A Corrected Sadowsky Functional for Inextensible Elastic Ribbons

A CORRECTED SADOWSKY FUNCTIONAL FOR INEXTENSIBLE ELASTIC RIBBONS

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Abstract. The classical theory of ribbons, developed by Sadowsky and Wunderlich, has recently received renewed attention. Here, by means of Γ-convergence, we re-examine the derivation of the limit energy of an inextensible, isotropic, elastic strip as the width goes to zero. We find that this rigorously derived functional agrees with the classical Sadowsky functional only for "large" curvature of the centerline of the strip.

1. Introduction

In 1930 Sadowsky [12] established the existence of a developable Möbius strip and stated that the configuration assumed by the strip can be computed by minimizing the bending energy. He further argued that the bending energy density is proportional to the square of the mean curvature of the surface.

The (bending) energy of an inextensible elastic strip $S_\varepsilon = (-\ell/2, \ell/2) \times (-\varepsilon/2, \varepsilon/2)$, where $\ell > 0$ and $\varepsilon > 0$ are the length and the thickness of the strip, is

$$u \mapsto \frac{1}{\varepsilon} \int_{S_\varepsilon} Q(A_u(x))\, dx, \tag{1.1}$$

where Q is the bending energy density and A_u denotes the second fundamental form of $u : S_\varepsilon \to \mathbb{R}^3$. For isotropic strips the energy density Q depends only on the determinant and the trace of A_u: the Gaussian curvature $K_u := \det A_u$ and the mean curvature $H_u := 1/2\, \mathrm{tr}\, A_u$. Since developable surfaces have null Gaussian curvature we have that Q only depends on H_u. By assuming, as tacitly done by Sadowsky [12], the material to be isotropic and the energy density to be quadratic, the bending energy is proportional to

$$u \mapsto \frac{1}{\varepsilon} \int_{S_\varepsilon} |H_u(x)|^2\, dx. \tag{1.2}$$

Sadowsky, still in [12], also argued that as $\varepsilon \to 0$ the energy of the ribbon reduces to

$$y \mapsto \int_{-\ell/2}^{\ell/2} \frac{(\kappa^2 + \tau^2)^2}{\kappa^2}\, ds \tag{1.3}$$

where κ and τ are the curvature and the torsion, respectively, of the centerline y of the strip, and s is its arc length. This functional is now known as Sadowsky's functional. The seminal paper [12] and also the subsequent paper by Sadowsky [13] have recently been translated into English [8, 9].

Wunderlich in [17] gave a formal justification of the energy (1.3), see [16] for a translation. His analysis is based on the fact that every smooth developable surface is a ruled surface, in particular, as pointed out in [14], it follows that if the centerline y is smooth and with *non vanishing curvature* κ, then the surface $u(S_\varepsilon)$ has a parametrization of the form

$$(s, z) \mapsto y(s) + z[b(s) + \eta(s)t(s)],$$

where t and b are the tangent and the binormal of the centerline y and $\eta := \tau/\kappa$. With this parametrization one may compute the mean curvature H_u and then rewrite the energy (1.2) as

$$y \mapsto \int_{-\ell/2}^{\ell/2} \frac{(\kappa^2 + \tau^2)^2}{\kappa^2} \frac{1}{\varepsilon\eta'} \ln \frac{1 + \varepsilon\eta'/2}{1 - \varepsilon\eta'/2}\, ds. \tag{1.4}$$

The energy (1.3) is then obtained as the pointwise limit, as $\varepsilon \to 0$, of this sequence of energies. Wunderlich's analysis makes it clear that the Sadowsky functional is derived under the assumption of non vanishing curvature of the centerline of the strip. Kirby and Fried [10] recently investigated if also the Γ-limit, under an appropriate topology, of the sequence of functionals (1.4) is the Sadowsky functional. They gave a positive answer after restricting the domain of the functional (1.4) to a space of curves with non vanishing curvature and with certain regularity.

Other interesting papers addressing the Sadowsky functional are [2, 3, 4, 11, 15].

In this paper we study the Γ-limit, with respect to a topology that ensures the convergence of the minimizers, of the sequence of energies (1.2): we therefore re-examine the same problem studied by Sadowsky in [12]. In our analysis, however, we make no a priori assumptions on the curvature of the centerline. The obtained limit functional depends on three orthonormal vectors d_1, d_2, and d_3. The first director d_1 is the tangent to the centerline y, whereas d_2 represents the "transversal" orientation of the strip, and $d_3 = d_1 \wedge d_2$. The limit problem describes an inextensible beam (since $y' = d_1$) which cannot bend within the plane of the strip (since the directors have to satisfy the constraint $d_1' \cdot d_2 = 0$). Its energy is given by

$$J(d_1, d_2, d_3) = \int_{-\ell/2}^{\ell/2} \overline{Q}(d_1' \cdot d_3, d_2' \cdot d_3)\, ds,$$

where the energy density $\overline{Q}$ is

$$\overline{Q}(\alpha, \beta) = \begin{cases} \dfrac{(\alpha^2 + \beta^2)^2}{\alpha^2} & \text{if } |\alpha| > |\beta|, \\ 4\beta^2 & \text{if } |\alpha| \le |\beta|. \end{cases} \tag{1.5}$$

We also show that if the curvature κ of the centerline y is everywhere different from zero, than $\kappa = |d_1' \cdot d_3|$ and the torsion $\tau = d_2' \cdot d_3$, so that the limit functional can be rewritten, using the notation adopted earlier, as

$$y \mapsto \int_{-\ell/2}^{\ell/2} \bar{Q}(\kappa, \tau)\, ds.$$

In view of this expression, we see that Sadowsky's functional (1.3) only agrees with the rigorously derived asymptotic functional when the curvature exceeds the absolute value of the torsion.

