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A bridge theorem for the entropy of semigroup actions

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Abstract: The topological entropy of a semigroup action on a totally disconnected locally compact abelian group coincides with the algebraic entropy of the dual action. This relation holds both for the entropy relative to a net and for the receptive entropy of finitely generated monoid actions.

Keywords: topological entropy, algebraic entropy, receptive entropy, bridge theorem, semigroup action, dynamical system, Pontryagin duality, continuous endomorphism, locally compact abelian group

MSC: 37B40, 22B05, 22D40, 20K30, 54H20.

Dedicated to Dikran Dikranjan on the occasion of his 70th birthday

1 Introduction

The notion of topological entropy h_{top} for continuous endomorphisms of locally compact groups is obtained by specializing the general notion of topological entropy given by Hood [29] for uniformly continuous self-maps of uniform spaces, which was inspired by the classical topological entropy of Bowen [4] and Dinaburg [19] (see [18, 27] for the details). For compact groups h_{top} coincides with the first notion of topological entropy by Adler, Konheim and McAndrew [1].

Recently, Virili [39] (see also [12]) gave a notion of algebraic entropy h_{alg} for continuous endomorphisms of locally compact groups, extending and appropriately modifying the existing notions by Weiss [43] and by Peters [35, 36] (see also [15, 17]).

Since its origin, the algebraic entropy was introduced in connection to the topological entropy by means of Pontryagin duality. For a locally compact abelian group G we denote its Pontryagin dual group by $\widehat{G}$, and for a continuous endomorphism $\phi : G \to G$ its dual endomorphism is $\widehat{\phi} : \widehat{G} \to \widehat{G}$.

In case G is a totally disconnected compact abelian group, and so $\widehat{G}$ is a torsion discrete abelian group, it is known from [43] that

$$h_{top}(\phi) = h_{alg}(\widehat{\phi}). \tag{1.1}$$

The equality in (1.1), namely, a so-called Bridge Theorem, holds also when G is a metrizable locally compact abelian group and ϕ is an automorphism [36], and in case G is a compact abelian group [13].

In this paper we are interested in generalizing the following Bridge Theorem.

Theorem 1.1 (See [14]). *Let G be a totally disconnected locally compact abelian group and $\phi : G \to G$ a continuous endomorphism. Then $h_{top}(\phi) = h_{alg}(\widehat{\phi})$.*

The Pontryagin dual groups of the totally disconnected locally compact abelian groups are precisely the compactly covered locally compact abelian groups. Recall that a topological group G is compactly covered if each

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element of G is contained in some compact subgroup of G. Examples of such groups are the p-adic numbers $\mathbb{Q}_p$, for p a prime number.

Let S be a semigroup that acts on the left on a locally compact abelian group G by continuous endomorphisms and denote this left action by

$$S \overset{\gamma}{\curvearrowright} G.$$

In case G is abelian, γ induces the right action

$$\widehat{G} \overset{\widehat{\gamma}}{\curvearrowleft} S,$$

defined by $\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{G} \to \widehat{G}$ for every $s \in S$. We call $\widehat{\gamma}$ the dual action of γ; see §4 for more details on the dual action and the relation of an action with its dual.

Roughly speaking, the aim of this paper is to extend Theorem 1.1 to this general setting of semigroup actions, for which there are several extensions of the previously recalled notions of topological entropy and algebraic entropy.

First of all, the topological entropy is widely studied for amenable group actions and even in more general settings (e.g., see [5, 9, 33, 34, 37, 42]). On the other hand, a notion of algebraic entropy for amenable group actions on modules is developed by Virili [41], and an algebraic entropy for sofic group actions by Liang [30] who proves also an instance of the Bridge Theorem in that setting.

Recently, the topological entropy of amenable semigroup actions on compact spaces was introduced by Ceccherini-Silberstein, Coornaert and Krieger [6], extending the classical notion from [1]. Analogously, in [10] the algebraic entropy of amenable semigroup actions on discrete abelian groups was defined and investigated, generalizing classical notions and results from [15, 35, 43]. The definition of these entropies of amenable semigroup actions is based on nets of non-empty finite subsets of the acting semigroup, namely, on Følner sent. The extension given by Ceccherini-Silberstein, Coornaert and Krieger [6] (see Theorem 2.1 below) of the celebrated Ornstein-Weiss Lemma [34] shows that the definition does not depend on the choice of the Følner net.

The same approach based on nets was used by Virili [40] to introduce topological entropy and algebraic entropy for actions on locally compact abelian groups. We consider these entropies in the case of semigroup actions on locally compact (abelian) groups by continuous endomorphisms; in this case they depend on the choice of the net $\mathfrak{s}$ of non-empty finite subsets of the acting semigroup S, so we denote them respectively by $h^{\mathfrak{s}}_{top}$ and $h^{\mathfrak{s}}_{alg}$. They extend in a natural way the topological entropy h_{top} and the algebraic entropy h_{alg} of a single continuous endomorphism recalled above, by taking $S = \mathbb{N}$ and $\mathfrak{s} = (\{0, \dots, n\})_{n\in\mathbb{N}}$. In §2 we give the definitions, some useful properties and we connect these entropies with the ones for amenable semigroup actions from [6, 10].

The main result in [40] is the Bridge Theorem $h^{\mathfrak{s}}_{top}(\gamma) = h^{\mathfrak{s}}_{alg}(\widehat{\gamma})$ under the assumption that $\gamma(s)$ is an automorphism for every $s \in S$. This covers the result announced in [36] in the case $S = \mathbb{N}$.

