Bubbles with constant mean curvature, and almost constant mean curvature, in the hyperbolic space

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Bubbles with constant mean curvature, and almost constant mean curvature, in the hyperbolic space

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Abstract Given a constant $k > 1$, let Z be the family of round spheres of radius $\operatorname{artanh}(k^{-1})$ in the hyperbolic space $\mathbb{H}^3$, so that any sphere in Z has mean curvature k. We prove a crucial nondegeneracy result involving the manifold Z. As an application, we provide sufficient conditions on a prescribed function ϕ on $\mathbb{H}^3$, which ensure the existence of a $\mathcal{C}^1$-curve, parametrized by $\varepsilon \approx 0$, of embedded spheres in $\mathbb{H}^3$ having mean curvature $k + \varepsilon\phi$ at each point.

Keywords Hyperbolic geometry · prescribed mean curvature

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1 Introduction

Let K be a given function on the hyperbolic space $\mathbb{H}^3$. The K-bubble problem consists in finding a K-*bubble*, which is an immersed surface $u : \mathbb{S}^2 \to \mathbb{H}^3$ having mean curvature K at each point. Besides its independent interest, the significance of the K-bubble problem is due to its connection with the Plateau problem for disk-type parametric surfaces having prescribed mean curvature K and contour Γ, see for instance [1,13]. In the Euclidean case, the impact of K-bubbles on nonexistence and lack of compactness phenomena in the Plateau problem has been investigated in [5,8,9].

To look for K-bubbles in the hyperbolic setting one can model $\mathbb{H}^3$ via the Poincaré upper half-space $(\mathbb{R}^3_+, p_3^{-2}\delta_{hj})$ and consider the elliptic system

$$\Delta u - 2u_3^{-1}G(\nabla u) = 2u_3^{-1}K(u)\,\partial_x u \wedge \partial_y u \tag{1.1}$$

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for functions $u=(u_1,u_2,u_3)\in\mathcal{C}^2(\mathbb{S}^2,\mathbb{H}^3)$. Here we used the stereographic projection to introduce local coordinates on $\mathbb{S}^2\equiv\mathbb{R}^2\cup\{\infty\}$ and put

$$G_\ell(\nabla u):=\nabla u_3\cdot\nabla u_\ell-\frac{1}{2}|\nabla u|^2\delta_{\ell 3}=-\frac{1}{2}u_3\sum_{h,j=1}^{3}\mathit{\Gamma}_{hj}^{\ell}(u)\nabla u_h\cdot\nabla u_j\,,\quad \ell=1,2,3\,, \tag{1.2}$$

where $\mathit{\Gamma}_{hj}^{\ell}$ are the Christoffel symbols. Any nonconstant solution u to (1.1) is a *generalized* K-bubble in $\mathbb{H}^3$ (see Lemma A.2 in the Appendix and [14, Chapter 2]), that is, u is a conformal parametrization of a surface having mean curvature $K(u)$, apart from a finite number of branch points. Once found a solution to (1.1), the next step should concern the study of the geometric regularity of the surface u, which might have self-intersections and branch points.

A remarkable feature of (1.1) is its variational structure, which means that its solutions are critical points of a certain energy functional E, see the Appendix for details. Because of their underlying geometrical meaning, both (1.1) and E are invariant with respect to the action of Möbius transformations. This produces some lack of compactness phenomena, similar to those observed in the largely studied K-bubble problem, raised by S.T. Yau [22], for surfaces in $\mathbb{R}^3$ (see for instance [7,10,12,20] and references therein; see also the pioneering paper [23] by R. Ye and [3,6,19,21] for related problems). However, the hyperbolic K-bubble problem is definitively more challenging, due to the homogeneity properties that characterize the hyperbolic-area and the hyperbolic-volume functionals.

The main differences between the Euclidean and the hyperbolic case are already evident when the prescribed curvature is a constant $k>0$ (the case $k<0$ is recovered by a change of orientation). Any round sphere of radius $1/k$ in $\mathbb{R}^3$ can be parameterized by an embedded k-bubble, which minimizes the energy functional

$$E_{\mathrm{Eucl}}(u)=\frac{1}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,dz+\frac{2k}{3}\int_{\mathbb{R}^2}u\cdot\partial_x u\wedge\partial_y u\,dz$$

on the Nehari manifold $\{\,u\neq\text{const.}\mid E'_{\mathrm{Eucl}}(u)u=0\,\}$, see [7, Remark 2.6]. In contrast, no immersed hyperbolic k-bubble exists if $k\in(0,1]$, see for instance [16, Theorem 10.1.3]. If $k>1$, then any sphere in $\mathbb{H}^3$ of radius

$$\rho_k:=\operatorname{artanh}\frac{1}{k}=\frac{1}{2}\ln\frac{k+1}{k-1}$$

can be parameterized by an embedding $U:\mathbb{S}^2\to\mathbb{H}^3$, which solves

$$\mathit{\Delta} u-2u_3^{-1}G(\nabla u)=2u_3^{-1}k\,\partial_x u\wedge\partial_y u\quad\text{on }\mathbb{R}^2\,, \tag{$\mathcal{P}_0$}$$

and which is a critical point of the energy functional

$$E_{\mathrm{hyp}}(u)=\frac{1}{2}\int_{\mathbb{R}^2}u_3^{-2}|\nabla u|^2\,dz-k\int_{\mathbb{R}^2}u_3^{-2}e_3\cdot\partial_x u\wedge\partial_y u\,dz\,. \tag{1.3}$$

As in the Euclidean case, the functional E_{hyp} is unbounded from below (see Remark A.1). Therefore U does not minimize the energy E_{hyp} on the Nehari manifold, which in fact fills $\{\,u\neq\text{const.}\,\}$.

Besides their invariance with respect to Möbius transformations, both system $(\mathcal{P}_0)$ and the related energy $E_{\rm hyp}$ are invariant with respect to the 3-dimensional group of hyperbolic translations as well. Thus, any k-bubble generates a smooth 9-dimensional manifold Z of solutions to $(\mathcal{P}_0)$. We explicitly describe the tangent space T_UZ at $U \in Z$ in formula (3.5).

As a further consequence of the invariances of problem $(\mathcal{P}_0)$, any tangent direction $\varphi \in T_UZ$ solves the elliptic system

$$\Delta\varphi - 2U_3^{-1}G'(\nabla U)\nabla\varphi = -U_3^{-1}\varphi_3\Delta U + 2U_3^{-1}k\big(\partial_x\varphi \wedge \partial_y U + \partial_x U \wedge \partial_y\varphi\big)\,, \tag{1.4}$$

which is obtained by linearizing $(\mathcal{P}_0)$ at U.

The next one is the main result of the present paper.

Theorem 1.1 (Nondegeneracy) *Let $U \in Z$. If $\varphi \in \mathcal{C}^2(\mathbb{S}^2, \mathbb{R}^3)$ solves the linear system (1.4), then $\varphi \in T_UZ$.*

Different proofs of the nondegeneracy in the Euclidean case can be found in [11,15,17]. The proof of Theorem 1.1 (see Section 3), is considerably more involved. It requires the choice of a suitable orthogonal frame for functions in $\mathcal{C}^2(\mathbb{S}^2, \mathbb{R}^3)$ and the complete classification of solutions of two systems of linear elliptic differential equations, which might have an independent geometrical interest (see Lemmata 3.3, 3.4).

As an application of Theorem 1.1, we provide sufficient conditions on a prescribed smooth function $\phi : \mathbb{H}^3 \to \mathbb{R}$ that ensure the existence of embedded surfaces $\mathbb{S}^2 \to \mathbb{H}^3$ having nonconstant mean curvature $k + \varepsilon\phi$. Our existence results involve the notion of *stable critical point* already used in [18] and inspired from [2, Chapter 2] (see Subsection 2.2). The main tool is a Lyapunov-Schmidt reduction technique combined with variational arguments, in the spirit of [2].

Theorem 1.2 *Let $k > 1$ and $\phi \in \mathcal{C}^1(\mathbb{H}^3)$ be given. Assume that the function*

$$F_k^\phi(q) := \int\limits_{B^{\mathbb{H}}_{\rho_k}(q)} \phi(p)\, d\mathbb{H}^3_p\ , \quad F_k^\phi : \mathbb{H}^3 \to \mathbb{R} \tag{1.5}$$

has a stable critical point in an open set $A \Subset \mathbb{H}^3$. For every $\varepsilon \in \mathbb{R}$ close enough to 0 there exist a point $q^\varepsilon \in A$, a conformal parametrization $\mathrm{U}_{q^\varepsilon}$ of a sphere of radius ρ_k about q^ε, and a conformally embedded $(k + \varepsilon\phi)$-bubble u^ε, such that $\|u^\varepsilon - \mathrm{U}_{q^\varepsilon}\|_{\mathcal{C}^2} = O(\varepsilon)$ as $\varepsilon \to 0$.

Moreover, any sequence $\varepsilon_h \to 0$ has a subsequence ε_{h_j} such that $q^{\varepsilon_{h_j}}$ converges to a critical point for F_k^ϕ. In particular, if $\hat q \in A$ is the unique critical point for F_k^ϕ in $\overline{A}$, then $u^\varepsilon \to \mathrm{U}_{\hat q}$ in $\mathcal{C}^2(\mathbb{S}^2, \mathbb{H}^3)$.

Theorem 1.3 *Assume that $\phi \in \mathcal{C}^1(\mathbb{H}^3)$ has a stable critical point in an open set $A \Subset \mathbb{H}^3$. Then there exists $k_0 > 1$ such that for any $k > k_0$ and for every ε close enough to 0, there exists a conformally embedded $(k + \varepsilon\phi)$-bubble.*

In Section 4 we first show that the existence of a critical point for $F_k^\phi(q)$ is a necessary condition in Theorem 1.2. Then we perform the dimension reduction and prove Theorems 1.2, 1.3. With respect to correspondent Euclidean results in [7], a different choice of the functional setting allows us to weaken the regularity assumption on ϕ (from $\mathcal{C}^2$ to $\mathcal{C}^1$).

We conclude the paper with an Appendix in which we collect some partially known results about the variational approach to (1.1) and prove a nonexistence result for (1.1) which, in particular, justifies the assumption on the existence of a critical point for ϕ in Theorem 1.3.

2 Notation and preliminaries

The vector space $\mathbb{R}^n$ is endowed with the Euclidean scalar product $\xi \cdot \xi'$ and norm $|\xi|$. We denote by $\{e_1, e_2, e_3\}$ the canonical basis and by $\wedge$ the exterior product in $\mathbb{R}^3$.

We will often identify the complex number $z = x + iy$ with the vector $z = (x, y) \in \mathbb{R}^2$. Thus, $iz \equiv (-y, x)$, $z^2 \equiv (x^2 - y^2, 2xy)$ and $z^{-1} \equiv |z|^{-2}(x, -y)$ if $z \neq 0$.

In local coordinates induced by the stereographic projections from the north and the south poles, the round metric on the sphere $\mathbb{S}^2$ is given by $g_{hj} = \mu^2 \delta_{hj}$, $d\mathbb{S}^2 = \mu^2 dz$, where

$$\mu(z) = \frac{2}{1 + |z|^2}\,.$$

We identify the compactified plane $\overline{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$ with the sphere $\mathbb{S}^2$ through the inverse of the stereographic projection from the north pole, which is explicitly given by

$$\omega(x, y) = (\mu x, \mu y, 1 - \mu)\,. \tag{2.1}$$

The identity $|\omega|^2 \equiv 1$ trivially gives $\omega \cdot \partial_x \omega \equiv 0$, $\omega \cdot \partial_y \omega \equiv 0$. We also notice that ω is a conformal (inward-pointing) parametrization of the unit sphere and satisfies

$$\begin{cases} \Delta\omega = 2\partial_x\omega \wedge \partial_y\omega\ , \quad -\Delta\omega = 2\mu^2\omega \\ \partial_x\omega \cdot \partial_y\omega = 0 \\ |\partial_x\omega|^2 = |\partial_y\omega|^2 = \frac{1}{2}|\nabla\omega|^2 = \mu^2\,. \\ \partial_x\omega \wedge \omega = \partial_y\omega\ , \quad \omega \wedge \partial_y\omega = \partial_x\omega\ , \quad \partial_x\omega \wedge \partial_y\omega = -\mu^2\omega. \end{cases} \tag{2.2}$$

2.1 The Poincaré half-space model

We adopt as model for the three dimensional hyperbolic space $\mathbb{H}^3$ the upper half-space $\mathbb{R}^3_+ = \{(p_1, p_2, p_3) \in \mathbb{R}^3 \mid p_3 > 0\}$ endowed with the Riemannian metric $g_{hj} = p_3^{-2}\delta_{hj}$.

The hyperbolic distance $d_{\mathbb{H}}(p, q)$ in $\mathbb{H}^3$ is related to the Euclidean one by

$$\cosh d_{\mathbb{H}}(p, q) = 1 + \frac{|p - q|^2}{2p_3q_3}\,,$$

and the hyperbolic ball $B^{\mathbb{H}}_\rho(p)$ centered at $p = (p_1, p_2, p_3)$ is the Euclidean ball of center $(p_1, p_2, p_3 \cosh\rho)$ and radius $p_3 \sinh\rho$.

