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Quotients of locally minimal groups ☆

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ABSTRACT

A topological group G is called *locally minimal* if there exists a neighbourhood V of 1 in G such that if H is a Hausdorff group and $f : G \to H$ is a continuous isomorphism such that $f(V)$ is a neighbourhood of 1 in H, then f is open. This paper is focused on the study of quotients of locally minimal groups.
A topological group G is called *locally q-minimal* if there exists a neighbourhood V of the identity of G such that whenever H is a Hausdorff group and $f : G \to H$ is a continuous surjective homomorphism such that $f(V)$ is a neighbourhood of 1 in H, then f is open. We find a close connection between locally q-minimality and divisibility, by showing that a dense subgroup of $\mathbb{R}^n$ is locally q-minimal if and only if it is divisible. A description of the locally q-minimal subgroups of the n-dimensional torus $\mathbb{T}^n$ is also given.
Two weaker versions of local q-minimality are proposed – a topological group G is:

(a) *locally t-minimal*, if all Hausdorff quotients of G are locally minimal;
(b) *locally q^*-minimal*, if there exists a neighbourhood V of 1 in G such that whenever H is a Hausdorff group and $f : G \to H$ is a continuous surjective homomorphism such that $f(V)$ is a neighbourhood of 1 in H and $\ker f \subseteq V$, then f is open.

While local q-minimality and local t-minimality are preserved by taking quotients, local q^*-minimality fails to have this property (so does not coincide with local t-minimality), yet it has a relevant advantage. By means of an appropriate notion of local t-density, one can show that a dense subgroup H of a Hausdorff group G is locally q^*-minimal if and only if G is locally q^*-minimal and H is locally t-dense in G. Similar criterion for local q-minimality is not available (examples are given to show that a topological group with a dense locally q-minimal subgroup need not be locally q-minimal). The interrelations of these three versions of local minimality, as well as their prominence properties are studied in detail.

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1. Introduction

Call a Hausdorff topological group G *minimal*, if G admits no properly coarser Hausdorff group topology. Obviously, G is minimal precisely when G satisfies the open mapping theorem with respect to continuous isomorphisms with domain G. Compact groups are minimal, the first examples of non-compact minimal groups were found by Doïchinov [23] and Stephenson [37]. The research in this field was inspired by a challenging problem set by G. Choquet at the ICM in Nice 1970; it was quite intensive for almost five decades (see [4,10,16–19,9,24,29,31,33–36,38], as well as the surveys or monographs [5,6,11,12,14,22]).

As noticed by Stephenson, locally compact groups need not be minimal (actually, a locally compact abelian group is minimal precisely when it is compact). This motivated Morris and Pestov [30] (see also Banakh [3]) to introduce the notion of locally minimal groups: a Hausdorff topological group (G, τ) is *locally minimal with respect to a neighbourhood* V of the identity of G if every Hausdorff group topology $\sigma \leq \tau$ and such that V is a σ-neighbourhood of the identity, coincides with τ (sometimes, for simplicity, we simply speak of local minimality without mentioning an explicit neighbourhood V). Locally compact groups and normed vector spaces are locally minimal [30]. Further details on locally minimal groups can be found in [15,1,2,40,41]. The relevant permanence properties of local minimality related to the passage to closed or dense subgroups were largely studied in these papers. Nevertheless, perhaps the most relevant one (from the point of view of the open mapping theorem) was not discussed so far, namely the preservation of local minimality under taking quotients. The aim of this paper is to fill this gap by a careful study of the stability of the class of locally minimal groups under taking quotients.

1.1. Local q-minimality and local t-minimality

Minimality fails to be preserved under taking quotients. This is why the smaller class of *totally minimal groups*, namely the minimal groups that are minimal along with all their Hausdorff quotients, was introduced in [16] (somewhat later also in [36], under the name *q-minimal groups*). Equivalently, a topological group G is totally minimal if every surjective continuous homomorphism of G onto a Hausdorff topological group is open. Inspired by the latter formulation (and the definition of local minimality, depending on a fixed neighbourhood V of the identity), the locally q-minimal groups were introduced in [15] as a kind of "local open mapping theorem":

Definition 1.1. [15] A topological group G is called *locally q-minimal with respect to a neighbourhood* V of the identity of G if every continuous surjective homomorphism $f : G \to H$ onto a Hausdorff group H such that $f(V)$ is a neighbourhood of 1 in H, is open.

Often we say briefly G *is locally q-minimal* if there exists such a neighbourhood V. Locally compact groups and totally minimal groups are locally q-minimal ([15, Lemma 2.8]).

The point of view adopted in the first of the above two equivalent formulations of total minimality (*every Hausdorff quotient of the group is minimal*) provides a natural and obvious alternative way to obtain a "local" version of the open mapping theorem as follows:

Definition 1.2. A topological group G is called *locally t-minimal* if each Hausdorff quotient group of G is locally minimal.

This property was given and used in [39] under the term local q-minimality. We prefer to use a different term (namely, local t-minimality), since one of the aims of this paper is to show that *these two notions differ substantially*.

It is clear that a topological group G is locally q-minimal with respect to a neighbourhood V of the identity of G iff G/N is locally minimal with respect to VN/N for each closed normal subgroup N of G. Hence,

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a locally q-minimal group is obviously locally t-minimal, but the converse is not true (see Examples 4.12 and 4.15). In particular, locally compact groups and totally minimal groups are locally t-minimal.

We provide a general sufficient condition for local q-minimality (Lemma 4.1 and Corollary 4.2). It provides a remarkable connection between local q-minimality and the algebraic structure of the underlying group, in particular, its divisibility. More precisely, we show that a dense subgroup of $\mathbb{R}^n$ is locally q-minimal if and only if it is divisible (see Theorem 4.14 for a sharper statement involving *all* subgroups of $\mathbb{R}^n$). This provides examples showing that a topological group with a dense locally q-minimal subgroup need not be locally q-minimal. In Theorem 4.9 we show that for a locally q-minimal subgroup G of $\mathbb{T}^n$ the torsion part $t(G)$ is totally minimal. This condition becomes also sufficient when either $t(G)$ is dense in G or $n = 1$, in the latter case it is equivalent to ask that either G is totally minimal or $t(G)$ is finite.

1.2. Local q^*-minimality

The third generalisation of total minimality (the local q^*-minimality, see Definition 1.4) is closely related to another relevant property of total minimality that we recall first.

The following notion was proposed in [16] (somewhat later also in [36]): a subgroup H of a topological group G is called *totally dense* if $H \cap N$ is dense in N for every closed normal subgroup N of G. This notion was used to provide the following crucial criterion for total minimality of dense subgroups:

Theorem 1.3. [16] *A dense subgroup H of a topological group G is totally minimal iff G is totally minimal and H is totally dense in G.*

All tentatives to find a criterion for local q-minimality of dense subgroups in this line have failed so far. We show that such a criterion for local q-minimality simply cannot be available, as a group containing a dense locally q-minimal subgroup need not be locally q-minimal itself (Example 4.15). This suggests to use the following weaker version of local q-minimality that still remains in the same spirit of "local open mapping theorem":

Definition 1.4. A Hausdorff topological group G is called *locally q^*-minimal with respect to a neighbourhood* V of the identity of G if every continuous surjective homomorphism $f : G \to H$ onto a Hausdorff group H such that $f(V)$ is a neighbourhood of the identity in H with $\ker f \subset V$, is open.

Often we say briefly G is *locally q^*-minimal* if there exists such a neighbourhood V. Obviously, local q-minimality implies local q^*-minimality, but the converse implication may fail even for subgroups of $\mathbb{R}$ (see Examples 4.12 and 4.15). The advantage of the notion of local q^*-minimality is that it allows for a criterion for local q^*-minimality of dense subgroups in the line of Theorem 1.3 (where total density is replaced by *local t-density*, see Definition 5.2 and Theorem 5.4). This criterion provides numerous applications (§6).

The paper is organised as follows. In §2 we give some background on minimal and locally minimal groups. Section 3 investigates permanence properties of locally q-, q^*- and t-minimal groups with respect to taking closed or open subgroups or extensions. We often refer to this last permanence property as "the three space property" intending classes $\mathcal{P}$ of topological groups, such that if a closed normal subgroup K of a topological group G, as well as the quotient G/K belong to $\mathcal{P}$, then also G belongs to $\mathcal{P}$.

Section 4 is dedicated to the connection of local q-minimality to divisibility mentioned above. Here we describe the locally q-minimal subgroups of $\mathbb{T}^n$ and $\mathbb{R}^n$, providing a necessary condition for local q-minimality of subgroups of $\mathbb{T}^n$. In Theorem 4.3 we prove that the precompact and locally q-minimal abelian groups with dense torsion part are actually totally minimal. This imposes very rigid algebraic restraint on the torsion precompact and locally q-minimal abelian groups.

In §5 we give the local q^*-minimality criterion and its applications. It allows us to show, among others, that an appropriate dense subgroup of the Hilbert space ℓ^2 is locally q^*-minimal, but has non-locally minimal quotients (Example 6.5). Consequently, local q^*-minimality, unlike local q-minimality and local t-minimality, is not preserved by taking quotients (so it does not imply local t-minimality).

In §6 we explore the connection of local q^*-minimality to other compactness-like properties, as pseudocompactness, sequential completeness and countable compactness. We show that in combination with some of these properties (i.e., pseudocompactness and sequential completeness) local q^*-minimality becomes equivalent to compactness (Theorem 7.1).

In §7 we collect some final remarks and open questions. It contains also a diagram connecting all properties studied in the paper.

Notation and terminology We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive natural numbers and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$, so written additively. The cardinality of the continuum 2^ω will be also denoted by $\mathfrak{c}$. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$. For a prime p the symbol $\mathbb{Z}(p^\infty)$ stands for the quasicyclic p-group and $\mathbb{Z}_p$ stands for the p-adic integers.

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The abbreviation $K \leq G$ is used to denote a subgroup K of G.

For a group G and $n \in \mathbb{N}$ let

$$G[n] := \{x \in G : nx = 0\} \ \text{ and } \ t(G) := \bigcup_n G[n] = \{g \in G : ng = 0 \text{ for some } n \in \mathbb{N}\},$$

the *torsion part* of G. Clearly, $G[n]$ and $t(G)$ are subgroups of G, in case G is abelian. The subgroup $Soc(G) = \bigoplus_{p\in\mathbb{P}} G[p]$ is called the *socle* of G. For a prime p, the *p-primary component* G_p of G is the subgroup of G that consists of all $x \in G$ satisfying $p^n x = 0$ for some positive integer n. An abelian group G is *divisible* if $nG = G$ for every $n \in \mathbb{N}$.

Throughout this paper all topological groups are assumed to be Hausdorff, unless otherwise stated explicitly. We denote by $\mathcal{V}_\tau(1)$ (or simply by $\mathcal{V}(1)$) the filter of neighbourhoods of 1 in a topological group (G, τ).

For a subset X of a topological group G we denote by $\overline{X}$ the closure of X and by $\widetilde{G}$ the Raĭkov completion of G. A group G is *precompact* (some authors prefer "totally bounded") if $\widetilde{G}$ is compact. The centre $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$ of G is a closed subgroup of G.

All unexplained topological terms can be found in [26]. For background on Abelian groups, see [27].

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2. Background on minimal and locally minimal groups

In order to formulate the minimality criterion from [4,31,37] and the local minimality criterion from [2], we need to recall first the following notions:

Definition 2.1. Let H be a subgroup of a topological group G. We say that

- [4,31,37] H is *essential* in G if $H \setminus \{0\}$ meets each nontrivial closed normal subgroup N of G;
- [2] H is *locally essential* in G if there exists a neighbourhood V of 0 in G such that $H \setminus \{0\}$ meets each nontrivial closed normal subgroup N of G which is contained in V.

When necessary, we shall say H is locally essential with respect to V to indicate that V witnesses local essentiality. Note that if V witnesses local essentiality, then any smaller neighbourhood of zero does too.

A topological group G is said to have *no small subgroups* (or shortly, to be an *NSS group*), if G has a neighbourhood of the identity element that contains no non-trivial subgroups.

Remark 2.2. Obviously, every subgroup of an NSS group is locally essential. On the other hand, locally minimal NSS groups are obviously locally q^*-minimal (see Example 4.13 for a natural class of groups with this property, containing all subgroups of normed space).