We do not require the strong assumption on the action to be by automorphisms, but we assume the locally compact abelian group to be totally disconnected, and we see that the Bridge Theorem holds, obtaining the announced extension of Theorem 1.1:

Theorem 1.2. *Let S be an infinite monoid, $\mathfrak{s} = (F_i)_{i\in I}$ a net of non-empty finite subsets of S such that $|F_i| \to \infty$ and $1 \in F_i$ for every $i \in I$, G a totally disconnected locally compact abelian group, and consider the left action $S \overset{\gamma}{\curvearrowright} G$. Then $h^{\mathfrak{s}}_{top}(\gamma) = h^{\mathfrak{s}}_{alg}(\widehat{\gamma})$.*

See Corollary 4.3 for an application to the classical setting of amenable monoid actions.

Another kind of entropy, called receptive entropy, for semigroup actions was considered in [2, 3, 16, 20, 28]. In §3 we introduce the receptive topological entropy and the receptive algebraic entropy of actions of finitely generated monoids S on locally compact (abelian) groups. These entropies depend on the choice of a regular system Γ of S, which is a special sequence of non-empty finite subsets of S, so we denote them respectively by $\widetilde{h}^{\Gamma}_{top}$ and $\widetilde{h}^{\Gamma}_{alg}$. Also the receptive topological entropy and the receptive algebraic entropy extend precisely the topological entropy h_{top} and the algebraic entropy h_{alg} of a single continuous endomorphism by taking $S = \mathbb{N}$ and $\Gamma = (\{0, \dots, n\})_{n\in\mathbb{N}}$.

We find a Bridge Theorem also for the receptive entropies generalizing Theorem 1.1:

Theorem 1.3. *Let S be a finitely generated monoid, Γ a regular system of S, G a totally disconnected locally compact abelian group, and consider the left action $S \overset{\gamma}{\curvearrowright} G$. Then $\widetilde{h}^{\Gamma}_{top}(\gamma) = \widetilde{h}^{\Gamma}_{alg}(\widehat{\gamma})$.*

We end the paper with some examples from [11, 16].

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Notation and terminology

For a semigroup S, denote by $\mathcal{P}(S)$ the power set of S and by $\mathcal{P}_{fin}(S)$ the family of all non-empty finite subsets of S. In case S is a monoid, we denote by 1 the neutral element of S.

For a locally compact group G, we denote by e its neutral element. Moreover, we denote by $\mathcal{C}(G)$ the family of all compact neighborhoods of e in G and by $\mathcal{B}(G)$ the family of all compact open subgroups of G; clearly, $\mathcal{B}(G) \subseteq \mathcal{C}(G)$.

When we consider an action γ of a semigroup S on a group G we mean that this action is by continuous endomorphisms, that is, $\gamma(s) : G \to G$ is a continuous endomorphism for every $s \in S$.

For every $F \in \mathcal{P}_{fin}(S)$ and $U \in \mathcal{C}(G)$, the *F-cotrajectory* of $U \in \mathcal{C}(G)$ with respect to γ is

$$C_F(\gamma, U) = \bigcup_{s\in F} \gamma(s)^{-1}(U);$$

if $U \in \mathcal{B}(G)$, then $C_F(\gamma, U) \in \mathcal{B}(G)$.

If G is abelian, we define also the *F-trajectory* of U with respect to γ to be

$$T_F(\gamma, U) = \sum_{s\in F} \gamma(s)(U);$$

analogously, if $U \in \mathcal{B}(G)$, then $T_F(\gamma, U) \in \mathcal{B}(G)$.

2 Entropy relative to a net

In this section we consider the topological entropy and the algebraic entropy from [40]. First we recall some basic definitions related to amenability.

Let S be a semigroup. For every $s \in S$ denote by $L_s : S \to S$ the left multiplication $x \mapsto sx$ and by $R_s : S \to S$ the right multiplication $x \mapsto xs$. The semigroup S is *left cancellative* (respectively, *right cancellative*) if L_s (respectively, R_s) is injective for every $s \in S$.

A semigroup S is *left amenable* if there exists a map $\mu : \mathcal{P}(S) \to [0, 1]$ such that:

(L1) $\mu(S) = 1$;
(L2) $\mu(F \cup E) = \mu(F) + \mu(E)$ for every $F, E \in \mathcal{P}(S)$ with $F \cap E = \emptyset$;
(L3) $\mu(L_s^{-1}(F)) = \mu(F)$ for every $s \in S$ and every $F \in \mathcal{P}(S)$.

The semigroup S is *right amenable* if S^{op} is left amenable. The semigroup S is *amenable* if it is both left amenable and right amenable. Every commutative semigroup is amenable (see [7, 8]).

A *left Følner net* of a semigroup S is a net $(F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$ such that for every $s \in S$

$$\lim_{i\in I} \frac{|sF_i \setminus F_i|}{|F_i|} = 0.$$

Analogously, a *right Følner net* of S is a left Følner net of S^{op}. By [32, Corollary 4.3], a left cancellative semigroup S is left amenable if and only if S admits a left Følner net.

Let S be a semigroup. A function $f : \mathcal{P}_{fin}(S) \to \mathbb{R}_{\geq 0}$ is:

(1) *subadditive* if $f(F_1 \cup F_2) \leq f(F_1) + f(F_2)$ for every $F_1, F_2 \in \mathcal{P}_{fin}(S)$;
(2) *right subinvariant* (respectively, *left subinvariant*) if $f(Fs) \leq f(F)$ (respectively, if $f(sF) \leq f(F)$) for every $s \in S$ and every $F \in \mathcal{P}_{fin}(S)$;
(3) *uniformly bounded on singletons* if there exists a real number $M \geq 0$ with $f(\{s\}) \leq M$ for every $s \in S$.

