldots,d_2\}$. This is more complicated since the generators of $A_{(d_1,j)}$ are not monomials. Let $\gamma$ be the homogeneous ideal generated by the monomials of the cases (1), (2), (3), (4) and (5), i.e. the generators that we have found so far. We have $\beta\subset\gamma\subset I$ and the exact sequence (1) given by evaluation at $f$ becomes

$$0\to\left(\frac{I}{\gamma}\right)_{(d_1,j)}\to\left(\frac{T}{\gamma}\right)_{(d_1,j)}\to A_{(0,d_2-j)}\to 0,$$

since we identify $A \cong \frac{T}{\gamma}/\frac{I}{\gamma}$. Then, if $\rho_1+\cdots+\rho_m = \sigma_1+\cdots+\sigma_m = j$, $X_r^{d_1}U_1^{\rho_1}\cdots U_m^{\rho_m} - X_s^{d_1}U_1^{\sigma_1}\cdots U_m^{\sigma_m} \in \left(\frac{T}{\gamma}\right)_{(d_1,j)}$ is in $\left(\frac{I}{\gamma}\right)_{(d_1,j)}$ if and only if $X_r^{d_1}U_1^{\rho_1}\cdots U_m^{\rho_m} = X_s^{d_1}U_1^{\sigma_1}\cdots U_m^{\sigma_m} \in A_{(0,d_2-j)}$, which means $U_1^{\rho_1}\cdots U_m^{\rho_m}(g_r) = U_1^{\sigma_1}\cdots U_m^{\sigma_m}(g_s)$. Since $A_{(0,d_2-j)}$ is generated by the monomials $\Omega_s := U_1^{s_1}\cdots U_m^{s_m}$, with $s_1+\cdots+s_m = d_2-j$, associated to the elements of the $(d_2-j-1)$-skeleton of $\Delta_f$, we obtain case (6). □

As we have done for Theorem 3.16, we give, as a corollary of Theorem 3.18 the case of the simplicial Nagata polynomial, giving an improvement of the second part of [1, Theorem 3.5]; we also correct that statement, since the authors forgot the generators $X_iX_j$, $i \neq j$.

**Corollary 3.19.** *Let* $f = \sum_{r=0}^{n} x_r^{d_1} g_r \in R_{(d_1,d_2)}$, *with* $g_r = x_{r_1}\cdots x_{r_{d_2}}$, *be a simplicial Nagata polynomial whose associated simplicial complex is* $\Delta_f$, *as in the preceding theorem.*

*Then* $I := \mathrm{Ann}(f)$ *is generated by:*

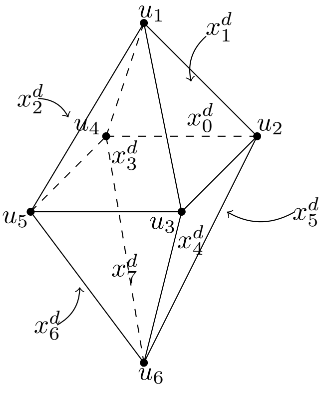
*(1)* $X_iX_j$ *and* $X_k^{d_1+1}$, *for* $i,j,k \in \{0,\dots,n\}$, $i<j$;
*(2)* $U_1^2,\dots,U_m^2$;
*(3) The monomials* $U_{s_1}\cdots U_{s_j}$, *where* $\{u_{s_1},\dots,u_{s_j}\}$ *is a (minimal)* $(j-1)$*-cell of* $2^{\{u_1,\dots,u_m\}}$ *not contained in* $\Delta_f$ *(for* $j \in \{1,\dots,d_2\}$*);*
*(4) The monomials* $X_rU_i$, *where* $u_i$ *does not divide* $g_s$ *(i.e.* $\{u_i\} \notin \Delta g_r$*);*
*(5) The binomials* $X_r^{d_1}U_{\rho_1}\cdots U_{\rho_j} - X_s^{d_1}U_{\sigma_1}\cdots U_{\sigma_j}$ *such that* $g_{r,s}\,\mathrm{GCD}(g_r,g_s)$, $g_r = u_{\rho_1}\cdots u_{\rho_j}g_{r,s}$, $g_s = u_{\sigma_1}\cdots u_{\sigma_j}g_{r,s}$ *(i.e.* $g_{r,s}$ *represents the* $(d_2-j-1)$*-face given by the intersection* $\Delta_{g_r}\cap\Delta_{g_s}$ *of the facets of* $g_r$ *and* $g_s$*:* $\Delta_{g_{r,s}} = \Delta_{g_r}\cap\Delta_{g_s}$*).*

**Proof.** We note only that we have to add the squares of case (2) although they do not correspond to cells, since the polynomials $g_i$ are square-free. The rest follows from Theorem 3.18. We observe that these squares are in case (2) of Theorem 3.18. □

