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*Original*

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*Publisher:*

*Published*

DOI:10.1007/s10884-018-9666-y

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# Rate-Independent Damage in Thermo-Viscoelastic Materials with Inertia

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Received: 24 October 2014 / Revised: 22 February 2018  
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**Abstract** We present a model for rate-independent, unidirectional, partial damage in viscoelastic materials with inertia and thermal effects. The damage process is modeled by means of an internal variable, governed by a rate-independent flow rule. The heat equation and the momentum balance for the displacements are coupled in a highly nonlinear way. Our assumptions on the corresponding energy functional also comprise the case of the Ambrosio–Tortorelli phase-field model (without passage to the brittle limit). We discuss a suitable weak formulation and prove an existence theorem obtained with the aid of a (partially) decoupled time-discrete scheme and variational convergence methods. We also carry out the asymptotic analysis for vanishing viscosity and inertia and obtain a fully rate-independent limit model for displacements and damage, which is independent of temperature.

**Keywords** Partial damage · Rate-independent systems · Elastodynamics · Phase-field models · Heat equation · Energetic solutions · Local solutions

**Mathematics Subject Classification** 35Q74 · 74H20 · 74R05 · 74C05 · 74F05

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14 **1 Introduction**

15 Gradient damage models have been extensively studied in recent years, in particular in order  
 16 to understand the behavior of brittle or quasi-brittle materials. In this paper we present a  
 17 model for rate-independent, unidirectional, partial damage in visco-elastic materials with  
 18 inertia and thermal effects. Thus we deal with a PDE system composed of the (damped)  
 19 equation of elastodynamics, a rate-independent flow rule for the damage variable, and the  
 20 heat equation, coupled in a highly nonlinear way. We prove an existence result basing on  
 21 time-discretization and variational convergence methods, where the analytical difficulties  
 22 arise from the interaction of rate-independent and rate-dependent phenomena. We study also  
 23 the relationship of our model with a fully rate-independent system by time rescaling.

24 Following Frémond’s approach [24], damage is represented through an internal variable,  
 25 in the context of generalized standard materials [29]. The damage process is unidirectional,  
 26 meaning that no healing is allowed; we do not use the term “irreversibility” to avoid confusion  
 27 with thermodynamical notions. In our model the evolution of this variable is rate-independent:  
 28 this choice is due to the consideration that, to damage a certain portion of the material, one  
 29 needs a quantity of energy that is independent of the rate of damage, see e.g. [32]. Rate-  
 30 independent damage has been widely explored over the last years, cf. e.g. [6, 18, 19, 26, 34,  
 31 44, 67, 68]. For different studies on rate-dependent damage we refer to e.g. [8, 9, 22] in the  
 32 isothermal case and [3, 28, 62, 63] for temperature-dependent systems.

33 Energy can be dissipated not only by damage growth, but also by viscosity and heat,  
 34 both phenomena having a rate-dependent nature. Rate-independent processes coupled with  
 35 viscosity, inertia, and also temperature have first been analyzed in the two pioneering papers  
 36 [56, 57], cf. also [45, Chapter 5]. Under the assumption of small strains, the momentum equation  
 37 is linearized and is formulated using Kelvin–Voigt rheology and inertia. The nonlinear  
 38 heat equation is coupled with the momentum balance through a thermal expansion term:  
 39 this reflects the fact that temperature changes produce additional stresses. Here, we extend  
 40 Roubíček’s ansatz for the temperature-dependent setting to a unidirectional process, thus  
 41 dealing with a discontinuous rate-independent dissipation potential, cf. (1.2) below. Existence  
 42 results for an Ambrosio–Tortorelli-type system with unidirectional damage, inertia,  
 43 and damping were already provided in [37] in the isothermal case.

44  
 45 *The PDE system.* More precisely, we address the analysis of the following PDE system:

46 
$$\rho \ddot{u} - \operatorname{div} (\mathbb{D}(z, \theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta \mathbb{B}) = f_V \quad \text{in } (0, T) \times \Omega, \tag{1.1a}$$

47 
$$\partial R_1(\dot{z}) + D_z G(z, \nabla z) - \operatorname{div} (D_\xi G(z, \nabla z)) + \frac{1}{2} \mathbb{C}'(z)e(u) : e(u) \ni 0 \quad \text{in } (0, T) \times \Omega, \tag{1.1b}$$

48 
$$\dot{\theta} - \operatorname{div} (\mathbb{K}(z, \theta)\nabla \theta) = R_1(\dot{z}) + \mathbb{D}(z, \theta)e(\dot{u}) : e(\dot{u}) - \theta \mathbb{B} : e(\dot{u}) + H \quad \text{in } (0, T) \times \Omega, \tag{1.1c}$$

50 where the unknowns are the displacement vector field  $u$ , the damage variable  $z$ , and the  
 51 absolute temperature  $\theta$ , all the three being functions of the time  $t \in (0, T)$  and of the position  
 52  $x$  in the reference configuration of a material  $\Omega$ , a bounded subset of  $\mathbb{R}^d$ , with  $d \in \{2, 3\}$ .  
 53 Here,  $e(u) := \frac{1}{2}(\nabla u + \nabla u^\top)$  denotes the linearized strain tensor.

54 In (1.1a), the constant  $\rho > 0$  is the mass density. Moreover,  $\mathbb{D}(z, \theta)$  and  $\mathbb{C}(z)$  are the  
 55 viscous and the elastic stress tensors and are both bounded, symmetric, and positive definite  
 56 on symmetric matrices, uniformly in  $z$  and  $\theta$ . This reflects two hypotheses of the model,

57 motivated by analytical reasons: first, we cannot renounce the presence of some damping in  
 58 the momentum balance; second, we restrict ourselves to the case of partial damage, assuming  
 59 that even in its most damaged state the material keeps some elastic properties. In order to  
 60 account for the phenomenological effect that an increase of damage reduces the stored elastic  
 61 energy, see e.g. [35], it is assumed that the elastic tensor  $\mathbb{C}(z)$  depends monotonically on the  
 62 internal variable  $z$ , cf. also [22, 24, 52].

63 According to the rate-independent and unidirectional nature of the damage process,  $R_1$  is  
 64 a 1-homogeneous dissipation potential of the form

$$65 \quad R_1(v) := \begin{cases} |v| & \text{if } v \leq 0, \\ +\infty & \text{otherwise,} \end{cases} \quad (1.2)$$

66 which enforces the internal variable  $z$  to be nonincreasing in time. Indeed, we assume that  
 67  $z = 1$  marks the sound material and  $z = 0$  the most damaged state.

68 The gradient term  $G(z, \nabla z)$  is needed to regularize damage; in particular, this term also  
 69 allows for a nonconvex dependence on  $z$  as in many phase-field models. Moreover, for  
 70 suitable choices we retrieve the Modica–Mortola term appearing in the Ambrosio–Tortorelli  
 71 functional, see Remark 2.2. The flow rule (1.1b) is given as a subdifferential inclusion, where  
 72  $\partial$  denotes the subdifferential in the sense of convex analysis of  $R_1$  while  $D_z$  and  $D_\xi$  stand for  
 73 the Gâteaux derivatives of  $G(\cdot, \xi)$  and  $G(z, \cdot)$ , respectively. This is a compact way to write  
 74 a (semi)-stability condition of Kuhn–Tucker type.

75 The term  $\theta \mathbb{B}$ , where  $\mathbb{B}$  is a fixed symmetric matrix, derives from thermodynamical consid-  
 76 erations and is a coupling term between the momentum (1.1a) and the heat equation (1.1c).  
 77 The information on the heat conductivity of the material is contained in the symmetric matrix  
 78  $\mathbb{K}(z, \theta)$ . We suppose that  $\mathbb{K}(z, \cdot)$  satisfies subquadratic growth conditions uniformly in  $z$ ,  
 79 which are borrowed from [63] and which are in the same spirit as in [23]. These conditions  
 80 are fundamental in the proof of some a priori estimates; see the discussion below (1.4) for  
 81 appropriate examples from materials science.

82 All the aforementioned quantities are independent of time and space, whilst the external  
 83 force  $f_V$  and the heat source  $H$  are functions of both. The system is complemented with the  
 84 natural boundary conditions

$$85 \quad (\mathbb{D}(z, \theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta \mathbb{B}) \nu = f_S \quad \text{on } (0, T) \times \partial_N \Omega, \quad (1.3a)$$

$$86 \quad u = 0 \quad \text{on } (0, T) \times \partial_D \Omega, \quad (1.3b)$$

$$87 \quad D_\xi G(z, \nabla z) \nu = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (1.3c)$$

$$88 \quad \mathbb{K}(z, \theta) \nabla \theta \cdot \nu = h \quad \text{on } (0, T) \times \partial \Omega, \quad (1.3d)$$

90 where  $\partial_D \Omega$  and  $\partial_N \Omega := \partial \Omega \setminus \partial_D \Omega$  are the Dirichlet and the Neumann part of the boundary,  
 91  $\nu$  denotes the outer unit normal vector to  $\partial \Omega$ , and  $f_S$  and  $h$  are prescribed external data  
 92 depending on time and space. As for the Dirichlet data, we restrict to homogeneous boundary  
 93 conditions, see Remark 2.7 for a discussion on this choice. Moreover, Cauchy conditions are  
 94 given on  $u(0)$ ,  $\dot{u}(0)$ ,  $z(0)$ , and  $\theta(0)$ . We refer to Sect. 2.1 for the precise assumptions on the  
 95 domain and the given data.

97 *The energetic formulation.* Due to the rate-independent character of the flow rule (1.1b)  
 98 and to the nonconvexity of the underlying energy, proving the existence of solutions to the  
 99 PDE system (1.1) in its pointwise form seems to be out of reach. As customary in rate-  
 100 independent processes, we will resort to a weak solvability concept, based on the notion

of *energetic solution*, see [40] and references therein. For fully rate-independent systems, governed (in the classical PDE-formulation) by the static momentum balance for  $u$  and the rate-independent flow rule for  $z$ , the energetic formulation consists of two properties:

- *global stability*: at each time  $t$  the configuration  $(u(t), z(t))$  is a global minimizer of the sum of energy and dissipation;
- *energy-dissipation balance*: the sum of the energy at time  $t$  and of the dissipated energy in  $[0, t]$  equals the initial energy plus the work of external loadings.

Over the last decade, this approach has been extensively applied to several mechanical problems and in particular to fracture, see e.g. [13, 14, 20], and damage, see e.g. [44, 67, 68].

However, in a context where other rate-dependent phenomena are present, the global stability condition is too restrictive. Following [56, 57] we will replace it with a *semistability* condition, where the sum of energy and dissipation is minimized with respect to the internal variable  $z$  only, while the displacement  $u(t)$  is kept fixed, see also [7, 59, 61]. Accordingly, we will weakly formulate system (1.1) by means of

- semistability,
- the (dynamic) momentum equation in a weak sense,
- a suitable energy-dissipation balance,
- the heat equation in a weak sense.

*Existence result.* Theorem 2.6 states the existence of energetic solutions to the initial-boundary value problem for system (1.1). For the proof we rely on a well-established method for showing existence for rate-independent processes [40], adjusted to the coupling with viscosity, inertia, and temperature in [57]. Although we follow the approach of the latter paper, let us point out that the results therein do not account for some properties of our model, namely,

- the unidirectionality of damage, see (1.2),
- the dependence of the viscous tensor  $\mathbb{D}(z, \theta)$  on damage and temperature.

These features are important for the modeling of volume-damage, as well as for the phase-field approximation of fracture and surface damage models, see also Remark 2.2, and cause some analytical difficulties.

As in many works on rate-independent systems, our existence proof is based on time-discretization and approximation by means of solutions to incremental problems. Differently from [57], in our discrete scheme the approximate flow rule is decoupled from the other two equations, which may produce more efficient numerical simulations. Moreover, the assumption of a constant heat capacity allows us to avoid a so-called enthalpy transformation and, together with the subquadratic growth of the heat conductivity, to deduce a priori estimates and the positivity of the temperature by carefully adapting the methods developed in [23, 63].

When taking the time discrete-to-continuous limit, we first pass to the limit in the weak momentum balance. From this we also deduce a (time-continuous) mechanical energy *inequality* by lower semicontinuity arguments. Next we pass to the limit in the semistability inequality using so-called mutual recovery sequences. As a further step we verify that the mechanical energy balance is satisfied as an *equality*: this follows from the momentum balance and the semistability so far obtained. This result allows us to conclude the convergence of the viscous dissipation terms, which, in turn, is crucial for the limit passage in the heat equation. See Sects. 4.1–4.3.

147 *Some remarks on the thermal properties of system (1.1) and its applicability.* For the ther-  
 148 modynamical derivation of the PDE system (1.1) one may follow the thermomechanical  
 149 modeling by Frémond in [24, Chapter 12] or Roubíček in [57]. In particular, the free energy  
 150 density associated with (1.1) is given by

$$151 \quad F(e(u), z, \nabla z, \theta) := \frac{1}{2} \mathbb{C}(z)e(u) : e(u) + G(z, \nabla z) + \varphi(\theta) - \theta \mathbb{B} : e(u), \quad (1.4)$$

152 which leads to the entropy density  $S$  and the internal energy density  $U$  of the form

$$153 \quad S(e(u), z, \nabla z, \theta) = -\partial_\theta F = \mathbb{B} : e(u) - \varphi'(\theta),$$

$$154 \quad U(e(u), z, \nabla z, \theta) = F + \theta S = \frac{1}{2} \mathbb{C}(z)e(u) : e(u) + G(z, \nabla z) + \varphi(\theta) - \theta \varphi'(\theta),$$

156 where  $\varphi$  is a function such that  $c_V(\theta) := \partial_\theta U = -\theta \varphi''(\theta)$  is the specific heat capacity,  
 157 and  $S$  and  $U$  satisfy a Gibbs' relation:  $\partial_\theta U = \theta \partial_\theta S$ . Starting from the entropy equation,  
 158 which balances the changes of entropy with the heat flux and the heat sources given by the  
 159 dissipation rate and the external sources  $H$ ,

$$160 \quad \theta \partial_\theta S \dot{\theta} + \operatorname{div} j = R_1(\dot{z}) + (\mathbb{D}(z, \theta)e(\dot{u}) - \theta \mathbb{B}) : e(\dot{u}) + H,$$

161 and then invoking Fourier's law  $j = -\mathbb{K}(z, \theta)\nabla\theta$  as well as the above Gibbs' relation, the  
 162 choice  $\varphi(\theta) = \theta(1 - \log \theta)$  indeed results in the heat equation (1.1c) with  $c_V(\theta) = \operatorname{const.} = 1$ .

163 In fact, the temperature dependence of the heat capacity can be described by the classical  
 164 Debye model, see e.g. [69, Sect. 4.2, p. 761]. In a first approximation it predicts a cubic growth  
 165 of  $c_V$  with respect to temperature up to a certain, material-specific temperature, the so-called  
 166 Debye temperature  $\theta_D$ , whereas for  $\theta \gg \theta_D$  it can be approximated by  $c_V \equiv \operatorname{const.}$  Thus,  
 167 the use of (1.1c) with  $c_V(\theta) = \operatorname{const.}$  (normalized to  $c_V(\theta) = 1$  for shorter presentation) is  
 168 justified if the temperature range of application is assumed to be above Debye temperature,  
 169 i.e.,  $\theta \gg \theta_D$ . Indeed, our main existence Theorem 2.6, see also Proposition 3.2, contains  
 170 an enhanced positivity estimate, which ensures that the temperature  $\theta$ , as a component of  
 171 an energetic solution  $(u, z, \theta)$ , always stays above a tunable threshold (to be tuned to  $\theta_D$ ),  
 172 provided that the initial temperature and the heat sources  $H$  are suitably large, see (2.16).

173 In this context, let us here also allude to our hypothesis on the heat conductivity tensor  
 174  $\mathbb{K}(z, \theta)$ , which is assumed to have subquadratic growth in  $\theta$ , see (2.6b). According to experi-  
 175 mental findings, cf. [16, 31], polymers such as e.g. polymethylmethacrylate (PMMA), exhibit  
 176 such a subquadratic growth of the heat conductivity. In contrast, for metals the heat conduc-  
 177 tivity is ruled by the electron thermal conductivity. For this, the Wiedemann–Franz law states  
 178 a linear dependence on the temperature, cf. [11, Chapter 17]. Moreover, let us mention that  
 179 the analytical results in [23] are obtained under the assumption of superquadratic growth,  
 180 which is justified by the examples on nonlinear heat conduction given in [70], that are related  
 181 to radiation heat conduction or electron/ion heat conduction in a plasma. Thus, in conclusion,  
 182 the thermal properties of our model rather comply with polymers than with metals.

184 *Vanishing viscosity and inertia.* Finally, in Sect. 5 ahead we will address the analysis of  
 185 system (1.1) as the rates of the external load and of the heat sources become slower and  
 186 slower. Therefore, we will rescale time by a factor  $\varepsilon$  and perform the asymptotic analysis  
 187 as  $\varepsilon \downarrow 0$  of the rescaled system, i.e. with vanishing viscosity and inertia in the momentum  
 188 equation, and vanishing viscosity in the heat equation. Before entering into the details of our  
 189 result, let us briefly overview some related literature.

190 On the one hand, the asymptotic analysis for vanishing viscosity and inertia of the sole  
 191 momentum balance has been the subject of earlier work: we refer, e.g., to [50] for study of the

192 purely elastic limit of dynamic viscoelastic solutions to a frictional contact problem, in terms  
 193 of a graph solution notion. This problem was approached from a more abstract viewpoint  
 194 in [46], with applications to finite-dimensional mechanical systems featuring elastic–plastic  
 195 behavior with linear hardening in [42]. On the other hand, a well-established approach to  
 196 fully rate-independent systems consists in viscously regularizing the rate-independent flow  
 197 rule for the internal variable (typically coupled with a purely elastic equilibrium equation for  
 198 the displacements), and taking the vanishing-viscosity limit. This leads to *parameterized/BV*  
 199 solutions, encoding information on the energetic behavior of the system at jumps, see e.g. [12,  
 200 17,48,49], as well as e.g. [33,34,39] for applications to fracture and damage. We also mention  
 201 [1,53] for finite-dimensional singularly perturbed second order potential-type equations. The  
 202 convergence of kinetic variational inequalities to rate-independent quasistatic variational  
 203 inequalities was tackled in [43].

204 Let us point out that our analysis is substantially different from the “standard” vanishing-  
 205 viscosity approach to rate-independent systems, since in our context viscosity (and inertia  
 206 for the momentum equation) vanish in the heat and momentum balances, only, while we  
 207 keep the flow rule for the damage parameter rate-independent. In fact, our study is akin  
 208 to the vanishing-viscosity and inertia analysis that has been addressed, in the momentum  
 209 equation only, for isothermal, rate-independent processes with dynamics in [56,58], leading  
 210 to an energetic-type notion of solution. We also refer to [15,66] for a combined vanishing-  
 211 viscosity limit in the momentum equation and in the flow rule, in the cases of perfect plasticity  
 212 and delamination, respectively.

213 The coupling with the temperature equation attaches an additional difficulty to our own  
 214 vanishing-viscosity analysis. Because of this, it will be essential to assume an appropriate  
 215 scaling of the tensor of heat conduction coefficients: in fact, we shall require that the conduc-  
 216 tivity matrix ( $\mathbb{K}$  in (1.1c)) diverges as inertia and viscosity vanish. This reflects the fact that in  
 217 the slow-loading regime heat propagates at infinite speed. Thus, in the slow-loading limit we  
 218 will obtain that the temperature is spatially constant and its evolution is fully decoupled from  
 219 the one of the mechanical variables. Indeed, in Theorem 5.3. we will prove convergence as  
 220  $\varepsilon \downarrow 0$  of energetic solutions  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$  of the rescaled system to a triple  $(u, z, \Theta)$  such that  
 221 –  $(u, z)$  is *local solution* (according to the notion introduced in [41,58]) to the (fully  
 222 rate-independent) system consisting of the static momentum balance and of the rate-  
 223 independent flow rule for damage;  
 224 – under a suitable scaling condition on the heat sources, the spatially constant function  $\Theta$   
 225 satisfies an ODE that involves a nonnegative defect measure arising from the limit of the  
 226 viscoelastic dissipation term.

227 *Plan of the paper.* The assumptions on the material quantities and the statement of the  
 228 existence results for energetic solutions are given in Sect. 2. In Sect. 3 we present the properties  
 229 of time-discrete solutions, hence in Sect. 4 we prove the main theorem by passing to the  
 230 time-continuous limit by variational convergence techniques. Finally, Sect. 5 is devoted to  
 231 the asymptotics for vanishing viscosity and inertia.

## 232 2 Setup and Main Result

233 **Notation:** Throughout this paper, for a given Banach space  $X$  we will denote by  $\langle \cdot, \cdot \rangle_X$  the  
 234 duality pairing between  $X^*$  and  $X$ , and by  $\text{BV}([0, T]; X)$ , resp.  $C_{\text{weak}}^0([0, T]; X)$ , the space  
 235 of the bounded variation, resp. weakly continuous, functions with values in  $X$ . Notice that we  
 236 shall consider any  $v \in \text{BV}([0, T]; X)$  to be defined *at all*  $t \in [0, T]$ . We also mention that

the symbols  $c, C, C' \dots$  will be used to denote a positive constant depending on given data, and possibly varying from line to line. Furthermore in proofs, the symbols  $I_i, i = 1, \dots$ , will be place-holders for several integral terms popping up in the various estimates. We warn the reader that we will not be self-consistent with the numbering so that, for instance, the symbol  $I_1$  will occur in several proofs with different meanings.

### 2.1 Assumptions

We now specify the assumptions on the domain  $\Omega$ , on the nonlinear functions featured in (1.1), on the initial data, and on the loading and source terms, under which our existence result, Theorem 2.6, holds. Let us mention in advance that, in order to simplify the exposition in Sects. 2–4, and in view of the analysis for vanishing viscosity and inertia in Sect. 5, cf. (5.32), we will suppose that the matrix of thermal expansion coefficients is a given symmetric matrix  $\mathbb{B} \in \mathbb{R}_{\text{sym}}^{d \times d}$ . We instead allow the elasticity and viscosity tensors to depend on the state variables  $z$  and  $(z, \theta)$ , respectively, thus we need to impose suitable growth and coercivity conditions. We will also make growth assumptions for the matrix of heat conduction coefficients, which are suited for our analysis and which are in the line of [23, 63]. These growth conditions will play a key role in the derivation of estimates for the temperature  $\theta$ , in that it will allow us to cope with the quadratic right-hand side of (1.1c). Before detailing the standing assumptions of this paper, let us mention that, to ease the presentation, we will assume the functions of the temperature featuring in the model to be defined also for nonpositive values of  $\theta$ . At any rate, later on we will prove the existence of solutions such that the temperature is bounded from below by a positive constant, see (2.14)–(2.16).

*Assumptions on the domain.* We assume that

$$\Omega \subset \mathbb{R}^d, \quad d \in \{2, 3\}, \quad \text{is a bounded domain with Lipschitz-boundary } \partial\Omega \text{ such that} \tag{2.1}$$

$$\partial_D\Omega \subset \partial\Omega \text{ is nonempty and relatively open and } \partial_N\Omega := \partial\Omega \setminus \partial_D\Omega.$$

Moreover, we will use the following notation for the state spaces for  $u$  and  $z$ :

$$H_D^1(\Omega; \mathbb{R}^d) := \{v \in H^1(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \partial_D\Omega \text{ in the trace sense}\}, \tag{2.2}$$

$$\mathcal{Z} := \{z \in W^{1,q}(\Omega) : z \in [0, 1] \text{ a.e. in } \Omega\},$$

with fixed  $q > 1$ , cf. (2.5d). Analogous notation will be employed for the Sobolev spaces  $W_D^{1,\gamma}$ ,  $\gamma \geq 1$ .

*Assumptions on the material tensors.* We require that the tensors  $\mathbb{B} \in \mathbb{R}^{d \times d}$ ,  $\mathbb{C} : \mathbb{R} \rightarrow \mathbb{R}^{d \times d \times d \times d}$ , and  $\mathbb{D} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d \times d \times d}$  fulfill

$$\mathbb{B} \in \mathbb{R}_{\text{sym}}^{d \times d} \text{ and set } C_{\mathbb{B}} := |\mathbb{B}|, \tag{2.3a}$$

$$\mathbb{C} \in C^{0,1}(\mathbb{R}; \mathbb{R}^{d \times d \times d \times d}) \text{ and } \mathbb{D} \in C^0(\mathbb{R} \times \mathbb{R}; \mathbb{R}^{d \times d \times d \times d}), \tag{2.3b}$$

$$\mathbb{C}(z), \mathbb{D}(z, \theta) \in \mathbb{R}_{\text{sym}}^{d \times d \times d \times d} \text{ and are positive definite for all } z \in \mathbb{R}, \theta \in \mathbb{R}, \tag{2.3c}$$

$$\exists C_C^1, C_C^2 > 0 \quad \forall z \in \mathbb{R} \quad \forall A \in \mathbb{R}_{\text{sym}}^{d \times d} : \quad C_C^1 |A|^2 \leq \mathbb{C}(z)A : A \leq C_C^2 |A|^2, \tag{2.3d}$$

$$\exists C_D^1, C_D^2 > 0 \quad \forall z \in \mathbb{R} \quad \forall \theta \in \mathbb{R} \quad \forall A \in \mathbb{R}_{\text{sym}}^{d \times d} : \quad C_D^1 |A|^2 \leq \mathbb{D}(z, \theta)A : A \leq C_D^2 |A|^2. \tag{2.3e}$$



272 In the expressions above,  $\mathbb{R}_{\text{sym}}^{d \times d}$  denotes the subset of symmetric matrices in  $\mathbb{R}^{d \times d}$  and  
 273  $\mathbb{R}_{\text{sym}}^{d \times d \times d \times d}$  is the subset of symmetric tensors in  $\mathbb{R}^{d \times d \times d \times d}$ . In particular,

274  $\mathbb{C}(z)_{ijkl} = \mathbb{C}(z)_{jikl} = \mathbb{C}(z)_{ijlk} = \mathbb{C}(z)_{klij}$  and  $\mathbb{D}(z, \theta)_{ijkl} = \mathbb{D}(z, \theta)_{jikl} = \mathbb{D}(z, \theta)_{ijlk} = \mathbb{D}(z, \theta)_{klij}$ .