The following is the counterpart of Ornstein-Weiss Lemma for cancellative left amenable semigroups.

Theorem 2.1 ([6, Theorem 1.1]). *Let S be a cancellative left amenable semigroup and let $f : \mathcal{P}_{fin}(S) \to \mathbb{R}$ be a subadditive right subinvariant map uniformly bounded on singletons. Then there exists $\lambda \in \mathbb{R}_{\geq 0}$ such that, for every left Følner net $(F_i)_{i\in I}$ of S, $\lim_{i\in I} f(F_i)/|F_i|$ exists and equals λ.*

2.1 Topological entropy

Definition 2.2 (See [40]). Let S be a semigroup, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$, G a locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. For $U \in \mathcal{C}(G)$, let

$$H_{top}^{\mathfrak{s}}(\gamma, U) = \limsup_{i\in I} \frac{-\log \mu(C_{F_i}(\gamma, U))}{|F_i|}.$$

The *topological entropy relative to* $\mathfrak{s}$ of γ is $h_{top}^{\mathfrak{s}}(\gamma) = \sup\{H_{top}^{\mathfrak{s}}(\gamma, U) \mid U \in \mathcal{C}(G)\}$.

The map $H_{top}^{\mathfrak{s}}(\gamma, -)$ is antimonotone, that is, if $U, V \in \mathcal{C}(G)$ and $U \subseteq V$, then $H_{top}^{\mathfrak{s}}(\gamma, V) \leq H_{top}^{\mathfrak{s}}(\gamma, U)$. Therefore, in order to compute $h_{top}^{\mathfrak{s}}$, it suffices to consider a local base at 1 of G contained in $\mathcal{C}(G)$. When the locally compact group G is totally disconnected, $\mathcal{B}(G)$ is a local base at 1 of G by van Dantzig Theorem [38], so we have the following useful property.

Proposition 2.3. *Let S be a semigroup, G a totally disconnected locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. Consider a net $\mathfrak{s} = (F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$. Then*

$$h_{top}^{\mathfrak{s}}(\gamma) = \sup\{H_{top}^{\mathfrak{s}}(\gamma, U) \mid U \in \mathcal{B}(G)\}.$$

The following result was given in [18] for the case $S = \mathbb{N}$. It shows that for $U \in \mathcal{B}(G)$ one can avoid the use of Haar measure to compute the topological entropy.

Proposition 2.4. *Let S be an infinite monoid, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$ such that $|F_i| \to \infty$ and $1 \in F_i$ for every $i \in I$, G a locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. If $U \in \mathcal{B}(G)$, then*

$$H_{top}^{\mathfrak{s}}(\gamma, U) = \limsup_{i\in I} \frac{\log[U : C_{F_i}(\gamma, U)]}{|F_i|}.$$

Proof. For every $i \in I$, since $C_{F_i}(\gamma, U) \in \mathcal{B}(G)$, we have that $[U : C_{F_i}(\gamma, U)]$ is finite and

$$\mu(U) = [U : C_{F_i}(\gamma, U)]\mu(C_{F_i}(\gamma, U)).$$

Hence,

$$\begin{aligned} H^{\mathfrak{s}}_{top}(\gamma, U) &= \limsup_{i\in I} \frac{-\log \mu(C_{F_i}(\gamma, U))}{|F_i|} \\ &= \limsup_{i\in I} \frac{\log[U : C_{F_i}(\gamma, U)] - \log \mu(U)}{|F_i|} \\ &= \limsup_{i\in I} \frac{\log[U : C_{F_i}(\gamma, U)]}{|F_i|}. \end{aligned}$$ □

If in addition the group G is compact, we obtain the following result.

Corollary 2.5. *Let S be an infinite monoid, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$ such that $|F_i| \to \infty$ and $1 \in F_i$ for every $i \in I$, K a compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. If $U \in \mathcal{B}(K)$, then*

$$H^{\mathfrak{s}}_{top}(\gamma, U) = \limsup_{i\in I} \frac{\log[K : C_{F_i}(\gamma, U)]}{|F_i|}.$$

Proof. This follows from Proposition 2.4 and the fact that, since K is compact, $[K : U]$ is finite. □

This result implies that, when a cancellative left amenable monoid acts on a compact group, this topological entropy coincides with the one introduced in [6], that in this paper we call h_{cov} (see Proposition 2.8).

We start recalling the definition of h_{cov}. Let S be a cancellative left amenable semigroup, let C be a compact topological space, and consider a left action $S \overset{\gamma}{\curvearrowright} C$ by continuous maps, that is, $\gamma(s) : C \to C$ is a continuous selfmap for every $s \in S$.

Let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_k\}_{k\in K}$ be two open covers of C. One says that $\mathcal{V}$ refines $\mathcal{U}$, denoted by $\mathcal{U} \prec \mathcal{V}$, if for every $k \in K$ there exists $j \in J$ such that $V_k \subseteq U_j$. Moreover,

$$\mathcal{U} \vee \mathcal{V} = \{U_j \cap V_k \mid (j, k) \in J \times K\}.$$

If $f : X \to X$ is a continuous selfmap, then $f^{-1}(\mathcal{U}) = \{f^{-1}(U_j)\}_{j\in J}$. Let

$$N(\mathcal{U}) = \min\{n \in \mathbb{N}_+ \mid \mathcal{U} \text{ admits a subcover of size } n\}.$$

We use that

$$\text{if } \mathcal{U} \prec \mathcal{V} \text{ then } N(\mathcal{U}) \le N(\mathcal{V}). \tag{2.1}$$

For an open cover $\mathcal{U}$ of C and for every $F \in \mathcal{P}_{fin}(S)$, let $\mathcal{U}_{\gamma,F} = \bigvee_{s\in F} \gamma(s)^{-1}(\mathcal{U})$. So, fixed an open cover $\mathcal{U}$ of C, consider the function

$$\mathcal{P}_{fin}(S) \to \mathbb{R}_{\ge 0}, \ F \mapsto \log N(\mathcal{U}_{\gamma,F}).$$

For every $\mathcal{U}$, this function is non-decreasing, subadditive, right subinvariant and uniformly bounded on singletons (see [6, Proposition 5.2]). So by applying Theorem 2.1, we have that the following definition is well posed.

