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The homogeneous Hénon-Lane-Emden system

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Abstract. We use variational methods to study the existence of a principal eigenvalue for the homogeneous Hénon-Lane-Emden system on a bounded domain. Then we provide a detailed insight into the problem in the linear case.

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1. Introduction

The Hénon-Lane-Emden system

$$\begin{cases} -\Delta u = |x|^a|v|^{q-2}v & \text{in } \Omega \\ -\Delta v = |x|^b|u|^{p-2}u & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

includes the second and fourth order Lane-Emden equations and the Hénon equation in astrophysics. Here Ω is a domain in $\mathbb{R}^n$ containing the origin, $p, q \in (1, \infty)$, and the weights are locally integrable, that is, $a, b > -n$.

Since the celebrated papers [15] by P.L. Lions and [17] by Mitidieri, where $a = b = 0$ and $\Omega = \mathbb{R}^n$ are assumed, large efforts have been made in investigating (1.1) and related problems. It is difficult to give a complete list of references on this topic. We limit ourselves to cite [2, 3, 4, 5, 8, 11, 12, 13, 18, 22, 25, 27, 28, 29, 30] and the references therein.

Most of the above mentioned papers require $n \geq 3$, deal with the so-called *anticoercive case* $(p-1)(q-1) > 1$, and underline the role of the

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"*critical hyperbola*"

$$\frac{a+n}{q} + \frac{b+n}{p} = n-2$$

in existence and nonexistence phenomena, as it separates the "subcritical case"

$$\frac{a+n}{q} + \frac{b+n}{p} > n-2\,, \tag{1.2}$$

from the supercritical one.

Formally letting $q \searrow p' = \frac{p}{p-1}$ in (1.1), and taking the homogeneities involved into account, one gets in the limit the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda_1 |x|^a |v|^{p'-2} v & \text{in } \Omega \\ -\Delta v = \lambda_2 |x|^b |u|^{p-2} u & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega\,, \end{cases} \tag{$\mathcal{P}$}$$

and (1.2) reduces to

$$\frac{a}{p'} + \frac{b}{p} + 2 > 0\,. \tag{1.3}$$

In the present paper we focus our attention on Problem ($\mathcal{P}$). We emphasize the fact that we include the lower dimensional cases $n = 1, 2$, that actually present some remarkable peculiarities.

Nonexistence results have been obtained in [1], see also the recent papers [8, 10] for the case $\Omega = \mathbb{R}^n$. Montenegro [20] used degree theory to face Problem ($\mathcal{P}$) in a more general setting that includes non-self-adjoint elliptic operators. We adopt a variational approach that allows us to weaken the integrability assumptions on the coefficients from Montenegro's $L^n(\Omega)$ to $L^1(\Omega)$.

We look for *finite energy solutions* and for a *principal eigenvalue* to ($\mathcal{P}$), according to the next definitions.

Definition 1.1. The pair (u, v) is a finite-energy solution to ($\mathcal{P}$) if:

- $u, v \in W^{2,1}(\Omega) \cap W_0^{1,1}(\Omega)$;
- $u \in L^p(\Omega, |x|^b dx)$, $v \in L^{p'}(\Omega, |x|^a dx)$, that is,

$$\int_\Omega |x|^b\, |u|^p\ dx < \infty\ , \quad \int_\Omega |x|^a\, |v|^{p'}\ dx < \infty\,; \tag{1.4}$$

- u, v are weak solutions to the elliptic equations in ($\mathcal{P}$). That is,

$$\int_\Omega \nabla u \cdot \nabla \varphi\, dx = \lambda_1 \int_\Omega |x|^a |v|^{p'-2} v\varphi\, dx, \quad \int_\Omega \nabla v \cdot \nabla \varphi\, dx = \lambda_2 \int_\Omega |x|^a |u|^{p-2} u\varphi\, dx$$

for any test function $\varphi \in C_c^\infty(\Omega)$.

Definition 1.2. A real number μ is a principal eigenvalue for ($\mathcal{P}$) if for any pair of real numbers (λ_1, λ_2) satisfying

$$|\lambda_1|^{p-1}\, \lambda_1\, |\lambda_2|^{p'-1}\, \lambda_2 = \mu^{p'}, \tag{1.5}$$

Problem ($\mathcal{P}$) has a finite-energy solution (u, v) such that $u, v > 0$ in Ω.

In Section 3 we prove the following result.

Theorem 1.3. *Let $a, b > -n$ and let $\Omega \ni 0$ be a bounded and smooth domain in $\mathbb{R}^n$. If (1.3) holds, then Problem $(\mathcal{P})$ has a positive principal eigenvalue.*

Notice that (1.3) is automatically satisfied if $n = 1, 2$ and $a, b > -n$. If $n \geq 3$ then assumption (1.3) can not be improved, see [7].