The following criteria for minimality and for local minimality was established in [4,31,37] and [2], respectively:

Fact 2.3. Let H be a dense subgroup of a topological group G.

- [Criterion for minimality] Then H is minimal iff G is minimal and H is essential in G.
- [Criterion for local minimality] Then H is locally minimal iff G is locally minimal and H is locally essential in G.

Remark 2.4. The proof of Fact 2.3 in [2, Theorem 3.5] shows more. Namely, for a dense subgroup H of G:

(1) When H is locally minimal and if W is a closed neighbourhood of 1 in G such that $W \cap H$ witnesses local minimality of H, then each neighbourhood W_1 of 1 in G satisfying $W_1^2 \subset W$ witnesses local essentiality of H in G and W witnesses local minimality of G.

(2) When G is locally minimal and if the neighbourhood V of 1 in G witnesses both local minimality of G and local essentiality of H in G, then for every neighbourhood V_1 of 1 in G with $V_1^2 \subset V$ the neighbourhood $V_1 \cap H$ witnesses local minimality of H.

Fact 2.5. [15,39] If a group G has an open locally minimal subgroup, then G itself is locally minimal.

In item (b) we see some examples of locally minimal groups without open locally minimal subgroups.

Example 2.6.

(a) According to [15,39,30] every subgroup G of a Lie group L is locally minimal. To see that G is also locally t-minimal, assume that G is dense in L (obviously, this is not a restrictive assumption). Let G/N be a quotient of G with respect to a closed normal subgroup N of G. Then G/N is isomorphic to a (dense) subgroup of the quotient $L/\overline{N}$ of L, hence G/N is locally minimal.
On the other hand, G is also locally q^*-minimal, witnessed by any neighbourhood of e_G containing no non-trivial subgroups.

(b) Consider the subgroup $H = \mathbb{Z}(p^\infty)$ of $\mathbb{T}$. Then H is locally t-minimal by (a), but H has no proper open subgroups and H itself is not minimal. Thus H has no open minimal subgroup. Analogous argument shows that any dense embedding of $\mathbb{Z}$ in $\mathbb{T}$ induces on $\mathbb{Z}$ a locally t-minimal topology on $\mathbb{Z}$ without open minimal subgroups.

3. Some general properties of the local q-, q^*- and t-minimality

3.1. *Invariance under taking closed central subgroups*

We start by recalling a result on closed central subgroups from [1].

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Fact 3.1. [1, Proposition 2.5] If G is a locally minimal topological group and H is a closed central subgroup of G, then H is locally minimal. More precisely, if local minimality of G is witnessed by V, then H is locally minimal with respect to $V_1 \cap H$ for any neighbourhood V_1 of the identity of G such that $V_1^2 \subset V$.

Even if the next fact can be found in [15, Lemma 2.3], we give here a short proof for the sake of completeness.

Lemma 3.2. *Let G be a locally minimal group. Then there exists a neighbourhood U of the identity in G such that each closed central subgroup N of G contained in U is minimal (so, precompact).*

Proof. Assume that G is locally minimal with respect to U^2, where U is a neighbourhood of the identity of G. Let N be a closed central subgroup of G contained in U. According to Fact 3.1, N is locally minimal with respect to $U \cap N = N$, hence, N is minimal. For the last assertion recall that according to the celebrated Prodanov-Stoyanov Theorem, every minimal abelian group is precompact. □

Corollary 3.3. *Let G be a locally minimal abelian group. Then there exists a neighbourhood U of the identity in G such that each closed subgroup N of G contained in U is minimal (so, precompact).*

Now we extend the lemma to its natural counterpart for local q^*-minimality.

Proposition 3.4. *Let G be a locally q^*-minimal group. Then there exists a neighbourhood U of the identity in G such that each closed central subgroup N of G contained in U is totally minimal (so, precompact).*

Proof. Assume that G is locally q^*-minimal with respect to U^2, where U is a neighbourhood of the identity of G. Let N be a closed central subgroup of G contained in U. To prove that N is totally minimal, take a closed subgroup K of N contained in U. It is central, so normal in G. Moreover, G/K is locally minimal with respect to U^2K/K, as K contained in U. According to Fact 3.1, the subgroup N/K of G/K is locally minimal with respect to $UK/K \cap N/K = N/K$, as $NK \subseteq UK$. Hence, N/K is minimal. The last assertion follows, as above, from Prodanov-Stoyanov Theorem. □

Obviously, the conclusion of the proposition remains true for the stronger property of local q-minimality. We are not aware if this holds true for local t-minimality (see Question 8.3). One can briefly resume these results as follows: "small" closed central subgroups of locally (q^*-) minimal group are (totally) minimal (hence, precompact). We give also a corollary of this proposition below (see Corollary 3.7).

Now we show that a closed central subgroup of a locally q-minimal (resp., locally q^*-minimal, locally t-minimal) group is locally q-minimal (resp., locally q^*-minimal, locally t-minimal).

Proposition 3.5. *Let H be a closed central subgroup of a topological group G.*

(a) *If G is locally q-minimal then also H is locally q-minimal.*
(b) *If G is locally q^*-minimal then also H is locally q^*-minimal.*
(c) *If G is locally t-minimal then also H is locally t-minimal.*

Proof. (a) Suppose that G is locally q-minimal with respect to U, a neighbourhood of the identity of G. Let H be a closed central subgroup of G. Take a neighbourhood V of the identity such that $V^2 \subset U$, we are going to prove that $V \cap H$ witnesses local q-minimality of H.

Let N be a closed subgroup of H, then it is a closed normal subgroup of G as well. Denote by π the natural quotient mapping of G onto G/N. By our assumption, G/N is locally minimal with respect to $\pi(U)$.

Since $\pi(H)$ is a closed subgroup of G/N and $\pi(V)^2 = \pi(V^2) \subset \pi(U)$, we apply Fact 3.1 and get that $\pi(H)$ is locally minimal with respect to $\pi(V) \cap \pi(H)$, so to $\pi(V \cap H) \subseteq \pi(V) \cap \pi(H)$ as well.

The proofs of (b) and (c) are similar. $\square$

3.2. Invariance under taking open subgroups

The next proposition shows that the implications of Proposition 3.5 can be inverted in case "closed central subgroup" is replaced by "open subgroup" (see also Remark 3.8 below):

Proposition 3.6. *A Hausdorff topological group with an open locally q-minimal (resp., q^*-minimal, t-minimal) subgroup is locally q-minimal (resp., q^*-minimal, t-minimal).*

Proof. We first prove the case of local q-minimality, the case of local q^*-minimality is similar. Let H be a locally q-minimal group witnessed by $U \in \mathcal{V}_H(1)$ and suppose that H is an open subgroup of (G, τ). Then U is a neighbourhood of 1 in G. Assume that $f : G \to G_1$ is a surjective homomorphism such that $f(U)$ is a neighbourhood of 1 in G_1. Let $f \restriction_H$ be the restriction of f to H, considered as a surjective homomorphism of H onto $f(H)$. Since $U \subseteq H$, we have that $f(U) \subseteq f(H)$, then $f(U)$ is a neighbourhood of 1 in $f(H)$ and $f(H)$ is open in G_1 (because it contains a neighbourhood $f(U)$ of the identity of G_1). By the U-local minimality of H, one readily gets that $f \restriction_H : H \to f(H)$ is open. Since that H is open in G and $f(H)$ is open in G_1, $f : G \to G_1$ is also open.

Now suppose that the open subgroup H of G is locally t-minimal. Let N be an arbitrary closed normal subgroup of G, it suffices to show that G/N is locally minimal. Obviously, $M = H \cap N$ is a closed normal subgroup of H. By local t-minimality of H, H/M is locally minimal.

Let $\pi : G \to G/N$ and $\xi : H \to H/M$ be the canonical maps. They are continuous and open; moreover, there exists a continuous isomorphism $j : H/M \to \pi(H)$ with $\pi \restriction_H = j \circ \xi$. Pick an open neighbourhood U of the identity in H, such that $\xi(U)$ witnesses local minimality of H/M. Since H is open in G, this yields that U is an open neighbourhood of the identity in G as well. Then $\pi(U)$ is an open neighbourhood of the identity in G/N contained in $\pi(H)$. To the continuous isomorphism $j : H/M \to \pi(H)$ we can apply the local minimality of H/M (with respect to $\xi(U)$) to conclude that j is a topological isomorphism, as $j(\xi(U)) = \pi(U)$ is open in $\pi(H)$. Hence, the open subgroup $\pi(H)$, being topologically isomorphic to H/M, is locally minimal. Hence, G/N is locally minimal as well. $\square$

By Proposition 3.4, if the neighbourhood $U \in \mathcal{V}_G(1)$ witnessing local (q^*-)minimality is a central subgroup, then U itself must be (totally) minimal. From this observation and the above proposition one obtains:

Corollary 3.7. *Suppose that G is a topological group with a local base at 1 of open central subgroups. Then the followings are equivalent:*

(a) *G is locally q^*-minimal;*
(b) *G has an open totally minimal subgroup;*
(c) *G is locally q-minimal.*

Remark 3.8. Let H be an open subgroup of a topological group G. We do not know if any of the three implication in Proposition 3.6 or in Fact 2.5 can be inverted in general. We show in Theorem 3.11 and in Corollary 3.12 that this is true under some restraint on H (e.g., when H is central, or simply when G is abelian).

Definition 3.9. [20, Definition 3.1] A subgroup H of a group G is called:

(i) *Hausdorff embedded* in G if for every Hausdorff group topology τ on H there exists a Hausdorff group topology τ' on G such that $\tau = \tau' \restriction_H$ (and in this case we say that τ' extends τ);
(ii) *super-normal* (in G) if $G = c_G(H)H$, i.e., for every $x \in G$ there exists $y \in H$ such that $x^{-1}hx = y^{-1}hy$ for every $h \in H$.

Super-normal subgroups are Hausdorff embedded, hence central subgroups, as well as direct summands are Hausdorff embedded [20]. As far as extension of a *fixed* Hausdorff group topology is concerned, one has the following:

Fact 3.10. A normal subgroup H of a group G is Hausdorff embedded in G if and only if the automorphisms of H induced by conjugation by elements of G are continuous for any Hausdorff group topology λ on H [20, Theorem 3.4]. In such a case one can extended λ to a *finest* Hausdorff group topology λ^* on G in a standard way, by declaring the family $\mathcal{V}_{(H,\lambda)}(e)$ to form a local base at e of λ^*. This standard extension is uniquely determined with the properties $\lambda^* \restriction_H = \lambda$ and the λ^*-openness of H.

As far as invariance under taking open subgroups is concerned, we are not aware if the additional hypotheses on the open subgroup (being Hausdorff embedded, or even super-normal) are essential.

Theorem 3.11. *Let H be an open normal subgroup of a topological group G.*

(a) *If H is Hausdorff embedded, then H is locally minimal if and only if G is locally minimal.*
(b) *If H is super-normal, then H is locally q-minimal (locally q^*-minimal, locally t-minimal) if and only if G is locally q-minimal (locally q^*-minimal, locally t-minimal).*

Proof. Let τ be the topology of G.

(a) Let $V \in \mathcal{V}_G(1)$ witnesses local minimality of (G, τ). Since H is open, we can assume that $V \subseteq H$, so $V \in \mathcal{V}_H(1)$. To show that V witnesses local minimality of $(H, \tau \restriction_H)$ pick a Hausdorff group topology $\sigma \leq \tau \restriction_H$ with $V \in \sigma$. Since H is a Hausdorff embedded subgroup of G, the standard extension σ^* of σ is a Hausdorff group topology on G such that $\sigma^* \restriction_H = \sigma \leq \tau \restriction_H$ and H is σ^*-open in G. Since $V \in \sigma$, we deduce that $V \in \sigma^*$ as well. Since H is τ-open, we deduce that $\sigma^* \leq \tau$. Now the local minimality of (G, τ) implies that the identity $(G, \tau) \to (G, \sigma^*)$ is open. Hence, $\sigma^* = \tau$ and consequently $\sigma = \sigma^* \restriction_H = \tau \restriction_H$. The other implication is valid (for arbitrary open subgroup H of G), by Fact 2.5.

(b) Let us note first that the (stronger) assumption that H is super-normal implies that every normal subgroup of H is normal in G as well.