Definition 2.6 (See [6, Theorem 5.3]). Let S be a cancellative left amenable semigroup, C a compact space and $S \overset{\gamma}{\curvearrowright} C$ a left action. For an open cover $\mathcal{U}$ of C, the *topological entropy of γ with respect to $\mathcal{U}$* is

$$H_{cov}(\gamma, \mathcal{U}) = \lim_{i\in I} \frac{\log N(\mathcal{U}_{\gamma,F_i})}{|F_i|},$$

for any left Følner net $(F_i)_{i\in I}$ of S. The *topological entropy of γ* is $h_{cov}(\gamma) = \sup\{H_{cov}(\gamma, \mathcal{U}) \mid \mathcal{U} \text{ open cover of } C\}$.

When $C = K$ is a totally disconnected compact group, for every $U \in \mathcal{B}(K)$ we can consider the open cover $\zeta(U) = \{kU \mid k \in K\}$ of K; then

$$N(\zeta(U)) = [K : U]. \tag{2.2}$$

Lemma 2.7. *Let S be a semigroup, K a totally disconnected compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. If $F \in \mathcal{P}_{fin}(S)$ and $U \in \mathcal{B}(K)$, then $\zeta(U)_{\gamma,F} \setminus \{\emptyset\} = \zeta(C_F(\gamma, U))$.*

Proof. We have to prove that

$$\zeta(U)_{\gamma,F} \setminus \{\emptyset\} = \bigvee_{s\in F} \gamma(s)^{-1}(\zeta(U)) \setminus \{\emptyset\} = \left\{ \bigcap_{s\in F} \gamma(s)^{-1}(k_s U) \mid k_s \in K \right\} \setminus \{\emptyset\}$$

coincides with

$$\zeta(C_F(\gamma, U)) = \left\{ z \bigcap_{s\in F} \gamma(s)^{-1}(U) \mid z \in K \right\}.$$

Note that, for $s \in F$ and $k \in K$, if $k' \in \gamma(s)^{-1}(kU)$ then $\gamma(s)^{-1}(kU) = k'\gamma(s)^{-1}(U)$.

Consider $z\bigcap_{s\in F} \gamma(s)^{-1}(U) \in \zeta(C_F(\gamma, U))$. Then there exists $\bigcap_{s\in F} \gamma(s)^{-1}(k_s U) \in \zeta(U)_{\gamma,F}$ such that $z \in \bigcap_{s\in F} \gamma(s)^{-1}(k_s U)$. It follows that $z\bigcap_{s\in F} \gamma(s)^{-1}(U) = \bigcap_{s\in F} \gamma(s)^{-1}(k_s U)$, and so that $\zeta(C_F(\gamma, U)) \subseteq \zeta(U)_{\gamma,F}$.

Assume now that $\bigcap_{s\in F} \gamma(s)^{-1}(k_s U) \in \zeta(U)_{\gamma,F}$ is non-empty. It is straightforward to verify that $\bigcap_{s\in F} \gamma(s)^{-1}(k_s U) = z\bigcap_{s\in F} \gamma(s)^{-1}(U)$ for every $z \in \bigcap_{s\in F} \gamma(s)^{-1}(k_s U)$. This proves that $\zeta(U)_{\gamma,F} \setminus \{\emptyset\} \subseteq \zeta(C_F(\gamma, U))$, and so concludes the proof. □

In the following results we consider an infinite cancellative left amenable monoid S. So every left Følner net $\mathfrak{s} = (F_i)_{i\in I}$ of S necessarily satisfies $|F_i| \to \infty$, and then we can assume without loss of generality that $1 \in F_i$ for every $i \in I$.

Proposition 2.8. *Let S be an infinite cancellative left amenable monoid, K a totally disconnected compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. If $\mathfrak{s} = (F_i)_{i\in I}$ is a left Følner net of S such that $1 \in F_i$ for every $i \in I$, then for $U \in \mathcal{B}(K)$,*

$$H_{cov}(\gamma, \zeta(U)) = H^{\mathfrak{s}}_{top}(\gamma, U),$$

and so $h_{cov}(\gamma) = h^{\mathfrak{s}}_{top}(\gamma)$.