275 In addition to (2.3), we impose that  $\mathbb{C}(\cdot)$  is monotonically nondecreasing, i.e.,

276 
$$\forall A \in \mathbb{R}_{\text{sym}}^{d \times d} \quad \forall 0 \leq z_1 \leq z_2 \leq 1 : \quad \mathbb{C}(z_1)A : A \leq \mathbb{C}(z_2)A : A. \quad (2.4)$$

277 *Assumptions on the damage regularization.* We require that  $G : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  fulfills

278 **Indicator:** For every  $(z, \xi) \in \mathbb{R} \times \mathbb{R}^d : \quad G(z, \xi) < \infty \Rightarrow z \in [0, 1]; \quad (2.5a)$

279 **Continuity:**  $G$  is continuous on its domain  $\text{dom}(G)$ ,  $G \geq 0$ , and  $G(0, 0) = 0$ ;  
 280  $(2.5b)$

281 **Convexity:** For every  $z \in \mathbb{R}$ ,  $G(z, \cdot)$  is convex;  $(2.5c)$

282 **Growth:** There exist constants  $q > 1$  and  $C_G^1, C_G^2 > 0$  such that for every  $(z, \xi) \in \text{dom}(G)$   
 283  $C_G^1(|\xi|^q - 1) \leq G(z, \xi) \leq C_G^2(|\xi|^q + 1)$ .  $(2.5d)$

284  
 285 *Remark 2.1* (Properties of the regularizing term) Since we are encompassing the feature that  
 286  $z(\cdot, x)$  is decreasing for almost all  $x \in \Omega$ , starting from an initial datum  $z_0 \in [0, 1]$  a.e. in  $\Omega$ ,  
 287 the  $z$ -component of any energetic solution to (1.1) will fulfill  $z(t, x) \leq 1$  a.e. in  $\Omega$ . Therefore,  
 288 we could weaken (2.5a) and just require that the domain of  $G$  is a subset of  $[0, \infty)$ .

289 Furthermore, we may require the third of (2.5b) without loss of generality, since adding  
 290 a constant to  $G$  shall not affect our analysis.

291 Further observe that the above assumptions (2.5) ensure that the integral functional

292 
$$\mathcal{G} : L^r(\Omega) \times L^q(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}, \quad \mathcal{G}(z, \xi) := \int_{\Omega} G(z, \xi) \, dx$$

293 is lower semicontinuous with respect to strong convergence in  $L^r(\Omega)$  for any  $r \in [1, \infty)$   
 294 and weak convergence in  $L^q(\Omega; \mathbb{R}^d)$ , cf. e.g. [21, Theorem 7.5, p. 492]. In addition,  $\mathcal{G}$  is  
 295 continuous with respect to strong convergence in  $(L^r(\Omega) \times L^q(\Omega; \mathbb{R}^d)) \cap \text{dom}(G)$ .

296 *Remark 2.2* (Example: Phase-field approximation of fracture) Starting from the work of  
 297 Ambrosio and Tortorelli [2], gradient damage models have been extensively used in recent  
 298 years to predict crack propagation in brittle or quasi-brittle materials, by means of phase-  
 299 field approximation [4]. In this approach, a sharp crack is regularized by defining an internal  
 300 variable that interpolates continuously between sound and fractured material. In the mathe-  
 301 matical literature, evolutionary problems for phase-field models were considered for instance  
 302 in the fully quasistatic case [25], in viscoelasticity as a gradient flow [5], and in dynamics  
 303 [37], always for isothermal systems. A thermodynamical model for regularized fracture with  
 304 inertia was proposed and treated numerically e.g. in [52]. The passage to the limit from phase-  
 305 field to sharp crack, though successfully treated in the quasistatic [25] and in the viscous case  
 306 [5], is by now an open problem in dynamics and is outside the scope of this contribution.

307 In this context, typical examples for the regularizing term are functionals of Modica-  
 308 Mortola type,

309 
$$\mathcal{G}_{\text{MM}}^q(z, \nabla z) = \int_{\Omega} G_{\text{MM}}^q(z, \nabla z) \, dx \quad \text{with } G_{\text{MM}}^q(z, \nabla z) := |\nabla z|^q + W(z) + I_{[0,1]}(z),$$

310 where  $q > 1$ ,  $W$  is a suitable potential, and  $I_{[0,1]}(z) := 0$  if  $z \in [0, 1]$ ,  $I_{[0,1]}(z) := +\infty$   
 311 otherwise. Such regularization agrees with the above assumptions up to an additive constant.

312 Notice that in Sect. 3, to construct discrete solutions, we will consider unilateral minimum  
 313 problems of the type

$$314 \min_{z \in \mathcal{Z}} \left\{ \int_{\Omega} \frac{1}{2} \mathbb{C}(z) e(u) : e(u) \, dx + \int_{\Omega} G(z, \nabla z) \, dx + \mathcal{R}_1(z - \bar{z}) \right\}$$

315 for given  $u \in H_D^1(\Omega; \mathbb{R}^d)$  and a given  $\bar{z} \in \mathcal{Z}$  defined in (2.2). Setting  $\mathbb{C}(z) := (z^2 + \delta) I$  with  
 316  $\delta > 0$ , and  $G := G_{MM}^2$  with  $W(z) := \frac{1}{2}(1 + z^2)$ , the minimum problem is equivalent to

$$317 \min_{0 \leq z \leq \bar{z}} \left\{ \int_{\Omega} (\frac{1}{2}(z^2 + \delta) |e(u)|^2 \, dx + \int_{\Omega} \frac{1}{2\delta} (1 - z)^2 \, dx + \int_{\Omega} \delta |\nabla z|^2 \, dx \right\},$$

318 that is the classical minimization of the Ambrosio–Tortorelli functional, see [2,25]. The  
 319 generalization to  $G = G_{MM}^q$  with  $q > 1$  was considered in [30]. In this case one may want  
 320 an effective dependence of the viscous tensor on  $z$ , choosing  $\mathbb{D}(z, \theta) = \mathbb{C}(z)$  as in [37].

321 *Assumptions on the heat conductivity.* On  $\mathbb{K} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$  we assume that

$$322 \mathbb{K} \in C^0(\mathbb{R} \times \mathbb{R}; \mathbb{R}^{d \times d}), \quad \mathbb{K}(z, \theta) \in \mathbb{R}_{\text{sym}}^{d \times d} \text{ for all } z \in \mathbb{R}, \theta \in \mathbb{R}, \tag{2.6a}$$

$$323 \exists \kappa \in (1, \kappa_d) \exists c_1, c_2 > 0 \forall (z, \theta) \in \mathbb{R} \times \mathbb{R} \forall \xi \in \mathbb{R}^d : \begin{cases} c_1 (|\theta|^\kappa + 1) |\xi|^2 \leq \mathbb{K}(z, \theta) \xi \cdot \xi, \\ |\mathbb{K}(z, \theta)| \leq c_2 (|\theta|^\kappa + 1), \end{cases} \tag{2.6b}$$

324 where  $\kappa_d = 5/3$  for  $d=3$  and  $\kappa_d = 2$  for  $d=2$ .

325 The bound  $\kappa_d$  essentially comes into play in the derivation of the *Fifth a priori estimate*  
 326 (cf. the proof of Proposition 3.4), and when passing from time-discrete to continuous in  
 327 the heat equation, cf. Proposition 4.9. Essentially, it arises as a consequence of the enhanced  
 328 integrability of the approximating temperature variables obtained by interpolation in (3.32k).  
 329

330 *Assumptions on the initial data.* We impose that

$$331 u_0 \in H_D^1(\Omega; \mathbb{R}^d), \quad \dot{u}_0 \in L^2(\Omega; \mathbb{R}^d), \quad z_0 \in \mathcal{Z}, \tag{2.7a}$$

$$332 \theta_0 \in L^1(\Omega), \quad \text{and } \theta_0 \geq \theta_* > 0 \text{ a.e. in } \Omega, \tag{2.7b}$$

333 where the state spaces  $H_D^1(\Omega; \mathbb{R}^d)$  and  $\mathcal{Z}$  are defined in (2.2).

334 *Assumptions on the loading and source terms.* On the data  $f_V, f_S, H$ , and  $h$  we require that

$$335 f_V \in H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*), \quad f_S \in H^1(0, T; L^2(\partial_N \Omega; \mathbb{R}^d)), \tag{2.8a}$$

$$336 H \in L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)^*), \quad H \geq 0 \text{ a.e. in } (0, T) \times \Omega, \tag{2.8b}$$

$$337 h \in L^1(0, T; L^2(\partial \Omega)), \quad h \geq 0 \text{ a.e. in } (0, T) \times \partial \Omega. \tag{2.8b}$$

338 For later convenience, we also introduce  $f : [0, T] \rightarrow H_D^1(\Omega; \mathbb{R}^d)^*$  defined by

$$339 \langle f(t), v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} := \langle f_V(t), v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} + \int_{\partial_N \Omega} f_S \cdot v \, d\mathcal{H}^{d-1}(x) \quad \text{for all } v \in H_D^1(\Omega; \mathbb{R}^d). \tag{2.9}$$

340 It follows from (2.8a) that  $f \in H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$ .

343 **2.2 Weak Formulation and Main Existence Result**

344 As already mentioned, following [57], the *energetic* formulation of (the initial-boundary  
 345 value problem associated with) system (1.1) consists of the variational formulation of the  
 346 momentum and of the heat equations (1.1a) and (1.1c), with suitable test functions, and of a  
 347 semistability condition joint with a *mechanical energy* balance, providing the weak formula-  
 348 tion of the damage equation (1.1b). The latter relations feature the mechanical (quasistatic)  
 349 energy associated with (1.1), i.e.,

$$350 \quad \mathcal{E}(t, u, z) := \int_{\Omega} \left( \frac{1}{2} \mathbb{C}(z) e(u) : e(u) + G(z, \nabla z) \right) dx - \langle f(t), u \rangle_{H_D^1(\Omega; \mathbb{R}^d)},$$

351 as well as the rate-independent dissipation potential, given as the integrated version of (1.2)

$$352 \quad \mathcal{R}_1(\dot{z}) := \int_{\Omega} R_1(\dot{z}) dx. \tag{2.10}$$

353 In Definition 2.3 below, the choice of the test functions for the weak momentum equation  
 354 reflects the regularity (2.11a) required for  $u$ , which in turn will derive from the standard  
 355 energy estimates that can be performed on system (1.1). As we will see, such estimates only  
 356 yield  $\theta \in L^\infty(0, T; L^1(\Omega))$ . In fact, the further regularity (2.11c) for  $\theta$  shall result from  
 357 a careful choice of test functions for the time-discrete version of (1.1c), and from refined  
 358 interpolation arguments, drawn from [23]. Finally, the BV([0, T];  $W^{2,d+\delta}(\Omega)^*$ )-regularity  
 359 for  $\theta$  follows from a comparison argument. The choice of the test functions in (2.12d) is the  
 360 natural one in view of (2.11).

361 **Definition 2.3** (*Energetic solution* (2.11)–(2.12)) Given a quadruple of initial data  $(u_0, \dot{u}_0,$   
 362  $z_0, \theta_0)$  satisfying (2.7), we call a triple  $(u, z, \theta)$  an *energetic solution* of the Cauchy problem  
 363 for the PDE system (1.1) complemented with the boundary conditions (1.3) if

$$364 \quad u \in H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{2.11a}$$

$$365 \quad z \in L^\infty(0, T; W^{1,q}(\Omega)) \cap L^\infty((0, T) \times \Omega) \cap \text{BV}([0, T]; L^1(\Omega)), \tag{2.11b}$$

$$366 \quad z(t, x) \in [0, 1] \text{ for a.a. } (t, x) \in (0, T) \times \Omega, \tag{2.11c}$$

$$367 \quad \theta \in L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^1(\Omega)) \cap \text{BV}([0, T]; W^{2,d+\delta}(\Omega)^*), \tag{2.11c}$$

368 such that the triple  $(u, z, \theta)$  complies with the initial conditions

$$369 \quad u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0, \quad z(0) = z_0, \quad \theta(0) = \theta_0 \quad \text{a.e. in } \Omega,$$

370 and with the following properties:

- 371 • *unidirectionality*: for a.a.  $x \in \Omega$ , the function  $z(\cdot, x) : [0, T] \rightarrow [0, 1]$  is nonincreasing;
- 372 • *semistability*: for every  $t \in [0, T]$

$$373 \quad \forall \tilde{z} \in \mathcal{Z} : \quad \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}_1(\tilde{z} - z(t)), \tag{2.12a}$$

374 where  $\mathcal{Z}$  is defined in (2.2);

- 375 • *weak formulation of the momentum equation*: for all  $t \in [0, T]$

$$376 \quad \begin{aligned} & \rho \int_{\Omega} \dot{u}(t) \cdot v(t) dx - \rho \int_0^t \int_{\Omega} \dot{u} \cdot \dot{v} dx ds \\ & + \int_0^t \int_{\Omega} (\mathbb{D}(z, \theta) e(\dot{u}) + \mathbb{C}(z) e(u) - \theta \mathbb{B}) : e(v) dx ds \\ & = \rho \int_{\Omega} \dot{u}_0 \cdot v(0) dx + \int_0^t \langle f, v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} ds \end{aligned} \tag{2.12b}$$

377 for all test functions  $v \in L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^d))$ ;

378 • *mechanical energy equality*: for all  $t \in [0, T]$

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\dot{u}(t)|^2 \, dx + \mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z_0 - z(t)) \, dx \\ & + \int_0^t \int_{\Omega} (\mathbb{D}(z, \theta)e(\dot{u}) - \theta \mathbb{B}) : e(\dot{u}) \, dx \, ds \\ & = \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 \, dx + \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds, \end{aligned} \tag{2.12c}$$

380 where  $\partial_t \mathcal{E}(t, u, z) = -\langle \dot{f}(t), u \rangle_{H_D^1(\Omega; \mathbb{R}^d)}$ ;

381 • *weak formulation of the heat equation*: for all  $t \in [0, T]$

$$\begin{aligned} & (\theta(t), \eta(t))_{W^{2,d+\delta}(\Omega)} - \int_0^t \int_{\Omega} \theta \dot{\eta} \, dx \, ds + \int_0^t \int_{\Omega} \mathbb{K}(\theta, z) \nabla \theta \cdot \nabla \eta \, dx \, ds \\ & = \int_{\Omega} \theta_0 \eta(0) \, dx + \int_0^t \int_{\Omega} \eta |\dot{z}| \, dx \, ds + \int_0^t \int_{\Omega} (\mathbb{D}(z, \theta)e(\dot{u}) : e(\dot{u}) - \theta \mathbb{B}) : e(\dot{u}) \eta \, dx \, ds \\ & + \int_0^t \int_{\partial\Omega} h \eta \, d\mathcal{H}^{d-1}(x) \, ds + \int_0^t \int_{\Omega} H \eta \, dx \, ds \end{aligned} \tag{2.12d}$$

382 for all test functions  $\eta \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; W^{2,d+\delta}(\Omega))$ , for some fixed  
 383  $\delta > 0$ . Here and in what follows,  $|\dot{z}|$  denotes the total variation measure of  $z$  (i.e., the  
 384 heat produced by the rate-independent dissipation), which is defined on every closed set  
 385 of the form  $[t_1, t_2] \times C \subset [0, T] \times \bar{\Omega}$  by

$$|\dot{z}|([t_1, t_2] \times C) := \int_C R_1(z(t_2) - z(t_1)) \, dx,$$

386 and, for simplicity, we shall write  $\int_0^t \int_{\Omega} \eta |\dot{z}| \, dx \, ds$  instead of  $\iint_{[0,t] \times \Omega} \eta |\dot{z}| \, (ds \, dx)$ .

387 Since  $z$  has at most BV-regularity as a function of time, it may have (at most countably  
 388 many) jump points, where the left and right limits  $z(t_-), z(t_+) \in L^1(\Omega)$  differ. Indeed,  
 389 from  $z \in L^\infty(0, T; W^{1,q}(\Omega)) \cap \text{BV}([0, T]; L^1(\Omega))$  it is immediate to deduce that, at every  
 390  $t \in [0, T]$  (with the standard conventions  $z(0_-) := z(0)$  and  $z(T_+) := z(T)$ ), both  $z(t_-)$   
 391 and  $z(t_+)$  are elements in  $W^{1,q}(\Omega)$ , with  $z(t_-) = \lim_{s \uparrow t} z(s)$  and  $z(t_+) = \lim_{s \downarrow t} z(s)$  w.r.t.  
 392 the weak topology of  $W^{1,q}(\Omega)$ . In particular, the right limit  $z(0_+)$  exists, and it may be  
 393  $z(0_+) \neq z(0) = z_0$  (observe that, by (2.7) the initial condition is fulfilled as an equality  
 394 in  $W^{1,q}(\Omega)$ ). In that case, the mechanical energy balance (2.12c) records the jump of the  
 395 stored/dissipated energies at the initial time.

396 *Remark 2.4* (Total energy balance) Summing up the mechanical energy inequality (2.12c)  
 397 and the weak heat equation (2.12d) tested by  $\eta \equiv 1$ , yields the total energy balance

$$\begin{aligned} & \int_{\Omega} \frac{\rho}{2} |\dot{u}(t)|^2 \, dx + \mathcal{E}(t, u(t), z(t)) + \int_{\Omega} \theta(t) \, dx = \int_{\Omega} \frac{\rho}{2} |\dot{u}_0|^2 \, dx + \mathcal{E}(0, u_0, z_0) + \int_{\Omega} \theta_0 \, dx \\ & + \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds + \int_0^t \int_{\Omega} H \, dx \, ds + \int_0^t \int_{\partial\Omega} h \, d\mathcal{H}^{d-1}(x) \, ds. \end{aligned}$$

401 *Remark 2.5* (Improved regularity on  $\dot{u}$ ) From the definition of energetic solution we can gain  
 402 improved regularity for the time derivatives of the displacement. Indeed, let  $(u, z, \theta)$  be as

in (2.11) and such that the weak momentum equation (2.12b) holds. Then (1.1a) holds in the sense of distributions and

$$\rho \|\ddot{u}\|_{L^2(0,T;H_D^1(\Omega;\mathbb{R}^d)^*)} = \sup_{\|v\| \leq 1} \int_0^T \int_{\Omega} (\mathbb{D}(z, \theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta \mathbb{B}) : e(v) \, dx \, dt - \int_0^T \langle f, v \rangle_{H_D^1(\Omega;\mathbb{R}^d)} \, dt,$$

where the supremum is taken over all functions such that  $\|v\|_{L^2(0,T;H_D^1(\Omega;\mathbb{R}^d))} \leq 1$ . The left-hand side of the previous equality is uniformly bounded thanks to (2.3), (2.9), and (2.11), thus we deduce that  $\ddot{u} \in L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$ . Since the spaces  $H_D^1(\Omega; \mathbb{R}^d) \subset L^2(\Omega; \mathbb{R}^d) \subset H_D^1(\Omega; \mathbb{R}^d)^*$  form a Gelfand triple, in view of e.g. [36, Chapter 1, Sec. 2.4, Proposition 2.2], we conclude that

$$\begin{aligned} & \int_{t_1}^{t_2} \langle \ddot{u}, \dot{u} \rangle_{H_D^1(\Omega;\mathbb{R}^d)} \, dt \\ &= \frac{1}{2} \langle \dot{u}(t_2), \dot{u}(t_2) \rangle_{H_D^1(\Omega;\mathbb{R}^d)} - \frac{1}{2} \langle \dot{u}(t_1), \dot{u}(t_1) \rangle_{H_D^1(\Omega;\mathbb{R}^d)} \\ &= \frac{1}{2} \|\dot{u}(t_2)\|_{L^2(\Omega;\mathbb{R}^d)}^2 - \frac{1}{2} \|\dot{u}(t_1)\|_{L^2(\Omega;\mathbb{R}^d)}^2 \end{aligned} \tag{2.13}$$

for every  $t_1, t_2 \in [0, T]$ . Hence,  $\dot{u}$  can be used as a test function in (2.12b).

We are now in a position to state the main result of this paper. The last part of the assertion concerns the strict positivity of the absolute temperature  $\theta$ . In particular, under (2.15) below we are able to specify, in terms of the given data, the constant which bounds  $\theta$  from below.

**Theorem 2.6** (Existence of energetic solutions (2.11)–(2.12)) *Under assumptions (2.1)–(2.4), (2.5), and (2.6), and (2.8) on the data  $f_V, f_S, H$ , and  $h$ , for every quadruple  $(u_0, \dot{u}_0, z_0, \theta_0)$  fulfilling (2.7) with  $z_0$  satisfying (2.12a), there exists an energetic solution  $(u, z, \theta)$  to the Cauchy problem for system (1.1).*

Moreover, there exists  $\tilde{\theta} > 0$  such that

$$\theta(t, x) \geq \tilde{\theta} > 0 \quad \text{for a.a. } (t, x) \in (0, T) \times \Omega. \tag{2.14}$$

Furthermore, if in addition

$$\begin{aligned} & \exists H_* > 0 : H(t, x) \geq H_* \text{ for a.a. } (t, x) \in (0, T) \times \Omega \\ & \text{and } \theta_0(x) \geq \sqrt{H_*/\bar{c}} \text{ for a.a. } x \in \Omega, \end{aligned} \tag{2.15}$$

where  $\bar{c} := \frac{(C_{\mathbb{B}})^2}{2C_{\mathbb{D}}}$ , then

$$\theta(t, x) \geq \max \left\{ \tilde{\theta}, \sqrt{H_*/\bar{c}} \right\} \quad \text{for a.a. } (t, x) \in (0, T) \times \Omega. \tag{2.16}$$

The proof of Theorem 2.6 will be developed in Sects. 3 and 4 by time-discretization (see Propositions 4.1–4.2).

**Remark 2.7** (Time-dependent Dirichlet loadings) The existence of energetic solutions can be proven also when *time-dependent Dirichlet loadings* are considered for the displacement  $u$  instead of the homogeneous Dirichlet condition (1.3), in the case the viscous tensor  $\mathbb{D}$  is independent of  $z$  and  $\theta$ . This restriction is due to technical reasons, related to the derivation of suitable estimates for the approximate solutions to (1.1).

An alternative damage model, that still features a  $(z, \theta)$ -dependence of  $\mathbb{D}$ , is discussed in [38], where a time-dependent loading for  $u$  can be encompassed in the analysis, albeit under suitable stronger conditions.

438 *Remark 2.8* (Failure of “entropic” solutions) As already mentioned, the regularity for the  
 439 temperature  $\theta \in L^2(0, T; H^1(\Omega)) \cap BV([0, T]; W^{2,d+\delta}(\Omega)^*)$  results from careful estimates  
 440 on the heat equation (1.1c), tailored on the quadratic character of its right-hand side and  
 441 drawn from [23]. There, the analysis of the *full system* for phase transitions proposed by  
 442 Frémond [24], featuring a heat equation with an  $L^1$  right-hand side, was carried out.

443 The techniques from [23] have been recently extended in [63] to analyze a model for  
 444 *rate-dependent* damage in thermo-viscoelasticity. Namely, in place of the 1-homogeneous  
 445 dissipation potential  $R_1$  from (1.2), the flow rule for the damage parameter in [63] features  
 446 the quadratic dissipation  $R_2(\dot{z}) = \frac{1}{2}|\dot{z}|^2$  if  $\dot{z} \leq 0$ , and  $R_2(\dot{z}) = \infty$  else. Consequently, the  
 447 heat equation in [63] is of the type

$$448 \quad \dot{\theta} - \operatorname{div}(\mathbb{K}(z, \theta)\nabla\theta) = |\dot{z}|^2 + \mathbb{D}(z)e(\dot{u}) : e(\dot{u}) - \theta \mathbb{B} : e(\dot{u}) + H \quad \text{in } (0, T) \times \Omega. \quad (2.17)$$

449 In [63], under a weaker growth condition on  $\mathbb{K}$  than the present (2.6), it was possible to prove  
 450 an existence result for a weaker formulation of (2.17), consisting of an entropy inequality  
 451 and of a total energy inequality. The resulting notion of “entropic” solution, originally  
 452 proposed in [23], indeed reflects the strict positivity of the temperature, and the fact that  
 453 the entropy increases along solutions. Without going into details, let us mention that this  
 454 entropy inequality is (formally) obtained by testing (2.17) by  $\varphi\theta^{-1}$ , with  $\varphi$  a smooth test  
 455 function, and integrating in time. This procedure is fully justified because  $\theta$  can be shown to  
 456 be bounded away from zero by a positive constant, hence  $\varphi(t)\theta^{-1}(t) \in L^\infty(\Omega)$  for almost  
 457 all  $t \in (0, T)$ , and the integrals  $\int_0^T \int_\Omega |\dot{z}|^2 \varphi\theta^{-1} \, dx \, dt$  and  $\int_0^T \int_\Omega \mathbb{D}(z)e(\dot{u}) : e(\dot{u})\varphi\theta^{-1} \, dx \, dt$   
 458 resulting from the first and second terms on the right-hand side of (2.17) are well-defined.

459 In the present *rate-independent* context, proving an existence result for the entropic  
 460 formulation of (1.1c) seems to be out of reach. Indeed, in such formulation the term  
 461  $\int_0^T \int_\Omega |\dot{z}|^2 \varphi\theta^{-1} \, dx \, dt$  would have to be replaced by  $\int_{[0,T] \times \Omega} \varphi\theta^{-1} |\dot{z}| \, (dx \, dt)$ , with  $|\dot{z}|$  the  
 462 total variation measure of  $z$ , cf. (2.12d), but the above integral is not well defined since  $\varphi\theta^{-1}$   
 463 is not a continuous function.