The techniques employed in our analysis make a strong use of the isotropy assumption. A Sadowsky type of functional for anisotropic elastic ribbons will be derived, with a more involved argument, in a forthcoming paper [7].

2. Main results

Let $\ell > 0$ and $I := (-\ell/2, \ell/2)$. For $0 < \varepsilon \ll 1$, we take $S_\varepsilon = I \times (-\varepsilon/2, \varepsilon/2)$ as the reference configuration of an inextensible elastic narrow strip. For a smooth deformation $u : S_\varepsilon \to \mathbb{R}^3$ we have, due to the inextensibility constraint, that $\partial_i u \cdot \partial_j u = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta. The second fundamental form of u, denoted by $A_u : S_\varepsilon \to \mathbb{R}^{2\times 2}_{\rm sym}$, is defined by

$$(A_u)_{ij} = n_u \cdot \partial_i \partial_j u,$$

where

$$n_u = \partial_1 u \wedge \partial_2 u$$

is the unit normal to u. We recall that by Gauss's Theorema Egregium we have $\det A_u = 0$, because the Gaussian curvature is everywhere equal to zero. We assume the energy density of the strip, $Q : \mathbb{R}^{2\times 2}_{\rm sym} \to [0, +\infty)$, to be an isotropic and quadratic function of the second fundamental form. Under these assumptions we may take

$$Q(A) = |A|^2 \text{ for every } A \in \mathbb{R}^{2\times 2}_{\rm sym}.$$

We note that $Q(A_u) = |A_u|^2 = (\operatorname{tr} A_u)^2$ since for every $A \in \mathbb{R}^{2\times 2}_{\rm sym}$ we have that $|A|^2 + 2 \det A = (\operatorname{tr} A)^2$; hence, the energy density considered here is equivalent with that discussed in the introduction.

Denoting the space of $W^{2,2}$ isometries of S_ε by $W^{2,2}_{iso}(S_\varepsilon;\mathbb{R}^3)$, that is,

$$W^{2,2}_{iso}(S_\varepsilon;\mathbb{R}^3) := \{u \in W^{2,2}(S_\varepsilon;\mathbb{R}^3) :\ \partial_i u \cdot \partial_j u = \delta_{ij} \text{ a.e. in } S_\varepsilon\},$$

we have that the bending energy of the strip is

$$E_\varepsilon(u) = \frac{1}{\varepsilon}\int_{S_\varepsilon} |A_u(x)|^2\,dx,$$

for any $u \in W^{2,2}_{iso}(S_\varepsilon;\mathbb{R}^3)$.

We now change variables in order to rewrite the energy over the fixed domain

$$S = I \times \big(-\tfrac{1}{2},\tfrac{1}{2}\big).$$

We introduce the rescaled version $y : S \to \mathbb{R}^3$ of u by setting

$$y(x_1,x_2) = u(x_1,\varepsilon x_2).$$

With the scaled gradient defined by

$$\nabla_\varepsilon \cdot = (\partial_1 \cdot \,|\, \varepsilon^{-1}\partial_2 \cdot),$$

we have that

$$\nabla_\varepsilon y(x) = \nabla u(x_1,\varepsilon x_2).$$

In particular, if $u \in W^{2,2}_{iso}(S_\varepsilon;\mathbb{R}^3)$, the map y belongs to the space of *scaled isometries*

$$W^{2,2}_{iso,\varepsilon}(S;\mathbb{R}^3) := \big\{y \in W^{2,2}(S;\mathbb{R}^3) :\ |\partial_1 y| = \varepsilon^{-1}|\partial_2 y| = 1,\ \partial_1 y \cdot \varepsilon^{-1}\partial_2 y = 0 \text{ a.e. in } S\big\}.$$

Defining the scaled unit normal to y by

$$n_{y,\varepsilon} = \partial_1 y \wedge \varepsilon^{-1}\partial_2 y$$

and the scaled second fundamental form of y by

$$A_{y,\varepsilon} = \begin{pmatrix} n_{y,\varepsilon}\cdot\partial_1\partial_1 y & \varepsilon^{-1} n_{y,\varepsilon}\cdot\partial_1\partial_2 y \\ \varepsilon^{-1} n_{y,\varepsilon}\cdot\partial_1\partial_2 y & \varepsilon^{-2} n_{y,\varepsilon}\cdot\partial_2\partial_2 y \end{pmatrix},$$

we have $A_u(x_1,\varepsilon x_2) = A_{y,\varepsilon}(x_1,x_2)$. Introducing

$$J_\varepsilon(y) = \int_S |A_{y,\varepsilon}(x)|^2\,dx,$$

we have $J_\varepsilon(y) = E_\varepsilon(u)$.