Our approach is based on the formal equivalence, already noticed for instance by Wang [31] and Calanchi-Ruf [6] in the anticoercive case, between $(\mathcal{P})$ and the fourth order eigenvalue problem

$$\begin{cases} \Delta\left(|x|^{-a(p-1)}\,|\Delta u|^{p-2}\,\Delta u\right) = \mu\,|x|^b\,|u|^{p-2}\,u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where μ, λ_1 and λ_2 satisfy (1.5). We will look for nontrivial solutions to (1.6) as constrained critical points for the energy

$$\mathcal{E}(u) = \int_\Omega |x|^{-a(p-1)}\,|\Delta u|^p\,dx \ \text{ on the constraint } M = \left\{ \int_\Omega |x|^b |u|^p\,dx = 1 \right\},$$

in a suitably defined space $W^{2,p}_N(\Omega, |x|^{-a(p-1)}\,dx)$ of functions such that $u = 0$ on $\partial\Omega$.

In spite of the apparent plainness of this program, its rigorous implementation needs a good understanding of some non trivial facts.

First of all one has to prove appropriate integral inequalities for smooth functions vanishing at the boundary of Ω. This will be done in Lemma 2.4. If $n \geq 3$ we take advantage of the weighted Rellich-type inequality in [19] and [21, Lemma 2.14]. The lower dimensional cases $n = 1, 2$ require an *ad hoc* argument and a preliminary result, that can be found in the appendix.

Secondly, one needs to detect the "right" function space. The weighted space $W^{2,p}_N(\Omega, |x|^{-a(p-1)}\,dx)$ has to be "small enough" to be compactly embedded in $L^p(\Omega; |x|^b dx)$. But the equivalence between weak solutions to (1.6) and finite-energy solutions to $(\mathcal{P})$ only holds if $W^{2,p}_N(\Omega, |x|^{-a(p-1)}\,dx)$ is "large enough".

The convenient definition of $W^{2,p}_N(\Omega, |x|^{-a(p-1)}\,dx)$ and details are given in Section 2. The above mentioned equivalence and Theorem 1.3 are proved in Section 3.

The last part of the paper is focused on the linear case $p = 2$, so that (1.3) becomes

$$a + b + 4 > 0\,. \tag{1.7}$$

In Section 4 we prove that the linear system

$$\begin{cases} -\Delta u = \lambda_1 |x|^a v & \text{in } \Omega \\ -\Delta v = \lambda_2 |x|^b u & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.8}$$

has a unique and simple principal eigenvalue $\mu_1 > 0$, and a discrete spectrum $\{\mu_k\}_{k\in\mathbb{N}}$. More precisely, the following facts hold.

Theorem 1.4. *Let $a, b > -n$ and let Ω be a bounded and smooth domain in $\mathbb{R}^n$. If $n \geq 3$ assume also that* (1.7) *holds.*

(a) *There exists an increasing, unbounded sequence of eigenvalues $\{\mu_k\}_{k\in\mathbb{N}}$ such that Problem* (1.8) *has a nontrivial and finite-energy solution (u, v) if and only if $\lambda_1\lambda_2 = \mu_k$ for some integer $k \geq 1$.*

(b) *The first eigenvalue μ_1 is the unique principal eigenvalue. In addition μ_1 is simple, that is, if (u, v) and $(\tilde{u}, \tilde{v})$ solve* (1.8) *and $\lambda_1\lambda_2 = \mu_1$, then $\tilde{u} = \alpha u$ and $\tilde{v} = \beta v$ for some $\alpha, \beta \in \mathbb{R}$.*

2. The functional setting

In this section we introduce and study certain second order weighted Sobolev spaces with Navier boundary conditions that are suitable for studying (1.6) via variational methods.

To simplify notation we set $s = a(p-1)$. Thus, from now on we assume that s, b are given exponents such that

$$s > n - np\ , \quad b > -n$$

even if not explicitly stated. In addition, $\Omega \subset \mathbb{R}^n$ will always denote a bounded and smooth domain. We denote by c any universal positive constant.