Proof. By definition, by Lemma 2.7, by (2.2) and by Corollary 2.5, we have

$$H_{cov}(\gamma, \zeta(U)) = \lim_{i\in I} \frac{\log N(\zeta(U)_{\gamma,F_i})}{|F_i|} = \lim_{i\in I} \frac{\log N(\zeta(C_{F_i}(\gamma, U)))}{|F_i|} = \lim_{i\in I} \frac{\log[K : C_{F_i}(\gamma, U)]}{|F_i|} = H^{\mathfrak{s}}_{top}(\gamma, U).$$

To prove the second assertion, let $\mathcal{U}$ be an open cover of K. Since $\mathcal{B}(K)$ is a local base at 1 of K by van Dantzig Theorem [38], there exists $U \in \mathcal{B}(K)$ such that $\mathcal{U} \prec \zeta(U)$. Therefore, in view of (2.1), we have the required equality. □

Corollary 2.9. *Let S be an infinite cancellative left amenable monoid, K a totally disconnected compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. For every left Følner net $\mathfrak{s} = (F_i)_{i\in I}$ of S with $1 \in F_i$ for every $i \in I$,*

$$h_{cov}(\gamma) = \sup\{H^{\mathfrak{s}}_{top}(\gamma, U) \mid U \in \mathcal{B}(K)\}.$$

2.2 Algebraic entropy

Definition 2.10 (See [40]). Let S be a semigroup, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$, G a locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. For $U \in \mathcal{C}(G)$, let

$$H^{\mathfrak{s}}_{alg}(\alpha, U) = \limsup_{i\in I} \frac{\log \mu(T_{F_i}(\alpha, U))}{|F_i|}.$$

The *algebraic entropy relative to* $\mathfrak{s}$ of α is $h^{\mathfrak{s}}_{alg}(\alpha) = \sup\{H^{\mathfrak{s}}_{alg}(\alpha, U) \mid U \in \mathcal{C}(G)\}$.

The map $H^{\mathfrak{s}}_{alg}(\gamma, -)$ is monotone, that is, if $U, V \in \mathcal{C}(G)$ and $U \subseteq V$, then $H^{\mathfrak{s}}_{alg}(\alpha, U) \leq H^{\mathfrak{s}}_{alg}(\alpha, V)$. Therefore, in order to compute $h^{\mathfrak{s}}_{alg}$, it suffices to consider a cofinal subfamily of $\mathcal{C}(G)$. When the locally compact abelian group G is compactly covered, $\mathcal{B}(G)$ is a cofinal in $\mathcal{C}(G)$ by [14, Proposition 2.2], so we have the following result.

Proposition 2.11. *Let S be a semigroup, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$, G a compactly covered locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. Then*

$$h^{\mathfrak{s}}_{alg}(\alpha) = \sup\{H^{\mathfrak{s}}_{alg}(\alpha, U) \mid U \in \mathcal{B}(G)\}.$$

The following result was proved for the case $S = \mathbb{N}$ in [22]. It shows that for $U \in \mathcal{B}(G)$ one can avoid the use of Haar Haar measure to compute the algebraic entropy. As above, we note that, since S is an infinite cancellative left amenable monoid, every left Følner net $\mathfrak{s} = (F_i)_{i\in I}$ of S necessarily satisfies $|F_i| \to \infty$, and so we can assume without loss of generality that $1 \in F_i$ for every $i \in I$.

Proposition 2.12. *Let S be an infinite monoid, $\mathfrak{s} = (F_i)_{i\in I}$ a net in $\mathcal{P}_{fin}(S)$ such that $|F_i| \to \infty$ and $1 \in F_i$ for every $i \in i$, G a locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. If $U \in \mathcal{B}(G)$, then*

$$H^{\mathfrak{s}}_{alg}(\alpha, U) = \limsup_{i\in I} \frac{\log[T_{F_i}(\alpha, U) : U]}{|F_i|}.$$

Proof. For every $i \in I$, since $T_{F_i}(\alpha, U) \in \mathcal{B}(G)$, we have that $[T_{F_i}(\alpha, U) : U]$ is finite and

$$\mu(T_{F_i}(\alpha, U)) = [T_{F_i}(\alpha, U) : U]\mu(U).$$

Hence,

$$\begin{aligned} H^{\mathfrak{s}}_{alg}(\alpha, U) &= \limsup_{i\in I} \frac{\log \mu(T_{F_i}(\alpha, U))}{|F_i|} \\ &= \limsup_{i\in I} \frac{\log[T_{F_i}(\alpha, U) : U] + \log \mu(U)}{|F_i|} \\ &= \limsup_{i\in I} \frac{\log[T_{F_i}(\alpha, U) : U]}{|F_i|}. \end{aligned}$$ □

In the discrete case we compare this algebraic entropy with the algebraic entropy ent introduced in [10] for a right action α of a cancelletive left amenable semigroup on a discrete abelian group A. For $U \in \mathcal{B}(A)$ (i.e., U is a finite subgroup of A), the function

$$\mathcal{P}_{fin}(S) \to \mathbb{R}_{\geq 0},\; F \mapsto \log |T_F(\alpha, U)|$$

is non-decreasing, subadditive, right subinvariant and uniformly bounded on singletons (see [10, Lemma 4.1]). So, by applying Theorem 2.1, we have that the following definition is well posed.

Definition 2.13 (See [10, Definition 4.2]). Let S be a cancellative left amenable semigroup, A a discrete abelian group and $A \overset{\alpha}{\curvearrowleft} S$ a right action. For $U \in \mathcal{B}(A)$, the *algebraic entropy of α with respect to U* is

$$\mathrm{ent}(\alpha, U) = \lim_{i\in I} \frac{\log |T_{F_i}(\alpha, U)|}{|F_i|},$$

for any left Følner net $\mathfrak{s} = (F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$. The *algebraic entropy* of α is $\mathrm{ent}(\alpha) = \sup\{\mathrm{ent}(\alpha, U) \mid U \in \mathcal{B}(A)\}$.

The next result follows immediately from the definitions and Proposition 2.11.

Proposition 2.14. *Let S be a cancellative left amenable semigroup, A a discrete abelian group and $A \overset{\alpha}{\curvearrowleft} S$ a right action. If $\mathfrak{s} = (F_i)_{i\in I}$ is a left Følner net of S, then for $U \in \mathcal{B}(A)$,*

$$\mathrm{ent}(\alpha, U) = H^{\mathfrak{s}}_{alg}(\alpha, U),$$

and so $\mathrm{ent}(\alpha) = h^{\mathfrak{s}}_{alg}(\alpha)$.