Our first result, whose proof will be postponed to the next section, is about compactness.

Lemma 2.1. *Let $(y_\varepsilon) \subset W^{2,2}_{iso,\varepsilon}(S;\mathbb{R}^3)$ be a sequence of scaled isometries such that*

$$\sup_\varepsilon J_\varepsilon(y_\varepsilon) < \infty. \tag{2.1}$$

Then, up to a subsequence and additive constants, there exist a deformation $y \in W^{2,2}(I;\mathbb{R}^3)$ and an orthonormal frame field $(d_1\,|\,d_2\,|\,d_3) \in W^{1,2}(I;SO(3))$ satisfying

$$d_1 = y' \quad \text{and} \quad d_1' \cdot d_2 = 0 \quad \text{a.e. in } I, \tag{2.2}$$

such that

$$y_\varepsilon \rightharpoonup y \text{ in } W^{2,2}(S;\mathbb{R}^3), \qquad \nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1\,|\,d_2) \text{ in } W^{1,2}(S;\mathbb{R}^{3\times 2}), \tag{2.3}$$

and

$$A_{y_\varepsilon,\varepsilon} \rightharpoonup \begin{pmatrix} d_1'\cdot d_3 & d_2'\cdot d_3 \\ d_2'\cdot d_3 & \gamma \end{pmatrix} \text{ in } L^2(S;\mathbb{R}^{2\times 2}_{\mathrm{sym}}) \tag{2.4}$$

with $\gamma \in L^2(S)$.

Hence a sequence (y_ε) of scaled isometries with equi-bounded energy weakly converges in $W^{2,2}$ to a deformation y that depends only on x_1. The orthonormal vectors d_1, d_2, and d_3 are the directors of the "limit beam", with d_1 tangent to the deformation y, d_2 representing the "transversal" orientation of the strip, and $d_3 = d_1 \wedge d_2$. The limiting values of the 11 and 12 components of the second fundamental form are measures of flexure and twist, respectively, cf. [1]. The 22 component instead cannot be expressed in terms of the directors. We also note that $A_{y_\varepsilon,\varepsilon}$ has null determinant while its limit, i.e., the limit matrix appearing in (2.4), does not need to have null determinant. The first constraint in (2.2) states that the "limit beam" is inextensible, while the second states that the strip does not bend within its plane. The same constraints were also obtained in [5, 6].

The previous lemma motivates the next definition. We set

$$\mathcal{A} = \{(d_1, d_2, d_3) : (d_1 \,|\, d_2 \,|\, d_3) \in W^{1,2}(I; SO(3)) \text{ and } d_1' \cdot d_2 = 0 \text{ a.e. in } I\}.$$

In order to state our next result we need to first introduce some definitions. Let $\overline{Q} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ and $J : \mathcal{A} \to \mathbb{R}$ be defined by

$$\overline{Q}(\kappa, \tau) := \min_{\gamma \in \mathbb{R}} \left\{ |M|^2 + 2|\det M| : M = \begin{pmatrix} \kappa & \tau \\ \tau & \gamma \end{pmatrix} \right\}$$

and

$$J(d_1, d_2, d_3) := \int_I \overline{Q}(d_1' \cdot d_3, d_2' \cdot d_3)\, dx_1.$$

A simple computation shows that $\overline{Q}$ can be written as in (1.5).

Theorem 2.2. *As $\varepsilon \to 0$, the functionals J_ε are Γ-converging to the functional J in the following sense:*

(i) (liminf inequality) *for every sequence $(y_\varepsilon) \subset W^{2,2}_{iso,\varepsilon}(S; \mathbb{R}^3)$, $y \in W^{2,2}(I; \mathbb{R}^3)$, and $(d_1, d_2, d_3) \in \mathcal{A}$ such that $y' = d_1$ a.e. in I, $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S; \mathbb{R}^3)$, and $\nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 \,|\, d_2)$ in $W^{1,2}(S; \mathbb{R}^{3\times 2})$, we have that*

$$\liminf_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon) \geq J(d_1, d_2, d_3);$$

(ii) (recovery sequence) *for every $(d_1, d_2, d_3) \in \mathcal{A}$ there exists a sequence $(y_\varepsilon) \subset W^{2,2}_{iso,\varepsilon}(S; \mathbb{R}^3)$ such that $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S; \mathbb{R}^3)$, $\nabla_\varepsilon y_\varepsilon \rightharpoonup (d_1 \,|\, d_2)$ weakly in $W^{1,2}(S; \mathbb{R}^{3\times 2})$, and*

$$\limsup_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon) \leq J(d_1, d_2, d_3),$$

where y is defined up to a constant by $y' = d_1$ a.e. in I.

