On hypercyclic fully zero-simple semihypergroups

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Abstract

Let $\mathfrak{I}$ the class of fully zero-simple semihypergroup $(H, \circ)$ generated by a hyperproduct of elements in H. In this paper we study some properties of residual semihypergroup $(H_+, \star)$ of $(H, \circ)$. Moreover, we find sufficint conditions for $(H, \circ)$ and $(H_+, \star)$ to be cyclic.

Keywords: semihypergroups, simple semihypergroups, fully semihypergroups.

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1 Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Many authors have been working on this field and in [5] numerous applications are recalled on algebraic hyperstructures such as: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays some scholars have studied different aspects of semihypergroups [2, 3, 8, 9, 19, 20, 22, 23, 24] and interesting problems arise in the study of their so called fundamental relations [1, 7, 16, 21, 25], which lead to analyze conditions for their transitivity, and minimal cardinality problems. In [16] the authors find all simple and zero-simple semihypergroups of size 3, whose the fundamental relation β is not transitive, apart of isomorphisms. This semihypergroups of size 3 are used in

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[8, 9, 10, 11, 12] to characterize the fully simple semihypergroups and the fully zero-simple semihypergroups having all hyperproducts of size ≤ 2. In particular, in paper [11] the authors have proved that if $(H, \circ)$ is a hypercyclic simple semihypergroup, that is generated by a hyperproduct of elements in H, the relation β is transitive. Consequently we have that in every fully simple semihypergroup the size of every hyperproduct is ≤ 2. This fact is not true for the fully zero-simple semihypergroups, many examples are found in this paper.

The plan of this paper is the following: After introducing some basic definitions and notations to be used throughout the paper, in Section 2, we prove that if $(H, \circ)$ is a hypercyclic fully zero-simple semihypergroups generated by hyperproduct P and $(H_+, \star)$ is the residual semihypergroup of $(H, \circ)$ then the relation β_{H_+} is transitive. Moreover, if $(H, \circ)$ is generated by hyperproduct P then $(P \cap P^2) - \{0\} = \emptyset$. In Section 3, we introduce the definition of *rank* for a hyperproduct P, that is the smallest positive integer k such that $P \cap P^{k+1} - \{0\} \neq \emptyset$. By means of this notion, we characterize the subsemihypergroup $\widehat{P}$ generated by a special hyperproduct P, called strong, and in Section 4 we analize properties of the fully zero-simple semihypergroup generated by a strong hyperproduct. In particular, we prove that if $(H, \circ)$ is a fully zero-simple semihypergroup generated by a strong hyperproduct P of rank a prime number r then $(H, \circ)$ is cyclic. In this case, the rank can be seen as a generalization of the concept of period in group theory. It is known that if G is a cyclic group of size a prime number r then every element different from identity is a generator of G. The same property is true for semihypergroups in Theorem 4.1, but the commutative property of cyclic groups does not generally hold, see example in Remark 4.1.

1.1 Basic definitions and results

Let H be a non-empty set and $P^*(H)$ be the set of all non-empty subsets of H. A hyperoperation $\circ$ on H is a map from $H \times H$ to $P^*(H)$. For all $x, y \in H$, the subset $x \circ y$ is called the hyperproduct of x and y. If A, B are non-empty subsets of H then $A \circ B = \bigcup_{x \in A, y \in B} x \circ y$.

A *semihypergroup* is a non-empty set H endowed with an associative hyperproduct $\circ$, that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

A non-empty subset K of a semihypergroup $(H, \circ)$ is called a *subsemihypergroup* of $(H, \circ)$ if it is closed with respect to multiplication, that is, $x \circ y \subseteq K$ for all $x, y \in K$. If $(H, \circ)$ is a semihypergroup, then the intersection $\bigcap_{i \in I} S_i$ of a family $\{S_i\}_{i \in I}$ of subsemihypergroups of $(H, \circ)$, if it is non-empty, is again a subsemihypergroup of $(H, \circ)$. For every non-empty subset $A \subseteq H$ there is at least one subsemihypergroup of $(H, \circ)$ containing A, e.g., H itself. Hence the intersection of all subsemihypergroups of $(H, \circ)$ containing A is a subsemihypergroup. We denote it by $\widehat{A}$, and note that it is defined by two properties:

1. $A \subseteq \widehat{A}$;
2. if S is a subsemihypergroup of H and $A \subseteq S$, then $\widehat{A} \subseteq S$.

Furthermore, $\widehat{A}$ is characterized as the algebraic closure of A under the hyperproduct in $(H, \circ)$, namely we have $\widehat{A} = \bigcup_{n\geq 1} A^n$. Moreover, if H is finite, the set $\left\{ r \in \mathbb{N} - \{0\} \mid \bigcup_{k=1}^{r} A^k = \bigcup_{k=1}^{r+1} A^k \right\}$ has minimum $m \leq |H|$ and then, it is known that

$$\widehat{A} = \bigcup_{k=1}^{m} A^k = \bigcup_{k=1}^{m+1} A^k = ... = \bigcup_{k=1}^{|H|} A^k. \tag{1}$$

If $x \in H$, we suppose $\circ x^1 = \{x\}$ and $\circ x^n = \underbrace{x \circ \ldots \circ x}_{\text{n times}}$ for all integer $n > 1$. We refer to $\widehat{x} = \bigcup_{n\geq 1} \circ x^n$ as the *cyclic subsemihypergroup of* $(H, \circ)$ *generated by the element* x. It is the smallest subsemihypergroup containing x.

If K is a subsemihypergroup of $(H, \circ)$, it is said *hypercyclic* if there exists a hyperproduct P of elements in K such that $K = \widehat{P}$.