If $F : \mathbb{H}^3 \to \mathbb{R}$ is a differentiable function, then $\nabla^{\mathbb{H}} F(p) = p_3^2 \nabla F(p)$, where $\nabla^{\mathbb{H}}$, ∇ are the hyperbolic and the Euclidean gradients, respectively. In particular, $\nabla^{\mathbb{H}} F(p) = 0$ if and only if $\nabla F(p) = 0$. The hyperbolic volume form is related to the Euclidean one by $d\mathbb{H}^3_p = p_3^{-3}dp$.

2.2 Stable critical points

Let $X \in \mathcal{C}^1(\mathbb{H}^3)$ and let $\Omega \Subset \mathbb{H}^3$ be open. We say that X has a stable critical point in Ω if there exists $r > 0$ such that any function $G \in \mathcal{C}^1(\overline{\Omega})$ satisfying $\|G - X\|_{\mathcal{C}^1(\overline{\Omega})} < r$ has a critical point in Ω.

As noticed in [18], conditions to have the existence of a stable critical point $p \in \Omega$ for X are easily given via elementary calculus. For instance, one can use Browder's topological degree theory or can assume that

$$\min_{\partial\Omega} X > \min_{\Omega} X \quad \text{or} \quad \max_{\partial\Omega} X < \max_{\Omega} X .$$

Finally, if X is of class $\mathcal{C}^2$ and Ω contains a nondegenerate critical point p_0 (i.e. the Hessian matrix of X at p_0 is invertible), then p_0 is stable.

2.3 Function spaces

Any function f on $\overline{\mathbb{R}}^2$ is identified with $f \circ \omega^{-1}$, which is a function on $\mathbb{S}^2$. If no confusion can arise, from now on we write f instead of $f \circ \omega^{-1}$.

The Hilbertian norm on $L^2(\overline{\mathbb{R}}^2, \mathbb{R}^n) \equiv L^2(\mathbb{S}^2, \mathbb{R}^n)$ is given by

$$\|f\|_2^2 = \int_{\mathbb{R}^2} |f|^2 \, \mu^2 dz < \infty .$$

Let $m \geq 0$. We endow

$$\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^n) = \{\, u \in \mathcal{C}^m(\mathbb{R}^2, \mathbb{R}^n) \mid u(z^{-1}) \in \mathcal{C}^m(\mathbb{R}^2, \mathbb{R}^n) \,\} \equiv \mathcal{C}^m(\mathbb{S}^2, \mathbb{R}^n)$$

with the standard Banach space structure (we agree that $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^n) = \mathcal{C}^{\lfloor m \rfloor, m - \lfloor m \rfloor}(\overline{\mathbb{R}}^2, \mathbb{R}^n)$ if m is not an integer). If m is an integer, a norm in $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^n)$ is given by

$$\|u\|_{\mathcal{C}^m} = \sum_{j=0}^{m} \|\mu^{-j} \nabla^j u\|_\infty . \tag{2.3}$$

Since we adopt the upper half-space model for $\mathbb{H}^3$, we are allowed to write

$$\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{H}^3) = \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3_+) = \{u \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3) \mid u_3 > 0\} ,$$

so that $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{H}^3)$ is an open subset of $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$.

If $\psi, \varphi \in \mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ and $\tau \in \mathbb{R}^2$, we put

$$\nabla\psi \cdot \nabla\varphi = \partial_x\psi \cdot \partial_x\varphi + \partial_y\psi \cdot \partial_y\varphi \; , \qquad \tau\nabla\varphi = \tau_1 \partial_x\varphi + \tau_2 \partial_y\varphi$$

(notice that $\tau\nabla\varphi(z) = d\varphi(z)\tau$ for any $z \in \mathbb{R}^2$). For instance, we have

$$z^h \nabla\varphi = \begin{cases} \partial_x\varphi & \text{if } h = 0 \\ x\partial_x\varphi + y\partial_y\varphi & \text{if } h = 1 \end{cases} , \qquad iz^h \nabla\varphi = \begin{cases} \partial_y\varphi & \text{if } h = 0 \\ -y\partial_x\varphi + x\partial_y\varphi & \text{if } h = 1 . \end{cases}$$

For future convenience we notice, without proof, that the next identities hold:

$$\begin{cases} \partial_x\omega = e_1 - \omega_1\omega - e_2 \wedge \omega \\ z\nabla\omega = e_3 - \omega_3\omega \\ z^2\nabla\omega = -(e_1 - \omega_1\omega + e_2 \wedge \omega) \end{cases} \qquad \begin{cases} \partial_y\omega = e_2 - \omega_2\omega + e_1 \wedge \omega \\ iz\nabla\omega = e_3 \wedge \omega, \\ iz^2\nabla\omega = e_2 - \omega_2\omega - e_1 \wedge \omega\,. \end{cases} \tag{2.4}$$

The monograph [4] is our reference text for the theory of Sobolev spaces on Riemannian manifolds. In view of our purposes, it is important to notice that

$$H^1(\overline{\mathbb{R}}^2, \mathbb{R}^n) = \{\, u \in H^1_{\rm loc}(\mathbb{R}^2, \mathbb{R}^n) \mid |\nabla u| + |u|\,\mu \in L^2(\mathbb{R}^2) \,\} \equiv H^1(\mathbb{S}^2, \mathbb{R}^n)\,.$$

We simply write $L^2(\overline{\mathbb{R}}^2)$, $\mathcal{C}^m(\overline{\mathbb{R}}^2)$ and $H^1(\overline{\mathbb{R}}^2)$ instead of $L^2(\overline{\mathbb{R}}^2, \mathbb{R})$, $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R})$ and $H^1(\overline{\mathbb{R}}^2, \mathbb{R})$, respectively.

2.4 Möbius transformations and hyperbolic translations

Transformations in $PGL(2, \mathbb{C})$ are obtained by composing translations, dilations, rotations and complex inversion. Its Lie algebra admits as a basis the transforms

$$z \mapsto 1\,, \quad z \mapsto i\,, \quad z \mapsto z\,, \quad z \mapsto iz\,, \quad z \mapsto z^2\,, \quad z \mapsto iz^2\,.$$

Therefore, for any $u \in \mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3)$, the functions

$$z^h\nabla u\,, \quad iz^h\nabla u\,, \qquad h = 0, 1, 2\,,$$

span the tangent space to the manifold $\{\, u \circ g \mid g \in PGL(2, \mathbb{C}) \,\}$ at u.

Hyperbolic translations are obtained by composing a horizontal (Euclidean) translation $p \mapsto p + ae_1 + be_2$, $a, b \in \mathbb{R}$ with an Euclidean homothety $p \mapsto tp$, $t > 0$. Therefore, for any $u \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{H}^3)$, the functions

$$e_1\,, \quad e_2\,, \quad u\,,$$

span the tangent space to the manifold $\{\, u_q \mid q \in \mathbb{H}^3 \,\}$ at u, where

$$u_q := q_3 u + q - (q \cdot e_3)e_3\,. \tag{2.5}$$

3 Nondegeneracy of hyperbolic k-bubbles

The proof of Theorem 1.1 needs some preliminary work. We put

$$\mathrm{U} = r_k(\omega + ke_3)\,, \quad r_k := \sinh\rho_k = \frac{1}{k}\cosh\rho_k = \frac{1}{\sqrt{k^2-1}}\,,$$

where ω is given by (2.1). Since U is a conformal parametrization of the Euclidean sphere of radius r_k about kr_ke_3, which coincides with the hyperbolic sphere of radius ρ_k about e_3, then U has curvature k and in fact it solves $(\mathcal{P}_0)$. Accordingly with (2.5), we put

$$\mathrm{U}_q := q_3\mathrm{U} + q - (q \cdot e_3)\, e_3 \tag{3.1}$$

(notice that $\mathrm{U}_{e_3} = \mathrm{U}$), and introduce the 9-dimensional manifold

$$Z = \left\{ \mathrm{U}_q \circ g \mid g \in PGL(2, \mathbb{C}) \ , \ q \in \mathbb{H}^3 \right\} . \tag{3.2}$$

Remark 3.1 *Any surface $U \in Z$ is an embedding and solves $(\mathcal{P}_0)$. Conversely, let $U \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$ be an embedding. If U solves $(\mathcal{P}_0)$, then it is a k-bubble by Lemma A.2 and, thanks to an Alexandrov' type argument (see for instance [16, Corollary 10.3.2]) it parametrizes a sphere of hyperbolic radius ρ_k and Euclidean radius r_k. Since U is conformal, then $\Delta U = 2r_k^{-1}\partial_x U \wedge \partial_y U$. Therefore $U \in Z$ by the uniqueness result in [5].*

By the remarks in Subsection 2.4 and since $\nabla \mathrm{U}_q$ is proportional to $\nabla \omega$, we have that $T_{\mathrm{U}_q} Z = T_{\mathrm{U}} Z$ for any $q \in \mathbb{H}^3$, and

$$T_{\mathrm{U}} Z = \left\langle \{ z^h \nabla \omega \ , \ i z^h \nabla \omega \mid h = 0, 1, 2 \} \right\rangle \oplus \left\langle e_1, e_2, \mathrm{U} \right\rangle . \tag{3.3}$$

Moreover, any tangent direction $\tau \in T_{\mathrm{U}} Z$ solves (1.4).

It is convenient to split $\mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ in the direct sum of its closed subspaces

$$\langle \omega \rangle_{\mathcal{C}^m}^{\perp} := \{ \varphi \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3) \mid \varphi \cdot \omega \equiv 0 \text{ on } \mathbb{R}^2 \} , \quad \langle \omega \rangle_{\mathcal{C}^m} := \{ \eta \omega \mid \eta \in \mathcal{C}^m(\overline{\mathbb{R}}^2) \} . \tag{3.4}$$

Since $T_{\mathrm{U}} Z = \left(T_{\mathrm{U}} Z \cap \langle \omega \rangle_{C^2}^{\perp} \right) \oplus \left(T_{\mathrm{U}} Z \cap \langle \omega \rangle_{C^2} \right)$, from (2.4) we infer another useful description of the tangent space, that is

$$T_{\mathrm{U}} Z = \left\{ s - (s \cdot \omega)\omega + t \wedge \omega \mid s, t \in \mathbb{R}^3 \right\} \oplus \left\{ (\alpha \cdot (k\omega + e_3))\, \omega \mid \alpha \in \mathbb{R}^3 \right\} . \tag{3.5}$$

We now introduce the differential operator

$$J_0(u) = -\mathrm{div}(u_3^{-2} \nabla u) - u_3^{-3} |\nabla u|^2 e_3 + 2k u_3^{-3} \partial_x u \wedge \partial_y u .$$

Notice that $Z \subset \{J_0 = 0\}$. Further, let $I(z) = z^{-1}$. Since $J_0(u \circ I) = |z|^{-4} J_0(u) \circ I$ for any $u \in \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{H}^3)$, $m \ge 0$, then J_0 is a $\mathcal{C}^1$ map

$$J_0 : \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{H}^3) \to \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3) .$$

We denote by $J_0'(u) : \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{R}^3) \to \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ its differential at u.

Finally, $J_0(\mathrm{U}_q \circ g) = 0$ for any $g \in PGL(2, \mathbb{C})$, $q \in \mathbb{H}^3$, that implies $T_{\mathrm{U}} Z \subseteq \ker J_0'(\mathrm{U})$. In order to prove Theorem 1.1 it suffices to show that

$$\ker J_0'(\mathrm{U}) \subseteq T_{\mathrm{U}} Z .$$

Main computations
Recall that $\mathrm{U} = r_k(\omega + ke_3)$ solves $J_0(\mathrm{U}) = 0$ to check that

$$J_0'(\mathrm{U})\varphi = -\mathrm{div}\big(\mathrm{U}_3^{-2}\nabla\varphi\big) \\ + 2\mathrm{U}_3^{-3}\big[G'(\nabla\mathrm{U})\nabla\varphi - \nabla U_3\nabla\varphi - \frac{1}{2}\varphi_3\Delta\mathrm{U} + k(\partial_x\varphi\wedge\partial_y\mathrm{U} + \partial_x\mathrm{U}\wedge\partial_y\varphi)\big]\,,$$

where G is given by (1.2). Since $\nabla\omega_3 = -\nabla\mu = \mu^2 z$, thanks to (2.2) we have

$$r_k^2 J_0'(\mathrm{U})\varphi = -\mathrm{div}\big((\omega_3+k)^{-2}\nabla\varphi\big) \\ + 2(\omega_3+k)^{-3}\big[\big(G'(\nabla\omega)\nabla\varphi - \mu^2 z\nabla\varphi\big) + \mu^2\varphi_3\omega + k(\partial_x\varphi\wedge\partial_y\omega + \partial_x\omega\wedge\partial_y\varphi)\big]\,, \tag{3.6}$$

$$G'(\nabla\omega)\nabla\varphi - \mu^2 z\nabla\varphi = \nabla\varphi_3\nabla\omega - (\nabla\varphi\cdot\nabla\omega)e_3\,. \tag{3.7}$$

To rewrite (3.6) in a less obscure form, we decompose any $\varphi \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$, $m \ge 0$, as

$$\varphi = \mathcal{P}\varphi + (\varphi\cdot\omega)\omega\,, \qquad \mathcal{P}\varphi := \varphi - (\varphi\cdot\omega)\omega = \mu^{-2}\big((\varphi\cdot\partial_x\omega)\partial_x\omega + (\varphi\cdot\partial_y\omega)\partial_y\omega\big)\,, \tag{3.8}$$

compare with (3.4). Accordingly, for $\varphi \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ we have

$$J_0'(\mathrm{U})\varphi = \mathcal{P}\big(J_0'(\mathrm{U})\varphi\big) + (J_0'(\mathrm{U})\varphi\cdot\omega)\omega\,,$$

so that we can reconstruct $J_0'(\mathrm{U})\varphi \in \mathcal{C}^0(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ by providing explicit expressions for $\mathcal{P}\big(J_0'(\mathrm{U})\varphi\big)$ and $J_0'(\mathrm{U})\varphi\cdot\omega$, separately. This will be done in the next Lemma.