We conclude the section by comparing the obtained Γ-limit with Sadowsky's functional. For $(d_1, d_2, d_3) \in \mathcal{A}$, let y be the function defined, up to a constant, by $y' = d_1$. Until the end of the section we suppose that y is smooth enough and the curvature

$$\kappa = |d_1'| = |y''|$$

is a strictly positive function. Under this assumption, the normal $n = d_1'/\kappa$, the binormal $b = t \wedge n$, and the torsion

$$\tau = -n \cdot b'$$

are well defined at every point of I. Set $\tilde{\kappa} = d_1' \cdot d_3$ and $\tilde{\tau} = d_2' \cdot d_3$. We now study the relation between $\kappa, \tau, \tilde{\kappa}$, and $\tilde{\tau}$. Since d_1' is orthogonal to d_1 and d_2 it follows that $d_1' = \tilde{\kappa} d_3$. Thus

$$\kappa = |d_1'| = |\tilde{\kappa}|, \quad n = \frac{\tilde{\kappa}}{\kappa} d_3 = \operatorname{sgn}(\tilde{\kappa}) d_3, \quad b = d_1 \wedge n = \operatorname{sgn}(\tilde{\kappa}) d_1 \wedge d_3 = -\operatorname{sgn}(\tilde{\kappa}) d_2,$$

where sgn is the sign function. Since n and d_3 are continuous functions, the second equality implies that $\operatorname{sgn}(\tilde{\kappa})$ is continuous, hence constant. Thus, differentiating the third equality above we have that $b' = -\operatorname{sgn}(\tilde{\kappa}) d_2'$, hence

$$\tau = -n \cdot b' = -\operatorname{sgn}(\tilde{\kappa}) d_3 \cdot (-\operatorname{sgn}(\tilde{\kappa}) d_2') = d_3 \cdot d_2' = \tilde{\tau}.$$

Thus, if the curvature κ of y is strictly positive everywhere, then the Γ-limit can be rewritten as

$$J(d_1, d_2, d_3) = \int_I \overline{Q}(\kappa, \tau)\, dx_1,$$

since $\kappa = |\tilde{\kappa}|$ and $\tau = \tilde{\tau}$. Therefore, the Γ-limit coincides with Sadowsky's functional for $\kappa > |\tau|$ and $\kappa > 0$.

3. Proofs

Here we prove the results stated in the previous section.

Proof of Lemma 2.1. Let $(y_\varepsilon) \subset W^{2,2}_{iso,\varepsilon}(S;\mathbb{R}^3)$ be a sequence satisfying (2.1), that is

$$\sup_\varepsilon \|A_{y_\varepsilon,\varepsilon}\|_{L^2(S)} < +\infty.$$

Since y_ε is a scaled isometry, we have that

$$\partial_1\partial_1 y_\varepsilon = (A_{y_\varepsilon,\varepsilon})_{11} n_{y_\varepsilon,\varepsilon}, \quad \varepsilon^{-1}\partial_1\partial_2 y_\varepsilon = (A_{y_\varepsilon,\varepsilon})_{12} n_{y_\varepsilon,\varepsilon}, \quad \varepsilon^{-2}\partial_2\partial_2 y_\varepsilon = (A_{y_\varepsilon,\varepsilon})_{22} n_{y_\varepsilon,\varepsilon},$$

where $n_{y_\varepsilon,\varepsilon}$ is the scaled unit normal to y_ε. For instance, the last equation can be checked by differentiating with respect to x_1 and x_2 the identities $\varepsilon^{-1}|\partial_2 y_\varepsilon| = 1$ and $\partial_1 y_\varepsilon \cdot \varepsilon^{-1}\partial_2 y_\varepsilon = 0$. Thus

$$\sup_\varepsilon \big(\|\partial_1\partial_1 y_\varepsilon\|_{L^2(S)} + \|\varepsilon^{-1}\partial_1\partial_2 y_\varepsilon\|_{L^2(S)} + \|\varepsilon^{-2}\partial_2\partial_2 y_\varepsilon\|_{L^2(S)}\big) < +\infty. \tag{3.1}$$

Moreover, $(y_\varepsilon) \subset W^{2,2}_{iso,\varepsilon}(S;\mathbb{R}^3)$ also implies that

$$\|\partial_1 y_\varepsilon\|_{L^\infty(S)} = 1, \qquad \|\partial_2 y_\varepsilon\|_{L^\infty(S)} = \varepsilon, \tag{3.2}$$

and hence, up to additive constants, the sequence (y_ε) is uniformly bounded in $W^{2,2}(S;\mathbb{R}^3)$. Therefore, up to subsequences, we have that $y_\varepsilon \rightharpoonup y$ in $W^{2,2}(S;\mathbb{R}^3)$ and strongly in $W^{1,p}(S;\mathbb{R}^3)$ for every $p < \infty$. Identities (3.2) imply that y is independent of x_2 and $|y'| = 1$ a.e. in I.