If $(H, \circ)$ is a semihypergroup, an element $0 \in H$ such that $x \circ 0 = \{0\}$ (resp., $0 \circ x = \{0\}$) for all $x \in H$ is called *right zero scalar element* or *right absorbing element* (resp., *left zero scalar element* or *left absorbing element*) of $(H, \circ)$. If 0 is both right and left zero scalar element, then 0 is called *zero scalar element* or *absorbing element*.

A semihypergroup $(H, \circ)$ is called *simple* if $H \circ x \circ H = H$, for all $x \in H$.

A semihypergroup $(H, \circ)$ with an absorbing element 0 is called *zero-simple* if $H \circ x \circ H = H$, for all $x \in H - \{0\}$.

Given a semihypergroup $(H, \circ)$, the relation β^* of H is the transitive closure of the relation $\beta = \cup_{n\geq 1}\beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined recursively as follows:

$$x\beta_n y \Longleftrightarrow \exists (z_1, \ldots, z_n) \in H^n \;:\; \{x, y\} \subseteq z_1 \circ z_2 \circ \ldots \circ z_n.$$

The relations β, β^* are called *fundamental relations* on H [25]. Their relevance in semihypergroup theory stems from the following facts [21]: The quotient set H/β^*, equipped with the operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $x, y \in H$ and $z \in x \circ y$, is a semigroup. Moreover, the relation β^* is the smallest strongly regular equivalence on H such that the quotient H/β^* is a semigroup.

The interested reader can find all relevant definitions, many properties and applications of fundamental relations, even in more abstract contexts, also in [4, 5, 6, 14, 15, 17, 18, 21, 25].

A semihypergroup $(H, \circ)$ is said to be *fully zero-simple* if it fulfills the following conditions:

1. All subsemihypergroups of $(H, \circ)$ ($(H, \circ)$ itself included) are zero-simple.

2. The relation β in $(H, \circ)$ and the relation β_K in all subsemihypergroups $K \subset H$ of size ≥ 3 are not transitive.

Since in all semihypergroups of size ≤ 2 the relation β is transitive, it follows that every fully zero-simple semihypergroup has size ≥ 3.

We denote by $\mathfrak{F}_0$ the class of fully zero-simple semihypergroups. We use 0 to denote the zero scalar element of each semihypergroup $(H, \circ) \in \mathfrak{F}_0$. Moreover, we use the notation H_+ to indicate the set of nonzero elements in H, that is, $H_+ = H - \{0\}$. Finally, for reader's convenience, we collect in the following lemma some preliminary results from [9].

Lemma 1.1. *Let $(H, \circ) \in \mathfrak{F}_0$ then we have:*

1. *$H \circ H = H$;*
2. *if S is a subsemihypergroup of H such that $0 \notin S$, then $|S| = 1$. Moreover, if $|S| \geq 2$ then the zero element of S is 0;*
3. *there exist $x, y \in H_+$ such that $0 \in x \circ y$;*
4. *for every sequence $z_1, \ldots, z_n$ of elements in H_+ we have $\prod_{i=1}^{n} z_i \neq \{0\}$;*
5. *the set H_+ equipped with hyperproduct $a \star b = (a \circ b) \cap H_+$, for all $a, b \in H_+$, is a semihypergroup.*

By points 2. and 4. of Lemma 1.1 we deduce the following result:

Corollary 1.1. *Let S be a subsemihypergroup of $H \in \mathfrak{F}_0$, then we have:*

1. *if $0 \notin S$ then there exists $a \in H_+$ such that $S = \{a\}$ and $a \circ a = \{a\}$;*
2. *if $|S| = 2$ then there exists $a \in H_+$ such that $S = \{a, 0\}$ and $\{a\} \subseteq a \circ a \subseteq \{0, a\}$.*

From point 5. of Lemma 1.1, we know that the set of nonzero element H_+ of a fully 0-simple semihypergroup $(H, \circ)$ is a simple semihypergroup equipped with hyperoperation $a \star b = (a \circ b) \cap H_+$, for all $a, b \in H_+$. This semihypergroup is called *residual semihypergroup* of $(H, \circ)$.

The following results have been proved in [13]:

Theorem 1.1. *Let $(H, \circ) \in \mathfrak{F}_0$. For all $x \in H$, we have $(x, 0) \in \beta$. Moreover H/β^* is trivial.*

Lemma 1.2. *Let A, B be two non-empty subsets of $(H, \circ) \in \mathfrak{F}_0$ different from the singleton $\{0\}$. We have:*

1. *$(A - \{0\}) \star (B - \{0\}) = A \circ B - \{0\}$.*
2. *If $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$ then $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$.*
3. *If $0 \in A$ and $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$ then $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$.*

4. *If $A_+ = A - \{0\}$ and $(\widehat{A}, \circ), (\widehat{A}_+, \star)$ are the subsemihypergroups of $(H, \circ)$ and $(H_+, \star)$ generated from A and A_+ respectively, then $\widehat{A}_+ = \widehat{A} - \{0\}$.*

Proposition 1.1. *Let $(H_+, \star)$ the residual semihypergroup of $(H, \circ) \in \mathfrak{F}_0$ and $[0,0]_H = \{(a,b) \in H \times H \mid a = 0$ or $b = 0\}$. Then we have $\beta_{H_+} = \beta - [0,0]_H$.*

2 Hypercyclic semihypergroup in $\mathfrak{F}_0$

In paper [11] the authors introduced the definition of hypercyclic semihypergroup and studied a class of semihypergroups $(H, \circ)$ such that for all hyperproducts P of elements in H the subsemihypergroup $\widehat{P}$ is hypercyclic. In this section we study some properties of the hypercyclic semihypergroups in $\mathfrak{F}_0$. For reader's convenience we denote with $\mathfrak{I}_0$ the subclass of hypercyclic semihypergroups in $\mathfrak{F}_0$.