We introduce the function space

$$C^2_N(\overline{\Omega}) := \{u \in C^2(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}.$$

Let $W^{2,p}_N(\Omega, |x|^{-s}\, dx)$ be the reflexive Banach space defined as the completion of the set

$$D_0 := \{u \in C^2_N(\overline{\Omega}) \mid \Delta u \equiv 0 \text{ on a neighborhood of the origin}\}\,,$$

with respect to the uniformly convex norm

$$\|u\|_s \equiv \|u\|_{p,s} := \left(\int_\Omega |x|^{-s}\, |\Delta u|^p\ dx\right)^{\frac{1}{p}}.$$

We begin to study the spaces $W^{2,p}_N(\Omega, |x|^{-s}\, dx)$ by pointing out some embedding results. Firstly, notice that the boundedness of the domain Ω implies

$$W^{2,p}_N(\Omega, |x|^{-s}\ dx) \hookrightarrow W^{2,p}_N(\Omega, |x|^{-s_0}\, dx) \quad \text{ if } s_0 \leq s. \tag{2.1}$$

Lemma 2.1. *Assume $s > n - np$. Then*

$$W^{2,p}_N(\Omega, |x|^{-s}\, dx) \hookrightarrow W^{2,\tau}(\Omega) \cap W^{1,\tau}_0(\Omega),$$

where $\tau < \frac{np}{n-s} < p$ if $s < 0$, or $\tau = p$ otherwise.

Proof. If $s \geq 0$ the conclusion is immediate. Assume $s < 0$. For any $u \in D_0$ and $\tau \in [1, np/(n-s))$ we use elliptic regularity estimates, see for instance [14, Lemma 9.17], to get

$$\|u\|^\tau_{W^{2,\tau}(\Omega)} \leq c\int_\Omega |\Delta u|^\tau\ dx \leq c\left(\int_\Omega |x|^{-s}\, |\Delta u|^p\ dx\right)^{\frac{\tau}{p}} \left(\int_\Omega |x|^{\frac{s\tau}{p-\tau}}\ dx\right)^{\frac{p-\tau}{p}}.$$

The last integral is finite as $s > n - np$, and the lemma is proved. □

The next lemma will be used in Section 3 to rigorously prove the equivalence between the second order system ($\mathcal{P}$) and the fourth order equation (1.6).

Lemma 2.2. *If $s > n - np$, then $u \in W^{2,p}_N(\Omega, |x|^{-s}\,dx)$ if and only if*

$$u \in W^{2,1} \cap W^{1,1}_0(\Omega) \quad \textit{and} \quad -\Delta u \in L^p(\Omega, |x|^{-s}dx). \tag{2.2}$$

Proof. Clearly, any $u \in W^{2,p}_N(\Omega, |x|^{-s}\,dx)$ satisfies (2.2) by Lemma 2.1.

Conversely, fix u satisfying (2.2). Assume in addition that $-\Delta u = 0$ almost everywhere on a ball B_r about 0, so that $-\Delta u \in L^p(\Omega)$. Hence, $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ by elliptic regularity theory. Extend u to a function u in $W^{2,p}(\mathbb{R}^n)$ with compact support and take a sequence of mollifiers $\{\rho_k\}_{k\in\mathbb{N}}$. Since, for k large enough, $-\Delta(\rho_k * u) \equiv 0$ on $B_{r/2}$ and $\rho_k * u \to u$ in $W^{2,p}(\Omega)$, then $-\Delta(\rho_k * u) \to -\Delta u$ in $L^p(\Omega, |x|^{-s}dx)$. Let u_k be the solution to

$$\begin{cases} -\Delta u_k = -\Delta(\rho_k * u) & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

It turns out that $u_k \in D_0 \cap W^{2,p}_N(\Omega)$, as u_k is smooth up to the boundary of Ω by regularity theory, and $-\Delta u_k \equiv 0$ in $B_{r/2}$. In addition, $u_k \to u$ in $W^{2,p}(\Omega)$ and $-\Delta u_k \to -\Delta u$ in $L^p(\Omega, |x|^{-s}dx)$, that is sufficient to conclude that $u \in W^{2,p}_N(\Omega, |x|^{-s}\,dx)$.

For a general u satisfying (2.2) let u_k be the unique solution to

$$\begin{cases} -\Delta u_k = \chi_{\Omega_k}(-\Delta u) & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega_k := \Omega \setminus \overline{B}_{\varepsilon_k}$ and $\varepsilon_k \to 0$. Then $u_k \in W^{2,p} \cap W^{1,p}_0(\Omega)$ and $u_k \in W^{2,p}_N(\Omega, |x|^{-s}\,dx)$ by the first part of the proof. Clearly, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{2,p}_N(\Omega, |x|^{-s}\,dx)$, and we can assume that $u_k \to \bar{u}$ weakly in $W^{2,p}_N(\Omega, |x|^{-s}\,dx)$. On the other hand, the sequence $-\Delta u_k$ converges to $-\Delta u$ in $L^p(\Omega, |x|^{-s}\,dx)$ by Lebesgue's theorem. Thus $\bar{u} = u$, that is, $u \in W^{2,p}_N(\Omega, |x|^{-s}\,dx)$. $\square$

The next corollary is an immediate consequence of Lemma 2.2.