Let $V \in \mathcal{V}_G(1)$ witnesses local q-minimality of (G, τ). Since H is open, we can assume without loss of generality that $V \subseteq H$, so $V \in \mathcal{V}_H(1)$. To show that V witnesses local q-minimality of $(H, \tau \restriction_H)$ pick a normal closed subgroup N of H, it will be a normal subgroup in G. Let $f : H \to H/N$ and $h : G \to G/N$ be the quotient maps and let σ be a Hausdorff group topology on H/N such that $f : (H, \tau \restriction_H) \to (H/N, \sigma)$ is continuous and $f(V) \in \sigma$. Let $j : H/N \to G/N$ be the obvious identification of H/N with a subgroup of the abstract group G/N. As H is open in G, $j(H/N)$ will be open in $(G/N, \bar{\tau})$, where $\bar{\tau}$ denotes the quotient topology of G/N. Since $j(H)$ is $\bar{\tau}$-open, we deduce that $\sigma^* \leq \bar{\tau}$, so the local q-minimality of (G, τ) implies that $h : (G, \tau) \to (G/N, \sigma^*)$ is open. Since H is τ-open, this yields that $f = h \restriction_H$ is open as well. The other implication is valid (for arbitrary open subgroup H of G), by Proposition 3.6.

Similar arguments work for the remaining two cases. □

The next corollary follows from Theorem 3.11 using the fact that central subgroups are super-normal.

Corollary 3.12. *Let H be an open central subgroup of a topological group G. Then H is locally (q, q^*-, t-)minimal iff G is locally (q, q^*-, t-)minimal.*

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3.3. The 3-space property

In case the open subgroup H as in Proposition 3.6 is also normal, one can formulate Proposition 3.6 in a way similar to the 3-space problem: if a group G has a normal subgroup H such that G/H is discrete (hence, locally q-minimal), then G is locally q-minimal (resp., locally q^*-minimal, locally t-minimal) whenever the subgroup H is locally q-minimal (resp., locally q^*-minimal, locally t-minimal). We shall see in Example 4.12 that the counterpart of this property, when H is supposed to be discrete and G/H locally q-minimal, fails.

In the sequel, for a group X, a subgroup $G \leq X$ and a topology τ on G the symbol τ/G stands for the quotient topology on X/G with respect to τ.

Lemma 3.13. *Let (G, τ) be a Hausdorff topological group and K a closed normal subgroup of G. Suppose that for some neighbourhood U of the identity of G the group K is locally minimal with respect to $U \cap K$ and UK/K witnesses local minimality of G/K. Then for any Hausdorff group topology σ which is coarser than τ, if U is a σ-neighbourhood of the identity and K is σ-closed, then $\sigma = \tau$.*

Proof. Local minimality of $(K, \tau\restriction_K)$ with respect to $U \cap K$ and $U \cap K \in \sigma\restriction_K \leq \tau\restriction_K$ entail $\sigma\restriction_K = \tau\restriction_K$. Now consider $\sigma/K \leq \tau/K$. Since K is σ-closed, σ/H is Hausdorff. Moreover, as $U \in \sigma$, we deduce that $UK/K \in \sigma/K$. By the local minimality of G/K with respect to UK/K we deduce that $\sigma/K = \tau/K$. Applying a well-known fact (known as *Merson lemma*, see [9, Lemma 1]), from $\sigma \leq \tau$, $\sigma\restriction_K = \tau\restriction_K$, and $\sigma/K = \tau/K$ we deduce $\sigma = \tau$. □

Now we show that the 3-space property is available both for local minimality and for local q^*-minimality under an appropriate natural condition. Recall that a group is called *totally complete* if every Hausdorff quotient group is complete.

Theorem 3.14. *If H is a totally complete normal subgroup of G such that both H and G/H are locally (q^*-)minimal, then also G is locally (q^*-)minimal.*

Proof. First we prove the version of local minimality and in this case it is enough to assume that H is complete.

Suppose that U is a neighbourhood of the identity of G such that H is locally minimal with respect to $U \cap H$ and UH/H witnesses local minimality of G/H. We are going to prove that G is U-locally minimal, by using Lemma 3.13. Denote by τ the original topology on G and σ another Hausdorff group topology on G such that $\sigma \subset \tau$ and U is a σ-neighbourhood of the identity. Clearly $U \cap H$ is a neighbourhood of the identity of $H \leq (G, \sigma)$, hence the local minimality of $(H, \tau\restriction_H)$ yields that $\sigma\restriction_H = \tau\restriction_H$. Furthermore, since $(H, \tau\restriction_H)$ is complete, H is σ-closed in (G, σ). Hence, we can apply Lemma 3.13 to complete this part of the proof.

Now we pass to the proof of the stronger property of local q^*-minimality. Without loss of generality, we can choose a neighbourhood U of 1 in G such that $U^2 \cap H$ witnesses local q^*-minimality of H and UH/H witnesses local q^*-minimality of G/H. We will show that G is locally q^*-minimal with respect to U. Take a closed normal subgroup N of G contained in U, it suffices to prove that G/N is locally minimal with respect to $\pi(U)$, where $\pi : G \to G/N$ is the quotient homomorphism. By local q^*-minimality of H, $\pi\restriction_H$ is open, i.e. $\pi(H) \cong H/(H \cap N)$. Therefore, $\pi(H)$ is locally minimal with respect to $\pi(U^2 \cap H) \supset \pi(UN \cap H) = \pi(U) \cap \pi(H)$. Since H is totally complete, $\pi(H)$ is complete. Hence, $\pi(H)$ is a closed normal subgroup of $\pi(G) = G/N$. Note that $\pi(G)/\pi(H) \cong G/NH \cong (G/H)/(NH/H)$. Since $N \subset U$, $NH/H \subset UH/H$. By local q^*-minimality of G/H, $\pi(G)/\pi(H)$ is locally minimal with respect to $\pi(U)\pi(H)/\pi(H)$. So, according to the proof of the case of local minimality, $\pi(G)$ is locally minimal with respect to $\pi(U)$. □

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To see why "totally complete" cannot be omitted in the above theorem (as well as the next one) recall that for the totally minimal group K of the integers provided with the p-adic topology the product $G = K \times K$ is not even locally minimal [2], even if $K \cong G/K$ is totally minimal.

Now we prove a somewhat weaker version of Theorem 3.14 for the remaining two properties: local q-minimality and local t-minimality.

Theorem 3.15. *Let G be a group with a totally complete and totally minimal normal subgroup K.*

(a) *If G/K is locally q-minimal, then so is G.*
(b) *If G/K is locally t-minimal, then so is G.*

Proof. (a) Suppose that G/K is locally q-minimal with respect to VK/K, where V is a neighbourhood of 1 in G. Let N be an arbitrary closed normal subgroup of G, it suffices to show that G/N is locally minimal with respect to VN/N. The subgroup $(NK)/N$ of the quotient group G/N is isomorphic to a quotient of the totally complete group K, hence $(NK)/N$ is complete, so closed in G/N. Therefore, the subgroup NK of G is closed. Denote by $f : G/K \to G/NK$ and $\pi : G/N \to G/NK$ the natural homomorphisms. Clearly both f and π are open and continuous. Hence, G/NK is locally minimal with respect to $VNK/NK = \pi(VN/N)$, by local q-minimality of G/K. Furthermore, since K is totally minimal, $\ker \pi = NK/N$ is minimal and $NK/N \cong K/(N \cap K)$, so locally minimal with respect to $VN/N \cap NK/N$. By assumption, K is totally complete, so $NK/K \cong K/(N \cap K)$ is complete. According to the first part of the proof of Theorem 3.14, G/N is locally minimal with respect to VN/N.

(b) Let N be a closed normal subgroup of G, it suffices to show that G/N is locally minimal. Since G/NK is locally minimal by locally t-minimality of G/K, we only need to prove that NK/N is locally minimal and complete, according to Theorem 3.14. This follows from similar argument with the proof of (a). □

The hypotheses on K in the above theorems are satisfied when K is totally minimal and locally compact, in particular, compact:

Corollary 3.16. *If a group G has a compact normal subgroup K such that G/K is locally q-minimal (resp., locally q^*-minimal, locally t-minimal), then G is also locally q-minimal (resp., locally q^*-minimal, locally t-minimal).*

4. Local q-minimality vs divisibility of abelian groups

This section is dedicated mainly to local q-minimality and its distinction from local q^*-minimality and local t-minimality. All three properties coincide with local minimality when the underlying group is "small" (e.g., finite or infinite cyclic). More generally, if G is a topologically simple group (i.e., has no proper closed normal subgroups), then again all four properties coincide, as local minimality yields local q-minimality on such G.

4.1. A necessary condition for local q-minimality

For an abelian group G, we define the subgroups

$$\nu(G) = \bigcap_{n\in\mathbb{N}} nG \quad \text{and} \quad \pi(G) = \bigcap_{p\in\mathbb{P}} pG.$$

Obviously,

$$d(G) \subseteq \nu(G) \subseteq \pi(G),$$

where $d(G)$ is the maximum divisible subgroup of G. Moreover, $\nu(G) = d(G)$ when G is torsion-free. On the other hand, $G = \pi(G)$ implies that $G = d(G)$ is divisible.

The subgroup $\pi(G)$ provides a nice necessary condition for local q-minimality:

Lemma 4.1. *Suppose that G is a locally q-minimal abelian group with respect to $U \in \mathcal{V}(0)$ and N is closed subgroup of G such that $N + U = G$. Then:*

(a) *G/N is totally minimal;*
(b) *if G is a dense subgroup of a divisible topological group G_1, such that N is a closed subgroup of G_1 as well, then $N \subseteq \pi(G)$.*

Proof. (a) Since every quotient group of G/N is topologically isomorphic to some G/H with $N \leq H$, it suffices to check that G/H is minimal for every closed subgroup H of G containing N. This follows from the definition of local q-minimality since the equality $H + U = G$ holds.

(b) Since N is a closed subgroup of G_1, we can consider the quotient group G_1/N and identify the quotient group G/N with a dense topological subgroup of G_1/N. Since G/N is minimal by (a), G/N contains $Soc(G_1/N)$, by Fact 2.3(a). To prove that $N \subseteq \pi(G)$ it is necessary to check that $N \subseteq pG$ for every prime p. Assume that $g \in N \setminus pG$. Then $g = pg_1$ for some $g_1 \in G_1 \setminus G$, by the divisibility of G_1. Hence, the element $\bar{g}_1 := g_1 + N \in G_1/N$ has order p, so $\bar{g}_1 \in Soc(G_1/N)$. By what was observed above, this yields $\bar{g}_1 \in G/N$, hence $\bar{g}_1 = x + N$, for some $x \in G$. Then $g_1 - x \in N \leq G$. As $x \in G$, we conclude that $g_1 \in G$ as well, a contradiction as $g \notin pG$. □

As $\pi(G) = G$ (precisely) for divisible groups, this criterion is non-vacuous only for non-divisible groups G.

Since connected locally compact abelian groups are divisible, we obtain from Lemma 4.1 the following

Corollary 4.2. *Suppose that G is a dense locally q-minimal subgroup of a connected locally compact abelian group G_1. If local q-minimality is witnessed by some $U \in \mathcal{V}(0)$ such that $U + N = G$ for some closed subgroup N of G_1, then $N \leq G$, G_1/N is compact, G/N is totally dense in G_1/N and $N \subseteq \pi(G)$.*

Proof. The equality $U + N = G$ obviously implies $N \leq G$. By the above lemma, G/N is totally minimal, hence precompact according to Prodanov's precompactness theorem for totally minimal abelian groups [32]. Since G/N is isomorphic to a dense topological subgroup of the locally compact group G_1/N, the latter group must be compact. The total density of G/N in G_1/N follows from Theorem 1.3. □

In the sequel we apply only the last part, $N \subseteq \pi(G)$, of this corollary in order to prove that dense non-divisible subgroups G of $\mathbb{R}^n$ are not locally q-minimal. To this end we need to find for every $U \in \mathcal{V}(0)$ an appropriate closed subgroup N_U of $\mathbb{R}^n$ contained in G, such that $U + N_U = G$, such that $N_U \subseteq \pi(G)$ fails. Then G cannot be locally q-minimal with respects to U.