Proposition 2.1. *If $(H, \circ) \in \mathfrak{I}_0$ is generated by hyperproduct P, then $(H_+, \star)$ is hypercyclic generated by $P_+ = P - \{0\}$ and β_{H_+} is transitive.*

Proof. If $(H, \circ) \in \mathfrak{F}_0$ is generated from the set P, then $H = \widehat{P}$ and, for Lemma 1.2, the residual semihypergroup $(H_+, \star)$ is generated from $P_+ = P - \{0\}$. Therefore $(H_+, \star)$ is a simple hypercyclic semihypergroup. By Theorem 3.1 in [11], the relation β_{H_+} is transitive. □

Corollary 2.1. *If $(H, \circ) \in \mathfrak{I}_0$ and a, b, c are three elements in H such that $(a,b) \in \beta$, $(b,c) \in \beta$ and $(a,c) \notin \beta$, then $b = 0$.*

Proof. By Theorem 1.1, we have that $a \neq 0$ and $c \neq 0$, otherwise $(a,c) \in \beta$. If, for absurd, $b \neq 0$, then $a, b, c \in H_+$ and, for Proposition 1.1, $(a,b) \in \beta_{H_+}$ and $(b,c) \in \beta_{H_+}$. Now, for Proposition 2.1, we obtain that $(a,c) \in \beta_{H_+}$, that is impossible because $\beta_{H_+} \subseteq \beta$ and $(a,c) \notin \beta$. Therefore, $b = 0$. □

Theorem 2.1. *If $(H, \circ) \in \mathfrak{I}_0$ then $\left| H_+/\beta^*_{H_+} \right| \geq 2$.*

Proof. For absurd, let $|H_+/\beta^*_{H_+}| = 1$. If $a, b \in H$, we can distinguish two cases: 1) $a = 0$ or $b = 0$; 2) $a \neq 0$ and $b \neq 0$. In the first case, by Theorem 1.1, we have that $(a,b) \in \beta$. In the second case, for the hypothesis $|H_+/\beta^*_{H_+}| = 1$ and Proposition 2.1, we obtain that $(a,b) \in \beta_{H_+} \subseteq \beta$. Thus, we have that $(a,b) \in \beta$, for all $a, b \in H$. Therefore, we conclude that β is transitive, that is an absurdity. □

By Corollary 1.1, if $(H, \circ) \in \mathfrak{F}_0$ and $K \subset H$ is a subsemihypergroup of size $|K| < 3$ then there exists an element $c \in K$ such that $c \in cc$. Now we will prove that if $|K| \geq 3$ and K is hypercyclic generated by hyperproduct P then $(P \cap P^2) - \{0\} = \emptyset$. We give the following result:

Lemma 2.1. *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a hyperproduct of elements in H such that $(P \cap P^2) - \{0\} \neq \emptyset$, then $(P^k \cap P^{k+1}) - \{0\} \neq \emptyset$ for every integer $k \geq 1$.*

Proof. By hypothesis the thesis is true for $k = 1$. Therefore, we suppose it is true for $k \geq 1$ and let $a \in (P^k \cap P^{k+1}) - \{0\} \neq \emptyset$. Obviously we have $aP \subseteq P^k P = P^{k+1}$ and $aP \subseteq P^{k+1}P = P^{k+2}$, hence $aP \subseteq P^{k+1} \cap P^{k+2}$. From Lemma 1.1(4), we obtain that $aP \neq \{0\}$ since $a \neq 0$ and $P \neq \{0\}$. Thus, we have that $(P^{k+1} \cap P^{k+2}) - \{0\} \neq \emptyset$. □

Proposition 2.2. *If $(H, \circ) \in \mathfrak{I}_0$ is generated by hyperproduct P then we have $(P \cap P^2) - \{0\} = \emptyset$.*

Proof. For absurd we suppose that $(P \cap P^2) - \{0\} \neq \emptyset$. By Lemma 2.1 we have $(P^k \cap P^{k+1}) - \{0\} \neq \emptyset$ for every integer $k \geq 1$. From Lemma 1.2 (1), if $P_+ = P - \{0\}$ then we obtain

$$\star P_+^k \cap \star P_+^{k+1} = (P^k - \{0\}) \cap (P^{k+1} - \{0\}) = (P^k \cap P^{k+1}) - \{0\} \neq \emptyset.$$

Moreover, by Proposition 2.1, the semihypergroup $(H_+, \star)$ is hypercyclic generated from P_+ and β_{H_+} is transitive. Now, if $x, y \in H_+$ then there exist two integers $m, n \geq 1$ such that $x \in \star P_+^m$ and $y \in \star P_+^n$. If $m = n$ then $(x, y) \in \beta_{H_+}$. If $m \neq n$ we can suppose that $m < n$ and $(\star P_+^{m+k} \cap \star P^{m+k+1}) - \{0\} \neq \emptyset$, for every $k \in \{0, 1, \ldots, n - m - 1\}$. Therefore, there exist $n - m$ elements $z_0, z_1, \ldots, z_{n-m-1} \in H_+$ such that

$$\{x, z_0\} \subseteq \star P_+^m, \{z_0, z_1\} \subseteq \star P_+^{m+1}, \ \ldots \ , \{z_{n-m-1}, y\} \subseteq \star P_+^n.$$

In consequence, $x\beta_{H_+}z_0\beta_{H_+}z_1\beta_{H_+} \ \ldots \ \beta_{H_+}z_{n-m-1}\beta_{H_+}y$ and $(x, y) \in \beta_{H_+}$ since β_{H_+} is transitive. Thus, for every $x, y \in H_+$ we have $(x, y) \in \beta_{H_+}$ and $|H_+/\beta_{H_+}^*| = 1$. This fact is impossible by Theorem 2.1. □

As immediate consequence of the preceding proposition, we can state the following result:

