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Abstract

Departing from a suitable categorical description of closure operators, this paper dualizes this notion and introduces some basic properties of dual closure operators. Usually these operators act on quotients rather than subobjects, and much attention is being paid here to their key examples in algebra and topology, which include the formation of monotone quotients (Eilenberg-Whyburn) and concordant quotients (Collins). In fair categorical generality, these constructions are shown to be factors of the fundamental correspondence that relates connectednesses and disconnectednesses in topology, as well as torsion classes and torsion-free classes in algebra. Depending on a given cogenerator, the paper also establishes a non-trivial correspondence between closure operators and dual closure operators in the category of R-modules. Dual closure operators must be carefully distinguished from interior operators that have been studied by other authors.

Key words: *closure operator, dual closure operator, preradical, monotone map, concordant map, Eilenberg-Whyburn dual closure operator, Cassidy-Hébert-Kelly dual closure operator, multi-monocoreflective subcategory.*
Mathematics classification: 18A32, 18A40, 54C10, 16D90, 20K40.

1 Introduction

A categorical *closure operator* C in the sense of [21] assigns to every subobject $m : M \to X$ in a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{X}$ a subobject $Cm : C_X M \to X$ in $\mathcal{M}$, and this operation is expansive, monotone and compatible with taking images or, equivalently, inverse images, in the same way as the usual topological closure is compatible with continuous maps. This notion, originally designed to help characterize epimorphisms in subcategories of topological spaces and to determine whether such subcategories are cowellpowered (so that every object X allows for only a set of non-isomorphic epimorphisms with domain X), has enjoyed considerable attention; see in particular the monographs [21, 8]. Its applications range from topology to algebra and theoretical computer science; see, for example, [22, 25, 6, 14, 19, 20]. What is the categorically dual notion of closure operator?

Starting with [49], in recent years several authors have investigated categorical *interior operators*, with the formation of the interior of a subspace of a topological space providing the role model; see [9, 10, 30, 19, 35]. While in Section 6 of this paper we make precise in which sense this notion is an *order*-dualization of the notion of closure operator, it certainly does not address the quest for the categorical dual of the notion of closure operator. There also seems to be a lack of striking examples of interior operators that do not already arise from closure operators via two-fold complementation, in the same way as its topological role model is expressible in terms of closure.

However, the categorical dualization of the notion of closure operator becomes quite obvious once it is expressed as a pointed endofunctor $(m \mapsto Cm)$ of the *category* $\mathcal{M}$, considered as a full subcategory of the morphism category $\mathcal{X}^2$. This approach to closure operators was already taken in the follow-up paper [22] to [21] and then expanded upon in [23, 48]. It allows one to minimize the conditions imposed upon the class $\mathcal{M}$ on which the closure operator acts; in fact, as we show in this paper, there is a priori no need for any restrictions on the class $\mathcal{M}$, although it is convenient to assume well-behaviour of $\mathcal{M}$ vis-á-vis isomorphisms in $\mathcal{X}$.

Once a closure operator of a class $\mathcal{M}$ of morphisms in $\mathcal{X}$ is presented as an endofunctor C of the category $\mathcal{M}$ pointed by a natural transformation $1_{\mathcal{M}} \to C$ that is compatible with the codomain functor of $\mathcal{M}$ (to make sure that the closure stays in the same ambient object), the dual notion is necessarily given by a copointed endofunctor of $\mathcal{M}$ that is compatible with the domain functor of $\mathcal{M}$. While already in [48] we pointed out that closure operators presented as pointed endofunctors have a formally equivalent presentation as copointed endofunctors, in this paper we study the thus emerging notion of *dual closure operator* more seriously, replacing their domain $\mathcal{M}$ of operation (that normally is given by a class of monomorphisms) by a clas $\mathcal{E}$ that is normally to

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be taken as a class of epimorphisms in the ambient category, and give a range of examples that entail a number of classical constructions. Of course, when $\mathcal{E}$ is the class of regular epimorphisms of the category, these morphisms are equivalently described by their kernelpairs and, hence, often by (normal) subobjects, as it is the case in the categories of R-modules or of groups. Consequently, in such categories dual closure operators may be considered as acting on certain subobjects, rather than on quotient maps, and they then become directly comparable with interior operators. With these they then share two of the three characteristic properties, but not the third, and this difference is significant: none of the major (groups of) examples of dual closure operators presented in this paper may simultaneously be considered as interior operators. Actually, the category Mod_R has no non-trivial interior operators at all; as a matter of fact this remains true in any category where all subobjects are normal, e.g., all abelian categories. More on the comparison between dual closure operators and interior operators can be found in the forthcoming paper [24].

It was observed already in [21] that a closure operator C of a class $\mathcal{M}$ of monomorphisms in a category $\mathcal{X}$ gives rise to two interesting subcategories, namely

$$\Delta(C) = \{X : (\delta_X : X \to X \times X) \text{ is } C\text{-closed}\},$$

$$\Delta^*(C) = \{X : (\delta_X : X \to X \times X) \text{ is } C\text{-dense}\},$$

defining respectively order-reversing or order-preserving maps from the conglomerate $CO(\mathcal{X}, \mathcal{M})$ of closure operators to that of all full subcategories, $SUB(\mathcal{X})$. In [46, 23] we showed that $\Delta : CO(\mathcal{X}, \mathcal{M}) \to SUB(\mathcal{X})^{op}$ has a right adjoint that assigns to a full subcategory B its *regular* closure operator (which has its roots in [31, 40]; see also [47, 16]). The paper [15] gave a categorical context for $\Delta^* : CO(\mathcal{X}, \mathcal{M}) \to SUB(\mathcal{X})$ to admit a left adjoint, assigning to a full subcategory A its *coregular* closure operator. Moreover, the composition of these two adjunctions gives precisely the Herrlich-Preuß-Arhangel'skii-Wiegandt (HPAW) "left-right-constant" correspondence

$$SUB(\mathcal{X}) \underset{l}{\overset{r}{\underset{\perp}{\rightleftarrows}}} SUB(\mathcal{X})^{op}$$

(see [28, 38, 39, 2, 15, 42, 43]) which, in the categories Top (Mod_R), links connectednesses (torsion classes) with disconnectednesses (torsion-free classes, respectively). In this paper we establish an analogous result for dual closure operators which also exhibits two fundamental types of dual closure operators, just like the regular and coregular closure operators.

Having laid the groundwork on dual closure operators in Sections 2 and 3, by faithfully dualizing the basic notions for closure operators and exhibiting in particular the fact that, in Mod_R, dual closure operators interact with preradicals like closure operators do, in Section 4 we start off by showing that, under mild categorical hypotheses and with a refined notion of constant morphism, the HPAW-correspondence may be restricted to a correspondence between the strongly epireflectve subcategories and the strongly multi-monocoreflective subcategories. Here multi-coreflectivity is to be understood as introduced in Diers' thesis (under a different name) [18]; more general predecessors of the notion were presented in [33, 5], with [5] establishing the crucial property of closure under connected colimits; see also [44, 41]. (Note that none of these works uses the nowadays common "multi" terminology.)

The restricted HPAW correspondence may now be factored through the conglomerate $DCO(\mathcal{X}, \mathcal{E})$ of all dual closure operators of the class $\mathcal{E}$ of strong epimorphisms in $\mathcal{X}$, as we may indicate here in the case of the prototypical example $\mathcal{X} = \mathsf{Top}, \mathcal{E} = \{\text{strong epimorphisms}\}$. Here a dual closure operator assigns to a quotient map $p : X \to P$ a quotient map $Dp : X \to D_X P$ through which p factors; one calls p D-*closed* if $Dp \cong p$, and D-*sparse* if $Dp \cong 1_X$. There are two subcategories of interest associated with D, namely

$$\mathrm{Shriek}(D) = \{X : X \neq \emptyset, (!_X : X \to 1) \text{ is } D\text{-closed}\},$$

$$\mathrm{Shriek}^*(D) = \{X : X \neq \emptyset \Rightarrow (!_X : X \to 1) \text{ is } D\text{-sparse}\},$$

defining order-preserving and -reversing maps Shriek and Shriek* to $SUB(\mathcal{X})$, respectively, analogously to Δ^* and Δ. The important point now is that Shriek has a left adjoint, ew, which assigns to a strongly multi-monocoreflective subcategory A the *Eilenberg-Whyburn* dual closure operator ew$^{\mathsf{A}}$. Indeed, in the guiding example A = {(non-empty) connected spaces}, $Dp : X \to D_X P = X/\sim$ is obtained by declaring $x, y \in X$ to be equivalent if they belong to the same connected component of the same fibre of p. This is precisely the *monotone* quotient map related to p as first considered by Eilenberg [27] in a metric and by Whyburn [50, 51] in a topological context, with the resulting factorization studied further in [37, 3, 36] and other papers; for categorical treatments, see also [26, 29, 5].

When forming the monotone factor $Dp : X \to X/\sim$ of $p : X \to P$, the second factor $X/\sim \to P$ generally fails to be *light*, i.e., to have totally disconnected fibres, unless the space P is T1. Unfortunately, even when both X and P are T1, the quotient space $X/\sim$ may fail to be T1, which is why Collins [17] exhibited the (*concordant, dissonant*) factorization of p in Top. We find it rewarding that Collins' concordant factor of a quotient map p may be provided by a dual closure operator, as follows. Shriek* has a right adjoint which, to a strongly epireflective subcategory B, assigns the dual closure operator $\mathrm{chk}^{\mathsf{B}}$; for B = {totally disconnected spaces}, $\mathrm{chk}^{\mathsf{B}}p$ produces Collins' construction. Here chk is named after the seminal Cassidy-Hébert-Kelly paper [11] which gives the general categorical construction behind the (concordant, dissonant)-factorization, although this application has been exhibited only subsequently: see [32, 7]. The factorization of the HPAW-correspondence through the adjunctions given by Shriek and Shriek* is displayed in Corollary 4.18.

In Section 5, depending on a cogenerator K of Mod_R, through double dualization with respect to K we establish a non-trivial correspondence between closure operators (of subobjects) and dual closure operators (of quotients) in the category of $R-$modules. We illustrate this correspondence in a particular instance in terms of its effect on the associated preradicals, thus linking the torsion radical of Abelian groups with the first-Ulm-subgroup preradical.

2 Closure operators and dual closure operators

2.1 Background on closure operators

Let $\mathcal{M}$ be a class of morphisms in a category $\mathcal{X}$ which contains all isomorphisms and is closed under composition with isomorphisms. We consider $\mathcal{M}$ as a full subcategory of the category $\mathcal{X}^2$ of morphisms of $\mathcal{X}$, so that a morphism $(u, v) : m \to n$ in $\mathcal{M}$ is given by a commutative diagram

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\;u\;} & \bullet \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\
\bullet & \xrightarrow{\;v\;} & \bullet
\end{array}
\tag{1}
$$

of morphisms in $\mathcal{X}$ with $m, n \in \mathcal{M}$. One has

- the domain functor dom : $\mathcal{M} \to \mathcal{X}$, defined by $(u, v) \mapsto u$; and

- the codomain functor cod : $\mathcal{M} \to \mathcal{X}$, defined by $(u, v) \mapsto v$.

Extending the definitions given in [23, 5.2] and [48] we define:

Definition 2.1 A *closure operator* of $\mathcal{M}$ in $\mathcal{X}$ is an endofunctor $C : \mathcal{M} \to \mathcal{M}$ together with a natural transformation $\Gamma : 1_{\mathcal{M}} \to C$ with components in the class $\mathcal{M}$ such that

$$
\mathrm{cod}\, C = \mathrm{cod} \quad \text{and} \quad \mathrm{cod}\, \Gamma = 1_{\mathrm{cod}}.
$$

Hence, for every $m : M \to X$ in $\mathcal{X}$ lying in the class $\mathcal{M}$ one has a morphism $\Gamma_m : m \to Cm$ in the category $\mathcal{M}$ with $\mathrm{cod}\Gamma_m = 1_X$ and therefore a factorization

$$
\begin{array}{ccc}
M & \xrightarrow{\;\gamma_m\;} & C_X M \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Cm} \\
X & \xrightarrow{\;1_X\;} & X
\end{array}
\tag{2}
$$

in $\mathcal{X}$ with $\gamma_m := \mathrm{dom}\Gamma_m$ and all morphisms in the class $\mathcal{M}$; furthermore, diagram (1) gets decomposed as

$$
\begin{array}{ccc}
M & \xrightarrow{\;u\;} & N \\
{\scriptstyle \gamma_m}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\
C_X M & \xrightarrow{\;C^{u,v}\;} & C_Y N \\
{\scriptstyle Cm}\downarrow & & \downarrow{\scriptstyle Cn} \\
X & \xrightarrow{\;v\;} & Y
\end{array}
\tag{3}
$$

where we have written $C^{u,v}$ instead of $\mathrm{dom}C(u, v)$.

Remark 2.2 (1) The condition that Γ be componentwise in $\mathcal{M}$ comes for free if the class $\mathcal{M}$ satisfies the cancellation condition

$$n \cdot m \in \mathcal{M}, \ n \in \mathcal{M} \text{ or } n \text{ monic} \quad \Rightarrow \quad m \in \mathcal{M}.$$

Note that if $\mathcal{X}$ has right $\mathcal{M}$-factorizations (see (2) below) and, *a fortiori*, if $\mathcal{M}$ belongs to an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$, then $\mathcal{M}$ automatically satisfies this cancellation condition.

(2) Recall from [23] that $\mathcal{X}$ has *right $\mathcal{M}$-factorizations* if $\mathcal{M}$ is reflective in $\mathcal{X}^2$; equivalently, if every morphism f factors as $f = m \cdot e$ with $m \in \mathcal{M}$ such that, whenever $v \cdot f = n \cdot u$ with $n \in \mathcal{M}$, there is a unique t with $t \cdot e = u$ and $n \cdot t = v \cdot m$. The existence of right $\mathcal{M}$-factorizations amounts to $\mathcal{M}$ belonging to an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ precisely when $\mathcal{M}$ is closed under composition; in this case $\mathcal{E}$ contains precisely those morphisms f whose reflection into the category $\mathcal{M}$ is *trivial*, *i.e.*, for which in the notation above m is an isomorphism.

Note that when $\mathcal{X}$ has right $\mathcal{M}$-factorizations, the class $\mathcal{E}$ of morphisms in $\mathcal{X}$ with trivial reflection into $\mathcal{M}$ is always closed under composition, regardless of whether $\mathcal{M}$ is closed under composition (see [32]).

(3) If $\mathcal{M}$, in addition to satisfying the cancellation condition of (1), is a class of monomorphisms in $\mathcal{X}$, then the natural transformation Γ belonging to a closure operator of $\mathcal{M}$ in $\mathcal{X}$ is uniquely determined by the endofunctor C. As shown in [23] and [48], in this case a closure operator C may simply be given by a family of maps

$$C_X : \mathrm{sub}X \to \mathrm{sub}X \quad (X \in \mathrm{ob}\mathcal{X}),$$

where $\mathrm{sub}X = \mathrm{cod}^{-1}X$ is the preordered class of $\mathcal{M}$-subobjects of X, such that

1. $m \le C_X m$,
2. $m \le m' \Rightarrow C_X m \le C_X m'$,
3. $C_X(f^{-1}(n)) \le f^{-1}(C_Y n)$ or
3'. $f(C_X m) \le C_Y(f(m))$,

for all $m, m' \in \mathrm{sub}X$, $n \in \mathrm{sub}Y$ and $f : X \to Y$; here, in order to form the needed images or inverse images, we must assume the existence of right $\mathcal{M}$-factorizations or of pullbacks of $\mathcal{M}$-subobjects lying in $\mathcal{M}$, noting that, when both are available, conditions 3 and 3' are equivalent in the presence of 1 and 2.