Lemma 3.1 *Let $\varphi \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)$. Then*

$$r_k^2\mathcal{P}\big(J_0'(\mathrm{U})\varphi\big) = \mathcal{P}\Big(-\mathrm{div}\Big(\frac{\nabla\mathcal{P}\varphi}{(\omega_3+k)^2}\Big)\Big) + \frac{2\mu^2}{(\omega_3+k)^3}\,(iz\nabla\mathcal{P}\varphi)\wedge\omega - \frac{2\mu^2}{(\omega_3+k)^2}\,\mathcal{P}\varphi\,, \tag{3.9}$$

$$r_k^2(J_0'(\mathrm{U})\varphi)\cdot\omega = -\mathrm{div}\Big(\frac{\nabla(\varphi\cdot\omega)}{(\omega_3+k)^2}\Big) - \frac{2k\,\mu^2}{(\omega_3+k)^3}\,(\varphi\cdot\omega)\,. \tag{3.10}$$

Proof We introduce the differential operator $L = -\mathrm{div}\big((\omega_3+k)^{-2}\nabla\ \big)$ and start to prove (3.10) by noticing that

$$L\varphi\cdot\omega = L(\varphi\cdot\omega) + 2(\omega_3+k)^{-3}\big[(\omega_3+k)\nabla\varphi\cdot\nabla\omega - \mu^2\varphi\cdot(z\nabla\omega) - \mu^2(\omega_3+k)(\varphi\cdot\omega)\big]\,. \tag{3.11}$$

Recalling that ω is pointwise orthogonal to $\partial_x\omega, \partial_y\omega$, from (3.7) we obtain

$$\big(G'(\nabla\omega)\nabla\varphi - \mu^2 z\nabla\varphi\big)\cdot\omega = -(\nabla\varphi\cdot\nabla\omega)\omega_3\,.$$

Further, by (2.2) we have $(\partial_x\varphi\wedge\partial_y\omega + \partial_x\omega\wedge\partial_y\varphi)\cdot\omega = -\nabla\varphi\cdot\nabla\omega$. Finally, we obtain

$$r_k^2(J_0'(\mathrm{U})\varphi)\cdot\omega = L(\varphi\cdot\omega) - 2(\omega_3+k)^{-3}\mu^2\big[\varphi\cdot(z\nabla\omega) - \varphi_3 + (\omega_3+k)(\varphi\cdot\omega)\big]\,,$$

and (3.10) follows, because $e_3 = z\nabla\omega + \omega_3\omega$, see (2.4).

Next, using the equivalent formulation

$$\mathrm{U}_3^2 J_0'(\mathrm{U})\varphi = -\Delta\varphi + 2(\omega_3+k)^{-1}\,\big[G'(\nabla\omega)\nabla\varphi + \mu^2\omega\varphi_3 + k(\partial_x\varphi\wedge\partial_y\omega + \partial_x\omega\wedge\partial_y\varphi)\big]$$

we find that, for $\varphi = \eta\omega$, $\eta \in \mathcal{C}^2(\overline{\mathbb{R}}^2)$, it holds

$$\mathrm{U}_3^2 J_0'(\mathrm{U})(\eta\omega) \cdot \partial_x\omega = \mathrm{U}_3^2 J_0'(\mathrm{U})(\eta\omega) \cdot \partial_y\omega = 0\,,$$

whence we infer

$$\mathcal{P}\big(J_0'(\mathrm{U})(\varphi - \mathcal{P}\varphi)\big) = 0\,, \quad \text{for every } \varphi \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)\,. \tag{3.12}$$

Thanks to (3.10) and (3.12) we get $\mathcal{P}\big(J_0'(\mathrm{U})\varphi\big) = J_0'(\mathrm{U})(\mathcal{P}\varphi)$, thus to conclude the proof we can assume that $\varphi = \mathcal{P}\varphi$. Since φ is pointwise orthogonal to ω, we trivially have

$$\partial_x\varphi \cdot \omega = -\varphi \cdot \partial_x\omega\,, \quad \partial_y\varphi \cdot \omega = -\varphi \cdot \partial_y\omega\,.$$

We start to handle (3.7). From $e_3 = z\nabla\omega + \omega_3\omega$ we get

$$\begin{aligned}(G'(\nabla\omega)\nabla\varphi - \mu^2 z\nabla\varphi) + \omega_3(\nabla\varphi \cdot \nabla\omega)\omega &= \nabla\varphi_3\nabla\omega - (\nabla\varphi \cdot \nabla\omega)z\nabla\omega \\ &= \big(\partial_x\varphi_3 - x(\nabla\varphi \cdot \nabla\omega)\big)\partial_x\omega + \big(\partial_y\varphi_3 - y(\nabla\varphi \cdot \nabla\omega)\big)\partial_y\omega\,.\end{aligned}$$

Further,

$$\begin{aligned}\partial_x\varphi_3 - x(\nabla\varphi \cdot \nabla\omega) &= \partial_x\varphi \cdot (z\nabla\omega + \omega_3\omega) - x(\nabla\varphi \cdot \nabla\omega) \\ &= \big(\partial_x\varphi \cdot (z\nabla\omega) - x(\nabla\varphi \cdot \nabla\omega)\big) - \omega_3\varphi \cdot \partial_x\omega = -(iz\nabla\varphi) \cdot \partial_y\omega - \omega_3\varphi \cdot \partial_x\omega\,.\end{aligned}$$

In a similar way one can check that $\partial_y\varphi_3 - y(\nabla\varphi \cdot \nabla\omega) = (iz\nabla\varphi) \cdot \partial_x\omega - \omega_3\varphi \cdot \partial_y\omega$, thus

$$G'(\nabla\omega)\nabla\varphi - \mu^2 z\nabla\varphi = \mu^2(iz\nabla\varphi) \wedge \omega - \omega_3(\nabla\varphi \cdot \nabla\omega)\omega - \mu^2\omega_3\varphi.$$

Next, using (2.2) we can compute

$$\begin{aligned}\partial_x\varphi \wedge \partial_y\omega &= \partial_x\varphi \wedge (\partial_x\omega \wedge \omega) = -(\varphi \cdot \partial_x\omega)\partial_x\omega - (\partial_x\varphi \cdot \partial_x\omega)\omega \\ \partial_x\omega \wedge \partial_y\varphi &= (\omega \wedge \partial_y\omega) \wedge \partial_y\varphi = -(\varphi \cdot \partial_y\omega)\partial_y\omega - (\partial_y\varphi \cdot \partial_y\omega)\omega,\end{aligned}$$

which give the identity

$$\partial_x\varphi \wedge \partial_y\omega + \partial_x\omega \wedge \partial_y\varphi = -\mu^2\varphi - (\nabla\varphi \cdot \nabla\omega)\omega\,, \tag{3.13}$$

that holds for any $\varphi \in \langle\omega\rangle^{\perp}_{\mathcal{C}^m}$.

Putting together the above informations we arrive at

$$r_k^2 J_0'(\mathrm{U})\varphi = L\varphi + \frac{2\mu^2}{(\omega_3+k)^3}\,(iz\nabla\varphi) \wedge \omega - \frac{2\mu^2}{(\omega_3+k)^2}\,\varphi + \frac{2}{(\omega_3+k)^3}\big[\mu^2\varphi_3 - (\omega_3+k)\nabla\varphi \cdot \nabla\omega\big]\omega\,.$$

Using (3.11) and $\varphi_3 = \varphi \cdot (z\nabla\omega)$, we conclude the proof. $\square$

Thanks to Lemma 3.1 we can study the system $J_0'(\mathrm{U})\varphi = 0$ separately, on $\langle\omega\rangle^\perp_{\mathcal{C}^m}$ first, and on $\langle\omega\rangle_{\mathcal{C}^m}$ later. In fact, $\varphi \in \ker J_0'(\mathrm{U})$ if and only if the pair of functions

$$\psi := \mathcal{P}\varphi \in \langle\omega\rangle^\perp_{\mathcal{C}^2} \subset \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)\ , \qquad \eta := \varphi\cdot\omega \in \mathcal{C}^2(\overline{\mathbb{R}}^2)\,,$$

solves

$$\begin{cases} \mathcal{P}\Big(-\operatorname{div}\Big(\dfrac{\nabla\psi}{(\omega_3+k)^2}\Big)\Big) + \dfrac{2\mu^2}{(\omega_3+k)^3}\,(iz\nabla\psi)\wedge\omega = \dfrac{2\mu^2}{(\omega_3+k)^2}\,\psi\,, & \text{(3.14a)} \\ -\operatorname{div}\Big(\dfrac{\nabla\eta}{(\omega_3+k)^2}\Big) = \dfrac{2k\,\mu^2}{(\omega_3+k)^3}\,\eta\,. & \text{(3.14b)} \end{cases}$$

We begin by facing problem (3.14a). Firstly, we show that the quadratic form associated to the differential operator $J_0'(\mathrm{U})$ is nonnegative on $\langle\omega\rangle^\perp_{\mathcal{C}^2}$.

Lemma 3.2 *Let $\psi \in \langle\omega\rangle^\perp_{\mathcal{C}^2}$. Then*

$$\int_{\mathbb{R}^2} J_0'(\mathrm{U})\psi\cdot\psi\,dz = r_k^{-2}\int_{\mathbb{R}^2} \frac{(\partial_x\psi\cdot\partial_x\omega - \partial_y\psi\cdot\partial_y\omega)^2 + (\partial_x\psi\cdot\partial_y\omega + \partial_y\psi\cdot\partial_x\omega)^2}{\mu^2(\omega_3+k)^2}\,dz\,.$$

Proof Since $J_0'(\mathrm{U})\psi\cdot\psi = \mathcal{P}\big(J_0'(\mathrm{U})\psi\big)\cdot\psi$ and $\mathcal{P}\psi = \psi$, formula (3.9) gives

$$r_k^2\int_{\mathbb{R}^2} J_0'(\mathrm{U})\psi\cdot\psi\,dz = \int_{\mathbb{R}^2}\frac{|\nabla\psi|^2}{(\omega_3+k)^2}\,dz + 2\int_{\mathbb{R}^2}\frac{\psi\cdot(iz\nabla\psi)\wedge\omega}{(\omega_3+k)^3}\,\mu^2dz - 2\int_{\mathbb{R}^2}\frac{|\psi|^2}{(\omega_3+k)^2}\,\mu^2dz.$$

Now we prove the identity

$$B_\psi := 2\int_{\mathbb{R}^2}\frac{\psi\cdot(iz\nabla\psi)\wedge\omega}{(\omega_3+k)^3}\,\mu^2dz = 2\int_{\mathbb{R}^2}\frac{\omega\cdot\partial_x\psi\wedge\partial_y\psi}{(\omega_3+k)^2}\,dz + \int_{\mathbb{R}^2}\frac{|\psi|^2}{(\omega_3+k)^2}\,\mu^2dz\,. \tag{3.15}$$

We use polar coordinates ρ,θ on $\mathbb{R}^2$ and notice that $\partial_\theta\psi = iz\nabla\psi$. From $\rho\mu^2 = \partial_\rho\omega_3$ we get

$$\begin{aligned} B_\psi &= -\int_0^{2\pi}d\theta\int_0^\infty(\psi\cdot\partial_\theta\psi\wedge\omega)\partial_\rho(\omega_3+k)^{-2}\,d\rho \\ &= \int_0^\infty d\rho\int_0^{2\pi}\frac{\omega\cdot\partial_\rho\psi\wedge\partial_\theta\psi - \psi\cdot\partial_\rho\omega\wedge\partial_\theta\psi}{(\omega_3+k)^2}d\theta + \int_0^\infty d\rho\int_0^{2\pi}\frac{\partial_{\rho\theta}\psi\cdot\omega\wedge\psi}{(\omega_3+k)^2}d\theta \\ &= \int_0^\infty d\rho\int_0^{2\pi}\frac{\omega\cdot\partial_\rho\psi\wedge\partial_\theta\psi - \psi\cdot\partial_\rho\omega\wedge\partial_\theta\psi}{(\omega_3+k)^2}d\theta + \int_0^\infty d\rho\int_0^{2\pi}\frac{\omega\cdot\partial_\rho\psi\wedge\partial_\theta\psi - \psi\cdot\partial_\rho\psi\wedge\partial_\theta\omega}{(\omega_3+k)^2}d\theta\,. \end{aligned}$$

Using the elementary identity $\partial_\rho\alpha\wedge\partial_\theta\beta = \rho(\partial_x\alpha\wedge\partial_y\beta)$, we see that

$$B_\psi = 2\int_{\mathbb{R}^2}\frac{\omega\cdot\partial_x\psi\wedge\partial_y\psi}{(\omega_3+k)^2}\,dz - \int_{\mathbb{R}^2}\frac{\psi\cdot(\partial_x\omega\wedge\partial_y\psi + \partial_x\psi\wedge\partial_y\omega)}{(\omega_3+k)^2}\,dz\,,$$

and (3.15) follows from (3.13) (with φ replaced by ψ).