Corollary 2.3. *Assume $s > n - np$. For any $f \in L^p(\Omega, |x|^{-s}dx)$, the unique solution u to*

$$\begin{cases} -\Delta u = f & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

belongs to $W^{2,p}_N(\Omega, |x|^{-s}\,dx)$.

Next we deal with embeddings in weighted L^p spaces.

Lemma 2.4. *If $s + b + 2p \geq 0$, then*

$$\Lambda(s,b) := \inf_{\substack{u \in W^{2,p}_N(\Omega, |x|^{-s}dx) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |x|^{-s} |\Delta u|^p \, dx}{\displaystyle\int_\Omega |x|^b |u|^p \, dx} > 0.$$

Proof. First of all, notice that $L^p(\Omega, |x|^{b_0}\ dx) \hookrightarrow L^p(\Omega, |x|^b\ dx)$ if $b_0 \le b$, that together with (2.1) implies

$$\Lambda(s, b) \ge c\Lambda(s_0, b_0) \quad \text{if } s_0 \le s \text{ and } b_0 \le b. \tag{2.3}$$

We start with the lowest dimensions $n = 1, 2$. Fix an exponent $s_0 \le s$, such that $n - np < s_0 \le b(p-1)$. Then $\Lambda(s, b) \ge c\Lambda\big(s_0, \frac{s_0}{p-1}\big) > 0$ by (2.3) and Lemma 4.2 in the Appendix.

Now assume $n \ge 3$. In addition, assume first that $s < n - 2p$. By the weighted Rellich inequality in [19] (see also [21, Lemma 2.14]) and using [21, Lemma 2.9], one has that there exists a positive and explicitly known constant $c = c(n, p, s)$, such that

$$c \int_\Omega |x|^{-s-2p} |u|^p\ dx \le \int_\Omega |x|^{-s} |\Delta u|^p\ dx \quad \text{for any } u \in C^2_N(\overline{\Omega}), \tag{2.4}$$

that in particular gives $c = \Lambda(s, -s-2p) > 0$. Thus

$$\Lambda(s, b) \ge c\Lambda(s, -s-2p) > 0$$

by (2.3). Finally, if $s \ge n - 2p$, we fix a parameter s_0 such that

$$\max\{n - np, -2p - b\} < s_0 < n - 2p \le s,$$

that is possible as $b > -n$ and $n \ge 3$. Then (2.3) and (2.4) with s replaced by s_0 give $\Lambda(s, b) \ge c\Lambda(s_0, -s_0 - 2p) > 0$, and the lemma is proved. □

Remark 2.5. If Ω contains the origin and $s + b + 2p < 0$, then $\Lambda(s, b) = 0$. Indeed, fix a nontrivial $\psi \in C^\infty_c(B_1 \setminus \{0\})$. For k large enough the function $\psi_k(x) = \psi(kx)$ belongs to D_0. Thus

$$\Lambda(s, b) \le \frac{\displaystyle\int_\Omega |x|^{-s} |\Delta\psi_k|^p\ dx}{\displaystyle\int_\Omega |x|^b |\psi_k|^p\ dx} = ck^{s+2p+b} = o(1) \quad \text{as } k \to \infty.$$

Remark 2.6. If $n - np < s < n - 2p$, then $C^2_N(\overline{\Omega}) \subset W^{2,p}_N(\Omega, |x|^{-s}\, dx)$ and the space

$$C^2_N(\overline{\Omega} \setminus \{0\}) := \{u \in C^2_N(\overline{\Omega}) \mid u \equiv 0 \text{ on a neighborhood of the origin}\}$$

is dense in $W^{2,p}_N(\Omega, |x|^{-s}\, dx)$, see Lemma 2.14 in [21].

Remark 2.7. By Lemma 2.2, the set D_0 is dense in the standard Sobolev space $W^{2,p}_N(\Omega) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. The smaller set $C^2_N(\overline{\Omega} \setminus \{0\})$ is dense in $W^{2,p}_N(\Omega)$ if $n > 2p$, compare with Remark 2.6.

The next compactness result is a crucial point for studying the eigenvalue problem (1.6).