3 Receptive entropy

Here we are in the setting of [2, 3, 16, 20, 28].

Definition 3.1. For a finitely generated monoid S, a *regular system* of S is a sequence $\Gamma = (N_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{P}_{fin}(S)$ such that

$$N_0 = \{1\}, \quad \text{and} \quad N_i \cdot N_j \subseteq N_{i+j} \text{ for every } i, j \in \mathbb{N}. \tag{3.1}$$

In particular, for a regular system $\Gamma = (N_n)_{n\in\mathbb{N}}$ of a finitely generated monoid S, we have that

$$N_n \subseteq N_{n+1} \text{ for every } n \in \mathbb{N}.$$

Clearly, if S is a finitely generated monoid and N_1 is a finite set of generators of S with $1 \in N_1$, then $(N_1^n)_{n\in\mathbb{N}}$ of S, with $N_1^0 = \{1\}$, is a regular system such that $S = \bigcup_{n\in\mathbb{N}} N_n$. This is the inspiring fundamental example of a regular system of a finitely generated monoid.

3.1 Topological receptive entropy

We start introducing the following notion of receptive topological entropy, imitating the topological entropy in Definition 2.2.

Definition 3.2. Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. For $U \in \mathcal{C}(G)$, let

$$\widetilde{H}^{\Gamma}_{top}(\alpha, U) = \limsup_{n\to\infty} \frac{-\log \mu(C_{N_n}(\gamma, U))}{n}.$$

The *receptive topological entropy with respect to Γ* of γ is $\widetilde{h}^{\Gamma}_{top}(\gamma) = \sup\{\widetilde{H}^{\Gamma}_{top}(\gamma, U) \mid U \in \mathcal{C}(G)\}$.

Proceeding as above we obtain the following results, we omit their proofs since they are the same of the proofs of Proposition 2.3, Proposition 2.4, Corollary 2.5 and Proposition 2.8, respectively.

Proposition 3.3. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a totally disconnected locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. Then*

$$\widetilde{h}^{\Gamma}_{top}(\gamma) = \sup\{\widetilde{H}^{\Gamma}_{top}(\gamma, U) \mid U \in \mathcal{B}(G)\}.$$

In the following result we see that, for $U \in \mathcal{B}(G)$, we can compute $\widetilde{H}^{\Gamma}_{top}(\gamma, U)$ avoiding the use of Haar measure.

Proposition 3.4. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a locally compact group, μ a Haar measure on G and $S \overset{\gamma}{\curvearrowright} G$ a left action. If $U \in \mathcal{B}(G)$, then*

$$\widetilde{H}^{\Gamma}_{top}(\gamma, U) = \limsup_{n\to\infty} \frac{\log[U : C_{N_n}(\gamma, U)]}{n}.$$

If in addition the group is compact, we obtain the following result.

Corollary 3.5. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, K a compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. If $U \in \mathcal{B}(K)$, then*

$$\widetilde{H}^{\Gamma}_{top}(\gamma, U) = \limsup_{n\to\infty} \frac{\log[K : C_{N_n}(\gamma, U))]}{n}.$$

We recall now the definition of receptive topological entropy from [3], which naturally extends the classical topological entropy h_{cov} from [1].

Definition 3.6. Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, C a compact space and $S \overset{\gamma}{\curvearrowright} C$ a left action. For an open cover $\mathcal{U}$ of C, let

$$\widetilde{H}^{\Gamma}_{cov}(\gamma, \mathcal{U}) = \limsup_{n\to\infty} \frac{\log N(\mathcal{U}_{\gamma,N_n})}{n}.$$

The *receptive topological entropy with respect to Γ* of γ is $\widetilde{h}^{\Gamma}_{cov}(\gamma) = \sup\{\widetilde{H}^{\Gamma}_{cov}(\gamma, \mathcal{U}) \mid \mathcal{U}$ open cover of $C\}$.

As a consequence of Corollary 3.5, the two notions of receptive topological entropy recalled in this section coincide when they are both available:

Proposition 3.7. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, K a totally disconnected compact group and $S \overset{\gamma}{\curvearrowright} K$ a left action. If $U \in \mathcal{B}(K)$, then*

$$\widetilde{H}^{\Gamma}_{top}(\gamma, U) = \widetilde{H}^{\Gamma}_{cov}(\gamma, \zeta(U)).$$

Therefore, $\widetilde{h}^{\Gamma}_{top}(\gamma) = \widetilde{h}^{\Gamma}_{cov}(\gamma)$.

3.2 Algebraic receptive entropy

We give a notion of receptive algebraic entropy, imitating the algebraic entropy in Definition 2.10.

Definition 3.8. Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. For $U \in \mathcal{C}(G)$, let

$$\widetilde{H}^{\Gamma}_{alg}(\alpha, U) = \limsup_{n\to\infty} \frac{\log \mu(T_{N_n}(\gamma, U))}{n}.$$

The *receptive algebraic entropy with respect to Γ* of α is $\widetilde{h}^{\Gamma}_{alg}(\alpha) = \sup\{\widetilde{H}^{\Gamma}_{alg}(\alpha, U) \mid U \in \mathcal{C}(G)\}$.

Proceeding as above we obtain the following results. We omit their proofs since they are the same of the proofs of Proposition 2.11 and Proposition 2.12.