Now we propose another application of Lemma 4.1(a) towards precompact locally q-minimal abelian groups. According to Proposition 3.4, a locally (q^*-)minimal abelian group contains plenty of (totally) minimal hence precompact subgroups. Precompact topologies on "small" abelian groups that are both locally q^*-minimal and locally t-minimal are easy to come by, just embed the group in the finite powers $\mathbb{T}^n$ (so "smallness" here refers to embeddability into the powers $\mathbb{T}^n$). These are precisely the abelian groups having free-rank $\leq \mathfrak{c}$ and all p-ranks finite and uniformly bounded. Yet, such embeddings do not allow one to find a locally q-minimal group topology on some small groups like $\mathbb{Z}(p^\infty)$, as we shall see now.

Let us recall first that an *exotic torus* is a compact abelian group K having subgroups isomorphic to $\mathbb{Z}_p$ for no prime p [17]. In particular, $t(K)$ is totally dense for such a group K [17].

Theorem 4.3. *For a topological abelian group G with dense $t(G)$ consider the following properties:*

(a) *G is precompact and locally q-minimal;*
(b) *G is totally minimal;*
(c) *$t(G)$ is totally minimal;*
(d) *the completion $\widetilde{G}$ of G is an exotic torus and $t(G) = t(\widetilde{G})$.*

Then (d) $\Leftrightarrow$ (c) $\Rightarrow$ (b) $\Leftrightarrow$ (a). If $t(G)$ is totally dense in G, then all the properties are equivalent, and

$$t(G) \cong (\mathbb{Q}/\mathbb{Z})^n \oplus \bigoplus_p B_p, \tag{1}$$

where $n \in \mathbb{N}$ and each B_p is a p-group admitting a compact group topology.

Proof. The implication (c) $\to$ (b) follows from Theorem 1.3 and the denseness of $t(G)$ in G. Furthermore, if $t(G)$ is totally dense in G, the same theorem also implies that (b) $\to$ (c). The implication (b) $\to$ (a) directly follows from Prodanov's precompactness theorem [32]. Since $\widetilde{t(G)} = \widetilde{G}$ in view of the density of $t(G)$ in G, the equivalence of (c) and (d) follows from the main results in [17,32]. Finally, the same source (see also [22, Chap. 5]) ensures the validity of the last assertion. Therefore, it remains only to check the implication (a) $\to$ (b).

Assume that local q-minimality of G is witnessed by the neighbourhood $U + U$ of 0 in G. Since $t(G)$ is dense, $t(G) + U = G$. On the other hand, since $t(G)$ is precompact, we can find a finite set $F \subsetneq t(G)$, such that $t(G) \subseteq U + F$. As $t(G)$ is torsion, we can assume without loss of generality that F is a finite subgroup of $t(G)$. Now we have $U + U + F = G$.

The local q-minimality of G with respect to $U + U$ yields that G/F is totally minimal, according to Lemma 4.1. As F is finite, from the three space property for total minimality ([24], see also [22, Theorem 7.3.1]), we deduce that G is totally minimal as well. □

Theorem 4.3 gives as an immediate corollary a relevant necessary condition of local q-minimality for subgroups of exotic tori:

Corollary 4.4. *If G is a locally q-minimal subgroup of an exotic torus, then $t(G)$ is totally minimal.*

Proof. Let K be an exotic torus and G be a locally q-minimal subgroup of K. Then the closure G_1 of $t(G)$ in G is locally q-minimal by Proposition 3.5 and $t(G_1) = t(G)$ is dense in G_1. By Theorem 4.3, G_1 is totally minimal, thus G_1 is totally dense in its completion L, according to Theorem 1.3. In particular, G_1 contains $t(L)$, so $t(L) = t(G_1) = t(G)$. Moreover, L is a closed subgroup of the exotic torus K, hence L is an exotic torus as well ([17]), thus $t(L) = t(G_1) = t(G)$ is totally minimal. □

It is worth noting that if $t(G)$ is not totally dense in G, then (b) cannot imply (c) in Theorem 4.3. For example, let $G = \mathbb{T}^{\mathbb{N}}$. Then G is compact with dense subgroup $t(G)$, yet $t(G)$ is not even minimal, according to Fact 2.3 (a) (being non-essential in G: note that G contains copies of all compact metric groups; in particular of $\mathbb{Z}_p$, which is torsion-free, so meets $t(G)$ trivially).

Theorem 4.3 allows us to find the first examples of groups that are locally t-minimal and locally q^*-minimal, but not locally q-minimal, others will follow in §4.2.

Remark 4.5. Theorem 4.3 shows that many torsion abelian groups (e.g., finite direct sums of copies of $\mathbb{Z}(p^\infty)$ regardless whether they are distinct or not) do not admit precompact locally q-minimal topologies (since they do not admit totally minimal ones).

(a) More precisely, the torsion abelian groups admitting totally minimal group topologies are precisely the groups described in (1), where each B_p has the form $B_p = \bigoplus_{i=1}^{m_p} \mathbb{Z}(p^i)^{\alpha_i}$, for arbitrary cardinals α_i [17]. Therefore, the only countable torsion abelian groups admitting totally minimal group topologies are these of the form $(\mathbb{Q}/\mathbb{Z})^n \oplus \bigoplus_p F_p$, where $n \in \mathbb{N}$ and each F_p is a finite p-group.
(b) Using Theorem 4.3 one can see that if G is a dense subgroup of $\mathbb{T}^n$ with dense $t(G)$, then G is locally q-minimal if and only if G is totally minimal as well (equivalently, $(\mathbb{Q}/\mathbb{Z})^n \subset G$). In particular, $(\mathbb{Q}/\mathbb{Z})^n$ is the only dense torsion subgroup of $\mathbb{T}^n$ that is locally q-minimal. All other dense torsion subgroups are locally t-minimal and locally q^*-minimal, but not locally q-minimal.

Remark 4.6. For a topological group G denote by $o(G)$ the intersection of all open subgroups of G. This is a closed normal subgroup of G containing the connected component $c(G)$ of G.

(a) The subgroup $o(G)$ coincides with G precisely when G has no proper open subgroup (in particular, when G is connected).
(b) If G is a dense subgroup of a topological group H, then $o(G) = G \cap o(H)$. Indeed, this follows from the fact that if U is an open subgroup of G, then $\bar{U}$ is on open subgroup of H with $U = \bar{U} \cap G$; while $G \cap O$ is an open subgroup of G for every open subgroup O of H. In particular, $o(G) = G$ if G is dense subgroup of a connected group.
(c) Suppose that G is locally precompact, then $o(G) = G$ precisely when the locally compact completion K of G is connected. More generally, $o(G)$ is an open subgroup of G precisely when $G \cap c(K)$ is dense in $c(K)$ and the latter is an open subgroup of K. This follows from item (b), as $o(G) = G$ for a locally compact group precisely when G is connected.

The following lemma will be needed in the sequel:

Lemma 4.7. *If a topological abelian group G with $o(G) = G$ is not divisible, then for every $U \in \mathcal{V}(0)$, there exists a prime p with*

$$U \not\subseteq pG. \tag{2}$$

Proof. Since G is not divisible, there exists a prime p such that $pG \neq G$. Then the proper subgroup pG cannot be open, so it has empty interior. This yields (2) for every $U \in \mathcal{V}(0)$. □

4.2. Local q-minimality in $\mathbb{R}^n$ and $\mathbb{T}^n$

We start with a characterisation of the dense locally q-minimal subgroups of $\mathbb{R}$. Even if this can be obtained from the more general Theorem 4.14, we prefer to give separately this much more transparent case with a short direct proof and use it to produce the first relevant examples.

Proposition 4.8. *A dense subgroup G of $\mathbb{R}$ endowed with the usual topology is locally q-minimal iff it is divisible.*

Proof. First, assume that G is locally q-minimal with respect to $U = W \cap G$, where W is a neighbourhood of 0 in $\mathbb{R}$. We can assume without loss of generality that $W = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Assume that G is not divisible. As $o(G) = G$ by Remark 4.6(b), we can apply Lemma 4.7 to find a prime p such that (2) holds. Hence, we can pick a $g \in U$ such that

$$g \notin pG \tag{3}$$

Let $N = \langle g \rangle$. Then $N + W = \mathbb{R}$ and this easily implies $N + U = G$. Since N is a closed subgroup of the divisible group $\mathbb{R}$, we can apply Corollary 4.2 to deduce that $N \leq \pi(G)$ and in particular, $g \in pG$. This contradicts (3).

Conversely, suppose that $G \leq \mathbb{R}$ is divisible. We claim that G is locally q-minimal with respect to $U := (-1, 1) \cap G$. Let N be a closed subgroup of G, we may assume that $\{0\} \neq N \neq G$. Since N is not dense in $\mathbb{R}$, N is an infinite cyclic subgroup of $\mathbb{R}$. We identify G/N with a subgroup of $\mathbb{R}/N \cong \mathbb{T}$, then the divisibility of G and $N \cong \mathbb{Z}$ imply that G/N contains the torsion part of $\mathbb{R}/N$. Hence G/N is minimal, by Fact 2.3(a) (as $t(\mathbb{T})$ is essential in $\mathbb{T}$). This implies that G/N is locally minimal with respect to $(U + N)/N$. □

In the next theorem we apply Corollary 4.4 to provide a necessary condition for local q-minimality of subgroups of $\mathbb{T}^n$. In case $n = 1$ it becomes a characterisation of the locally q-minimal subgroups of $\mathbb{T}$ (as the subgroups having totally minimal torsion part), but for $n > 1$ it need not be sufficient (see Example 4.11). Recall that an infinite subgroup of $\mathbb{T}$ is totally minimal precisely when it contains $t(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$.

Theorem 4.9. *Let $n \geq 1$. For a subgroup G of $\mathbb{T}^n$ consider the following conditions:*

(1) *G is locally q-minimal;*
(2) *$t(G)$ is totally minimal;*
(3) *either G is totally minimal or $t(G)$ is finite.*

Then (1) $\Rightarrow$ (2) $\Leftarrow$ (3). If either $n = 1$ or $\overline{t(G)} = G$, then all three conditions are equivalent.

Proof. The implication (1) $\Rightarrow$ (2) follows from Corollary 4.4, as $\mathbb{T}^n$ is an exotic torus. To prove the implication (3) $\Rightarrow$ (2) assume that G is totally minimal (otherwise there is nothing to prove). Since the closure K of G in $\mathbb{T}^n$ is a Lie group (so, an exotic torus), $t(K)$ is totally minimal. By Theorem 1.3, G is totally dense in K, so G contains $t(K)$, i.e., $t(G) = t(K)$. This proves that $t(G)$ is totally minimal.

Now assume that $n = 1$. First we prove that (1) $\Leftrightarrow$ (3). Theorem 4.3 yields that if $t(G)$ is dense in G, then G is locally q-minimal precisely when it is totally minimal. It remains the case when $t(G)$ is not dense in G. Then it is not dense in $\mathbb{T}$ as well, so $t(G)$ is finite. Consider first the case when $t(G) = \{0\}$, i.e., G is torsion-free. Then G has no proper closed subgroups. Since all subgroups of $\mathbb{T}$ are locally minimal, we conclude that G is locally q-minimal. In the general case when $t(G)$ is finite, the quotient $G/t(G)$ is isomorphic to a torsion-free subgroup of $\mathbb{T}/t(G) \cong \mathbb{T}$. By the above argument, $G/t(G)$ is locally q-minimal. Now Corollary 3.16 applies to ensure local q-minimality of G.

In view of the above equivalence established for $n = 1$, we are left with the proof of the implication (2) $\Rightarrow$ (3). Assume that $t(G)$ is totally minimal. If $t(G)$ is finite there is nothing left to prove. If $t(G)$ is infinite, then it is dense, in $\mathbb{T}$, hence in G as well. Now total minimality of G follows from Theorem 1.3.

Finally, assume that $\overline{t(G)} = G$. Then the second part of (3) becomes obsolete, unless G is finite – in this case all three conditions are equivalent. The implication (2) $\Rightarrow$ (3) becomes obvious in view of our assumption $\overline{t(G)} = G$ and Theorem 1.3. Since total minimality implies local q-minimality, this proves also the implication (2) $\Rightarrow$ (1). □

The following two examples show that the implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) both fail in the case $n \geq 2$ and $t(G)$ is not dense in G. Hence, condition (3) is neither necessary nor sufficient for local q-minimality of G in the case $n \geq 2$.