We note that also the morphisms $C^{u.v}$ rendering diagram (3) commutative are uniquely determined if $\mathcal{M}$ is a class of monomorphisms.

(4) The prototypical example of a closure operator is the Kuratowski closure operator in the category Top, assigning to a subspace M of a space X its ordinary topological closure $\overline{M}$ in X.

2.2 Dual closure operators

The point of Definition 2.1 is that it lends itself to easy dualization. For psychological reasons we denote the given class of morphisms in $\mathcal{X}$ not by $\mathcal{M}$ but $\mathcal{E}$ in the dual situation, continuing to assume that $\mathcal{E}$ contains all isomorphisms of $\mathcal{X}$ and be closed under composition with them.

Definition 2.3 A *dual closure operator* (*dco*) of $\mathcal{E}$ in $\mathcal{X}$ is a closure operator of $\mathcal{E}$ in $\mathcal{X}^{op}$, that is: an endofunctor $D : \mathcal{E} \to \mathcal{E}$ together with a natural transformation $\Delta : D \to 1_{\mathcal{X}}$ componentwise in the class $\mathcal{E}$ such that

$$\mathrm{dom}\, D = \mathrm{dom} \text{ and } \mathrm{dom}\, \Delta = 1_{\mathrm{dom}}.$$

Hence, for all $p : X \to P$ in $\mathcal{X}$ with $p \in \mathcal{E}$ one obtains $\Delta_p : Dp \to p$ in the category $\mathcal{E}$, so that

$$
\begin{array}{ccc}
X & \xrightarrow{\ 1_X\ } & X \\
{\scriptstyle Dp}\downarrow & & \downarrow{\scriptstyle p} \\
D_X P & \xrightarrow{\ \delta_p\ } & P
\end{array}
\qquad (2^*)
$$

commutes in $\mathcal{X}$, with $\delta_p = \text{cod}\Delta_p$ and all arrows in the class $\mathcal{E}$; furthermore, a morphism $(u,v): p \to q$ in $\mathcal{E}$ gives the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\;u\;} & Y \\
{\scriptstyle Dp}\downarrow & & \downarrow{\scriptstyle Dq} \\
D_X P & \xrightarrow{\;D_{u,v}\;} & D_Y Q \\
{\scriptstyle \delta_p}\downarrow & & \downarrow{\scriptstyle \delta_q} \\
P & \xrightarrow{\;v\;} & Q
\end{array} \qquad (3^*)$$

in $\mathcal{X}$, with $D_{u,v} = \text{cod}D(u,v)$.

The condition that Δ be componentwise in the class $\mathcal{E}$ comes for free if $\mathcal{E}$ satisfies the cancellation condition

$$q \cdot p \in \mathcal{E}, \ p \in \mathcal{E} \text{ or } p \text{ epic} \quad \Rightarrow \quad q \in \mathcal{E},$$

in particular if $\mathcal{X}$ has *left $\mathcal{E}$-factorizations* (so that $\mathcal{E}$ is coreflective in $\mathcal{X}^2$) and, *a fortiori*, if $\mathcal{E}$ belongs to an orthogonal factorization system $(\mathcal{E}, \mathcal{M})$ in $\mathcal{X}$ (see (2.2(1)).

If, in addition to the cancellation condition, $\mathcal{E}$ is a class of epimorphisms in $\mathcal{X}$, then Δ is uniquely determined by D, and we may dualize Remark 2.2(3) as follows. Writing

$$\text{quot } X = \text{dom}^{-1} X$$

for the preordered class of $\mathcal{E}$-quotients of X, for every $f: X \to Y$ in $\mathcal{X}$ we have the monotone map

$$f^-(-): \text{quot } Y \to \text{quot } X$$

which, existence of the needed factorizations granted, assigns to $q: Y \to Q$ in the class $\mathcal{E}$ the $\mathcal{E}$-part $f^-(q): X \to f^-Q$ of a left $\mathcal{E}$-factorization of $q \cdot f$.

Proposition 2.4 *If, for a class $\mathcal{E}$ of epimorphisms, the category $\mathcal{X}$ has left $\mathcal{E}$-factorizations (in particular, if $\mathcal{E}$ belongs to an orthogonal factorization system in $\mathcal{X}$), then a dco of $\mathcal{E}$ in $\mathcal{X}$ may equivalently be given by a family of maps*

$$D_X: \text{quot}X \to \text{quot}X \quad (X \in \text{ob}\mathcal{X}),$$

satisfying

1. $D_X p \le p$,

2. $p \le p' \Rightarrow D_X(p) \le D_X(p')$,

3. $D_X(f^-q) \le f^-(D_Y q)$,

for all $p, p' \in \text{quot}X$, $q \in \text{quot}Y$ and $f: X \to Y$ in $\mathcal{X}$. $\qquad\square$

Remark 2.5 (1) Condition 3 of Proposition 2.4 dualizes condition 3' of Remark 2.2(3). One may, of course, also dualize condition 3, using pushouts instead of pullbacks.

(2) The prototypical example of a dual closure operator is given by forming the torsion part $\text{tor}A$ of a subgroup A of an abelian group X, to be regarded as assigning to the quotient map $X \to X/A$ in AbGrp the map $X \to X/\text{tor}A$. Hence, when for simplicity we regard the dual closure operator as operating on their kernels rather than on the quotient maps, it simply assigns to A its torsion part. This simplified viewpoint can be adopted more generally, as we show next.

Let the category $\mathcal{X}$ have kernelpairs (= pullbacks of pairs of equal morphisms) and coequalizers. Then $\mathcal{X}$ has left $\mathcal{E}$-factorizations (so that $\mathcal{E}$ is coreflective in $\mathcal{X}^2$), for $\mathcal{E} = \mathcal{R}eg\mathcal{E}pi\mathcal{X}$ the class of regular epimorphisms: simply factor a morphism through the coequalizer of its kernelpair. The preordered class $\text{quot}X$ is equivalent to the class $\text{kerp}X$ of all kernelpairs with codomain X, and for a morphism $f: X \to Y$ in $\mathcal{X}$ one now has the monotone map

$$f^-(-): \text{kerp } Y \to \text{kerp } X$$

which assigns to a kernelpair $(q_1, q_2: L \to Y)$ the kernelpair of the composite $q \cdot f$, with q the coequalizer of q_1, q_2. Note that f^- has a left adjoint

$$f(-): \text{kerp } X \to \text{kerp } Y$$

which assigns to a kernelpair $p_1, p_2: K \to X$ the kernel pair of the coequalizer of $f \cdot p_1, f \cdot p_2$.

Remark 2.6 A kernel pair $p_1, p_2 : K \to X$ gives rise to a particular regular subobject of $X \times X$. In the category Grp of groups these correspond to the normal subgroups of X, and in the category Mod_R of $R-$modules, they are equivalently described by all submodules of X. Guided by these examples, in what follows, we laxly write just K for a kernel pair $p_1, p_2 : K \to X$ and denote its coequalizer by $p : X \to X/K$. However, the reader must keep in mind that the notion of dual closure operator be carefully distinguished from that of an interior operator (see Section 6), even in the category Mod_R where we may let a dual closure operator act on subjects of X rather than on quotients of X.

Corollary 2.7 *If $\mathcal{X}$ has kernelpairs and coequalizers, a dco of $\mathcal{R}eg\mathcal{E}pi\mathcal{X}$ may equivalently be given by a family of maps*

$$D_X : \mathrm{kerp}X \to \mathrm{kerp}X \quad (X \in \mathrm{ob}\mathcal{X}),$$

satisfying

1. *$D_X K \leq K$,*

2. *$K \leq K' \Rightarrow D_X K \leq D_X K'$,*

3. *$D_X(f^-(L)) \leq f^-(D_Y L)$ or, equivalently,*

3'. *$f(D_X K) \leq D_Y f(K)$,*

for all $K, K' \in \mathrm{kerp}X$, $L \in \mathrm{kerp}Y$ and $f : X \to Y$ in $\mathcal{X}$.
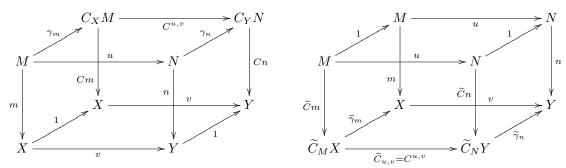
Key examples are presented in the following sections.

2.3 A correspondence between closure operators and dual closure operators

We conclude this section with an initially surprising but in fact easy observation. As done in [48] when $\mathcal{M}$ is a class of monomorphisms, using the notation of 2.2 and 2.3 one may also in the general case define the *companion* $(\widetilde{C}, \widetilde{\Gamma})$ of a closure operator (C, Γ) by putting

$$\widetilde{C}m := \gamma_m \quad \text{and} \quad \widetilde{\gamma}_m := Cm$$

for all $m \in \mathcal{M}$. Since the diagram below on the right displaying naturality of $\widetilde{\Gamma}$ is just a re-drawn version of the commutative diagram on the left displaying the naturality of Γ, it is clear that $(\widetilde{C}, \widetilde{\Gamma})$ is in fact a dual closure operator of $\mathcal{M}$:

Furthermore, since the passage $(C, \Gamma) \mapsto (\widetilde{C}, \widetilde{\Gamma})$ is facilitated by switching the roles of the factors in $m = Cm \cdot \gamma_m = \widetilde{\gamma}_m \cdot \widetilde{C}m$, it is obviously bijective. These are the essentials of the proof of the following Proposition:

Proposition 2.8 *For a class $\mathcal{M}$ of morphisms in a category $\mathcal{X}$, assigning to a closure operator its companion constitutes a bijective correspondence between closure operators (C, Γ) of $\mathcal{M}$ and dual closure operators (D, Δ) of $\mathcal{M}$.*

Remark 2.9 (1) We stress the fact that, even in the presence of an orthogonal $(\mathcal{E}, \mathcal{M})$ factorization system, Proposition 2.8 does not give a correspondence between closure operators of $\mathcal{M}$ and dual closure operators of $\mathcal{E}$, but between closure operators of $\mathcal{M}$ and dual closure operators (D, Δ) of $\mathcal{M}$ (!), and dually between dual closure operators of $\mathcal{E}$ and closure operators (C, Γ) of $\mathcal{E}$ (!). Hence, even if $(\mathcal{E}, \mathcal{M})$ is proper, so that $\mathcal{M}$ is a class of monomorphisms and $\mathcal{E}$ is a class of epimorphisms, the convenient description given in Proposition 2.4 will generally *not* apply to the companion of a closure operator of $\mathcal{M}$ (since $\mathcal{M}$ generally fails to be a class of epimorphisms in $\mathcal{X}$); likewise for the companion of a dco of $\mathcal{E}$.

(2) In the notation of 2.1 and under the conditions of 2.2(3), since CCm is monic as a morphism in $\mathcal{X}$ one obtains $\gamma_{Cm} = C^{\gamma_m, 1_X}$ for all $m \in \mathrm{sub}X$; equivalently, $\Gamma C = C\Gamma$. In the terminology of [34], that means: the pointed endofunctor (C, Γ) is *well-pointed*. From Cm monic one also obtains the equality $C\gamma_m = C^{1_M, Cm}$ for all $m : M \to X$ in $\mathcal{M}$, which precisely means that the companion of (C, Γ) is also well-pointed (as a co-pointed endofunctor).

(3) In Section 5.2, depending on a chosen cogenerator in Mod_R, we establish a Galois correspondence between closure operators of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$ and dual closure operators of $\mathcal{R}eg\mathcal{E}pi\mathsf{Mod}_R$ of the category of R-modules for a commutative unital ring R.

3 General properties of (dual) closure operators

As in the previous section, $\mathcal{M}$ and $\mathcal{E}$ denote classes of morphisms in a category $\mathcal{X}$ containing all isomorphisms and being closed under composition with isomorphisms. For simplicity, we also assume throughout that they satisfy the cancellation conditions of Remark 2.2(1)and Definition 2.3, respectively.

3.1 Idempotency and weak heredity

Recall that, for a closure operator (C, Γ) of $\mathcal{M}$ in $\mathcal{X}$, a morphisms m in the class $\mathcal{M}$ is called

- (C, Γ)-*closed* if γ_m is an isomorphism;

- (C, Γ)-*dense* if Cm (considered as a morphism in $\mathcal{X}$) is an isomorphism.

Let $\mathrm{Cl}_{C,\Gamma}$ and $\mathrm{Ds}_{C,\Gamma}$ denote the respective subclasses of $\mathcal{M}$. If $\mathcal{M}$ is a class of monomorphisms, the redundant parameter Γ may be omitted from these notations.

Expanding on Kelly's terminology for pointed and co-pointed endofunctors (see [34]) we call the closure operator (C, Γ)

- *well-pointed* if $C\Gamma = \Gamma C$, that is: if $\gamma_{Cm} = C^{\gamma_m, 1_X}$ for all $m : M \to X$ in the class $\mathcal{M}$ (see 2.8(2));

- *well-bipointed* if (C, Γ) and its companion (see 2.3) are well-pointed and cowell-pointed, respectively, so that in addition to the above identity one has $C\gamma_m = C^{1_M, Cm}$ for all $m : M \to X$ in $\mathcal{M}$ (see 2.8(2));

- *idempotent* if (C, Γ) is well-pointed and ΓC is an isomorphism, that is: if $\gamma_{Cm} = C^{\gamma_m, 1_X}$ is an isomorphism for all $m : M \to X$ in $\mathcal{M}$;

- *weakly hereditary (wh)* if the companion of (C, Γ) (see 2.3) is idempotent, that is: if $C\gamma_m = C^{1_M, Cm}$ for all $m : M \to X$ in $\mathcal{M}$ is an isomorphism.

Note that, for $\mathcal{M}$ a class of monomorphisms, (C, Γ) is always well-bipointed (see 2.8(2)), and in that case

- C is idempotent if Cm is C-closed for all $m \in \mathcal{M}$, and

- C is wh if γ_m is C-dense for all $m \in \mathcal{M}$.

The assertions of the following Proposition are well known in the case that $\mathcal{M}$ is a class of monomorphisms, but they hold also in the absence of this provision. Here we let $\mathcal{X}$ have right $\mathcal{M}$-factorizations and let $\mathcal{E}$ be the class of morphisms in $\mathcal{X}$ whose reflection into $\mathcal{M}$ is trivial (see Remark 2.2(2))

Proposition 3.1 *For a well-pointed closure operator (C, Γ) of $\mathcal{M}$ in $\mathcal{X}$, the class $\mathrm{Cl}_{C,\Gamma}$ (considered as a full subcategory of $\mathcal{X}^2$) is closed under limits. In particular, the class $\mathrm{Cl}_{C,\Gamma}$ is stable under (multiple) pullback in $\mathcal{X}$ and satisfies the cancellation condition*

$$n \cdot m \in \mathrm{Cl}_{C,\Gamma}, n \text{ monic} \implies m \in \mathrm{Cl}_{C,\Gamma},$$

while the class $\mathrm{Ds}_{C,\Gamma}$ satisfies

$$n \cdot m \in \mathrm{Ds}_{C,\Gamma}, Cn \text{ monic} \implies n \in \mathrm{Ds}_{C,\Gamma}.$$

Furthermore, if (C, Γ) is idempotent, $\mathcal{X}$ has right $\mathrm{Cl}_{C,\Gamma}$-factorizations, and the class $\mathrm{Ds}_{C,\Gamma} \cdot \mathcal{E}$ (of composites of morphisms in $\mathcal{E}$ followed by (C, Γ)-dense morphisms in $\mathcal{M}$) is closed under composition in $\mathcal{X}$. Consequently, when the class $\mathcal{M}$ is closed under composition, the category $\mathcal{X}$ has orthogonal $(\mathrm{Ds}_{C,\Gamma} \cdot \mathcal{E}, \mathrm{Cl}_{C,\Gamma})$-factorizations precisely when (C, Γ) is idempotent and weakly hereditary.