Proposition 3.9. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a compactly covered locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. Then*

$$\widetilde{h}^{\Gamma}_{alg}(\alpha) = \sup\{\widetilde{H}^{\Gamma}_{alg}(\alpha, U) \mid U \in \mathcal{B}(G)\}.$$

In the following result we see that, for $U \in \mathcal{B}(G)$, we can compute $\widetilde{H}^{\Gamma}_{alg}(\alpha, U)$ avoiding the use of Haar measure.

Proposition 3.10. *Let S be a finitely generated monoid, $\Gamma = (N_n)_{n\in\mathbb{N}}$ a regular system of S, G a locally compact abelian group, μ a Haar measure on G and $G \overset{\alpha}{\curvearrowleft} S$ a right action. If $U \in \mathcal{B}(G)$, then*

$$\widetilde{H}^{\Gamma}_{alg}(\alpha, U) = \limsup_{n\to\infty} \frac{\log[T_{N_n}(\alpha, U) : U]}{n}.$$

4 The entropy of the dual action

Let G be a locally compact abelian group and denote by $\widehat{G}$ its Pontryagin dual group. For a continuous homomorphism $\phi : G \to H$, where H is another locally compact abelian group, let $\widehat{\phi} : \widehat{H} \to \widehat{G}$ be the dual of ϕ, defined by $\widehat{\phi}(\chi) = \chi \circ \phi$ for every $\chi \in \widehat{H}$.

If G is a locally compact abelian group and U is a closed subgroup of G, the *annihilator* of U in $\widehat{G}$ is the closed subgroup

$$U^\perp = \{\chi \in \widehat{G} \mid \chi(U) = 0\}$$

of $\widehat{G}$. Moreover, G is topologically isomorphic to $\widehat{\widehat{G}}$, so in the sequel we shall simply identify $\widehat{\widehat{G}}$ with G; under this identification we have that

$$(U^\perp)^\perp = U \tag{4.1}$$

for every closed subgroup U of G.

We give a proof of the following basic fact for reader's convenience.

Lemma 4.1. *Let G be a locally compact abelian group. If $U \in \mathcal{B}(G)$, then $U^\perp \in \mathcal{B}(\widehat{G})$.*

Proof. Since U is open in G, the quotient G/U is discrete, therefore $U^\perp \cong \widehat{G/U}$ is compact. Moreover, U is compact, so $G/U^\perp \cong \widehat{U}$ is discrete and hence $U^\perp$ is open in G. □

Let S be a semigroup and G a locally compact abelian group and consider the left action $S \overset{\gamma}{\curvearrowright} G$. Then γ induces the right action $\widehat{G} \overset{\widehat{\gamma}}{\curvearrowleft} S$, defined by

$$\widehat{\gamma}(s) = \widehat{\gamma(s)} : \widehat{G} \to \widehat{G} \quad \text{for every } s \in S.$$

In fact, fixed $s, t \in S$, since $\gamma(st) = \gamma(s)\gamma(t)$, we have that $\widehat{\gamma}(st) = \widehat{\gamma(st)} = \widehat{\gamma(s)\gamma(t)} = \widehat{\gamma(t)}\widehat{\gamma(s)} = \widehat{\gamma}(t)\widehat{\gamma}(s)$. Analogously, let S be a semigroup and G a locally compact abelian group and consider the right action $G \overset{\alpha}{\curvearrowleft} S$. Then α induces the left action $S \overset{\widehat{\alpha}}{\curvearrowright} \widehat{G}$, defined by

$$\widehat{\alpha}(s) = \widehat{\alpha(s)} : \widehat{G} \to \widehat{G}, \quad \text{for every } s \in S.$$

In fact, fixed $s, t \in S$, since $\alpha(st) = \alpha(t)\alpha(s)$, we have that $\widehat{\alpha}(st) = \widehat{\alpha(st)} = \widehat{\alpha(t)\alpha(s)} = \widehat{\alpha(s)}\widehat{\alpha(t)} = \widehat{\alpha}(s)\widehat{\alpha}(t)$. By Pontryagin duality, $\widehat{\widehat{\gamma}} = \gamma$ and $\widehat{\widehat{\alpha}} = \alpha$.

The following technical lemma is a key step in the proof of the Bridge Theorem, see [14] for more details in the case $S = \mathbb{N}$.

Lemma 4.2. *Let S be a semigroup, G a locally compact abelian group and $S \overset{\gamma}{\curvearrowright} G$ a left action. If $F \in \mathcal{P}_{fin}(S)$ and $U \in \mathcal{B}(G)$, then*

$$[U : C_F(\gamma, U)] = [T_F(\widehat{\gamma}, U^\perp) : U^\perp].$$

Proof. Let $U \in \mathcal{B}(G)$. By Pontryagin duality theory,

$$C_F(\gamma, U)^\perp = \Big(\bigcap_{s\in F} \gamma(s)^{-1}(U)\Big)^\perp = \sum_{s\in F} (\gamma(s)^{-1}(U))^\perp = \sum_{s\in F} \widehat{\gamma(s)}(U^\perp) = \sum_{s\in F} \widehat{\gamma}(s)(U^\perp) = T_F(\widehat{\gamma}, U^\perp).$$

Since $U/C_F(\gamma, U)$ is finite, $U/C_F(\gamma, U) \cong U/\widehat{C_F(\gamma, U)}$, and so

$$U/C_F(\gamma, U) \cong U/\widehat{C_F(\gamma, U)} \cong C_F(\gamma, U)^\perp/U^\perp = T_F(\widehat{\gamma}, U^\perp)/U^\perp.$$

This gives the required equality. □

We are in position to prove the two versions of the Bridge Theorem stated in the introduction.