Thanks to (3.15), we have the identity

$$r_k^2 \int\limits_{\mathbb{R}^2} J_0'(\mathrm{U})\psi \cdot \psi \, dz = \int\limits_{\mathbb{R}^2} \frac{|\nabla\psi|^2 + 2\omega \cdot \partial_x\psi \wedge \partial_y\psi - \mu^2|\psi|^2}{(\omega_3 + k)^2} \, dz,$$

so that we only need to handle the function

$$b_\psi := |\nabla\psi|^2 + 2\omega \cdot \partial_x\psi \wedge \partial_y\psi - \mu^2|\psi|^2.$$

We decompose $\partial_x\psi$ and $\partial_y\psi$ accordingly with (3.8), to obtain

$$\mu^2\partial_x\psi = (\partial_x\psi \cdot \partial_x\omega)\omega_x + (\partial_x\psi \cdot \partial_y\omega)\omega_y - \mu^2(\psi \cdot \partial_x\omega)\omega \,,$$
$$\mu^2\partial_y\psi = (\partial_y\psi \cdot \partial_x\omega)\omega_x + (\partial_y\psi \cdot \partial_y\omega)\omega_y - \mu^2(\psi \cdot \partial_y\omega)\omega \,,$$

respectively. Since $|\nabla\psi|^2 = |\partial_x\psi|^2 + |\partial_y\psi|^2$, we infer

$$\mu^2\big(|\nabla\psi|^2 - \mu^2|\psi|^2\big) = (\partial_x\psi \cdot \partial_x\omega)^2 + (\partial_x\psi \cdot \partial_y\omega)^2 + (\partial_y\psi \cdot \partial_x\omega)^2 + (\partial_y\psi \cdot \partial_y\omega)^2 \,.$$

Writing $\mu^2\omega = -\partial_x\omega \wedge \partial_y\omega$, see (2.2), we get

$$\mu^2\omega \cdot (\partial_x\psi \wedge \partial_y\psi) = -(\partial_x\psi \cdot \partial_x\omega)(\partial_y\psi \cdot \partial_y\omega) + (\partial_x\psi \cdot \partial_y\omega)(\partial_y\psi \cdot \partial_x\omega) \,,$$

from which it readily follows that $\mu^2 b_\psi = (\partial_x\psi \cdot \partial_x\omega - \partial_y\psi \cdot \partial_y\omega)^2 + (\partial_x\psi \cdot \partial_y\omega + \partial_y\psi \cdot \partial_x\omega)^2$. The proof is complete. □

Lemma 3.3 *Let $\psi \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ be a solution to (3.14a). There exist $s, t \in \mathbb{R}^3$ such that*

$$\psi = s - (s \cdot \omega)\omega + t \wedge \omega \,,$$

and thus $\psi \in T_{\mathrm{U}}Z \cap \langle\omega\rangle^\perp_{\mathcal{C}^2} = \{\, s - (s \cdot \omega)\omega + t \wedge \omega \mid s, t \in \mathbb{R}^3 \,\}$.

Proof From (3.14a) it immediately follows that ψ is pointwise orthogonal to ω, which implies $\psi \in \langle\omega\rangle^\perp_{\mathcal{C}^2}$. Since $\mathcal{P}\psi = \psi$, then $J_0'(\mathrm{U})\psi = 0$ by (3.9) and (3.10), hence

$$\begin{cases} \partial_x\psi \cdot \partial_x\omega - \partial_y\psi \cdot \partial_y\omega = 0 \\ \partial_x\psi \cdot \partial_y\omega + \partial_y\psi \cdot \partial_x\omega = 0 \end{cases} \tag{3.16}$$

by Lemma 3.2. Since $\psi \in \langle\partial_x\omega, \partial_y\omega\rangle$ pointwise on $\mathbb{R}^2$, we can write

$$\psi = f\nabla\omega \,, \quad \text{where} \quad f := \mu^{-2}(\psi \cdot \partial_x\omega, \psi \cdot \partial_y\omega) \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{R}^2) \,.$$

We identify f with a complex valued function. A direct computation based on (2.2) shows that ψ solves (3.16) if and only if f solves $\partial_x f + i\partial_y f = 0$ on $\mathbb{R}^2$. In polar coordinates we have that

$$\rho\partial_\rho f + i\partial_\theta f = 0. \tag{3.17}$$

For every $\rho > 0$ we expand the periodic function $f(\rho, \cdot)$ in Fourier series,

$$f(\rho, \theta) = \sum_{h \in \mathbb{Z}} \gamma_h(\rho) e^{ih\theta} \;, \quad \gamma_h(\rho) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(\rho, \theta) e^{-ih\theta} d\theta \,.$$

The coefficients γ_h are complex-valued functions on the half-line $\mathbb{R}_+$ and solve

$$\gamma_h' - h\gamma_h = 0\,,$$

because of (3.17). Thus for every $h \in \mathbb{Z}$ there exists $a_h \in \mathbb{C}$ such that $\gamma_h(\rho) = a_h\rho^h$. Now recall that $\mu\psi \in L^2(\mathbb{R}^2, \mathbb{R}^3)$. Since

$$\int\limits_{\mathbb{R}^2} \mu^2|\psi|^2\,dz = \int\limits_{\mathbb{R}^2} \mu^4|f|^2\,dz \geq 2\pi\int_0^\infty \mu^4\rho|\gamma_h|^2\,d\rho = a_h^2\int\limits_{\mathbb{R}^2} \mu^4|z|^{2h}\,dz\,, \quad \forall\, h \in \mathbb{Z}\,,$$

we infer that $\gamma_h = 0$ for every $h \neq 0, 1, 2$. Thus $f(z) = \sum\limits_{h=0}^{2} a_h z^h$, that is $\psi = \sum\limits_{h=0}^{2} a_h z^h \nabla\omega$, and in particular the space of solutions of (3.14a) has (real) dimension 6. The conclusion of the proof follows from the relations (2.4). $\square$

Lemma 3.4 *Let $\eta \in \mathcal{C}^2(\overline{\mathbb{R}}^2)$ be a solution to (3.14b). There exists $\alpha \in \mathbb{R}^3$ such that*

$$\eta = \alpha \cdot (k\omega + e_3)\,,$$

and thus $\eta\omega \in T_{\mathrm{U}}Z \cap \langle\omega\rangle_{\mathcal{C}^2} = \{\, (\alpha \cdot (k\omega + e_3))\,\omega \mid \alpha \in \mathbb{R}^3\,\}$.

Proof First of all, we notice that $\alpha \cdot (k\omega + e_3)$ solves (3.14b) for any $\alpha \in \mathbb{R}^3$.

By the Hilbert–Schmidt theorem, the eigenvalue problem

$$-\mathrm{div}\Big(\frac{\nabla\eta}{(\omega_3 + k)^2}\Big) = \frac{\lambda\,\mu^2}{(\omega_3 + k)^3}\,\eta \quad \text{on } \mathbb{R}^2\,, \quad \eta \in \mathcal{C}^2(\mathbb{R}^2)\,, \tag{3.18}$$

has a non decreasing, divergent sequence $(\lambda_h)_{h\geq 0}$ of eigenvalues which correspond to critical levels of the quotient

$$R(\eta) := \frac{\displaystyle\int_{\mathbb{R}^2} \frac{|\nabla\eta|^2}{(\omega_3 + k)^2}\,dz}{\displaystyle\int_{\mathbb{R}^2} \frac{|\eta|^2}{(\omega_3 + k)^3}\,\mu^2 dz}\,, \quad \eta \in H^1(\overline{\mathbb{R}}^2) \setminus \{0\}\,.$$

Clearly, $\lambda_0 = 0$ is simple, and its eigenfunctions are constant functions. We claim that the next eigenvalue is $2k$, and that its eigenspace has dimension 3, which concludes the proof.

To this goal, we use the functional change

$$\eta(z) = \frac{\mu(z)}{\mu(c_k z)}\varPhi(c_k z)\,, \quad c_k := e^{\rho_k} = \sqrt{\frac{k+1}{k-1}}\,.$$

By a direct computation involving the identity $(\omega_3(z) + k)\mu(c_k z) = (k-1)\mu(z)$ and integration by parts, one gets

$$\lambda_1 = \inf_{\substack{\eta \in \mathcal{C}^2(\overline{\mathbb{R}}^2)\setminus\{0\} \\ \int_{\mathbb{R}^2} \frac{\eta\;\mu^2 dz}{(\omega_3+k)^3} = 0}} R(\eta) = 2k + \inf_{\substack{\varPhi \in \mathcal{C}^2(\overline{\mathbb{R}}^2)\setminus\{0\} \\ \int_{\mathbb{R}^2} \varPhi\mu^2 dz = 0}} \frac{\displaystyle\int_{\mathbb{R}^2} |\nabla\varPhi|^2 dz - 2\int_{\mathbb{R}^2} |\varPhi|^2\mu^2 dz}{\displaystyle\int_{\mathbb{R}^2} \frac{|\varPhi|^2}{(k - \omega_3)}\,\mu^2 dz}\,.$$

On the other hand, it is well known that

$$\min_{\substack{\Phi\in\mathcal{C}^2(\overline{\mathbb{R}}^2)\setminus\{0\}\\ \int_{\mathbb{R}^2}\Phi\mu^2dz=0}} \frac{\displaystyle\int_{\mathbb{R}^2}|\nabla\Phi|^2dz}{\displaystyle\int_{\mathbb{R}^2}|\Phi|^2\mu^2dz}=2$$

is the first nontrivial eigenvalue for the Laplace-Beltrami operator on the sphere and that its eigenspace has dimension 3, see for instance [4]. The proof is complete. □

Remark 3.2 *The third eigenvalue λ_2 of (3.18) verifies $\lambda_2>2k$ by Lemma 3.4, and*

$$\lambda_2=\min\Big\{\,R(\eta)\;\Big|\;\int_{\mathbb{R}^2}\frac{\eta}{(\omega_3+k)^3}\,\mu^2dz=\int_{\mathbb{R}^2}\frac{\eta(k\omega_j+\delta_{j3})}{(\omega_3+k)^3}\,\mu^2dz=0\;,\quad j=1,2,3\Big\}\,.$$

Proof of Theorem 1.1 In fact, we only have to sum up the argument. Let $U\in Z$. Thanks to (3.2), $U=\mathrm{U}_q\circ g$ for some $q\in\mathbb{H}^3$, $g\in PGL(2,\mathbb{C})$. Since

$$T_{\mathrm{U}_q\circ g}Z=T_{\mathrm{U}}Z\circ g\,,\quad \ker J_0'(\mathrm{U}_q\circ g)=\ker J_0'(\mathrm{U})\circ g\,,\quad \text{for every } q\in\mathbb{H}^3\,,\ g\in PGL(2,\mathbb{C})\,,$$

it suffices to consider the case $U=\mathrm{U}$.

If $\varphi\in\mathcal{C}^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)$ solves (1.4) then $J_0'(\mathrm{U})\varphi=0$, which means $\mathcal{P}\left(J_0'(\mathrm{U})\varphi\right)=0$ and $(J_0'(\mathrm{U})\varphi)\cdot\omega=0$. From Lemma 3.1 we infer that $\mathcal{P}\varphi$ solves (3.14a) and that $\varphi\cdot\omega$ solves (3.14b). Therefore, Lemmata 3.3, 3.4 give the existence of $s,t,\alpha\in\mathbb{R}^3$ such that

$$\mathcal{P}\varphi=s-(s\cdot\omega)\omega+t\wedge\omega\ ,\quad \varphi\cdot\omega=\alpha\cdot(k\omega+e_3)\,.$$

Thus $\varphi=\mathcal{P}\varphi+(\varphi\cdot\omega)\omega\in T_{\mathrm{U}}Z$ by (3.5), which concludes the proof. □

3.1 Further results on the operator $J_0'(\mathrm{U})$

To shorten notation we put

$$H^1=H^1(\overline{\mathbb{R}}^2,\mathbb{R}^3)\,.$$

Since integration by parts gives

$$\int_{\mathbb{R}^2}-\mathrm{div}\Big(\frac{\nabla\varphi}{(\omega_3+k)^2}\Big)\cdot\psi\,dz=\int_{\mathbb{R}^2}\frac{\nabla\varphi\cdot\nabla\psi}{(\omega_3+k)^2}\,dz\ ,\quad \varphi,\psi\in\mathcal{C}^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)\,,$$

the quadratic form

$$(\varphi,\psi)\mapsto\int_{\mathbb{R}^2}J_0'(\mathrm{U})\varphi\cdot\psi\,dz \tag{3.19}$$

can be extended to a continuous bilinear form $H^1\times H^1\to\mathbb{R}$ via a density argument. It can be checked by direct computation (see also Remark 4.2) that the quadratic form in (3.19) is self-adjoint on H^1, that is,

$$\int_{\mathbb{R}^2}J_0'(\mathrm{U})\varphi\cdot\psi\,dz=\int_{\mathbb{R}^2}J_0'(\mathrm{U})\psi\cdot\varphi\,dz\quad\text{for any }\varphi,\psi\in H^1. \tag{3.20}$$

Since $T_{\rm U}Z$ is a subspace of $L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)\equiv L^2(\mathbb{S}^2,\mathbb{R}^3)$, we are allowed to put

$$T_{\rm U}Z^\perp := \Big\{\, f\in L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3) \ \Big| \ \int\limits_{\mathbb{R}^2} f\cdot\tau\,\mu^2dz=0\,,\ \forall\,\tau\in T_{\rm U}Z\,\Big\}.$$

Moreover, we introduce on $L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)$ the equivalent scalar product

$$(f,\psi)_*=\int\limits_{\mathbb{R}^2}\frac{\mathcal{P}f\cdot\mathcal{P}\psi}{(\omega_3+k)^2}\,\mu^2dz+\int\limits_{\mathbb{R}^2}\frac{(f\cdot\omega)(\psi\cdot\omega)}{(\omega_3+k)^3}\,\mu^2dz$$

and the subspaces

$$\begin{aligned} T_{\rm U}Z^\perp_* :=&\big\{\, f\in L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)\mid (f,\tau)_*=0,\ \forall\,\tau\in T_{\rm U}Z\,\big\}\,,\\ N_* :=&\langle\omega\rangle^\perp_*=\big\{\, f\in L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)\mid (f,\omega)_*=0\,\big\}\,.\end{aligned}$$

We are in position to state the main result of this section.