Proof. For any well-pointed endofunctor (T, η) of a category A, the full subcategory of objects A with $\eta_A : A \to TA$ an isomorphism ("A is T-*fixed*") is closed under limits in A (see [34], [32]). Applying this general fact to the well-pointed endofunctor (C, Γ) of $\mathcal{M}$, and noting that $\mathcal{M}$ is reflective in $\mathcal{X}^2$ by hypothesis, one obtains closure of $\mathrm{Cl}_{C,\Gamma}$ under limits in $\mathcal{X}^2$ as well as the stated consequences of this fact. The stated cancellation property for $\mathrm{Ds}_{C,\Gamma}$ is elementary to check. If the well-pointed endofunctor (T, η) is idempotent (so that $T\eta = \eta T$ is an isomorphism), the subcategory of fixed objects is even reflective in A. Consequently, in our situation, if (C, Γ) is idempotent, $\mathrm{Cl}_{C,\Gamma}$ is reflective in $\mathcal{M}$ and, hence, in $\mathcal{X}^2$. Therefore, $\mathcal{X}$ has right $\mathrm{Cl}_{C,\Gamma}$-factorizations.

For the last assertions, note that if, for any class C, $\mathcal{X}$ has right C-factorizations, then the class of morphisms with trivial reflection into C is always closed under composition in $\mathcal{X}$. In the case at hand, it is easy to see that $\mathrm{Cl}_{C,\Gamma} \cdot \mathcal{E}$ is precisely the class of morphisms with trivial reflection into $\mathrm{Cl}_{C,\Gamma}$ (see Remark 2.2(2)). $\qquad\square$

For dual closure operators of a class $\mathcal{E}$ in $\mathcal{X}$ we adopt the following terminology which will become plausible once we have presented the key examples in the following sections.

Definition 3.2 Let (D, Δ) be a dual closure operator of $\mathcal{E}$ in $\mathcal{X}$. Then a morphism $p \in \mathcal{E}$ is called (D, Δ)-*closed* if it has this property with respect to the closure operator (D^{op}, Δ^{op}) in $\mathcal{X}^{op}$, that is: if δ_p is an isomorphism in $\mathcal{X}$; and p is called (D, Δ)-*sparse* if it is (D^{op}, Δ^{op})-dense in $\mathcal{X}^{op}$, that is: if Dp is an isomorphism in $\mathcal{X}$.

The dual closure operator (D, Δ) is called *idempotent* if (D^{op}, Δ^{op}) is idempotent, and it is called *weakly cohereditary (wch)* if (D^{op}, Δ^{op}) is wh in $\mathcal{X}^{op}$, that is: if respectively $\delta_{Dp} = D_{1_X, \delta_p}$ or $D\delta_p = D_{Dp, 1_Y}$ is an isomorphism for all $p : X \to Y$ in $\mathcal{E}$. Hence, when $\mathcal{E}$ is a class of epimorphisms, so that D is well-bipointed, D is idempotent if Dp is D-closed for all $p \in \mathcal{E}$, and wch if δ_p is D-sparse for all $p \in \mathcal{E}$.

$\mathrm{Cl}^*_{D,\Delta}$ denotes the class of (D, Δ)-closed morphisms in $\mathcal{E}$, and $\mathrm{Ds}^*_{D,\Delta}$ the class of (D, Δ)-sparse morphisms in $\mathcal{E}$.

Remark 3.3 (1) A closure operator is idempotent (wh) if, and only if, its companion dual closure operator is wch (idempotent, respectively). Likewise, a dual closure operator is idempotent (wch) if, and only if, its companion closure operator is wh (idempotent, respectively).

(2) For the prototypical example of a dual closure operator D given by the formation of the torsion part of a subgroup A of an Abelian group X, the projection $X \to X/A$ is $D-$closed precisely when A is a torsion subgroup, and D-sparse when A is torsion-free. Of course, D is both idempotent and wch.

A straight dualization of Proposition 3.1 gives the following Corollary:

Corollary 3.4 *Let $\mathcal{X}$ have left $\mathcal{E}$-factorizations, and let $\mathcal{M}$ be the class of morphisms whose coreflection into $\mathcal{E}$ is trivial. For a well-pointed dual closure (D, Δ) of $\mathcal{E}$ in $\mathcal{X}$, the class $\mathrm{Cl}^*_{D,\Delta}$ is closed under colimits in $\mathcal{X}^2$. Hence, it is stable under (multiple) pushout in $\mathcal{X}$ and satisfies the cancellation condition*

$$q \cdot p \in \mathrm{Cl}^*_{D,\Delta}, p \ epic \implies q \in \mathrm{Cl}^*_{D,\Delta},$$

while the class $\mathrm{Ds}^*_{D,\Delta}$ satisfies

$$q \cdot p \in \mathrm{Ds}^*_{D,\Delta}, Dp \ epic \implies p \in \mathrm{Ds}^*_{D,\Delta}.$$

Furthermore, if (D, Δ) is idempotent, $\mathcal{X}$ has left $\mathrm{Cl}^*_{D,\Delta}$-factorizations, and the class $\mathcal{M} \cdot \mathrm{Ds}^*_{D,\Delta}$ is closed under composition in $\mathcal{X}$. Consequently, when $\mathcal{E}$ is closed under composition, the category $\mathcal{X}$ has orthogonal $(\mathrm{Cl}^*_{D,\Delta}, \mathcal{M} \cdot \mathrm{Ds}^*_{D,\Delta})$-factorizations precisely when (D, Δ) is idempotent and weakly cohereditary.

When applied to our prototypical example, the assertions of the Corollary amount to closure under colimits of the full subcategory of torsion groups in the category of Abelian groups, as well as closure under subobjects and quotients; also the least subgroup generated by a family of torsion subgroups of a group is torsion again. The associated factorization system lets a morphism $f : X \to Y$ factor through $X/\mathrm{tor}(\ker f)$.

3.2 Minimality and heredity, and their dualizations

Some closure operators allow us to compute from the closure of a composite subobject

$$M \xrightarrow{m} N \xrightarrow{n} X$$

the closure of m or of n. Indeed, when $\mathcal{M}$ is a class of monomorphisms and closed under composition, one calls a closure operator C of $\mathcal{M}$ in $\mathcal{X}$

- *hereditary* if $n \cdot Cm \cong n \wedge C(n \cdot m)$ for all composable $m, n \in \mathrm{sub}X$, and

- *minimal* if $Cn \cong n \vee C(n \cdot m)$ for all composable $m, n \in \mathrm{sub}X$.

Hence, in the presence of pullbacks of morphisms in $\mathcal{M}$ belonging to $\mathcal{M}$ again, for C hereditary one obtains Cm as a pullback of $C(n \cdot m)$ along n:

$$Cm = n^{-1}(C(n \cdot m)).$$

In the presence of a least subobject 0_X, for C minimal one obtains Cn as the join of n and $C0_X$ in $\mathrm{sub}X$:

$$Cn = n \vee C0_X.$$

More importantly, as shown in [23, Theorem 2.5], one has:

- C is hereditary if and only if C is wh and

$$n \cdot m \in \mathrm{Ds}_C \Longrightarrow m \in \mathrm{Ds}_C \tag{4}$$

  for all composable $m, n \in \mathrm{sub}X$,

- C is minimal if and only if C is idempotent and

$$n \cdot m \in \mathrm{Cl}_C \Longrightarrow n \in \mathrm{Cl}_C \tag{5}$$

  for all composable $m, n \in \mathrm{sub}X$.

Without imposing $\mathcal{M}$ to be a class of monomorphisms or $\mathcal{E}$ to be a class of epimorphisms a priori, as long as these classes are closed under composition one may therefore define in general:

Definition 3.5 A closure operator (C, Γ) of $\mathcal{M}$ in $\mathcal{X}$ is *hereditary* if it is wh and $\mathrm{Ds}_{C,\Gamma}$ (in lieu of Ds_C) satisfies the cancellation condition (4), and it is *minimal* if it is idempotent and $\mathrm{Cl}_{C,\Gamma}$ (in lieu of Cl_C) satisfies the cancellation condition (5).

A dual closure operator (D, Δ) of $\mathcal{E}$ in $\mathcal{X}$ is *cohereditary* or *maximal* if (D^{op}, Δ^{op}) is, respectively, hereditary or minimal as a closure operator in $\mathcal{X}^{op}$, that is: (D, Δ) is

- cohereditary if and only it is wch and satisfies

$$q \cdot p \in \mathrm{Ds}^*_{D,\Delta} \Longrightarrow q \in \mathrm{Ds}^*_{D,\Delta} \tag{4*}$$

  for all composable $p, q \in \mathcal{E}$,

- maximal if and only if it is idempotent and satisfies

$$q \cdot p \in \mathrm{Cl}^*_{D,\Delta} \Longrightarrow p \in \mathrm{Cl}^*_{D,\Delta} \tag{5*}$$

  for all composable $p, q \in \mathcal{E}$.

Remark 3.6 (1) Since for the companion closure operator $(\widetilde{D}, \widetilde{\Delta})$ of a dco (D, Δ) one trivially has

$$\mathrm{Ds}^*_{D\Delta} = \mathrm{Cl}_{\widetilde{D},\widetilde{\Delta}} \quad and \quad \mathrm{Cl}^*_{D\Delta} = \mathrm{Ds}_{\widetilde{D},\widetilde{\Delta}},$$

we can conclude:
(D, Δ) cohereditary $\Leftrightarrow$ $(\widetilde{D}, \widetilde{\Delta})$ minimal,
(D, Δ) maximal $\Leftrightarrow$ $(\widetilde{D}, \widetilde{\Delta})$ hereditary.

(2) While the Kuratowski closure operator K in Top is hereditary but not minimal, the dco in AbGrp given by torsion is maximal but not cohereditary. Indeed, considering for subgroups $A \leq B \leq X$ the composite projections

$$X \xrightarrow{p} X/A \xrightarrow{q} X/B$$

we see that closedness of $q \cdot p$ trivially implies closedness of p since B torsion implies A torsion. However, when $q \cdot p$ is sparse, so that B is torsion-free, we generally cannot at all conclude that $\ker q = B/A$ stays torsion-free.

For a category $\mathcal{X}$ with kernelpairs, coequalizers and a terminal object let us now consider the class $\mathcal{E} = \mathcal{R}eg\mathcal{E}pi\mathcal{X}$ and assume that pullbacks of regular epimorphisms along arbitrary morphisms are epic (although not necessarily regular). Then $\mathcal{X}$ has (orthogonal) $(\mathcal{R}eg\mathcal{E}pi\mathcal{X}, \mathcal{M}ono\mathcal{X})$-factorizations, and for a dco D of $\mathcal{E}$ in $\mathcal{X}$, to be considered as acting on kernelpairs rather than on their coequalizers (see Corollary 2.6), we obtain the following handy characterizations of the special properties discussed so far:

Proposition 3.7 *Under the stated hypotheses on $\mathcal{X}$, a dual closure operator D of $\mathcal{R}eg\mathcal{E}pi\mathcal{X}$ is*

- *idempotent if $D_X D_X K \cong D_X K$;*

- *weakly cohereditary if $D_{X/D_X K}(p(K))$ is the diagonal kernel pair on $X/D_X K$ (i.e., the kernelpair of $1_{X/D_X K}$; here $p : X \to X/D_X K$ is the coequalizer of $D_X K$);*

- *maximal if $D_X K \cong K \wedge D_X(X \times X)$;*

- *cohereditary if $D_{X/K}(p(L)) \cong p(D_X L)$ (with $p : X \to X/K$ the coequalizer of K),*

for all $K \leq L$ in kerp$X$ in each case.

Proof. The assertion regarding idempotency is obvious. The given characterization of maximality is a straight dualization of the corresponding characterization of minimal closure operators in the presence of a least element in sub$X$. Indeed, the product $X \times X$ (which exists as the kernel pair of $X \to 1$, with 1 terminal in $\mathcal{X}$) is a largest element in kerp$X$.

Dualizing the characterization of hereditary closure operators using pullbacks gives that the dco D of $\mathcal{E}$ in $\mathcal{X}$ is cohereditary if
$$Dq = p(D(q \cdot p)),$$
for all composable $p, q \in \mathcal{E}$; here $p(D(q \cdot p))$ denotes the pushout of $D(q \cdot p)$ along p. For $K \leq L$ in kerp$X$ and
$$p : X \to X/K, q : X/K \to X/L$$
the corresponding regular epimorphisms, the needed pushout exists, and the characterization of heredity translates into the stated condition on kernelpairs. The characterization of weak coheredity of D follows again by dualization of the characterization of wh closure operators; it amounts to specializing $K \leq L$ in the characterization of cohereditary dcos to $D_X K \leq K$. $\qquad\square$

3.3 Closure operators and dual closure operators induced by preradicals

Closure operators are known to be closely related to preradicals (see [23]), and so are dual closure operators, as we show next. Although these connections may be established much more generally (as has been done for closure operators in Section 5.5 of [23]), for simplicity, here we restrict ourselves to considering the category $\mathcal{X} = \mathsf{Mod}_R$ of R-modules (for a commutative unital ring R) with $\mathcal{M}$ and $\mathcal{E}$ the classes of mono- and epimorphisms, both being automatically regular.

Recall that a *preradical* $\mathbf{r}$ is simply a subfunctor of 1_{Mod_R}, so $\mathbf{r}$ assigns to every module X a submodule $\mathbf{r}X$ such that $f(\mathbf{r}X) \leq \mathbf{r}Y$ for every R-linear map $f : X \to Y$. Every closure operator C of $\mathcal{M}$ induces the preradical $\pi(C) = \mathbf{r}$, with $\mathbf{r}X = C_X 0$, *i.e.*, the C-closure of $0 \to X$. Ordering both the conglomerate of all closure operators and of all preradicals "objectwise" (so that $C \leq C'$ if $C_X M \leq C'_X M$ for all $M \leq X \in \mathsf{Mod}_R$, and $r \leq r'$ if $r_X \leq r'_X$ for all $X \in \mathsf{Mod}_R$), one obtains a monotone map
$$\pi : CO(\mathcal{X}, \mathcal{M}) \to PRAD(\mathcal{X}, \mathcal{M}).$$

As shown in [23], π has both a left adjoint (min) and a right adjoint (max), assigning to a predaical $\mathbf{r}$ the least and largest closure operators with π-image $\mathbf{r}$, $\min^{\mathbf{r}}$ and $\max^{\mathbf{r}}$, defined by
$$\min_X^{\mathbf{r}} M = M + \mathbf{r}X \text{ and } \max_X^{\mathbf{r}} M = p^{-1}(\mathbf{r}(X/M)),$$
respectively, for every $R-$module X and submodule $M \leq X$, with $p : X \to X/M$ the projection.