### 464 3 Time-Discretization

#### 465 3.1 The Time-Discrete Scheme

466 Given a partition

$$467 \quad 0 = t_n^0 < \dots < t_n^n = T \quad \text{with} \quad t_n^k - t_n^{k-1} = \frac{T}{n} =: \tau_n,$$

468 we construct a family of discrete solutions  $(u_n^k, z_n^k, \theta_n^k)_{k=1, \dots, n}$  by solving recursively the  
 469 time-discretization scheme (3.3) below, where the data  $f$ ,  $H$ , and  $h$  are approximated by  
 470 *local means* as follows

$$471 \quad f_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} f(s) \, ds, \quad H_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} H(s) \, ds, \quad h_n^k := \frac{1}{\tau_n} \int_{t_n^{k-1}}^{t_n^k} h(s) \, ds, \quad (3.1)$$

472 and the above integrals need to be understood in the Bochner sense.

473 We mention in advance that we have to add the regularizing term  $-\tau_n \operatorname{div}(|e(u_n^k)|^\gamma e(u_n^k))$   
 474 in the discrete momentum equation, with  $\gamma > 4$ . Basically, the reason for this is that we need  
 475 to compensate the quadratic term in  $e(u_n^k)$  on the right-hand side of the discrete heat equation  
 476 (3.3c). In practice, the term  $-\tau_n \operatorname{div}(|e(u_n^k)|^\gamma e(u_n^k))$  will have a key role in proving that  
 477 the pseudomonotone operator in terms of which the (approximate) discrete system can be

reformulated is coercive, and thus such system admits solutions. Because of this additional regularization, it will be necessary to further approximate the initial datum  $u_0$  from (2.7a) by a sequence (cf. [10, p. 56, Corollary 2])

$$(u_n^0)_n \subset W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \text{ such that } u_n^0 \rightarrow u_0 \text{ in } H_D^1(\Omega; \mathbb{R}^d) \text{ as } n \rightarrow \infty, \quad (3.2)$$

where  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d) = \{v \in W^{1,\gamma}(\Omega; \mathbb{R}^d) : v = 0 \text{ on } \partial_D \Omega \text{ in the trace sense}\}$ .

For the weak formulation of the discrete heat equation, we also need to introduce the function space appropriate for  $\theta$ , dependent on a given  $\bar{z} \in L^\infty(\Omega)$

$$X_{\bar{z}} := \left\{ \vartheta \in H^1(\Omega) : \int_{\Omega} \mathbb{K}(\bar{z}, \vartheta) \nabla \vartheta \cdot \nabla v \, dx \text{ is well defined for all } v \in H^1(\Omega) \right\}.$$

In fact, the above space encodes the sharpest property that we will be able to obtain for our discrete solutions  $(u_n^k, z_n^k, \theta_n^k)_{k=1}^n$ . This will be proven by approximating system (3.3) by truncations, so that in the truncated system the heat equation is standardly weakly formulated in  $H^1(\Omega)^*$ . Passing to the limit as the truncation parameter tends to infinity, with a careful comparison argument in the discrete heat equation (cf. the proof of [63, Lemma 4.4] for all details), it is possible to prove that  $\theta_n^k \in X_{z_n^k}$ .

We consider the following weakly-coupled discretization scheme (in fact, only the momentum and the heat equation are coupled, while the discrete equation for  $z$  is decoupled from them):

**Problem 3.1** Starting from

$$u_n^0, \quad z_n^0 := z_0, \quad \theta_n^0 := \theta_0,$$

and setting  $u_n^{-1} := u_n^0 - \tau_n \dot{u}_0$ , find  $(u_n^k, z_n^k, \theta_n^k)_{k=1}^n \subset W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times W^{1,q}(\Omega) \times X_{z_n^k}$  such that the following hold:

– Minimality of  $z_n^k$ :

$$z_n^k \in \operatorname{argmin} \left\{ \mathcal{R}_1(z - z_n^{k-1}) + \mathcal{E}(u_n^k, u_n^{k-1}, z) : z \in \mathcal{Z} \right\}; \quad (3.3a)$$

– Time-discrete weak formulation of the coupled momentum balance and the heat equation:

Find  $u_n^k \in W_D^{1,\gamma}(\Omega; \mathbb{R}^d)$  and  $\theta_n^k \in X_{z_n^k}$  such that

$$\begin{aligned} & \rho \int_{\Omega} \frac{u_n^k - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2} \cdot v \, dx \\ & + \int_{\Omega} \left( \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e \left( \frac{u_n^k - u_n^{k-1}}{\tau_n} \right) + \mathbb{C}(z_n^k) e(u_n^k) - \theta_n^k \mathbb{B} + \tau_n |e(u_n^k)|^{\gamma-2} e(u_n^k) \right) : e(v) \, dx \\ & = \left\langle f_n^k, v \right\rangle_{H_D^1(\Omega; \mathbb{R}^d)} \quad \text{for all } v \in W_D^{1,\gamma}(\Omega; \mathbb{R}^d), \end{aligned} \quad (3.3b)$$

$$\begin{aligned} & \int_{\Omega} \frac{\theta_n^k - \theta_n^{k-1}}{\tau_n} \eta \, dx + \int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \eta \, dx \\ & = \int_{\Omega} \frac{z_n^{k-1} - z_n^k}{\tau_n} \eta \, dx + \int_{\Omega} \left( \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e \left( \frac{u_n^k - u_n^{k-1}}{\tau_n} \right) - \theta_n^k \mathbb{B} \right) : e \left( \frac{u_n^k - u_n^{k-1}}{\tau_n} \right) \eta \, dx \\ & + \int_{\partial \Omega} h_n^k \eta \, d\mathcal{H}^{d-1}(x) + \left\langle H_n^k, \eta \right\rangle_{H^1(\Omega)} \quad \text{for all } \eta \in H^1(\Omega). \end{aligned} \quad (3.3c)$$

510 The above time-discrete problem has been carefully designed in such a way as to be  
 511 weakly-coupled in that, for each  $k \in \{1, \dots, n\}$ , it can be solved successively starting from  
 512 (3.3a) and then solving the system (3.3b)–(3.3c). See [63, Remark 4.3] for similar ideas.

513 Our existence result for Problem 3.1 reads:

514 **Proposition 3.2** *Let the assumptions of Theorem 2.6 hold true. Then there exists a solution*

$$(u_n^k, z_n^k, \theta_n^k)_{k=1}^n \subset W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times W^{1,q}(\Omega) \times H^1(\Omega)$$

515 to Problem 3.1, satisfying the following properties: There exists  $\tilde{\theta} > 0$  such that

$$\theta_n^k \geq \tilde{\theta} > 0 \quad \text{for all } k = 1, \dots, n, \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

516 Furthermore, if in addition (2.15) holds, then

$$\theta_n^k \geq \max \left\{ \tilde{\theta}, \sqrt{H_*/\bar{c}} \right\} > 0 \quad \text{for all } k = 1, \dots, n, \quad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

517 with  $H_*$  and  $\bar{c}$  from (2.15).

521 While the existence of solutions for (3.3a) follows from the direct method of the calculus  
 522 of variations in a straightforward manner, the existence proof for system (3.3b)–(3.3c) is  
 523 more involved, due to the quasilinear character of the discrete heat equation. This is due to  
 524 the fact that the viscous dissipation  $\mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e\left(\frac{u_n^k - u_n^{k-1}}{\tau_n}\right) : e\left(\frac{u_n^k - u_n^{k-1}}{\tau_n}\right)$  as well as the  
 525 thermal stresses  $\theta_n^{k-1} \mathbb{B} : e\left(\frac{u_n^k - u_n^{k-1}}{\tau_n}\right)$  only happen to be of  $L^1$ -summability as a consequence  
 526 of (3.3b). Observe in particular that  $\mathbb{C}(z_n^k), \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) \in (L^\infty(\Omega) \cap W^{1,q}(\Omega))^{d \times d \times d \times d}$ ,  
 527 and we do not impose the assumption  $q > d$ , which would guarantee the continuity of the  
 528 coefficients. As it is demonstrated by the counterexample in [54], in absence of continuous  
 529 coefficients, it is not ensured that the solution of (3.3b) enjoys elliptic regularity. Because of  
 530 this expected lack of additional regularity, the existence of solutions for the coupled system  
 531 (3.3b)–(3.3c) will be verified by means of an approximation procedure, in which the  $L^1$  right-  
 532 hand side in (3.3c) is replaced by a sequence of truncations. For this we proceed along the lines  
 533 of [63] where the analysis of a time-discrete system analogous to (3.3a)–(3.3c) was carried  
 534 out. The existence of solutions to the approximate discrete system in turn follows from an  
 535 existence result for a wide class of elliptic equations, in the framework of the Leray-Schauder  
 536 theory of pseudo-monotone operators. We will then conclude the existence of solutions to  
 537 (3.3b)–(3.3c) by passing to the limit with the truncation parameter. In such a step, we shall  
 538 exploit the strict positivity of the approximate discrete temperatures, cf. (3.15) below. This  
 539 property and the convergence of the approximate discrete temperatures clearly imply the  
 540 strict positivity (3.4). Arguing directly on the non-truncated discrete heat equation, we will  
 541 also obtain the enhanced positivity property (3.5) which, unlike (3.13), in fact provides a  
 542 tunable threshold from below to the discrete temperatures.

543 In the forthcoming proof, we will use that for any convex (differentiable) function  $\psi : \mathbb{R} \rightarrow (-\infty, +\infty]$

$$\psi(x) - \psi(y) \leq \psi'(x)(x - y) \quad \text{for all } x, y \in \text{dom}(\psi). \tag{3.6}$$

546 *Proof Existence of a minimizer to (3.3a):* We first verify the coercivity of the functional  
 547  $z \mapsto \mathcal{E}(t_n^k, u_n^{k-1}, z) + \mathcal{R}_1(z - z_n^{k-1}) : W^{1,q}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ , where  $\mathcal{R}_1$  is the dissipation  
 548 potential (2.10). Indeed, by the positivity of  $\mathcal{R}_1(\cdot)$  and assumption (2.5d) on the density  $G$   
 549 we have

$$\mathcal{E}(t_n^k, u_n^{k-1}, z) + \mathcal{R}_1(z - z_n^{k-1}) \geq \int_\Omega G(z, \nabla z) \, dx - C \geq C_G^1 \|z\|_{W^{1,q}(\Omega)}^q - C_G^1 \mathcal{L}^d(\Omega) - C,$$



551 where we also used that  $G(z(x), \nabla z(x)) < \infty$  implies  $z(x) \in [0, 1]$ , cf. (2.5a). By the  
 552 convexity and the continuity assumptions (2.5b)–(2.5c) on  $G$  and by the properties of  $\mathcal{R}_1$  we  
 553 conclude that the functional

$$554 \quad \mathcal{E}(z_n^k, u_n^{k-1}, \cdot) + \mathcal{R}_1((\cdot) - z_n^{k-1}) : W^{1,q}(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$$

555 is weakly sequentially lower semicontinuous. Since  $\mathcal{Z} = \{z \in W^{1,q}(\Omega) : z \in$   
 556  $[0, 1]$  a.e. in  $\Omega\}$ , see (2.2), is a closed subset of a reflexive Banach space, the direct method  
 557 of the calculus of variations ensures the existence of a minimizer  $z_n^k \in \mathcal{Z}$ .

558 *Existence of an approximate solution to system (3.3b)–(3.3c):* As in [63, proof of Lemma  
 559 4.4], we approximate (3.3b)–(3.3c) by a suitable truncation of the heat conductivity matrix  
 560  $\mathbb{K}$ , in such a way as to reduce to an elliptic operator with *bounded* coefficients in the discrete  
 561 heat equation. In a similar manner we treat the  $L^1$  right-hand sides in order to improve  
 562 their integrability. Accordingly, we truncate all occurrences of  $\theta_n^k$  in the respective terms of  
 563 system (3.3b)–(3.3c). We show that the approximate system thus obtained admits solutions by  
 564 resorting to an existence result from the theory of elliptic systems featuring pseudo-monotone  
 565 operators drawn from [60]. Hence, we pass to the limit with the truncation parameter and  
 566 conclude the existence of solutions to (3.3b)–(3.3c).

567 Let  $z_n^k$  be a solution of (3.3a). In what follows, we shall denote by  $\bar{\mathbb{K}} = \bar{\mathbb{K}}(x, \theta)$  the  
 568 function  $\mathbb{K}(z_n^k(x), \theta)$ . Let  $M > 0$ . We introduce the truncation operator

$$569 \quad \mathcal{J}_M(\theta) := \begin{cases} 0 & \text{if } \theta < 0, \\ \theta & \text{if } 0 \leq \theta \leq M, \\ M & \text{if } \theta > M, \end{cases}$$

570 and we set

$$571 \quad \bar{\mathbb{K}}_M : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}, \quad \bar{\mathbb{K}}_M(x, \theta) := \bar{\mathbb{K}}(x, \mathcal{J}_M(\theta)).$$

572 Since  $\mathbb{K} \in C^0(\mathbb{R} \times \mathbb{R}; \mathbb{R}^{d \times d})$  and  $0 \leq z_n^k(x) \leq 1$  for almost all  $x \in \Omega$ , it is immediate to  
 573 check that there exists a positive constant  $C_M$  such that  $|\bar{\mathbb{K}}_M(x, \theta)| \leq C_M$  for almost all  
 574  $x \in \Omega$  and  $\theta \in \mathbb{R}$ . The truncated version of system (3.3b)–(3.3c) thus reads: find  $(u, \theta) \in$   
 575  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$  such that

$$576 \quad \begin{aligned} & \rho \int_{\Omega} \frac{u - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2} \cdot v \, dx + \int_{\Omega} \left( \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e \left( \frac{u - u_n^{k-1}}{\tau_n} \right) + \mathbb{C}(z_n^k) e(u) \right. \\ & \quad \left. - \mathcal{J}_M(\theta) \mathbb{B} + \tau_n |e(u)|^{\gamma-2} e(u) \right) : e(v) \, dx \\ & = \left\langle f_n^k, v \right\rangle_{H_D^1(\Omega; \mathbb{R}^d)} \end{aligned} \quad \text{for all } v \in W_D^{1,\gamma}(\Omega; \mathbb{R}^d),$$

(3.7a)

$$577 \quad \begin{aligned} & \int_{\Omega} \frac{\theta - \theta_n^{k-1}}{\tau_n} \eta \, dx + \int_{\Omega} \bar{\mathbb{K}}_M(x, \theta) \nabla \theta \cdot \nabla \eta \, dx \\ & = \int_{\Omega} \frac{z_n^{k-1} - z_n^k}{\tau_n} \eta \, dx + \int_{\Omega} \left( \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e \left( \frac{u - u_n^{k-1}}{\tau_n} \right) - \mathcal{J}_M(\theta) \mathbb{B} \right) : e \left( \frac{u - u_n^{k-1}}{\tau_n} \right) \eta \, dx \\ & \quad + \int_{\partial \Omega} h_n^k \eta \, d\mathcal{H}^{d-1}(x) + \left\langle H_n^k, \eta \right\rangle_{H^1(\Omega)} \end{aligned} \quad \text{for all } \eta \in H^1(\Omega).$$

(3.7b)

581 Observe that system (3.7) rewrites as

$$\begin{aligned}
 & \rho \int_{\Omega} u \cdot v \, dx \\
 & + \tau_n \int_{\Omega} \left( \mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e(u) + \tau_n \mathbb{C}(z_n^k)e(u) - \tau_n \mathcal{T}_M(\theta) \mathbb{B} + \tau_n^2 |e(u)|^{\gamma-2} e(u) \right) : e(v) \, dx \\
 & = \rho \int_{\Omega} (2u_n^{k-1} - u_n^{k-2}) \cdot v \, dx + \tau_n \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e(u_n^{k-1}) : e(v) \, dx + \tau_n^2 \langle f_n^k, v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \\
 & \hspace{20em} \text{for all } v \in W_D^{1,\gamma}(\Omega; \mathbb{R}^d),
 \end{aligned}
 \tag{3.8a}$$

$$\begin{aligned}
 & \int_{\Omega} \theta \, \eta \, dx + \tau_n \int_{\Omega} \overline{\mathbb{K}}_M(x, \theta) \nabla \theta \cdot \nabla \eta \, dx - \frac{1}{\tau_n} \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e(u) : e(u) \eta \, dx \\
 & + \int_{\Omega} \mathcal{T}_M(\theta) \mathbb{B} : e(u) \eta \, dx + \frac{2}{\tau_n} \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e(u) : e(u_n^{k-1}) \eta \, dx \\
 & - \int_{\Omega} \mathcal{T}_M(\theta) \mathbb{B} : e(u_n^{k-1}) \eta \, dx \\
 & = \int_{\Omega} \theta_n^{k-1} \eta \, dx + \frac{1}{\tau_n} \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1})e(u_n^{k-1}) : e(u_n^{k-1}) \eta \, dx \\
 & + \int_{\Omega} (z_n^{k-1} - z_n^k) \eta \, dx + \tau_n \int_{\partial \Omega} h_n^k \eta \, d\mathcal{J}^{d-1}(x) + \tau_n \langle H_n^k, \eta \rangle_{H^1(\Omega)},
 \end{aligned}
 \tag{3.8b}$$

for all  $\eta \in H^1(\Omega)$ ,

587 which in turn can be recast in the form

$$A_{k,M}(u, \theta) = B_{k-1}.$$

589 Here,  $A_{k,M} : W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times H^1(\Omega) \rightarrow W_D^{1,\gamma}(\Omega; \mathbb{R}^d)^* \times H^1(\Omega)^*$  is the elliptic operator,  
 590 acting on the unknown  $(u, \theta)$ , defined by the left-hand sides of (3.8a) and (3.8b), while  $B_{k-1}$   
 591 is the vector defined by the right-hand side terms in system (3.8). It can be verified that  
 592  $A_{k,M}$  is a pseudo-monotone operator in the sense of [60, Chapter II, Definition 2.1]: without  
 593 entering into details, we may in fact observe that  $A_{k,M}$  is given by the sum of either bounded,  
 594 radially continuous, monotone operators, or totally continuous operators, cf. [60, Chapter II,  
 595 Definition 2.3, Lemma 2.9, Cor. 2.12]. Furthermore, crucially exploiting the presence of the  
 596 regularizing term  $-\tau_n \operatorname{div}(|e(u)|^{\gamma-2} e(u))$ , with  $\gamma > 4$ , in the discrete momentum balance,  
 597 we may show that  $A_{k,M}$  is coercive on  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$ . This can be checked directly  
 598 on system (3.8), testing (3.8a) by  $u$  and (3.8b) by  $\theta$  and adding the resulting equations: it is  
 599 then sufficient to deduce from these calculations an estimate for  $\|u\|_{W_D^{1,\gamma}(\Omega; \mathbb{R}^d)}$  and  $\|\theta\|_{H^1(\Omega)}$ .  
 600 We refer to [63, proof of Lemma 4.4] for all the detailed calculations, which show that, since  
 601  $\gamma > 4$ , the term  $-\tau_n \operatorname{div}(|e(u)|^{\gamma-2} e(u))$  can absorb the quadratic terms in  $e(u)$  on the right-  
 602 hand side of (3.7b). In this way, it is possible to carry out the test of (3.8b) by  $\theta$  and obtain  
 603 the bound for  $\|\theta\|_{H^1(\Omega)}$ : for this, one also exploits that the operator with coefficients  $\overline{\mathbb{K}}_M$   
 604 is uniformly elliptic thanks to (2.6b). Since  $A_{k,M}$  is pseudo-monotone and coercive, we are  
 605 in a position to apply [60, Chapter II, Theorem 2.6] to system (3.8), for every  $M \in \mathbb{N}$  thus  
 606 deducing the existence of a solution  $(u, \theta)$  which shall be hereafter denoted as  $(u_{n,M}^k, \theta_{n,M}^k)$ .  
 607 *Positivity of  $\theta_{n,M}^k$* : First of all, we show that  $\theta_{n,M}^k \geq 0$  a.e. in  $\Omega$ . To this end, we test the  
 608 (approximate) discrete heat equation (3.7b) by  $-(\theta_{n,M}^k)^- = \min\{\theta_{n,M}^k, 0\}$ . We thus obtain

$$\begin{aligned}
 & \int_{\Omega} \frac{1}{\tau_n} |(\theta_{n,M}^k)^-|^2 dx + \int_{\Omega} \frac{1}{\tau_n} \theta_n^{k-1} (\theta_{n,M}^k)^- dx + \int_{\Omega} \overline{\mathbb{K}}_M(x, \theta_{n,M}^k) \nabla (\theta_{n,M}^k)^- \cdot \nabla (\theta_{n,M}^k)^- dx \\
 609 \quad & = - \int_{\Omega} \frac{z_n^{k-1} - z_n^k}{\tau_n} \theta_{n,M}^k dx - \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e\left(\frac{u - u_n^{k-1}}{\tau_n}\right) : e\left(\frac{u - u_n^{k-1}}{\tau_n}\right) \theta_{n,M}^k dx \\
 & \quad - \int_{\Omega} \mathcal{T}_M(\theta_{n,M}^k) \mathbb{B} : e\left(\frac{u - u_n^{k-1}}{\tau_n}\right) \theta_{n,M}^k dx + \int_{\partial\Omega} h_n^k \theta_{n,M}^k d\mathcal{J}^{d-1}(x) + \left\langle H_n^k, \theta_{n,M}^k \right\rangle_{H^1(\Omega)}.
 \end{aligned}$$

610 Now, the second term on the left-hand side is non-negative, since we may suppose, by  
 611 induction, that  $\theta_n^{k-1} \geq 0$  a.e. in  $\Omega$  (in fact, for  $k = 0$  the *strict* positivity (3.4) holds with  
 612  $\tilde{\theta} = \theta_*$ , thanks to (2.7b)). The third term is also non-negative, by ellipticity of  $\overline{\mathbb{K}}_M$ . As for  
 613 the right-hand side, the first, second, fourth, and fifth terms are negative, since  $z_n^{k-1} \geq z_n^k$   
 614 a.e. in  $\Omega$ , and by the positivity properties of the data  $\mathbb{D}$ ,  $H$ , and  $h$ . The very definition of the  
 615 truncation operator  $\mathcal{T}_M$  does ensure that the third term is null. All in all, we conclude that  
 616  $\int_{\Omega} |(\theta_{n,M}^k)^-|^2 dx \leq 0$ , whence  $(\theta_{n,M}^k)^- = 0$  a.e. in  $\Omega$ , i.e. the desired positivity. Let us now  
 617 prove that  $\theta_{n,M}^k$  fulfills (3.4), namely

$$618 \quad \theta_{n,M}^k \geq \tilde{\theta} > 0 \quad \text{a.e. in } \Omega. \tag{3.9}$$

619 Following the lines of [63, proof of Lemma 4.4] we develop a comparison argument drawn  
 620 from [23]. In this context, we will use the following estimate

$$621 \quad \mathbb{D}(\bar{z}, \bar{\theta}) \bar{e} : \bar{e} - \mathcal{T}_M(\bar{\theta}) \mathbb{B} : \bar{e} \geq C_{\mathbb{D}}^1 |\bar{e}|^2 - |\bar{e}| C_{\mathbb{B}} |\bar{\theta}| \geq \frac{C_{\mathbb{D}}^1}{2} |\bar{e}|^2 - \frac{(C_{\mathbb{B}})^2}{2C_{\mathbb{D}}^1} |\bar{\theta}|^2. \tag{3.10}$$

622 Exploiting (3.10) and also using that  $z_{k-1} \geq z_k$  a.e. in  $\Omega$ , the positivity (2.8b) of the data  $H$   
 623 and  $h$  and of  $\theta_n^{k-1}$ , we deduce from (3.3c) that  $\theta_{n,M}^k$  fulfills

$$624 \quad \int_{\Omega} \theta_{n,M}^k \eta dx + \tau_n \int_{\Omega} \overline{\mathbb{K}}_M(z_n^k, \theta_{n,M}^k) \nabla \theta_{n,M}^k \cdot \nabla \eta dx \geq \int_{\Omega} \theta_n^{k-1} \eta dx - \tau_n \bar{c} \int_{\Omega} (\theta_{n,M}^k)^2 \eta dx \tag{3.11}$$

625 for all  $\eta \in H^1(\Omega) \cap L^\infty(\Omega)$  with  $\eta \geq 0$  a.e. in  $\Omega$ , with the constant  $\bar{c} = \frac{(C_{\mathbb{B}})^2}{2C_{\mathbb{D}}^1}$  independent  
 626 of  $k$ . Hence, we compare  $\theta_{n,M}^k$  with the solution  $v_k \in \mathbb{R}$  of the finite difference equation

$$627 \quad v_k = v_{k-1} - \tau_n \bar{c} v_k^2, \quad k = 1, \dots, n, \quad \text{with } v_0 := \theta_* > 0. \tag{3.12}$$

628 Now, it is possible to show that

$$629 \quad v_k \geq \tilde{\theta} := \left( \bar{c} T + \frac{1}{\theta_*} \right)^{-1}. \tag{3.13}$$

630 We test the difference of (3.11) and (3.12) by the function  $L_\varepsilon(v_k - \theta_{n,M}^k)$ , with

$$631 \quad L_\varepsilon(x) := \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{x}{\varepsilon} & \text{if } 0 < x < \varepsilon, \\ 1 & \text{if } x \geq \varepsilon, \end{cases}$$

632 and we conclude that

$$633 \quad \int_{\Omega} (v_k - v_{k-1}) - (\theta_{n,M}^k - \theta_n^{k-1}) H_\varepsilon(v_k - \theta_{n,M}^k) dx = \tau_n \bar{c} \int_{\Omega} \left( (\theta_{n,M}^k)^2 - v_k^2 \right) H_\varepsilon(v_k - \theta_{n,M}^k) dx \leq 0. \tag{3.14}$$

634 Observe that, in order to conclude that the above integral is negative, it was essential to  
 635 preliminarily show that  $\theta_{n,M}^k \geq 0$  a.e. in  $\Omega$ . Assume now that  $\theta_n^{k-1} \geq v_{k-1}$  (which is true for

636  $k = 0$ , cf. (2.7b)). Letting  $\varepsilon \downarrow 0$  in (3.14) yields that  $\theta_{n,M}^k \geq v_k$  a.e. in  $\Omega$ . Hence, in view of  
 637 (3.13) we conclude the desired (3.9).