Example 4.10. Let $\alpha \in \mathbb{T}$ be a non-torsion element and $C = \langle \alpha \rangle$. Consider the subgroup $G = \mathbb{Q}/\mathbb{Z} \times C$ of $\mathbb{T}^2$. We see below that G is locally q-minimal. As $t(G) = \mathbb{Q}/\mathbb{Z}$ is infinite (even if, totally minimal), and

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G is not totally minimal (as $G/t(G) \cong C$ is not minimal), the group G fails to satisfy item (3) of the above theorem.

First we see that G is locally minimal. Indeed, let U be a neighbourhood of 0 in $\mathbb{T}$ witnessing the fact that $\mathbb{T}$ is NSS and pick a neighbourhood U_1 of 0 in $\mathbb{T}$ with $U_1 + U_1 \subseteq U$. As $\mathbb{Q}/\mathbb{Z} \times \{0\}$ is totally minimal, the subgroup $\mathbb{Q}/\mathbb{Z} \times \{0\}$ is essential (actually, totally dense), in $\mathbb{T} \times \{0\}$. On the other hand, every closed subgroup of $\mathbb{T}^2$ contained in $\mathbb{T} \times U$ must be contained actually in $\mathbb{T} \times \{0\}$. Hence, we deduce that G is locally essential in $\mathbb{T}^2$ with respect to $\mathbb{T} \times U$. As $\mathbb{T}^2$ is compact (so also locally minimal with respect to $\mathbb{T} \times U$) we deduce from Remark 2.4(2) that G is locally minimal with respect to $W := \mathbb{Q}/\mathbb{Z} \times V$, where $V = C \cap U_1$. Our next aim is to see that W also witnesses local q-minimality of G.

First we show that all proper closed subgroups of G have the form $\mathbb{Q}/\mathbb{Z} \times \{0\}$, $F \times \{0\}$, or $F \times C$, where F is a finite subgroup of $\mathbb{Q}/\mathbb{Z}$. To this end denote by $p : G \to C$ the projection, pick a proper closed subgroup H of G and consider the following two cases.

(i) $p(H) = 0$. Then $H = F \times \{0\}$, where F is a closed subgroup of $\mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ has no proper infinite subgroups, we conclude that either $F = \mathbb{Q}/\mathbb{Z}$ or F is a finite subgroup of $\mathbb{Q}/\mathbb{Z}$.

(ii) $p(H) \neq 0$. Then there exists an element $a = (x, n\alpha) \in H$ such that $n\alpha \neq 0$. Since $x \in \mathbb{Q}/\mathbb{Z}$, it is torsion. So there exists a non-zero integer m such that $mx = 0$. Hence $ma = (0, mn\alpha) \in H$. So $\overline{\langle ma \rangle} = \{0\} \times \overline{\langle mn\alpha \rangle} = \{0\} \times C$, is also contained in H. Consider the restriction map $p \restriction_H : H \to C$, by 6.22(a) in [28], we conclude that $H = F \times C$, where F is a closed subgroup of $\mathbb{Q}/\mathbb{Z}$. Since H is proper in G, F must be a proper closed subgroup of $\mathbb{Q}/\mathbb{Z}$, so finite (as above).

It remains to see that G is local q-minimal with respect to W. Take a proper closed subgroup H of G, consider the following three cases:

Case 1, H has the form $\mathbb{Q}/\mathbb{Z} \times \{0\}$. Then $G/H \cong C$ and this isomorphism takes the neighbourhood $(H + W)/H$ to V. This proves that $G/H \cong C$ is locally minimal with respect to $(H + W)/H$;

Case 2, H has the form $F \times \{0\}$. Then there exists an obvious isomorphism $f : G/H \to G$ with $f((H + W)/H) = W$. Hence, G/H is locally minimal with respect to $(H + W)/H$.

Case 3, H has the form $F \times C$. Then $(G/H) \cong \mathbb{Q}/\mathbb{Z}$ is totally minimal, so it is also locally minimal with respect to $(H + W)/H$.

Example 4.11. Let $\{x_n : n \in \mathbb{N}\}$ be an independent subset of $\mathbb{T}$ consisting of non-torsion elements. Take a prime p. Set $g_0 = (x_0, 0) \in \mathbb{T}^2$ and $g_n = (x_n, y_n) \in \mathbb{T}^2$ for all $n \geq 1$, where $y_n \in \mathbb{T}$ with order p^n. Let G be the subgroup of $\mathbb{T}^2$ generated by all g_n's. We claim that G is dense in $\mathbb{T}^2$ with trivial torsion part and fails to be locally q-minimal.

We denote $\mathbb{T}^2$ by $\mathbb{T}_1 \times \mathbb{T}_2$ to distinguish these two coordinates. Since $\langle x_0 \rangle$ is dense in $\mathbb{T}_1$, $\overline{G}$ contains $\mathbb{T}_1 \times \{0\}$, so $\overline{G} = \mathbb{T}_1 \times K$, where K is the projection of $\overline{G}$ to $\mathbb{T}_2$. Then K certainly contains the projection of G to $\mathbb{T}_2$, which is the dense subgroup $\mathbb{Z}(p^\infty)$ of $\mathbb{T}_2$. Since K is compact, we obtain that $K = \mathbb{T}_2$. This yields that G is dense in $\mathbb{T}_1 \times \mathbb{T}_2$. To see that G is torsion-free, we note that each g_n is non-torsion and $\{g_n : n \in N\}$ is independent as $\{x_n : n \in \mathbb{N}\}$ is independent. Let $\pi : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{T}_2$ be the second projection. Since $N := (\mathbb{T}_1 \times \{0\}) \cap G$ is dense in $\mathbb{T}_1 \times \{0\}$, the restriction $\pi \restriction_G : G \to \pi(G)$ remains open. Hence $G/N \cong \pi(G)$. By the construction of G, the quotient G/N is p-torsion. Hence, $\pi(G)$ is a p-torsion subgroup of $\mathbb{T}$. Since it is not of bounded exponent, $\pi(G)$ is exactly the quasicyclic subgroup $\mathbb{Z}(p^\infty)$ of $\mathbb{T}_2$. According to Theorem 4.3, it is not locally q-minimal, so neither G is locally q-minimal.

Now we are in position to produce new examples of groups that are both locally t-minimal and locally q^*-minimal, but not locally q-minimal. Similar examples were already produced in Remark 4.5, but those examples were *precompact*.

Example 4.12. The examples we provide now are not only non-precompact, their only precompact subgroup is $\{0\}$. On the other hand, they have an additional property (see item (b)), that cannot be obtained using Remark 4.5.

(a) If G is a dense non-divisible subgroup of $\mathbb{R}$, G is not locally q-minimal, by Proposition 4.8. On the other hand, G is both locally t-minimal and locally q^*-minimal, by Example 2.6 (a).
(b) Proposition 4.8 also produces an example which shows that a topological group with a dense locally q-minimal subgroup need not be locally q-minimal. In fact, let H be the discrete subgroup $\langle\sqrt{2}\rangle$ of $\mathbb{R}$ and $G = \mathbb{Q} \oplus H < \mathbb{R}$. Then H is a totally complete and locally q-minimal subgroup of G, while G is not locally q-minimal since it is not divisible. On the other hand, $\mathbb{Q}$ is dense in G and locally q-minimal. Note that the group G is both locally t-minimal and locally q^*-minimal.
(c) The group G from item (b) shows that local q-minimality has not the 3-space property in the format of Theorem 3.14. Indeed, H is a totally complete and locally q-minimal subgroup of G, while G/H is topologically isomorphic to a torsion-free dense subgroup of $\mathbb{T}$, so G/H is also locally q-minimal.

This example shows that a subgroup of a Banach space need not be locally q-minimal.

In the next example we recall a group analog of a normed space, introduced by Enflo [25], providing an ample source of locally q^*-minimal groups.

Example 4.13. For a symmetric subset U of a group $(G, +)$ with $0 \in U$, and $n \in \mathbb{N}$ let

$$(1/n)U := \{x \in G : kx \in U \text{ for all } k \in \{1, 2, \dots, n\}\}.$$

A Hausdorff topological group (G, τ) is said to be *uniformly free from small subgroups* (UFSS for short) if for some neighbourhood U of 0, the sets $(1/n)U$ form a neighbourhood basis at 0 for τ. The class of UFSS groups is stable under taking subgroups, completions, local isomorphisms and has the three space property (and so stability under finite direct product). Finally, UFSS groups are both NSS and locally minimal [1, Proposition 3.12]. Therefore, they are locally q^*-minimal, according to Remark 2.2.

The property UFSS is not stable under taking quotients, nevertheless this cannot exclude a priori that (subgroups of) UFSS groups are locally q-minimal. In (b) of Example 4.12 we saw that $\mathbb{R}$ has subgroups that are not locally q-minimal, so UFSS groups need not be locally q-minimal.

Proposition 4.8 gives a necessary and sufficient condition for a dense subgroup of $\mathbb{R}$ to be locally q-minimal. A natural question is to consider the high-dimensional situation, i.e. $\mathbb{R}^n$, $n > 1$. The following theorem gives a complete description of all (not necessarily dense) locally q-minimal subgroup of $\mathbb{R}^n$.

Theorem 4.14. *Let n be a positive integer. A subgroup G of $\mathbb{R}^n$ is locally q-minimal iff $o(G)$ is divisible. In particular, a dense subgroup of $\mathbb{R}^n$ is locally q-minimal iff it is divisible.*

Proof. We start with the case when G is dense in $\mathbb{R}^n$. It follows from the general case, as $o(G) = G$ when G is dense in $\mathbb{R}^n$, by Remark 4.6.

Necessity. Suppose that G is a dense subgroup of $\mathbb{R}^n$ that is not divisible. According to Remark 4.6, G has no proper open subgroups.

Assume that G is locally q-minimal with respect to $V = W \cap G$, where W is an open neighbourhood of 0 in $\mathbb{R}^n$. Take a convex open neighbourhood U of $\mathbb{R}^n$ such that

$$\underbrace{U + U + \ldots + U}_{n \text{ times}} \subset W.$$

By Lemma 4.7, there exists a prime p with $U \cap G \nsubseteq pG$. Pick an element $g_1 \in (U \cap G) \setminus pG$. Since $\overline{G \cap U} = \overline{U}$ has a non-empty interior in $\mathbb{R}^n$, we can choose $g_2, g_3, \dots, g_n \in G \cap U$ such that $\{g_1, g_2, \dots, g_n\}$ forms a basis of the vector space $\mathbb{R}^n$. Let $N_i = \langle g_i \rangle$ and denote by L_i the linear hull of g_i for $i = 1, 2, \dots, n$. Then $N_i \subset L_i$ and $\mathbb{R}^n$ can be identified with $\prod_{i=1}^n L_i$ and each point in $\mathbb{R}^n$ can be uniquely represented as $a_1 g_1 + a_2 g_2 + ... + a_n g_n$, where $a_i \in \mathbb{R}$ for each $i = 1, 2, \dots, n$. The group $N := \prod_{i=1}^n N_i$ is a closed subgroup of both $\mathbb{R}^n$ and G. Moreover,

$$\mathbb{R}^n / N = \prod_{i=1}^n T_i, \quad \text{where} \quad T_i = L_i / N_i \cong \mathbb{T} \quad \text{for each } i.$$

Claim 1. $(W \cap G) + N = G$.

First we prove that $W + N = \mathbb{R}^n$. It suffices to show that if $x = \sum_{i=1}^n a_i g_i$ with $a_i \in [0, 1)$ for each i, then $x \in W$. Since U is convex and $g_i \in U$, we have that $a_i g_i \in U$, for $i = 1, \dots, n$. Hence

$$x = a_1 g_1 + a_2 g_2 + \dots + a_n g_n \in \underbrace{U + U + \dots + U}_{n \text{ times}} \subset W.$$

We now prove that $(W \cap G) + N = G$. To show the equation we only need to prove that $G \subset (W \cap G) + N$. Fix $g \in G$, then there exist $x \in W$ and $y \in N$ such that $g = x + y$ since $W + N = \mathbb{R}^n$. So, $x \in G + N = G$, i.e. $x \in W \cap G$. Hence, $g = x + y \in (W \cap G) + N$. This proves the claim.