Lemma 2.8. *If $s + b + 2p > 0$ then $W^{2,p}_N(\Omega, |x|^{-s}\, dx)$ is compactly embedded into $L^p(\Omega, |x|^b\, dx)$.*

Proof. It suffices to show that any sequence $\{u_k\}_{k\in\mathbb{N}}$ that converges weakly to the null function in $W^{2,p}_N(\Omega,|x|^{-s}\,dx)$ is compact in $L^p(\Omega,|x|^b\,dx)$. Fix such a sequence, and take $\varepsilon>0$ small. Since clearly $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(\Omega\setminus\overline{B}_\varepsilon)$, then $|x|^b|u_k|^p\to 0$ in $L^1(\Omega\setminus\overline{B}_\varepsilon)$ by Rellich theorem. Therefore, for any $b_0\in(-n,b)$ we have that

$$\int_\Omega |x|^b|u_k|^p\,dx=\int_{B_\varepsilon}|x|^b|u_k|^p\,dx+o(1)\le\varepsilon^{b-b_0}\int_\Omega|x|^{b_0}|u_k|^p\,dx+o(1)\,.$$

Now, if b_0 is close enough to b, then $s+b_0+2p>0$. Hence

$$\int_\Omega |x|^b|u_k|^p\,dx\le c\varepsilon^{b-b_0}+o(1)$$

by Lemma 2.4. The conclusion follows, as $\varepsilon>0$ was arbitrarily chosen. □

3. Two equivalent problems

In this section we provide the rigorous proof of the equivalence between the eigenvalue problems $(\mathcal{P})$ and (1.6). We start with a preliminary result.

Lemma 3.1. *Assume that $a,b>-n$ and that* (1.3) *holds. For any $f\in L^p(\Omega,|x|^b\,dx)$, the problem*

$$\begin{cases}-\Delta u=|x|^a\,|v|^{p'-2}\,v & (3.1a)\\ -\Delta v=|x|^b\,|f|^{p-2}\,f & (3.1b)\\ u\in W^{2,p}_N(\Omega,|x|^{-a(p-1)}\;dx),\;\; v\in W^{2,p'}_N(\Omega,|x|^{-b(p'-1)}\;dx)\end{cases}$$

admits a unique solution.

Proof. First of all, notice that $a(p-1)>n-np$, $b(p'-1)>n-np'$. Thus the results in Section 2 apply to the spaces $W^{2,p}_N(\Omega,|x|^{-a(p-1)}\;dx)$ and $W^{2,p'}_N(\Omega,|x|^{-b(p'-1)}\;dx)$.

Since $|x|^b\,|f|^{p-2}\,f\in L^{p'}(\Omega,|x|^{-b(p'-1)}\;dx)$, then Corollary 2.3 guarantees (3.1b) has a unique solution $v\in W^{2,p'}_N(\Omega,|x|^{-b(p'-1)}\;dx)$. The embedding Lemma 2.4 gives that $|x|^a\,|v|^{p'-2}\,v\in L^p(\Omega,|x|^{-a(p-1)}\;dx)$. Thus there exists a unique solution $u\in W^{2,p}_N(\Omega,|x|^{-a(p-1)}\;dx)$ to (3.1a), thanks again to Corollary 2.3. □

We are ready to prove the equivalence result we need.

Lemma 3.2. *Assume that $a,b>-n$ and that* (1.3) *holds. Let $\mu,\lambda_1,\lambda_2\in\mathbb{R}$ satisfying* (1.5)*. Then the following statements are equivalent.*

(*a*) *$u\in W^{2,p}_N(\Omega,|x|^{-a(p-1)}\;dx)$ is a weak solution to* (1.6).
(*b*) *The pair $u, v:=-|x|^{-a(p-1)}|\Delta u|^{p-2}\Delta u$ solves the system*

$$\begin{cases}-\Delta u=|x|^a\,|v|^{p'-2}\,v & (3.2a)\\ -\Delta v=\mu\,|x|^b\,|u|^{p-2}\,u, & (3.2b)\\ u\in W^{2,p}_N(\Omega,|x|^{-a(p-1)}\;dx)\;,\;\; v\in W^{2,p'}_N(\Omega,|x|^{-b(p'-1)}\;dx).\end{cases}$$

(c) *The pair* $u, v := -|x|^{-a(p-1)}|\Delta u|^{p-2}\Delta u$ *is a finite-energy solution to* $(\mathcal{P})$, *in the sense of Definition* 1.1.

Proof. If $u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx)$, then $u \in L^p(\Omega, |x|^b dx)$ by Lemma 2.4. Thus we can apply Lemma 3.1 to find a unique pair u_0, v_0 such that

$$\begin{cases} -\Delta u_0 = |x|^a\, |v_0|^{p'-2}\, v_0 \\ -\Delta v_0 = \mu\, |x|^b\, |u|^{p-2}\, u, \\ u_0 \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx), v_0 \in W^{2,p'}(\Omega, |x|^{-b(p'-1)}\ dx). \end{cases}$$