Corollary 2.2. *Let $(H, \circ) \in \mathfrak{F}_0$ and let $K \subseteq H$ be a hypercyclic subsemihypergroup of size $|K| \geq 3$. If P is a hyperproduct of elements in $K - \{0\}$ such that $K = \widehat{P}$ then $(P \cap P^2) - \{0\} = \emptyset$.*

3 Strong hyperproduct

Let $(H, \circ) \in \mathfrak{F}_0$ and let $K \subseteq H$ be a subsemihypergroup generated by P with $|K| \geq 3$. Since $(K, \circ) \in \mathfrak{F}_0$, by Lemma 1.1 we have $K = K \circ K = \bigcup_{n \geq 2} P^n$, hence there exists an integer $s \geq 2$ such that $(P \cap P^s) - \{0\} \neq \emptyset$. This fact suggests the following definition:

Definition 3.1. Let $(H, \circ)$ be a semihypergroup and let P be a hyperproduct of elements in H. The smallest positive integer k such that $P \cap P^{k+1} - \{0\} \neq \emptyset$ is called to be the *rank* of P. If no such k exists, then we say P has rank 0.

Clearly, by Corollary 2.2, if K is a hypercyclic subsemihypergroup of $(H, \circ) \in \mathfrak{F}_0$, with size $|K| \geq 3$, and P is a hyperproduct of elements in K such that $K = \widehat{P}$ then the rank of P is ≥ 2 .

In this section we will use the notion of rank to determine a sufficient condition for a hypercyclic semihypergroup $(H, \circ) \in \mathfrak{I}_0$ to be cyclic.

Definition 3.2. Let $(H, \circ) \in \mathfrak{F}_0$, an element $c \in H$ is called *quasi-idempotent* if $c \neq 0$ and $\{c\} \subseteq c \circ c \subseteq \{0, c\}$.

Definition 3.3. Let $(H, \circ) \in \mathfrak{F}_0$. A hyperproduct P of elements in H is called *strong* if it fulfills the following conditions:

1. P does not contain any quasi-idempotent element of H.
2. The subsemihypergroup $\widehat{P}$ owns a quasi-idempotent element.
3. If $c \in \widehat{P}$ is a quasi-idempotent element then $P^s - \{0\} = \{c\}$, for all integers s such that $c \in P^s$.

An immediate consequence of the previous definition and points 2., 4. of Lemma 1.1 is the following result:

Proposition 3.1. *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a strong hyperproduct then $0 \in \widehat{P}$ and $|\widehat{P}| \geq 3$.*

Proposition 3.2. *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct. The semihypergroup $\widehat{P}$ owns one and only one quasi-idempotent element.*

Proof. Since P is a strong hyperproduct $\widehat{P}$ owns a quasi-idempotent element c_1. If there exists another quasi-idempotent element $c_2 \in \widehat{P}$, then there exist two positive integers s_1 and s_2 such that $P^{s_1} - \{0\} = \{c_1\}$ and $P^{s_2} - \{0\} = \{c_2\}$. Obviously we have

$$\{c_1\} = c_1^{s_2} - \{0\} = (P^{s_1})^{s_2} - \{0\} = (P^{s_2})^{s_1} - \{0\} = c_2^{s_1} - \{0\} = \{c_2\}$$

and so $c_1 = c_2$. □

Corollary 3.1. *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of elements in H. If c is the quasi-idempotent element in $\widehat{P}$ and $s \in \mathbb{N} - \{0\}$ then P^s is a strong hyperproduct if and only if $c \notin P^s$.*

Next table shows a fully zero-simple semihypergroup with two quasi-idempotent elements c_1, c_2 and two strong hyperproducts P and Q such that $c_1 \in \widehat{P}$ and $c_2 \in \widehat{Q}$.

Example 3.1. Let $H = \{0, 1, 2, 3, 4, 5, 6\}$ and let $\circ$ be the hyperproduct defined in the following table:

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0, 3	0, 3	0, 1, 2	5, 6	0, 4, 6	0, 4, 5, 6
2	0	0, 3	0, 3	0, 1, 2	0, 4, 6	5, 6	0, 4, 5, 6
3	0	0, 1, 2	0, 1, 2	3	0, 4, 5, 6	0, 4, 5, 6	0, 4, 5, 6
4	0	2, 3	0, 1, 3	0, 1, 2, 3	0, 6	0, 6	0, 4, 5
5	0	0, 1, 3	2, 3	0, 1, 2, 3	0, 6	0, 6	0, 4, 5
6	0	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3	0, 4, 5	0, 4, 5	0, 6

We have $(H, \circ) \in \mathfrak{F}_0$. The elements $3, 6$ are quasi-idempotent, the hyperproducts $P = 1 \circ 3$, $Q = 4 \circ 6$ are strong of rank two and we have $3 \in \widehat{P}$, $6 \in \widehat{Q}$. Moreover, we note that $(H_+, \star)$ is a not commutative simple semihypergroup and $H_+/\beta^*_{H_+}$ is isomorphic to semigroup

	1	2
1	1	2
2	1	2

Proposition 3.3. *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of rank r. Then we have $r \geq 2$.*

Proof. Let c be the quasi-idempotent element in $\widehat{P}$ and $s \geq 2$ the minimum integer such that $P^s - \{0\} = \{c\}$. If for absurd we suppose $P \cap P^2 \neq \emptyset$ then $P^{s-1} \cap P^s - \{0\} \neq \emptyset$ and so $c \in P^{s-1}$. By definition of strong hyperproduct, we have $P^{s-1} - \{0\} = \{c\}$. That is a contradiction for the minimality of s. □

We are ready to prove the following result

Proposition 3.4. *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of rank r. Then r is the minimum positive integer such that $P^r - \{0\} = \{c\}$, where c is the quasi-idempotent element in $\widehat{P}$.*