Lemma 3.5 *Let $q\in\mathbb{H}^3$. For any $v\in T_{\rm U}Z^\perp$, there exists $\varphi_v\in H^1\cap T_{\rm U}Z^\perp_*\cap N_*$ such that*

$$J_0'({\rm U}_q)\varphi_v=v\,\mu^2\qquad\text{on } \mathbb{R}^2. \tag{3.21}$$

If in addition $v\in\mathcal{C}^m(\overline{\mathbb{R}}^2,\mathbb{R}^3)$ for some $m\in(0,1)$, then $\varphi_v\in\mathcal{C}^{2+m}(\overline{\mathbb{R}}^2,\mathbb{R}^3)$.

In view of Lemma 3.1, we split the proof of Lemma 3.5 in few steps.

Lemma 3.6 *Let $v\in T_{\rm U}Z^\perp$ be such that $v\cdot\omega\equiv0$ on $\mathbb{R}^2$. There exists $\varphi\in H^1\cap T_{\rm U}Z^\perp_*$ such that $\varphi\cdot\omega\equiv0$ on $\mathbb{R}^2$ and*

$$J_0'({\rm U})\varphi=v\,\mu^2\qquad\text{on } \mathbb{R}^2. \tag{3.22}$$

Proof We introduce

$$X:=\big\{\,\psi\in H^1\ \big|\ \psi\cdot\omega\equiv0\ \ \text{on }\mathbb{R}^2\,\big\}\cap T_{\rm U}Z^\perp_*\,,$$

which is a closed subspace of H^1. Notice that $\psi=\mathcal{P}\psi$ for any $\psi\in X$ and moreover

$$\int\limits_{\mathbb{R}^2}J_0'({\rm U})\psi\cdot\psi\,dz=\int\limits_{\mathbb{R}^2}\frac{|\nabla\psi|^2}{(\omega_3+k)^2}\,dz+2\int\limits_{\mathbb{R}^2}\Big(\frac{(\psi\cdot iz\nabla\psi)\wedge\omega}{(\omega_3+k)^3}-\frac{|\psi|^2}{(\omega_3+k)^2}\Big)\,\mu^2dz\,,$$

use (3.9) and a density argument. Next we put

$$\lambda:=\inf_{\substack{\psi\in X\\ \psi\neq0}}\frac{\displaystyle\int_{\mathbb{R}^2}J_0'({\rm U})\psi\cdot\psi\,dz}{\displaystyle\int_{\mathbb{R}^2}(\omega_3+k)^{-2}|\psi|^2\ \mu^2dz}\,,$$

and notice that $\lambda\ge0$ by Lemma 3.2. On the other hand, λ is achieved by Rellich theorem. Thus $\lambda>0$, because of Lemma 3.3. It follows that the energy functional $I:X\to\mathbb{R}$,

$$I(\psi)=\frac12\int\limits_{\mathbb{R}^2}J_0'({\rm U})\psi\cdot\psi\,dz-\int\limits_{\mathbb{R}^2}v\cdot\psi\ \mu^2dz\,,$$

is weakly lower semicontinuous and coercive. Thus its infimum is achieved by a function $\varphi \in X$ which satisfies

$$\int_{\mathbb{R}^2} J_0'(\mathrm{U})\varphi \cdot \psi \, dz = \int_{\mathbb{R}^2} v \cdot \psi \; \mu^2 dz \,, \quad \forall\, \psi \in X \,. \tag{3.23}$$

If $\psi \in H^1$ we write

$$\psi = (\mathcal{P}\psi^\top + \mathcal{P}\psi^\perp) + \eta\omega \,,$$

where $\eta = \psi \cdot \omega$, $\mathcal{P}\psi^\top \in T_{\mathrm{U}}Z = \ker J_0'(U)$ is the orthogonal projection of $\mathcal{P}\psi = \psi - \eta\omega$ onto $T_{\mathrm{U}}Z$ in the scalar product $(\cdot,\cdot)_*$ and $\mathcal{P}\psi^\perp := \psi - \mathcal{P}\psi^\top - \eta\omega \in X$. We use (3.20) and (3.10) to compute

$$\int_{\mathbb{R}^2} J_0'(\mathrm{U})\varphi \cdot \mathcal{P}\psi^\top \, dz = \int_{\mathbb{R}^2} J_0'(\mathrm{U})\mathcal{P}\psi^\top \cdot \psi \, dz = 0 \,,$$

$$\int_{\mathbb{R}^2} J_0'(\mathrm{U})\varphi \cdot (\eta\omega) \, dz = \int_{\mathbb{R}^2} \frac{\nabla(\varphi\cdot\omega)\cdot\nabla\eta}{(\omega_3+k)^2} \, dz - 2k \int_{\mathbb{R}^2} \frac{(\varphi\cdot\omega)\eta}{(\omega_3+k)^3} \; \mu^2 dz = 0 \,,$$

because $\varphi \cdot \omega \equiv 0$. Therefore, (3.23) gives

$$\int_{\mathbb{R}^2} J_0'(\mathrm{U})\varphi \cdot \psi \, dz = \int_{\mathbb{R}^2} J_0'(\mathrm{U})\varphi \cdot \mathcal{P}\psi^\perp \, dz = \int_{\mathbb{R}^2} v \cdot \mathcal{P}\psi^\perp \, \mu^2 dz = \int_{\mathbb{R}^2} v \cdot \psi \, \mu^2 dz \,,$$

as v is orthogonal to $T_{\mathrm{U}}Z \ni \mathcal{P}\psi^\top$ and to $\eta\omega$ in $L^2(\overline{\mathbb{R}}^2, \mathbb{R}^3)$. We showed that φ solves (3.22), and thus the proof is complete. □

Lemma 3.7 *Let $f \in H^1(\overline{\mathbb{R}}^2)$ be such that $f\omega \in T_{\mathrm{U}}Z^\perp$. There exists $\eta \in H^1(\overline{\mathbb{R}}^2)$ such that $\eta\omega \in H^1 \cap T_{\mathrm{U}}Z_*^\perp \cap N_*$ and*

$$J_0'(\mathrm{U})(\eta\omega) = f\omega\,\mu^2 \quad \text{on } \mathbb{R}^2. \tag{3.24}$$

Proof We introduce the space

$$Y := \left\{ \eta \in H^1(\overline{\mathbb{R}}^2) \;\Big|\; \int_{\mathbb{R}^2} \frac{\eta}{(\omega_3+k)^3} \; \mu^2 dz = \int_{\mathbb{R}^2} \frac{\eta(\tau\cdot\omega)}{(\omega_3+k)^3} \; \mu^2 dz \; = 0 \,, \; \forall\, \tau \in T_{\mathrm{U}}Z \right\},$$

so that $\eta\omega \in H^1 \cap T_{\mathrm{U}}Z_*^\perp \cap N_*$ for any $\eta \in Y$, and the energy functional $I : Y \to \mathbb{R}$,

$$\begin{aligned} I(\varphi) &= \frac{1}{2} \int_{\mathbb{R}^2} J_0'(\mathrm{U})(\eta\omega)\cdot(\eta\omega) \, dz - \int_{\mathbb{R}^2} f\eta \; \mu^2 dz \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \frac{|\nabla\eta|^2}{(\omega_3+k)^2} \, dz - k \int_{\mathbb{R}^2} \frac{|\eta|^2}{(\omega_3+k)^3} \, \mu^2 dz - \int_{\mathbb{R}^2} \eta f \, \mu^2 dz \,, \end{aligned}$$

compare with (3.10). The functional I is weakly lower semicontinuous with respect to the $H^1(\overline{\mathbb{R}}^2)$ topology and coercive by Remark 3.2. Thus its infimum is achieved by a function $\eta \in Y$. To conclude, argue as in the proof of Lemma 3.6 to show that η solves (3.24). □

Proof of Lemma 3.5. Since $J_0'(\mathrm{U}_q) = q_3^{-2} J_0'(\mathrm{U})$, we can assume that $q = e_3$, that is, $\mathrm{U}_q = \mathrm{U}$. We take any $v \in T_{\mathrm{U}} Z^{\perp}$, and write

$$v = \mathcal{P}v + (v \cdot \omega)\omega \,,$$

where $\mathcal{P}v = v - (v \cdot \omega)\omega$, as before. Since $\mathcal{P}v \in T_{\mathrm{U}} Z^{\perp}$, by Lemma 3.6 there exists a unique $\hat{\varphi} \in H^1 \cap T_{\mathrm{U}} Z_*^{\perp}$ such that $\hat{\varphi} \cdot \omega \equiv 0$ on $\mathbb{R}^2$ and

$$\int\limits_{\mathbb{R}^2} J_0'(\mathrm{U})\hat{\varphi} \cdot \psi \, dz \, = \int\limits_{\mathbb{R}^2} \mathcal{P}v \cdot \psi \, \mu^2 dz \,, \quad \text{for any } \psi \in H^1.$$

Next, notice that $(v \cdot \omega)\omega \in T_{\mathrm{U}} Z^{\perp}$, so we can use Lemma 3.7 to find $\eta \in H^1(\overline{\mathbb{R}}^2)$ such that $\eta\omega \in H^1 \cap T_{\mathrm{U}} Z_*^{\perp} \cap N_*$ solves

$$\int\limits_{\mathbb{R}^2} J_0'(\mathrm{U})(\eta\omega) \cdot \psi \, dz = \int\limits_{\mathbb{R}^2} (v \cdot \omega)(\psi \cdot \omega) \, \mu^2 dz, \quad \text{for any } \psi \in H^1.$$

The function $\varphi_v = \hat{\varphi} + \eta\omega$ solves (3.21).

To conclude the proof we have to show that if $v \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ then $\varphi_v \in \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{R}^3)$. Since $\omega \in \mathcal{C}^\infty(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ and $\omega_3 + k$ is bounded and bounded away from zero, φ_v solves a linear system of the form

$$-\Delta\varphi_v = A(z)\varphi_v + B(z)\nabla\varphi_v + \mu^2(\omega_3 + k)^2 v \,,$$

for certain smooth matrices on $\overline{\mathbb{R}}^2$. A standard bootstrap argument and Schauder regularity theory plainly imply that $\varphi_v \in \mathcal{C}^{2+m}_{loc}(\mathbb{R}^2, \mathbb{R}^3)$. The function $z \mapsto \varphi_v(z^{-1})$ satisfies a linear system of the same kind, hence $\varphi_v \in \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{R}^3)$, as desired. □

4 The perturbed problem

In this Section we perform the finite dimensional reduction and prove Theorems 1.2, 1.3. By the results in the Appendix, any critical point of the $\mathcal{C}^2$-functional $E_\varepsilon : \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3) \to \mathbb{R}$,

$$E_\varepsilon(u) := \frac{1}{2} \int\limits_{\mathbb{R}^2} u_3^{-2} |\nabla u|^2 \, dz - k \int\limits_{\mathbb{R}^2} u_3^{-2} e_3 \cdot \partial_x u \wedge \partial_y u \, dz + 2\varepsilon \, V_\phi(u) = E_0(u) + 2\varepsilon \, V_\phi(u)$$

(notice that $E_0 = E_{\mathrm{hyp}}$, compare with (1.3)), solves

$$\Delta u - 2u_3^{-1} G(\nabla u) = 2u_3^{-1}(k + \varepsilon\phi(u)) \, \partial_x u \wedge \partial_y u \quad \text{on } \mathbb{R}^2 \tag{$\mathcal{P}_\varepsilon$}$$

and has mean curvature $(k + \varepsilon\phi)$, apart from a finite set of branch points.