638 *Passage to the limit as  $M \rightarrow \infty$ :* We now consider a family  $(u_{n,M}^k, \theta_{n,M}^k)_M$  of solutions to  
 639 the truncated system (3.7): we shall derive some a priori estimates on  $(u_{n,M}^k, \theta_{n,M}^k)_M$  which  
 640 will allow us to extract a (not relabeled) subsequence converging as  $M \rightarrow \infty$  to a solution of  
 641 system (3.3b)–(3.3c). For the ensuing calculations, it is crucial to observe that

642 
$$\exists \tilde{\theta} \text{ such that } \theta_{n,M}^k \geq \tilde{\theta} > 0 \text{ for all } M > 0. \tag{3.15}$$

643 This follows from the very same arguments as for (3.4): indeed, notice that  $\tilde{\theta}$  does not depend  
 644 on  $M$ .

645 Hence, let us first test (3.7a) by  $(u_{n,M}^k - u_n^{k-1})/\tau_n$ , (3.7b) by 1, and add the resulting  
 646 relations. Taking into account the cancellation of the coupling terms between (3.7a) and  
 647 (3.7b), by convexity, cf. (3.6), we obtain

648 
$$\begin{aligned} & \frac{\rho}{2\tau_n^3} \int_{\Omega} |u_{n,M}^k - u_n^{k-1}|^2 dx + \frac{1}{2\tau_n} \int_{\Omega} \mathbb{C}(z_n^k) e(u_{n,M}^k) : e(u_{n,M}^k) dx \\ & \quad + \frac{1}{\gamma} \int_{\Omega} |e(u_{n,M}^k)|^\gamma dx + \frac{1}{\tau_n} \int_{\Omega} \theta_{n,M}^k dx \\ & \leq \frac{\rho}{2\tau_n^3} \int_{\Omega} |u_n^{k-1} - u_n^{k-2}|^2 dx + \frac{1}{2\tau_n} \int_{\Omega} \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1}) dx \\ & \quad + \frac{1}{\gamma} \int_{\Omega} |e(u_n^{k-1})|^\gamma dx + \frac{1}{\tau_n} \int_{\Omega} \theta_n^{k-1} dx \\ & \quad + \left\langle f_n^k, \frac{u_{n,M}^k - u_n^{k-1}}{\tau_n} \right\rangle_{H_D^1(\Omega; \mathbb{R}^d)} + \int_{\Omega} \left( \frac{z_n^{k-1} - z_n^k}{\tau_n} + H_n^k \right) dx \\ & \quad + \int_{\partial\Omega} h_n^k d\mathcal{H}^{d-1}(x) \leq C_{k,n}, \end{aligned}$$

649 where the constant  $C_{k,n}$  is uniform with respect to the truncation parameter  $M$  (but depends  
 650 on  $k$  and  $n$ ). Therefore, also on account of (3.15) we infer that

651 
$$\|u_{n,M}^k\|_{W^{1,\gamma}(\Omega; \mathbb{R}^d)} + \|\theta_{n,M}^k\|_{L^1(\Omega)} \leq C_{k,n}, \tag{3.16}$$

652 for a (possibly different) constant  $C_{k,n}$  uniform w.r.t.  $M$  but depending on  $k$  and  $n$ . From now  
 653 till the end of the discussion of the limit passage  $M \rightarrow \infty$ , we will omit the dependence of  
 654 such constants on  $k$  and  $n$ . As a straightforward consequence of (3.16), if we define

655 
$$S_M = \{x \in \Omega : \theta_{n,M}^k \leq M\},$$

656 using Markov's inequality, it is not difficult to infer from (3.16) that

657 
$$|\Omega \setminus S_M| \rightarrow 0 \text{ as } M \rightarrow \infty. \tag{3.17}$$

658 Secondly, we test (3.7b) by  $\mathcal{J}_M(\theta_{n,M}^k)$ . Using that

659 
$$\theta \mathcal{J}_M(\theta) \geq |\mathcal{J}_M(\theta)|^2 \text{ and } \overline{\mathbb{K}}_M(x, \theta) \nabla \theta \cdot \nabla \mathcal{J}_M(\theta) = \overline{\mathbb{K}}(x, \mathcal{J}_M(\theta)) \nabla \mathcal{J}_M(\theta) \cdot \nabla \mathcal{J}_M(\theta),$$

660 we obtain

661 
$$\begin{aligned} & \frac{1}{2\tau_n} \int_{\Omega} |\mathcal{J}_M(\theta_{n,M}^k)|^2 dx + \int_{\Omega} \overline{\mathbb{K}}(x, \mathcal{J}_M(\theta_{n,M}^k)) \nabla \mathcal{J}_M(\theta_{n,M}^k) \cdot \nabla \mathcal{J}_M(\theta_{n,M}^k) dx \\ & \leq \frac{1}{2\tau_n} \int_{\Omega} |\theta_n^{k-1}|^2 dx + I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{3.18}$$

662 where, taking into account (2.3e) and the previously obtained (3.16), we have

$$\begin{aligned}
 I_1 &:= \left| \int_{\Omega} \mathbb{D}(z_n^{k-1}, \theta_n^{k-1}) e\left(\frac{u_{n,M}^k - u_n^{k-1}}{\tau_n}\right) : e\left(\frac{u_{n,M}^k - u_n^{k-1}}{\tau_n}\right) \mathcal{T}_M(\theta_{n,M}^k) \, dx \right| \\
 &\leq C \left\| e\left(\frac{u_{n,M}^k - u_n^{k-1}}{\tau_n}\right) \right\|_{L^4(\Omega; \mathbb{R}^{d \times d})}^4 + \frac{1}{8\tau_n} \int_{\Omega} |\mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx, \\
 I_2 &:= \left| \int_{\Omega} \mathcal{T}_M(\theta_{n,M}^k) \mathbb{B} : e\left(\frac{u_{n,M}^k - u_n^{k-1}}{\tau_n}\right) \mathcal{T}_M(\theta_{n,M}^k) \, dx \right| \\
 &\leq C \left\| e\left(\frac{u_{n,M}^k - u_n^{k-1}}{\tau_n}\right) \right\|_{L^2(\Omega; \mathbb{R}^{d \times d})} \|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^4(\Omega)}^2 \\
 &\leq C \|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^4(\Omega)}^2 \leq \frac{c_1}{4} \int_{\Omega} |\nabla \mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx + \|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^1(\Omega)}^2, \\
 I_3 &:= \left| \int_{\Omega} \frac{z_n^k - z_n^{k-1}}{\tau_n} \mathcal{T}_M(\theta_{n,M}^k) \, dx \right| \leq C + \frac{1}{8\tau_n} \int_{\Omega} |\mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx, \\
 I_4 &:= \left| \left\langle H_n^k, \mathcal{T}_M(\theta_{n,M}^k) \right\rangle_{H^1(\Omega)} + \int_{\partial\Omega} h_n^k \mathcal{T}_M(\theta_{n,M}^k) \, d\mathcal{H}^{d-1}(x) \right| \\
 &\leq \frac{1}{16\tau_n} \int_{\Omega} |\mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx + \frac{c_1}{2} \int_{\Omega} |\nabla \mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx + C.
 \end{aligned}$$

668 where in the estimate for  $I_2$  we have used the previously obtained bound (3.16), the  
 669 Gagliardo-Nirenberg inequality  $\|v\|_{L^4(\Omega)} \leq C \|v\|_{H^1(\Omega)}^\sigma \|v\|_{L^1(\Omega)}^{1-\sigma}$  for  $\sigma = 9/10$ , and the  
 670 Young inequality. As by (2.6b) it is  $\overline{\mathbb{K}}_M \xi \cdot \xi \geq c_1 |\xi|^2$ , combining the above estimates with  
 671 (3.18) and taking into account (3.16), we conclude that

$$\|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^2(\Omega)} + \int_{\Omega} \overline{\mathbb{K}}(x, \mathcal{T}_M(\theta_{n,M}^k)) \nabla \mathcal{T}_M(\theta_{n,M}^k) \cdot \nabla \mathcal{T}_M(\theta_{n,M}^k) \, dx \leq C.$$

673 Now, the coercivity (2.6b) implies

$$\begin{aligned}
 &\int_{\Omega} \overline{\mathbb{K}}(x, \mathcal{T}_M(\theta_{n,M}^k)) \nabla \mathcal{T}_M(\theta_{n,M}^k) \cdot \nabla \mathcal{T}_M(\theta_{n,M}^k) \, dx \\
 &\geq c_1 \int_{\Omega} |\mathcal{T}_M(\theta_{n,M}^k)|^\kappa |\nabla \mathcal{T}_M(\theta_{n,M}^k)|^2 \, dx = c \int_{\Omega} |\nabla (\mathcal{T}_M(\theta_{n,M}^k))^{(\kappa+2)/2}|^2 \, dx.
 \end{aligned}$$

675 From this, recalling the continuous embedding  $H^1 \subset L^6$  we infer

$$\|\mathcal{T}_M(\theta_{n,M}^k)\|_{H^1(\Omega)} + \|\mathcal{T}_M(\theta_{n,M}^k)\|_{L^{3\kappa+6}(\Omega)} \leq C. \tag{3.19}$$

677 Thirdly, we test (3.7b) by  $\theta_{n,M}^k$ . Relying on estimate (3.19) to bound the second term on  
 678 the right-hand side of (3.7b) and mimicking the above calculations, we obtain

$$\|\theta_{n,M}^k\|_{H^1(\Omega)} + \|\theta_{n,M}^k\|_{L^{3\kappa+6}(\mathcal{S}_M)} \leq C. \tag{3.20}$$

680 With estimates (3.16), (3.19), and (3.20), combined with well-known compactness arguments,  
 681 we find a pair  $(u, \theta)$  such that, along a not relabeled subsequence,  $(u_{n,M}^k, \theta_{n,M}^k) \rightharpoonup (u, \theta)$   
 682 in  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d) \times H^1(\Omega)$ . The argument for passing to the limit as  $M \rightarrow \infty$  in (3.7), also  
 683 based on (3.17), is completely analogous to the one developed in the proof of [63, Lemma  
 684 4.4], therefore we refer to the latter paper for all details.

685 *Positivity of the discrete temperature, ad (3.4):* The strict positivity (3.4) is now inherited  
 686 by  $\theta_n^k$  in the limit passage, as  $M \rightarrow \infty$ , in (3.4).

687 *Refined positivity estimate for the discrete temperature, ad (3.5):* Under the additional  
 688 strict positivity (2.15) of  $H$ , arguing as in the above lines we infer that  $\theta_n^k$  fulfills

$$689 \int_{\Omega} \theta_n^k \eta \, dx + \tau_n \int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \eta \, dx \geq \int_{\Omega} \theta_n^{k-1} \eta \, dx + \int_{\Omega} \tau_n (H_* - \bar{c} (\theta_n^k)^2) \eta \, dx$$

690 for all  $\eta \in L^\infty(\Omega)$  with  $\eta \geq 0$  a.e. in  $\Omega$ , with  $\bar{c} > 0$  the same constant as in (3.11). Hence,  
 691 we compare  $\theta_n^k$  with the solution  $\tilde{v}_k \in \mathbb{R}$

$$692 \tilde{v}_k = \tilde{v}_{k-1} + \tau_n (H_* - \bar{c} \tilde{v}_k^2), \quad k = 1, \dots, n, \quad \text{with } \tilde{v}_0 := \max \left\{ \theta_*, \sqrt{H_* / \bar{c}} \right\} > 0, \quad (3.21)$$

693 The very same arguments from [63, proof of Lemma 4.4], cf. also the previous discussion,  
 694 allow us to show for all  $k = 0, \dots, n$  that  $\theta_n^k(x) \geq \tilde{v}_k$  for almost all  $x \in \Omega$ . Since  $\tilde{v}_k >$   
 695  $\tilde{v}_{k-1} - \tau_n \bar{c} \tilde{v}_k^2$ , and  $\tilde{v}_0 \geq v_0 = \theta_*$ , a comparison with the solution  $v_k$  of the finite-difference  
 696 equation (3.12) and induction over  $k$  yield that  $\tilde{v}_k \geq v_k$ . Hence  $\tilde{v}_k \geq \tilde{\theta} > 0$ . We now aim to  
 697 prove that

$$698 \tilde{v}_k \geq \sqrt{H_* / \bar{c}} \quad \text{for all } k = 1, \dots, n. \quad (3.22)$$

699 We proceed by contradiction and suppose that  $H_* > \bar{c} \tilde{v}_{\bar{k}}^2$  for a certain  $\bar{k} \in \{1, \dots, n\}$ . Then,  
 700 we read from (3.21) that  $\tilde{v}_{\bar{k}} > \tilde{v}_{\bar{k}-1}$ . Since  $\tilde{v}_{\bar{k}-1} > 0$ , we then conclude that  $H_* > \bar{c} \tilde{v}_{\bar{k}}^2 >$   
 701  $\bar{c} \tilde{v}_{\bar{k}-1}^2$ . Proceeding by induction, we thus conclude that  $H_* > \bar{c} \tilde{v}_0^2$ , which is a contradiction to  
 702 (3.21). Therefore, (3.22) ensues. This concludes the existence proof for system (3.3b)–(3.3c).  
 703 □

### 704 3.2 Time-Discrete Version of the Energetic Formulation

705 We now define the approximate solutions to the energetic formulation of the initial-boundary  
 706 value problem for system (1.1) by suitably interpolating the discrete solutions  $(u_n^k, z_n^k, \theta_n^k)_{k=1}^n$   
 707 from Proposition 3.2. Namely, for  $t \in (t_n^{k-1}, t_n^k]$ ,  $k = 1, \dots, n$ , we set

$$708 \bar{u}_n(t) := u_n^k, \quad \bar{\theta}_n(t) := \theta_n^k, \quad \bar{z}_n(t) := z_n^k, \quad (3.23a)$$

$$709 \underline{u}_n(t) := u_n^{k-1}, \quad \underline{\theta}_n(t) := \theta_n^{k-1}, \quad \underline{z}_n(t) := z_n^{k-1}, \quad (3.23b)$$

711 and we also consider the piecewise linear interpolants, defined by

$$712 u_n(t) := \frac{t-t_n^{k-1}}{\tau_n} u_n^k + \frac{t_n^k-t}{\tau_n} u_n^{k-1}, \quad z_n(t) := \frac{t-t_n^{k-1}}{\tau_n} z_n^k + \frac{t_n^k-t}{\tau_n} z_n^{k-1}, \quad \theta_n(t) := \frac{t-t_n^{k-1}}{\tau_n} \theta_n^k + \frac{t_n^k-t}{\tau_n} \theta_n^{k-1}. \quad (3.23c)$$

713 In what follows, we shall understand the time derivative of the piecewise linear interpolant  
 714  $u_n$  to be defined also at the nodes of the partition by

$$715 \dot{u}_n(t_n^k) := \frac{u_n^k - u_n^{k-1}}{\tau_n}, \quad \text{for } k = 1, \dots, n. \quad (3.23d)$$

716 This will allow us, for instance, to state (3.27) for all  $t \in [0, T]$ . We also introduce the  
 717 piecewise constant and linear interpolants of the discrete data  $(f_n^k, H_n^k, h_n^k)_{k=1}^n$  in (3.1) by  
 718 setting for  $t \in (t_n^{k-1}, t_n^k]$

$$719 \bar{f}_n(t) := f_n^k, \quad \bar{H}_n(t) := H_n^k, \quad \bar{h}_n(t) := h_n^k,$$

720 and  $f_n(t) := \frac{t-t_n^{k-1}}{\tau_n} f_n^k + \frac{t_n^k-t}{\tau_n} f_n^{k-1}$  with time derivative  $\dot{f}_n(t) := \frac{f_n^k-f_n^{k-1}}{\tau_n}$ . It follows  
 721 from (2.8) that, as  $n \rightarrow \infty$ ,

722  $\bar{f}_n \rightarrow f$  in  $L^p(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$  for all  $1 \leq p < \infty$ ,

723  $\bar{f}_n \overset{*}{\rightharpoonup} f$  in  $L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$ , (3.24a)

724  $\bar{f}_n(t) \rightarrow f(t)$  in  $H_D^1(\Omega; \mathbb{R}^d)^*$  for all  $t \in [0, T]$ , (3.24b)

725  $f_n \rightharpoonup f$  in  $H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$ , (3.24c)

726  $\bar{H}_n \rightarrow H$  in  $L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)^*)$ ,

727  $\bar{h}_n \rightarrow h$  in  $L^1(0, T; L^2(\partial\Omega))$ . (3.24d)

729 Finally, we consider the piecewise constant interpolants associated with the partition, i.e.,

730  $\bar{\tau}_n(t) := t_n^k$  and  $\underline{\tau}_n(t) := t_n^{k-1}$  for  $t \in (t_n^{k-1}, t_n^k]$ .

731 In Proposition 3.3 we show that the approximate solutions introduced above indeed fulfill  
 732 the discrete version of the energetic formulation from Definition 2.3. In order to check the  
 733 discrete momentum equation (3.27b) and (3.27e), we shall make use of the following *discrete*  
 734 *by-part integration* formula, for every  $(r_k)_{k=1}^n \subset X$  and  $(s_k)_{k=1}^n \subset X^*$ , with  $X$  a given Banach  
 735 space:

736 
$$\sum_{k=1}^n \langle s_k, r_k - r_{k-1} \rangle_X = \langle s_n, r_n \rangle_X - \langle s_0, r_0 \rangle_X - \sum_{k=1}^n \langle s_k - s_{k-1}, r_{k-1} \rangle_X. \quad (3.25)$$

737 In the discrete mechanical energy inequality (3.27c) below, the mechanical energy  $\mathcal{E}$  will be  
 738 replaced by

739 
$$\mathcal{E}_n(t, u, z) := \int_{\Omega} \left( \frac{1}{2} \mathbb{C}(z) e(u) : e(u) + \frac{\tau_n}{\gamma} |e(u)|^\gamma \right) dx + \mathcal{G}(z, \nabla z) - \langle \bar{f}_n(t), u \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \quad \text{with } \tau_n = \frac{T}{n}. \quad (3.26)$$

740 **Proposition 3.3** (Time-discrete version of the energetic formulation (2.12) & total energy  
 741 inequality) *Let the assumptions of Theorem 2.6 hold true. Then the interpolants of the time-*  
 742 *discrete solutions  $(\bar{u}_n, \underline{u}_n, u_n, \bar{z}_n, \underline{z}_n, z_n, \bar{\theta}_n, \underline{\theta}_n, \theta_n)$  obtained via Problem 3.1 and (3.23)*  
 743 *satisfy the following properties:*

- 744 • *unidirectionality: for a.a.  $x \in \Omega$ , the functions  $\bar{z}_n(\cdot, x) : [0, T] \rightarrow [0, 1]$  are nonincreas-*  
 745 *ing;*
- 746 • *discrete semistability: for all  $t \in [0, T]$*

747 
$$\forall \tilde{z} \in \mathcal{Z} : \mathcal{E}_n(t, \underline{u}_n(t), \bar{z}_n(t)) \leq \mathcal{E}_n(t, \underline{u}_n(t), \tilde{z}) + \mathcal{R}_1(\tilde{z} - \bar{z}_n(t)); \quad (3.27a)$$

- 748 • *discrete formulation of the momentum equation: for all  $t \in [0, T]$  and for every  $(n +$   
 749  $1)$ -tuple  $(v_n^k)_{k=0, \dots, n} \subset W_D^{1,\gamma}(\Omega; \mathbb{R}^d)$ , setting  $\bar{v}_n(s) := v_n^k$  and  $v_n(s) := \frac{s-t_n^{k-1}}{\tau_n} v_n^k +$   
 750  $\frac{t_n^k-s}{\tau_n} v_n^{k-1}$  for  $s \in (t_n^{k-1}, t_n^k]$ ,*

$$\begin{aligned} & \rho \int_{\Omega} (\dot{u}_n(t) \cdot \bar{v}_n(t) - \dot{u}_0 \cdot v_n(0)) \, dx - \rho \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \dot{u}_n(s - \tau_n) \cdot \dot{v}_n(s) \, dx \, ds \\ & \quad + \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) + \mathbb{C}(\bar{z}_n)e(\bar{u}_n) - \bar{\theta}_n \mathbb{B} + \tau_n |e(\bar{u}_n)|^{\gamma-2} e(\bar{u}_n)) : e(\bar{v}_n) \, dx \, ds \\ & = \int_0^{\bar{\tau}_n(t)} \langle \bar{f}_n, \bar{v}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds, \end{aligned} \tag{3.27b}$$

751

where we have extended  $u_n$  to  $(-\tau_n, 0]$  by setting  $u_n(t) := u_n^0 + t\dot{u}_0$ ;

752

- discrete mechanical energy inequality: for all  $t \in [0, T]$

753

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\dot{u}_n(t)|^2 \, dx + \mathcal{E}_n(t, \bar{u}_n(t), \bar{z}_n(t)) + \int_{\Omega} (z_0 - \bar{z}_n(t)) \, dx \\ & \quad + \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) - \bar{\theta}_n \mathbb{B}) : e(\dot{u}_n) \, dx \, ds \\ & \leq \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 \, dx + \mathcal{E}_n(0, u_n^0, z_0) - \int_0^{\bar{\tau}_n(t)} \langle \dot{f}_n, \underline{u}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds; \end{aligned} \tag{3.27c}$$

754

- discrete total energy inequality: for all  $t \in [0, T]$

755

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\dot{u}_n(t)|^2 \, dx + \mathcal{E}_n(t, \bar{u}_n(t), \bar{z}_n(t)) + \int_{\Omega} \bar{\theta}_n(t) \, dx \\ & \leq \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 \, dx + \mathcal{E}_n(0, u_n^0, z_0) + \int_{\Omega} \theta_0 \, dx \\ & \quad - \int_0^{\bar{\tau}_n(t)} \langle \dot{f}_n, \underline{u}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds + \int_0^{\bar{\tau}_n(t)} \left[ \int_{\partial\Omega} \bar{h}_n \, d\mathcal{H}^{d-1}(x) + \int_{\Omega} \bar{H}_n \, dx \right] \, ds; \end{aligned} \tag{3.27d}$$

756

- discrete formulation of the heat equation: for all  $t \in [0, T]$  and for every  $(n + 1)$ -tuple  $(\eta_n^k)_{k=0}^n \subset H^1(\Omega)$ , setting  $\bar{\eta}_n(s) := \eta_n^k$  and  $\eta_n(s) := \frac{s-t_n^{k-1}}{\tau_n} \eta_n^k + \frac{t_n^k - s}{\tau_n} \eta_n^{k-1}$  for  $s \in (t_n^{k-1}, t_n^k]$ ,

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759

$$\begin{aligned} & \int_{\Omega} \bar{\theta}_n(t) \bar{\eta}_n(t) \, dx - \int_{\Omega} \theta_0 \eta_n(0) \, dx - \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \underline{\theta}_n(s) \dot{\eta}_n(s) \, dx \, ds \\ & \quad + \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla \bar{\theta}_n) \cdot \nabla \bar{\eta}_n \, dx \, ds \\ & = \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \bar{\eta}_n |\dot{z}_n| \, dx \, ds - \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) - \bar{\theta}_n \mathbb{B}) : e(\dot{u}_n) \bar{\eta}_n \, dx \, ds \\ & \quad + \int_0^{\bar{\tau}_n(t)} \left[ \int_{\partial\Omega} \bar{h}_n \eta_n \, d\mathcal{H}^{d-1}(x) + \langle \bar{H}_n, \eta_n \rangle_{H^1(\Omega)} \right] \, ds. \end{aligned} \tag{3.27e}$$

760

*Proof* The discrete momentum and heat equations (3.27b) and (3.27e) follow from testing (3.3b) and (3.3c) by the discrete test functions  $(v_n^k)_{k=0}^n \subset W_D^{1,\gamma}(\Omega; \mathbb{R}^d)$  and  $(\eta_n^k)_{k=0}^n \subset H^1(\Omega)$ , respectively, and applying the discrete by-part integration formula (3.25). From the discrete minimum problem (3.3a) we infer

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$$\begin{aligned} & \mathcal{E}(t_n^k, u_n^{k-1}, z_n^k) \leq \mathcal{E}(t_n^k, u_n^{k-1}, \tilde{z}) + \int_{\Omega} (z_n^{k-1} - \tilde{z}) \, dx \\ & \quad - \int_{\Omega} (z_n^{k-1} - z_n^k) \, dx \leq \mathcal{E}(t_n^k, u_n^{k-1}, \tilde{z}) + \int_{\Omega} (z_n^k - \tilde{z}) \, dx \end{aligned}$$

765

766



767 for all  $\tilde{z} \in \mathcal{Z}$  with  $\tilde{z} \leq z_n^{k-1}$ . By (3.3a) and the definition of the dissipation  $\mathcal{R}_1$  we have  
 768  $z_n^k \leq z_n^{k-1}$ , whence the unidirectionality and the discrete semistability (3.27a) hold.  
 769 To deduce the mechanical energy inequality (3.27c) we choose  $z_n^{k-1}$  as a competitor in  
 770 (3.3a) and get

$$\int_{\Omega} (z_n^{k-1} - z_n^k) \, dx + \int_{\Omega} \left( \frac{1}{2} \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1}) + G(z_n^k, \nabla z_n^k) \right) \, dx$$

$$\leq \int_{\Omega} \left( \frac{1}{2} \mathbb{C}(z_n^{k-1}) e(u_n^{k-1}) : e(u_n^{k-1}) + G(z_n^{k-1}, \nabla z_n^{k-1}) \right) \, dx. \tag{3.28}$$

772 Moreover, we test (3.3b) by  $v = u_n^k - u_n^{k-1}$ . To this aim, we observe that by convexity (3.6)

$$\rho \int_{\Omega} \frac{u_n^k - 2u_n^{k-1} + u_n^{k-2}}{\tau_n^2} \cdot (u_n^k - u_n^{k-1}) \, dx \geq \rho \int_{\Omega} \left( \frac{1}{2} \frac{|u_n^k - u_n^{k-1}|^2}{\tau_n^2} - \frac{1}{2} \frac{|u_n^{k-1} - u_n^{k-2}|^2}{\tau_n^2} \right) \, dx, \tag{3.29a}$$

$$\int_{\Omega} \mathbb{C}(z_n^k) e(u_n^k) : (e(u_n^k) - e(u_n^{k-1})) \, dx \geq \int_{\Omega} \frac{1}{2} \left( \mathbb{C}(z_n^k) e(u_n^k) : e(u_n^k) - \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1}) \right) \, dx, \tag{3.29b}$$

$$\int_{\Omega} \tau_n |e(u_n^k)|^{\gamma-2} e(u_n^k) : (e(u_n^k) - e(u_n^{k-1})) \, dx \geq \int_{\Omega} \left( \frac{\tau_n}{\gamma} |e(u_n^k)|^{\gamma} - \frac{\tau_n}{\gamma} |e(u_n^{k-1})|^{\gamma} \right) \, dx. \tag{3.29c}$$

777 Further, let  $t \in (0, T]$  be fixed, and let  $1 \leq j \leq n$  fulfill  $t \in (t_n^{j-1}, t_n^j]$ . We sum (3.29a)–  
 778 (3.29c) over the index  $k = 1, \dots, j$ . Applying the by-part integration formula (3.25) we  
 779 conclude that

$$\sum_{k=1}^j \left\langle f_n^k, u_n^k - u_n^{k-1} \right\rangle_{H_D^1(\Omega; \mathbb{R}^d)} = \int_0^{\bar{\tau}_n(t)} \langle \bar{f}_n, \dot{u}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds$$

$$= \langle \bar{f}_n(t), \bar{u}_n(t) \rangle_{H_D^1(\Omega; \mathbb{R}^d)} - \langle f(0), u_0 \rangle_{H_D^1(\Omega; \mathbb{R}^d)} - \int_0^{\bar{\tau}_n(t)} \langle \dot{f}_n, \underline{u}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds. \tag{3.30}$$

781 All in all we infer

$$\frac{\rho}{2} \int_{\Omega} |\dot{u}_n(t)|^2 \, dx + \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\mathbb{D}(z_n, \varrho_n) e(\dot{u}_n) - \varrho_n \mathbb{B}) : e(\dot{u}_n) \, dx \, ds$$

$$+ \int_{\Omega} \frac{1}{2} \mathbb{C}(\bar{z}_n(t)) e(\bar{u}_n(t)) : e(\bar{u}_n(t)) \, dx + \int_{\Omega} \frac{\tau_n}{\gamma} |e(\bar{u}_n(t))|^{\gamma} \, dx - \langle \bar{f}_n(t), \bar{u}_n(t) \rangle_{H_D^1(\Omega; \mathbb{R}^d)}$$

$$\leq \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 \, dx + \int_{\Omega} \frac{\tau_n}{\gamma} |e(u_0)|^{\gamma} \, dx - \langle f(0), u_0 \rangle_{H_D^1(\Omega; \mathbb{R}^d)} - \int_0^{\bar{\tau}_n(t)} \langle \dot{f}_n, \underline{u}_n \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds$$

$$+ \sum_{k=1}^j \int_{\Omega} \frac{1}{2} \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1}) \, dx.$$

783 We add the above inequality to (3.28), summed over  $k = 1, \dots, j$ . Observing the cancelation  
 784 of the term  $\sum_{k=1}^j \int_{\Omega} \frac{1}{2} \mathbb{C}(z_n^k) e(u_n^{k-1}) : e(u_n^{k-1}) \, dx$ , we conclude (3.27c).