Proof. Since $c \in \widehat{P}$ there exists a minimum positive integer s such that $P^s - \{0\} = \{c\}$. Hence $\{c\} = (P^s)^r - \{0\} = (P^r)^s - \{0\}$ and $c \in \widehat{P^r}$. Clearly, there exists a minimum positive integer t such that $P^{rt} - \{0\} = \{c\}$. Suppose, for absurd, that $t \geq 2$. By point 4. of Lemma 1.1, we have $P^{(t-1)r-1} \neq \{0\}$. Moreover, since $(P \cap P^{r+1}) - \{0\} \neq \emptyset$, we obtain

$$\emptyset \neq ((P \cap P^{r+1}) - \{0\}) \circ P^{(t-1)r-1} - \{0\} \subseteq (P^{(t-1)r} \cap P^{tr}) - \{0\} \subseteq P^{(t-1)r} \cap \{c\}.$$

By Definition 3.3, it follows $P^{(t-1)r} - \{0\} = \{c\}$, that is a contradiction for the minimality of element t. Therefore $t = 1$ and $P^r - \{0\} = \{c\}$. Now, let s a positive integer such that $P^s - \{0\} = \{c\}$, then $\emptyset \neq P \cap P^{r+1} - \{0\} = P \cap cP - \{0\} = P \cap P^{s+1} - \{0\}$ and so $s \geq r$. Therefore, r is the minimum positive integer such that $P^r - \{0\} = \{c\}$. □

Proposition 3.5. *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a strong hyperproduct of elements in H of rank r then there exists a positive integer $t \leq 2r$ such that $0 \in P^t$.*

Proof. Let c be the quasi-idempotent element in $\widehat{P}$. From Definition 3.2 and Proposition 3.4, we have $c \in c \circ c \subseteq \{0, c\}$ and $c \in P^r \subseteq \{0, c\}$. Moreover, by Proposition 3.1, we know that $0 \in \widehat{P}$, hence we can distinguish two cases: $0 \in P^{2r}$ or $0 \notin P^{2r}$. In the first case we have the thesis. In the second case, we obtain $c \circ c = \{c\} = P^r$. Now, there exists an integer $m \geq 1$ such that $0 \in P^m$. If $m > 2r$, by euclidean division, there exist two non-negative integers q, n such that $m = qr + n$ with $q \neq 0$ and $0 \leq n < r$. We have $n \neq 0$ otherwise $0 \in P^m = P^{qr} = (P^r)^q = c^q = \{c\}$. Hence we deduce $0 \in P^m = P^{qr+n} = (P^r)^q \circ P^n = c^q \circ P^n = c \circ P^n = P^r \circ P^n = P^{n+r}$ and so $0 \in P^{n+r}$ with $n + r < 2r$. □

Corollary 3.2. *Let* $(H, \circ) \in \mathfrak{F}_0$*. If* P *is a strong hyperproduct of elements in* H *of rank* r *then* $\widehat{P} = \bigcup_{k=1}^{2r} P^k$*.*

Proof. Let c be the quasi-idempotent element in $\widehat{P}$. For all $m > 2r$ there exist $q, n \in \mathbb{N}$ such that $m = qr + n$, $q \geq 2$ and $0 \leq n < r$. By Proposition 3.4, if $n = 0$ then $P^m = (P^r)^q \subseteq \{0, c\}^q \subseteq \{0, c\} \subseteq \{0\} \cup P^r$. Otherwise, if $n \neq 0$ then we have $P^m = (P^r)^q \circ P^n \subseteq \{0, c\}^q \circ P^n \subseteq (\{0\} \cup c^q) \circ P^n = \{0\} \cup (c^q \circ P^n) = \{0\} \cup (\{0, c\} \circ P^n) = \{0\} \cup c \circ P^n = \{0\} \cup P^r \circ P^n = \{0\} \cup P^{n+r}$. Hence, for Proposition 3.5, we obtain $P^m \subseteq \bigcup_{k=1}^{2r} P^k$ and $\widehat{P} = \bigcup_{k=1}^{2r} P^k$. □

Lemma 3.1. *Let* $(H, \circ) \in \mathfrak{F}_0$ *and let* P *be a strong hyperproduct of elements in* H *of rank* r*. If* c *is the quasi-idempotent element of* $\widehat{P}$ *then we have*

1. $\widehat{P} = (\{c\} \cup c \circ P \cup c \circ P^2 \cup ... \cup c \circ P^{r-1})$;
2. $c \circ P^i - \{0\} = P^i \circ c - \{0\}$ *for every* $i \in \{1, 2, ..., r\}$;
3. $(c \circ P)^i - \{0\} = c \circ P^i - \{0\}$ *for every* $i \in \{1, 2, ..., r\}$;
4. $(c \circ P^i) \circ (c \circ P^j) - \{0\} = c \circ P^{i+j} - \{0\}$*, for every* $i, j \in \{1, 2, ..., r\}$;
5. $(P^i \cap P^j) - \{0\} = \emptyset$*, for every* $i, j \in \{1, 2, ..., r\}$ *and* $i \neq j$;
6. $c \notin c \circ P^i - \{0\}$*, for every* $i \in \{1, 2, ..., r - 1\}$;
7. $(c \circ P^i \cap c \circ P^j) - \{0\} = \emptyset$*, for all* $i, j \in \{1, 2, ..., r - 1\}$ *and* $i \neq j$;
8. $P^i \subseteq c \circ P^i$*, for every* $i \in \{1, 2, ..., r - 1\}$.

Proof. For Corollary 3.2, we can put $\widehat{P} = P \cup P^2 \cup ... \cup P^{2r}$.