For every dual closure operator D of $\mathcal{E}$, to be thought of as acting on submodules $K \leq X$ rather than on their quotient maps $X \to X/K$, we trivially obtain a preradical $\pi^*(D) = \mathbf{r}$ with $\mathbf{r}X = D_X X$, *i.e.*, the D-closure of

$X \to 0 \cong X/X$. With the conglomerate of all dual closure operators ordered "objectwise" (so that $D \leq D'$ if $D_X K \leq D'_X K$ for all $K \leq X \in \mathsf{Mod}_R$), we again obtain a monotone map

$$\pi^* : DCO(\mathcal{X}, \mathcal{E}) \to PRAD(\mathcal{X}, \mathcal{M}) \tag{6}$$

which has both a left adjoint (min*) and a right adjoint (max*), assigning to a preradical $\mathbf{r}$ the dual closure operators min*$^{\mathbf{r}}$ and max*$^{\mathbf{r}}$, respectively, defined as follows:

$$\min_X^{*\mathbf{r}} K = \mathbf{r}K \text{ and } \max_X^{*\mathbf{r}} K = K \cap \mathbf{r}X,$$

for all $K \leq X \in \mathsf{Mod}_R$.

We can now illustrate the special properties of closure operators and dual closure operators discussed in Sections 3.1 and 3.2 in terms of properties of preradicals. Recall that a preradical $\mathbf{r}$ is

- idempotent if $\mathbf{rr}X = \mathbf{r}X$ for all R-modules X;

- a radical if $\mathbf{r}(X/\mathbf{r}X) = 0$ for all R-modules X;

- hereditary if $\mathbf{r}M = M \cap \mathbf{r}X$ for all $M \leq X \in \mathsf{Mod}_R$;

- cohereditary if $\mathbf{r}(X/M) = (\mathbf{r}X + M)/M$ for all $M \leq X \in \mathsf{Mod}_R$.

Proposition 3.8 *Let $\mathbf{r}$ be a preradical.*

(1) min$^{\mathbf{r}}$ is minimal and max$^{\mathbf{r}}$ is maximal, hence both are idempotent. In fact, min$^{\mathbf{r}}$ is the only minimal closure operator with induced preradical $\mathbf{r}$, and max*$^{\mathbf{r}}$ is the only maximal dual closure operator with induced preradical $\mathbf{r}$.*

(2) $\mathbf{r}$ is idempotent $\Leftrightarrow$ min$^{\mathbf{r}}$ is idempotent $\Leftrightarrow$ max$^{\mathbf{r}}$ is wh $\Leftrightarrow$ min$^{\mathbf{r}}$ is wh.*

(3) $\mathbf{r}$ is a radical $\Leftrightarrow$ max$^{\mathbf{r}}$ is idempotent $\Leftrightarrow$ min$^{\mathbf{r}}$ is wch $\Leftrightarrow$ max*$^{\mathbf{r}}$ is wch.*

(4) $\mathbf{r}$ is hereditary $\Leftrightarrow$ max$^{\mathbf{r}}$ is hereditary $\Leftrightarrow$ min$^{\mathbf{r}}$ is hereditary $\Leftrightarrow$ min$^{\mathbf{r}}$ = max*$^{\mathbf{r}}$.*

(5) $\mathbf{r}$ is cohereditary $\Leftrightarrow$ min$^{\mathbf{r}}$ is cohereditary $\Leftrightarrow$ max*$^{\mathbf{r}}$ is cohereditary $\Leftrightarrow$ max$^{\mathbf{r}}$ = min$^{\mathbf{r}}$.*

Proof. The statements involving $\mathbf{r}$ vis-a-vis max$^{\mathbf{r}}$ and min$^{\mathbf{r}}$ are well known (see [23]), so we can concentrate on those involving $\mathbf{r}$ vis-a-vis max*$^{\mathbf{r}}$ and min*$^{\mathbf{r}}$. Of these (1), (2) and (4) are immediate. For (3), note that weak coheredity of a dual closure operator D means

$$D_{X/D_X X}(K/D_X X) = 0$$

for all $K \leq X \in \mathsf{Mod}_R$ (see Prop. 3.7) which, for $D = \min^{*\mathbf{r}}$, means just $\mathbf{r}(K/\mathbf{r}K) = 0$. For $D = \min^{*\mathbf{r}}$ the condition reads as

$$K/(K \cap \mathbf{r}X) \cap \mathbf{r}(X/(K \cap \mathbf{r}X)) = 0 \tag{7}$$

which, for $K = X$, reduces to $\mathbf{r}(K/\mathbf{r}K) = 0$ again. Conversely, if $\mathbf{r}$ is a radical, since the canonical map $X/(K \cap \mathbf{r}X) \to X/\mathbf{r}X$ restricts to the preradicals of its domain and codomain, one obtains $\mathbf{r}(X/(K \cap \mathbf{r}X)) \subseteq \mathbf{r}X/(K \cap \mathbf{r}X)$ and, therefore, (7).

Finally, for (5), the fact that coheredity of $\mathbf{r}$ translates to coheredity of min*$^{\mathbf{r}}$ is immediate. Coheredity of max*$^{\mathbf{r}}$ means, by definition,

$$L/K \cap \mathbf{r}(X/K) = ((L \cap \mathbf{r}X) + K)/K \tag{8}$$

for all $K \leq L \leq X$. For $L = X$, (8) implies coheredity of $\mathbf{r}$. Conversely, $\mathbf{r}$ being cohereditary, the left-hand side of (8) becomes $(L \cap (\mathbf{r}X + K))/K$ which equals the right-hand side. $\square$

Note that, for $R = \mathbb{Z}$ and $\mathbf{r} = $ tor, max*$^{\mathbf{r}}$ = min*$^{\mathbf{r}}$ is the maximal dual closure operator of 2.5(2). From Prop. 3.8(4) one deduces the following Corollary:

Corollary 3.9 *Every hereditary preradical is induced by a unique dual closure operator. Every non-hereditary preradical may be induced by a non-maximal dual closure operator.*

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3.4 Dual closure and torsion

For a preradical $\mathbf{r}$ of Mod_R (as in 3.3), let

- $\mathcal{T}_\mathbf{r} = \{X \in \mathsf{Mod}_R : \mathbf{r}X = X\}$ be the class of $\mathbf{r}$-*torsion* modules, and

- $\mathcal{F}_\mathbf{r} = \{X \in \mathsf{Mod}_R : \mathbf{r}X = 0\}$ be the class of $\mathbf{r}$-*torsion-free* modules.

As $\mathcal{F}_\mathbf{r}$ $(\mathcal{T}_\mathbf{r})$ is closed under products and subobjects (coproducts and quotients) in Mod_R, $\mathcal{F}_\mathbf{r}$ $(\mathcal{T}_\mathbf{r})$ is epireflective (monocoreflective, respectively) in Mod_R. Keeping the notation of 3.3 and denoting by $SER(\mathcal{X}, \mathcal{M})$ $(SMC(\mathcal{X}, \mathcal{M}))$ the conglomerate of all full epireflective (monocoreflective, respectively) subcategories of $\mathcal{X} = \mathsf{Mod}_R$, ordered by inclusion, one obtains monotone maps

$$SMC(\mathcal{X}) \longleftarrow PRAD(\mathcal{X}, \mathcal{M}) \longrightarrow SER(\mathcal{X})^{op}$$

defined by $\mathbf{r} \mapsto \mathcal{T}_\mathbf{r}$ and $\mathbf{r} \mapsto \mathcal{F}_\mathbf{r}$, respectively.

It is easy to see that:

- $(\mathbf{r} \mapsto \mathcal{T}_\mathbf{r})$ has a left adjoint which assigns to $\mathsf{A} \in SMC(\mathcal{X})$ the idempotent preradical which, for every R-module X, selects the mono-coreflection of X into A; in case $\mathsf{A} = \mathcal{T}_\mathbf{r}$, this is the *idempotent core* $\mathbf{r}_{idp}$ of $\mathbf{r}$;

- $(\mathbf{r} \mapsto \mathcal{F}_\mathbf{r})$ has a right adjoint which assigns to $\mathsf{B} \in SER(\mathcal{X})$ the radical which, for every R-module X, selects the kernel of the epireflection of X into B; in case $\mathsf{B} = \mathcal{F}_\mathbf{r}$, this is the *radical hull* $\mathbf{r}_{rad}$ of $\mathbf{r}$.

Note that, consequently, one has

$$\mathcal{T}_\mathbf{r} = \mathcal{T}_{\mathbf{r}_{idp}} \text{ and } \mathcal{F}_\mathbf{r} = \mathcal{F}_{\mathbf{r}_{rad}}$$

for every preradical $\mathbf{r}$.

If one composes the two adjunctions, one obtains the correspondence

$$SMC(\mathcal{X}) \underset{l}{\overset{r}{\underset{\perp}{\rightleftarrows}}} SER(\mathcal{X})^{op} \ ,$$

with

$$r(\mathsf{A}) = \{B \in \mathsf{Mod}_R : \forall A \in \mathsf{A}, f : A \to B \ (f = 0)\},$$
$$l(\mathsf{B}) = \{A \in \mathsf{Mod}_R : \forall B \in \mathsf{B}, f : A \to B \ (f = 0)\}.$$

Proposition 3.10 *There is a commutative triangle*

of adjunctions. Moreover,

$$r(\mathcal{T}_\mathbf{r}) = \mathcal{F}_{\mathbf{r}_{idp}} \ and \ l(\mathcal{F}_\mathbf{r}) = \mathcal{T}_{\mathbf{r}_{rad}}$$
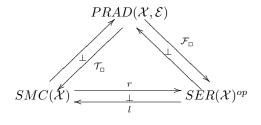
for all preradicals $\mathbf{r}$. Consequently, $r(\mathcal{T}_\mathbf{r}) = \mathcal{F}_\mathbf{r}$, whenever $\mathbf{r}$ is idempotent, and $l(\mathcal{F}_\mathbf{r}) = \mathcal{T}_\mathbf{r}$ whenever $\mathbf{r}$ is a radical.

Proof. The first statement has been shown above, and the second statement is a consequence of the first one. □

Let us now compose the two adjunctions on the left and the right sides of the triangle with the adjunction given by π^* of 3.3., with its left adjoint $\min^*$ and its right adjoint $\max^*$, respectively. Hence, for a dual closure operator D of $\mathcal{E}$ in Mod_R, let

Shriek$(D) := \{X \in \mathsf{Mod}_R : X \to 0 \text{ is } D\text{-closed}\}$ and Shriek$^*(D) := \{X \in \mathsf{Mod}_R : X \to 0 \text{ is } D\text{-sparse}\}$.

(These subcategories will be considered in a more general setting in 4.2 and 4.3). Considering D as operating on submodules rather than on quotients, we conclude that $X \in$ Shriek(D) $(X \in$ Shriek$^*(D))$ precisely when $D_X X = X$ $(D_X X = 0$, respectively). As $D_X X = \mathbf{r}X$, with $\mathbf{r} = \pi^*(D)$, we get

$$\text{Shriek}(D) = \mathcal{T}_{\pi^*(D)} \text{ and } \text{Shriek}^*(D) = \mathcal{F}_{\pi^*(D)}. \tag{9}$$

From this observation we obtain the following consequence of Prop. 3.10.

Corollary 3.11 *There is a commutative triangle*

$$
\begin{array}{c}
DCO(\mathcal{X}, \mathcal{E}) \\[1em]
\nearrow \min{}^{*\mathbf{r}} \quad \perp \quad \text{Shriek}^* \quad \nwarrow \\
\text{Shriek} \quad \max{}^{*\mathbf{r}} \\
SMC(\mathcal{X}) \xrightarrow[\;l\;]{\;r\;} SER(\mathcal{X})^{op}
\end{array}
$$

of adjunctions. The left adjoint of Shriek *assigns to a monocoreflective subcategory with coreflector* $\mathbf{r}$ *the dco* $\min{}^{*\mathbf{r}}$*, and the right adjoint of* Shriek* *assigns to an epireflective subcategory the dco* $\max{}^{*\mathbf{r}}$ *with* $\mathbf{r}$ *given by the kernels of the epireflections.* □

In the following section we will show that this commuting triangle may be established in a fairly abstract categorical context which includee its key application in topology. The adjoints of Shriek and Shriek* are being described in as concrete terms as possible.

4   Closure operators and their duals vis-á-vis subcategories

Throughout this section we consider a category $\mathcal{X}$ with a terminal object 1. We denote by $SUB(\mathcal{X})$ the conglomerate of all full subcategories of $\mathcal{X}$ that are replete (=closed under isomorphisms) and contain 1, ordered by inclusion.

4.1   The Preuß-Herrlich-Arhangel′skii-Wiegandt correspondence

Recall that an epimorphism in a category $\mathcal{X}$ is *strong* ([34]) if it is orthogonal to every monomorphism. Every regular epimorphism is strong, and both notions are equivalent if all morphisms factor into a regular epimorphism followed by a monomorphism. In deviation from the terminology used in other works, here we call a morphism $f : X \to Y$ *constant* if $!_X : X \to 1$ is a strong epimorphism and a factor of f; in other words: the strong epimorphic image of f (exists and) is isomorphic to 1. (As a particular consequence, in Set and any topological category over Set, a map with empty domain is never constant.) With $SUB(\mathcal{X})$ denoting the conglomerate of all full subcategories of $\mathcal{X}$, ordered by inclusion, the Preuß-Herrlich-Arhangel′skii-Wiegandt correspondence (see [39, 28, 2])) is given by the adjunction

$$
SUB(\mathcal{X}) \xrightarrow[\;l\;]{\;r\;} SUB(\mathcal{X})^{op}
$$

with

$$
r(\mathsf{A}) = \{B \in \mathcal{X} : \forall A \in \mathsf{A}, f : A \to B \ (f \text{ constant})\},
$$

$$
l(\mathsf{B}) = \{A \in \mathcal{X} : \forall B \in \mathsf{B}, f : A \to B \ (f \text{ constant})\},
$$

the *right* and *left-constant subcategories* induced by $\mathsf{A}, \mathsf{B} \in SUB(\mathcal{X})$, respectively. (We note that we have chosen to present r as *left* adjoint to l, for ease of composition of adjunctions later on: see Section 4.4.)

The prototypical example in the category Top is

$$
\mathsf{A} = l(r(\mathsf{A})) = \{(\text{non-empty}) \text{ connected spaces}\}
$$

$$
\mathsf{B} = r(\mathsf{A}) = \{\text{hereditarily disconnected spaces}\}.
$$

In AbGrp one has $\mathsf{A} = \{\text{torsion groups}\}$ and $\mathsf{B} = \{\text{torsion-free groups}\}$ being related by the correspondence.

Recall that a full subcategory A of $\mathcal{X}$ is *multicoreflective* if for every object X in $\mathcal{X}$,

- the distinct connected components of the comma category A/X (of morphisms with codomain X and domain in A) may be labelled by a (small) set, and

- every connected component of A/X has a terminal object.