785 Finally, the discrete total energy inequality ensues from adding the discrete mechanical  
 786 energy inequality (3.27c) with the discrete heat equation (3.3c), tested for  $\eta = \tau_n$  and added  
 787 up over  $k = 1, \dots, j$ . We observe the cancelation of some terms, and readily conclude  
 788 (3.27d). □

789 **3.3 A Priori Estimates**

790 The following result collects a series of a priori estimates on the approximate solutions,  
 791 uniform with respect to  $n \in \mathbb{N}$ . Let us mention in advance that, in its proof we will start from  
 792 the discrete total energy inequality (3.27d) and derive estimates (3.32a), (3.32b), (3.32d),  
 793 (3.32h), for  $\bar{u}_n, \dot{u}_n, \bar{z}_n$ , as well as estimate (3.32i) below for  $\|\bar{\theta}_n\|_{L^\infty(0,T;L^1(\Omega))}$ . The next  
 794 crucial step will be to obtain a bound for the  $L^2(0, T; H^1(\Omega))$ -norm of  $\bar{\theta}_n$ . For this, we will  
 795 make use of a technique developed in [23], cf. also [63]. Namely, we will test the discrete  
 796 heat equation (3.3c) by  $(\theta_n^k)^\alpha$ , with  $\alpha \in (0, 1)$ . Exploiting the concavity of the function  
 797  $F(\theta) = \theta^\alpha/\alpha$ , we will deduce that

$$798 \int_0^T \int_\Omega \mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla(\bar{\theta}_n^{\alpha/2}) \cdot \nabla(\bar{\theta}_n^{\alpha/2}) \, dx \, dt + \int_\Omega \frac{\theta_0^\alpha}{\alpha} \, dx \leq \int_\Omega \frac{\bar{\theta}_n^\alpha(T)}{\alpha} \, dx + C \int_0^T \int_\Omega \bar{\theta}_n^{\alpha+1}(t) \, dx \, dt,$$

799 where the positive and quadratic terms on the right-hand side of (3.3c) have been confined to  
 800 the *left-hand side* and thus can be neglected. Hence, relying on the growth (2.6b) of  $\mathbb{K}$ , we will  
 801 end up with an estimate for  $\bar{\theta}_n^{\alpha/2}$  in  $L^2(0, T; H^1(\Omega))$ , from which we will ultimately infer  
 802 the desired bound (3.32j), whence (3.32k) by interpolation. We will be then in a position to  
 803 exploit the mechanical energy inequality in order to recover the *dissipative* estimate (3.32c).  
 804 Estimate (3.32l) will finally ensue from a comparison in (3.3c).

805 In the following proof we will also use the concave counterpart to inequality (3.6), namely  
 806 that for any *concave* (differentiable) function  $\psi : \mathbb{R} \rightarrow (-\infty, +\infty]$

$$807 \psi(x) - \psi(y) \leq \psi'(y)(x-y) \quad \text{for all } x, y \in \text{dom}(\psi). \tag{3.31}$$

808 **Proposition 3.4** (A priori estimates) *Let the assumptions of Theorem 2.6 hold true and*  
 809 *consider a sequence  $(\bar{u}_n, \underline{u}_n, u_n, \bar{z}_n, \underline{z}_n, \bar{\theta}_n, \underline{\theta}_n, \theta_n)_n$  complying with Proposition 3.3. Then*  
 810 *there exists a constant  $C > 0$  such that the following estimates hold uniformly with respect*  
 811 *to  $n \in \mathbb{N}$ :*

$$812 \|\bar{u}_n\|_{L^\infty(0,T;H_D^1(\Omega;\mathbb{R}^d))} \leq C, \tag{3.32a}$$

$$813 \tau_n^{1/\gamma} \|\bar{u}_n\|_{L^\infty(0,T;W_D^{1,\gamma}(\Omega;\mathbb{R}^d))} \leq C, \tag{3.32b}$$

$$814 \|u_n\|_{H^1(0,T;H_D^1(\Omega;\mathbb{R}^d))} \leq C, \tag{3.32c}$$

$$815 \|\dot{u}_n\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \leq C, \tag{3.32d}$$

$$816 \|\dot{u}_n\|_{\text{BV}([0,T];W_D^{1,\gamma}(\Omega;\mathbb{R}^{d*}))} \leq C, \tag{3.32e}$$

$$817 \mathcal{R}_1(\bar{z}_n(T) - z_0) \leq C, \tag{3.32f}$$

$$818 \|\bar{z}_n\|_{L^\infty((0,T)\times\Omega)} \leq 1, \tag{3.32g}$$

$$819 \|\bar{z}_n\|_{L^\infty(0,T;W^{1,q}(\Omega))} \leq C, \tag{3.32h}$$

$$820 \|\bar{\theta}_n\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \tag{3.32i}$$

$$821 \|\bar{\theta}_n\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{3.32j}$$

$$822 \|\bar{\theta}_n\|_{L^p((0,T)\times\Omega)} \leq C \quad \text{for any } p \in \begin{cases} [1, 8/3] & \text{if } d=3, \\ [1, 3] & \text{if } d=2, \end{cases} \tag{3.32k}$$

$$823 \|\bar{\theta}_n\|_{\text{BV}([0,T];W^{1,\infty}(\Omega)^*)} \leq C, \tag{3.32l}$$

825 where  $\mathcal{R}_1$  is from (2.10).

826 Observe that estimate (3.32c) implies (3.32a), and that (3.32k) is a consequence of (3.32i) and  
 827 (3.32j). Nonetheless, we have chosen to highlight (3.32a) and (3.32k) for ease of exposition,  
 828 both in the proof of Proposition 3.4 and for the compactness arguments of Proposition 4.1.

829 *Proof* Estimate (3.32f) follows from (2.5a), (2.7a), the definition of  $\mathcal{R}_1$ , and the monotonicity  
 830 of  $\bar{z}_n$  and  $\underline{z}_n$ . We divide the proof of the other estimates in subsequent steps.

831 *First a priori estimates, ad (3.32a), (3.32b), (3.32d), (3.32g), (3.32h), (3.32i):* We start  
 832 from the discrete total energy inequality (3.27d). For its left-hand side, we observe that the  
 833 first and the third term are nonnegative. For the second one, we use that, in view of (2.3d),  
 834 (2.5d), and (2.8a), we have

$$\begin{aligned} \mathcal{E}_n(t, \bar{u}_n(t), \bar{z}_n(t)) &\geq C_G^1 \int_{\Omega} |e(\bar{u}_n(t))|^2 dx + C_G^1 \int_{\Omega} |\nabla \bar{z}_n(t)|^q dx + \frac{\tau_n}{\gamma} \int_{\Omega} |e(\bar{u}_n(t))|^\gamma dx \\ &\quad - \|\bar{f}_n\|_{L^\infty(0,T;H_D^1(\Omega;\mathbb{R}^d)^*)} \|\bar{u}_n(t)\|_{H_D^1(\Omega;\mathbb{R}^d)} - C \\ &\geq C \left( \|\bar{u}_n(t)\|_{H_D^1(\Omega;\mathbb{R}^d)}^2 + \tau_n \|\bar{u}_n(t)\|_{W_D^{1,\gamma}(\Omega;\mathbb{R}^d)}^\gamma + \|\bar{z}_n(t)\|_{W^{1,q}(\Omega)}^q \right) - C, \end{aligned} \tag{3.33}$$

835 for almost all  $t \in (0, T)$ , where we have also used Poincaré’s and Korn’s inequal-  
 836 ities. Concerning the right-hand side of (3.27d), we use that  $|\partial_t \mathcal{E}_n(t, \underline{u}_n(t), \underline{z}_n(t))| \leq$   
 837  $\|\dot{f}_n\|_{H_D^1(\Omega;\mathbb{R}^d)^*} \|\underline{u}_n(t)\|_{H_D^1(\Omega;\mathbb{R}^d)}$  for almost all  $t \in (0, T)$ . The remaining terms on the right-  
 838 hand side are bounded, uniformly with respect to  $n \in \mathbb{N}$ , in view of the properties of the  
 839 initial and given data (2.7) and (3.2), and of (3.24d). All in all, from (3.27d) we deduce  
 840

$$841 \quad C \|\bar{u}_n(t)\|_{H_D^1(\Omega;\mathbb{R}^d)}^2 \leq C + \frac{1}{2} \int_0^{\bar{\tau}_n(t)} \|\underline{u}_n(s)\|_{H_D^1(\Omega;\mathbb{R}^d)}^2 ds + \frac{1}{2} \int_0^{\bar{\tau}_n(t)} \|\dot{f}_n\|_{H_D^1(\Omega;\mathbb{R}^d)^*}^2 ds.$$

842 Also in view of the bounds on  $\dot{f}_n$  by (3.24c), estimate (3.32a) then follows from the Gronwall  
 843 Lemma. As a by-product, we conclude that

$$844 \quad \int_0^{\bar{\tau}_n(t)} |\partial_t \mathcal{E}_n(s, \underline{u}_n(s), \underline{z}_n(s))| ds \leq C \int_0^{\bar{\tau}_n(t)} \|\dot{f}_n(s)\|_{H_D^1(\Omega;\mathbb{R}^d)^*} ds \leq C. \tag{3.34}$$

845 Inserting this into (3.27d) we also infer estimates (3.32d), (3.32i), and that  
 846  $|\mathcal{E}_n(t, \bar{u}_n(t), \bar{z}_n(t))| \leq C$  for a constant independent of  $n \in \mathbb{N}$  and  $t \in (0, T)$ . This implies  
 847 (3.32b) and the first estimate in (3.32h) via (3.33). Then the second estimate in (3.32h)  
 848 immediately follows from the very definition of the interpolants (3.23). Moreover, (3.32g) is  
 849 a direct consequence of the boundedness of the energy, which implies  $\bar{z}_n, \underline{z}_n \in [0, 1]$  a.e. in  
 850  $\Omega$ , for a.e.  $t \in (0, T)$ .

851 *Second a priori estimate:* We fix  $\alpha \in (0, 1)$ . Exploiting that  $\theta_n^k \geq \tilde{\theta} > 0$ , we may test  
 852 (3.3c) by  $(\theta_n^k)^{\alpha-1}$ , thus obtaining

$$\begin{aligned} &\frac{4(1-\alpha)}{\alpha^2} \int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla(\theta_n^k)^{\alpha/2} \cdot \nabla(\theta_n^k)^{\alpha/2} dx + \int_{\Omega} \mathbb{D}(z_n^k) e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) : e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) (\theta_n^k)^{\alpha-1} dx \\ &\quad + \int_{\Omega} \frac{z_n^{k-1} - z_n^k}{\tau} (\theta_n^k)^{\alpha-1} dx + \left\langle H_n^k, (\theta_n^k)^{\alpha-1} \right\rangle_{H^1(\Omega)} + \int_{\partial\Omega} h_n^k (\theta_n^k)^{\alpha-1} d\mathcal{H}^{d-1} \\ &= \int_{\Omega} \frac{\theta_n^k - \theta_n^{k-1}}{\tau} (\theta_n^k)^{\alpha-1} dx + \int_{\Omega} \theta_n^k \mathbb{B} : e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) (\theta_n^k)^{\alpha-1} dx \doteq I_1 + I_2, \end{aligned} \tag{3.35}$$

853

854 where we used that

$$\begin{aligned}
 855 \quad \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla (\theta_n^k)^{\alpha-1} &= (\alpha - 1)(\theta_n^k)^{\alpha-2} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \theta_n^k \\
 856 \quad &= \frac{4(\alpha-1)}{\alpha^2} \mathbb{K}(z_n^k, \theta_n^k) \nabla (\theta_n^k)^{\alpha/2} \cdot \nabla (\theta_n^k)^{\alpha/2}
 \end{aligned}$$

857 and moved the term  $\int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla (\theta_n^k)^{\alpha-1} \, dx$  to the opposite side. It follows from (3.31)  
 858 with  $\psi(x) := \frac{x^\alpha}{\alpha}$  that

$$859 \quad I_1 \leq \int_{\Omega} \psi(\theta_n^k) \, dx - \int_{\Omega} \psi(\theta_n^{k-1}) \, dx,$$

860 whereas we estimate  $I_2$  by

$$861 \quad I_2 \leq \frac{C_{\mathbb{D}}^1}{2} \int_{\Omega} \left| e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) \right|^2 (\theta_n^k)^{\alpha-1} \, dx + C \int_{\Omega} |\theta_n^k|^2 (\theta_n^k)^{\alpha-1} \, dx \doteq I_3 + I_4,$$

862 where  $C_{\mathbb{D}}^1$  from (2.3e) is such that  $\int_{\Omega} \mathbb{D}(z_n^k) e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) : e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) (\theta_n^k)^{\alpha-1} \, dx$  on the left-  
 863 hand side of (3.35) is bounded from below by  $C_{\mathbb{D}}^1 \int_{\Omega} \left| e\left(\frac{u_n^k - u_n^{k-1}}{\tau}\right) \right|^2 (\theta_n^k)^{\alpha-1} \, dx$ , which in  
 864 turn dominates  $I_3$ . Taking into account that the second, the third and the fourth integrals on  
 865 the left-hand side of (3.35) are nonnegative also thanks to (2.8b) and summing up over the  
 866 index  $k$ , we end up with

$$\begin{aligned}
 867 \quad &\frac{4(1-\alpha)}{\alpha^2} \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla (\bar{\theta}_n^{\alpha/2}) \cdot \nabla (\bar{\theta}_n^{\alpha/2}) \, dx \, ds + \int_{\Omega} \frac{\theta_n^\alpha}{\alpha} \, dx \\
 868 \quad &\leq \int_{\Omega} \frac{\bar{\theta}_n(t)^\alpha}{\alpha} \, dx + C \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \bar{\theta}_n(t)^{\alpha+1} \, dx \, ds. \tag{3.36}
 \end{aligned}$$

869 Since  $\alpha \in (0, 1)$  and  $\theta_n^k \geq \tilde{\theta} > 0$ , we have

$$870 \quad \int_{\Omega} \frac{\bar{\theta}_n(t)^\alpha}{\alpha} \, dx \leq \frac{1}{\alpha} \int_{\Omega} \bar{\theta}_n(t) \, dx + C \leq C,$$

871 where the latter estimate follows by (3.32i). From (2.6b) we deduce that

$$\begin{aligned}
 &\int_0^{\bar{\tau}_n(t)} \int_{\Omega} \mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla (\bar{\theta}_n^{\alpha/2}) \cdot \nabla (\bar{\theta}_n^{\alpha/2}) \, dx \, ds \geq c_1 \int_0^{\bar{\tau}_n(t)} \int_{\Omega} (\bar{\theta}_n)^\kappa |\nabla (\bar{\theta}_n^{\alpha/2})|^2 \, dx \, ds \\
 872 \quad &= C \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |(\bar{\theta}_n)^{\kappa+\alpha-2}| |\nabla \bar{\theta}_n|^2 \, dx \, ds = C \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\nabla (\bar{\theta}_n^{(\kappa+\alpha)/2})|^2 \, dx \, ds. \tag{3.37}
 \end{aligned}$$

873 In order to clarify the estimate for the second term on the right-hand side of (3.36), we now  
 874 use the placeholder

$$875 \quad w_n := (\bar{\theta}_n)^{(\kappa+\alpha)/2},$$

876 so that  $(\bar{\theta}_n)^{\alpha+1} = (w_n)^{2(\alpha+1)/(\kappa+\alpha)}$ . Hence, neglecting the (positive) second term on the  
 877 left-hand side of (3.36), we infer

$$878 \quad \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\nabla w_n|^2 \, dx \, ds \leq C + C \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |w_n|^\omega \, dx \, ds \quad \text{with } \omega = 2\frac{\alpha+1}{\alpha+\kappa}. \tag{3.38}$$

879 We now proceed exactly in the same way as in [23], cf. also [63]. Namely, the Gagliardo-  
 880 Nirenberg inequality for  $d=3$  (for  $d=2$  even better estimates hold true) yields

$$881 \quad \|w_n\|_{L^\omega(\Omega)} \leq C \|\nabla w_n\|_{L^2(\Omega; \mathbb{R}^d)}^\sigma \|w_n\|_{L^r(\Omega)}^{1-\sigma} + C' \|w_n\|_{L^r(\Omega)}$$

882 for suitable constants  $C$  and  $C'$ , and for  $1 \leq r \leq \omega$  and  $\sigma$  satisfying  $1/\omega = \sigma/6 + (1 - \sigma)/r$ .  
 883 Hence  $\sigma = 6(\omega - r)/\omega(6 - r)$ . Observe that  $\sigma \in (0, 1)$  since  $\omega = 2(\alpha + 1)/(\alpha + \kappa) < 6$ ,  
 884 which is satisfied because  $\kappa > 1$ . Hence we transfer the Gagliardo-Nirenberg estimate into  
 885 (3.38) and use Young's inequality in the estimate of the term

$$\begin{aligned}
 & C \int_0^{\bar{\tau}_n(t)} \|\nabla w_n\|_{L^2(\Omega; \mathbb{R}^d)}^{\omega\sigma} \|w_n\|_{L^r(\Omega)}^{\omega(1-\sigma)} \, ds \\
 & \leq \frac{1}{2} \int_0^{\bar{\tau}_n(t)} \|\nabla w_n\|_{L^2(\Omega; \mathbb{R}^d)}^2 \, ds \\
 & + C' \int_0^{\bar{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^{2\omega(1-\sigma)/(2-\omega\sigma)} \, ds .
 \end{aligned}$$

889 In the previous inequality we have used the fact that  $\omega\sigma < 2$ , which holds since  $\omega < 2$  and  
 890  $\sigma < 1$  by (3.38). The term  $\frac{1}{2} \int_0^{\bar{\tau}_n(t)} \|\nabla w_n\|_{L^2(\Omega; \mathbb{R}^d)}^2 \, ds$  may be absorbed into the left-hand  
 891 side of (3.38). All in all, we conclude

$$\int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\nabla w_n|^2 \, dx \, ds \leq C + C \int_0^{\bar{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^{2\omega(1-\sigma)/(2-\omega\sigma)} \, ds + C' \int_0^{\bar{\tau}_n(t)} \|w_n\|_{L^r(\Omega)}^{\omega} \, ds . \tag{3.39}$$

892 Now, let us choose

$$1 \leq r \leq 2/(\alpha + \kappa) . \tag{3.40}$$

893 Then, we have for almost all  $t \in (0, T)$  that

$$\|w_n(t)\|_{L^r(\Omega)} = \left( \int_{\Omega} (\bar{\theta}_n(t))^{r(\kappa + \alpha)/2} \, dx \right)^{1/r} = \left( \int_{\Omega} \bar{\theta}_n(t) \, dx \right)^{1/r} \leq C \tag{3.40}$$

897 for a constant independent of  $t$ , where again we have used estimate (3.32i). Observe that,  
 898 since we have previously imposed  $\kappa + \alpha - 2 \geq 0$ , we ultimately find that (3.40) must hold  
 899 for  $r = 1$  and that, moreover,  $\alpha = 2 - \kappa \in (2 - \kappa_d, 1)$ , with  $\kappa_d = 5/3$  if  $d=3$  and  $\kappa_d = 2$  if  
 900  $d=2$ , so that  $w_n = \bar{\theta}_n$ . From (3.39)–(3.40) we then infer

$$\int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\nabla \bar{\theta}_n|^2 \, dx \, ds \leq C . \tag{3.41}$$

902 *Third a priori estimate, ad (3.32j) and (3.32k):* From (3.41) we deduce (3.32j) in view  
 903 of the previously obtained (3.32i) via Poincaré's inequality. Estimate (3.32k) ensues by  
 904 interpolation between  $L^2(0, T; H^1(\Omega))$  and  $L^\infty(0, T; L^1(\Omega))$ , relying on (3.32j) and (3.32i)  
 905 and exploiting the Gagliardo-Nirenberg inequality. For later convenience, let us also point  
 906 out that, we indeed recover the following bound

$$\left\| (\bar{\theta}_n)^{(\kappa + \alpha)/2} \right\|_{L^2(0, T; H^1(\Omega))} \leq C \tag{3.42}$$

908 for arbitrary  $\alpha \in (0, 1)$ . For this, it is sufficient to observe that second term on the right-hand  
 909 side of (3.36) now fulfills  $\int_0^{\bar{\tau}_n(t)} \int_{\Omega} \bar{\theta}_n(t)^{\alpha+1} \, dx \, ds \leq C$  thanks to estimate (3.32k). Then,  
 910 by (3.37) we find that  $\int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\nabla (\bar{\theta}_n^{(\kappa + \alpha)/2})|^2 \, dx \, ds \leq C$ , whence (3.42) via Poincaré's  
 911 inequality.

912 *Fourth a priori estimate, ad (3.32c) and (3.32e):* From the discrete mechanical energy  
 913 inequality (3.27c) we infer

$$C_{\mathbb{D}}^1 \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |e(\dot{u}_n)|^2 \, dx \, ds \leq C + \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \bar{\theta}_n \mathbb{B} : e(\dot{u}_n) \, dx \, ds \tag{3.43}$$

915 where we have used (3.33), (3.34), and the fact that the terms  $\int_{\Omega} |u_0|^2 dx$  and  $\mathcal{E}(0, u_n^0, z_0)$   
 916 are bounded, uniformly with respect to  $n \in \mathbb{N}$ , in view of (2.7a) and (3.2). Exploiting the  
 917 previously obtained estimate (3.32j) we find

$$\int_0^{\bar{\tau}_n(t)} \int_{\Omega} \bar{\theta}_n \mathbb{B} : e(\dot{u}_n) dx dt \leq \frac{C_{\mathbb{D}}^1}{2} \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |e(\dot{u}_n)|^2 dx dt + C \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |\bar{\theta}_n|^2 dx ds$$

$$\leq \frac{C_{\mathbb{D}}^1}{2} \int_0^{\bar{\tau}_n(t)} \int_{\Omega} |e(\dot{u}_n)|^2 dx dt + C.$$

918  
 919 Inserting this into (3.43) we conclude (3.32c) via Korn’s inequality, again exploiting the  
 920 definition of the interpolants (3.23). Finally, estimate (3.32e) ensues from a comparison  
 921 argument in (3.3b), taking into account the previously proven (3.32b), (3.32c), (3.32j), as  
 922 well as (3.24a).