Notice that $v_0 = |x|^{-a(p-1)}|\Delta u_0|^{p-2}(-\Delta u_0)$ almost everywhere in Ω. Therefore, if u solves (1.6), then for any $\varphi \in D_0$ it holds that

$$\begin{aligned} \int_\Omega |x|^{-a(p-1)}\, |\Delta u|^{p-2}\, \Delta u \Delta\varphi\, dx &= \mu \int_\Omega |x|^b\, |u|^{p-2}\, u\varphi\, dx \\ &= \int_\Omega (-\Delta v_0)\varphi\, dx = \int_\Omega v_0(-\Delta\varphi)\, dx \\ &= \int_\Omega |x|^{-a(p-1)}\, |\Delta u_0|^{p-2}\, \Delta u_0 \Delta\varphi\, dx, \end{aligned}$$

that readily gives that $u = u_0$, since $u, u_0 \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx)$ and D_0 is dense in $W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx)$. Hence also $v = v_0$, the pair u, v solves (3.2), and the first implication is proved.

The equivalence between (b) and (c) is immediate, thanks to Lemma 2.1 and Corollary 2.3. It remains to be shown that (b) implies (a). If (u, v) solves (3.2), then for every $\varphi \in D_0$ it holds that

$$\begin{aligned} \mu \int_\Omega |x|^b\, |u|^{p-2}\, u\varphi\, dx &= \int_\Omega v(-\Delta\varphi)\, dx \\ &= \int_\Omega |x|^{-a(p-1)}\, |\Delta u|^{p-2}\, \Delta u \Delta\varphi\, dx, \end{aligned}$$

that is, u solves (1.6). □

Lemma 3.2 shows that finite energy solutions to $(\mathcal{P})$ are the stationary points of the functional

$$u \mapsto \int_\Omega |x|^{-a(p-1)}\, |\Delta u|^p\, dx$$

on the constraint

$$M = \left\{ u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx) \ \middle|\ \int_\Omega |x|^b |u|^p\, dx = 1 \right\}.$$

If $a, b > -n$ and (1.3) holds, then M is weakly compact by Lemma 2.8. Thus the infimum

$$\mu := \Lambda(a(p-1), b) = \inf_{\substack{u \in M \\ u \neq 0}} \int_\Omega |x|^{-a(p-1)}\, |\Delta u|^p\ dx \tag{3.3}$$

is positive and attained.

Remark 3.3. Let (u, v) be a finite energy solution to $(\mathcal{P})$. Clearly, if $n = 1$ then u, v are continuous on $\overline{\Omega}$. If $n \geq 2$ a standard bootstrap argument can be used to check that $u, v \in W^{2,q}_{\mathrm{loc}}(\Omega \setminus \{0\})$ for any $q \in [1, \infty]$. In particular, u, v are continuous on $\Omega \setminus \{0\}$.

The next lemma deals with minimizers for μ.

Lemma 3.4. *Assume that $a, b > -n$ and that* (1.3) *holds.*
If $u \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx)$ achieves μ, then, up to a change of sign, u is superharmonic and positive on Ω.

Proof. Let $v = |x|^{-a(p-1)}|\Delta u|^{p-2}(-\Delta u)$, so that the pair (u, v) solves (3.2). Use Corollary 2.3 to introduce u_0 via

$$\begin{cases} -\Delta u_0 = |x|^a\, |v|^{p'-1} \\ u_0 \in W^{2,p}_N(\Omega, |x|^{-a(p-1)}\ dx). \end{cases}$$

In particular, u_0 is superharmonic and positive on Ω. Next, put

$$g = |x|^a\, |v|^{p'-2}\, v.$$

Thus u and u_0 solve, for some $\tau \in [1, p)$,

$$\begin{cases} -\Delta u = g \\ u \in W^{2,\tau} \cap W^{1,\tau}_0(\Omega), \end{cases} \qquad \begin{cases} -\Delta u_0 = |g| \\ u_0 \in W^{2,\tau} \cap W^{1,\tau}_0(\Omega). \end{cases}$$

Since $-\Delta(u_0 \pm u) \geq 0$ and $u_0 \pm u = 0$ on the boundary of Ω, then $u_0 \pm u \geq 0$, that is, $u_0 \geq |u|$. On the other hand, $|\Delta u_0| = |g| = |\Delta u|$. Therefore

$$\mu \leq \frac{\displaystyle\int_\Omega |x|^{-a(p-1)}\, |\Delta u_0|^p\ dx}{\displaystyle\int_\Omega |x|^b\, |u_0|^p\ dx} \leq \frac{\displaystyle\int_\Omega |x|^{-a(p-1)}\, |\Delta u|^p\ dx}{\displaystyle\int_\Omega |x|^b\, |u|^p\ dx} = \mu,$$

that is, u_0 attains μ and $u_0 = |u|$. Since u_0 is positive in Ω, then u and $-\Delta u$ have constant sign (use Remark 3.3), as desired. □

Proof of Theorem 1.3. Assume that (1.3) holds. The existence of a principal eigenvalue is immediate, thanks to Lemmata 2.8 and 3.4 and the equivalence given by Lemma 3.2. □

Remark 3.5. Assume that $a, b > -n$ and

$$\frac{a}{p'} + \frac{b}{p} + 2 < 0\,,$$

so that, in particular, $n \geq 3$. Then Problem $(\mathcal{P})$ has no positive principal eigenvalue. This is a consequence of [1, Theorem 2.2].

