# Variational Inequalities for the Fractional Laplacian

# Variational inequalities for the fractional Laplacian

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**Abstract**

In this paper we study the obstacle problems for the fractional Lapalcian of order $s \in (0,1)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, under mild assumptions on the data.

## 1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 1$. Given $s \in (0,1)$, a measurable function $\psi$ and a distribution $f$ on $\Omega$, we consider the problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{1.1}$$

Our interest is motivated by the noticeable paper [19], where Louis E. Silvestre investigated (1.1) in case $\Omega = \mathbb{R}^n$, $f = 0$ and $\psi$ smooth. His results apply also to Dirichlet's problems on balls, see [19, Section 1.3]. Besides remarkable results, in [19] the interested reader can find stimulating motivations for (1.1), arising from

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mathematical finance. In addition, Signorini's problem, also known as the lower dimensional obstacle problem for the classical Laplacian, can be recovered from (1.1) by taking $s = \frac{1}{2}$.

Among the papers dealing with (1.1) and related problems we cite also [1, 3, 4, 7, 15, 18] and references there-in, with no attempt to provide a complete reference list.

In the present paper we show that the free boundary problem (1.1) admits a solution under quite mild assumptions on the data, see Theorems 1.1 and 1.2 below. However, our starting interest included broader questions concerning the variational inequality

$$u \in K^s_\psi \,, \qquad \langle (-\Delta)^s u - f, v - u \rangle \geq 0 \quad \forall v \in K^s_\psi \,, \qquad (\mathcal{P}(\psi, f))$$

where $f \in \widetilde{H}^s(\Omega)'$ and

$$K^s_\psi = \Big\{ v \in \widetilde{H}^s(\Omega) \mid v \geq \psi \ \textit{a.e. on} \ \Omega \ \Big\} .$$

Notation and main definitions are listed at the end of this introduction. We will always assume that the closed and convex set $K^s_\psi$ is not empty, also when not explicitly stated.

Problem $\mathcal{P}(\psi, f)$ admits a unique solution $u$, that can be characterized as the unique minimizer for

$$\inf_{v \in K^s_\psi} \ \frac{1}{2} \langle (-\Delta)^s v, v \rangle - \langle f, v \rangle \,. \qquad (1.2)$$

The variational inequality $\mathcal{P}(\psi, f)$ and the free boundary problem (1.1) are naturally related. Any solution $u \in \widetilde{H}^s(\Omega)$ to (1.1) coincides with the unique solution to $\mathcal{P}(\psi, f)$, see Remark 3.5. Conversely, if $u$ solves $\mathcal{P}(\psi, f)$ then $(-\Delta)^s u - f$ is a nonnegative distribution on $\Omega$, compare with Theorem 3.2. By analogy with the local case $s = 1$ one can guess that $(-\Delta)^s u = f$ outside the coincidence set $\{u = \psi\}$, at least when $u$ is regular enough. This is essentially the content of Section 3 in [19], where $f = 0$ and $\psi$ is a smooth, rapidly decreasing function on $\Omega = \mathbb{R}^n$, and of Theorems 1.1, 1.2 below.

To study the variational inequality $\mathcal{P}(\psi, f)$ we took inspiration from the classical theory about the local case $s = 1$. In particular, we refer to the fundamental monograph [9] by Kinderlehrer and Stampacchia, and to the pioneering papers [2, 10, 11, 12, 13, 20, 21], among others.

Standard techniques do not apply directly in the fractional case, mostly because of the different behavior of the truncation operator $v \mapsto v^+$, $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$. Section 2 is entirely devoted to this subject; we collect there some lemmata that might have an independent interest.

We take advantage of the results in Section 2 to obtain equivalent and useful formulations for $\mathcal{P}(\psi, f)$, and to prove continuous dependence theorems upon the data $f$ and $\psi$, see Sections 3 and 4, respectively.

Some extra difficulties arise from having settled a nonlocal problem on a bounded domain, producing at least, but not only, the same (partially solved) technical difficulties as for the unconstrained problem $(-\Delta)^s u = f$, $u \in \widetilde{H}^s(\Omega)$ (see for instance [6], [16], [17] and references there-in, for regularity issues).

Our main results proved in Section 5. They involve the unique solution $\omega_f$ to

$$(-\Delta)^s \omega_f = f \quad \text{in } \Omega \,, \qquad \omega_f \in \widetilde{H}^s(\Omega). \tag{1.3}$$

**Theorem 1.1** *Assume that $\psi$ and $f \in \widetilde{H}^s(\Omega)'$ satisfy the following conditions:*

*A1)* $(\psi - \omega_f)^+ \in \widetilde{H}^s(\Omega)$*;*

*A2)* $(-\Delta)^s(\psi - \omega_f)^+ - f$ *is a locally finite signed measure on* $\Omega$*;*

*A3)* $((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \in L^p_{\mathrm{loc}}(\Omega)$ *for some* $p \in [1, \infty]$*.*

*Let $u \in \widetilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then the following facts hold.*

*i)* $(-\Delta)^s u - f \in L^p_{\mathrm{loc}}(\Omega)$*;*

*ii)* $0 \le (-\Delta)^s u - f \le ((-\Delta)^s(\psi - \omega_f)^+ - f)^+$ *a.e. on* $\Omega$*;*

*iii)* $(-\Delta)^s u = f$ *a.e. on* $\{u > \psi\}$*.*

*In particular, $u$ solves the free boundary problem (1.1).*

**Theorem 1.2** *Assume that $\Omega$ is a bounded Lipschitz domain satisfying the exterior ball condition. Let $\psi \in C^0(\overline{\Omega})$ be a given obstacle, such that $K^s_\psi$ is not empty, $\psi \le 0$ on $\partial\Omega$ and $f \in L^p(\Omega)$, for some exponent $p > n/2s$.*

*Then the unique solution $u$ to $\mathcal{P}(\psi, f)$ is continuous on $\mathbb{R}^n$ and solves the free boundary problem (1.1).*

Our results plainly cover the non-homogeneous Dirichlet's free boundary problem

$$\begin{cases} u \geq \psi & \text{in } \Omega \\ (-\Delta)^s u \geq f & \text{in } \Omega \\ (-\Delta)^s u = f & \text{in } \{u > \psi\} \\ u = g & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

under appropriate assumptions on the datum $g$. Notice indeed that $u$ solves the related variational inequality if and only if $u - g$ solves $\mathcal{P}(\psi - g, f + (-\Delta)^s g)$.