In other words, for every object X there is a (small) family

$$(\rho_i : A_i \to X)_{i \in I}$$

of morphisms in $\mathcal{X}$ with codomain X and domain in A, such that every morphism $f : A \to X$ with $A \in \mathsf{A}$ factors as $\rho_i \cdot g = f$, for a unique pair (i, g). A is *(strongly) multi-monocoreflective* if all morphisms ρ_i are (strong) monomorphisms. Recall that a category is *weakly (co)well-powered* if, for every object X, the non-isomorphic strong monomorphisms into X (strong epimorphisms out of X) may be labelled by a set. In the prototypical example of $\mathcal{X} = \mathsf{Top}$, $\mathsf{A} = \{(\text{non-empty) connected spaces}\}$ the multicoreflection of X is given by the family of connected components, considered as subspaces of X.

Proposition 4.1 *Let A and B be in $SUB(\mathcal{X})$.*

(1) *If all sources in $\mathcal{X}$ have (strong epi, mono-source)-factorizations, then $r(\mathsf{A})$ is strongly epireflective in $\mathcal{X}$.*

(2) *If all sinks in $\mathcal{X}$ have (epi-sink, strong mono)-factorizations and if $\mathcal{X}$ is weakly well-powered, then $l(\mathsf{B})$ is strongly multi-monocoreflective in $\mathcal{X}$.*

(3) *If $\mathcal{X}$ is complete and cocomplete, weakly well-powered and weakly cowell-powered, then $r(\mathsf{A})$ is strongly epireflective in $\mathcal{X}$ and $l(\mathsf{B})$ is strongly multi-monocoreflective in $\mathcal{X}$.*

Proof. (1) It suffices to show that, for a monosource $(p_j : B \to B_j)_{j \in J}$ with all $B_j \in r(\mathsf{A})$, also $B \in r(\mathsf{A})$. Indeed, for any $f : A \to B$ with $A \in \mathsf{A}$ and all $j \in J$ one has a commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B \\
\downarrow & & \downarrow{\scriptstyle p_j} \\
1 & \xrightarrow[\ x_j\ ]{} & B_j
\end{array}
$$

with $A \to 1$ strongly epic, so that the diagonalization property makes f constant.

(2) Let us first observe that, for an epimorphism $e : A \to A'$ with $A \to 1$ strongly epic, also $A' \to 1$ is strongly epic; therefore, when $A \in l(\mathsf{B})$, also $A' \in l(\mathsf{B})$. Since every morphism $f : A \to X$ factors as

$$f = (A \xrightarrow{\ e\ } A' \xrightarrow{\ m\ } X),$$

with e epic and m strongly monic, we know that every connected component of the comma-category A/X contains (among its objects) a strong monomorphism of $\mathcal{X}$. But since $\mathcal{X}$ is weakly wellpowered, there is only a small set of non-isomorphic strong monomorphisms with codomain X. Consequently, A/X has only a small family of connected components. Let us label the family of connected components of A/X bijectively by the set I. The class of objects of the connected component with label $i \in I$ forms a sink in $\mathcal{X}$ with codomain X that has an (epi-sink, strong mono)-factorization. It now suffices to show that the strong monomorphism

$$\rho_i : A_i \to X$$

of the factorization has its domain lying in $l(\mathsf{B})$, and for that it suffices to show that $l(\mathsf{B})$ is closed under connected epic cocones. So, let

$$(u_j : A_j \to A)_{j \in J}$$

be epic, all $A_j \in \mathsf{A}$, where j runs through the object class of the (non-empty) connected category J, and consider a morphism $f : A \to B$ with $B \in \mathsf{B}$. Then, for every $j \in J$, one has a factorization

$$f \cdot u_j = (A_j \longrightarrow 1 \xrightarrow{\ x_j\ } B)$$

Furthermore, when there is a morphism $A_j \to A_k$, since $A_k \to 1$ is epic, one must have $x_j = x_k$. Consequently, since J is connected, the family (x_j) is given by a single morphism x, and since $(u_j)_{j \in J}$ is epic, we see that f factors through that morphism.

Moreover, as at the beginning of this proof, we see that $A \to 1$ is strongly epic.

(3) gives well-known sufficient conditions for the existence of the factorizations needed in (1) and (2) (see [1]).
$\square$

We note that, since $1 \in \mathsf{B}$, every $A \in l(\mathsf{B})$ has the property that $A \to 1$ is strongly epic. Therefore, when putting

$$SMC(\mathcal{X}) = \{\mathsf{A} \subseteq \mathcal{X} : \mathsf{A} \text{ strongly multi-monocoreflective in } \mathcal{X}, \ 1 \in \mathsf{A}, \ \& \ \forall A \in \mathsf{A} \ (A \to 1 \text{ is strongly epic})\},$$

$$SER(\mathcal{X}) = \{\mathsf{B} \subseteq \mathcal{X} : \mathsf{A} \text{ strongly epireflective in } \mathcal{X}\},$$

we obtain the following Corollary from Prop. 4.1:

Corollary 4.2 *The adjunction $r \dashv l$ restricts to an adjunction*

$$SMC(\mathcal{X}) \underset{l}{\overset{r}{\underset{\perp}{\rightleftarrows}}} SER(\mathcal{X})^{op},$$

provided that $\mathcal{X}$ is complete and cocomplete, weakly wellpowered and weakly cowellpowered.

A full characterization of subcategories A, B closed under this Galois correspondence may be obtained as in [15], which generalizes the work of [2] for $\mathcal{X} = \mathsf{Top}$. Here we give an easy characterization for objects to lie in $l(\mathsf{B})$ or $r(\mathsf{A})$, in terms of their B-reflections or A-multicoreflections, respectively.

Proposition 4.3 *For an object X in $\mathcal{X}$, let $\rho : X \to B$ be the strong epireflection morphism of $X \in \mathcal{X}$ into the subcategory B, and $(\rho_i : A_i \to X)_{i \in I}$ the multicoreflection of X into the subcategory A. Assume that $A \to 1$ is strongly epic, for all $A \in \mathsf{A}$. Then:*

(1) $X \in l(\mathsf{B}) \Leftrightarrow B \cong 1$,

(2) $X \in r(\mathsf{A}) \Leftrightarrow \forall i \in I (A_i \cong 1)$.

Proof. (1) "$\Rightarrow$": By hypothesis, ρ factors as

$$\rho = (X \xrightarrow{!_X} 1 \xrightarrow{x} B),$$

and since $x \cdot !_B \cdot \rho = x \cdot !_X = \rho$, one has $x \cdot !_B = 1_B$ and, hence, $B \cong 1$.
"$\Leftarrow$": Every morphism $X \to B' \in \mathsf{B}$ factors through ρ with $B \cong 1$ and, hence, is constant.
(2) "$\Rightarrow$": By hypothesis, every ρ_i factors through 1, and one concludes $A_i \cong 1$ similarly as in (1).
"$\Leftarrow$": Every morphism $A \to X$ with $A \in \mathsf{A}$ factors through one ρ_i with $A_i \cong 1$ and, since $A \to 1$ is strongly epic, is therefore constant. $\qquad \square$

Remark 4.4 Of course, for $\mathcal{X} = \mathsf{Mod}_R$, the Proposition reproduces the simple fact that $l(\mathsf{B})$ contains precisely those modules with trivial reflection into B, and $r(\mathsf{A})$ those modules with trivial coreflection into A, for $\mathsf{B} \in SER(\mathcal{X})$ and $\mathsf{A} \in SMC(\mathcal{X})$; see 3.4.

4.2 The Eilenberg-Whyburn dual closure operator

We consider a class $\mathcal{E} \subseteq Epi\mathcal{X}$ and assume, for simplicity, that $\mathcal{X}$ be $\mathcal{E}$-*cocomplete*, that is ([23]):

- pushouts of morphisms in $\mathcal{E}$ along arbitrary morphisms exist and belong to $\mathcal{E}$ again;

- multiple (= wide) pushouts of arbitrary sources of morphisms in $\mathcal{E}$ exists and belong to $\mathcal{E}$ again.

$\mathcal{E}$-cocompleteness guarantees in particular, the existence of left $\mathcal{E}$-factorizations in $\mathcal{X}$. If $\mathcal{X}$ is cocomplete and (weakly) cowell-powered, $\mathcal{X}$ is $\mathcal{E}$-cocomple with $\mathcal{E}$ the class of (strong) epimorphisms.

For a dual closure operator D of $\mathcal{E}$ in $\mathcal{X}$, we consider the full subcategory

$$\text{Shriek}(D) = \{X : (!_X : X \to 1) \in \text{Cl}_D^*\}$$

of $\mathcal{X}$. As we showed in Section 3.4, Shriek(D) coincides with the torsion class $\mathcal{T}_\mathbf{r}$ of the preradical $\mathbf{r} = \pi^*(D)$ for every dual closure operator D of the class $\mathcal{E}$ of (strong) epimorphisms in Mod_R.

Note that $!_X \in \text{Cl}_X^*$ forces in particular $!_X \in \mathcal{E}$. Let us call a full subcategory A of $\mathcal{X}$ $\mathcal{E}$-*admissible*, if

- $1 \in \mathsf{A}$ and $(A \to 1) \in \mathcal{E}$ for all $A \in \mathsf{A}$;

- $A' \in \mathsf{A}$ whenever $(A \to A') \in \mathcal{E}$ with $A \in \mathsf{A}$.

With the cancellation property of Corollary 3.4 one sees that $\mathrm{Shriek}(D)$ is $\mathcal{E}$-admissible. Consequently, denoting by $SUB(\mathcal{X}, \mathcal{E})$ the conglomerate of all $\mathcal{E}$-admissible full subcategories, we obtain a monotone map

$$\mathrm{Shriek} : DCO(\mathcal{X}, \mathcal{E}) \longrightarrow SUB(\mathcal{X}, \mathcal{E}). \qquad (10)$$

Here $DCO(\mathcal{X}, \mathcal{E})$ is the conglomeratee of all dcos of $\mathcal{E}$ in $\mathcal{X}$, ordered "objectwise", so that $D \leq D'$ whenever $D_X p \leq D'_X p$ for all $X \in \mathcal{X}, p \in \mathrm{quot}\, X$; see Prop. 2.4. It is easy to see that Shriek preserves infima, i.e.,

$$\mathrm{Shriek}\left(\bigwedge_i D_i\right) = \bigcap_i \mathrm{Shriek}(D_i).$$

Proposition 4.5 *For every dual closure operator D of $\mathcal{E}$ in $\mathcal{X}$, $\mathrm{Shriek}(D)$ is strongly multi-monocoreflective in $\mathcal{X}$.*

Proof. For the same reasons as in the proof of Proposition 4.1(2), it suffiices to show closure of $\mathrm{Shriek}(D)$ under connected epic cocones. So, we consider again an epic cocone

$$(u_j : A_j \to A)_{j \in J},$$

with J connected and all $A_j \to 1$ strongly epic and D-closed. Then also $A \to 1$ is strongly epic, and for every j one obtains a commutative square

$$\begin{array}{ccc}
A_j & \xrightarrow{u_j} & A \\
{\scriptstyle !_{A_j}}\downarrow & & \downarrow{\scriptstyle D!_A} \\
1 & \xrightarrow{v_j} & D_A 1
\end{array}$$

Since $(u_j)_{j \in J}$ is epic, so is $(v_j)_{j \in J}$. Moreover, whenever there is a morphism $t : A_j \to A_k$ with $u_k \cdot t = u_j$, we must have $v_j = v_k$ since $!_{A_j}$ is epic and

$$v_j \cdot !_{A_j} = D!_A \cdot u_j = D!_A \cdot u_k \cdot t = v_k \cdot !_{A_k} \cdot t = v_k \cdot !_{A_j}.$$

Consequently, since J is (non-empty and) connected, $v_j = v$ for all $j \in J$, and the split monomorphism v is epic since $(v_j)_{j \in J}$ is and, hence, an isomorphism. This makes $!_A$ D-closed. $\qquad \square$
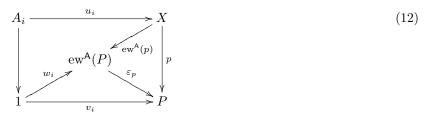
Now give an explicit construction of the left adjoint of Shriek which, to $\mathsf{A} \in SUB(\mathcal{X}, \mathcal{E})$, assigns the *Eilenberg-Whyburn dual closure operator* ew^{A}. Given $p : X \to P$ in $\mathcal{E}$, we consider the sink of all morphisms

$$u_i : A_i \to X, A_i \in \mathsf{A}\ (i \in I),$$

with $p \cdot u_i$ factoring through 1, i.e., we consider all commutative squares

$$\begin{array}{ccc}
A_i & \xrightarrow{u_i} & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
1 & \xrightarrow{v_i} & P
\end{array} \qquad (11)$$

We then form, for every $i \in I$, the pushout of $(A_i \to 1)$ along u_i, and then the multiple pushout $\mathrm{ew}^{\mathsf{A}}(p)$ of all these pushouts, which will complete every square as follows:

$$(12)$$

Because of $\mathcal{E}$'s (multiple) pushout stability, $\mathrm{ew}^{\mathsf{A}}(p) \in \mathcal{E}$. Also, $\varepsilon_p \in \mathcal{E}$ by the cancellation property of Definition 2.3.

Theorem 4.6 *For every $\mathcal{E}$-admissible full subcategory A of the $\mathcal{E}$-cocomplete category $\mathcal{X}$, ew^A is an idempotent dual closure operator of $\mathcal{X}$ with the property that, for every dual closure operator D of $\mathcal{E}$, one has*

$$\mathrm{ew}^\mathsf{A} \leq D \Leftrightarrow \mathsf{A} \subseteq \mathrm{Shriek}(D);$$

that is, ew *is left adjoint to* Shriek*, and one has*

$$\mathrm{ew}^{\mathrm{Shriek}(D)} \leq D \ \textit{and} \ \mathsf{A} \subseteq \mathrm{Shriek}(\mathrm{ew}^\mathsf{A}) \tag{13}$$

for all $D \in DCO(\mathcal{X},\mathcal{E})$ and $\mathsf{A} \in SUB(\mathcal{X},\mathcal{E})$.

Proof. The essential part of the proof that ew^A is a dual closure operator is to confirm that, given a morphism $(u,v) : p \to q$ in the category $\mathcal{E} \subseteq \mathcal{X}^2$, one obtains a morphism (omitting the superscript A in ew^A for ease of notation)

$$\mathrm{ew}_{u,v} : \mathrm{ew}_X P \to \mathrm{ew}_Y Q$$

that leads to a commutative diagram as in (3^*) of 2.3. For that, consider left $\mathcal{E}$-factorizations $m_i \cdot e_i = u \cdot u_i$, $n_i \cdot t_i = v \cdot v_i$ for all $i \in I$, with $e_i : A_i \to A_i'$, $t_i : 1 \to E_i$ in $\mathcal{E}$; since t_i is split mono we may assume $E_i = 1$.

By $\mathcal{E}$-admissibility of A, $A_i' \in \mathsf{A}$ and, hence, the back square of the cube above is a "contributing" square to the formation of the pushout $\mathrm{ew}(q)$. Consequently, for every $i \in I$, there is a morphism $k_i : 1 \to \mathrm{ew}_Y Q$ with

$$k_i \cdot !_{A'} = \mathrm{ew}(q) \cdot m_i \ \textit{and} \ \cdot k_i = n_i.$$

With the pushout property of $\mathrm{ew}(p)$ one now obtains the desired morphism $\mathrm{ew}_{u,v}$.