We conclude the section pointing out a symmetry result about the infimum in (3.3). It is convenient to use the notation $\mu(a, b, p)$ to emphasize the dependence of μ on the exponents a, b and p.

Proposition 3.6. *If $a, b > -n$ and* (1.3) *holds, then $\mu(b, a, p')^p = \mu(a, b, p)^{p'}$.*

Proof. Let u be an extremal for $\mu(a,b,p)$. By Lemma 3.2, the pair u,v, where

$$v := -|x|^{-a(p-1)}|\Delta u|^{p-2}\Delta u\,,$$

solves (3.2) with respect to the eigenvalue $\mu = \mu(a,b,p)$. Hence

$$\begin{aligned}\int_\Omega |x|^{-b(p'-1)}\,|\Delta v|^{p'}\,dx &= \mu(a,b,p)^{p'}\int_\Omega |x|^b\,|u|^p\,dx \\ &= \mu(a,b,p)^{p'-1}\int_\Omega |x|^{-a(p-1)}\,|\Delta u|^p\,dx = \mu(a,b,p)^{p'-1}\int_\Omega |x|^a\,|v|^{p'}\,dx.\end{aligned}$$

Thus $\mu(b,a,p') \le \mu(a,b,p)^{p'-1}$, or equivalently $\mu(b,a,p')^p \le \mu(a,b,p)^{p'}$. The opposite inequality follows by exchanging the roles of u and v. □

4. The linear problem

In this section we prove Theorem 1.4. First of all we notice that if $p = 2$, then equation (1.6) reduces to

$$\Delta\left(|x|^{-a}\,\Delta u\right) = \mu\,|x|^b\,u. \tag{4.1}$$

We denote by X_a the Hilbert space $W^{2,2}_N(\Omega,|x|^{-a}\,dx)$, endowed with norm $\|\cdot\|_a$ and scalar product $(\,\cdot\mid\cdot\,)_a$.

Proof of Theorem 1.4 (a). We formally introduce the "solution operator" to (4.1) under Navier boundary conditions. More precisely, we define the linear operator

$$T\colon L^2(\Omega,|x|^b\,dx) \to X_a\ ,\quad (Tf|w)_a = \int_\Omega |x|^b\,fw\,dx \quad \text{for } w \in X_a.$$

Then T is continuous, positive and self-adjoint. Let $j\colon X_a \to L^2(\Omega,|x|^b\,dx)$ be the embedding in Lemma 2.4. Then the operator

$$T \equiv j\circ T\ ,\ T\colon L^2(\Omega,|x|^b\,dx) \to L^2(\Omega,|x|^b\,dx)$$

is compact. Thus the point spectrum $\sigma_p(T)$ of T is a non-increasing sequence $\{\nu_k\}_{k\in\mathbb{N}}$ of positive numbers converging to 0, and

$$\frac{1}{\nu_k} = \min\left\{\frac{\displaystyle\int_\Omega |x|^{-a}\,|\Delta u|^2\,dx}{\displaystyle\int_\Omega |x|^b\,u^2\,dx}\ \Big|\ u \in \Lambda_i^\perp, 1 \le i \le k-1\right\},$$

where Λ_i is the eigenspace relative to the eigenvalue ν_i. Thus (a) readily follows by Lemma 3.2. □

Proof of Theorem 1.4 (b). We will use the theory of abstract positive operators on Banach lattices, for which we refer to the monograph [26]. Recall that $L^2(\Omega,|x|^b\,dx)$ has a natural Banach lattice structure induced by the cone P_+ of nonnegative functions. We will show that T is positive and irreducible. Then, the conclusion will follow thanks to an adaptation of Theorem V.5.2 in [26], that guarantees that the following facts hold:

- the spectral radius $r(T) \in \mathbb{R}_+$ is an eigenvalue;
- the eigenspace $\Lambda(r(T))$ has dimension one, and is spanned by a (unique, normalized) quasi-interior point of P_+;
- $r(T)$ is the unique eigenvalue of T with a positive eigenvector.