Free boundary problems for the operator $(-\Delta)^s u + u$ can be considered as well, with minor modifications in the statements and in the proofs.

**Notation** The definition of the fractional Laplacian $(-\Delta)^s$ involves the Fourier transform:

$$\mathcal{F}\left[(-\Delta)^s u\right] = |\xi|^{2s}\mathcal{F}[u]\,, \quad \mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx\,.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We adopt the standard notation

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n)\ \},$$
$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}\}.$$

We endow $H^s(\mathbb{R}^n)$ and $\widetilde{H}^s(\Omega)$ with their natural Hilbertian structures. We recall that the norm of $u$ in $\widetilde{H}^s(\Omega)$ is given by the $L^2(\mathbb{R}^n)$-norm of $(-\Delta)^{\frac{s}{2}} u$.

We do not make any assumption on $\Omega$. Thus $\partial\Omega$ might be very irregular, even a fractal, and $C^\infty_0(\Omega)$ might be not dense in $\widetilde{H}^s(\Omega)$. Notice that $\widetilde{H}^s(\Omega)$ coincides with $\widetilde{H}^s(\Omega')$, whenever $\overline{\Omega} = \overline{\Omega}'$.

We denote by $\langle \cdot, \cdot \rangle$ the duality product between $\widetilde{H}^s(\Omega)$ and its dual $\widetilde{H}^s(\Omega)'$. In particular, $(-\Delta)^s u \in \widetilde{H}^s(\Omega)'$ for any $u \in \widetilde{H}^s(\Omega)$, and

$$\langle (-\Delta)^s u, v \rangle = \int\limits_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u \cdot (-\Delta)^{\frac{s}{2}} v\, dx = \int\limits_{\mathbb{R}^n} |\xi|^{2s} \mathcal{F}[u]\, \overline{\mathcal{F}[v]}\, d\xi.$$

# 2 Truncations

For measurable functions $v, w$ we put, as usual,

$$v \vee w = \max\{v, w\}\ , \quad v \wedge w = \min\{v, w\}\ , \quad v^+ = v \vee 0\ , \quad v_- = -(v \wedge 0),$$

so that $v = v^+ - v^-$. It is well known that $v \vee w \in H^s(\mathbb{R}^n)$ and $v \wedge w \in H^s(\mathbb{R}^n)$ if $v, w \in H^s(\mathbb{R}^n)$.

**Lemma 2.1** *Let $v \in H^s(\mathbb{R}^n)$. Then*

*i)* $\langle(-\Delta)^s v^+, v^-\rangle = \langle(-\Delta)^s v^-, v^+\rangle \le 0$ *;*

*ii)* $\langle(-\Delta)^s v, v^-\rangle + \int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^-|^2\, dx \le 0$ *;*

*iii)* $\langle(-\Delta)^s v, v^+\rangle - \int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v^+|^2\, dx \ge 0$ *.*

*In addition, if $v \in H^s(\mathbb{R}^n)$ does not have constant sign, then all the above inequalities are strict.*

**Proof.** In [14, Theorem 6], the Caffarelli-Silvestre extension argument [5] has been used to check that

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} |v||^2\, dx < \int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v|^2\, dx\,,$$

whenever $v$ changes sign. That is,

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ + v^-)|^2\, dx < \int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} (v^+ - v^-)|^2\, dx\,.$$

The conclusion is immediate. □

**Remark 2.2** *One can use ii) in Lemma 2.1 to get the well known weak maximum principle, that is, if $u \in \widetilde{H}^s(\Omega)$ and $(-\Delta)^s u \ge 0$ in $\Omega$ then $u \ge 0$ in $\Omega$.*

**Corollary 2.3** *Let $v_h$ be a sequence in $H^s(\mathbb{R}^n)$ such that $v_h$ converges to a nonpositive function in $H^s(\mathbb{R}^n)$. Then $v_h^+ \to 0$ in $H^s(\mathbb{R}^n)$.*

**Proof.** Statement *iii)* in Lemma 2.1 provides the estimate

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} v_h^+|^2\, dx \le \langle(-\Delta)^s v_h, v_h^+\rangle, \tag{2.1}$$

that gives us the boundedness of the sequence $v_h^+$ in $H^s(\mathbb{R}^n)$. Since $v_h^+ \to 0$ in $L^2(\mathbb{R}^n)$, we have $v_h^+ \to 0$ weakly in $H^s(\mathbb{R}^n)$. Thus $\langle(-\Delta)^s v_h, v_h^+\rangle \to 0$, as $(-\Delta)^s v_h$ converges in $H^s(\mathbb{R}^n)'$, and the conclusion follows from (2.1). □

**Lemma 2.4** *Let $v \in \widetilde{H}^s(\Omega)$ and $m > 0$. Then $(v+m)^-, (v-m)^+, v \wedge m \in \widetilde{H}^s(\Omega)$ and*

*i)* $$\langle (-\Delta)^s v, (v+m)^- \rangle + \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v+m)^- |^2 \, dx \le 0;$$

*ii)* $$\langle (-\Delta)^s v, (v-m)^+ \rangle - \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v-m)^+ |^2 \, dx \ge 0;$$

*iii)* $$\int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v \wedge m)|^2 \, dx \le \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} v|^2 \, dx - \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v-m)^+ |^2 \, dx.$$

**Proof.** Clearly, $(v+m)^- \in L^2(\mathbb{R}^n)$ and $(v+m)^- \equiv 0$ outside $\Omega$. Fix a cutoff function $\eta \in C^\infty_0(\mathbb{R}^n)$, with $0 \le \eta \le 1$, and such that $\eta \equiv 1$ in a ball containing $\overline{\Omega}$. Then $(v+m)^- = (v+m\eta)^- \in \widetilde{H}^s(\Omega)$, as trivially $m\eta \in H^s(\mathbb{R}^n)$.