By the previous bounds the sequence $(\varepsilon^{-1}\partial_2 y_\varepsilon)$ is bounded in $L^\infty(S;\mathbb{R}^3)$ and $(\nabla_\varepsilon(\varepsilon^{-1}\partial_2 y_\varepsilon))$ is bounded in $L^2(S;\mathbb{R}^{3\times 2})$. Hence, up to subsequences, $\varepsilon^{-1}\partial_2 y_\varepsilon \rightharpoonup d_2$ weakly in $W^{1,2}(S;\mathbb{R}^3)$ and strongly in $L^p(S;\mathbb{R}^3)$ for every $p < \infty$, with d_2 independent of x_2 and $|d_2| = 1$ a.e. in S. Moreover, by passing to the limit in the relation $\partial_1 y_\varepsilon \cdot (\varepsilon^{-1}\partial_2 y_\varepsilon) = 0$, we deduce that $y' \cdot d_2 = 0$ a.e. in I. Since $n_{y_\varepsilon,\varepsilon} = \partial_1 y_\varepsilon \wedge \varepsilon^{-1}\partial_2 y_\varepsilon$ we have that $n_{y_\varepsilon,\varepsilon} \to d_3$ in $L^p(S;\mathbb{R}^3)$ for every $p < \infty$, where $d_3 = y' \wedge d_2$.

By differentiating the equality $\partial_1 y_\varepsilon \cdot \partial_1 y_\varepsilon = 1$ with respect to x_2 and scaling by ε, we obtain

$$\partial_1(\varepsilon^{-1}\partial_2 y_\varepsilon) \cdot \partial_1 y_\varepsilon = 0.$$

By letting ε go to zero we find $d_2' \cdot y' = 0$, from which, setting $d_1 := y'$ and using that $d_1 \cdot d_2 = 0$, we deduce that $d_1' \cdot d_2 = 0$.

Finally, up to subsequences, we have that $A_{y_\varepsilon,\varepsilon}$ weakly converges to a matrix field A in $L^2(S;\mathbb{R}^{2\times 2}_{\text{sym}})$. By using the convergences established above, it follows that $A_{11} = y'' \cdot d_3$ and $A_{12} = d_2' \cdot d_3$. The entry A_{22}, that cannot be identified in terms of y, d_2, and d_3, is denoted by γ in the statement. □

We now prove the liminf inequality inTheorem 2.2.

Proof of Theorem 2.2–(i), liminf inequality. We may suppose that $\liminf_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon) < \infty$, since otherwise there is nothing to prove. Then, by passing to a subsequence, we may suppose that $\sup_\varepsilon J_\varepsilon(y_\varepsilon) < \infty$. By Lemma 2.1 we have that $A_{y_\varepsilon,\varepsilon} \rightharpoonup A$ in $L^2(S;\mathbb{R}^{2\times 2}_{\text{sym}})$, where

$$A = \begin{pmatrix} d_1' \cdot d_3 & d_2' \cdot d_3 \\ d_2' \cdot d_3 & \gamma \end{pmatrix}$$

with $\gamma \in L^2(S)$. In the rest of the proof, to simplify the notation, we set $A^\varepsilon := A_{y_\varepsilon,\varepsilon}$. We note that $|A^\varepsilon|^2 = (\operatorname{tr} A^\varepsilon)^2$, since $|B|^2 + 2\det B = (\operatorname{tr} B)^2$ for every $B \in \mathbb{R}^{2\times 2}_{\text{sym}}$ and $\det A_\varepsilon = 0$, and also that

$$(\operatorname{tr} A^\varepsilon)^2 = (A^\varepsilon_{11} - A^\varepsilon_{22})^2 + 4A^\varepsilon_{11}A^\varepsilon_{22} = (A^\varepsilon_{11} - A^\varepsilon_{22})^2 + 4(A^\varepsilon_{12})^2.$$

Let $S^+ = S \cap \{\det A \geq 0\}$ and $S^- = S \cap \{\det A < 0\}$, then

$$\liminf_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon) = \liminf_{\varepsilon\to 0} \int_S (\operatorname{tr} A^\varepsilon)^2\,dx = \liminf_{\varepsilon\to 0} \Big\{ \int_{S^+} (\operatorname{tr} A^\varepsilon)^2\,dx + \int_{S^-} (A^\varepsilon_{11} - A^\varepsilon_{22})^2 + 4(A^\varepsilon_{12})^2\,dx \Big\}$$
$$\geq \int_{S^+} (\operatorname{tr} A)^2\,dx + \int_{S^-} (A_{11} - A_{22})^2 + 4(A_{12})^2\,dx,$$

where in the last inequality we used the lower semicontinuity of convex functionals with respect to L^2-weak convergence. Noticing again that $(\operatorname{tr} A)^2 = |A|^2 + 2\det A$ and that

$$(A_{11} - A_{22})^2 + 4(A_{12})^2 = (A_{11})^2 + (A_{22})^2 + 2(A_{12})^2 - 2(A_{11}A_{22} - (A_{12})^2) = |A|^2 - 2\det A,$$

we deduce that

$$\liminf_{\varepsilon\to 0} J_\varepsilon(y_\varepsilon) \geq \int_{S^+} |A|^2 + 2\det A\,dx + \int_{S^-} |A|^2 - 2\det A\,dx = \int_S |A|^2 + 2|\det A|\,dx$$
$$\geq \int_S \overline{Q}(d_1' \cdot d_3, d_2' \cdot d_3)\,dx = J(d_1, d_2, d_3),$$

where the last inequality follows from the definitions of A and of $\overline{Q}$. □

The following lemma plays a crucial role in the construction of the recovery sequence of Theorem 2.2.