**Example 3.20.** Let

$$f = x_0^d u_1u_2u_3 + x_1^d u_1u_2u_4 + x_2^d u_1u_4u_5 + x_3^d u_1u_3u_5 + x_4^d u_2u_3u_6 + x_5^d u_2u_4u_6 + x_6^d u_4u_5u_6 + x_7^d u_3u_5u_6$$

be a bihomogeneous bidegree $(d,3)$ polynomial with $d \geq 1$; it is a simplicial Nagata polynomial, whose associated simplicial complex is in the following figure:




We have:

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d+3}.$$

We want firstly to compute the Hilbert vector by applying Corollary 3.17; first of all,

$$a_{1,0} = 8 \qquad a_{0,1} = 6,$$

and therefore

$$h_0 = h_{d+3} = 1$$
$$h_1 = h_{d+2} = a_{1,0} + a_{0,1} = 8 + 6 = 14.$$

Then, we analyze the possible cases depending on the degree $d$:

- If $d = 1$, then

$$a_{1,1} = 8 \cdot 3 = 24$$
$$a_{0,2} = 12$$
$$h_2 = a_{1,1} + a_{0,2} = 36$$

  and the Hilbert vector is $(1, 14, 36, 14, 1)$.
- If $d = 2$, then, recalling bigraded Poincaré duality,

$$a_{2,0} = a_{0,3} = 8 \qquad a_{2,1} = a_{0,2} = 12$$

  and therefore

$$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 8 + 8 \cdot 3 + 12 = 44,$$
$$h_3 = 0 + a_{2,1} + a_{1,2} + a_{0,3} = 8 + 8 \cdot 3 + 8 = 44$$

in accordance with Poincaré duality; so the Hilbert vector is $(1, 14, 44, 44, 14, 1)$ (cfr. [1, Example 3.6]).

- If $d = 3$, then, again by bigraded Poincaré duality,

$$a_{3,0} = a_{0,3} = 8, \qquad a_{2,1} = a_{1,2} = 8 \cdot 3 = 24, \qquad a_{3,1} = a_{0,2} = 12,$$
$$a_{2,2} = a_{1,1} = 24, \qquad a_{1,3} = a_{2,0} = 8,$$

therefore

$$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44,$$
$$h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 64$$
$$h_4 = 0 + a_{3,1} + a_{2,2} + a_{1,3} = 44$$

$h_2 = h_4$ in accordance with Poincaré duality and the Hilbert vector is $(1, 14, 44, 64, 44, 14, 1)$.

- In general, let $d \geq 4$; by hypothesis

$$h_{d+1} = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 44,$$

and

$$h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} + a_{k-3,3} \qquad \forall k \in \{3, \dots, d\},$$

where

$$a_{k,0} = 8 \qquad a_{k-1,1} = 8 \cdot 3 = 24 \qquad a_{k-2,2} = 8 \cdot 3 = 24 \qquad a_{k,3} = 8.$$

Again using the Poincaré duality we have:

$$h_{d+3-k} = h_k = 64 \qquad \forall k \in \left\{3, \dots, \left\lfloor \frac{d+3}{2} \right\rfloor \right\}$$

and the Hilbert vector is $(1, 14, 44, 64, \dots, 64, 44, 14, 1)$.

Now, we want to find the generators of $\mathrm{Ann}(f)$, by applying Corollary 3.19. Behavior depends on $d$:

- If $d = 1$, by Corollary 3.19 $\mathrm{Ann}(f)$ is (minimally) generated by:

(1) $\langle X_0, \dots, X_7 \rangle^2 = X_0^2, X_0X_1, \dots;$
(2) $U_1^2, \dots, U_6^2;$
(3) $U_1U_6, U_2U_5, U_3U_4;$

(4) $X_0U_4, X_0U_5, X_0U_6, X_1U_3, X_1U_5, X_1U_6, X_2U_2, X_2U_3, X_2U_6, X_3U_2, X_3U_4, X_3U_6, X_4U_1, X_4U_4, X_4U_5, X_5U_1, X_5U_3, X_5U_5, X_6U_1, X_6U_2, X_6U_3, X_7U_1, X_7U_2, X_7U_4$;
(5) $X_0U_3 - X_1U_4, X_0U_2 - X_3U_5, X_0U_1 - X_4U_6, X_1U_2 - X_2U_5, X_1U_1 - X_5U_6, X_2U_4 - X_3U_3, X_2U_1 - X_6U_6, X_3U_1 - X_7U_6, X_4U_3 - X_5U_4, X_4U_2 - X_7U_5, X_5U_2 - X_6U_5, X_6U_4 - X_7U_3$.