Let us also see that ew^A is idempotent. In fact, the factorization (12) of diagram (11) shows that every commutative square contributing to the formation of $\mathrm{ew}^\mathsf{A}(p)$ gives a square contributing to the formation of $\mathrm{ew}^\mathsf{A}(\mathrm{ew}^\mathsf{A}(p))$, and vice versa, with the top and left left arrow of (11) staying the same. Consequently,

$$\mathrm{ew}^\mathsf{A}(p) \cong \mathrm{ew}^\mathsf{A}(\mathrm{ew}^\mathsf{A}(p)).$$

Let us now consider a dual closure operator D of $\mathcal{E}$ and first assume $\mathrm{ew}^\mathsf{A} \leq D$. For every $A \in \mathsf{A}$, the square

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle !_A}\downarrow & & \downarrow{\scriptstyle !_A} \\ 1 & \longrightarrow & 1 \end{array}$$

contributes to the formation of the pushout $\mathrm{ew}_A(!_A)$. There is therefore a morphism $z : 1 \to \mathrm{ew}_A(1)$ with $z \cdot !_A = \mathrm{ew}_A(!_A)$. Since $\mathrm{ew}_A(!_A) \leq D_A !_A$, we then obtain a morphism $w : 1 \to D_A(1)$ with $w \cdot !_A = D_A(!_A)$. In particular, $w \in \mathcal{E}$, so the split monomorphism w must be an isomorphism. Consequently, $!_A$ is D-closed, which proves that $\mathsf{A} \subseteq \mathrm{Shriek}(D)$.

Conversely, if $!_A$ is D-closed for all $A \in \mathsf{A}$, also any pushout of $!_A$ is D-closed, which then makes also the multiple pushout $\mathrm{ew}_X(p)$ (for any $p : X \to Y$) D-closed. Consequently, $\mathrm{ew}^\mathsf{A} \leq D$. $\qquad\square$

One can compute $\mathrm{ew}^\mathsf{A}(p)$ more conveniently imposing some mild natural conditions on $\mathcal{X}$ and A:

Corollary 4.7 *If the $\mathcal{E}$-admissible full subcategory A is multi-monocoreflective in $\mathcal{X}$ and $\mathcal{X}$ has pullbacks, then $\mathrm{ew}^\mathsf{A}_X(p)$ is obtained as the multiple pushout of $A_i \to 1$ along u_i, where u_i runs through all multicoreflections of the fibres of p, composed with the canonical morphisms of the fibres into X.*

Proof. For every commutative square (11) contributing to the construction of $\mathrm{ew}^{\mathsf{A}}(p)$, $u_i : A_i \to X$ factors through the fibre $f^{-1}(v_i)$ of p and, hence, through one of the multicoreflection morphisms of the fibre $f^{-1}(v)$ into A. An easy examination now shows that, in the contributing square (11), the morphism u_i may be replaced by that multicoreflection morphism composed with $f^{-1}(v)$, without affecting the value of $\mathrm{ew}^{\mathsf{A}}(p)$. Hence, $\mathrm{ew}^{\mathsf{A}}(p)$ coincides with the multiple pushout of $A_i \to 1$ along u_i, where u_i runs through all multicoreflections of the fibres of p, composed with the canonical morphisms of the fibres into X. □

For $\mathsf{A} \in SUB(\mathcal{X}, \mathcal{E})$ let us call $p \in \mathcal{E}$ A-*monotone* if p is ew^{A}-closed, and an arbitrary morphism in $\mathcal{X}$ is A-*light* if it factors through an ew^{A}-sparse morphism in $\mathcal{E}$ followed by a morphism with trivial coreflection into $\mathcal{E}$. With Theorem 4.6 we then deduce from Corollary 3.4:

Corollary 4.8 *For every* $\mathsf{A} \in SUB(\mathcal{X}, \mathcal{E})$, $\mathcal{X}$ *has left* A-*monotone-factorizations. If* $\mathcal{E}$ *is closed under composition, so that* $\mathcal{X}$ *has orthogonal* $(\mathcal{E}, \mathcal{M})$-*factorizations, then every morphism factors (*A-*monotone,* A-*light) precisely when* ew^{A} *is weakly cohereditary, and this then constitutes again an orthogonal factorization system.*

Example 4.9 (1) If in Top, with $\mathcal{E}$ the class of regular (=strong) epimorphisms, we let A be the class of all (non-empty) connected spaces, the construction of ew^{A} leads to the Eilenberg-Whyburn (monotone, light)-factorization of morphisms whose codomain is T1. Here A-monotone and A-light assume the classical meaning: a map $f : X \to Y$ in Top is *monotone* (resp., *light*), if all fibers of f are connected (resp., hereditarily disconnected)). Indeed, given a quotient map $p : X \to P$, for every square (11) we see that the space A_i must map into a connected component of some fiber of p. Hence, the equivalence relation $\sim$ describing the quotient map $\mathrm{ew}^{\mathsf{A}}(p) : X \to X/\sim$ is given by

$$x \sim y \Leftrightarrow (p(x) = p(y)) \ \&(\exists A \in \mathsf{A}, A \subseteq p^{-1}(p(x))(x, y \in A)).$$

In other words, the related equivalence classes in X are precisely the connected components of the fibres of p. Note, however, that ew^{A} fails to be weakly hereditary, i.e., monotone quotient maps fail to be closed under composition in Top, which is why the map δ_p with $p = \delta_p \cdot \mathrm{ew}^{\mathsf{A}}(p)$ may fail to be light, unless its codomain is T1.

However, for $\mathcal{X}$ the category of compact Hausdorff spaces and A the subcategory of (non-empty) connected spaces, ew^{A} is weakly cohereditary, and Cor. 4.8 produces the monotone-light factorization in this classical context.

(2) For $\mathcal{X} = \mathsf{AbGrp}$ and $\mathsf{A} = \{$torsion groups$\}$, ew^{A} reproduces the prototypical example 2.9 (2). More generally, for $\mathcal{X} = \mathsf{Mod}_R$ and $\mathsf{A} = \mathcal{T}_{\mathbf{r}}$ (see Cor. 3.11) the $\mathbf{r}$-torsion class for $\mathbf{r}$ an idempotent preradical, $\mathrm{ew}^{\mathsf{A}} K = \mathbf{r} K = \min^{*\mathbf{r}} K$ for all $K \leq X$.

4.3 The Cassidy-Hébert-Kelly dual closure operator

For simplicity we now let $\mathcal{E}$ be the class of strong epimorphisms of $\mathcal{X}$ and assume *sources in $\mathcal{X}$ to have (strong epi, mono-source)-factorizations* (which is guaranteed if $\mathcal{X}$ is $\mathcal{E}$-cocomplete), *as well as finite limits*.

For every $X \in \mathcal{X}$, let

$$!_X = (X \xrightarrow{\eta_X} TX \xrightarrow{\mu_X} 1)$$

be a (strong epi, mono)-factorization, which defines a pointed endofunctor (T, η) of $\mathcal{X}$. For a dual closure operator D of $\mathcal{E}$ in $\mathcal{X}$, we consider the full subcategory

$$\mathrm{Shriek}^*(D) := \{X : \eta_X \in \mathrm{Ds}_D^*\},$$

thus defining a monotone map

$$\mathrm{Shriek}^* : DCO(\mathcal{X}, \mathcal{E}) \to SUB(\mathcal{X})^{op}.$$

As we showed in Section 3.4, $\mathrm{Shriek}^*(D)$ coincides with the torsion-free class $\mathcal{F}_{\mathbf{r}}$ of the preradical $\mathbf{r} = \pi^*(D)$ for every dual closure operator D of the class $\mathcal{E}$ of (strong) epimorphisms in Mod_R.

It is easy to see that Shriek^* transforms suprema into intersections, and we will now embark on describing its right adjoint, after restricting the codomain of Shriek^*.

Proposition 4.10 $\mathrm{Shriek}^*(D)$ *is strongly epireflective in* $\mathcal{X}$.

Proof. Let $(p_i : X \to Y_i)_{i \in I}$ be a mono-source with $Y_i \in \mathrm{Shriek}^*(D)$ for all $i \in I$ in the commutative diagram

$$(14)$$

$$
\begin{array}{ccc}
X & \xrightarrow{\;p_i\;} & Y_i \\
{\scriptstyle D(\eta_X)}\downarrow & & \downarrow{\scriptstyle D(\eta_{Y_i})} \\
D_X T X & \xrightarrow[D_{p_i},T_{p_i}]{} & D_{Y_i} T Y_i
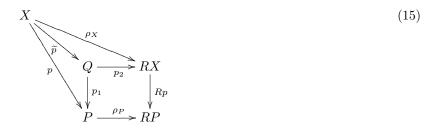\end{array}
$$

the morphisms $D(\eta_{Y_i})$ are isos, by hypothesis. Hence, $D(\eta_X)$ appears as a first factor of the mono-source $(p_i)_{i \in I}$ and is therefore monic, in addition to being strongly epic. Consequently, $X \in \mathrm{Shriek}^*(D)$.

Being closed under mono-sources, our hypotheses on $\mathcal{X}$ guarantee that $\mathrm{Shriek}^*(D)$ is strongly epireflective. $\square$

Keeping the notation of 4.2, we now construct a right adjoint to

$$\mathrm{Shriek}^* : DCO(\mathcal{X}, \mathcal{E}) \to SER(\mathcal{X})^{op},$$

employing a construction used in [11]. Given a strongly epireflective subcategory B of $\mathcal{X}$, with reflector R and unit ρ, for every $p : X \to P$ in $\mathcal{E}$ one forms the inscribed pullback diagram of the naturality diagram for p:

$$(15)$$

$$
\begin{array}{ccc}
X & & \\
& \searrow^{\rho_X} & \\
\widetilde{p}\searrow & & \\
p\searrow & Q \xrightarrow{\;p_2\;} & RX \\
& \downarrow{\scriptstyle p_1} & \downarrow{\scriptstyle Rp} \\
& P \xrightarrow[\rho_P]{} & RP
\end{array}
$$

The induced morphism $\widetilde{p}$ has a (strong epi, mono)-factorization

$$\widetilde{p} = m_p \cdot \mathrm{chk}^{\mathsf{B}}(p)$$

which defines the *Cassidy-Hébert-Kelly dual closure of p w.r.t.* B. To see that $\mathrm{chk}^{\mathsf{B}}$ is in fact a dual closure operator of $\mathcal{E}$ is a straightforward exercise.

Theorem 4.11 *For all $D \in DCO(\mathcal{X}, \mathcal{E})$, $\mathsf{B} \in SER(\mathcal{X})$, one has*

$$D \le \mathrm{chk}^{\mathsf{B}} \Leftrightarrow \mathrm{Shriek}^*(D) \supseteq \mathsf{B};$$

that is, chk *is right adjoint to* Shriek^*, *and*

$$D \le \mathrm{chk}^{\mathrm{Shriek}^*(D)} \quad and \quad \mathrm{Shriek}^*(\mathrm{chk}^{\mathsf{B}}) \supseteq \mathsf{B} \qquad (16)$$
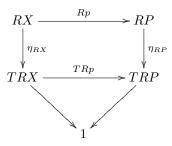
for all $D \in DCO(\mathcal{X}, \mathcal{E})$ and $\mathsf{B} \in SER(\mathcal{X}, \mathcal{E})$.

Proof. By construction, for every $p : X \to P$ in $\mathcal{E}$ one has $\mathrm{chk}^{\mathsf{B}}(p) \le \rho_X$ in $\mathrm{quot}X$. Hence, when $X \in \mathsf{B}$, so that ρ_X is an isomorphism, from $D \le \mathrm{chk}^{\mathsf{B}}$ one obtains

$$Dp \le \mathrm{chk}^{\mathsf{B}}(p) \le 1_X,$$

which makes Dp an isomorphism in $\mathcal{X}$. This is true in particular for $p = \eta_X$, which shows $X \in \mathrm{Shriek}^*(D)$.

Conversely, let us first observe that the ρ-naturality diagram for $p \in \mathcal{E}$ and the composition and cancellation properties of $\mathcal{E}$ show that Rp is strongly epi. In fact, if $\mathsf{B} \subseteq \mathrm{Shriek}^*(D)$, then Rp is D-sparse, as an examination of the following diagram shows:

$$
\begin{array}{ccc}
RX & \xrightarrow{\;Rp\;} & RP \\
{\scriptstyle \eta_{RX}}\downarrow & & \downarrow{\scriptstyle \eta_{RP}} \\
TRX & \xrightarrow[TRp]{} & TRP \\
& \searrow \quad \swarrow & \\
& 1 &
\end{array}
$$

Commutativity of its upper part shows that TRp is strongly epic, while the commutativity of the the lower part makes TRp monic and, hence, an isomorphism. But since η_{RX} (and η_{RP}) are D-sparse by hypothesis, also Rp is D-sparse. Considering the morphism $(\rho_X, \rho_P) : p \to Rp$ in the category $\mathcal{E}$, we see that there is a morphism

$$t : D_X P \to RX$$

with $t \cdot Dp = \rho_X$ and $Rp \cdot t = \rho_X \cdot \varepsilon_p$. In the notations of diagram (15) we therefore have a morphism $s : D_X R \to Q$ with $p_i \cdot s = \varepsilon_p$ and $p_2 \cdot s = t$, which actually must factor through m_p. This shows $Dp \leq \mathrm{chk}^{\mathsf{B}}(p)$, as desired. $\square$

In our next example we compute $\mathrm{chk}^{\mathsf{B}}$ in the category Mod_R, for R a unital ring.

Example 4.12 Let B be a (strongly) epireflective subcategory of Mod_R. Then there exists a radical $\mathbf{r}$ such that $\mathsf{B} = \mathcal{F}_{\mathbf{r}}$. Since right adjoints are unique (up to isomorphism), one has

$$\mathrm{chk}^{\mathsf{B}} K = K \cap \mathbf{r}X = \max{}^{*\mathbf{r}}(K).$$

for all $K \leq X \in \mathsf{Mod}_R$. In particular, $\mathrm{chk}^{\mathsf{B}}$ is a maximal dual closure operator which, for $\mathsf{B} = \{$torsion-free groups$\}$ in AbGrp, returns the prototypical example 2.5(2).

The argument just given works also in the general context of this section, provided that strong epimorphisms in $\mathcal{X}$ are regular, which holds when $\mathcal{X}$ has (regular epi, mono)-factorizations. Since in diagram (15) the kernelpair of $\widetilde{p}$ is the meet of the kernelpairs of p and of ρ_X, when letting the dual closure operator $\mathrm{chk}^{\mathsf{B}}$ operate on kernelpairs rather than on their regular quotient maps, we obtain the formula

$$\mathrm{chk}^{\mathsf{B}}_X(K) = K \wedge \mathrm{chk}^{\mathsf{B}}_X(X \times X).$$

and, therefore, the following Corollary.