923 *Fifth a priori estimate, ad (3.32i):* Let  $\kappa$  be as in (2.6). In (3.3c) we use a test function  
 924  $\eta \in W^{1,\infty}(\Omega)$ , thus we find

$$\left| \int_{\Omega} \frac{\theta_n^k - \theta_n^{k-1}}{\tau_n} \eta dx \right| \leq \left| \int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \eta dx \right| + \left| \langle \text{RHS}_n^k, \eta \rangle_{W^{1,\infty}(\Omega)} \right|, \tag{3.44}$$

925  
 926 where the terms on the right-hand side of (3.3c) are summarized in  $\text{RHS}_n^k$ . It follows from  
 927 assumptions (2.3) and (2.8b) that

$$\left| \langle \text{RHS}_n^k, \eta \rangle_{W^{1,\infty}(\Omega)} \right|$$

$$\leq C \left( \left\| e \left( \frac{u_n^k - u_n^{k-1}}{\tau_n} \right) \right\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \|\theta_n^k\|_{L^2(\Omega)}^2 + \left\| \frac{z_n^k - z_n^{k-1}}{\tau_n} \right\|_{L^1(\Omega)} \right)$$

$$+ \|\theta_n^k\|_{L^2(\partial\Omega)} + \|H_n^k\|_{L^1(\Omega)} \|\eta\|_{L^\infty(\Omega)}$$

$$\doteq \Lambda_n^k \|\eta\|_{L^\infty(\Omega)}.$$

928  
 929 Furthermore, with (2.6) we find for every  $\alpha \in (1/2, 1)$

$$\left| \int_{\Omega} \mathbb{K}(z_n^k, \theta_n^k) \nabla \theta_n^k \cdot \nabla \eta dx \right|$$

$$\leq \|\nabla \eta\|_{L^\infty(\Omega; \mathbb{R}^d)} c_2 \left( (\theta_n^k)^\kappa + 1 \right) \|\nabla \theta_n^k\|_{L^1(\Omega; \mathbb{R}^d)}$$

$$\leq \|\nabla \eta\|_{L^\infty(\Omega; \mathbb{R}^d)} c_2 \left( \|\theta_n^k\|^{(\kappa-\alpha+2)/2} \|\theta_n^k\|^{(\kappa+\alpha-2)/2} \|\nabla \theta_n^k\|_{L^2(\Omega; \mathbb{R}^d)} \right.$$

$$\left. + \mathcal{L}^d(\Omega)^{1/2} \|\nabla \theta_n^k\|_{L^2(\Omega; \mathbb{R}^d)} \right). \tag{3.46}$$

930  
 931  
 932  
 933  
 934  
 935 Inserting (3.45) and (3.46) into (3.44) and summing over the index  $k = 1, \dots, n$ , we find for  
 936 every time-dependent function  $\eta \in C^0([0, T]; W^{1,\infty}(\Omega))$  that

$$\left| \int_0^{\bar{\tau}_n(t)} \int_{\Omega} \dot{\theta}_n \eta dx ds \right|$$

$$\leq C \|\nabla \eta\|_{L^\infty((0,T) \times \Omega; \mathbb{R}^d)} \left( \|\bar{\theta}_n\|_{L^{\kappa-\alpha+2}((0,T) \times \Omega)}^{(\kappa-\alpha+2)/2} \|\bar{\theta}_n\|^{(\kappa+\alpha)/2} \|\bar{\theta}_n\|_{L^2(0,T; H^1(\Omega))} \right.$$

$$\left. + \|\nabla \bar{\theta}_n\|_{L^2((0,T) \times \Omega; \mathbb{R}^d)} \right)$$

$$+ \|\eta\|_{L^\infty((0,T) \times \Omega)} \int_0^{\bar{\tau}_n(t)} \bar{\Lambda}_n ds, \tag{3.47}$$

942 where  $\bar{\Lambda}_n$  denotes the piecewise constant interpolant of the values  $(\Lambda_n^k)_k$ . Note that the  
 943 estimate on  $\|(\theta_n^k)^{(\kappa+\alpha-2)/2} \nabla \theta_n^k\|_{L^2(\Omega; \mathbb{R}^d)}$  ensues from (3.37) and (3.42). Now, observe that

944 
$$\|\bar{\theta}_n\|_{L^{\kappa-\alpha+2}((0,T)\times\Omega)}^{(\kappa-\alpha+2)/2} \leq C$$

945 thanks to (3.32k) if  $p = \kappa - \alpha + 2$  satisfies the constraints in (3.32k). Recall that the parameter  
 946  $\alpha$  for which (3.42) holds can be chosen arbitrarily close to 1. Therefore, such constraints for  
 947  $p = \kappa - \alpha + 2$  are valid since, by (2.6b),  $\kappa \in (1, \kappa_d)$  with  $\kappa_d = 5/3$  if  $d=3$  and  $\kappa_d = 2$   
 948 if  $d=2$ . Finally, it follows from (3.24d), (3.32c), (3.32f), and (3.32j) that  $\int_0^T \bar{\Lambda}_n dt \leq C$ .  
 949 Ultimately, from (3.47) we conclude (3.32l). □

### 950 4 Passage from Time-Discrete to Continuous

951 Based on the a priori bounds deduced in Proposition 3.4, exploiting compactness results à la  
 952 Aubin–Lions as well as a version of Helly’s selection principle, we are now in a position to  
 953 extract a subsequence of solutions of the time-discrete problems converging to a limit triple  
 954  $(u, z, \theta)$  in suitable topologies. In (4.1) below we have collected all of these convergences with  
 955 some redundancies: for example, (4.1g) and (4.1i) imply (4.1h) and (4.1j), but the latter are  
 956 stated for later reference. Subsequently, we will verify that the triple  $(u, z, \theta)$  is an energetic  
 957 solution of the time-continuous problem as stated in Definition 2.3.

958 **Proposition 4.1** (Convergence of the time-discrete solutions) *Let the assumptions of Theo-*  
 959 *rem 2.6 be satisfied. Then, there exists a triple  $(u, z, \theta) : [0, T] \times \Omega \rightarrow \mathbb{R}^d \times \mathbb{R} \times [0, \infty)$  of*  
 960 *regularity (2.11) such that for a.a.  $x \in \Omega$  the function  $t \mapsto z(t, x) \in [0, 1]$  is nonincreasing,*  
 961 *(2.14) holds, as well as (2.16) under the assumption (2.15), and there exists a subsequence*  
 962 *of the time-discrete solutions  $(\bar{u}_n, \underline{u}_n, u_n, \bar{z}_n, \underline{z}_n, \bar{\theta}_n, \underline{\theta}_n)_n$  from (3.23) such that*

963 
$$\bar{u}_n \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d)), \tag{4.1a}$$

964 
$$u_n \rightharpoonup u \quad \text{in } H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)), \tag{4.1b}$$

965 
$$\dot{u}_n \overset{*}{\rightharpoonup} \dot{u} \quad \text{in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)), \tag{4.1c}$$

966 
$$\bar{u}_n(t), u_n(t) \rightharpoonup u(t) \quad \text{in } H_D^1(\Omega; \mathbb{R}^d) \text{ for all } t \in [0, T], \tag{4.1d}$$

967 
$$\dot{u}_n(t) \rightharpoonup \dot{u}(t) \quad \text{in } L^2(\Omega; \mathbb{R}^d) \text{ for all } t \in [0, T], \tag{4.1e}$$

968 
$$\bar{z}_n, \underline{z}_n \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty(0, T; W^{1,q}(\Omega)) \cap L^\infty((0, T) \times \Omega), \tag{4.1f}$$

969 
$$\bar{z}_n(t) \rightharpoonup z(t) \quad \text{in } W^{1,q}(\Omega) \text{ for all } t \in [0, T], \tag{4.1g}$$

970 
$$\bar{z}_n(t) \rightarrow z(t) \quad \text{in } L^r(\Omega) \text{ for all } r \in [1, \infty) \text{ and for all } t \in [0, T], \tag{4.1h}$$

971 
$$\underline{z}_n(t) \rightharpoonup z(t) \quad \text{in } W^{1,q}(\Omega) \text{ for all } t \in [0, T] \setminus J, \tag{4.1i}$$

972 
$$\underline{z}_n(t) \rightarrow z(t) \quad \text{in } L^r(\Omega) \text{ for all } r \in [1, \infty) \text{ and for all } t \in [0, T] \setminus J, \tag{4.1j}$$

973 
$$\bar{\theta}_n, \underline{\theta}_n \rightharpoonup \theta \quad \text{in } L^2(0, T; H^1(\Omega)), \tag{4.1k}$$

974 
$$\bar{\theta}_n, \underline{\theta}_n, \theta_n \rightarrow \theta \quad \text{in } L^2(0, T; Y) \text{ for all } Y \text{ such that } H^1(\Omega) \Subset Y \subset W^{2,d+\delta}(\Omega)^*, \tag{4.1l}$$

975 
$$\bar{\theta}_n, \underline{\theta}_n, \theta_n \rightarrow \theta \quad \text{in } L^p((0, T) \times \Omega) \text{ for all } p \in \begin{cases} [1, 8/3) & \text{if } d=3, \\ [1, 3) & \text{if } d=2, \end{cases} \tag{4.1m}$$

976 
$$\theta_n(t) \rightharpoonup \theta(t) \quad \text{in } W^{2,d+\delta}(\Omega)^* \text{ for all } t \in [0, T], \tag{4.1n}$$

978 The set  $J \subset [0, T]$  appearing in (4.1i)–(4.1j) denotes the jump set of  $z \in \text{BV}([0, T]; L^1(\Omega))$ .  
 979 Finally,

$$980 \quad |\dot{z}_n| \rightarrow |\dot{z}| \text{ in the sense of measures on } [0, T] \times \bar{\Omega}. \quad (4.1o)$$

981 *Proof Convergence of the displacements:* The convergences (4.1a), (4.1b), and (4.1c) follow  
 982 by compactness from (3.32a), (3.32c), and (3.32d). As  $u_n(t) - \bar{u}_n(t) = (t - t_n^k)\dot{u}_n(t)$  and  
 983  $u_n(t) - \underline{u}_n(t) = (t - t_n^{k-1})\dot{u}_n(t)$ , we immediately deduce from (4.1b) that the sequences  $u_n, \bar{u}_n,$   
 984 and  $\underline{u}_n$  have the same limit in  $L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d))$ , and the pointwise weak convergences  
 985 (4.1d) ensue. Furthermore, due to estimate (3.32e), by compactness, there exists a further  
 986 subsequence such that  $\dot{u}_n \rightharpoonup \dot{u}$  in  $\text{BV}([0, T]; W_D^{1,\gamma}(\Omega; \mathbb{R}^d)^*)$  as well as  $\dot{u}_n(t) \rightharpoonup \dot{u}(t)$  in  
 987  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d)^*$  for all  $t \in [0, T]$ . Thanks to (3.32d), arguing by contradiction and using that  
 988  $L^2(\Omega; \mathbb{R}^d)$  is dense in  $W_D^{1,\gamma}(\Omega; \mathbb{R}^d)^*$ , we may also conclude that  $\dot{u}_n(t) \rightharpoonup \dot{u}(t)$  in  $L^2(\Omega; \mathbb{R}^d)$   
 989 for all  $t \in [0, T]$ , i.e. (4.1e).

990 *Convergence of the damage variables:* From estimates (3.32f) on the  $\mathcal{R}_1$ -total varia-  
 991 tion of  $(\bar{z}_n)_n$  (by monotonicity of  $\bar{z}_n$ ), combined with (3.32h), a generalized version of  
 992 Helly’s selection principle, cf. e.g. [51, Theorem 6.1], allows us to extract a subsequence  
 993 such that  $\bar{z}_n(t) \rightharpoonup z(t)$  and  $\underline{z}_n(t) \rightharpoonup \underline{z}(t)$  weakly in  $W^{1,q}(\Omega)$  for all  $t \in [0, T]$ , and  
 994  $z, \underline{z} \in L^\infty(0, T; W^{1,q}(\Omega))$ . Moreover, the limit functions  $z$  and  $\underline{z}$  inherit the monotonic-  
 995 ity in time from  $\bar{z}_n$  and  $\underline{z}_n$ , hence  $z, \underline{z} \in \text{BV}([0, T]; L^1(\Omega))$ , and their jump sets  $J$   
 996 and  $\underline{J}$  are at most countable. Let  $t \in [0, T] \setminus (J \cup \underline{J})$  fixed. Then, by (3.23), for every  
 997  $n \in \mathbb{N}$  we have  $\bar{z}_n(t - \tau_n) = z_n(t)$  and therefore as  $n \rightarrow \infty$  we get  $z(t) = \underline{z}(t)$ . Let now  
 998  $t \in J \cup \underline{J}$  and let  $(t_j^-)_j, (t_j^+)_j \subset [0, T] \setminus (J \cup \underline{J})$  be such that  $t_j^- \nearrow t$  and  $t_j^+ \searrow t$ .  
 999 Since  $z$  and  $\underline{z}$  coincide on  $[0, T] \setminus (J \cup \underline{J})$ , we deduce that the left and the right limit satisfy  
 1000  $z^-(t) = \lim_j z(t_j^-) = \lim_j \underline{z}(t_j^-) = \underline{z}^-(t)$  and  $z^+(t) = \lim_j z(t_j^+) = \lim_j \underline{z}(t_j^+) = \underline{z}^+(t)$ .  
 1001 Therefore  $J = \underline{J}$  and the convergences (4.1f), (4.1g), (4.1i) hold. From this, using (3.32g)  
 1002 we conclude that (4.1h) and (4.1j) hold true as well. In this line, we conclude by observing  
 1003 that (4.1o) follows from the fact that  $\int_\Omega (z_n(0) - z_n(T)) \, dx$ , i.e. the total variation of  $\dot{z}_n$  on  
 1004  $[0, T] \times \bar{\Omega}$ , converges to the total variation  $\int_\Omega (z(0) - z(T)) \, dx$  of  $\dot{z}$ , also relying on the  
 1005 argument from [57, Proposition 4.3, proof of (4.80)].

1006 *Convergence of the temperature variables:* Due to estimate (3.32j) we have  $\bar{\theta}_n \rightharpoonup \theta$   
 1007 in  $L^2(0, T; H^1(\Omega))$ . Exploiting the definition of the interpolants (3.23), similarly to the  
 1008 arguments for the damage variables, we conclude that also  $\underline{\theta}_n \rightharpoonup \theta$  in  $L^2(0, T; H^1(\Omega))$ ,  
 1009 thus (4.1k) is proven. From this, convergences (4.1l) and (4.1m) for  $(\bar{\theta}_n, \underline{\theta}_n)_n$  follow by a  
 1010 generalized Aubin–Lions Lemma, cf. [60, Corollary 7.9, p. 196], making use of the estimates  
 1011 (3.32j), (3.32k), and (3.32l). Taking into account that  $|\theta_n(t, x)| \leq \max\{|\bar{\theta}_n(t, x)|, |\underline{\theta}_n(t, x)|\}$   
 1012 for almost all  $(t, x) \in (0, T) \times \Omega$ , (a generalized version of) the Lebesgue Theorem yields  
 1013 convergence (4.1m) for  $(\theta_n)_n$  as well. All in all, we conclude the weak convergence (4.1k),  
 1014 as well as (4.1l), for  $(\theta_n)_n$ . Convergence (4.1n) is a consequence of [51, Theorem 6.1]. The  
 1015 positivity properties (2.14) and (2.16) (under the additional (2.15)) then follow from their  
 1016 discrete analogues (3.4) and (3.5), respectively, combined with (3.32k).  $\square$

1017 The fact that the limit triple  $(u, z, \theta)$  is an energetic solution of the limit problem will be  
 1018 verified in Sects. 4.1–4.3 right below. For this, in Sect. 4.1, we first pass from time-discrete to  
 1019 continuous in the weak momentum balance (3.27b) using suitably chosen time-discrete test  
 1020 functions and deduce a time-continuous limit inequality for the mechanical energy balance  
 1021 (3.26) by lower semicontinuity arguments. Secondly, in Sect. 4.2 we pass to the limit in the  
 1022 semistability inequality (3.27a) using mutual recovery sequences. As a further step in Sect.  
 1023 4.3 it has to be verified that the limit triple  $(u, z, \theta)$  indeed satisfies the mechanical energy  
 1024 balance as an equality by deducing the reverse inequality from the momentum balance and



1025 the semistability so far obtained. This result allows us to conclude the convergence of the  
 1026 viscous dissipation terms, which, in turn, is crucial for the limit passage in the heat equation  
 1027 (3.27e).

1028 Altogether, these steps amount to the following

1029 **Proposition 4.2** (Energetic solution of the limit problem) *Let the assumptions of Theo-*  
 1030 *rem 2.6 be satisfied and let  $(u, z, \theta)$  be a triple of regularity (2.11) obtained as a limit, in*  
 1031 *the sense of convergences (4.1), of a sequence of solutions to Problem 3.1. Then,  $(u, z, \theta)$  is*  
 1032 *an energetic solution of the time-continuous problem (1.1), supplemented with the boundary*  
 1033 *conditions (1.3), in the sense of Definition 2.3.*

1034 *Proof* The statement of the proposition follows directly by combining Propositions 4.3, 4.6,  
 1035 and 4.9 and Theorem 4.5. □

### 1036 4.1 Limit Passage in the Momentum Balance and the Energy Inequalities

1037 Based on the convergence properties (4.1) we now pass from time-discrete to time-continuous  
 1038 in the weak momentum balance. By lower semicontinuity we will then carry out the limit  
 1039 passage in the mechanical as well as in the total energy inequality and obtain their analogues  
 1040 for the limit problem.

1041 Let us mention in advance that, while the passage to the limit in most of the terms of the  
 1042 momentum balance can be treated in a straightforward way by exploiting the convergence  
 1043 properties (4.1), the quadratic terms arising from the stored elastic energy and the viscous  
 1044 dissipation, which involve the state-dependent coefficients  $\mathbb{D}(\underline{z}_n, \underline{\theta}_n)$  and  $\mathbb{C}(\bar{z}_n)$ , need special  
 1045 attention. For these terms the limit will be deduced by exploiting the  $L^\infty$ -bounds (2.3) on  $\mathbb{C}$   
 1046 and  $\mathbb{D}$  and the dominated convergence theorem.

1047 **Proposition 4.3** (Limit passage in the weak momentum balance) *Let the assumptions of*  
 1048 *Theorem 2.6 be satisfied. Then, a limit triple  $(u, z, \theta)$  extracted as in Proposition 4.1 solves*  
 1049 *the time-continuous momentum balance (2.12b) at every  $t \in [0, T]$ . In particular, it holds*  
 1050  *$\dot{u} \in H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*) \cap C_{\text{weak}}^0([0, T]; L^2(\Omega; \mathbb{R}^d))$ .*

1051 *Proof* Let  $v \in L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap H^1(0, T; L^2(\Omega; \mathbb{R}^d))$  be a test function for (2.12b).  
 1052 It follows from, e.g., [10, p. 56, Corollary 2] and [60, p. 189, Lemma 7.2], that for every  
 1053  $\varepsilon > 0$  there exists

$$1054 \begin{aligned} v^* &\in L^2(0, T; C^1(\bar{\Omega}; \mathbb{R}^d)) \cap L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap H^1(0, T; L^2(\Omega; \mathbb{R}^d)) : \\ &\|v - v^*\|_{L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap H^1(0, T; L^2(\Omega; \mathbb{R}^d))} \leq \varepsilon \text{ and } v^* = v \text{ on } \partial_D \Omega \text{ in the trace sense.} \end{aligned} \tag{4.2}$$

1055 In particular,  $v^* \in L^2(0, T; W^{1,\gamma}(\Omega; \mathbb{R}^d))$ , with  $\gamma > 4$  the same exponent as in the regular-  
 1056 izing term  $-\tau_n \operatorname{div}(|e(u)|^{\gamma-2} e(u))$  in time-discrete momentum balance (3.27b). Therefore,  
 1057 the discrete test functions  $(v^*)_n^k := \frac{1}{\tau_n} \int_{t_{n-1}^k}^{t_n^k} v^*(s) \, ds$  for all  $k = 0, \dots, n$  fulfill  $(v^*)_n^k \in$   
 1058  $W^{1,\gamma}(\Omega; \mathbb{R}^d)$ , so that they are admissible test functions for (3.27b). We now consider the  
 1059 piecewise constant and linear interpolants  $\bar{v}_n^*$  and  $v_n^*$  of the elements  $((v^*)_n^k)_{k=0}^n$ . In view of  
 1060 (4.2), it can be checked that

$$1061 \begin{aligned} \bar{v}_n^* &\rightarrow v^* \text{ in } L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \text{ and } v_n^* \rightarrow v^* \text{ in } H^1(0, T; L^2(\Omega; \mathbb{R}^d)), \\ \tau_n^{1/\gamma} \|e(\bar{v}_n^*)\|_{L^\gamma(0, T; L^\gamma(\Omega; \mathbb{R}^{d \times d}))} &\rightarrow 0. \end{aligned} \tag{4.3a}$$

1062 Observe that (4.3a) implies

$$1063 v_n^*(t) \rightarrow v^*(t) \text{ in } L^2(\Omega; \mathbb{R}^d) \text{ for all } t \in [0, T]. \tag{4.3b}$$

Using such sequences  $(\bar{v}_n^*, v_n^*)_n$  of interpolants of smooth, dense test functions, we can now carry out the limit passage in (3.27b). By the convergence properties of the given data (3.24a) and for the smooth test functions (4.3), together with the convergence results (4.1e), (4.1b) and (4.1k) we immediately find

$$\begin{aligned} & \rho \int_{\Omega} (\dot{u}_n(t) \cdot v_n^*(t) - \dot{u}_0 \cdot v_n^*(0)) \, dx \\ & - \int_0^{\bar{\tau}_n(t)} \left( \int_{\Omega} (\rho \dot{u}_n(s - \tau_n) \cdot \dot{v}_n^* - \bar{\theta}_n \mathbb{B} : e(\bar{v}_n^*)) \, dx - \langle \bar{f}_n, \bar{v}_n^* \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \right) \, ds \\ & \rightarrow \rho \int_{\Omega} (\dot{u}(t) \cdot v^*(t) - \dot{u}_0 \cdot v^*(0)) \, dx \\ & - \int_0^t \left( \int_{\Omega} (\rho \dot{u} \cdot \dot{v}^* - \theta \mathbb{B} : e(v^*)) \, dx - \langle f, v^* \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \right) \, ds. \end{aligned}$$

Moreover, the convergence of the term involving the  $\gamma$ -Laplacian follows from the estimate

$$\begin{aligned} & \left| \int_0^t \int_{\Omega} \tau_n |e(\bar{u}_n)|^{\gamma-2} e(\bar{u}_n) : e(\bar{v}_n^*) \, dx \, ds \right| \\ & \leq \tau_n^{\frac{\gamma-1}{\gamma}} \|e(\bar{u}_n)\|_{L^{\gamma}((0,T) \times \Omega; \mathbb{R}^{d \times d})}^{\gamma-1} \tau_n^{\frac{1}{\gamma}} \|e(\bar{v}_n^*)\|_{L^{\gamma}((0,T) \times \Omega; \mathbb{R}^{d \times d})} \rightarrow 0, \end{aligned}$$

due to the uniform bound (3.32b) and the convergence of  $(v_n^*)_n$  by (4.3).

Finally, in order to handle the remaining quadratic terms with state-dependent coefficients in (3.27b), we will prove that

$$(\mathbb{D}(z_n, \underline{\theta}_n) + \mathbb{C}(\bar{z}_n))e(\bar{v}_n^*) \rightarrow (\mathbb{D}(z, \theta) + \mathbb{C}(z))e(v^*) \text{ strongly in } L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}). \tag{4.4}$$

Then, the convergence of the quadratic terms with state-dependent coefficients follows from weak-strong convergence, using that both  $e(\dot{u}_n) \rightharpoonup e(\dot{u})$  and  $e(u_n) \rightharpoonup e(u)$  weakly in  $L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))$  by (4.1b). Now, to verify (4.4) we are going to apply the dominated convergence theorem. For this, we observe that for a.e.  $t \in (0, T)$  we have  $|(\mathbb{D}(z_n(t), \underline{\theta}_n(t)) + \mathbb{C}(\bar{z}_n(t))) : e(\bar{v}_n^*(t))| \rightarrow |(\mathbb{D}(z(t), \theta(t)) + \mathbb{C}(z(t))) : e(v(t))|$  pointwise a.e. in  $\Omega$ , by assumption (2.3b) and since by convergence results (4.1j) and (4.1i) we can resort to a subsequence  $(z_n(t), \bar{z}_n(t), \underline{\theta}_n)_n$  that converges pointwise a.e. in  $\Omega$  for a.e.  $t \in (0, T)$ . Moreover, by assumption (2.3) we find an integrable, convergent majorant, i.e.,

$$|(\mathbb{D}(z_n, \underline{\theta}_n) + \mathbb{C}(\bar{z}_n))e(\bar{v}_n^*)| \leq (C_D^2 + C_C^2)|e(\bar{v}_n^*)| \rightarrow (C_D^2 + C_C^2)|e(v^*)|$$

pointwise a.e. in  $(0, T) \times \Omega$  and with respect to the strong  $L^2((0, T) \times \Omega)$ -topology by (4.3). Hence, a generalized version of the Dominated Convergence Theorem, cf. e.g., [55, Sect. 4.4, Theorem 19], yields (4.4). This concludes the limit passage in the momentum balance for smooth test function as in (4.2). By density this result carries over to all test functions  $v \in L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)) \cap H^1(0, T; L^2(\Omega; \mathbb{R}^d))$ . As by (4.1e) we have  $\dot{u}(t) \in L^2(\Omega; \mathbb{R}^d)$  for every  $t \in [0, T]$ , we immediately deduce that (2.12b) holds true at all  $t \in [0, T]$ .