By Corollary 4.2, $N \leq \pi(G) \leq pG$, this contradicts our choice $g_1 \in N \setminus pG$.

Sufficiency. Let G be a divisible dense subgroup of $\mathbb{R}^n$ and U a bounded neighbourhood of 0 in $\mathbb{R}^n$. Take a neighbourhood U_1 of 0 in $\mathbb{R}^n$ such that $U_1 + U_1 \subset U$. Let $V = U_1 \cap G$. We will see that G is locally q-minimal with respect to V. Let N be a closed subgroup of G. Consider the linear hull X of N, clearly it is topologically linearly isomorphic to $\mathbb{R}^m$ for some positive integer $m \leq n$.

If $m = n$, then N contains a subset $P = \{x_1, x_2, \dots, x_n\}$ that is a basis of $X = \mathbb{R}^n$, i.e. each element in $\mathbb{R}^n$ can be represented as the form $a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ uniquely, where $a_1, a_2, \dots, a_n \in \mathbb{R}$. This implies that $\mathbb{R}^n$ can be identified with $\prod_{i=1}^n L_i$, where $L_i = \mathbb{R} x_i$. Put $N' = \langle P \rangle$, then N' is a discrete (hence closed) subgroup of both G and $\mathbb{R}^n$. Note that $\mathbb{R}^n / N'$ is naturally topologically isomorphic to $\prod_{i=1}^n T_i$, where $T_i = L_i / \langle x_i \rangle \cong \mathbb{T}$ for each i. Since G is divisible, G/N' contains the torsion part of $\mathbb{R}^n / N'$. So G/N' is totally minimal. Then $G/N \cong (G/N')/(N/N')$ is minimal, so locally minimal with respect to $(V + N)/N$.

Now we consider the case $m < n$. Similarly, we can choose a subset $P = \{x_1, x_2, \dots, x_m\}$ of N such that P is a basis of X. Since G is dense in $\mathbb{R}^n$, the linear hull of G is exactly $\mathbb{R}^n$, so we can choose $Q = \{y_1, y_2, \dots, y_{n-m}\} \subset G$ such that $P \cup Q$ is a basis of $\mathbb{R}^n$. Let Y be the linear hull of Q, then $\mathbb{R}^n$ can be identified with $X \times Y$. Since G is divisible, the subgroup $P' = \{q_1 x_1 + q_2 x_2 + \dots + q_m x_m : q_i \in \mathbb{Q}\}$ of X is contained in G. Clearly P' is dense in X, so $G_1 := G \cap X$ is dense in X. Similarly, $G_2 := G \cap Y$ is dense in Y. Further, both G_1 and G_2 are divisible since G, X and Y are divisible and $\mathbb{R}^n$ is torsion-free. Let $N' = \langle P \rangle$, then $N' \subset N \subset G \cap X = G_1$. A similar argument with the case $m = n$ shows that G_1/N' is totally minimal, so G_1/N is minimal. Hence, G_1/N is essential in X/N. Let π be the natural projection of $X \times Y$ onto Y. Then $W = \pi(U)$ is bounded since U is bounded. Clearly, $U \subset X \times W$.

We claim that $G_1/N \times G_2$ is locally essential with respect to $X/N \times W$ in $X/N \times Y$. Indeed, Y is NSS with respect to W, so any closed subgroup K of $X/N \times Y$ contained in $X/N \times Y$ is also contained in $X/N \times \{0\}$. Then the essentiality of G_1/N in X/N implies that K intersects $G_1/N \times \{0\}$ non-trivially. Since $G_1 \times G_2 \subset G$, G/N is also locally essential with respect to $X/N \times W$. Note that $X/N \times W$ also witnesses local minimality of $X/N \times Y$ (since X/N is compact as a quotient group of X/N' and $Y \cong \mathbb{R}^{n-m}$). Further, by the choice of V, we obtain that

$$(V + N)/N + (V + N)/N = (V + V + N)/N \subset (U + N)/N \subset (X \times W + N)/N = X/N \times W.$$

According to (1) of Remark 2.4, G/N is locally minimal with respect to $(V+N)/N$.

Let us consider now the case when the subgroup G of $\mathbb{R}^n$ is not necessarily dense. Then its closure K in $\mathbb{R}^n$ is locally compact, so isomorphic to $\mathbb{R}^m \times \mathbb{Z}^k$ by the structure theory of the closed subgroups of $\mathbb{R}^n$. In particular, $c(K) \cong \mathbb{R}^m$ is open. Hence, $o(G) = G \cap K$ and $o(G)$ is dense in $c(K)$ (see Remark 4.6). As $o(G)$ is dense in the group $K \cong \mathbb{R}^m$, so $o(G)$ is locally q-minimal if and only if $o(G)$ is divisible, by the first part of the proof. On the other hand, $o(G)$ is locally q-minimal if and only if G is locally q-minimal, in view of Corollary 3.12. Therefore, G is locally q-minimal if and only if $o(G)$ is divisible. □

According to [10, Proposition 2.1], minimal abelian groups that are also divisible, are totally minimal. A similar phenomenon can be observed in Proposition 4.8 and Theorem 4.14, where we prove the counterpart of this property for local q-minimality for the dense subgroups of $\mathbb{R}$ and $\mathbb{R}^n$. Moreover, we see that the implication can be inverted, namely local q-minimality for these subgroups implies divisibility. (Such a phenomenon is not present for arbitrary minimal abelian groups, More precisely, the compact Pontryagin dual $K = \mathbb{Q}^\wedge$ of the discrete group $\mathbb{Q}$ is divisible and has dense totally minimal subgroups that are not divisible. Same applies to $\mathbb{T}$, it has dense totally minimal subgroups that are not divisible.)

Example 4.15. Let G be a quasi-cyclic subgroup of $\mathbb{T}$, the latter group is endowed with the usual compact topology. Proposition 4.9 and Remark 4.5 show that G is not locally q-minimal. According to Example 2.6(a), the group G is also locally t-minimal and locally q^*-minimal. This example shows that a precompact divisible group that is both locally t-minimal and locally q^*-minimal, need not be locally q-minimal.

In Example 6.4(b) we build a non-precompact locally minimal divisible abelian group which is neither locally q^*-minimal nor locally t-minimal.

5. Local q^*-minimality criterion

The criterion of total minimality 1.3 implies that a topological group containing a dense totally minimal subgroup must be totally minimal on its own account. Example 4.12 (b) implies that a similar criterion for local q-minimality cannot be available.

We will give a criterion for local q^*-minimality.

It is easy to check that the Hausdorff group G is locally q^*-minimal iff there exists a neighbourhood V of the identity such that G/N is $\pi(V)$-locally minimal for each closed normal subgroup N of H contained in V, where π is the natural quotient mapping of G onto G/N.

Lemma 5.1. *If a topological group G is locally q-minimal (resp. locally q^*-minimal) with respect to U^2, then G/N is locally minimal with respect to $\overline{UN/N}$ for any closed normal subgroup N (resp. for any closed normal subgroup $N \subset U$) of G.*

Proof. We prove the case of local q-minimality, the other is similar.

Since G is locally q-minimal with respect to U^2, G/N is locally minimal with respect to $U^2N/N = (UN/N)^2 \supset \overline{UN/N}$. The last conclusion is from the openness of the quotient mapping of G onto G/N. □

Definition 5.2. A dense subgroup H of a topological group G is called *locally t-dense* if there exists a neighbourhood $V \in \mathcal{V}_G(1)$ such that $H \cap N$ is dense in N for every closed normal subgroup N of G contained in V.

In an NSS group every dense subgroup is obviously locally t-dense. In general, total density implies local t-dense. The next proposition shows that every locally t-dense subgroup of a compact torsion abelian group K is actually totally dense (consequently coincides with K).

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Proposition 5.3. *Let K be a compact torsion abelian group. Then every locally t-dense subgroup H of K coincides with K.*

Proof. Indeed, let U be the neighbourhood of 0 witnessing the local t-density of H. Since K has a local base of open subgroups, we can assume without loss of generality that U is an open subgroup of K. Moreover, as $K = \prod_{i\in I} C_i$ is a topological product of finite cyclic groups C_i, we can assume (by further shrinking U), that U is a direct summand of K, i.e., $K = F \times U$, where F is a finite group. Now $H_1 = H \cap U$ is dense in U and the local t-density of H with respect to U means that H_1 is totally dense in U. Since U_1 is torsion, total density of H_1 implies $H_1 = U$. This proves that H contains U. Consequently, H itself is open and consequently also closed. Therefore, $H = K$. $\square$

Compactness plays a relevant role in this proposition. Indeed, the torsion group $\mathbb{Q}/\mathbb{Z}$ has plenty of proper dense subgroups and they are all locally t-dense as $\mathbb{Q}/\mathbb{Z}$ is NSS.

Theorem 5.4. *A dense subgroup H of a Hausdorff group G is locally q^*-minimal iff G is locally q^*-minimal and H is locally t-dense in G.*

Proof. First we assume that H is locally q^*-minimal with respect to a neighbourhood V^2 of the identity e in H, where $V = W \cap H$ and W is a closed neighbourhood of the identity in G. Take a neighbourhood U of e in G such that $U^2 \subset W$. We prove that U witnesses both local q^*-minimality of G and local t-density of H in G.

Let N be a closed normal subgroup of G contained in U and N' the closure of $N \cap H$. Then N' is normal in G. Denote by ψ the quotient mapping of G onto G/N and by π the quotient mapping of G onto G/N'. Then we can identify G/N with the quotient group of G/N' with respect to the closed normal subgroup N/N' of G/N'. Let p be the above quotient mapping of G/N' onto G/N. Clearly, $\psi = p \circ \pi$. Since $H \cap N' = H \cap N$ is dense in N', the quotient mapping π remains open when restricted to H, hence we identify $H/(H \cap N)$ with the dense subgroup $\pi(H)$ of G/N'. By the local q^*-minimality assumption of H and Lemma 5.1, $\pi(H)$ is $\overline{\pi(V)} \cap \pi(H)$-locally minimal, where $\overline{\pi(V)}$ is the closure of $\pi(V)$ in G/N'. Clearly, $W \subset \overline{V}$ yields $\pi(W) \subset \pi(\overline{V}) \subset \overline{\pi(V)}$. According to Remark 2.4 (1), G/N' is locally minimal with respect to $\overline{\pi(V)}$, so locally minimal with respect to $\pi(W)$ and $\pi(U)$. Moreover, the inclusion $\pi(U)^2 = \pi(U^2) \subset \pi(W)$ implies that $\pi(H)$ is locally essential in G/N' with respect to $\pi(U)$. Therefore, the proofs of both the local q^*-minimality and local t-density will be complete if we show that $N = N'$ (i.e., p is a topological isomorphism). Denote by K the kernel N/N' of p. We aim to show that

$$K \cap \pi(H) = \{e_q\}. \tag{4}$$

Take $h \in H$ such that $\pi(h) \in K \cap \pi(H) = \pi(N) \cap \pi(H)$, then

$$h \in NN' \cap H = N \cap H \subset N' = \ker \pi,$$

hence $\pi(h) = \{e_q\}$, where $\{e_q\}$ is the identity of G/N'. Since $K = \pi(N) \subset \pi(U)$, this proves (4). Hence, by the $\pi(U)$-local essentiality of $\pi(H)$ in G/N', K is trivial, which implies that $N = N'$.

Now we assume that G is locally q^*-minimal with respect to a neighbourhood U of the identity and H is locally t-dense in G with respect to U. Take neighbourhoods W, W' of the identity in G such that $W^2 \subset U$, ${W'}^2 \subset W$, and let $V = W' \cap H$. We are going to prove that H is locally q^*-minimal with respect to V.