Corollary 4.13 *If strong epimorphisms in $\mathcal{X}$ are regular, the Cassidy-Hébert-Kelly dual closure operator $\mathrm{chk}^{\mathsf{B}}$ is maximal and, in particular, idempotent, for every strongly epireflective subcategory B of $\mathcal{X}$.*

For $\mathsf{B} \in SER(\mathcal{X}, \mathcal{E})$, let us call $p \in \mathcal{E}$ B -*concordant* if p is $\mathrm{chk}^{\mathsf{B}}$-closed, and an arbitrary morphism B -*dissonant* if it factors through a $\mathrm{chk}^{\mathsf{B}}$-sparse morphism in $\mathcal{E}$ followed by a monomorphism. It is well known (see [11, 23]) that $p \in \mathcal{E}$ is B-*concordant if, and only if, Rp is an isomorphism* (where R is the reflector into B). Indeed, in the notation of (15), if Rp is an isomorphism, so are p_1 and m_p, so that $p \cong \mathrm{chk}^{\mathsf{B}}p$; conversely, if $p_1 \cdot m_p$ is an isomorphism, an application of R to diagram (15) gives, with $d := p_2 \cdot (p_1 \cdot m_p)^{-1}$, that $Rd \cdot Rp = R\rho_X$ and $RRp \cdot Rd = R\rho_P$ are isomorphisms, and then so are Rd and finally Rp, as desired.

Corollary 4.14 *Let strong epimorphisms be regular in $\mathcal{X}$, and let $\mathsf{B} \in SER(\mathcal{X}, \mathcal{E})$. Then every morphism factors into a B-concordant morphism followed by a B-dissonant morphism precisely when $\mathrm{chk}^{\mathsf{B}}$ is weakly cohereditary, and in this case these factorizations constitute an orthogonal factorization system of $\mathcal{X}$.*

Example 4.15 Recall that a continuous map $f : X \to Y$ in Top is said to be *concordant* if every fibre of f is contained in a quasi-component of X, and it is *dissonant* if every quasi-component of X intersects a fibre of f only in at most one point. For B the strongly epireflective subcategory having as objects all totally disconnected spaces (i.e., spaces in which the quasi-components are trivial), the factorization described by Corollary 4.14 is precisely the (concordant,dissonant)-factorization established by [17].

4.4 Connecting the three correspondences

We now show that the Preuß-Herrlich-Arhangel′skii-Wiegandt correspondence is the composite of the adjunctions defining the Eilenberg-Whyburn and the Cassidy-Hébert-Kelly dual closure operators. For simplicity, in what follows, *the category $\mathcal{X}$ is assumed to be complete and cocomplete, weakly wellpowered and weakly cowell-powered, and $\mathcal{E}$ denotes the class of strong epimorphisms in $\mathcal{X}$.*
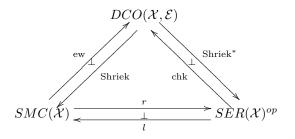
Proposition 4.16 *For every strongly epireflective subcategory B of $\mathcal{X}$,*

$$l(\mathsf{B}) = \mathrm{Shriek}(\mathrm{chk}^{\mathsf{B}}).$$

Proof. For $X \in \mathcal{X}$, let $\rho_X : X \to RX$ be the B-reflection. When $X \in l(\mathsf{B})$ one has $RX \cong 1$ (see Proposition 4.3), and also $R1 \cong 1$ (since $1 \in \mathsf{B}$). Consequently, all four arrows in the defining pullback square for $\mathrm{chk}^\mathsf{B}(!_B)$ are isomorphisms (see (15)), which shows that $!_X$ is chk^B-closed. Thus, $X \in \mathrm{Shriek}(\mathrm{chk}^\mathsf{B})$.

Conversely, assuming $X \in l(\mathsf{B})$, $!_X$ must be chk^B-closed and keeping the notation of (15) with $p = !_X$, we see that p_1 must be an isomorphism, since ρ_1 and therefore p_2 are. Consequently, $RX \cong 1$, as desired. □

Corollary 4.17 *The Preuß-Herrlich-Arhangel'skii-Wiegandt correspondence factors as*

Proof. By Proposition 4.5 the codomain of Shriek may be restricted as indicated, and by Proposition 4.16, the right adjoints in this diagram commute. Consequently, also the left adjoints do. □

Corollary 4.18 *For every strongly multi-monocoreflective subcategory* A *of* $\mathcal{X}$.

$$r(\mathsf{A}) = \mathrm{Shriek}^*(\mathrm{ew}^\mathsf{A}).$$

We are now ready to exhibit $DCO(\mathcal{X}, \mathcal{E})$ as an overarching environment for $SER(\mathcal{X})$ and, under some restrictive hypotheses on $\mathcal{X}$, also for $SMC(\mathcal{X})$.

Theorem 4.19 *(1) For every* $\mathsf{B} \in SER(\mathcal{X})$, *one has*

$$\mathrm{Shriek}^*(\mathrm{chk}^\mathsf{B}) = \mathsf{B} \tag{17}$$

for all $B \in SER(\mathcal{X})$. *Hence,* chk *embeds* $SER(\mathcal{X})^{op}$ *fully and reflectively into* $DCO(\mathcal{X}, \mathcal{E})$.

(2) Let $\mathcal{X}$ *be a topological category over* Set, *such that the terminal object is a generator of* $\mathcal{X}$. *Then, for every* $\mathsf{A} \in SMC(\mathcal{X})$, *one has*

$$\mathrm{Shriek}(\mathrm{ew}^\mathsf{A}) = \mathsf{A}. \tag{18}$$

Hence, ew *embeds* $SMC(\mathcal{X})$ *fully and coreflectively into* $DCO(\mathcal{X}, \mathcal{E})$.

Proof. (1) As "$\supseteq$" holds by adjunction, to prove "$\subseteq$" consider $X \in \mathcal{X}$ with $\mathrm{ew}_X^\mathsf{B}(\eta_X)$ an isomorphism in $\mathcal{X}$. In diagram (15), with $p = \eta_X$ and $P = TX$, the strong epimorphism $\rho_{TX} = \rho_P$ is monic (as a first factor of μ_X), hence an isomorphism. Consequently, also p_2 and m_p are isomorphisms, and then $\rho_X = p_2 \cdot m_p \cdot \mathrm{ew}^\mathsf{B}(\eta_X) : X \to RX$ is an isomorphism as well. Consequently, $X \in \mathsf{B}$.

(2) Again, "$\supseteq$" holds by adjunction. For "$\subseteq$", consider $X \in \mathrm{Shriek}(\mathrm{ew}^\mathsf{A})$ with A-multicoreflections $u_i : A_i \to X$ ($i \in I$), which may be taken as inclusion maps and will then form a partition of the set X. As remarked in 4.6(2), $\mathrm{ew}^\mathsf{A}(!_X)$ is the multiple pushout $P = \mathrm{ew}_X^\mathsf{A}(1)$ of all pushouts P_i of $A_i \to 1$ along u_i ($i \in I$). In a topological category over Set, every P_i is obtained from X by collapsing A_i into a singleton and keeping the remaining A_js unchanged; furthermore, $|P| = |I|$. But since $!_X$ is ew^A-closed, $|P| = 1$. Consequently, A is a strongly monocoreflective subcategory of $\mathcal{X}$. But since A contains the terminal object 1 of X, which is a generator of $\mathcal{X}$ by hypothesis, the strongly monic coreflection morphism of X is also epic and, hence, an isomorphism. This proves $X \in \mathsf{A}$. □

Remark 4.20 (1) By Prop. 4.16 and by adjunction, one has

$$\mathrm{ew}^{l(\mathsf{B})} = \mathrm{ew}^{\mathrm{Shriek}(\mathrm{chk}^\mathsf{B})} \le \mathrm{chk}^\mathsf{B}$$

for all $\mathsf{B} \in SER(\mathcal{X})$. The last inequality may be proper, even for $\mathcal{X} = \mathsf{Top}$. Indeed, consider the full subcategory B of all totally disconnected spaces in Top. Fix a hereditarily disconnected space that is not totally

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disconnected; the classical examples to this effect were given by Knaster, Kuratowski and Mazurkiewicz about ninety years ago. Then the reflexion map $p := \rho_X : X \to RX$ onto B is $\mathrm{chk}^{\mathsf{B}}$-closed since it is consonant. On the other hand, X is hereditarily disconnected, so that the map p is light, i.e., $\mathrm{ew}^{l(\mathsf{B})}$-sparse, as $l(\mathsf{B})$ is exactly the subcategory of connected spaces. Hence,

$$\mathrm{chk}^{\mathsf{B}}(p) = p \neq \mathrm{ew}^{l(\mathsf{B})}(p).$$

(2) For the same reasons as in (1) one also has

$$\mathrm{ew}^{\mathsf{A}} \leq \mathrm{chk}^{\mathrm{Shriek}^*(\mathrm{ew}^{\mathsf{A}})} = \mathrm{chk}^{r(\mathsf{A})}$$

for all $\mathsf{A} \in SMC(\mathcal{X})$. Again, the inequality may be proper. Indeed, in $\mathcal{X} = \mathsf{Mod}_R$ consider an idempotent non-hereditary radical $\mathbf{r}$ of $\mathcal{X}$, such as, in the case $R = \mathsf{Z}$, the radical defined by the maximal divisible subgroup of an abelian group. For $\mathsf{A} = \mathcal{T}_r$ one has $\mathrm{ew}^{\mathsf{A}} = \min^{*\mathbf{r}}$ by Example 4.9 (2). So, $\pi^*(\mathrm{ew}^{\mathsf{A}}) = \mathbf{r}$. Then $\mathrm{Shriek}^*(\mathrm{ew}^{\mathsf{A}}) = \mathcal{F}_{\mathbf{r}}$, by (9). From Example 4.12 one obtains

$$\mathrm{chk}^{\mathrm{Shriek}^*(\mathrm{ew}^{\mathsf{A}})} = \mathrm{chk}^{\mathcal{F}_{\mathbf{r}}} = \max^{*\mathbf{r}}.$$

Now $\mathrm{ew}^{\mathsf{A}} < \mathrm{chk}^{\mathrm{Shriek}^*(\mathrm{ew}^{\mathsf{A}})}$ follows from $\min^{*\mathbf{r}} < \max^{*\mathbf{r}}$, which is due to Proposition 3.8 (4).

5 Further examples

5.1 Some dual closure operators for groups and rings

We begin with a couple of examples of dual closure operators for surjective homomorphisms (i.e., of regular epimorphisms) in the categories Grp of groups and Rng of unital rings. We again describe them as operating on the kernels of homomorphisms, rather than on the respective quotient maps, that is: on normal subgroups and ideals, respectively. Our examples in Grp are of the form $\min^{*\mathbf{r}}$ and $\max^{*\mathbf{r}}$ for a preradical $\mathbf{r}$, defined like in Section 3.3 as a subfunctor of the identity functor. Since $\mathbf{r}G$ stays invariant under endomorphisms of the group G, it is a characteristic subgroup of G and, hence, normal; moreover, for K normal in G, $\mathbf{r}K$ is even normal in G, which makes $\min_G^{*\mathbf{r}} K = \mathbf{r}K$ well defined. This, of course, is trivially true for $\max_G^{*\mathbf{r}} K = K \cap \mathbf{r}G$.

Example 5.1 Assigning to a group G its commutator subgroup $\mathbf{c}G = G'$ defines a preradical of Grp that is actually a radical, but not idempotent. Hence, $\min^{*\mathbf{c}}$ and $\max^{*\mathbf{c}}$ are both weakly cohereditary, but only $\max^{*\mathbf{c}}$ is idempotent.

We recall that a group G is perfect when $G = \mathbf{c}G$ coincides with its commutator subgroup. The idempotent hull of the (pre)radical $\mathbf{c}$ assigns to G its largest perfect normal subgroup $\mathbf{p}G$ of G. (To confirm that such a subgroup exists, note that the subgroup generated by any family of normal perfect subgroups is still a normal perfect subgroup.) The resulting idempotent preradical $\mathbf{p}$ is again a radical, so the minimal dco $\min^{*\mathbf{p}}$ is idempotent and weakly cohereditary.

Example 5.2 One defines a family of dual closure operators D_n, $n \in \mathbb{N}$, in the category Rng of unital rings by assigning to an ideal I of a ring R the ideal I^n of finite sums of n-fold products of elements in I. (Note that ideals represent quotient maps but are general not subobjects of the ambient rings.) These dcos are all weakly cohereditary but generally fail to be idempotent.

5.2 Correspondences for closure operators and their duals for R-modules

Here, for a fixed R-module K, we establish a Galois correspondence between closure operators of subobjects and dual closure operators of quotient maps in Mod_R, under some restrictions on K that will come into play only in Proposition 5.7 below. Again, dual closure operators are (like closure operators) presented as operating on kernels of homomorphisms, i.e. on submodules.

For an R-module X, let $X^* = \hom(X, K)$ be its K-dual and

$$\eta_X : X \to X^{**}, \quad x \mapsto (\lambda \mapsto \lambda(x)),$$

be the (co)unit of the self-adjoint endofunctor $\hom(-, K)$. For a submodule $A \leq X$ one has the restriction map

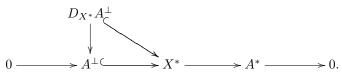$$X^* \to A^*, \lambda \mapsto \lambda \restriction_A,$$

which is surjective if K is injective. We denote its kernel by

$$A^\perp = \{\lambda \in X^* : \lambda \upharpoonright_A = 0\}.$$

Let now D be a dco of $\mathcal{R}eg\mathcal{E}pi\mathsf{Mod}_R$, to be thought of as operating on the kernels of the quotient maps rather than on the quotient maps themselves (see 2.6), and define

$$\check{D}_X A = \eta_X^{-1}((D_{X^*} A^\perp)^\perp).$$

(Note that in this formula the inner $^\perp$ operates on a submodule A of X while the outer $^\perp$ operates on a submodule of X^*.) One has the following diagram in which the bottom row is short exact if K is injective (which guarantees surjectivity of $X^* \to A^*$):

$$D_{X^*} A^\perp$$

$$0 \longrightarrow A^\perp \lhook\joinrel\longrightarrow X^* \longrightarrow A^* \longrightarrow 0.$$

We deduce that $A^{\perp\perp} \leq (D_{X^*} A^\perp)^\perp$, since $(-)^\perp$ is order-reversing. Since obviously $\eta_X(A) \leq A^{\perp\perp}$, we obtain

$$A \leq \eta_X^{-1}(A^{\perp\perp}) \leq \eta_X^{-1}((D_{X^*} A^\perp)^\perp) = \check{D}_X A. \tag{19}$$

In case K is injective, one has also the following short exact sequence:

$$0 \longrightarrow (D_{X^*} A^\perp)^\perp \lhook\joinrel\longrightarrow X^{**} \longrightarrow (D_{X^*} A^\perp)^* \longrightarrow 0.$$

Remarks 5.3 (1) For D the discrete dual closure operator, one has

$$\check{D}_X A = \eta_X(A^{\perp\perp}) = \{x \in X : \lambda \in X^*(\lambda \upharpoonright_A = 0 \Rightarrow \lambda(x) = 0)\}.$$

Hence, $\check{D}$ is precisely the K-regular closure operator of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$ (see [23]). *This closure operator is discrete again precisely when K is a cogenerator of Mod_R.* Indeed, recall that K is a cogenerator if, and only if, for every $0 \neq x \in X \in \mathsf{Mod}_R$ one can find an R-linear map $\varphi : X \to K$ with $\varphi(x) \neq 0$. Trading X for X/A we see that, for every $x \notin A \leq X$, one finds $\lambda : X \to K$ with $\lambda(x) \neq 0$ but $\lambda \upharpoonright_A = 0$, which means $\check{D}_X A = A$.