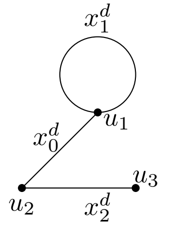
- If $d \geq 2$, by Corollary 3.19 $\mathrm{Ann}(f)$ is (minimally) generated by

(1) $\langle X_0, \dots, X_7\rangle^{d+1}$ and $X_hX_k$ where $h, k \in \{0, \dots, 7\}$, $h < k$;
(2) $U_1^2, \dots, U_6^2$;
(3) $U_1U_6, U_2U_5, U_3U_4$;
(4) $X_0^dU_4, X_0^dU_5, X_0^dU_6, X_1^dU_3, X_1^dU_5, X_1^dU_6, X_2^dU_2, X_2^dU_3, X_2^dU_6, X_3^dU_2, X_3^dU_4, X_3^dU_6, X_4^dU_1, X_4^dU_4, X_4^dU_5, X_5^dU_1, X_5^dU_3, X_5^dU_5, X_6^dU_1, X_6^dU_2, X_6^dU_3, X_7^dU_1, X_7^dU_2, X_7^dU_4$;
(5) $X_0^dU_3 - X_1^dU_4, X_0^dU_2 - X_3^dU_5, X_0^dU_1 - X_4^dU_6, X_1^dU_2 - X_2^dU_5, X_1^dU_1 - X_5^dU_6, X_2^dU_4 - X_3^dU_3, X_2^dU_1 - X_6^dU_6, X_3^dU_1 - X_7^dU_6, X_4^dU_3 - X_5^dU_4, X_4^dU_2 - X_7^dU_5, X_5^dU_2 - X_6^dU_5, X_6^dU_4 - X_7^dU_3$.

**Example 3.21.** Let

$$f = x_0^du_1u_2 + x_1^du_1^2 + x_2^du_2u_3$$

be a bihomogeneous bidegree $(d, 2)$ polynomial, with $d \geq 1$; it is a CW-Nagata polynomial whose CW-complex is the following:




We have:

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_{d+2}$$

and we want to find its Hilbert vector; first of all,

$$a_{1,0} = 3 \qquad a_{0,1} = 3$$

and therefore

$$h_0 = h_{d+2} = 1 \qquad h_1 = h_{d+1} = a_{1,0} + a_{0,1} = 6.$$

Therefore, if $d = 1$, then Hilbert vector is $(1, 6, 6, 1)$.

If $d = 2$, we have

$$a_{1,1} = 2 + 1 + 2 = 5,$$

so

$$h_2 = \dim A_2 = a_{2,0} + a_{1,1} + a_{0,2} = 3 + 5 + 3 = 11$$

and the Hilbert vector is $(1, 6, 11, 6, 1)$.

If $d = 3$ then, by bigraded Poincaré duality

$$a_{3,0} = a_{0,2} = 3 \qquad a_{0,3} = 3$$

so

$$h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11$$
$$h_3 = a_{3,0} + a_{2,1} + a_{1,2} + a_{0,3} = 3 + 5 + 3 = 11$$

and the Hilbert vector is $(1, 6, 11, 11, 6, 1)$.

In general, let $d \geq 4$; by hypothesis

$$h_d = h_2 = a_{2,0} + a_{1,1} + a_{0,2} = 11,$$

and

$$h_k = \dim A_{(k,0)} + \dim A_{(k-1,1)} + \dim A_{(k-2,2)} \qquad \forall k \in \{3, \dots, d\},$$

so, since

$$a_{k,0} = 3 \qquad a_{k-1,1} = 5 \qquad a_{k-2,2} = 3$$

using Poincaré duality we have:

$$h_{d+2-k} = h_k = a_{k,0} + a_{k-1,1} + a_{k-2,2} = 11 \qquad \forall k \in \left\{3, \dots, \left\lfloor \frac{d+2}{2} \right\rfloor \right\},$$

and the Hilbert vector is $(1, 6, 11, \dots, 11, 6, 1)$.

Let $d = 1$, by Theorem 3.18 $\mathrm{Ann}(f)$ is (minimally) generated by:

- $\langle X_0, X_1, X_2 \rangle^2, U_2^2, U_3^2, U_1U_3, U_1^3$;
- $X_0U_1^2, X_0U_3, X_1U_2, X_1U_3, X_2U_1$;
- $X_0U_2 - X_1U_1, X_0U_1 - X_3U_3$.

Let $d \geq 2$, by Theorem 3.18 $\mathrm{Ann}(f)$ is (minimally) generated by:

- $\langle X_0, X_1, X_2 \rangle^{d+1}, X_0X_1, X_0X_2, X_1X_2, U_2^2, U_3^2, U_1U_3, U_1^3$;
- $X_0^dU_1^2, X_0^dU_3, X_1^dU_2, X_1^dU_3, X_2^dU_1$;
- $X_0^dU_2 - X_1^dU_1, X_0^dU_1 - X_3^dU_3$.

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