Due to the action of the Möbius transformations and of the hyperbolic translations, for any $u \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$ we have the identities

$$E_\varepsilon'(u)(z^h \nabla u) = 0 \ , \quad E_\varepsilon'(u)(iz^h \nabla u) = 0 \ , \quad \text{for } h = 0, 1, 2 \,, \ \ \varepsilon \in \mathbb{R} \,, \tag{4.1}$$

$$E_0'(u)e_1 = 0 \,, \quad E_0'(u)e_2 = 0 \,, \quad E_0'(u)u = 0 \,. \tag{4.2}$$

Now we prove that

$$E_\varepsilon(\mathrm{U}_q) = E_0(\mathrm{U}) - 2\varepsilon F_k^\phi(q)\,, \tag{4.3}$$

where F_k^ϕ is the Melnikov-type function in (1.5). The above mentioned invariances give $E_0(\mathrm{U}_q) = E_0(\mathrm{U})$. Since the hyperbolic ball $B^{\mathbb{H}}_{\rho_k}(q)$ coincides with the Euclidean ball of radius $q_3 r_k$ about the point $q^k := (q_1, q_2, k r_k q_3)$, the divergence theorem gives

$$F_k^\phi(q) = \int\limits_{B^{\mathbb{H}}_{\rho_k}(q)} \phi(p)\, d\mathbb{H}^3_p \;=\; \int\limits_{B_{q_3 r_k}(q^k)} p_3^{-3}\phi(p)\, dp \;=\; \int\limits_{\partial B_{q_3 r_k}(q^k)} Q_\phi(p)\cdot \nu_p\,.$$

Here $Q_\phi \in \mathcal{C}^1(\mathbb{R}^3_+, \mathbb{R}^3)$ is any vectorfield such that $\mathrm{div} Q_\phi(p) = p_3^{-3}\phi(p)$ and ν_p is the outer normal to $\partial B_{q_3 r_k}(q^k)$ at p. The function U_q in (3.1) parameterizes the Euclidean sphere $\partial B q_3 r_k(q^k)$. Since $\partial_x \mathrm{U}_q \wedge \partial_y \mathrm{U}_q$ is inward-pointing, we have

$$F_k^\phi(q) = -\int\limits_{\mathbb{R}^2} Q_\phi(p)\cdot \partial_x \mathrm{U}_q \wedge \partial_y \mathrm{U}_q \, dz = -V_\phi(\mathrm{U}_q), \tag{4.4}$$

and (4.3) is proved. Before going further, let us show that the existence of critical points for F_k^ϕ is a necessary condition for the conclusion in Theorem 1.2.

Theorem 4.1 *Let $k > 1$, $\phi \in \mathcal{C}^1(\mathbb{H}^3)$. Assume that there exist sequences $\varepsilon_h \subset \mathbb{R}\setminus\{0\}$, $\varepsilon_h \to 0$, $u^h \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$ and a point $q \in \mathbb{H}^3$ such that u_h solves $(\mathcal{P}_{\varepsilon_h})$, and $u^h \to \mathrm{U}_q$ in $\mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3)$. Then q is a stationary point for F_k^ϕ.*

Proof The function u^h is a stationary point for the energy functional $E_{\varepsilon_h} = E_0 + 2\varepsilon_h V_\phi$. From (4.2) we have $V'_\phi(u^h)e_j = 0$ for $j = 1, 2$ and $V'_\phi(u^h)u^h = 0$. We can plainly pass to the limit to obtain $V'_\phi(\mathrm{U}_q)e_j = 0$ for $j = 1, 2$ and $V'_\phi(\mathrm{U}_q)\mathrm{U}_q = 0$. To conclude, use (4.4) and recall that $\partial_{q_j}\mathrm{U}_q = e_j$ for $j = 1, 2$, and $\partial_{q_3}\mathrm{U}_q = \mathrm{U} = q_3^{-1}(\mathrm{U}_q - q_1 e_1 - q_2 e_2)$. □

Now we fix $m \in (0, 1)$. The operator $J_\varepsilon : \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{H}^3) \to \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ defined by

$$J_\varepsilon(u) = -\mathrm{div}(u_3^{-2}\nabla u) - u_3^{-3}|\nabla u|^2 e_3 + 2(k + \varepsilon\phi)u_3^{-3}\partial_x u \wedge \partial_y u\,,$$

is related to the differential of E_ε via the identity

$$E'_\varepsilon(u)\varphi = \int\limits_{\mathbb{R}^2} J_\varepsilon(u)\cdot \varphi\, dz\,, \quad u \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{H}^3)\,,\ \varphi \in \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)\,. \tag{4.5}$$

Remark 4.2 *Since E_ε is of class $\mathcal{C}^2$ and*

$$E''_\varepsilon(u)[\varphi, \psi] = \int_{\mathbb{R}^2} J'_\varepsilon(u)\psi \cdot \varphi\, dz\,,$$

then the quadratic form in the right hand side is a self-adjoint form on H^1.

We are in position to state and prove the next Lemma, which is the main step towards the proofs of Theorems 1.2, 1.3.

Lemma 4.1 (Dimension reduction) *Let $\Omega \Subset \mathbb{H}^3$ be an open set. There exist $\hat{\varepsilon} > 0$ and a unique $\mathcal{C}^1$-map*

$$[-\hat{\varepsilon}, \hat{\varepsilon}] \times \overline{\Omega} \to \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{H}^3), \quad (\varepsilon, q) \mapsto u_q^\varepsilon,$$

such that the following facts hold:

i) u_q^ε *parameterizes an embedded $\mathbb{S}^2$-type surface, and* $u_q^0 = \mathrm{U}_q$ *;*
ii) $u_q^\varepsilon - \mathrm{U}_q \in T_{\mathrm{U}}Z^\perp \cap \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ *and* $E'_\varepsilon(u_q^\varepsilon)\varphi = 0$ *for any* $\varphi \in T_{\mathrm{U}}Z^\perp \cap \mathcal{C}^0(\overline{\mathbb{R}}^2, \mathbb{R}^3)$ *;*
iii) for any $\varepsilon \in [-\hat{\varepsilon}, \hat{\varepsilon}]$*, the manifold* $\{ u_q^\varepsilon \mid q \in \Omega \}$ *is a natural constraint for* E_ε*, that is, if* $\nabla_q E_\varepsilon(u_{q^\varepsilon}^\varepsilon) = 0$ *for some* $q^\varepsilon \in \Omega$*, then* $u_{q^\varepsilon}^\varepsilon$ *is a* $(k + \varepsilon\phi)$*-bubble;*
iv) $\|E_\varepsilon(u_q^\varepsilon) - E_\varepsilon(\mathrm{U}_q)\|_{\mathcal{C}^1(\overline{\Omega})} = o(\varepsilon)$ *as* $\varepsilon \to 0$*, uniformly on* $\overline{\Omega}$*.*

Proof To shorten the notation, we put $\mathcal{C}^m := \mathcal{C}^m(\overline{\mathbb{R}}^2, \mathbb{R}^3)$. For $s > 0$ and $\delta > 0$ we write

$$\Omega_s := \{ p \in \mathbb{H}^3 \mid \mathrm{dist}(p, \Omega) < s \}, \text{ and } \mathcal{U}_\delta := \{ \nu \in \mathcal{C}^{2+m} \mid |\nu(z)| < \delta \text{ for every } z \in \mathbb{R}^2 \}.$$

We fix s and $\delta = \delta(s)$ such that $\overline{\Omega}_{2s} \subset \mathbb{H}^3$ and $(\mathrm{U}_q + \nu) \cdot e_3 > 0$ for $q \in \Omega_{2s}$, $\nu \in \mathcal{U}_\delta$.

We define

$$\begin{aligned} &\tau_1 := c_0 \partial_x \omega, \ \tau_3 := c_0 \sqrt{2} z \nabla\omega, \ \tau_5 := c_0 z^2 \nabla\omega, \\ &\tau_2 := c_0 \partial_y \omega, \ \tau_4 := c_0 \sqrt{2} i z \nabla\omega, \ \tau_6 := c_0 i z^2 \nabla\omega, \end{aligned} \qquad \gamma := 2c_0(k\omega + e_3), \tag{4.6}$$

where $c_0 := \sqrt{\frac{3}{2^4 \pi}}$ is a normalization constant. Thanks to (3.3), (3.5), we have

$$T_{\mathrm{U}}Z = \langle \tau_1, \dots \tau_6 \rangle \oplus \ \{ (\alpha \cdot \gamma)\, \omega \mid \alpha \in \mathbb{R}^3 \}.$$

Trivially, $\tau_j \cdot \omega \equiv 0$ on $\mathbb{R}^2$. Elementary computations give

$$\int_{\mathbb{R}^2} \tau_i \cdot \tau_j \ \mu^2 dz = \delta_{ij}, \quad \int_{\mathbb{R}^2} \gamma_h \gamma_\ell \ \mu^2 dz = 0 \quad \text{if } h \neq \ell,$$

for $i, j \in \{1, \dots, 6\}$, $h, \ell \in \{1, 2, 3\}$, and moreover

$$\int_{\mathbb{R}^2} \gamma_1^2 \ \mu^2 dz = \int_{\mathbb{R}^2} \gamma_2^2 \ \mu^2 dz = k^2, \quad \int_{\mathbb{R}^2} \gamma_3^2 \ \mu^2 dz = k^2 + 3.$$

Construction of $\mathbf{u}_\mathbf{q}^\varepsilon$ *satisfying* ***i), ii).*** By our choices of s and δ, the functions

$$\begin{aligned} &\mathcal{F}_1(\varepsilon, q; \nu, \xi, \alpha) := \mu^{-2} J_\varepsilon(\mathrm{U}_q + \nu) - \sum_{j=1}^{6} \xi_j \tau_j - (\alpha \cdot \gamma)\, \omega \ \in \mathcal{C}^m, \\ &\mathcal{F}_2(\varepsilon, q; \nu, \xi, \alpha) := \Big(\int_{\mathbb{R}^2} \nu \cdot \tau_1 \ \mu^2 dz, \dots, \int_{\mathbb{R}^2} \nu \cdot \tau_6 \ \mu^2 dz \ ; \ \int_{\mathbb{R}^2} \gamma\, (\nu \cdot \omega) \ \mu^2 dz \Big) \ \in \mathbb{R}^6 \times \mathbb{R}^3, \end{aligned}$$

are well defined and continuously differentiable on $\mathbb{R} \times \Omega_{2s} \times \mathcal{U}_\delta \times (\mathbb{R}^6 \times \mathbb{R}^3)$. Thus

$$\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2) : \mathbb{R} \times \Omega_{2s} \times \mathcal{U}_\delta \times (\mathbb{R}^6 \times \mathbb{R}^3) \to \mathcal{C}^m \times (\mathbb{R}^6 \times \mathbb{R}^3)$$

is of class $\mathcal{C}^1$ on its domain. Notice that $\mathcal{F}(0,q;0,0,0)=0$ for every $q\in\Omega_{2s}$ because $J_0(\mathrm{U}_q)=0$. Now we solve the equation $\mathcal{F}(\varepsilon,q;\nu,\xi,\alpha)=0$ in a neighborhood of $(0,q;0,0,0)$ via the implicit function theorem. Let

$$\mathcal{L}:=(\mathcal{L}_1,\mathcal{L}_2):\mathcal{C}^{2+m}\times(\mathbb{R}^6\times\mathbb{R}^3)\to\mathcal{C}^m\times(\mathbb{R}^6\times\mathbb{R}^3)$$

given by

$$\mathcal{L}_1(\varphi;\zeta,\beta):=\mu^{-2}J_0'(\mathrm{U}_q)\varphi-\sum_{j=1}^{6}\zeta_j\tau_j-(\beta\cdot\gamma)\,\omega\,,$$
$$\mathcal{L}_2(\varphi;\zeta,\beta):=\mathcal{L}_2(\varphi)=\Big(\int_{\mathbb{R}^2}\varphi\cdot\tau_1\;\mu^2dz\;,\dots,\int_{\mathbb{R}^2}\varphi\cdot\tau_6\;\mu^2dz\;\;;\;\int_{\mathbb{R}^2}\gamma\,(\varphi\cdot\omega)\;\mu^2dz\;\Big)\,,$$

so that $\mathcal{L}=(\mathcal{L}_1,\mathcal{L}_2)$ is the differential of $\mathcal{F}(0,q;\cdot,\cdot,\cdot)$ evaluated in $(\nu,\xi,\alpha)=(0,0,0)$.

To prove that $\mathcal{L}$ is injective we assume that $\mathcal{L}(\varphi,\zeta,\beta)=0$ and put

$$v=\mu^{-2}J_0'(\mathrm{U}_q)\varphi\in T_{\mathrm{U}}Z\,.$$

From (3.20) we find

$$\int_{\mathbb{R}^2}|v|^2\;\mu^2dz=\int_{\mathbb{R}^2}\big(\mu^{-2}J_0'(\mathrm{U}_q)\varphi\big)\cdot v\;\mu^2dz=\int_{\mathbb{R}^2}J_0'(\mathrm{U}_q)\varphi\cdot v\;dz=\int_{\mathbb{R}^2}J_0'(\mathrm{U}_q)v\cdot\varphi\;dz=0\,,$$

which implies $J_0'(\mathrm{U}_q)\varphi=0$, that is, $\varphi\in T_{\mathrm{U}}Z$. On the other hand, $\varphi\in T_{\mathrm{U}}Z^\perp$ because $\mathcal{L}_2(\varphi)=0$. Thus $\varphi=0$ and therefore also $\beta=\zeta=0$.