The last assertion follows from Remark 2.5. □

**Lemma 4.4** (Energy inequalities by lower semicontinuity) *Let the assumptions of Theorem 2.6 be satisfied and let  $(u, z, \theta)$  be a limit triple given by Proposition 4.1. Then for every  $t \in [0, T]$  we have*

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\dot{u}(t)|^2 dx + \mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z(t) - z_0) dx + \int_0^t \int_{\Omega} (\mathbb{D}(z, \theta)e(\dot{u}) - \theta \mathbb{B}) : e(\dot{u}) dx ds \\ & \leq \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 dx + \mathcal{E}(0, u_0, z_0) - \int_0^t \langle \dot{f}, v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} ds. \end{aligned} \tag{4.5}$$

1099

1100 *Proof* It is enough to pass to the limit in (3.27c) taking into account (3.24b), (4.1d), (4.1e),  
1101 (4.1j), and (4.1l). □

## 1102 4.2 Limit Passage in the Semistability Inequality

1103 In order to carry out the passage from time-discrete to continuous in the semistability inequality  
1104 we follow the well-established method of circumventing a direct passage to the limit on the  
1105 left- and on the right-hand side of the semistability inequality (3.27a). Instead, it is enough  
1106 to prove a limsup inequality for the difference, cf. also [44, 47], using a so-called mutual  
1107 recovery sequence. This procedure, which allows one to take advantage of some cancela-  
1108 tions in the regularizing terms for the internal variable  $\mathcal{G}(z, \nabla z)$ , has been already employed  
1109 in [44, 67, 68] in problems concerned with (fully) rate-independent, partial, isotropic and  
1110 unidirectional damage, featuring a  $W^{1,q}(\Omega)$ -gradient regularization, with  $q > d$  in [44], any  
1111  $q > 1$  in [68] as in the present context, and  $q = 1$  in [67]. In what follows, we verify that the  
1112 recovery sequence constructed in [68], where  $\mathcal{G}(z, \nabla z) = |\nabla z|^q$ , is also suited in our setting  
1113 of semistability with a general gradient term.

1114 More precisely, let us fix  $t \in [0, T]$  in the energy functionals  $\mathcal{E}_n$  from (3.26), and a  
1115 sequence  $(v_n, \zeta_n)_n \subset H_D^1(\Omega; \mathbb{R}^d) \times \mathcal{Z}$  such that

$$\begin{aligned} & v_n \rightharpoonup v \quad \text{weakly in } H_D^1(\Omega; \mathbb{R}^d), \quad \zeta_n \rightharpoonup \zeta \quad \text{weakly in } W^{1,q}(\Omega), \\ & \mathcal{E}_n(t, v_n, \zeta_n) \leq \mathcal{E}_n(t, v_n, \hat{\zeta}) + \mathcal{R}_1(\hat{\zeta} - \zeta_n) \quad \text{for all } \hat{\zeta} \in \mathcal{Z}, \end{aligned} \tag{4.6}$$

1116

1117 i.e.,  $\zeta_n$  is semistable for  $\mathcal{E}_n(t, v_n, \cdot)$ . Given  $\tilde{\zeta} \in \mathcal{Z}$  let the recovery sequence  $(\tilde{\zeta}_n)_n \subset \mathcal{Z}$  be  
1118 defined by

$$\tilde{\zeta}_n := \min \{ \zeta_n, \max \{ (\tilde{\zeta} - \delta_n), 0 \} \} = \begin{cases} (\tilde{\zeta} - \delta_n) & \text{on } A_n = \{ 0 \leq (\tilde{\zeta} - \delta_n) \leq \zeta_n \}, \\ \zeta_n & \text{on } B_n = \{ \tilde{\zeta} - \delta_n > \zeta_n \}, \\ 0 & \text{on } C_n = \{ \tilde{\zeta} - \delta_n < 0 \}, \end{cases} \tag{4.7}$$

1119

$$\text{where } \delta_n := \|\zeta_n - \zeta\|_{L^q(\Omega)}^{1/q}.$$

1120 The sequence  $(\tilde{\zeta}_n)_n$  was introduced in [68] where it was shown that

$$\tilde{\zeta}_n \rightarrow \tilde{\zeta} \quad \text{in } W^{1,q}(\Omega) \quad \text{for } q \in (1, \infty) \text{ from (2.5d) fixed.} \tag{4.8}$$

1121

1122 Note however that strong convergence in  $W^{1,q}(\Omega)$  cannot be expected, since  $\zeta_n \rightharpoonup \zeta$  weakly  
1123 in  $W^{1,q}(\Omega)$ , only. This makes it impossible to show directly that  $\mathcal{G}(\tilde{\zeta}_n, \nabla \tilde{\zeta}_n) \rightarrow \mathcal{G}(\tilde{\zeta}, \nabla \tilde{\zeta})$ ,  
1124 since this would require the strong convergence of the gradients. Nevertheless the following  
1125 result holds.

1126 **Theorem 4.5** *Let the assumptions of Theorem 2.6 be satisfied. Let  $t \in [0, T]$  be fixed and*  
1127 *consider a sequence  $(v_n, \zeta_n)_n \subset H_D^1(\Omega; \mathbb{R}^d) \times \mathcal{Z}$  such that (4.6) holds. Given  $\tilde{\zeta} \in \mathcal{Z}$ , let*  
1128  *$(\tilde{\zeta}_n)_n \subset \mathcal{Z}$  as in (4.7). Then*

$$0 \leq \limsup_{n \rightarrow \infty} \left( \mathcal{E}_n(t, v_n, \tilde{\zeta}_n) - \mathcal{E}_n(t, v_n, \zeta_n) + \mathcal{R}_1(\tilde{\zeta}_n - \zeta_n) \right) \leq \mathcal{E}(t, v, \tilde{\zeta}) - \mathcal{E}(t, v, \zeta) + \mathcal{R}_1(\tilde{\zeta} - \zeta). \tag{4.9}$$

1129

1130 Therefore the limit  $\zeta$  is semistable for  $\mathcal{E}(t, v, \cdot)$ .

1131 *Proof* First of all note that, if  $\tilde{\zeta} \in \mathcal{Z}$  does not satisfy  $0 \leq \tilde{\zeta} \leq \zeta$ , then (4.9) trivially holds,  
 1132 since in this case  $\mathcal{R}_1(\tilde{\zeta} - \zeta) = +\infty$ .

1133 Assume now  $0 \leq \tilde{\zeta} \leq \zeta$  for a.e.  $x \in \Omega$ . Let us estimate the left-hand side of (4.9) as  
 1134 follows:

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \left( \mathcal{E}_n(t, v_n, \tilde{\zeta}_n) - \mathcal{E}_n(t, v_n, \zeta_n) + \mathcal{R}_1(\tilde{\zeta}_n - \zeta_n) \right) \\
 & \leq \limsup_{n \rightarrow \infty} \int_{\Omega} (\mathbb{C}(\tilde{\zeta}_n) - \mathbb{C}(\zeta_n))e(v_n) : e(v_n) \, dx \\
 & \quad + \limsup_{n \rightarrow \infty} (\mathcal{G}(\tilde{\zeta}_n, \nabla \tilde{\zeta}_n) - \mathcal{G}(\zeta_n, \nabla \zeta_n)) + \limsup_{n \rightarrow \infty} \mathcal{R}_1(\tilde{\zeta}_n - \zeta_n) \tag{4.10}
 \end{aligned}$$

1139 and then treat each of the terms on the right-hand side of (4.10) separately. Since  $\zeta_n \rightarrow \zeta$  in  
 1140  $W^{1,q}(\Omega)$ , we may choose a (not relabeled) subsequence that converges pointwise a.e. in  $\Omega$ .

1141 *Estimation of  $\limsup_{n \rightarrow \infty} (\mathcal{G}(\tilde{\zeta}_n, \nabla \tilde{\zeta}_n) - \mathcal{G}(\zeta_n, \nabla \zeta_n))$ :* Note that  $G(\tilde{\zeta}_n, \nabla \tilde{\zeta}_n) = G(\zeta_n,$   
 1142  $\nabla \zeta_n)$  on  $B_n$ . If  $\|\zeta_n - \zeta\|_{L^q(\Omega)} > 0$ , by Markov's inequality

$$\mathcal{L}^d(B_n) \leq \mathcal{L}^d([\delta_n \leq |\zeta_n - \zeta|]) \leq \frac{1}{\delta_n} \int_{\Omega} |\zeta_n - \zeta| \, dx \leq \frac{1}{\delta_n} \|\zeta_n - \zeta\|_{L^q(\Omega)} \rightarrow 0,$$

1144 with  $\delta_n$  from (4.7), while for  $\|\zeta_n - \zeta\|_{L^q(\Omega)} = 0$  it is indeed  $\mathcal{L}^d(B_n) = 0$ , thus

$$\mathcal{L}^d(A_n \cup C_n) \rightarrow \mathcal{L}^d(\Omega). \tag{4.11}$$

1146 In what follows,  $\mathcal{X}_D$  will denote the characteristic function of a set  $D$ . By (2.5b), (2.5d) and  
 1147 (4.7), we deduce

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} (\mathcal{G}(\tilde{\zeta}_n, \nabla \tilde{\zeta}_n) - \mathcal{G}(\zeta_n, \nabla \zeta_n)) \\
 & = \limsup_{n \rightarrow \infty} \int_{A_n} G((\tilde{\zeta} - \delta_n), \nabla \tilde{\zeta}) \, dx + \int_{C_n} G(0, 0) \, dx - \int_{A_n \cup C_n} G(\zeta_n, \nabla \zeta_n) \, dx \\
 & \leq \limsup_{n \rightarrow \infty} \left( \int_{\Omega} G(\mathcal{X}_{A_n}(\tilde{\zeta} - \delta_n), \mathcal{X}_{A_n} \nabla \tilde{\zeta}) \, dx + \int_{\Omega} G(0, \mathcal{X}_{C_n} \nabla \tilde{\zeta}) \, dx \right. \\
 & \quad \left. - \int_{\Omega} G(\mathcal{X}_{A_n \cup C_n} \zeta_n, \mathcal{X}_{A_n \cup C_n} \nabla \zeta_n) \, dx \right) \\
 & = \limsup_{n \rightarrow \infty} \left( \int_{\Omega} G(\mathcal{X}_{A_n \cup C_n}(\tilde{\zeta}_n), \mathcal{X}_{A_n \cup C_n} \nabla \tilde{\zeta}) \, dx - \int_{\Omega} G(\mathcal{X}_{A_n \cup C_n} \zeta_n, \mathcal{X}_{A_n \cup C_n} \nabla \zeta_n) \, dx \right) \\
 & \leq \mathcal{G}(\tilde{\zeta}, \nabla \tilde{\zeta}) - \liminf_{n \rightarrow \infty} \mathcal{G}(\mathcal{X}_{A_n \cup C_n} \zeta_n, \mathcal{X}_{A_n \cup C_n} \nabla \zeta_n) \tag{4.12a}
 \end{aligned}$$

$$\leq \mathcal{G}(\tilde{\zeta}, \nabla \tilde{\zeta}) - \mathcal{G}(\zeta, \nabla \zeta), \tag{4.12b}$$

1158 where in the second integral term in the third line we have used the obvious identity  $\mathcal{X}_{C_n} 0 = 0$ .  
 1159 To obtain (4.12a) we have used the dominated convergence theorem, while in order to prove  
 1160 (4.12b) we employed the lower semicontinuity of  $\mathcal{G} : L^q(\Omega) \times L^q(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$ ,  
 1161 since, by (4.8) and (4.11), we have  $\mathcal{X}_{A_n \cup C_n} \zeta_n \rightarrow \zeta$  strongly in  $L^q(\Omega)$  and  $\mathcal{X}_{A_n \cup C_n} \nabla \zeta_n \rightarrow \nabla \zeta$   
 1162 weakly in  $L^q(\Omega; \mathbb{R}^d)$ .

1163 *Estimation of the remaining terms in (4.10):* Since construction (4.7) ensures  $\tilde{\zeta}_n \leq \zeta_n$   
 1164 for every  $n \in \mathbb{N}$ , as well as  $\tilde{\zeta}_n \rightarrow \tilde{\zeta}$  in  $L^q(\Omega)$ , due to  $\zeta_n \rightarrow \zeta$  in  $L^q(\Omega)$ , we immediately  
 1165 conclude that  $\mathcal{R}_1(\tilde{\zeta}_n - \zeta_n) \rightarrow \mathcal{R}_1(\tilde{\zeta} - \zeta)$ .

1166 We now estimate the difference of the quadratic terms in the mechanical energy. As  $\tilde{\zeta}_n \leq$   
 1167  $\zeta_n$ , by the monotonicity assumption (2.4) we have that  $(\mathbb{C}(\tilde{\zeta}_n) - \mathbb{C}(\zeta_n))e(v_n) : e(v_n) \leq 0$ .

1168 Since both  $\zeta_n \rightarrow \zeta$  and  $\tilde{\zeta}_n \rightarrow \tilde{\zeta}$  in  $L^q(\Omega)$ , the Lipschitz-continuity of  $\mathbb{C}$ , cf. (2.3b), implies  
 1169 that  $\mathbb{C}(\tilde{\zeta}_n) - \mathbb{C}(\zeta_n) \rightarrow (\mathbb{C}(\tilde{\zeta}) - \mathbb{C}(\zeta))$  in  $L^q(\Omega; \mathbb{R}^{d \times d \times d \times d}_{\text{sym}})$ . Let us consider the auxiliary  
 1170 functional  $\mathcal{C} : L^q(\Omega) \times L^q(\Omega) \times L^2(\Omega; \mathbb{R}^{d \times d}) \rightarrow \mathbb{R}$  defined by

1171 
$$\mathcal{C}(\zeta, \tilde{\zeta}, e) := \int_{\Omega} (\mathbb{C}(\zeta(x)) - \mathbb{C}(\min\{\zeta(x), \tilde{\zeta}(x)\}))e(x) : e(x) \, dx .$$

1172 By e.g. [21, Theorem 7.5, p. 492] the functional  $\mathcal{C}$  is lower semicontinuous with respect to  
 1173 the strong convergence in  $L^q(\Omega) \times L^q(\Omega)$  and the weak convergence in  $L^2(\Omega; \mathbb{R}^{d \times d})$ . Thus,  
 1174 the first term on the right-hand side of (4.10) can be rewritten and estimated as follows, using  
 1175 (3.32c) and the lower semicontinuity of  $\mathcal{C}$ ,

1176 
$$\limsup_{n \rightarrow \infty} \int_{\Omega} (\mathbb{C}(\tilde{\zeta}_n) - \mathbb{C}(\zeta_n))e(v_n) : e(v_n) \, dx \leq \int_{\Omega} (\mathbb{C}(\tilde{\zeta}) - \mathbb{C}(\zeta))e(v) : e(v) \, dx .$$

1178 Combining the above established estimates for the three terms on the right-hand side of  
 1179 (4.10) shows that condition (4.9) is satisfied. □

1180 **4.3 Energy Equalities and Limit Passage in the Heat Equation**

1181 We now show that the limit triple  $(u, z, \theta)$  satisfies the mechanical energy equality (2.12c).  
 1182 The inequality ( $\leq$ ) has been proven in Lemma 4.4. The opposite inequality is found by  
 1183 approximation with Riemann sums, as common in existence proofs of rate-independent and  
 1184 rate-dependent evolutions, see e.g. [13].

1185 **Proposition 4.6** (Mechanical energy equality) *Let the assumptions of Theorem 2.6 be satis-*  
 1186 *fied, let  $(u, z, \theta)$  be a triple given by Proposition 4.1, and let  $t \in [0, T]$ . Then (2.12c) holds.*

1188 *Proof* We fix a sequence of subdivisions  $(s_n^k)_{0 \leq k \leq k_n}$  of the interval  $[0, t]$ , with  $0 = s_n^0 <$   
 1189  $s_n^1 < \dots < s_n^{k_n-1} < s_n^{k_n} = t$ ,  $\lim_n \max_k (s_n^k - s_n^{k-1}) = 0$ , and

1190 
$$\left| \sum_{k=1}^{k_n} \int_{s_n^{k-1}}^{s_n^k} \int_{\Omega} [\mathbb{C}(z(s_n^k)) - \mathbb{C}(z(s))] e(u(s)) : e(\dot{u}(s)) \, dx \, ds \right| \rightarrow 0 . \quad (4.13)$$

1191 The existence of such a sequence is guaranteed by [27], see also [57, Proposition 4.3, Step  
 1192 7]. Taking  $z(s_n^k)$  as test function in the time-continuous semistability inequality (2.12a) at  
 1193 time  $s_n^{k-1}$  we get

1194 
$$\begin{aligned} & \mathcal{E}(s_n^{k-1}, u(s_n^{k-1}), z(s_n^{k-1})) \\ & \leq \mathcal{E}(s_n^{k-1}, u(s_n^{k-1}), z(s_n^k)) + \int_{\Omega} (z(s_n^{k-1}) - z(s_n^k)) \, dx \\ & = \mathcal{E}(s_n^k, u(s_n^k), z(s_n^k)) + \int_{\Omega} (z(s_n^{k-1}) - z(s_n^k)) \, dx - \int_{s_n^{k-1}}^{s_n^k} \partial_t \mathcal{E}(s, u(s), z(s)) \, ds \\ & \quad + \int_{s_n^{k-1}}^{s_n^k} \langle f(s), \dot{u}(s) \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds - \int_{s_n^{k-1}}^{s_n^k} \int_{\Omega} \mathbb{C}(z(s_n^k))e(u(s)) : e(\dot{u}(s)) \, dx \, ds . \end{aligned}$$

1199 Next we sum up the previous inequality over  $k = 1, \dots, k_n$  and we pass to the limit in  $n$  in  
 1200 the last term thanks to (4.13), obtaining

$$\begin{aligned} \mathcal{E}(0, u_0, z_0) &\leq \mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z_0 - z(t)) \, dx - \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds \\ &\quad + \int_0^t \langle f(s), \dot{u}(s) \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds - \int_0^t \int_{\Omega} \mathbb{C}(z(s))e(u(s)) : e(\dot{u}(s)) \, dx \, ds. \end{aligned} \tag{4.14}$$

1201 Further, thanks to Remark 2.5 we can test (2.12b) by  $\dot{u}$  and get

$$\begin{aligned} \frac{\rho}{2} \|\dot{u}(t)\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \int_0^t \int_{\Omega} (\mathbb{D}(z, \theta)e(\dot{u}) + \mathbb{C}(z)e(u) - \theta \mathbb{B}) : e(\dot{u}) \, dx \, ds \\ = \frac{\rho}{2} \|\dot{u}_0\|_{L^2(\Omega; \mathbb{R}^d)}^2 + \int_0^t \langle f, \dot{u} \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds, \end{aligned} \tag{4.15}$$

1204 where we applied the by-part integration formula (2.13), as allowed by [60, Lemma 7.3].  
 1205 Summing up (4.15) with (4.14) we obtain

$$\begin{aligned} \mathcal{E}(0, u_0, z_0) &\leq \mathcal{E}(t, u(t), z(t)) + \frac{\rho}{2} \int_{\Omega} |\dot{u}(t)|^2 \, dx + \int_{\Omega} (z_0 - z(t)) \, dx \\ &\quad - \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds \\ &\quad - \frac{\rho}{2} \int_{\Omega} |\dot{u}_0|^2 \, dx + \int_0^t \int_{\Omega} (\mathbb{D}(z(s), \theta(s))e(\dot{u}(s)) - \theta(s) \mathbb{B}) : e(\dot{u}(s)) \, dx \, ds. \end{aligned}$$

1210 Combining this estimate with the reverse inequality (4.5) concludes the proof of (2.12c).  $\square$

1211 In order to prove a stronger convergence of the displacements we shall repeatedly make  
 1212 use of the following result. Given two constants  $C_1, C_2$  with  $0 < C_1 \leq C_2$ , let  $\mathcal{T}_{C_1, C_2}$  denote  
 1213 the class of tensors  $\mathbb{A} \in \mathbb{R}^{d \times d \times d \times d}$  that are symmetric, i.e.,

$$\mathbb{A}_{ijkl} = \mathbb{A}_{jikl} = \mathbb{A}_{ijlk} = \mathbb{A}_{klij},$$

1215 positive definite and bounded:

$$C_1 |A|^2 \leq \mathbb{A} A : A \leq C_2 |A|^2 \quad \text{for every } A \in \mathbb{R}_{\text{sym}}^{d \times d}. \tag{4.16}$$

1218 **Lemma 4.7** *Let  $\mathcal{K}_n$  be the functional defined by*

$$\mathcal{K}_n(e) := \int_0^T \int_{\Omega} \mathbb{A}_n(t, x)e(t, x) : e(t, x) \, dx \, dt \quad \text{for every } e \in L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}),$$

1220 where  $\mathbb{A}_n \in L^\infty((0, T) \times \Omega; \mathcal{T}_{C_1, C_2})$  are such that

$$\mathbb{A}_n(t, x) \rightarrow \mathbb{A}_\infty(t, x) \quad \text{for a.e. } t \in (0, T) \text{ and a.e. } x \in \Omega, \tag{4.17a}$$

$$e_n \rightharpoonup e_\infty \quad \text{weakly in } L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}), \tag{4.17b}$$

$$\limsup_{n \rightarrow \infty} \mathcal{K}_n(e_n) \leq \mathcal{K}_\infty(e_\infty), \tag{4.17c}$$

1225 and  $\mathcal{K}_\infty$  is defined by

$$\mathcal{K}_\infty(e) := \int_0^T \int_{\Omega} \mathbb{A}_\infty(t, x)e(t, x) : e(t, x) \, dx \, dt \quad \text{for every } e \in L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}).$$

1227 Then,  $\lim_{n \rightarrow \infty} \mathcal{K}_n(e_n) = \mathcal{K}_\infty(e_\infty)$  and

1228 
$$e_n \rightarrow e_\infty \text{ strongly in } L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}). \tag{4.18}$$

1229 *Proof* It is enough to observe that under the above hypotheses  $\mathbb{A}_\infty \in L^\infty((0, T) \times \Omega; \mathcal{T}_{C_1, C_2})$   
 1230 and

1231 
$$\begin{aligned} \mathcal{K}_n(e_n - e_\infty) &= \int_0^T \int_\Omega \mathbb{A}_n(t, x)(e_n(t, x) - e_\infty) : (e_n(t, x) - e_\infty(t, x)) \, dx \, dt \\ &= \mathcal{K}_n(e_n) - 2 \int_0^T \int_\Omega \mathbb{A}_n(t, x)e_\infty(t, x) : e_n(t, x) \, dx \, dt + \mathcal{K}_n(e_\infty). \end{aligned}$$

1232 By (4.16) and (4.17a) we obtain  $\limsup_n \mathcal{K}_n(e_n - e_\infty) \leq 0$ . Since  $\mathbb{A}_n(t, x) \in \mathcal{T}_{C_1, C_2}$  we  
 1233 have  $\mathcal{K}_n(e_n - e_\infty) \geq C_1 \|e_n - e_\infty\|_{L^2((0, T) \times \Omega; \mathbb{R}^{d \times d})}^2$ , so that (4.18) holds.  $\square$

1234 Thanks to the mechanical energy inequality proven above, we may deduce strong conver-  
 1235 gence of the displacements, as provided in the following lemma.