Lemma 3.1. *Let $M \in L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$. There exists a sequence $(M_n) \subset L^2\left(I;\mathbb{R}^{2\times 2}_{\rm sym}\right)$ such that $\det M_n = 0$ for every n, $M_n \rightharpoonup M$ in $L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$, and*

$$\int_I |M_n|^2\,dx \to \int_I |M|^2 + 2|\det M|\,dx.$$

Proof. If $\det M = 0$, then we can choose all M_n to be identically equal to M. Thus, hereafter we assume $\det M \neq 0$. We subdivide the proof into three steps.

Step 1: assume M is constant. Further assume, for the moment, that there exist $\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0\}$ such that $M = \operatorname{diag}(\lambda_1, \lambda_2) = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2$. Let

$$\theta = \frac{|\lambda_1|}{|\lambda_1| + |\lambda_2|} \in (0,1),$$

so that

$$\frac{\lambda_1^2}{\theta} + \frac{\lambda_2^2}{1-\theta} = \lambda_1^2 + \lambda_2^2 + 2|\lambda_1\lambda_2| = |M|^2 + 2|\det M|. \tag{3.3}$$

Let $\chi : \mathbb{R} \to \{0,1\}$ denote the 1-periodic extension of the characteristic function of the interval $(0,\theta)$. Define $M_n : I \to \mathbb{R}^{2\times 2}_{\rm sym}$ by setting

$$M_n(x) = \chi(nx_1)\frac{\lambda_1}{\theta} e_1 \otimes e_1 + (1 - \chi(nx_1))\frac{\lambda_2}{1-\theta}\; e_2 \otimes e_2.$$

Clearly $\det M_n = 0$ for every n and M_n converges weakly* in $L^\infty(I;\mathbb{R}^{2\times 2}_{\rm sym})$ to the constant matrix M, since $\chi(n\,\cdot) \rightharpoonup \theta$ weakly* in $L^\infty(I)$.

On the other hand, using (3.3), we compute

$$\int_I |M_n|^2\,dx = \int_I \theta \cdot \frac{\lambda_1^2}{\theta^2} + (1-\theta)\cdot\frac{\lambda_2^2}{(1-\theta)^2}\,dx = \int_I |M|^2 + 2|\det M|\,dx. \tag{3.4}$$

This concludes the proof of the step in the case that M is diagonal.

For an arbitrary constant matrix $M \in \mathbb{R}^{2\times 2}_{\rm sym}$ with $\det M \neq 0$ there exists $Q \in O(2)$ such that $Q^T M Q$ is diagonal. Applying the construction above to $Q^T M Q$ we obtain a sequence $\widetilde{M}_n$ with the properties stated in the lemma for $Q^T M Q$, and setting $M_n = Q\widetilde{M}_n Q^T$ we find the desired sequence.

Step 2: assume M is piecewise constant. It suffices to apply Step 1 on each interval on which M is constant.

Step 3: assume $M \in L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$. It suffices to approximate M by a sequence (M_k) of piecewise constant matrices in the strong topology of $L^2(I;\mathbb{R}^{2\times 2}_{\rm sym})$. For each M_k we apply Step 2 and obtain a sequence $(M_{k,n})$ with the required properties and such that $\|M_{k,n}\|_{L^2}^2 \leq 2\|M_k\|_{L^2}^2$ for every k and n, in view of (3.4). This allows one to apply a diagonal argument and conclude the proof. $\square$

Proof of Theorem 2.2–(ii), recovery sequence. Let $(d_1, d_2, d_3) \in \mathcal{A}$ and let $y \in W^{2,2}(I;\mathbb{R}^3)$ be such that $y' = d_1$ a.e. in I. We set $R := (y' \,|\, d_2 \,|\, d_3) \in SO(3)$ a.e. in I and

$$M := \begin{pmatrix} y'' \cdot d_3 & d_2' \cdot d_3 \\ d_2' \cdot d_3 & \gamma \end{pmatrix},$$

where $\gamma \in L^2(I)$ is such that

$$\overline{Q}(y'' \cdot d_3, d_2' \cdot d_3) = |M|^2 + 2|\det M| \qquad \text{a.e. in } I.$$

Such a γ can indeed be chosen measurable. Moreover, $\gamma \in L^2(I)$ because by minimality, comparing M to the same matrix with 0 instead of γ, we have

$$\gamma^2 \leq |M|^2 + 2|\det M| \leq M_{11}^2 + 4M_{12}^2 \qquad \text{a.e. in } I,$$

and the right-hand side is in $L^1(I)$.