To prove that $\mathcal{L}$ is surjective fix $v\in\mathcal{C}^m$ and $(\theta,b)\in\mathbb{R}^6\times\mathbb{R}^3$. We have to find $\varphi\in\mathcal{C}^{2+m}$ and $(\zeta,\beta)\in\mathbb{R}^6\times\mathbb{R}^3$ such that $\mathcal{L}_1(\varphi;\zeta,\beta)=v$ and $\mathcal{L}_2(\varphi)=(\theta,b)$. To this goal we introduce the minimal distance projection

$$P^\top:L^2(\overline{\mathbb{R}}^2,\mathbb{R}^3)\to T_{\mathrm{U}}Z\,,\quad w\mapsto P^\top w\,,$$

so that $\mathcal{L}_2(w)$ is uniquely determined by $P^\top w$, and vice-versa. We find ζ_j and β so that

$$\sum_{j=1}^{6}\zeta_j\tau_j+(\beta\cdot\gamma)\omega=-P^\top v\,.$$

Then, we use Lemma 3.5 to find $\widehat{\varphi}\in\mathcal{C}^{2+m}\cap T_{\mathrm{U}}Z_*^\perp\cap N_*$ such that

$$J_0'(\mathrm{U}_q)\widehat{\varphi}=(v-P^\top v)\,\mu^2.$$

Finally, we take the unique tangent direction $\varphi^\top\in T_{\mathrm{U}}Z$ such that $\mathcal{L}_2(\varphi^\top)=(\theta,b)-\mathcal{L}_2(\widehat{\varphi})$. The triple $(\varphi^\top+\widehat{\varphi};\zeta,\beta)$ satisfies $\mathcal{L}(\varphi^\top+\widehat{\varphi};\zeta,\beta)=(v;\theta,b)$ and surjectivity is proved. We are in the position to apply the implicit function theorem to $\mathcal{F}$, for any fixed $q\in\Omega_{2s}$. In fact, thanks to a standard compactness argument, we get that there exist $\varepsilon'>0$ and uniquely determined $\mathcal{C}^1$ functions

$$\begin{array}{lll}\nu:(-\varepsilon',\varepsilon')\times\Omega_s\to\mathcal{U}_\delta & \alpha:(-\varepsilon',\varepsilon')\times\Omega_s\to\mathbb{R}^3 & \xi:(-\varepsilon',\varepsilon')\times\Omega_s\to\mathbb{R}^6\\ \nu:(\varepsilon,q)\mapsto\nu_q^\varepsilon & \alpha:(\varepsilon,q)\mapsto\alpha^\varepsilon(q) & \xi:(\varepsilon,q)\mapsto\xi^\varepsilon(q)\end{array}$$

such that

$$\nu_q^0 \equiv 0, \qquad \alpha^0(q) = 0\,, \qquad \xi^0(q) = 0\,, \qquad \mathcal{F}(\varepsilon, q; \nu_q^\varepsilon, \xi^\varepsilon(q), \alpha^\varepsilon(q)) = 0\,. \tag{4.7}$$

By (4.7), the $\mathcal{C}^1$ function $(-\varepsilon', \varepsilon') \times \Omega_s \to \mathcal{C}^{2+m}(\overline{\mathbb{R}}^2, \mathbb{H}^3)$,

$$(\varepsilon, q) \ \mapsto \ u_q^\varepsilon := \mathrm{U}_q + \nu_q^\varepsilon = \big(q_3\mathrm{U} + q_1 e_1 + q_2 e_2\big) + \nu_q^\varepsilon\,,$$

satisfies *i*), if ε' is small enough. Further, using (4.5) (see also Lemma A.1) we rewrite the last identity in (4.7) as

$$\begin{aligned} E'_\varepsilon(u_q^\varepsilon)\varphi &= \int\limits_{\mathbb{R}^2} J'_\varepsilon(\mathrm{U}_q + \nu_q^\varepsilon) \cdot \varphi\, dz \\ &= \sum_{j=1}^6 \xi_j^\varepsilon(q) \int\limits_{\mathbb{R}^2} \tau_j \cdot \varphi\ \mu^2 dz + \int\limits_{\mathbb{R}^2} (\alpha^\varepsilon(q)\cdot\gamma)(\omega\cdot\varphi)\ \mu^2 dz \quad \forall\, \varphi \in \mathcal{C}^0, \end{aligned} \tag{4.8}$$
$$\int\limits_{\mathbb{R}^2} \nu_q^\varepsilon \cdot \tau_j\ \mu^2 dz = 0\,, \quad \forall\, j \in \{\,1, \dots, 6\,\}\,, \qquad \int\limits_{\mathbb{R}^2} \gamma_\ell(\nu_q^\varepsilon \cdot \omega)\ \mu^2 dz = 0\,, \quad \forall\, \ell \in \{\,1,2,3\,\}\,.$$

In particular, claim *ii*) holds true.

Proof of **iii**). As a straightforward consequence of (4.8) we have that

$$\int\limits_{\mathbb{R}^2} \partial_{q_i}\nu_q^\varepsilon \cdot \tau_j\ \mu^2 dz = 0\,, \qquad \int\limits_{\mathbb{R}^2} \gamma_\ell\,(\partial_{q_i}\nu_q^\varepsilon \cdot \omega)\ \mu^2 dz = 0\,,$$

hence $E'_\varepsilon(u_q^\varepsilon)\partial_{q_i}\nu_q^\varepsilon = 0$ for any $i = 1,2,3$. We infer the identities

$$\begin{aligned} \partial_{q_i} E_\varepsilon(u_q^\varepsilon) &= E'_\varepsilon(u_q^\varepsilon)(e_i + \partial_{q_i}\nu_q^\varepsilon) = E'_\varepsilon(u_q^\varepsilon)e_i\,, \quad i = 1,2\,, \\ \partial_{q_3} E_\varepsilon(u_q^\varepsilon) &= E'_\varepsilon(u_q^\varepsilon)(\mathrm{U} + \partial_{q_3}\nu_q^\varepsilon) = E'_\varepsilon(u_q^\varepsilon)\mathrm{U}\,. \end{aligned} \tag{4.9}$$

Now, from (2.4), (4.6) and (4.8) we find

$$2c_0 e_1 = \tau_1 - \tau_5 + k^{-1}\gamma_1\omega\,, \quad 2c_0 e_2 = \tau_2 + \tau_6 + k^{-1}\gamma_2\omega\,, \quad 2c_0\mathrm{U} = kr_k(\sqrt{2}\tau_3 + k^{-1}\gamma_3\omega)\,,$$
$$E'_\varepsilon(u_q^\varepsilon)\tau_j = \xi_j^\varepsilon(q)\,, \qquad E'_\varepsilon(u_q^\varepsilon)(\gamma_\ell\omega) = (k^2 + 3\delta_{\ell 3})\alpha_\ell^\varepsilon(q)\,,$$

for any $j = 1, \dots, 6$, $\ell = 1,2,3$. Thus by (4.9) we get

$$2c_0 \nabla_q E_\varepsilon(u_q^\varepsilon) = M_k \xi^\varepsilon(q) + \Theta_k \alpha^\varepsilon(q)\,, \tag{4.10}$$

where M_k and Θ_k are constant matrices, namely

$$M_k = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & \sqrt{2}kr_k & 0 & 0 & 0 \end{pmatrix}\,, \qquad \Theta_k = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & (k^2+3)r_k \end{pmatrix}\,.$$

On the other hand, from (4.1) and using $\nabla\mathrm{U}_q = r_k q_3 \nabla\omega$ we obtain

$$-q_3 r_k\, \xi_j^\varepsilon(q) = E'_\varepsilon(u_q^\varepsilon)(\tau_j^\varepsilon(q))\,, \tag{4.11}$$

where, in the spirit of (4.6), we have putted

$$\begin{aligned}&\tau_1^\varepsilon(q) := c_0\partial_x\nu_q^\varepsilon\,,\ \tau_3^\varepsilon(q) := c_0\sqrt{2}z\nabla\nu_q^\varepsilon\,,\ \ \tau_5^\varepsilon(q) := c_0 z^2\nabla\nu_q^\varepsilon\,,\\ &\tau_2^\varepsilon(q) := c_0\partial_y\nu_q^\varepsilon\,,\ \tau_4^\varepsilon(q) := c_0\sqrt{2}iz\nabla\nu_q^\varepsilon\,,\ \tau_6^\varepsilon(q) := c_0 iz^2\nabla\nu_q^\varepsilon\,.\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^2}|\tau_j^\varepsilon(q)|^2\ \mu^2dz \le 2\int_{\mathbb{R}^2}|\nabla_z\nu_q^\varepsilon|^2\ \mu\,dz \le 2\ \|\nu_q^\varepsilon\|^2_{\mathcal{C}^1}\int_{\mathbb{R}^2}\mu^3\,dz = o(1), \tag{4.12}$$

as $\varepsilon\to 0$, uniformly on $\overline{\Omega}$, see (2.3).

For the sake of clarity, we make now some explicit computations. We denote by $\sigma_{\ell h}$ the entries of the 3×6 constant matrix $\Theta_k^{-1}M_k$, and introduce the 6×6 matrix $A^\varepsilon(q) = (a_{jh}^\varepsilon(q))_{j,h=1,\dots,6}$, whose entries are given by

$$a_{jh}^\varepsilon(q) = \int_{\mathbb{R}^2}\tau_h\cdot\tau_j^\varepsilon(q)\ \mu^2dz - \sum_{\ell=1}^{3}\sigma_{\ell h}\int_{\mathbb{R}^2}\gamma_\ell(\omega\cdot\tau_j^\varepsilon(q))\ \mu^2dz\,.$$

Since $\tau_j^\varepsilon\mu\to 0$ in $L^2(\mathbb{R}^2,\mathbb{R}^3)$ by (4.12), then $A^\varepsilon\to 0$ uniformly on compact subsets of $(-\varepsilon',\varepsilon')\times\Omega_s$. In particular, if $\hat\varepsilon\in(0,\varepsilon')$ is small enough, then the determinant of the 6×6 matrix $(A^\varepsilon(q)+q_3r_k\mathrm{Id})$ is uniformly bounded away from 0 on $[-\hat\varepsilon,\hat\varepsilon]\times\overline{\Omega}$.

Assume that $\nabla_q E_\varepsilon(u^\varepsilon_{q^\varepsilon}) = 0$ for some $\varepsilon\in[-\hat\varepsilon,\hat\varepsilon]$, $q^\varepsilon\in\Omega$. From (4.10) we obtain $\alpha^\varepsilon(q^\varepsilon) = -\Theta_k^{-1}M_k\xi^\varepsilon(q^\varepsilon)$. Thus (4.8) and (4.11) give

$$-q_3^\varepsilon r_k\,\xi^\varepsilon(q^\varepsilon) = A^\varepsilon(q^\varepsilon)\xi^\varepsilon(q^\varepsilon),$$

and hence $\xi^\varepsilon(q^\varepsilon)=0$, because the matrix $(A^\varepsilon(q^\varepsilon)+q_3^\varepsilon r_k\mathrm{Id})$ is invertible. But then (4.10) and $\nabla_q E_\varepsilon(u^\varepsilon_{q^\varepsilon}) = 0$ imply that $\alpha^\varepsilon(q^\varepsilon)=0$ as well, hence $E'(u^\varepsilon_{q^\varepsilon})=0$ by (4.8).

Proof of **iv**). The function $(\varepsilon,q)\mapsto\nu_q^\varepsilon$ is of class $\mathcal{C}^1$, and in particular $\partial_\varepsilon\nu_q^\varepsilon$ is uniformly bounded in $\mathcal{C}^2$ for $(\varepsilon,q)\in[-\hat\varepsilon,\hat\varepsilon]\times\overline{\Omega}$. Thus Taylor expansion formula for

$$\varepsilon\mapsto E_\varepsilon(u_q^\varepsilon)-E_\varepsilon(\mathrm{U}_q) = E_0(u_q^\varepsilon)-E_0(\mathrm{U}_q)+2\varepsilon\big(V_\phi(u_q^\varepsilon)-V_\phi(\mathrm{U}_q)\big)$$

gives $E_\varepsilon(u_q^\varepsilon)-E_\varepsilon(\mathrm{U}_q)=o(\varepsilon)$ as $\varepsilon\to 0$, uniformly on $\overline{\Omega}$.

Now we estimate $\nabla_q(E_\varepsilon(u_q^\varepsilon)-E_\varepsilon(\mathrm{U}_q))$. We use (4.2), (4.9) to obtain, for $j=1,2$,

$$\begin{aligned}\partial_{q_j}(E_\varepsilon(u_q^\varepsilon)-E_\varepsilon(\mathrm{U}_q)) &= \big(E_0'(u_q^\varepsilon)e_j - E_0'(\mathrm{U}_q)e_j\big)+2\varepsilon\big(V_\phi'(u_q^\varepsilon)e_j - V_\phi'(\mathrm{U}_q)e_j\big)\\ &= 2\varepsilon\big(V_\phi'(u_q^\varepsilon)e_j - V_\phi'(\mathrm{U}_q)e_j\big) = o(\varepsilon)\,,\end{aligned}$$

because $\|u_q^\varepsilon-\mathrm{U}_q\|_{\mathcal{C}^{2+m}}=o(1)$ and V_ϕ is a $\mathcal{C}^1$-functional.