For any integer $h \ge 1$ we set
$$\eta_h(x) = \eta\Big(\frac{x}{h}\Big),$$
so that $\eta_h \to 1$ pointwise. A direct computation shows that
$$(-\Delta)^s \eta_h(x) = h^{-2s} \Big( (-\Delta)^s \eta \Big) \Big(\frac{x}{h}\Big) \ \longrightarrow \ 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^n). \tag{2.2}$$
By *ii)* in Lemma 2.1 we have that
$$\begin{aligned} 0 \ &\ge \ \langle (-\Delta)^s (v + m\eta_h), (v+m)^- \rangle + \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v+m)^- |^2 \, dx \\ &= \ \langle (-\Delta)^s v, (v+m)^- \rangle + \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v+m)^- |^2 \, dx + m \int_\Omega ((-\Delta)^s \eta_h)(v+m)^- \, dx \\ &= \ \langle (-\Delta)^s v, (v+m)^- \rangle + \int\limits_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (v+m)^- |^2 \, dx + o(1), \end{aligned}$$
by (2.2) and since $(v+m)^-$ has compact support in $\Omega$. Claim *i)* is proved. To check *ii)* notice that $(v-m)^+ = ((-v)+m)^-$ and then use *i)* with $(-v)$ instead of $v$.

It remains to prove *iii)*. Notice that $v \wedge m = v - (v-m)^+$. Hence $v \wedge m \in \widetilde{H}^s(\Omega)$. Using *ii)* we get
$$\begin{aligned} \|\, (-\Delta)^{\frac{s}{2}} (v \wedge m)\|^2 \ &= \ \|\, (-\Delta)^{\frac{s}{2}} v\|^2 - 2\langle (-\Delta)^s v, (v-m)^+ \rangle + \|\, (-\Delta)^{\frac{s}{2}} (v-m)^+ \|^2 \\ &\le \ \|\, (-\Delta)^{\frac{s}{2}} v\|^2 - \|\, (-\Delta)^{\frac{s}{2}} (v-m)^+ \|^2 \,. \end{aligned}$$
The proof is complete. □

## 3 Equivalent formulations

We start this section by introducing a crucial notion.

**Definition 3.1** *A function $\mathcal{U} \in \widetilde{H}^s(\Omega)$ is a supersolution for $(-\Delta)^s v = f$ if*

$$\langle (-\Delta)^s \mathcal{U} - f, \varphi \rangle \ge 0 \qquad \text{for any } \varphi \in \widetilde{H}^s(\Omega),\ \varphi \ge 0.$$

The above definition extends the usually adopted one in the local case $s = 1$, see [9, Definition 6.3]. A different definition of supersolution is used in [19] for $f = 0$. We refer to [19, Subsection 2.10], for a stimulating discussion on this subject.

**Theorem 3.2** *Let $u \in K^s_\psi$. The following sentences are equivalent.*

*a) $u$ is the solution to problem $\mathcal{P}(\psi, f)$;*

*b) $u$ is the smallest supersolution for $(-\Delta)^s v = f$ in the convex set $K^s_\psi$. That is, $\mathcal{U} \ge u$ almost everywhere in $\Omega$, for any supersolution $\mathcal{U} \in K^s_\psi$;*

*c) $u$ is a supersolution for $(-\Delta)^s v = f$ and*

$$\langle (-\Delta)^s u - f, (v - u)^- \rangle = 0 \qquad \text{for any } v \in K^s_\psi.$$

*d) $\langle (-\Delta)^s v - f, v - u \rangle \ge 0$ for any $v \in K^s_\psi$.*

**Proof.** *a)* $\Longleftrightarrow$ *b)*. Assume that $u$ solves $\mathcal{P}(\psi, f)$. Fix any nonnegative $\varphi \in \widetilde{H}^s(\Omega)$. Testing $\mathcal{P}(\psi, f)$ with $u + \varphi \in K^s_\psi$ one gets $\langle (-\Delta)^s u - f, \varphi \rangle \ge 0$, that proves that $u$ is a supersolution.

Next, take any supersolution $\mathcal{U} \in K^s_\psi$. Then $u - (u - \mathcal{U})^+ = \mathcal{U} \wedge u \in K^s_\psi$. Thus

$$\langle (-\Delta)^s u - f, -(u - \mathcal{U})^+ \rangle \ge 0.$$

On the other hand, from $(-\Delta)^s \mathcal{U} - f \ge 0$ we get

$$\langle (-\Delta)^s \mathcal{U} - f, (u - \mathcal{U})^+ \rangle \ge 0.$$

Adding the above inequalities we arrive at

$$0 \ge \langle (-\Delta)^s (u - \mathcal{U}), (u - \mathcal{U})^+ \rangle \ge \int_{\mathbb{R}^n} |\, (-\Delta)^{\frac{s}{2}} (u - \mathcal{U})^+|^2 \, dx,$$

thanks to $iii$) in Lemma 2.1. Thus $(u-\mathcal{U})^+ = 0$ almost everywhere in $\Omega$, that is, $u \le \mathcal{U}$ and proves that $a$) implies $b$).

Conversely, assume that $u$ satisfies $b$) and let $\tilde{u}$ be the solution to $\mathcal{P}(\psi, f)$. We already know that $a) \Rightarrow b$). Thus $u$ and $\tilde{u}$ must coincide, because both obey the condition of being the smallest supersolution to $(-\Delta)^s v = f$ in $K^s_\psi$. Hence, $a$) holds.

$a) \iff c$). Let $u$ be the solution to $\mathcal{P}(\psi, f)$. We already know that $u$ is supersolution. Fix any function $v \in K^s_\psi$. Notice that

$$u + (v-u)^- \ge u \ge \psi \ , \quad u - (v-u)^- = v \wedge u \ge \psi.$$

Thus, testing $\mathcal{P}(\psi, f)$ with $u \pm (v-u)^-$ we get $\langle (-\Delta)^s u - f, \pm(v-u)^- \rangle \ge 0$, that is, $c$) holds.