Let N be a closed normal subgroup of H such that $N \subset V$. Denote by $\overline{N}$ the closure of N in G. Then the natural quotient mapping of H onto H/N can be extended to the quotient mapping $\pi : G \to G/\overline{N}$ when we identify H/N with the dense subgroup $\pi(H)$ of $G/\overline{N}$. The assumption that $N \subset V$ implies that $\overline{N} \subset \overline{V} = \overline{W'} \subset W \subset U$. Then $G/\overline{N}$ is locally minimal with respect to $\pi(U)$, hence, with respect to $\pi(W)$, by our assumption. We claim that $\pi(H)$ is locally essential in $G/\overline{N}$ with respect to $\pi(W)$. Take a closed normal subgroup $K \subset \pi(W)$ of $\pi(H)$ such that $K \cap \pi(H) = \{e_q\}$, where e_q is the identity of $G/\overline{N}$. Then $\pi^{-1}(K)$ is a closed normal subgroup of G and

$$\pi^{-1}(K) \subset W\overline{N} \subset WW \subset U.$$

So, $\pi^{-1}(K) \cap H$ is dense in $\pi^{-1}(K)$. Therefore $\{e_q\} = K \cap \pi(H)$ is dense in K, which implies that $K = \{e_q\}$. Hence we finish the proof of the local essentiality. Moreover, it is clear that $\pi(W')^2 = \pi(W'^2) \subset \pi(W)$. Again, Remark 2.4 (2) shows that $\pi(H)$ is locally minimal with respect to $\pi(W') \cap \pi(H)$, so to $\pi(W' \cap H) = \pi(V)$. □

Item (b) of Example 4.12 and Example 4.15 together show that we can not obtain a criterion of local q-minimality for dense subgroups (at least in the same format as the criterions for (total) minimality). Anyway, one can prove the following about totally dense subgroups.

Proposition 5.5. *A totally dense subgroup H of G is locally q-minimal iff G is locally q-minimal.*

Proof. We first assume that U is a neighbourhood of the identity of G such that H is locally q-minimal with respect to V^2, where V denotes $U \cap H$. Let W be a neighbourhood of the identity in G such that $W^2 \subset U$, we are going to prove that G is locally q-minimal with respect to W. Let N be a closed normal subgroup of G and $N' = H \cap N$. By the total density of H in G, we can identity H/N' with the dense subgroup HN/N of G/N. Denote by π the natural quotient mapping of G on to G/N.

Since H is locally q-minimal with respect to V^2, Lemma 5.1 implies that $\pi(H)$ is locally minimal with respect to $\overline{\pi(V)} \cap \pi(H)$, where $\overline{\pi(V)}$ is the closure of $\pi(V)$ in G/N. According to Remark 2.4 (1), G/N is locally minimal with respect to $\overline{\pi(V)}$, so it suffices to prove that $\pi(W) \subset \overline{\pi(V)}$. Since $\pi(H)$ is dense in G/N, $\pi(W) \subset \overline{\pi(W) \cap \pi(H)}$. Therefore, it is enough to check that $\pi(W) \cap \pi(H) \subset \pi(V) \subset \overline{\pi(V)}$. Since, $N = \overline{N'}$ and $N' \subset H$, we have the following chain of inclusions:

$$\pi(W) \cap \pi(H) = \pi(WN \cap H) \subset \pi(WWN' \cap H) = \pi((W^2 \cap H)N') \subset \pi((W^2 \cap H)N) = \pi(W^2 \cap H) \subset \pi(V).$$

Conversely, assume that G is locally q-minimal with respect to U. Choose a neighbourhood W of the identity in G such that $W^2 \subset U$, let $V = W \cap H$. We claim that V witness local q-minimality of H. Let N' be a closed normal subgroup of H and N the closure of N', then N is a closed normal subgroup of G. Denote by π the natural quotient mapping of G onto G/N, clearly $\pi(H) = H/N'$ is dense in G/N. Moreover, since H is totally dense in G, $\pi(H)$ is also totally dense, hence locally essential with respect to any neighbourhood of the identity, in G/N. By the U-local q-minimality assumption of G we know that G/N is locally minimal with respect to $\pi(U)$. Since $\pi(V) \subset \pi(W) \cap \pi(H)$, according to Remark 2.4 (2), it suffices to prove that $\pi(W)^2 \subset \pi(U)$. This obviously follows from the choice of W. □

Since local minimality and local q^*-minimality coincide on NSS-groups and since every subgroup of a Lie group is locally minimal, we conclude (as in Example 2.6 (a)) that every subgroup of a Lie group is locally q^*-minimal.

6. Applications of the local q^*-minimality criterion

In this subsection we give various consequences of the local q^*-minimality criterion. Since any locally q-minimal group is locally q^*-minimal, we get the following immediate corollary of Theorem 5.4:

Corollary 6.1. *If H is a dense locally q-minimal subgroup of a Hausdorff topological group G, then H is locally t-dense in G.*

Notice that the converse of Corollary 6.1 is not true, Example 4.15 provides a counterexample.

Another corollary is obtained by making use of Proposition 5.3.

Corollary 6.2. *A locally q^*-minimal precompact bounded torsion abelian group is compact.*

According to [7, Lemma 7.4], every pseudocompact torsion abelian group is bounded, we have the following result.

Corollary 6.3. *A locally q^*-minimal pseudocompact torsion abelian group is compact.*

The next example was given in [15, Example 2.10] to show the difference between local minimality and local q-minimality. We will see that the group in this example is not even locally q^*-minimal either. Moreover, it can be used to produce a divisible locally minimal abelian group that is neither locally q^*-minimal nor locally t-minimal.

Example 6.4. Let $c = (a_p)_{p\in\mathbb{P}}$ be a topological generator of the compact monothetic group $K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$.

(a) Consider the subgroups $N = \prod_{p\in\mathbb{P}} p\mathbb{Z}_p$ and $G = \langle c\rangle + N$ of K. Then G is dense in K and minimal, hence, locally minimal ([15, Example 2.10]). Let us see that G is not locally q^*-minimal. Indeed, if G were locally q^*-minimal, then G would be locally t-dense with respect to some neighbourhood U of the identity in K. One can choose $p \in \mathbb{P}$ such that $\mathbb{Z}_p \subset U$. Theorem 5.4 implies that $G \cap \mathbb{Z}_p$ is dense in $\mathbb{Z}_p$. While, $G \cap \mathbb{Z}_p = p\mathbb{Z}_p$, a contradiction. To see that G is not locally t-minimal, it is enough to note that G/N, algebraically isomorphic to $\mathbb{Z}$, is not locally minimal, as G/N is a dense subgroup of $K/N \cong \prod_p \mathbb{Z}(p)$ that is not locally essential.

(b) Consider the topological group (G, τ) introduced in (a). Let H be divisible hull of G and let τ^* be the topology (standard extension of τ) on H defined in Fact 3.10. By Proposition 3.5 (b), local q^*-minimality and locally t-minimal are stable under taking open subgroups in abelian groups. Hence, we deduce that (H, τ^*) is neither locally q^*-minimal nor locally t-minimal, as G is neither locally q^*-minimal nor locally t-minimal, by item (a). However, (H, λ) is locally minimal since it contains the open locally minimal subgroup G (see [1, Proposition 2.4]). The group H is not precompact, as the open subgroup G has infinite index.

Local t-minimality and local q^*-minimality are both strictly weaker than local q-minimality (by Proposition 4.8 and Theorem 4.14, any dense non-divisible subgroup G of $\mathbb{R}$ is not locally q-minimal, while it is both locally t-minimal and locally q^*-minimal). The following example shows that a locally q^*-minimal abelian group needs not to be locally t-minimal, i.e., has a non-locally minimal quotient. This shows that local q^*-minimality, unlike local q-minimality and local t-minimality, is not preserved by taking quotients.

Example 6.5. The Hilbert space ℓ^2 considered as a topological abelian group ℓ^2 is UFSS (see [1, Example 3.14]). Hence, every subgroup of ℓ^2 is also UFSS (see [1, Lemma 3.12(b)]), so locally q^*-minimal. Let $\{e_n : n \in \mathbb{N}\}$ be the canonical basis of ℓ^2. Take a prime p and let P be the dense subgroup of the $\mathbb{R}$ generated by $\{\frac{1}{p^n}, n \in \mathbb{N}\}$. Then the group $G = \{(x_n) \in \ell^2 : x_n \in P\} = P^{\mathbb{N}} \cap \ell^2$ is locally q^*-minimal. Following [1, Example 3.14], let $H = \overline{\langle\{\frac{1}{p^n}e_n : n \in \mathbb{N}\}\rangle}$. We prove that for the closed subgroup $N := H \cap G$ of G the quotient G/N is not locally minimal.

(a) We prove first that G is dense in ℓ^2. Indeed, fix $y = (y_n) \in \ell^2$ and $\varepsilon > 0$, by $\overline{P} = \mathbb{R}$, we can choose $x_n \in P$ such that $|x_n - y_n| < \frac{\varepsilon}{2^n}$ for each $n \in \mathbb{N}$. Since

$$\sqrt{\sum_{n\in\mathbb{N}} (x_n - y_n)^2} < \sqrt{\sum_{n\in N} \left(\frac{\varepsilon}{2^n}\right)^2} \leq \frac{\varepsilon}{\sqrt{3}} < \varepsilon,$$

we deduce that $z := (x_n - y_n) \in \ell^2$, so $x = (x_n) = z + y \in \ell^2$, and hence, $x \in G$. The former inequality also implies that $||x - y|| < \varepsilon$, thus G is dense in ℓ^2.

(b) Denote by π the natural projection of ℓ^2 onto ℓ^2/H. As N is dense in H, the subgroup $\pi(G)$ of ℓ^2/N is naturally topologically isomorphic to G/N, according to [22, Lemma 4.3.2].

(c) We now show that $\pi(G)$ is not locally minimal. Indeed, if $\pi(G)$ were locally minimal, there must exist $\varepsilon > 0$ such that each closed subgroup of $\pi(G)$ contained in $\pi(\varepsilon B)$ is minimal, by Corollary 3.3, where B is the unit ball in ℓ^2.

An argument similar to that in [1, Example 3.14] shows that there exists a positive integer k_0 such that $\pi(S) \subset \pi(\varepsilon B)$, where S is the linear hull of the set $\{e_k : k > k_0\}$. Let $k > k_0$ be an integer. Then $\pi(Pe_k) \subset \pi(\mathbb{R}e_k) \subset \pi(S) \subset \pi(\varepsilon B)$ and $\pi(\mathbb{R}e_k)$ is closed in $\pi(\ell^2)$, as $\pi(\mathbb{R}e_k) \cong \mathbb{T}$. Moreover, $\ker(\pi\restriction_{\mathbb{R}e_k}) = \langle \frac{1}{p^k} e_k \rangle \subset G$. So $\pi(Pe_k) = \pi(G) \cap \pi(\mathbb{R}e_k)$, hence, it is a closed subgroup of $\pi(G)$. This implies that $\pi(Pe_k)$ is minimal, by Corollary 3.3. Hence, $\pi(Pe_k)$ is essential in its completion $\pi(\mathbb{R}e_k) \cong \mathbb{T}$, by Fact 2.3(a). Since $\pi(Pe_k)$ is p-torsion, so $\pi(Pe_k)$ is not essential in $\pi(\mathbb{R}e_k)$, a contradiction.

7. Local q^*-minimality combined with other compactness properties

A Tychonov topological space X is said to be *pseudocompact* when every continuous real valued function on X is bounded. By a celebrated Comfort-Ross' criterion [8] for pseudocompactness, a topological group G is pseudocompact if and only if $K = \widetilde{G}$ is compact and G is G_δ-dense in K.

A topological group G is said to be *sequentially complete* if every Cauchy sequence in G is convergent (equivalently, when G is sequentially closed in its Raĭkov completion); G is said to be *sequentially q-complete*, if every Hausdorff quotient of G is sequentially complete [21]. Clearly, complete groups are sequentially complete, so the latter is a rather weak compactness-like property. On the other hand, countably compact groups are sequentially q-complete.

Now we shall combine sequentially completeness with pseudocompactness and local q^*-minimality.