To handle the derivative with respect to q_3 we first argue as before to get

$$\begin{aligned}\partial_{q_3}(E_\varepsilon(u_q^\varepsilon)-E_\varepsilon(\mathrm{U}_q)) &= \big(E_0'(u_q^\varepsilon)\mathrm{U} - E_0'(\mathrm{U}_q)\mathrm{U}\big)+2\varepsilon\big(V_\phi'(u_q^\varepsilon)\mathrm{U} - V_\phi'(\mathrm{U}_q)\mathrm{U}\big)\\ &= E_0'(u_q^\varepsilon)\mathrm{U}+o(\varepsilon)\,,\end{aligned}$$

uniformly on $\overline{\Omega}$. Next, from $q_3\mathrm{U} = u_q^\varepsilon - (q_1e_1 + q_2e_2) - \nu_q^\varepsilon$ and (4.2) we obtain

$$\begin{aligned} q_3E_0'(u_q^\varepsilon)\mathrm{U} =& E_0'(u_q^\varepsilon)(u_q^\varepsilon - (q_1e_1 + q_2e_2) - \nu_q^\varepsilon) \\ =& -E_0'(u_q^\varepsilon)\nu_q^\varepsilon = -E_\varepsilon'(u_q^\varepsilon)\nu_q^\varepsilon + 2\varepsilon V_\phi'(u_q^\varepsilon)\nu_q^\varepsilon = 2\varepsilon V_\phi'(u_q^\varepsilon)\nu_q^\varepsilon \end{aligned}$$

because of (4.8). Since $\nu_q^\varepsilon \to 0$ in $\mathcal{C}^{2+m}$ we infer that $E_0'(u_q^\varepsilon)u_q^\varepsilon = o(\varepsilon)$ uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$, which concludes the proof. □

Proof of Theorem 1.2. Take an open set $\Omega \Subset \mathbb{R}^3_+$ containing the closure of A, let u_q^ε be the function given by Lemma 4.1 and notice that, by (4.4), $E_\varepsilon(\mathrm{U}_q) = E_0(\mathrm{U}_q) - 2\varepsilon F_k^\phi(q)$. Thus for $\varepsilon \in [-\hat\varepsilon, \hat\varepsilon], \varepsilon \neq 0$ we can estimate

$$\Big\| \frac{1}{2\varepsilon}\big(E_\varepsilon(u_q^\varepsilon) - E_0(\mathrm{U}_q)\big) + F_k^\phi(q)\Big\|_{\mathcal{C}^1(\overline{A})} = \frac{1}{2|\varepsilon|}\|E_\varepsilon(u_q^\varepsilon) - E_\varepsilon(\mathrm{U}_q)\|_{\mathcal{C}^1(\overline{A})} = o(1)\,,$$

uniformly on $\overline{\Omega}$ by *iv)* in Lemma 4.1. Recalling the definition of stable critical point presented in Subsection 2.2, we infer that for any $\varepsilon \approx 0$ the function $\frac{1}{2\varepsilon}\big(E_\varepsilon(u_q^\varepsilon) - E_0(\mathrm{U}_q)\big)$ has a critical point $q^\varepsilon \in A$, to which corresponds the embedded $(k + \varepsilon\phi)$-bubble $u^\varepsilon := u_{q^\varepsilon}^\varepsilon$ by *iii)* in Lemma 4.1. The continuity of $(\varepsilon, q) \mapsto u_q^\varepsilon$ gives the continuity of $\varepsilon \mapsto u^\varepsilon$.

The last conclusion in Theorem 1.2 follows via a simple compactness argument and thanks to Theorem 4.1. □

Proof of Theorem 1.3. Recalling that $q^k := (q_1, q_2, kr_kq_3)$, we write

$$F_k^\phi(q) = \int\limits_{B_{r_k}(0)} (p_3 + kr_k)^{-3}\phi(q_3p + q^k)\,dp\,.$$

Since $r_k \to 0$ and $kr_k = k(k^2-1)^{-1/2} \to 1$ as $k \to \infty$, we infer that $q^k \to q$ uniformly on compact sets of $\mathbb{R}^3_+$ and

$$\frac{3}{4\pi r_k^3}F_k^\phi \to \phi \qquad \text{as } k \to \infty\,,$$

uniformly on $\overline{\Omega}$. Next, we easily compute

$$\begin{aligned} \partial_{q_j}F_k^\phi(q) &= \int\limits_{B_{r_k}(0)} (p_3 + kr_k)^{-3}\partial_{q_j}\phi(q_3p + q^k)\,dp\,, \qquad j = 1, 2\,, \\ \partial_{q_3}F_k^\phi(q) &= \int\limits_{B_{r_k}(0)} (p_3 + kr_k)^{-3}\nabla\phi(q_3p + q^k)\cdot(p + kr_ke_3)\,dp\,, \end{aligned}$$

and thus we obtain, by the same argument,

$$\frac{3}{4\pi r_k^3}\nabla F_k^\phi \to \nabla\phi \qquad \text{as } k \to \infty,$$

uniformly on $\overline{\Omega}$. It follows that for k large enough, F_k^ϕ has a stable critical point in $\Omega \Subset \mathbb{H}^3$, since having a stable critical point is a $\mathcal{C}^1$-open condition. Hence Theorem 1.1 applies and gives the conclusion of the proof. □

Appendix

Let $K \in \mathcal{C}^0(\mathbb{H}^3)$. Take any vectorfield $Q_K \in \mathcal{C}^1(\mathbb{R}^3_+, \mathbb{R}^3)$ such that $\mathrm{div} Q_K(p) = p_3^{-3} K(p)$ for any $p \in \mathbb{R}^3_+$ (here $\mathrm{div} = \sum_j \partial_j$ is the Euclidean divergence). The functional

$$V_K(u) := \int_{\mathbb{R}^2} Q_K(u) \cdot \partial_x u \wedge \partial_y u \, dz \,, \qquad u \in \mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3) \,,$$

measures the signed (hyperbolic) volume enclosed by the surface u, with respect to the weight K. In fact, if u parameterizes the boundary of a smooth open set $\Omega \Subset \mathbb{R}^3_+$ and if $\partial_x u \wedge \partial_y u$ is inward-pointing, then the divergence theorem gives

$$V_K(u) = -\int_{\partial\Omega} Q_K(u) \cdot \nu \, du = -\int_{\Omega} p_3^{-3} K \, dp = -\int_{\Omega} K \, d\mathbb{H}^3 \,.$$

Clearly, the functional V_K does not depend on the choice of the vectorfield Q. Notice that if $K \equiv k$ is constant, then

$$V_k(u) = -\frac{k}{2} \int_{\mathbb{R}^2} u_3^{-2} e_3 \cdot \partial_x u \wedge \partial_y u \, dz \;, \quad u \in \mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3) \,.$$

In the next Lemma we collect few simple remarks about the energy functional

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} u_3^{-2} |\nabla u|^2 \, dz + 2 V_K(u) \,. \tag{A.1}$$

Lemma A.1 *Let $K \in \mathcal{C}^0(\mathbb{H}^3)$.*

i) The functional $E : \mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3) \to \mathbb{R}$ is of class $\mathcal{C}^1$, and its differential is given by

$$E'(u)\varphi = \int_{\mathbb{R}^2} (u_3^{-2} \nabla u \cdot \nabla \varphi - u_3^{-3} |\nabla u|^2 e_3 \cdot \varphi) \, dz + 2 \int_{\mathbb{R}^2} u_3^{-3} K(u) \varphi \cdot \partial_x u \wedge \partial_y u \, dz \;;$$

ii) If $u \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$, then $E'(u)$ extends to a continuous form on $\mathcal{C}^0(\overline{\mathbb{R}}^2, \mathbb{R}^3)$, namely

$$E'(u)\varphi = \int_{\mathbb{R}^2} (-\mathrm{div}(u_3^{-2} \nabla u) - u_3^{-3} |\nabla u|^2 e_3 + 2 u_3^{-3} K(u) \partial_x u \wedge \partial_y u) \cdot \varphi \, dz \;;$$

iii) If $K \in \mathcal{C}^1(\mathbb{H}^3)$, then E is of class $\mathcal{C}^2$ on $\mathcal{C}^1(\overline{\mathbb{R}}^2, \mathbb{H}^3)$.

In the next Lemma we show that critical points for E are in fact hyperbolic K-bubbles.

Lemma A.2 *Let $K \in \mathcal{C}^0(\mathbb{H}^3)$ and let $u \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$ be a nonconstant critical point for E. Then u is conformal, that is,*

$$|\partial_x u| = |\partial_y u| \,, \quad \partial_x u \cdot \partial_y u = 0 \,,$$

hence it parameterizes an $\mathbb{S}^2$ type surface in $\mathbb{H}^3$, having mean curvature K, apart from a finite number of branch points.

Proof Put $\alpha=\frac{1}{2}u_3^{-2}(|\partial_x u|^2-|\partial_y u|^2)$, $\beta=-u_3^{-2}\partial_x u\cdot\partial_y u$, $\varphi=\alpha+i\beta$ and notice that $|\varphi|\le c_u|\nabla u|^2\in L^\infty(\mathbb{R}^2)$. By direct computation we find

$$\begin{aligned}(\partial_x\alpha-\partial_y\beta)u_3^3&=u_3\partial_x u\cdot\Delta u-(|\partial_x u|^2-|\partial_y u|^2)\partial_x u_3-2(\partial_x u\cdot\partial_y u)\partial_y u_3\,,\\(\partial_y\alpha+\partial_x\beta)u_3^3&=-u_3\partial_y u\cdot\Delta u-(|\partial_x u|^2-|\partial_y u|^2)\partial_y u_3+2(\partial_x u\cdot\partial_y u)\partial_x u_3\,.\end{aligned}\tag{A.2}$$

Since u solves (1.1), it holds that

$$\begin{aligned}u_3\partial_x u\cdot\Delta u=2G(\nabla u)\cdot\partial_x u&=2(\partial_x u\cdot\partial_y u)\partial_y u_3+(|\partial_x u|^2-|\partial_y u|^2)\partial_x u_3\,,\\u_3\partial_y u\cdot\Delta u=2G(\nabla u)\cdot\partial_y u&=2(\partial_x u\cdot\partial_y u)\partial_x u_3-(|\partial_x u|^2-|\partial_y u|^2)\partial_y u_3\,.\end{aligned}\tag{A.3}$$

Putting together (A.2) and (A.3) we obtain $\partial_x\alpha-\partial_y\beta=\partial_y\alpha+\partial_x\beta=0$, namely, φ is an holomorphic function. Since φ is bounded and vanishes at infinity then $\varphi\equiv 0$ on $\mathbb{R}^2$, hence u is conformal.

The last conclusion follows from Proposition 2.4 and Example 2.5(4) in [14]. □

Remark A.1 *Here we take $K\equiv k$ constant and point out two simple facts about the energy functional E_{hyp} in (1.3).*

By (4.2), the Nehari manifold contains any nonconstant function. Secondly, E_{hyp} is unbounded from below. In fact, for $t>1$ we have

$$E_{hyp}(\omega+te_3)=\frac{1}{2}\int\limits_{\mathbb{R}^2}(\omega_3+t)^{-2}\ \mu^2dz+k\int\limits_{\mathbb{R}^2}(\omega_3+t)^{-2}\omega_3\ \mu^2dz=4\pi\Big(-\frac{kt-1}{t^2-1}+\frac{k}{2}\ln\frac{t+1}{t-1}\Big).$$

Notice that $\omega+te_3$ approaches a horosphere as $t\to1$, and that $\lim\limits_{t\to1}E_{hyp}(\omega+te_3)=-\infty$.

Remark A.2 *Differently from the Euclidean case, see for instance [5], the geometric and compactness properties of the energy functional E are far from being understood (also in the case of a constant curvature), and would deserve a careful analysis.*

We conclude the paper by pointing out a necessary condition for the existence of embedded K bubbles.

Let $K\in\mathcal{C}^1(\mathbb{H}^3)$ be given, and let $u\in\mathcal{C}^2(\overline{\mathbb{R}}^2,\mathbb{H}^3)$ be an embedded solution to (1.1). By Lemma A.2, u is a conformal parametrization of the open set $\Omega\subset\mathbb{R}^3_+$, which is the bounded connected component of $\mathbb{R}^3_+\setminus u(\mathbb{S}^2)$. We can assume that the nowhere vanishing normal vector $\partial_x u\wedge\partial_y u$ is inward pointing. Since u is a critical point of the energy functional in (A.1), then for $j=1,2$ we have that

$$0=E'(u)e_j=V_K'(u)e_j=\int\limits_{\mathbb{R}^2}u_3^{-3}K(u)e_j\cdot\partial_x u\wedge\partial_y u\,dz=-\int\limits_{\Omega}\mathrm{div}(p_3^{-3}K(p)e_j)\,dp$$

by the divergence theorem. Thus

$$\int\limits_{\Omega}p_3^{-3}\partial_{p_j}K(p)\,dp=0.$$

In a similar way, from $E'(u)u = 0$ and since $\operatorname{div}(p_3^{-3}K(p)p) = p_3^{-3}\nabla K(p)\cdot p$, one gets

$$\int_{\Omega} p_3^{-3}\nabla K(p)\cdot p\,dp = 0.$$

In particular, $\partial_{p_1}K, \partial_{p_2}K$ and the radial derivative of K can not have constant sign in Ω. We infer the next nonexistence result (see [7, Proposition 4.1] for the Euclidean case).

Theorem A.3 *Assume that $K \in \mathcal{C}^1(\mathbb{H}^3)$ satisfies one of the following conditions,*

i) $K(p) = f(\nu\cdot p)$ *for some direction ν orthogonal e_3, where f is strictly monotone;*
ii) $K(p) = f(|p|)$, *where f is strictly monotone.*

Then (1.1) has no embedded solution $u \in \mathcal{C}^2(\overline{\mathbb{R}}^2, \mathbb{H}^3)$.

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