Conversely, assume that $u$ satisfies $c$). Let $\tilde{u} \in K^s_\psi$ be the solution to $\mathcal{P}(\psi, f)$. We already proved that $\tilde{u}$ is the smallest supersolution in $K^s_\psi$. In particular, $\tilde{u} \le u$ and thus

$$\langle (-\Delta)^s u - f, u - \tilde{u} \rangle = \langle (-\Delta)^s u - f, (\tilde{u} - u)^- \rangle = 0$$

by the assumption $c$) on $u$. Since $\tilde{u}$ solves $\mathcal{P}(\psi, f)$, we also get

$$\langle (-\Delta)^s \tilde{u} - f, u - \tilde{u} \rangle \ge 0 \,.$$

Substracting, we infer $\langle (-\Delta)^s (u - \tilde{u}), u - \tilde{u} \rangle \le 0$, that is, $u = \tilde{u}$.

$a) \iff d$). Clearly $a$) implies $d$) because

$$\begin{aligned} &\langle (-\Delta)^s v - f, v - u \rangle \\ &\qquad = \langle (-\Delta)^s u - f, v - u \rangle + \langle (-\Delta)^s (v-u), v-u \rangle \ge \langle (-\Delta)^s u - f, v - u \rangle \,. \end{aligned}$$

Now assume that $u$ satisfies $d$) and fix any $v \in K^s_\psi$. From $\frac{v+u}{2} \in K^s_\psi$ and $d$) we obtain

$$\begin{aligned} 0 \ &\le \ 2\langle (-\Delta)^s \Big(\frac{v+u}{2}\Big) - f, \frac{v+u}{2} - u \rangle = \frac{1}{2}\langle (-\Delta)^s (v+u), v-u \rangle - \langle f, v-u \rangle \\ &= \ \left(\frac{1}{2}\langle (-\Delta)^s v, v \rangle - \langle f, v \rangle\right) - \left(\frac{1}{2}\langle (-\Delta)^s u, u \rangle - \langle f, u \rangle\right) \,. \end{aligned}$$

Thus $u$ solves the minimization problem (1.2), that is, $u$ solves $\mathcal{P}(\psi, f)$. □

**Remark 3.3** *In the local case $s = 1$, the equivalence between a) and d) is commonly known as Minty's lemma, see [13].*

**Corollary 3.4** *Let $f_1, f_2 \in \widetilde{H}^s(\Omega)'$ and let $u_i$ be the solution to $\mathcal{P}(\psi, f_i)$, $i = 1, 2$. If $f_1 \ge f_2$ in the sense of distributions, then $u_1 \ge u_2$ a.e. in $\Omega$.*

**Proof.** The function $u_1$ is a supersolution for $(-\Delta)^s v = f_2$ and $u_1 \in K^s_\psi$. Hence $u_1 \ge u_2$, by statement $b)$ in Theorem 3.2. □

**Remark 3.5** *Let $u \in \widetilde{H}^s(\Omega)$ be a solution to (1.1). Then $(-\Delta)^s u - f$ can be identified with a nonnegative Radon measure on $\Omega$ having support in $\{u = \psi\}$. If $v \in K^s_\psi$, then $(v - u)^-$ vanishes on $\{u = \psi\}$. Thus $\langle(-\Delta)^s u - f, (v - u)^-\rangle = 0$, hence $u$ solves $\mathcal{P}(\psi, f)$ by Theorem 3.2.*

## 4 Continuous dependence results

**Theorem 4.1** *Let $\psi_1, \psi_2$ be given obstacles, $f \in \widetilde{H}^s(\Omega)'$ and let $u_i$ be the solution to $\mathcal{P}(\psi_i, f)$, $i = 1, 2$. If $\psi_1 - \psi_2 \in L^\infty(\Omega)$, then $u_1 - u_2$ is bounded, and*

$$i)\ \ \|(u_1 - u_2)^+\|_\infty \le \|(\psi_1 - \psi_2)^+\|_\infty\ , \quad ii)\ \ \|(u_1 - u_2)^-\|_\infty \le \|(\psi_1 - \psi_2)^-\|_\infty.$$

**Proof.** Put $m := \|(\psi_1 - \psi_2)^+\|_\infty$. Since $(u_2 - u_1 + m)^- \in \widetilde{H}^s(\Omega)$ by Lemma 2.4, then

$$v_1 := u_1 - (u_2 - u_1 + m)^- = (u_2 + m) \wedge u_1 \in K^s_{\psi_1}\,.$$

Hence we can use $v_1$ as test function in $\mathcal{P}(\psi_1, f)$ to get

$$\langle(-\Delta)^s u_1 - f, -(u_2 - u_1 + m)^-\rangle \ge 0.$$

On the other hand, we can test $\mathcal{P}(\psi_2, f)$ with $u_2 + (u_2 - u_1 + m)^- \in K^s_{\psi_2}$. Hence

$$\langle(-\Delta)^s u_2 - f, (u_2 - u_1 + m)^-\rangle \ge 0.$$

Adding and taking $i)$ of Lemma 2.4 into account, we arrive at

$$-\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{s}{2}}(u_2 - u_1 + m)^-|^2\,dx \ge \langle(-\Delta)^s(u_2 - u_1), (u_2 - u_1 + m)^-\rangle \ge 0.$$

Hence, $(u_2 - u_1 + m)^- = 0$. We have proved that $(u_1 - u_2)^+ \le m$ a.e. in $\Omega$, hence $i)$ holds. Inequality $ii)$ can be proved in the same way. □

**Corollary 4.2** *Let $\psi \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$, with $p \in (1,\infty)$, $p > n/2s$. Let $u \in \widetilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then $u \in L^\infty(\Omega)$ and*

$$\psi \vee \omega_f \le u \le \|\psi^+\|_\infty + c\|f^+\|_p \quad \text{a.e. in } \Omega, \tag{4.1}$$

*where $\omega_f$ solves (1.3) and $c$ depends only on $n, s, p$ and $\Omega$. In particular, if $f = 0$ then*

$$\psi^+ \le u \le \|\psi^+\|_\infty .$$

**Proof.** First of all, notice that $f \in \widetilde{H}^s(\Omega)'$ by Sobolev embedding theorem. Since $u$ is supersolution of (1.3), the first inequality in (4.1) follows by the maximum principle in Remark 2.2.