By Lemma 3.1 there exist $M_n \in L^2(I;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ with $\det M_n = 0$ for every n and such that $M_n \rightharpoonup M$ weakly in $L^2(I;\mathbb{R}^{2\times2}_{\mathrm{sym}})$ and

$$\mathcal{F}(M_n) := \int_I |M_n|^2\, dx_1 \to \overline{\mathcal{F}}(M) := \int_I |M|^2 + 2|\det M|\, dx_1,$$

as $n \to \infty$. Denote by $\lambda_n \in L^2(I)$ the trace of M_n. Since M_n is symmetric with $\det M_n = 0$, there exists $\beta_n(x_1) \in (-\pi/2, \pi/2]$ such that

$$M_n = \begin{pmatrix} \cos\beta_n & -\sin\beta_n \\ \sin\beta_n & \cos\beta_n \end{pmatrix} \begin{pmatrix} \lambda_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta_n & \sin\beta_n \\ -\sin\beta_n & \cos\beta_n \end{pmatrix},$$

and β_n is uniquely determined if $\lambda_n \neq 0$. When $\lambda_n(x_1) = 0$, we set $\beta_n(x_1) = 0$. After truncating λ_n in modulus by n, we may assume without loss of generality that $\lambda_n \in L^\infty(I)$, while M_n still enjoys the same properties as before. By mollification, we can find $\lambda_{n,k} \in C^\infty(\bar{I})$ and $\beta_{n,k} \in C^\infty(\bar{I})$ such that

- $\beta_{n,k}(x_1) \in (-\pi/2, \pi/2)$ for every $x_1 \in \bar{I}$ and every n, k (this condition is achieved after possibly multiplying each mollification by a constant smaller than 1);
- $\lambda_{n,k} \to \lambda_n$ in $L^p(I)$ for every $p < +\infty$, as $k \to \infty$;
- $\beta_{n,k} \to \beta_n$ in $L^p(I)$ for every $p < +\infty$, as $k \to \infty$.

Set

$$M_{n,k} := \begin{pmatrix} \cos\beta_{n,k} & -\sin\beta_{n,k} \\ \sin\beta_{n,k} & \cos\beta_{n,k} \end{pmatrix} \begin{pmatrix} \lambda_{n,k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta_{n,k} & \sin\beta_{n,k} \\ -\sin\beta_{n,k} & \cos\beta_{n,k} \end{pmatrix}.$$

Then, $\det M_{n,k} = 0$ for every n, k and $M_{n,k} \to M_n$ in $L^2(I;\mathbb{R}^{2\times2}_{\mathrm{sym}})$, as $k \to \infty$.

Thus, by a diagonal argument, we may assume that there exist $\lambda^j \in C^\infty(\bar{I})$ and $\beta^j \in C^\infty(\bar{I})$ such that $|\beta^j| < \pi/2$ on $\bar{I}$, and with

$$\begin{aligned} M^j &:= \begin{pmatrix} \cos\beta^j & -\sin\beta^j \\ \sin\beta^j & \cos\beta^j \end{pmatrix} \begin{pmatrix} \lambda^j & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\beta^j & \sin\beta^j \\ -\sin\beta^j & \cos\beta^j \end{pmatrix} \\ &= \lambda^j \begin{pmatrix} \cos^2\beta^j & \sin\beta^j\cos\beta^j \\ \sin\beta^j\cos\beta^j & \sin^2\beta^j \end{pmatrix} \end{aligned}$$

we have that $\det M^j = 0$ for every j, $M^j \rightharpoonup M$ in $L^2(I;\mathbb{R}^{2\times2}_{\mathrm{sym}})$, and $\mathcal{F}(M_j) \to \overline{\mathcal{F}}(M)$, as $j \to \infty$.

Set $\vartheta^j = \frac{\pi}{2} + \beta^j$ and define

$$\widetilde{b}^j(\xi_1) := \cos\vartheta^j(\xi_1)e_1 + \sin\vartheta^j(\xi_1)e_2$$

and $\Phi^j(\xi_1, \xi_2) := \xi_1 e_1 + \xi_2 \widetilde{b}^j(\xi_1)$. Since $\beta^j \in (-\pi/2, \pi/2)$, we can argue as in [7] to see that, for ε small enough, $(\Phi^j)^{-1} : S_\varepsilon \to \mathbb{R}^2$ is well defined.