1. Since $\widehat{P} = (\widehat{P})^r = P^r \cup P^{r+1} \cup ... \cup P^{2r^2}$, by Proposition 3.4 it results:

   $P^r - \{0\} = \{c\}$
   $P^{r+1} - \{0\} = c \circ P - \{0\}$
   . . . .
   $P^{2r-1} - \{0\} = c \circ P^{r-1} - \{0\}$
   $P^{2r} - \{0\} = c \circ P^r - \{0\} = \{c\} = P^r - \{0\}$.

   At this point, taking in account Proposition 3.5, the assertion follows immediately.

2. Since $\{c\} = P^r - \{0\}$ we have

   $$c \circ P^i - \{0\} = P^r \circ P^i - \{0\} = P^i \circ P^r - \{0\} = P^i \circ c - \{0\}.$$

3. By item 2. we have $(c \circ P)^i - \{0\} = \underbrace{(c \circ P) \circ (c \circ P) \circ \ldots \circ (c \circ P)}_{\text{i times}} - \{0\} =$ $\underbrace{c \circ c \circ \ldots \circ c}_{\text{i times}} \circ \underbrace{P \circ P \circ \ldots \circ P}_{\text{i times}} - \{0\} = c \circ P^i - \{0\}$.

4. By item 2., $(c\circ P^i)\circ(c\circ P^j)-\{0\}=c\circ c\circ P^i\circ P^j-\{0\}=(c\circ P^{i+j})-\{0\}$.

5. For absurd let $(P^i\cap P^j)-\{0\}\neq\emptyset$ for some $i,j\in\{1,2,...,r\}$ with $i\neq j$. Supposing $i<j$ we obtain $(P^{r-j+i}\cap P^{r-j+j})-\{0\}\neq\emptyset$, hence $(P^{r-j+i}\cap P^r)-\{0\}\neq\emptyset$. Since $P^r-\{0\}=\{c\}$ we have $c\in P^{r-j+i}-\{0\}$, that is a contradiction because r is the minimum integer such that $\{c\}\in P^r-\{0\}$.

6. Let $i\in\{1,2,...,r-1\}$. Since $c\circ P-\{0\}=P^{r+1}-\{0\}$, we have

   $$(P\cap P^{r+1})-\{0\}\neq\emptyset\Rightarrow(P\cap c\circ P)-\{0\}\neq\emptyset\Rightarrow(P^i\cap(c\circ P)^i)-\{0\}\neq\emptyset.$$

   Moreover, by item 3. we obtain $P^i\cap c\circ P^i-\{0\}\neq\emptyset$. Now, if $c\in(c\circ P)^i-\{0\}$ then $c\circ P^i-\{0\}=P^r\circ P^i-\{0\}=P^{r+i}-\{0\}$, hence $c\circ P^i-\{0\}=P^{r+i}-\{0\}=\{c\}$. Consequently we have $c\in P^i-\{0\}$, that is impossible because $i<r$. Thus $c\notin c\circ P^i-\{0\}$.

7. For absurd, we suppose that there exists $i,j\in\{1,2,...,r-1\}$, with $i\neq j$, such that $(c\circ P^i\cap c\circ P^j)-\{0\}\neq\emptyset$. Let $i<j$, by item 3. we obtain $(c\circ P^{r-j+i}\cap c\circ P^{r-j+j})-\{0\}\neq\emptyset$. Being $c\circ P^r-\{0\}=\{c\}$, it follows that $c\in c\circ P^{r-j+i}-\{0\}$. Since $r-j+i<r$, by item 6., we have a contradiction.

8. Let $i\in\{1,2,\ldots,r-1\}$ and $a\in P^i$. From item 1., there exists an integer s, with $1\leq s\leq r-1$, such that $a\in c\circ P^s$. Clearly it results $c\circ a\subseteq c\circ P^i$ and $c\circ a\subseteq c\circ P^s$. Therefore, by 4. of Lemma 1.1 we have $\emptyset\neq c\circ a\subseteq(c\circ P^i\cap c\circ P^s)$. Moreover, for item 7., $i=s$ and $a\in c\circ P^i$. □

Remark 3.1. From Lemma 3.1, if P is a strong hyperproduct of rank r, of elements in a fully zero-simple semihypergroup, then $\widehat{P}$ is partitioned by the family of subsets $\left\{\{c\},c\circ P,c\circ P^2,...,c\circ P^{r-1}\right\}$, where c is the quasi-idempotent element of $\widehat{P}$.

Lemma 3.2. *Let $(H,\circ)\in\mathfrak{F}_0$ and let P be a strong hyperproduct of rank r, with c quasi-idempotent element in $\widehat{P}$. If Q is a hyperproduct such that $\emptyset\neq Q-\{0\}\subseteq P$ then:*

1. Q is a strong hyperproduct with $c\in\widehat{Q}$, having the same rank r of P;

2. $c\circ Q^i-\{0\}=c\circ P^i-\{0\}$, for all $i\in\{1,2,\ldots,r-1\}$;

3. $\widehat{Q}=\widehat{P}$.

Proof.

1. From Proposition 3.4 we have $Q^r-\{0\}\subseteq P^r-\{0\}=\{c\}$ and so $Q^r-\{0\}=\{c\}$ and $c\in\widehat{Q}$. Clearly $c\notin Q$ because $Q\subseteq P$ and $c\notin P$. Moreover, by point 3. of Definition 3.3, if $c\in Q^s-\{0\}$ then $c\in P^s-\{0\}$ and $Q^s-\{0\}=P^s-\{0\}=\{c\}$. Hence Q is a strong hyperproduct of $(H,\circ)$ and c is quasi-idempotent element in $\widehat{Q}$. From Proposition 3.4, if t is the rank of Q then $t\leq r$ because $Q^r-\{0\}=\{c\}$. Moreover, since $\{c\}=Q^t-\{0\}\subseteq P^t-\{0\}$, by point 3. of Definition 3.3 and Proposition 3.4, we have $P^t-\{0\}=\{c\}$ and $r\leq t$, therefore $r=t$.