Denote by $\omega_{f^+}$ the unique solution to (1.3) with $f$ replaced by $f^+$. If $n > 2s$ we use convolution to define

$$U = c_1|x|^{2s-n} * (f^+ \cdot \chi_\Omega).$$

For proper choice of the constant $c_1$, $U$ solves $(-\Delta)^s U = f^+ \cdot \chi_\Omega$ in $\mathbb{R}^n$. Convolution estimates give $U \le c\|f^+\|_p$ on $\mathbb{R}^n$. By the maximum principle, $\omega_{f^+} \le U$ on $\Omega$, hence $\omega_{f^+} \le c\|f^+\|_p$. For $n = 1 \le 2s$ this inequality also holds, see, e.g., [16, Remark 1.5].

Now let $u_1$ be the unique solution of $\mathcal{P}(\psi, f^+)$. Then $u_1 \ge u$ by Corollary 3.4. Finally, we can consider $\omega_{f^+}$ as the solution of the problem $\mathcal{P}(\omega_{f^+}, f^+)$. Theorem 4.1 gives

$$u \le (u_1 - \omega_{f^+})^+ + \omega_{f^+} \le \|(\psi - \omega_{f^+})^+\|_\infty + \omega_{f^+},$$

and the last inequality in (4.1) follows. □

Roughly speaking, Theorem 4.1 concerns the continuity of $L^\infty \ni \psi \mapsto u \in L^\infty$. The next result gives the continuity of the arrow $L^\infty \ni \psi \mapsto u \in \widetilde{H}^s(\Omega)$.

**Theorem 4.3** *Let $\psi_h \in L^\infty(\Omega)$ be a sequence of obstacles and let $f \in \widetilde{H}^s(\Omega)'$ be given. Assume that there exists $v_0 \in \widetilde{H}^s(\Omega)$, such that $v_0 \ge \psi_h$ for any $h$.*

*Denote by $u_h$ the solution to the obstacle problem $\mathcal{P}(\psi_h, f)$. If $\psi_h \to \psi$ in $L^\infty(\Omega)$, then $u_h \to u$ in $\widetilde{H}^s(\Omega)$, where $u$ is the solution to the limiting problem $\mathcal{P}(\psi, f)$.*

**Proof.** Let $u$ be the solution to $\mathcal{P}(\psi, f)$. We already know from Theorem 4.1 that $\|u - u_h\|_\infty \le \|\psi - \psi_h\|_\infty$. Hence, in particular, $u_h \to u$ a.e. in $\Omega$. Now, test $\mathcal{P}(\psi_h, f)$

with $v_0$ to obtain that

$$\langle (-\Delta)^s u_h, u_h \rangle \le \langle (-\Delta)^s u_h - f, v_0 \rangle + \langle f, u_h \rangle.$$

Hence, the sequence $u_h$ is bounded in $\widetilde{H}^s(\Omega)$. Therefore, $u_h \to u$ weakly in $\widetilde{H}^s(\Omega)$. To prove that $u_h \to u$ in the $\widetilde{H}^s(\Omega)$ norm we only need to show that

$$\limsup_{h\to\infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2 \le \| (-\Delta)^{\frac{s}{2}} u \|_2.$$

For any $\varepsilon > 0$ we introduce the function

$$v_\varepsilon = u + (v_0 - u) \wedge \varepsilon.$$

Since $\psi_h \to \psi$ in $L^\infty(\Omega)$, we have $v_\varepsilon \ge \psi_h$ for $h$ large enough. Using $v_\varepsilon$ as test function in $\mathcal{P}(\psi_h, f)$ we get

$$\langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon - u_h \rangle \ge 0,$$

and hence

$$\| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 = \langle (-\Delta)^s u_h, u_h \rangle \le \langle (-\Delta)^s u_h - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u_h \rangle.$$

Letting $h \to \infty$ we infer

$$\begin{aligned} \limsup_{h\to\infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 &\le \langle (-\Delta)^s u - f, u + (v_0 - u) \wedge \varepsilon \rangle + \langle f, u \rangle \\ &= \| (-\Delta)^{\frac{s}{2}} u \|_2^2 + \langle (-\Delta)^s u - f, (v_0 - u) \wedge \varepsilon \rangle. \end{aligned} \tag{4.2}$$

Now we let $\varepsilon \to 0$. Clearly $(v_0 - u) \wedge \varepsilon \to -(v_0 - u)^-$ in $L^2(\Omega)$. In addition, the functions $(v_0 - u) \wedge \varepsilon$ are uniformly bounded in $\widetilde{H}^s(\Omega)$ by *iii)* in Lemma 2.4. Thus $(v_0 - u) \wedge \varepsilon \to -(v_0 - u)^-$ weakly in $\widetilde{H}^s(\Omega)$. Thus, from (4.2) we get

$$\limsup_{h\to\infty} \| (-\Delta)^{\frac{s}{2}} u_h \|_2^2 \le \| (-\Delta)^{\frac{s}{2}} u \|_2^2 - \langle (-\Delta)^s u - f, (v_0 - u)^- \rangle = \| (-\Delta)^{\frac{s}{2}} u \|_2^2$$

since $u$ solves $\mathcal{P}(\psi, f)$, and therefore it satisfies condition $c)$ in Theorem 3.2. Thus $u_h \to u$ in $\widetilde{H}^s(\Omega)$. □

Next we deal with the continuity of the arrow $H^s \ni \psi \mapsto u \in \widetilde{H}^s$.

**Theorem 4.4** *Let $\psi_h \in H^s(\mathbb{R}^n)$ be a sequence of obstacles such that $\psi_h^+ \in \widetilde{H}^s(\Omega)$, and let $f_h$ be a sequence in $\widetilde{H}^s(\Omega)'$. Assume that*

$$\psi_h \to \psi \quad \text{in } H^s(\mathbb{R}^n), \text{ and} \quad f_h \to f \quad \text{in } H^s(\Omega)'.$$

*Denote by $u_h$ the solution to the obstacle problem $\mathcal{P}(\psi_h, f_h)$. Then $u_h \to u$ in $\widetilde{H}^s(\Omega)$, where $u$ is the solution to the limiting obstacle problem $\mathcal{P}(\psi, f)$.*

**Proof.** We can assume that $f_h, f = 0$. If not, replace the obstacles $\psi_h$ and $\psi$ with $\psi_h - \omega_{f_h}$ and $\psi - \omega_f$, respectively, see (1.3).