(2) Although in some examples the choice of the cogenerator does not matter (see Example 5.5 below), as a quite canonical choice of a cogenerator one can take the minimal injective cogenerator U of Mod_R, defined as the injective hull of the direct sum $S = \bigoplus_{i \in I} S_i$ of all (up to isomorphism) simple R-modules taken with multiplicity one. Since each of these S_i is a cyclic R-module, isomorphic to R/M for some maximal ideal M, it is clear that there is indeed a set of representatives of these simple modules. To see that U is indeed a cogenerator, observe that, for every $X \in \mathsf{Mod}_R$ and every non-zero element $x \in X$, the submodule Rx of X, being isomorphic to a cyclic module of the form R/J for some proper ideal J of R, admits a homomorphism $f : Rx \to S_i \leq U$ for some $i \in I$ with $f(x) \neq 0$. Since U is injective, this homomorphism extends to a homomorphism $\lambda : X \to U$ with $\lambda(x) \neq 0$.

Proposition 5.4 *For every dco of $\mathcal{R}eg\mathcal{E}pi\mathsf{Mod}_R$, $\check{D}$ is a closure operator of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$.*

Proof. We have to verify conditions 1–3 of 2.2(3).
  1. Condition 1 follows from (19).
  2. Since the operation $(-)^\perp$ is order-reversing, $\check{D}_X$ is (like D_X) order-preserving.
  3. For $f : X \to Y$ in Mod_R and $A \leq X$ we first note that

$$(f(A))^\perp = (f^*)^{-1}(A^\perp), \tag{$*$}$$

with $f^* = \hom(f, K) : Y^* \to X^*$. Indeed,

$$\begin{aligned} \kappa \in (f(A))^\perp &\Leftrightarrow \kappa \upharpoonright_{f(A)} = 0 \\ &\Leftrightarrow f^*(\kappa) \upharpoonright_A = \kappa \cdot f \upharpoonright_A = 0 \\ &\Leftrightarrow f^*(\kappa) \in A^\perp. \end{aligned}$$

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Furthermore, for $B \leq Y$ one has

$$f^*(B^\perp) \leq \left(f^{-1}(B)\right)^\perp, \qquad\qquad (**)$$

since for $\kappa \in B^\perp$, $\kappa \restriction_B = 0$ implies $\kappa \cdot f \restriction_{f^{-1}(B)} = 0$, so that $f^*(\kappa) \in ((f^{-1}(B))^\perp)$.

Trading f for f^* and B for $D_{X^*}(A^\perp)$ the inclusion $(**)$ reads as

$$f^{**}(D_{X^*}(A^\perp)) \leq \left((f^*)^{-1}(D_{X^*}A^\perp)\right)^\perp. \qquad\qquad (***)$$

We now conclude:

$$
\begin{aligned}
f(\check{D}_X A) = f\left(\eta_X^{-1}\left(D_{X^*}(A^\perp)\right)^\perp\right) \\
\leq \eta_Y^{-1}\left(f^{**}\left((D_{X^*}(A^\perp))^\perp\right)\right) \\
\leq \eta_Y^{-1}\left(\left((f^*)^{-1}(D_{X^*}A^\perp)\right)^\perp\right) \\
\leq \eta_Y^{-1}\left(\left(D_{Y^*}\left((f^*)^{-1}(A^\perp)\right)\right)^\perp\right) \\
\leq \eta_Y^{-1}\left(\left(D_{Y^*}\left((f(A))^\perp\right)\right)^\perp\right) = \check{D}_Y(f(A)).
\end{aligned}
$$

$\square$

Example 5.5 Let $R = \mathbb{Z}$ be the ring of integers, and let K be a cogenerator of $\mathsf{Mod}_R = \mathsf{AbGrp}$. (For example, for $K = \mathbb{T}$ the circle group and $X \in \mathcal{X} = \mathsf{AbGrp}$, the group X^{**} is precisely the *Bohr compactification* of the discrete group X.) Let us explicitly compute the closure operator $\check{D}$ for our prototypical dco $D = \min^{*\mathrm{tor}} = \max^{*\mathrm{tor}}$ (see Section 3.3). Note that for an Abelian group X one has $\mathrm{tor}X = \bigcup_n X[n!]$, where $X[n!] = \{x \in X : n!x = 0\}$. Hence, for $A \leq X$, an easy calculation gives the following steps:

$$\check{D}_X(A) = \eta_X^{-1}(\mathbf{t}(A^\perp)^\perp) = \eta_X^{-1}\left(\left(\bigcup_n A^\perp[n!]\right)^\perp\right) = \eta_X^{-1}\left(\left(\bigcup_n A^\perp \cap (X^*[n!])\right)^\perp\right) =$$

$$\eta_X^{-1}\left(\left(\bigcup_n A^\perp \cap (n!X)^\perp\right)^\perp\right) = \eta_X^{-1}\left(\left(\bigcup_n (A + (n!X))^\perp\right)^\perp\right) = \eta_X^{-1}\left(\left(\bigcap_n A + (n!X)\right)^{\perp\perp}\right) = \bigcap_n A + (n!X).$$

We now see that the closure operator $\check{D}$ coincides with the maximal closure operator corresponding to the preradical given by the *first Ulm subgroup*

$$\mathbf{u}^1 X = \bigcap_n n!X.$$

Hence, if $D = \min^{*\mathrm{tor}} = \max^{*\mathrm{tor}}$, then $\check{D} = \max^{\mathbf{u}^1}$. In particular: $\check{D}$ *is independent of the choice of the cogenerator K!*

Remark 5.6 The previous example suggests that, for every preradical $\mathbf{r}$ of Mod_R one should first introduce and study the *dual preradical* $\mathbf{r}^*$, defined by

$$\mathbf{r}^*(X) := \eta_X^{-1}(\mathbf{r}(X^*)^\perp).$$

One observes that the correspondence $\mathbf{r} \mapsto \mathbf{r}^*$ between preradicals of Mod_R is order reversing. Furthermore, for $R = \mathbb{Z}$, $n \in \mathbb{Z}$, and the preradical $\mathbf{r}_n$ of AbGrp defined by $\mathbf{r}_n X = X[n]$, one has $\mathbf{r}_n^* X = nX$. Finally,

$$\mathrm{tor}^* = \mathbf{u}^1,$$

where $\mathbf{u}^1 X$ is the first Ulm subgroup of X (see Example 5.5).

With K a cogenerator of Mod_R we are able to show a converse statement to Proposition 5.4, using the same construction as before: for a closure operator C of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$, let

$$\check{C}_X A := \eta_X^{-1}((C_{X^*}A^\perp)^\perp).$$

Proposition 5.7 *Assume that the module K is a cogenerator of Mod_R. Then for every closure operator C of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$, $\check{C}$ is a dco of $\mathcal{R}eg\mathcal{E}pi\mathsf{Mod}_R$.*

Proof. We have to verify conditions 1–3 of Corollary 2.7. Of these, 2 is trivial and 1 follows from Remark 5.3(1). For 3, we consider $f : X \to Y$ in Mod_R and $B \leq Y$ and, using the notation of Prop. 5.4, conclude:

$$
\begin{aligned}
\check{C}_X(f^{-1}B) &= \eta_X^{-1}\left((C_{X^*}(f^{-1}B)^\perp)^\perp\right) \\
&\leq \eta_X^{-1}\left(\left(C_{X^*}(f^*(B^\perp))\right)^\perp\right) \\
&\leq \eta_X^{-1}\left(\left(f^*(C_{Y^*}B^\perp)\right)^\perp\right) \\
&= \eta_X^{-1}\left((f^{**})^{-1}\left((C_{Y^*}B^\perp)^\perp\right)\right) \\
&= f^{-1}\left(\eta_Y^{-1}\left((C_{Y^*}B^\perp)^\perp\right)\right) \\
&= f^{-1}(\check{C}_Y B).
\end{aligned}
$$

$\square$

Denoting by $\mathcal{E}$ and $\mathcal{M}$ the classes of epi- and monomorphisms in $\mathcal{X} = \mathsf{Mod}_R$, respectively, we can now state:

Theorem 5.8 *If the module K is a cogenerator of Mod_R, then the monotone map*

$$
DCO(\mathcal{X}, \mathcal{E}) \to CO(\mathcal{X}, \mathcal{M})^{op}, D \mapsto \check{D},
$$

has a right adjoint, given by $C \mapsto \check{C}$.

Proof. Trivially the correspondence $C \mapsto \check{C}$ reverses the order. For a closure operator C of $\mathcal{R}eg\mathcal{M}ono\mathsf{Mod}_R$, we show $C \leq \overset{\vee}{\check{C}}$, as follows.

For $A \leq X \in \mathsf{Mod}_R$ and $x \in C_X A$, we must verify $\eta_X(x) \in (\check{C}_{X^*}A^\perp)^\perp$. Since $\eta_X(A) \leq A^{\perp\perp}$, one has

$$
\eta_X(C_X A) \leq C_{X^{**}}(\eta_X(A)) \leq C_{X^{**}}(A^{\perp\perp}), \tag{+}
$$

hence $\eta_X(x) \in C_{X^{**}}(A^{\perp\perp})$. Consequently, for $\lambda \in \check{C}_{X^*}(A^\perp)$, from $\eta_{X^*}(\lambda) \restriction_{C_{X^{**}}(A^{\perp\perp})}$ we obtain

$$
0 = \eta X^*(\lambda)(\eta_X(x)) = \eta_X(x)(\lambda) = \lambda(x),
$$

that is: $\eta_X(x) \restriction_{\check{C}_{X^*}(A^\perp)} = 0$. This means $x \in \overset{\vee}{\check{C}}_X A$.

For a dco D the inequality $D \leq \overset{\vee}{\check{D}}$ follows analogously since we may rewrite $(+)$ as

$$
D_X A \leq D_X \eta_X^{-1}(A^{\perp\perp}) \leq \eta_X^{-1}(D_{X^{**}}(A^{\perp\perp})).
$$

$\square$

6 Comparison of dual closure operators with interior operators

If $\mathcal{M}$ is a pullback-stable class of monomorphisms of a category $\mathcal{X}$ with (the required) pullbacks, the assignment $X \mapsto \mathrm{sub}X$ of 2.2(3) is the object part of a pseudofunctor

$$
\mathrm{sub} : \mathcal{X}^{op} \to ORD
$$

which, to a morphism $f : X \to Y$ in $\mathcal{X}$, assigns the monotone map $\mathrm{sub}f : \mathrm{sub}Y \to \mathrm{sub}X$ of (pre-)ordered classes, sending $n : N \to Y$ in $\mathcal{M}$ to its pullback $f^{-1}(n) : f^{-1}N \to X$ along f. Expanding on this language, from conditions 1-3 of 2.2(3) we see directly the following description of closure operators:

Proposition 6.1 *A closure operator C of $\mathcal{M}$ in $\mathcal{X}$ is equivalently described as an op-lax natural transformation*

$$
(C : \mathrm{sub} \to \mathrm{sub}) = (C_X : \mathrm{sub}X \to \mathrm{sub}X)_{X \in \mathcal{X}}
$$

with $1_{\mathrm{sub}} \leq C$.

Proof. Condition 2 of 2.2(3) makes every map C_X live in ORD, condition 3 expresses op-laxness of C, and condition 1 means precisely $1_{\text{sub}} \leq C$. $\qquad\qquad\square$

Let us note now that ORD is in fact a 2-category, with 2-cells given by the pointwise defined order relation of the homs. By dualization with respect to the 2-cells, we arrive at the following definition.

Definition 6.2 An *interior operator* of $\mathcal{M}$ in $\mathcal{X}$ is a lax natural transformation $I : \text{sub} \to \text{sub}$ with $I \leq 1_{\text{sub}}$. Equivalently, I is given by a family of maps

$$(I_X : \text{sub}X \to \text{sub}X)_{X \in \mathcal{X}}$$

such that

1. $I_X m \leq m$,

2. $m \leq m' \Rightarrow I_X m \leq I_X m'$,

3. $f^{-1}(I_Y n) \leq I_X(f^{-1}(n))$,

for all $m, m' \in \text{sub}X, n \in \text{sub}Y$, and $f : X \to Y$ in $\mathcal{X}$.

One orders interior operators of $\mathcal{M}$ in $\mathcal{X}$ "objectwise", so that $I \leq I'$ whenever $I_X m \leq I'_X m$ for all $X \in \mathcal{X}, m \in \text{sub}X$.

Remarks 6.3 (1) The prototypical example of an interior operator of Top assigns to a subspace M of a topological space X its open interior $I_X M = M^o$. The fact that M^o is the complement of the topological closure of the complement of M in X leads to a much more general observation. As already observed in [49], if there is a natural involution

$$(-)' : \text{sub} \to \text{sub}^{co}$$

(where $\text{sub}^{co}X$ inverts the order of $\text{sub}X$), so that every $\text{sub}X$ is naturally (= compatibly with taking inverse images) "complemented" via $'$, then interior operators I of $\mathcal{M}$ in the category $\mathcal{X}$ are in bijective correspondence with closure operators C of $\mathcal{M}$ in $\mathcal{X}$, via

$$I_X m = (C_X(m'))', C_X m = (I_X(m'))',$$

for all $m \in \text{sub}X, X \in \mathcal{X}$. Of course, set-theoretic complementation for regular subobjects in Top and, more generally, in every topological category over Set, provides the needed involution. In such categories, every interior operator of regular subobjects is induced by a closure operator, and vice versa.

(2) In the category Grp one obtains an interior operator N (of the class of monomorphisms) by letting $N_G(A)$ be the largest normal subgroup of the group G contained in the subgroup A, for all $A \leq G \in$ Grp. Remarkably, N is in fact the least interior operator since *every interior operator of subobjects in the category* Grp *of groups contains the normal core N*. Indeed, let I be an interior operator of subobjects in Grp and consider $A \leq G \in$ Grp. Exploiting the defining properties for an interior operator with the quotient homomorphism $q : G \to H := G/N_G A$, we obtain

$$N_G A = q^{-1}(\{1_H\}) = q^{-1}(I_H(\{1_H\})) \leq I_G(q^{-1}(\{1_H\})) = I_G N_G A \leq I_G A,$$

as desired. As a consequence, *there is no non-identical interior operator of normal subobjects in* Grp. By contrast, there is an abundance of closure operators of normal subobjects in Grp that were considered in more general categorical contexts in [6, 14]). In fact, the scarcity of interior operators as demonstrated here for Grp prevails also in these general contexts; see [24].

(3) Let us expand on (2) and mention in particular the category Mod_R, or any abelian category $\mathcal{X}$. Since every subobject is normal, $\mathcal{X}$ has no other interior operator of subobjects than the identical operator. By contrast, there is an abundance of dual closure operators, as indicated in Sections 3.3 and 5.

(4) Some claims in the recent literature regarding examples of interior operators in Grp and AbGrp are faulty. Specifically, [9, Example 3.8(h)] (and consequenlty, also [9, Example 3.8(i)]) violates condition 1 of Definition 6.2 and therefore fails to constitute an interior operator of subobjects. (The same faulty example appears also at the end of [30, Example 3]). Example 3.8(j) from [9], deduced from [9, Example 3.8(i)], is invalid as well since, by (3), AbGrp admits no proper interior operators. Also [9, Example 3.8(k)] is invalid, since in the category Grp all interior operators must contain the normal core, by (2).

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