Theorem 7.1. *For a locally q^*-minimal topological abelian group G the followings are equivalent:*

(a) *G is sequentially complete and pseudocompact;*
(b) *G is sequentially q-complete and precompact;*
(c) *G is compact.*

Proof. (a) → (c) Assume that G is a sequentially complete locally q^*-minimal pseudocompact abelian group. Since pseudocompact groups are precompact, the completion K of G is compact. By Theorem 5.4, G is locally t-dense in K and let U be a neighbourhood of 0 in K witnessing that. By the structure theory of compact groups, one can find a closed subgroup N of K contained in U such that K/N is metrizable (actually, one can have it even a Lie group). Then the subgroup N of K is a G_δ-set. By Comfort-Ross' criterion for pseudocompactness, we deduce that G is G_δ-dense in K. In particular, the subgroup $G_1 = N \cap G$ of G is G_δ-dense in N. On the other hand, G_1 is closed in G, hence G_1 is sequentially complete. Next we note that G_1 is totally dense in its completion N, by the local t-density of G with respect to U. Hence, G_1 is totally minimal and sequentially complete, hence compact, according to [12, Theorem 3.4]. This proves that $G_1 = N$, so $N \leq G$. Since G is G_δ-dense and N is a G_δ-subgroup, we deduce that $K = G + N$, hence $G = K$.

(b) → (c) We assume that G is a sequentially q-complete locally q^*-minimal precompact abelian group. Let K be the completion of G and U be a neighbourhood of 0 in K witnessing local t-denseness in K. A similar argument as in the proof of the implication shows that there exists a closed subgroup N of K contained in U such that K/N is metrizable and $N \leq G$. So G/N is naturally topologically isomorphic to a dense subgroup of K/N. Since G/N is sequentially complete and K/N is metrizable, G/N is closed in K/N, which implies that $G/N = K/N$, hence, compact. Note that compactness is a three space property, so G is compact by compactness of N and G/N.

(c) $\to$ (a) and (c) $\to$ (b) are trivially verified. □

We are not aware whether "pseudocompact" can be replaced by "precompact" in item (a) Theorem 7.1, or equivalently, whether sequential q-completeness can be weakened to sequential completeness in item (b) (see Question 8.11). Clearly, one cannot completely omit both "pseudocompact" or "precompact" (every infinite discrete abelian group is a counterexample, less trivial examples are complete non-compact UFSS groups, e.g., the Hilbert space ℓ^2).

On the other hand, since countably compact groups are both sequentially complete and pseudocompact, we obtain:

Corollary 7.2. *Every countably compact locally q^*-minimal abelian group is compact.*

The compact abelian groups with a proper totally dense pseudocompact subgroup were described in [13] (see also [18] for a consistent result). We show now that these are also the groups with a proper locally t-dense pseudocompact subgroup:

Theorem 7.3. *For a compact abelian group K the followings are equivalent:*

(a) *K has no proper pseudocompact totally dense subgroup;*
(b) *K has no proper pseudocompact locally t-dense subgroup;*
(c) *there exists a torsion closed G_δ-subgroup of K;*
(d) *K has no proper dense pseudocompact and locally q^*-minimal subgroup;*
(e) *mK is a metrizable subgroup of K for some positive $m \in \mathbb{Z}$.*

Proof. The equivalence of (a) and (c) is contained in [13] (see also [18] for a consistent result), the implication (b) $\to$ (a) is trivial. To prove (c) $\to$ (b) assume that N is a torsion closed G_δ-subgroup of K. Assume that H is a pseudocompact locally t-dense subgroup of K. It suffices to show that $H = G$. Since dense pseudocompact subgroups are G_δ-dense by Comfort-Ross' criterion for pseudocompactness of dense subgroups of compact groups, the subgroup $H_1 := H \cap N$ is G_δ-dense in N. Moreover, the closed subgroup H_1 of H is locally q^*-minimal (by Proposition 3.5). Since the group H_1 is pseudocompact and torsion, we obtain that H_1 is compact according to Corollary 6.3, which implies that $H_1 = N$. Hence, $N \leq H$, i.e. $N + H = H$. So, by G_δ-denseness of H in G, $G = H + N = H$.

To prove (c) $\to$ (e) assume that N is a torsion closed G_δ-subgroup of K. Since compact torsion abelian groups are of finite exponent, there exists a positive $m \in \mathbb{Z}$ such that $mN = 0$, so $N \leq K[m]$. Therefore, metrizable the quotient group K/N projects onto the quotient group $K/K[m] \cong mK$. This proves that mK is a metrizable as well. For the implication (c) $\to$ (e) assume that mK is a metrizable subgroup of K for some positive $m \in \mathbb{Z}$. As $mK \cong K/K[m]$, we deduce that $N := K[m]$ is a G_δ-subgroup. Since N is torsion, we are done.

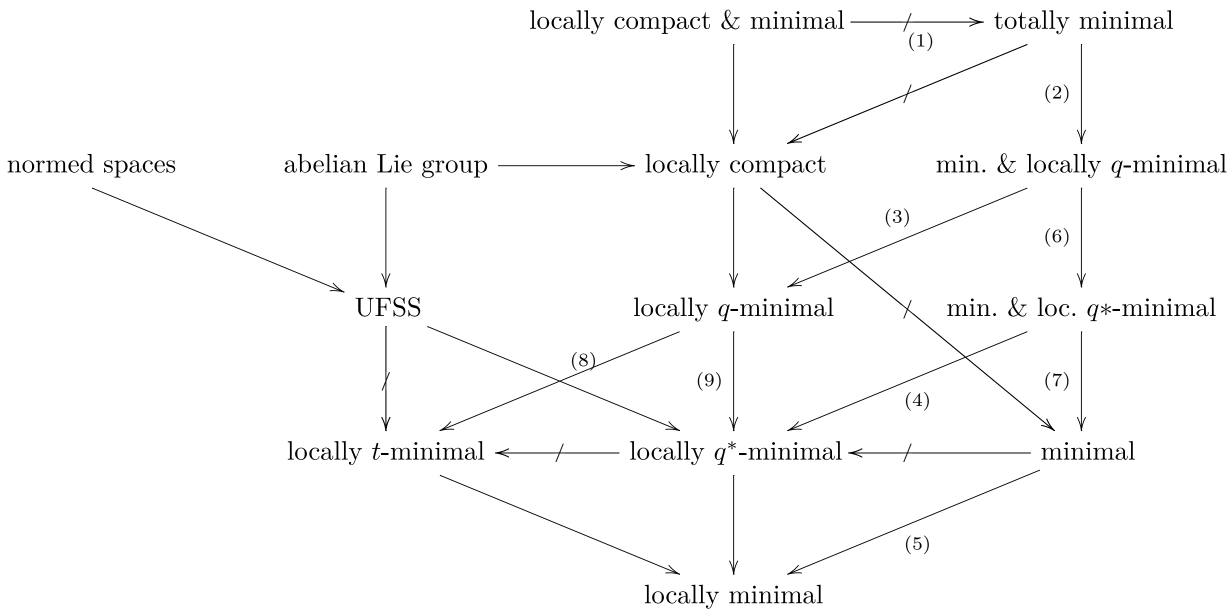
The equivalence of (b) and (d) follows from Theorem 5.4. □

8. Final comments and open problems

We have three types of notions to describe the three non-coinciding levels of local minimality of quotient groups – local t-minimality, local q-minimality and local q^*-minimality. We expect that local t-minimality does not imply local q^*-minimality, but we have no proof at hand:

Question 8.1. *Does local t-minimality imply local q^*-minimality?*

The following diagram shows the implication we have proved or disproved:



The non-implication "totally minimal$\not\to$ locally compact" (witnessed by the group $\mathbb{Q}/\mathbb{Z}$) yields also the non-implication "minimal & locally q-minimal$\not\to$ locally compact".

The non-implication "UFSS$\not\to$ locally t-minimal" follows from Example 6.5.

The non-implication (1) is witnessed by the following subgroup of the linear group $GL_2(\mathbb{R})$

$$\left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) : a, b \in \mathbb{R}, a \neq 0\right\} \cong (\mathbb{R}, +) \rtimes (\mathbb{R} \setminus \{0\}, \cdot),$$

which is minimal and locally compact, hence locally q-minimal, but not totally minimal.

The following example shows that the implication (2) in the above diagram cannot be inverted.

Example 8.2. Let p be a prime and $K = \mathbb{Z}_p \times \mathbb{Z}(p^2)$. Let $\xi \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ be independent with $1 \in \mathbb{Z}_p$ and let c be the generator of the group $\mathbb{Z}(p^2)$. The subgroup $G = \langle(\xi, c)\rangle + \mathbb{Z} \times \langle pc\rangle$ of K is dense, essential and contains an open totally minimal (hence, locally q-minimal) subgroup. Hence, G is minimal and locally q-minimal, but G is not totally minimal as G is not totally dense in K.

The non-reversibility of the "parallel" implications (3), (4) and (5) follows from the fact that the non-compact locally compact abelian groups are not minimal.

The non-reversibility of the implication (6) is witnessed by the socle of the group $\mathbb{T}$. The non-reversibility of the implication (7) is witnessed by the subgroup $G = K[2] + \bigoplus_\omega \mathbb{Z}(4)$ of the group $K = \mathbb{Z}(4)^\omega$.

The non-reversibility of the implications (8) and (9) follows from Example 2.6.

Question 8.3. *Does Proposition 3.4 remain true for local t-minimality, i.e., if G is a locally t-minimal group, does there exist a neighbourhood U of the identity in G such that each closed central subgroup N of G contained in U is totally minimal?*

Question 8.4. *We are not aware if one can add in Corollary 3.7 the following condition:*

(d) *G is locally t-minimal.*

By applying Proposition 3.6, one can see that a positive answer to Question 8.3 is also a positive answer to Question 8.4.

Question 8.5. *If H is an open subgroup of a locally t-minimal group, is then H itself locally t-minimal? What about locally q-minimal and locally q^*-minimal?*

A topological group is called *h-complete*, if all continuous homomorphic images of H are complete. Example 4.12 and Theorem 3.15 leave open the following:

Question 8.6. *Suppose that K is a closed normal subgroup of a topological group G.*

(a) *Is G necessarily locally q-minimal, if K and G/K are locally q-minimal and K is h-complete?*
(b) *Is G necessarily locally t-minimal, if K and G/K are locally t-minimal and K is totally complete?*

Theorem 4.14 gives a necessary and sufficient condition for a dense subgroup of $\mathbb{R}^n$ to be locally q-minimal. Theorem 4.3 gives a necessary and sufficient condition for a dense subgroup G of $\mathbb{T}^n$ with dense $t(G)$ to be locally q-minimal (namely, $t(\mathbb{T}^n) \leq G$, so G, as well as $t(G)$, are totally minimal), yet for $n > 1$ total minimality of $t(G)$ remains only *a necessary condition*. This leaves open the following:

Question 8.7. *Find a sufficient condition for a subgroup G of $\mathbb{T}^n$ to be locally q-minimal.*

According to Theorem 4.3 and Remark 4.5, the group $\mathbb{Z}(p^\infty)$ does not admit a precompact locally q-minimal group topologies. This leaves open the following:

Question 8.8. *Does the group $\mathbb{Z}(p^\infty)$ admit a non-discrete locally q-minimal group topology? Same about the groups $G = \mathbb{Z}(p_1^\infty) \oplus \ldots \oplus \mathbb{Z}(p_n^\infty)$, where $p_1, \ldots, p_n$ are (not necessarily distinct) primes.*

Question 8.9. *Let p be a prime and $S_p = \bigoplus_\omega \mathbb{Z}(p)$.*

(a) *Does the group S_p admit a non-discrete locally q-minimal group topology? Does it admit a non-discrete locally minimal group topology at all?*
(b) *Same about the group S_2.*

Note that S_p does not admit minimal group topologies [22]. Hence, if τ is a non-discrete group topology on S_p and $U \in \mathcal{V}(0)$ witnesses local minimality of τ, then U cannot contain infinite subgroups, by Corollary 3.3.

Call a property $\mathcal{P}$ of topological spaces *contagious*, if whenever X is a dense subspace of a space Y, then $X \in \mathcal{P}$ implies $Y \in \mathcal{P}$. When we speak of topological groups, we consider, of course, dense subgroups. Here is a list of contagious properties of topological groups: connectedness; pseudocompactness; minimality; total minimality; local minimality; local q^*-minimality; commutativity. On the other hand, we showed that local q-minimality is not a contagious property.

Question 8.10. *Is local t-minimality a contagious property?*

Our criterions for (total) minimality, of local (q^*-)minimality are designed for contagious properties, that's why we cannot produce such a criterion for local q-minimality. This circumstance determines our major interest in locally q^*-minimal groups (rather than locally q-minimal ones).

Question 8.11. *Can "pseudocompact" be replaced by "precompact" in (a) of Theorem 7.1?*

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