Let $u_h$ solve $\mathcal{P}(\psi_h, 0)$ and let $u$ be the solution to the limiting problem $\mathcal{P}(\psi, 0)$. Recall that $u$ is the unique minimizer for

$$\inf_{v \in K^s_\psi} \langle (-\Delta)^s v, v \rangle . \tag{4.3}$$

Since $u \vee \psi_h = u + (\psi_h - u)^+$ and $\psi_h - u \to \psi - u \le 0$, then

$$u \vee \psi_h \to u \quad \text{in } \widetilde{H}^s(\Omega) \tag{4.4}$$

by Corollary 2.3. Moreover, $u \vee \psi_h \in K^s_{\psi_h}$ and thus from $\mathcal{P}(\psi_h, 0)$ we infer

$$\langle (-\Delta)^s u_h, u_h \rangle \le \langle (-\Delta)^s u_h, u \vee \psi_h \rangle. \tag{4.5}$$

Inequality (4.5) guarantees the boundedness of the sequence $u_h$ in $\widetilde{H}^s(\Omega)$. Hence we can assume that $u_h \to \tilde{u}$ weakly in $\widetilde{H}^s(\Omega)$. Since $\psi_h \to \psi$ and $u_h \to \tilde{u}$ a.e. in $\Omega$, clearly $\tilde{u} \in K^s_\psi$.

Next, by weak lower semicontinuity, (4.5) and (4.4) we get

$$\langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \le \liminf_{h\to\infty} \langle (-\Delta)^s u_h, u_h \rangle \le \limsup_{h\to\infty} \langle (-\Delta)^s u_h, u_h \rangle \le \langle (-\Delta)^s \tilde{u}, u \rangle. \tag{4.6}$$

Thus

$$\| (-\Delta)^{\frac{s}{2}} \tilde{u} \|_2^2 \le \| (-\Delta)^{\frac{s}{2}} \tilde{u} \|_2 \| (-\Delta)^{\frac{s}{2}} u \|_2.$$

Hence, $\tilde{u} = u$, as the minimization problem (4.3) admits a unique solution, and (4.6) implies $\| (-\Delta)^{\frac{s}{2}} u_h \|_2 \to \| (-\Delta)^{\frac{s}{2}} u \|_2$. Hence $u_h \to u$ strongly in $\widetilde{H}^s(\Omega)$. $\square$

## 5 Proof of the main results

We start with a preliminary theorem of independent interest, that gives distributional bounds on $(-\Delta)^s u - f$ under mild assumptions on the data.

**Theorem 5.1** *Let $\psi$ and $f \in \widetilde{H}^s(\Omega)'$ satisfying assumptions A1) and A2) in Theorem 1.1. Let $u \in \widetilde{H}^s(\Omega)$ be the unique solution to $\mathcal{P}(\psi, f)$. Then*

$$0 \le (-\Delta)^s u - f \le ((-\Delta)^s(\psi - \omega_f)^+ - f)^+ \quad \textit{in the distributional sense on } \Omega.$$

**Proof.** The main tool was inspired by the penalty method by Lewy-Stampacchia [10] and already used for instance in [18] under smoothness assumptions on the data and on the solution.

In order to simplify notations we start the proof with some remarks. First, we can assume that $f = 0$, as we did in the proof of Theorem 4.4. Thus $(-\Delta)^s u \ge 0$ and $u \ge \psi$, that imply $u \ge \psi^+$, use the maximum principle in Remark 2.2. Clearly $u$ is the smallest supersolution to $(-\Delta)^s v = 0$ in $K^s_{\psi^+}$, and hence it solves the obstacle problem $\mathcal{P}(\psi^+, 0)$. In conclusion, it suffices to prove Theorem 5.1 in case $f = 0$ and $\psi \ge 0$ in $\mathbb{R}^n$. Our aim is to show that

$$0 \le (-\Delta)^s u \le ((-\Delta)^s \psi)^+ \quad \text{in the distributional sense on } \Omega, \tag{5.1}$$

for $\psi \in \widetilde{H}^s(\Omega)$, $\psi \ge 0$, such that $(-\Delta)^s \psi$ is a measure on $\Omega$.

The proof of (5.1) will be achieved in few steps.

**Step 1** *Assume $(-\Delta)^s \psi \in L^p(\Omega)$ for any large exponent $p > 1$. Then (5.1) holds.*

We take $p \ge \frac{2n}{n+2s}$, that is needed only if $n > 2s$. Then $\widetilde{H}^s(\Omega) \hookrightarrow L^{p'}(\Omega)$ and $L^p(\Omega) \subset \widetilde{H}^s(\Omega)'$ by Sobolev embeddings. In particular $((-\Delta)^s \psi)^+ \in \widetilde{H}^s(\Omega)'$.

Take a function $\theta_\varepsilon \in C^\infty(\mathbb{R})$ such that $0 \le \theta_\varepsilon \le 1$, and

$$\theta_\varepsilon(t) = 1 \ \text{ for } t \le 0, \quad \theta_\varepsilon(t) = 0 \ \text{ for } t \ge \varepsilon.$$

By standard variational methods we have that there exists a unique $u_\varepsilon \in \widetilde{H}^s(\Omega)$ that weakly solves

$$(-\Delta)^s u_\varepsilon = \theta_\varepsilon(u_\varepsilon - \psi)\,((-\Delta)^s \psi)^+ \quad \text{in } \Omega.$$

We claim that

$$u \le u_\varepsilon \le u + \varepsilon \quad \text{a.e. in } \Omega.$$

By *iii)* in Lemma 2.1 we can estimate

$$\begin{aligned} \|\,(-\Delta)^{\frac{s}{2}}(\psi - u_\varepsilon)^+\|_2^2 &\le \langle(-\Delta)^s(\psi - u_\varepsilon), (\psi - u_\varepsilon)^+\rangle \\ &\le \int_\Omega ((-\Delta)^s\psi)^+(1 - \theta_\varepsilon(u_\varepsilon - \psi))(\psi - u_\varepsilon)^+\,dx = 0\,. \end{aligned}$$