To check that T is irreducible we first recall that that the only closed ideals in $L^2(\Omega, |x|^b\, dx)$ are the ones of the form

$$I_A = \left\{ f \in L^2(\Omega, |x|^b\, dx) \middle| f = 0 \text{ on } A \right\},$$

where A is a measurable set, see for instance [26, p. 157]. Therefore, we have to show that if A satisfies

$$0 < \int_\Omega |x|^b \chi_A\, dx < \int_\Omega |x|^b\, dx,$$

then I_A is not fixed by T.

Let $f \in I_A$ be a nonnegative fixed function. Then the problem

$$\begin{cases} -\Delta v = |x|^b f \\ v \in X_b \end{cases}$$

admits a solution by Corollary 2.3, and $v \in W^{2,\tau} \cap W_0^{1,\tau}(\Omega)$ for $\tau > 1$ small enough. The minimum principle implies that v is strictly positive in Ω. For the same reason, the problem

$$\begin{cases} -\Delta u = |x|^a v \\ u \in X_a, \end{cases}$$

defines a function u that is strictly positive in Ω. Hence $u \equiv Tf \notin I_A$, and this proves the irreducibility property. The same argument proves also the positivity property. □

Remark 4.1. There are of course a number of possible generalizations of the above results. For instance, the weights $|x|^a$, $|x|^b$ may be replaced by more general measurable weights $\alpha(x), \beta(x)$, suitably pinched between power-type functions. More general differential operators might be considered as well. One could wonder what happens if $-\Delta$ is replaced by an m-order (possibly, fractional) differential operator.

Finally, one may wonder if a Faber-Krahn type inequality can hold, in particular in the non-linear case $p \neq 2$.

Appendix. An inequality in lower dimensions

We sketch here the proof of a second order integral inequality in low dimensions by using, in essence, the Rellich-type identity in [19].

Lemma 4.2. *Let $n \in \{1,2\}$ and let Ω be a a bounded domain in $\mathbb{R}^n$. If $n=2$, assume that Ω is of class C^2. For any $s > n - np$, there exists a constant $c > 0$ such that*

$$c \int_\Omega |x|^{\frac{s}{p-1}} |u|^p \, dx \le \int_\Omega |x|^{-s} |\Delta u|^p \, dx$$

for any $u \in C^2_N(\overline{\Omega})$ such that $\Delta u = 0$ in a neighborhood of 0.

Proof. We can assume that Ω is contained in the unit ball about the origin. Put

$$a = \frac{s}{p-1},$$

and notice that $a > -n$. We argue in a heuristic way. A more rigorous proof requires a suitable approximation of the weight $|x|^{a+2}$ by smooth functions. We omit details.

Fix $u \in C^2_N(\overline{\Omega})$ such that $\Delta u = 0$ in a neighborhood of 0. For $p \ge 2$ one clearly has

$$(p-1) \int_\Omega |\nabla u|^2 |u|^{p-2} \, dx = \int_\Omega (-\Delta u)|u|^{p-2} u \, dx.$$

For general $p > 1$, one can check that $|\nabla u|^2 |u|^{p-2} \in L^1(\Omega)$ and

$$(p-1) \int_\Omega |\nabla u|^2 |u|^{p-2} \, dx \le \int_\Omega |\Delta u| \, |u|^{p-1} \, dx \,. \tag{4.2}$$

Next, we are allowed to use integration by parts again and Hölder's inequality to estimate

$$\begin{aligned}(a+2)(a+n) \int_\Omega |x|^a |u|^p \, dx &= - \int_\Omega (\Delta |x|^{a+2}) \, |u|^p \, dx \\ &= p \int_\Omega (\nabla |x|^{a+2} \cdot \nabla u) \, |u|^{p-2} u \, dx \le p(a+2) \int_\Omega |x|^{a+1} |\nabla u| |u|^{p-1} \, dx \\ &\le p(a+2) \left(\int_\Omega |\nabla u|^2 |u|^{p-2} \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |x|^{2a+2} |u|^p \, dx \right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, from $a+2 \ge a+n > 0$ we infer

$$\left(\frac{a+n}{p} \right)^2 \int_\Omega |x|^a |u|^p \, dx \le \int_\Omega |\nabla u|^2 |u|^{p-2} \, dx \le \frac{1}{p-1} \int_\Omega |u|^{p-1} |\Delta u| \, dx$$

by (4.2). Then we use again Hölder's inequality to estimate

$$c \int_\Omega |x|^a |u|^p \, dx \le \left(\int_\Omega |x|^a |u|^p \, dx \right)^{\frac{1}{p'}} \left(\int_\Omega |x|^{-a(p-1)} |\Delta u|^p dx \right)^{\frac{1}{p}}$$

where $c = c(a,n,p) > 0$. Thus

$$c \int_\Omega |x|^a |u|^p \, dx \le \int_\Omega |x|^{-a(p-1)} |\Delta u|^p dx,$$

as desired. $\square$

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