Let $R^j : I \to SO(3)$ be the solution of the ODE

$$(R^j)' = R^j \begin{pmatrix} 0 & 0 & -M^j_{11} \\ 0 & 0 & -M^j_{12} \\ M^j_{11} & M^j_{12} & 0 \end{pmatrix} \tag{3.5}$$

with $R^j(0) = R(0) = (y'(0)\,|\,d_2(0)\,|\,d_3(0))$. Since M^j is smooth, so is R^j and, since $R(0) \in SO(3)$, R^j attains values in $SO(3)$. We set

$$d^j_k(t) = R^j(t)e_k \text{ for } k = 1,2,3, \qquad y^j(t) = y(0) + \int_0^t d^j_1(s)\,ds.$$

Then $y^j \rightharpoonup y$ weakly in $W^{2,2}(I;\mathbb{R}^3)$, see, for instance, the proof of Lemma 4.2 of [5]. It follows from (3.5) that

$$\begin{aligned} (d^j_1)' \cdot d^j_2 &= 0, \\ (d^j_2)' \cdot d^j_3 &= M^j_{12} = \lambda^j \sin\beta^j \cos\beta^j = -\lambda^j \sin\vartheta^j \cos\vartheta^j, \\ (d^j_1)' \cdot d^j_3 &= M^j_{11} = \lambda^j \cos^2\beta^j = \lambda^j \sin^2\vartheta^j. \end{aligned} \tag{3.6}$$

Define

$$\begin{aligned} b^j(\xi_1) &= \cos\vartheta^j(\xi_1) d^j_1(\xi_1) + \sin\vartheta^j(\xi_1) d^j_2(\xi_1), \\ v^j(\xi_1,\xi_2) &= y^j(\xi_1) + \xi_2 b^j(\xi_1), \\ u^j(x_1,x_2) &= v^j\Big((\Phi^j)^{-1}(x_1,x_2)\Big). \end{aligned}$$

Then

$$\nabla v^j = ((y^j)' + \xi_2 (b^j)' | b^j), \quad \nabla\Phi^j = \begin{pmatrix} 1 - \xi_2(\vartheta^j)'\sin\vartheta^j & \cos\vartheta^j \\ \xi_2(\vartheta^j)'\cos\vartheta^j & \sin\vartheta^j \end{pmatrix}, \quad (\nabla u^j)(\Phi^j)\nabla\Phi^j = \nabla v^j.$$

By means of (3.6) we can check that

$$(b^j)' \cdot d^j_3 = 0, \quad |(b^j)'| = |(\vartheta^j)'|.$$

With these identities we can show that $(\nabla v^j)^T \nabla v^j = (\nabla\Phi^j)^T\nabla\Phi^j$, that is, $(\nabla u^j)^T\nabla u^j = I$. Clearly $u^j(x_1,0) = y^j(x_1)$ and $\partial_1 u^j(x_1,0) = (y^j)'(x_1) = d^j_1(x_1)$. Moreover, since $\partial_2\Phi^j = \widetilde{b}^j$, we have

$$\nabla u^j(\Phi^j)\widetilde{b}^j = \partial_2 v^j = b^j. \tag{3.7}$$

From this one readily deduces that

$$\nabla u^j(\cdot,0) = (d^j_1\,|\,d^j_2). \tag{3.8}$$

Taking ξ_2-derivatives on both sides of (3.7), we see that $\nabla^2 u(\Phi^j)(\widetilde{b}^j,\widetilde{b}^j) = 0$, and therefore

$$A_{u^j}(\Phi^j)(\widetilde{b}^j,\widetilde{b}^j) = 0. \tag{3.9}$$

Taking derivatives in (3.8), we see that

$$(A_{u^j}(\cdot,0))_{11} = d^j_3 \cdot \partial_1\partial_1 u^j(\cdot,0) = d^j_3 \cdot (y^j)'' = M^j_{11}$$

and similarly that $(A_{u^j}(\cdot,0))_{12} = M^j_{12}$. Combining these with (3.9) and with the fact that $M^j\widetilde{b}^j = 0$, and recalling that $\widetilde{b}^j \cdot e_2 \neq 0$, we conclude that $A_{u^j}(x_1,0) = M^j(x_1)$, because both A_{u^j} and M^j are symmetric and have zero determinant.

For ε small enough, the maps $y^j_\varepsilon : S \to \mathbb{R}^3$ given by $y^j_\varepsilon(x_1,x_2) = u^j(x_1,\varepsilon x_2)$ are well-defined scaled C^2 isometries of S, such that

$$\nabla_\varepsilon y^j_\varepsilon = (\nabla u^j)(T_\varepsilon) \to \nabla u^j(\cdot,0) = (d^j_1\,|\,d^j_2) \qquad \text{strongly in } W^{1,2}(S;\mathbb{R}^{3\times 2}),$$

as $\varepsilon \to 0$; here $T_\varepsilon x = (x_1,\varepsilon x_2)$. Set $A^j_\varepsilon := A_{y^j_\varepsilon,\varepsilon}$. Then since $A_{u^j}(x_1,0) = M^j(x_1)$, we see that $A^j_\varepsilon \to M^j$ strongly in $L^2(S;\mathbb{R}^{2\times 2}_{\text{sym}})$, as $\varepsilon \to 0$. Hence,

$$\lim_{\varepsilon\to 0} J_\varepsilon(y^j_\varepsilon) = \lim_{\varepsilon\to 0}\int_S |A^j_\varepsilon|^2\,dx = \int_S |M^j|^2\,dx = \mathcal{F}(M^j).$$

Therefore, taking diagonal sequences we obtain the desired maps. □

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