Hence, $u_\varepsilon \ge \psi$. Since $(-\Delta)^s u_\varepsilon \ge 0$, then $u_\varepsilon \ge u$ by *b)* in Theorem 3.2. Next, we use *iii)* in Lemma 2.4 and $(-\Delta)^s u \ge 0$ to estimate

$$\begin{aligned} \|\,(-\Delta)^{\frac{s}{2}}(u_\varepsilon - u - \varepsilon)^+\|_2^2 &\le \langle(-\Delta)^s(u_\varepsilon - u), (u_\varepsilon - u - \varepsilon)^+\rangle \\ &\le \int_\Omega ((-\Delta)^s\psi)^+\,\theta_\varepsilon(u_\varepsilon - \psi)\,(u_\varepsilon - u - \varepsilon)^+\,dx = 0\,. \end{aligned}$$

Thus $u_\varepsilon \le u+\varepsilon$, and the claim is proved. In particular, we have that $\|u_\varepsilon - u\|_\infty \to 0$ as $\varepsilon \to 0$. Therefore, for any nonnegative test function $\varphi \in C_0^\infty(\Omega)$ we have that

$$\begin{aligned} \langle(-\Delta)^s u, \varphi\rangle &= \int_\Omega u\,(-\Delta)^s\varphi\,dx = \int_\Omega u_\varepsilon\,(-\Delta)^s\varphi\,dx + o(1) \\ &= \langle(-\Delta)^s u_\varepsilon, \varphi\rangle + o(1) \le \langle((-\Delta)^s\psi)^+, \varphi\rangle + o(1), \end{aligned}$$

that readily gives $(-\Delta)^s u \le ((-\Delta)^s\psi)^+$ in the distributional sense in $\Omega$.

**Step 2** *Approximation argument.*

Fix a small $\varepsilon > 0$ and put $\Omega_\varepsilon := \{x \in \Omega \mid \text{dist}(x, \Omega) < \varepsilon\}$. The convex set

$$K_\varepsilon = \{v \in \widetilde{H}^s(\Omega_\varepsilon) \mid v \ge \psi \;\text{ a.e. on } \mathbb{R}^n\;\}$$

contains $K^s_\psi$, hence it is not empty. We denote by $u_\varepsilon$ the unique solution to the variational inequality

$$u_\varepsilon \in K_\varepsilon\;, \qquad \langle(-\Delta)^s u_\varepsilon, v - u_\varepsilon\rangle \ge 0 \qquad \forall v \in K_\varepsilon\,, \tag{$\mathcal{P}_\varepsilon$}$$

so that $u_\varepsilon \in \widetilde{H}^s(\Omega_\varepsilon)$ and is nonnegative. Next we prove that

$$0 \le (-\Delta)^s u_\varepsilon \le ((-\Delta)^s\psi)^+ \quad \text{in the distributional sense on } \Omega. \tag{5.2}$$

For, we approximate $\psi$ in a standard way, via convolution. Let $(\rho_h)_h$ be a sequence of mollifiers such that $\text{supp}(\rho_h) \subset B_{\frac{1}{h}}$ and put $\psi_h = \psi * \rho_h$. Notice that for $h$ large enough, $\psi_h = 0$ outside $\Omega_\varepsilon$. Therefore

$$\psi_h \in \widetilde{H}^s(\Omega_\varepsilon) \ , \qquad \psi_h \to \psi \quad \text{in } H^s(\mathbb{R}^n). \tag{5.3}$$

The convex set $K_{\varepsilon,h} := \{v \in \widetilde{H}^s(\Omega_\varepsilon) \mid u \ge \psi_h\}$ is not empty, as it contains $\psi_h$. The variational inequality

$$u_h \in K_{\varepsilon,h} \ , \qquad \langle (-\Delta)^s u_h, v - u_h \rangle \ge 0 \qquad \forall v \in K_{\varepsilon,h} \, , \tag{$\mathcal{P}_{\varepsilon,h}$}$$

has a unique solution $u_h \in \widetilde{H}^s(\Omega_\varepsilon)$. Theorem 4.4 readily gives that $u_h \to u_\varepsilon$ in $\widetilde{H}^s(\Omega_\varepsilon)$. Since $(-\Delta)^s \psi_h \in L^p(\mathbb{R}^n)$ for any $p \ge 1$, then Step 1 applies. In particular

$$0 \le (-\Delta)^s u_h \le ((-\Delta)^s \psi_h)^+ \quad \text{in the distributional sense on } \Omega. \tag{5.4}$$

Next, $((-\Delta)^s \psi)^+ * \rho_h$ is a nonnegative smooth function, and

$$((-\Delta)^s \psi)^+ * \rho_h \ge ((-\Delta)^s \psi) * \rho_h = (-\Delta)^s \psi_h \, .$$

Thus $((-\Delta)^s \psi)^+ * \rho_h \ge ((-\Delta)^s \psi_h)^+$, and (5.4) implies

$$0 \le (-\Delta)^s u_h \le ((-\Delta)^s \psi)^+ * \rho_h \quad \text{in the distributional sense on } \Omega.$$

Claim (5.2) follows, since $((-\Delta)^s \psi)^+ * \rho_h \to ((-\Delta)^s \psi)^+$ in the sense of measures, and $(-\Delta)^s u_h \to (-\Delta)^s u_\varepsilon$ in the sense of distributions.

**Step 3** *Conclusion of the proof.*

The last step in the proof consists in passing to the limit along a sequence $\varepsilon \to 0$. First, we notice that $u \in \widetilde{H}^s(\Omega_\varepsilon)$ and in particular $u \in K_\varepsilon$. Therefore, using the variational characterization of the unique solution $u_\varepsilon$ to $(\mathcal{P}_\varepsilon)$ we find

$$\frac{1}{2} \langle (-\Delta)^s u_\varepsilon, u_\varepsilon \rangle \le \frac{1}{2} \langle (-\Delta)^s u, u \rangle \, . \tag{5.5}$$

Now we fix $\varepsilon_0 > 0$. Thanks to (5.5), we get that the sequence $u_\varepsilon$ is bounded in $\widetilde{H}^s(\Omega_{\varepsilon_0})$, and therefore we can assume that $u_\varepsilon \to \tilde{u}$ weakly in $\widetilde{H}^s(\Omega_{\varepsilon_0})$. From (5.5) we readily get

$$\frac{1}{2} \langle (-\Delta)^s \tilde{u}, \tilde{u} \rangle \le \frac{1}{2} \langle (-\Delta)^s u, u \rangle. \tag{5.6}$$