2. Let b be an element in $P-\{0\}$. We have $b\circ Q^{r-1}-\{0\}\subseteq b\circ P^{r-1}-\{0\}\subseteq P^r-\{0\}=\{c\}$ and so $b\circ Q^{r-1}-\{0\}=\{c\}$. Moreover, by item 1. we have $b\circ c-\{0\}=b\circ(Q^r-\{0\})-\{0\}=b\circ(Q^{r-1}-\{0\})\circ(Q-\{0\})-\{0\}=c\circ(Q-\{0\})-\{0\}=c\circ Q-\{0\}$ and so $b\circ c-\{0\}=c\circ Q-\{0\}$, for all $b\in P-\{0\}$. Thus, by point 2. of Lemma 3.1, we deduce that $c\circ P-\{0\}=P\circ c-\{0\}=(P-\{0\})\circ c-\{0\}=\bigcup_{b\in P-\{0\}}(b\circ c)-\{0\}=\bigcup_{b\in P-\{0\}}(b\circ c-\{0\})=c\circ Q-\{0\}$. Hence $c\circ P-\{0\}=c\circ Q-\{0\}$. The proof of item follows from point 4. of Lemma 3.1.

3. The result follows from previous item 2 and point 1. of Lemma 3.1. □

4 Semihypergroup in $\mathfrak{F}_0$ generated by a strong hyperproduct

In this section we consider hypercyclic fully simple semihypergroup generated by a strong hyperproduct. For reader's convenience we give the following

Definition 4.1. A semihypergroup $(H,\circ)\in\mathfrak{I}_0$ is called *S-hypercyclic* if there exists a strong hyperproduct P such that $H=\widehat{P}$.

Example 4.1. Next table shows a S-hypercyclic semihypergroup $(H,\circ)\in\mathfrak{I}_0$. For notational and descriptive simplicity we denote $A=\{0,1\}$, $B=\{0,2,3,4\}$, $C=\{0,5,6\}$ and $D=\{0,7,8\}$.

$\circ$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	B	B	B	C	C	D	D
2	0	B	$0,5$	C	C	D	D	A	A
3	0	B	C	C	C	D	D	A	A
4	0	B	C	C	C	D	D	A	A
5	0	C	D	D	D	A	A	$2,3,4$	B
6	0	C	D	D	D	A	A	$2,3$	$2,4$
7	0	D	A	A	A	B	B	C	C
8	0	D	A	A	A	B	$2,4$	C	C

The elements 1 is quasi-idempotent and, for example, $P=6\circ 7$ is a strong hyperproduct of rank four. Also the elements $2,3,4,7,8$ can be regarded as strong hyperproducts of rank four and $H=\widehat{P}=\widehat{a}$, for all $a\in\{2,3,4,7,8\}$. In this case $H_+/\beta^*_{H_+}$ is isomorphic to group $\mathbb{Z}_4$.

Proposition 4.1. *Let $(H,\circ)\in\mathfrak{I}_0$ be a S-hypercyclic semihypergroup generated by the strong hyperproduct P of rank r and let c the quasi-idempotent element of H, we have*

1. *If $a\in P-\{0\}$ then a is strong hyperproduct of rank r and $H=\widehat{a}$.*

2. If Q is a hyperproduct of elements in H_+ then $c \in \widehat{Q}$. Moreover, if $c \in Q$ then $Q - \{0\} = \{c\}$. Otherwise, if $c \notin Q$ then Q is strong and has rank $\leq r$.

3. For every Q and T strong hyperproducts of elements in H_+, we have $c \circ Q^i \cap c \circ T^i - \{0\} = \emptyset$ or $c \circ Q^i - \{0\} = c \circ T^i - \{0\}$, for all $i \in \{1, 2, ..., r-1\}$.

4. The element c is the only identity of $(H, \circ)$.

5. The residual semihypergroup $(H_+, \star)$ of $(H, \circ)$ is a cyclic semihypergroup with identity.

Proof.

1. Immediate consequence of Lemma 3.2.

2. Let $a \in P - \{0\}$ and let $Q = \prod_{i=1}^{n} \alpha_i$ be a hyperproduct of elements in H_+. From point 1., $H = \widehat{a}$ and for every element α_i there exists an integer q_i such that $\alpha_i \in a^{q_i}$. Clearly we have $Q = \prod_{i=1}^{n} \alpha_i \subseteq \prod_{i=1}^{n} a^{q_i} = a^u$, where $u = \sum_{i=1}^{n} q_i$. Hence, $Q^r - \{0\} \subseteq (a^u)^r - \{0\} = (a^r)^u - \{0\} = \{c\}$ and so $c \in \widehat{Q}$. Now, if $c \in Q$ then $c \in Q \subseteq a^u$ and we have $a^u - \{0\} = \{c\} = Q - \{0\}$. Moreover, if $c \notin Q$ and $c \in Q^s - \{0\}$ then $c \in Q^s - \{0\} \subseteq (a^u)^s - \{0\} = a^{us} - \{0\}$ and so $\{c\} = a^{us} - \{0\} = Q^s - \{0\}$. Hence Q is a strong hyperproduct and rank of Q is $\leq r$.

3. If $c \circ Q^i \cap c \circ T^i - \{0\} = \emptyset$ the thesis follows. Otherwise, if $c \circ Q^i \cap c \circ T^i - \{0\} \neq \emptyset$ then there exists $a \in H_+ - \{c\}$ such that $a \in c \circ Q^i$ and $a \in c \circ T^i$ and so, by point 2. of Lemma 3.2, $c \circ Q^i - \{0\} = c \circ a - \{0\} = c \circ T^i - \{0\}$.