Proof of Theorem 1.2. Let $U \in \mathcal{B}(G)$. For the net $\mathfrak{s} = (F_i)_{i\in I}$ in $\mathcal{P}_{fin}(S)$, Proposition 2.4, Proposition 2.12 and Lemma 4.2 give

$$H^{\mathfrak{s}}_{top}(\gamma, U) = \limsup_{i\in I} \frac{\log[U : C_{F_i}(\gamma, U)]}{|F_i|} = \limsup_{i\in I} \frac{\log[T_{F_i}(\widehat{\gamma}, U^\perp) : U^\perp]}{|F_i|} = H^{\mathfrak{s}}_{alg}(\widehat{\gamma}, U^\perp).$$

By (4.1) and Lemma 4.1, the map

$$\mathcal{B}(G) \to \mathcal{B}(\widehat{G}), \ U \mapsto U^\perp$$

is a bijection, so we conclude that $h^{\mathfrak{s}}_{top}(\gamma) = h^{\mathfrak{s}}_{alg}(\widehat{\gamma})$ in view of Proposition 2.3 and Proposition 2.11. □

As an application of Theorem 1.2, together with Proposition 2.8 and Proposition 2.14, we obtain the following counterpart of Weiss' Bridge Theorem for amenable semigroup actions.

Corollary 4.3. *Let S be an infinite cancellative left amenable monoid, K a totally disconnected compact abelian group and $S \overset{\gamma}{\curvearrowright} K$ a left action. Then* $h_{cov}(\gamma) = \mathrm{ent}(\widehat{\gamma})$.

The proof of Theorem 1.3 is analogous to that of Theorem 1.2, one applies Proposition 3.4, Proposition 3.10, Proposition 3.3 and Proposition 3.9.

We end with two important examples for the computation of the values of the entropies considered in this paper.

Example 4.4. Consider p a positive prime number, $d > 0$ and denote by $\mathbb{Z}(p)$ the finite group with p-many elements and by $\mathbb{Z}(p)^{(\mathbb{N}^d)}$ the direct sum of $\mathbb{N}^d$-copies of $\mathbb{Z}(p)$, defined by the family of all functions $\mathbb{N}^d \to \mathbb{Z}(p)$ with finite support.

(a) Consider the forward Bernoulli shift $\mathbb{N}^d \overset{\sigma}{\curvearrowright} \mathbb{Z}(p)^{(\mathbb{N}^d)}$, defined for every $s, y \in \mathbb{N}^d$ and $f \in \mathbb{Z}(p)^{(\mathbb{N}^d)}$ by

$$\sigma(s)(f)(y) = \begin{cases} f(y-s)) & \text{if } y \in s + \mathbb{N}^d, \\ 0 & \text{if } y \notin s + \mathbb{N}^d. \end{cases}$$

Then $h_{alg}(\sigma) = \log p$ (see [11]).

The dual action $\widehat{\sigma}$ is conjugated to the backward Bernoulli shift $\mathbb{N}^d \overset{\beta}{\curvearrowright} \mathbb{Z}(p)^{\mathbb{N}^d}$, that is, $\beta(s)(f)(y) = f(y+s)$ for every $f \in \mathbb{Z}(p)^{\mathbb{N}^d}$ and $y, s \in \mathbb{N}^d$. Since it is known that $h_{top}(\beta) = \log p$, from Theorem 1.2 we immediately get that $h_{alg}(\sigma) = \log p$ without any direct computation. Note that in this case, when $d > 1$ and for example we take the regular system $\Gamma = (\{0, \dots, n\}^d)_{n\in\mathbb{N}}$ of $\mathbb{N}^d$, we get $\widetilde{h}^{\Gamma}_{alg}(\sigma) = \infty = \widetilde{h}^{\Gamma}_{top}(\beta)$.

(b) Consider now the forward Bernoulli shift $\sigma : \mathbb{Z}(p)^{(\mathbb{N})} \to \mathbb{Z}(p)^{(\mathbb{N})}$, which has $h_{alg}(\sigma) = \log p$. It is known that $\widehat{\sigma}$ is conjugated to the backward Bernoulli shift $\beta : \mathbb{Z}(p)^{\mathbb{N}} \to \mathbb{Z}(p)^{\mathbb{N}}$ (e.g., see [21]), which indeed has $h_{top}(\beta) = \log p$. Define the action $\mathbb{N}^2 \overset{\sigma\times\sigma}{\curvearrowright} \mathbb{Z}(p)^{(\mathbb{N})} \times \mathbb{Z}(p)^{(\mathbb{N})} \cong \mathbb{Z}(p)^{(\mathbb{N}^2)}$ by letting $(\sigma\times\sigma)(n, m) = \sigma(n)\times\sigma(m)$ for every $(n, m) \in \mathbb{N}^2$. Then $h_{alg}(\sigma\times\sigma) = 0$ (see [10]). The dual action of $\sigma\times\sigma$ is conjugated to the action $\mathbb{N}^2 \overset{\beta\times\beta}{\curvearrowright} \mathbb{Z}(p)^{\mathbb{N}} \times \mathbb{Z}(p)^{\mathbb{N}} \cong \mathbb{Z}(p)^{(\mathbb{N}^2)}$, which therefore has $h_{top}(\beta\times\beta) = 0$ by Theorem 1.2. On the other hand, for the regular system $\Gamma = (\{0, \dots, n\}^2)_{n\in\mathbb{N}}$ of $\mathbb{N}^2$, we have $\widetilde{h}^{\Gamma}_{alg}(\sigma\times\sigma) = 2\log p$ and so also $\widetilde{h}^{\Gamma}_{top}(\beta\times\beta) = 2\log p$ by Theorem 1.3.

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