On the other hand, $u_\varepsilon \to \tilde{u}$ almost everywhere. Hence $\tilde{u} \in \widetilde{H}^s(\Omega)$ and $\tilde{u} \geq \psi$ on $\Omega$, that is, $\tilde{u} \in K^s_\psi$. Using the characterization of $u$ as the unique solution to the minimization problem (4.3), from (5.6), (5.5) we get that $\tilde{u} = u$ and $u_\varepsilon \to u$ in $\widetilde{H}^s(\Omega_{\varepsilon_0})$. In particular, $\langle(-\Delta)^s u_\varepsilon, \varphi\rangle \to \langle(-\Delta)^s u, \varphi\rangle$ for any $\varphi \in C^\infty_0(\Omega)$. Now, from (5.2) we know that $((-\Delta)^s\psi)^+ - (-\Delta)^s u_\varepsilon$ is a nonnegative distribution on $\Omega$. Thus $((-\Delta)^s\psi)^+ - (-\Delta)^s u$ is nonnegative as well, and (5.1) is proved. □

**Proof of Theorem 1.1**

Statements *i*) and *ii*) hold by Theorem 5.1. It remains to prove the last claim.

It is not restrictive to assume $f \equiv 0$. Hence $u$ solves $\mathcal{P}(\psi, 0)$, $(-\Delta)^s u \geq 0$ by Theorem 3.2, and $u$ is nonnegative in $\Omega$, see Remark 2.2. Actually $u$ is lower semicontinuous and positive by the strong maximum principle, see for instance [8, Theorem 2.5]. Thus $u \geq \psi^+$ and $\{u > \psi\} = \{u > \psi^+\}$.

Next we use *c*) in Theorem 3.2 with $v = \psi^+ \in \widetilde{H}^s(\Omega)$, to get

$$\langle(-\Delta)^s u, u - \psi^+\rangle = 0.$$

Let $\Omega'$ be any domain compactly contained in $\Omega$. We claim that

$$\int\limits_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+)\, dx = 0\,. \tag{5.7}$$

Since $(-\Delta)^s u \cdot (u - \psi^+)$ is a measurable nonnegative function then the integral in (5.7) is nonnegative. To prove the opposite inequality we put $g_m = (u - \psi^+) \wedge m$, $m \geq 1$. Let $\varphi$ be any nonnegative cut off function, with $\varphi \in C^\infty_0(\Omega)$ and $\varphi \equiv 1$ on $\Omega'$. Since $(-\Delta)^s u \geq 0$, $(-\Delta)^s u \in L^1_{\rm loc}(\Omega)$, $u - \psi^+ \geq \varphi g_m$ and $\varphi g_m \in L^\infty(\Omega)$ has compact support in $\Omega$, we have that

$$0 = \langle(-\Delta)^s u, u - \psi^+\rangle \geq \langle(-\Delta)^s u, \varphi g_m\rangle = \int\limits_\Omega (-\Delta)^s u \cdot (\varphi g_m) dx \geq \int\limits_{\Omega'} (-\Delta)^s u \cdot g_m dx.$$

Next, use the monotone convergence theorem to get

$$0 \geq \lim_{m\to\infty} \int\limits_{\Omega'} (-\Delta)^s u \cdot g_m\, dx = \int\limits_{\Omega'} (-\Delta)^s u \cdot (u - \psi^+)\, dx,$$

that concludes the proof of (5.7).

Now, since $\Omega'$ was arbitrarily chosen and $(-\Delta)^s u \cdot (u - \psi^+) \geq 0$, equality (5.7) implies that $(-\Delta)^s u \cdot (u - \psi^+) = 0$ a.e. in $\Omega$, and *iii*) is proved. □

**Remark 5.2** *Theorem 1.1 holds with the same proof also in the local case $s = 1$. Notice that no regularity assumptions on $\Omega$ are needed, and the cases $p = 1, p = \infty$ are included as well.*

**Remark 5.3** *To obtain better regularity results for $u$, one can apply the regularity theory for*

$$(-\Delta)^s u = g \in L^p(\Omega) \quad in\ \Omega\ , \qquad u \in \widetilde{H}^s(\Omega).$$

*In particular, if $p > \frac{n}{2s}$ and $\Omega$ is Lipschitz and satisfies the exterior ball condition, then $u$ is Hölder continuous in $\Omega$. See for example [16, Proposition 1.4] and [17, Proposition 1.1].*

**Proof of Theorem 1.2**

As usual, we can assume $f = 0$. Fix a small $\varepsilon > 0$, and let $\psi_h^\varepsilon$ be a mollification of $\psi - \varepsilon$. Then $\psi_h^\varepsilon$ is smooth on $\overline{\Omega}$, $\psi_h^\varepsilon < 0$ on $\partial\Omega$ and $\psi_h^\varepsilon \to \psi - \varepsilon$ uniformly on $\overline{\Omega}$, as $h \to \infty$.

By Theorem 1.1, the solution $u_h \in \widetilde{H}^s(\Omega)$ to $\mathcal{P}(\psi_h^\varepsilon, 0)$ satisfies $(-\Delta)^s u_h^\varepsilon \in L^p(\Omega)$ and therefore $u_h^\varepsilon$ is Hölder continuous, see Remark 5.3. Moreover, the estimates in Theorem 4.1 imply that $u_h^\varepsilon \to u^\varepsilon$ uniformly on $\Omega$, where $u^\varepsilon$ solves $\mathcal{P}(\psi - \varepsilon, 0)$. In particular, $u^\varepsilon \in C^0(\overline{\Omega})$. Finally, use again Theorem 4.1 to get that $u^\varepsilon \to u$ uniformly, where $u$ solves $\mathcal{P}(\psi, 0)$. In particular, $u$ is continuous on $\mathbb{R}^n$.

To check the last statement notice that the set $\{u > \psi\} \subseteq \Omega$ is open; for any test function $\varphi \in C^\infty(\{u > \psi\})$ we have that $u \pm t\varphi \in K_\psi^s$ and therefore $t\langle(-\Delta)^s u, \pm\varphi\rangle \geq 0$ for $|t|$ small enough. The conclusion is immediate. □

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