4. By item 2., the element b is a hyperproduct strong for every $b \in H_+ - \{c\}$. From Lemma 3.1 (1.), we have $\widehat{b} = \{c\} \cup c \circ b \cup ... \cup c \circ b^{s-1}$, where s is the rank of b. Hence there exists $i \in \{1, 2, \ldots, s-1\}$ such that $b \in c \circ b^i$. Clearly $c \circ b - \{0\} \subseteq c \circ b^i - \{0\}$ and, by item 7. of Lemma 3.1, we have $i = 1$ and $b \in c \circ b$. In the same way, by item 2. of Lemma 3.1, we obtain $b \in b \circ c$. Hence $b \in c \circ b \cap b \circ c$ for all $b \in H_+$. Obviously, we have also $c \circ 0 = 0 \circ c = \{0\}$, hence c is an identity of $(H, \circ)$. If $c' \in H - \{0, c\}$ is another identity and $Q = c \circ c'$, for item 2., we have $\{c, c'\} \subseteq Q - \{0\}$={c} and $c = c'$. Hence the element c is the only identity of $(H, \circ)$.

5. By point 1. and Lemma 1.2 (1) the semihypergroup $(H_+, \star)$ is cyclic. Moreover, from previous point 4., the element c is an identity of $(H_+, \star)$. □

Proposition 4.2. *Let $(H, \circ) \in \mathfrak{I}_0$ a S-hypercyclic semihypergroup generated by the strong hyperproduct P of rank r and suppose r is a prime number, then $\widehat{c \circ P^h} = H$ for all $h \in \{1, 2, ..., r-1\}$.*

Proof. By item 6. of Lemma 3.1 and the preceding proposition, $c \circ P^i$ is a strong hyperproduct for all $i \in \{1, 2, ..., r-1\}$. Let now $h \in \{1, 2, ..., r-1\}$.

By item 1. of Lemma 3.1, we need to prove that $c \circ P^j \subseteq \widehat{c \circ P^h}$, for every $j \in \{1, 2, ..., r-1\}$. Since r is a prime number, the congruence $hx = j(\text{mod r})$ has exactly one solution $s \neq 0$ (mod r). Thus $hs = j + kr$ and so

$$c \circ P^{j+kr} - \{0\} = c \circ P^{hs} - \{0\}.$$

By points 2., 3. and 4. of Lemma 3.1, we have $c \circ P^{hs} - \{0\} = c^s \circ P^{hs} - \{0\} = (c \circ P^h)^s - \{0\}$. Hence $c \circ P^{j+kr} - \{0\} = (c \circ P^h)^s - \{0\}$. Clearly, if $k = 0$ we have $c \circ P^j \subseteq \widehat{c \circ P^h}$, otherwise if $k \neq 0$ then $c \circ P^j - \{0\} \subseteq c \circ P^j \circ c - \{0\} = c \circ P^j \circ (P^r)^k - \{0\} = c \circ P^{hs} - \{0\} = (c \circ P^h)^s - \{0\}$ and also in this case $c \circ P^j \subseteq \widehat{c \circ P^h}$. □

Theorem 4.1. *Let $(H, \circ) \in \mathfrak{I}_0$ a $S-$ hypercyclic semihypergroup generated by a strong hyperproduct P of rank a prime number r and having c as quasi-idempotent element, then:*

1. *Every element $a \in H_+ - \{c\}$ is a strong hyperproduct of rank r and $H = \widehat{a}$.*
2. *For every strong hyperproduct Q of H, $\widehat{Q} = H$ and Q has rank r;*
3. *$(H_+, \star)$ is a cyclic semihypergroup generated by every $a \in H_+ - \{c\}$;*
4. *$H_+/\beta^*_{H_+}$ is a cyclic semigroup.*

Proof.

1. Let $a \in H_+ - \{c\}$, then by item 1. of Lemma 3.1, there exists $h \in \{1, 2, ..., r-1\}$, such that $a \in c \circ P^h$. By Proposition 4.2, we have $\widehat{c \circ P^h} = H$. Moreover $c \notin c \circ P^h$ and so, from Lemma 3.2, $H = \widehat{a}$.
2. Consequence of item 1.
3. Consequence of Lemma 1.2 (4) and item 1.
4. Immediate since $(H_+, \star)$ is a cyclic semihypergroup.

□

Remark 4.1. In Example 4.1, the S-hypercyclic semihypergroup $(H, \circ)$ is generated by a strong hyperproduct P of rank four, while the elements 5 and 6 are strong hyperproducts of rank two and do not generate $(H, \circ)$. This fact shows that the hypothesis *rank of P is a prime number* in Theorem 4.1 can not be deleted. The following product table shows a S-hypercyclic semihypergroup $(H, \circ)$ generated by a strong hyperproduct P of rank three.

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0, 2, 3, 4	0, 2, 3, 4	0, 2, 3, 4	0, 5, 6	0, 5, 6
2	0	0, 2, 3, 4	0, 5, 6	5, 6	0, 5, 6	0, 1	0, 1
3	0	0, 2, 3, 4	0, 5, 6	0, 5, 6	5, 6	0, 1	0, 1
4	0	0, 2, 3, 4	5, 6	0, 5, 6	0, 5, 6	0, 1	0, 1
5	0	0, 5, 6	0, 1	0, 1	0, 1	0, 2, 3	0, 4
6	0	0, 5, 6	0, 1	0, 1	0, 1	0, 2, 3, 4	0, 2, 3, 4

By previous theorem, we have $H = \widehat{a}$, for every $a \in H_+ - \{1\}$, where 1 is the quasi-idempotent element of H. We note that if G is a group then every element a is a strong product and if a is a torsion element its rank is the period of the element a. Therefore the rank can be seen as a generalization of the concept of period. Moreover it is known that if G is a cyclic group of size a prime number r then every element different from identity is a generator of G. The same property is true for semihypergroups in Theorem 4.1, but the commutative property of cyclic groups does not generally hold. The hyperoperation in previous example is not commutative.

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