1236 **Lemma 4.8** (Stronger convergences) *Let the assumptions of Theorem 2.6 be satisfied and*  
 1237 *let  $(u, z, \theta)$  be a triple given by Proposition 4.1. Then*

1238 
$$\lim_{n \rightarrow \infty} \int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{u}_n) \, dx \, dt = \int_0^T \int_\Omega \mathbb{D}(z, \theta)e(\dot{u}) : e(\dot{u}) \, dx \, dt \tag{4.19}$$

1239 and then

1240 
$$e(\dot{u}_n) \rightarrow e(\dot{u}) \text{ strongly in } L^2((0, T) \times \Omega; \mathbb{R}^{d \times d}). \tag{4.20}$$

1241 *Proof* By lower semicontinuity, taking into account the convergences already proven in  
 1242 Proposition 4.1, together with both the discrete mechanical energy inequality (3.27c) and the  
 1243 mechanical energy equality (2.12c), the following chain of inequalities holds:

1244 
$$\begin{aligned} &\int_0^T \int_\Omega \mathbb{D}(z, \theta)e(\dot{u}) : e(\dot{u}) \, dx \, dt + \int_\Omega (z_0 - z(T)) \, dx \\ &\leq \liminf_n \left( \int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{u}_n) \, dx \, dt + \int_\Omega (z_n(0) - z_n(T)) \, dx \right) \\ &\leq \limsup_n \left( \int_0^T \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n)e(\dot{u}_n) : e(\dot{u}_n) \, dx \, dt + \int_\Omega (z_n(0) - z_n(T)) \, dx \right) \\ &\leq \limsup_n \left( -\mathcal{E}_n(T, u_n(T), z_n(T)) + \mathcal{E}_n(0, u_0, z_0) - \frac{\rho}{2} \int_\Omega |\dot{u}_n(T)|^2 \, dx + \frac{\rho}{2} \int_\Omega |\dot{u}_0|^2 \, dx \right. \\ &\quad \left. + \int_0^T \int_\Omega \bar{\theta}_n \mathbb{B} : e(\dot{u}_n) \, dx \, dt + \int_0^T \partial_t \mathcal{E}_n(s, \underline{u}_n, \underline{z}_n) \, ds \right) \\ &\leq -\mathcal{E}(T, u(T), z(T)) + \mathcal{E}(0, u_0, z_0) - \frac{\rho}{2} \int_\Omega |\dot{u}(T)|^2 \, dx + \frac{\rho}{2} \int_\Omega |\dot{u}_0|^2 \, dx \\ &\quad + \int_0^T \int_\Omega \theta \mathbb{B} : e(\dot{u}) \, dx \, dt + \int_0^T \partial_t \mathcal{E}(s, u, z) \, ds \\ &= \int_0^T \int_\Omega \mathbb{D}(z, \theta)e(\dot{u}) : e(\dot{u}) \, dx \, dt + \int_\Omega (z_0 - z(T)) \, dx. \end{aligned}$$

1251  
1252

1253 Hence all inequalities above are actually equalities and we deduce that (4.19) holds.

1254 Next, we apply Lemma 4.7 with  $\mathbb{A}_n = \mathbb{D}(\underline{z}_n, \underline{\theta}_n)$ ,  $\mathbb{A}_\infty = \mathbb{D}(z, \theta)$ ,  $e_n = e(\dot{u}_n)$ , and  
 1255  $e_\infty = e(\dot{u})$ . Indeed, (4.17a) is obtained from the strong convergences (4.1j) and (4.11) up to  
 1256 the passage to a further subsequence converging pointwise; the weak convergence (4.17b) is  
 1257 given in (4.1b), while (4.17c) is provided by (4.19). Therefore we deduce that (4.20) holds  
 1258 (for the initial subsequence, since the limit is the same for all subsequences).  $\square$

1259 Finally, we pass to the limit in the heat equation.

1260 **Proposition 4.9** (Limit passage in the weak form of the heat equation) *Let the assumptions of*  
 1261 *Theorem 2.6 be satisfied, Let  $(u, z, \theta)$  be a triple given by Proposition 4.1, and let  $t \in [0, T]$ .*  
 1262 *Then the weak formulation of the heat equation (2.12d) holds.*

1263 *Proof* Let us fix  $\eta \in H^1(0, T; L^2(\Omega)) \cap C^0([0, T]; W^{2,d+\delta}(\Omega))$ , define  $\eta_n^k := \eta(t_n^k)$  for all  
 1264  $k = 0, \dots, n$ , and let  $\eta_n, \bar{\eta}_n$  be the piecewise linear and constant interpolations of the values  
 1265  $(\eta_n^k)$ . It can be checked that

$$\begin{aligned} \bar{\eta}_n &\rightarrow \eta \quad \text{in } L^p(0, T; W^{2,d+\delta}(\Omega)) \text{ for all } 1 \leq p < \infty, \\ \bar{\eta}_n &\overset{*}{\rightharpoonup} \eta \quad \text{in } L^\infty(0, T; W^{2,d+\delta}(\Omega)), \\ \eta_n &\rightarrow \eta \quad \text{in } H^1(0, T; L^2(\Omega)) \cap C^0(0, T; W^{2,d+\delta}(\Omega)). \end{aligned} \tag{4.21}$$

1267 We now pass to the limit in the discrete heat equation (3.27e) tested by  $\eta_n$ . The first three  
 1268 integral terms on the left-hand side of (3.27e) can be dealt with combining convergences  
 1269 (4.11)–(4.1n) with (4.21). In order to pass to the limit in the fourth one, we argue along the  
 1270 lines of [63, proof of Theorem 2.8] and derive a finer estimate for  $(\mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla \bar{\theta}_n)_n$ . Indeed,  
 1271 thanks to (2.6b) we have

$$1272 \quad |\mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla \bar{\theta}_n| \leq c_2 (|\bar{\theta}_n|^{\kappa-\alpha+2})^{1/2} |\bar{\theta}_n|^{\kappa+\alpha-2})^{1/2} |\nabla \bar{\theta}_n| + |\nabla \bar{\theta}_n| \quad \text{a.e. in } (0, T) \times \Omega,$$

1273 with  $\alpha$  as in (3.37). From this particular estimate we also gather that  $|\bar{\theta}_n|^{\kappa+\alpha-2})^{1/2} |\nabla \bar{\theta}_n|$  is  
 1274 bounded in  $L^2((0, T) \times \Omega)$ . Since  $(\bar{\theta}_n)_n$  is bounded in  $L^{8/3}((0, T) \times \Omega)$  if  $d=3$  (and in  
 1275  $L^3((0, T) \times \Omega)$  if  $d=2$ ), choosing  $\alpha \in (1/2, 1)$  such that  $\kappa - \alpha < 2/3$  (which can be done,  
 1276 since  $\kappa < 5/3$ ), we conclude that  $|\bar{\theta}_n|^{\kappa-\alpha+2})^{1/2}$  is bounded in  $L^{2+\delta}((0, T) \times \Omega)$  for some  
 1277  $\delta > 0$ . All in all, we have that  $\mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla \bar{\theta}_n$  is bounded in  $L^{1+\delta}((0, T) \times \Omega; \mathbb{R}^d)$  for some  
 1278  $\delta > 0$ . With the very same arguments as in [63, proof of Theorem 2.8], we show that

$$1279 \quad \mathbb{K}(\bar{z}_n, \bar{\theta}_n) \nabla \bar{\theta}_n \rightharpoonup \mathbb{K}(z, \theta) \nabla \theta \quad \text{in } L^{1+\delta}((0, T) \times \Omega; \mathbb{R}^d),$$

1280 which, combined with convergences (4.21) for  $\bar{\eta}_n$ , is enough to pass to the limit in the last  
 1281 term on the left-hand side of (3.27e).

1282 Combining (4.1b), (4.1m), and (4.21) yields  $\int_0^{\bar{\tau}_n(t)} \int_\Omega \bar{\theta}_n \mathbb{B} : e(\dot{u}_n) \bar{\eta}_n \, dx \, ds \rightarrow \int_0^t \int_\Omega \theta \mathbb{B} :$   
 1283  $e(\dot{u}) \eta \, dx \, ds$  as  $n \rightarrow \infty$ , while the passage to the limit in the term

$$1284 \quad \int_0^{\bar{\tau}_n(t)} \int_\Omega \mathbb{D}(\underline{z}_n, \underline{\theta}_n) e(\dot{u}_n) : e(\dot{u}_n) \bar{\eta}_n \, dx \, ds$$

1285 results from (4.20) combined with (4.21). Convergence (4.1o) allows us to deal with the  
 1286 second term on the right-hand side of (3.27e), and we handle the last two terms via (3.24d)  
 1287 and (4.21), again. This concludes the proof of the weak heat equation and of the main existence  
 1288 result Theorem 2.6.  $\square$



## 5 Asymptotic Behavior in the Slow Loading Regime: The Vanishing Viscosity and Inertia Limit

In this section we address the limiting behavior of system (1.1) as the rate of the external load and of the heat sources becomes slower and slower. Accordingly, we will rescale time by a factor  $\varepsilon > 0$ . For analytical reasons we restrict to the case of a Dirichlet problem in the displacement, namely within this section we shall suppose that

$$\partial_D \Omega = \partial \Omega. \tag{5.1}$$

Like in the previous sections, we assume that the Dirichlet datum is homogeneous, cf. (1.3b). As  $\varepsilon \downarrow 0$  we will *simultaneously* pass to

1. a rate-independent system for the limit displacement and damage variables  $(u, z)$ , which does not display any temperature dependence and which formally reads

$$\begin{aligned} -\operatorname{div} \mathbb{C}(z)e(u) &= f_V && \text{in } (0, T) \times \Omega, \\ \partial R_1(\dot{z}) + D_z G(z, \nabla z) - \operatorname{div} (D_\xi G(z, \nabla z)) + \frac{1}{2} \mathbb{C}'(z)e(u) : e(u) &\ni 0 && \text{in } (0, T) \times \Omega \end{aligned}$$

and will be weakly formulated through the concept of *local solution* to a rate-independent system;

2. a limit temperature  $\theta = \Theta$ , which is constant in space, but still time-dependent. The limit passage in the heat equation amounts to the trivial limit  $0 = 0$ , once more emphasizing that the limit system does not depend on temperature any more. A rescaling of the heat equation at level  $\varepsilon$ , however, reveals that  $\Theta$  evolves in time according to an ODE in the sense of measures and the evolution is driven by the rate-independent dissipation and a measure originating from the viscous dissipation.

Indeed, for the limit system we expect that, if a change of heat is caused at some spot in the material, then the heat must be conducted all over the material with infinite speed, so that the temperature is kept constant in space. This justifies a scaling of the tensor of heat conduction coefficients for the systems at level  $\varepsilon$ . More precisely, we will suppose that

$$\mathbb{K}_\varepsilon(z, \theta) := \frac{1}{\varepsilon^\beta} \mathbb{K}(z, \theta) \quad \text{with } \mathbb{K} \text{ satisfying (2.6) and } \beta > 0. \tag{5.2}$$

While Proposition 5.2 holds with  $\beta > 0$ , in Theorem 5.3 we shall require  $\beta \geq 2$ .

### 5.1 Time Rescaling

Let us now set up the vanishing viscosity analysis following [56], where this analysis was carried out for *isothermal* rate-independent processes in viscous solids, see also [15] in the context of perfect plasticity and [58, 66] for delamination, still in the isothermal case. We consider a family  $(f_{V,\varepsilon}, H_\varepsilon, h_\varepsilon)_\varepsilon$  of data for system (1.1) and we rescale  $f_{V,\varepsilon}, H_\varepsilon, h_\varepsilon$  by the factor  $\varepsilon > 0$ , hence we introduce

$$f^\varepsilon(t) := f_{V,\varepsilon}(\varepsilon t) \quad H^\varepsilon(t) := H_\varepsilon(\varepsilon t), \quad h^\varepsilon(t) := h_\varepsilon(\varepsilon t) \quad \text{for } t \in [0, \frac{T}{\varepsilon}].$$

Theorem 2.6 guarantees that for every  $\varepsilon > 0$  there exists an energetic solution  $(u^\varepsilon, z^\varepsilon, \theta^\varepsilon)$ , defined on  $[0, \frac{T}{\varepsilon}]$ , to (the Cauchy problem for) system (1.1) supplemented with the data  $f^\varepsilon, H^\varepsilon, h^\varepsilon$ , and with the matrix of heat conduction coefficients  $\mathbb{K}_\varepsilon$  from (5.2). For later convenience, let us recall that such solutions arise as limits of the time-discrete solutions to Problem 3.1. We now perform a rescaling of the solutions in such a way as to have them

1329 defined on the interval  $[0, T]$ . Namely, we set

1330 
$$u_\varepsilon(t) := u^\varepsilon\left(\frac{t}{\varepsilon}\right), \quad z_\varepsilon(t) := z^\varepsilon\left(\frac{t}{\varepsilon}\right), \quad \theta_\varepsilon(t) := \theta^\varepsilon\left(\frac{t}{\varepsilon}\right) \quad \text{for } t \in [0, T].$$

1331 It is not difficult to check that, after transforming the time scale, the triple  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$  (for-  
1332 mally) solves the following system in  $(0, T) \times \Omega$ :

1333 
$$\varepsilon^2 \rho \ddot{u}_\varepsilon - \operatorname{div}(\varepsilon \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) + \mathbb{C}(z_\varepsilon) e(u_\varepsilon) - \theta_\varepsilon \mathbb{B}) = f_\varepsilon, \tag{5.3a}$$

1334 
$$\partial R_1(\dot{z}_\varepsilon) + D_z G(z_\varepsilon, \nabla z_\varepsilon) - \operatorname{div}(D_\xi G(z_\varepsilon, \nabla z_\varepsilon)) + \frac{1}{2} \mathbb{C}'(z_\varepsilon) e(u_\varepsilon) : e(u_\varepsilon) \ni 0, \tag{5.3b}$$

1335 
$$\varepsilon \dot{\theta}_\varepsilon - \frac{1}{\varepsilon^\beta} \operatorname{div}(\mathbb{K}(z_\varepsilon, \theta_\varepsilon) \nabla \theta_\varepsilon) = \varepsilon R_1(\dot{z}_\varepsilon) + \varepsilon^2 \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)$$
  
1336 
$$- \varepsilon \theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon) + H_\varepsilon, \tag{5.3c}$$

1338 with the original data  $f_\varepsilon := f_{V,\varepsilon}$ ,  $H_\varepsilon$ , and  $h_\varepsilon$ , and complemented with the boundary con-  
1339 ditions (1.3). Since in the following we will be interested in the limit of (5.3) as  $\varepsilon \downarrow 0$ , for  
1340 notational simplicity we shall henceforth set  $\rho = 1$  in (5.3a).

1341 *Energetic solutions for the rescaled system (5.4)–(5.9).* For later reference in the limit passage  
1342 procedure as  $\varepsilon \downarrow 0$ , we recall the defining properties of energetic solutions. Given a quadruple  
1343 of initial data  $(u_\varepsilon^0, \dot{u}_\varepsilon^0, z_\varepsilon^0, \theta_\varepsilon^0)$  satisfying (2.7), a triple  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$  is an energetic solution of  
1344 the Cauchy problem for the PDE system (5.3) if it has the regularity (2.11), it complies with  
1345 the initial conditions

1346 
$$u_\varepsilon(0) = u_\varepsilon^0, \quad \dot{u}_\varepsilon(0) = \dot{u}_\varepsilon^0, \quad z_\varepsilon(0) = z_\varepsilon^0, \quad \theta_\varepsilon(0) = \theta_\varepsilon^0 \quad \text{a.e. in } \Omega, \tag{5.4}$$

1347 and fulfills

- 1348 • *semistability and unidirectionality*: for a.a.  $x \in \Omega$ ,  $z_\varepsilon(\cdot, x) : [0, T] \rightarrow [0, 1]$  is nonin-  
1349 creasing and for all  $t \in [0, T]$

1350 
$$\forall \tilde{z} \in \mathcal{Z}, \tilde{z} \leq z_\varepsilon(t) : \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) \leq \mathcal{E}_\varepsilon(t, u_\varepsilon(t), \tilde{z}) + \mathcal{R}_1(z_\varepsilon(t) - \tilde{z}), \tag{5.5}$$

1351 with the mechanical energy

1352 
$$\mathcal{E}_\varepsilon(t, u, z) := \int_\Omega \left( \frac{1}{2} \mathbb{C}(z) e(u) : e(u) + G(z, \nabla z) \right) dx - \langle f_\varepsilon(t), u \rangle_{H_D^1(\Omega; \mathbb{R}^d)}; \tag{5.6}$$

- 1353 • *weak formulation of the momentum equation*: for all test functions  $v \in L^2(0, T; H_D^1$   
1354  $(\Omega; \mathbb{R}^d)) \cap W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^d))$  and for all  $t \in [0, T]$

1355 
$$\begin{aligned} & \varepsilon^2 \int_\Omega \dot{u}_\varepsilon(t) \cdot v(t) dx - \varepsilon^2 \int_0^t \int_\Omega \dot{u}_\varepsilon \cdot \dot{v} dx dt \\ & + \int_0^t \int_\Omega (\varepsilon \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) + \mathbb{C}(z_\varepsilon) e(u_\varepsilon) - \theta_\varepsilon \mathbb{B}) : e(v) dx ds \\ & = \varepsilon^2 \int_\Omega \dot{u}_\varepsilon^0 \cdot v(0) dx + \int_0^t \langle f_\varepsilon, v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} ds; \end{aligned} \tag{5.7}$$

- 1356 • *mechanical energy equality*: for all  $t \in [0, T]$

1357 
$$\begin{aligned} & \frac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon(t)|^2 dx + \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \int_\Omega (z_\varepsilon^0 - z_\varepsilon(t)) dx \\ & + \int_0^t \int_\Omega (\varepsilon \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) - \theta_\varepsilon \mathbb{B}) : e(\dot{u}_\varepsilon) dx ds \\ & = \frac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon^0|^2 dx + \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) + \int_0^t \partial_t \mathcal{E}_\varepsilon(s, u(s), z(s)) ds; \end{aligned} \tag{5.8}$$

1358 • *weak formulation of the heat equation:* for all  $t \in [0, T]$

$$\begin{aligned} & \varepsilon \langle \theta_\varepsilon(t), \eta(t) \rangle_{W^{2,d+\delta}} - \varepsilon \int_0^t \int_\Omega \theta_\varepsilon \dot{\eta} \, dx \, ds + \frac{1}{\varepsilon^\beta} \int_0^t \int_\Omega \mathbb{K}(\theta_\varepsilon, z_\varepsilon) \nabla \theta_\varepsilon \cdot \nabla \eta \, dx \, ds \\ & = \varepsilon \int_\Omega \theta_\varepsilon^0 \eta(0) \, dx + \int_0^t \int_\Omega (\varepsilon^2 \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon) - \varepsilon \theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon)) \eta \, dx \, ds \\ & \quad + \varepsilon \int_0^t \int_\Omega \eta |\dot{z}_\varepsilon| \, dx \, ds + \int_0^t \int_{\partial\Omega} h_\varepsilon \eta \, d\mathcal{H}^{d-1}(x) \, ds + \int_0^t \int_\Omega H_\varepsilon \eta \, dx \, ds \end{aligned} \tag{5.9}$$

1359 for all test functions  $\eta \in H^1(0, T; L^2(\Omega)) \cap C^0(0, T; W^{2,d+\delta}(\Omega))$  (recall that  $|\dot{z}_\varepsilon|$   
1361 denotes the total variation measure of  $z_\varepsilon$ ).

1362 *Remark 5.1* Let us also observe that testing (5.9) by  $\frac{1}{\varepsilon}$  and summing up with (5.8) leads to  
1363 the rescaled total energy equality

$$\begin{aligned} & \frac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon(t)|^2 \, dx + \mathcal{E}_\varepsilon(t, u_\varepsilon(t), z_\varepsilon(t)) + \int_\Omega \theta_\varepsilon(t) \, dx \\ & = \frac{\varepsilon^2}{2} \int_\Omega |\dot{u}_\varepsilon^0|^2 \, dx + \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) + \int_\Omega \theta_\varepsilon^0 \, dx \\ & \quad + \int_0^t \partial_t \mathcal{E}_\varepsilon(s, u_\varepsilon(s), z_\varepsilon(s)) \, ds + \frac{1}{\varepsilon} \int_0^t \int_{\partial\Omega} h_\varepsilon \, d\mathcal{H}^{d-1}(x) \, ds + \frac{1}{\varepsilon} \int_0^t \int_\Omega H_\varepsilon \, dx \, ds. \end{aligned} \tag{5.10}$$

1365 **5.2 A Priori Estimates Uniform with Respect to  $\varepsilon$**

1366 As done in the proof of Theorem 2.6, we shall derive the basic a priori estimates on the  
1367 rescaled solutions  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$  from the total energy equality (5.10). Therefore, it is clear  
1368 that we shall have to assume that the families of data  $(H_\varepsilon)_\varepsilon$  and  $(h_\varepsilon)_\varepsilon$  converge to zero in the  
1369 sense that there exists  $C > 0$  such that for all  $\varepsilon > 0$

$$\int_0^t \int_\Omega H_\varepsilon \, dx \, ds \leq C\varepsilon, \quad \int_0^t \int_{\partial\Omega} h_\varepsilon \, d\mathcal{H}^{d-1}(x) \, ds \leq C\varepsilon. \tag{5.11}$$

1370 Furthermore, we shall suppose that there exists  $f$  such that

$$f_\varepsilon \rightarrow f \quad \text{in } H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*). \tag{5.12}$$

1373 We are now in a position to derive a priori bounds on the rescaled solutions  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$ ,  
1374 uniform with respect to  $\varepsilon > 0$ . These estimates are the time-continuous counterpart of  
1375 the *First–Third a priori estimates* in the proof of Proposition 3.4. Actually, the calcula-  
1376 tions underlying the *Second* and *Third* estimates can be performed only formally, when  
1377 arguing on the energetic formulation of system (5.3). Indeed, these computations are  
1378 based on testing the weak heat equation (5.9) by  $\theta_\varepsilon^{\alpha-1}$ , which is not admissible since  
1379  $\theta_\varepsilon^{\alpha-1} \notin C^0([0, T]; W^{2,d+\delta}(\Omega))$ .

1380 That is why Proposition 5.2 below will be stated not for *all* energetic solutions to the  
1381 rescaled system (5.3), but just for those arising from the discrete solutions to (5.3) constructed  
1382 in Sect. 3.1. More precisely, we shall call “approximable solution” to the rescaled system  
1383 (5.3) any triple obtained in the time-discrete to continuous limit, for which convergences  
1384 (4.1) of Proposition 4.1 hold; in Sect. 4 we have shown that any approximable solution is  
1385 an energetic solution. Now, it can be checked that some of the a priori estimates on the  
1386 discrete solutions in Proposition 3.4 (i.e. those corresponding to (5.14) below) are uniform

with respect to  $\tau$  and  $\varepsilon$  as well. Therefore, Proposition 4.1 ensures that they are inherited by the “approximable” solutions in the limit  $\tau \downarrow 0$ , still uniformly with respect to  $\varepsilon$ .

Nonetheless, to simplify the exposition, in the proof of Proposition 5.2 we will no longer work on the time-discrete scheme but rather develop the calculations directly (and sometimes only formally) on the time-continuous level.

**Proposition 5.2** (A priori estimates) *Assume (2.1)–(2.5), (5.2) with  $\beta > 0$ ,  $(H_\varepsilon)_\varepsilon \subset L^1(0, T; L^1(\Omega)) \cap L^2(0, T; H^1(\Omega)^*)$ ,  $(h_\varepsilon)_\varepsilon \subset L^1(0, T; L^2(\partial\Omega))$  fulfill (5.11), and  $(f_\varepsilon)_\varepsilon \subset H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)^*)$  comply with (5.12). In addition to (2.7), let the family of initial data  $(u_\varepsilon^0, \dot{u}_\varepsilon^0, z_\varepsilon^0, \theta_\varepsilon^0)_\varepsilon$  fulfill*

$$|\mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0)| + \varepsilon \|\dot{u}_\varepsilon^0\|_{L^2(\Omega; \mathbb{R}^d)} + \|\theta_\varepsilon^0\|_{L^1(\Omega)} \leq C \tag{5.13}$$

for a constant  $C$  independent of  $\varepsilon$ . Let  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$  be a family of approximable solutions to system (5.3). Then, there exists a constant  $C > 0$  such that the following estimates hold for all  $\varepsilon > 0$ :

$$\|u_\varepsilon\|_{L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d))} \leq C, \tag{5.14a}$$

$$\varepsilon \|\dot{u}_\varepsilon\|_{L^\infty(0, T; L^2(\Omega; \mathbb{R}^d))} \leq C, \tag{5.14b}$$

$$\mathcal{R}_1(z_\varepsilon(T) - z_\varepsilon^0) \leq C, \tag{5.14c}$$

$$\|z_\varepsilon\|_{L^\infty((0, T) \times \Omega)} \leq 1, \tag{5.14d}$$

$$\|z_\varepsilon\|_{L^\infty(0, T; W^{1,q}(\Omega))} \leq C, \tag{5.14e}$$

$$\|\theta_\varepsilon\|_{L^\infty(0, T; L^1(\Omega))} \leq C, \tag{5.14f}$$

$$\|\nabla \theta_\varepsilon\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^d))} \leq C\varepsilon^{\beta/2}, \tag{5.14g}$$

$$\|\theta_\varepsilon\|_{L^2(0, T; H^1(\Omega))} \leq C, \tag{5.14h}$$

$$\|\theta_\varepsilon\|_{L^p((0, T) \times \Omega)} \leq C \text{ for any } p \in \begin{cases} [1, 8/3] & \text{if } d=3, \\ [1, 3] & \text{if } d=2, \end{cases} \tag{5.14i}$$

with  $\mathcal{R}_1$  from (1.2).

*Sketch of the proof First a priori estimate: ad (5.14a), (5.14b), (5.14c), (5.14d), (5.14e), (5.14f):* Estimate (5.14d) is obvious. Estimate (5.14c) follows from the definition of  $\mathcal{R}_1$ , (2.5a), and (2.7a), and the fact that the functions  $z_\varepsilon(\cdot, x)$  are nonincreasing. We start from the total energy equality (5.10). Also thanks to (5.12), the energies  $\mathcal{E}_\varepsilon$  enjoy the coercivity property (3.33) with constants independent of  $\varepsilon$ . Therefore, relying on the uniform bound (5.12) for  $\dot{f}_\varepsilon$ , and using that  $\theta_\varepsilon > 0$  a.e. in  $(0, T) \times \Omega$  for every  $\varepsilon > 0$ , one can repeat the very same calculations as in the first step of the proof of Proposition 3.4, and conclude that the left-hand side of (5.10) is uniformly bounded from above and from below, whence (5.14a), (5.14b), (5.14e), (5.14f).

*Second and third a priori estimates: ad (5.14g), (5.14h), and (5.14i):* We (formally) test (5.9) by  $\theta_\varepsilon^{\alpha-1}$ , integrate in time, and arrive at the (formally written) analogue of (3.35), viz.

$$\begin{aligned} & \frac{c}{\varepsilon^\beta} \int_0^t \int_\Omega \mathbb{K}(z_\varepsilon, \theta_\varepsilon) \nabla(\theta_\varepsilon^{\alpha/2}) \cdot \nabla(\theta_\varepsilon^{\alpha/2}) \, dx \, ds + \varepsilon^2 \int_0^t \int_\Omega \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon) \theta_\varepsilon^{\alpha-1} \, dx \, ds \\ & + \varepsilon \int_0^t \int_\Omega \theta_\varepsilon^{\alpha-1} |\dot{z}_\varepsilon| \, dx \, ds + \int_0^t \int_{\partial\Omega} h_\varepsilon \theta_\varepsilon^{\alpha-1} \, d\mathcal{H}^{d-1}(x) \, ds + \int_0^t \int_\Omega H_\varepsilon \theta_\varepsilon^{\alpha-1} \, dx \, ds \\ & = \varepsilon \int_0^t \int_\Omega \dot{\theta}_\varepsilon \theta_\varepsilon^{\alpha-1} \, dx \, ds + \varepsilon \int_0^t \int_\Omega \theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon) \theta_\varepsilon^{\alpha-1} \, dx \, ds \doteq I_1 + I_2. \end{aligned} \tag{5.15}$$

1422 As in the proof of Proposition 3.4, we estimate

$$1423 \quad I_1 = \varepsilon \int_\Omega \frac{(\theta_\varepsilon(t))^\alpha}{\alpha} \, dx - \varepsilon \int_\Omega \frac{(\theta_\varepsilon^0)^\alpha}{\alpha} \, dx, \tag{5.16}$$

1424 whereas we estimate  $I_2 = \int \varepsilon \theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon) \theta_\varepsilon^{\alpha-1}$  by

$$1425 \quad I_2 \leq \varepsilon^2 \frac{C_1}{2} \int_0^t \int_\Omega |e(\dot{u}_\varepsilon)|^2 \theta_\varepsilon^{\alpha-1} \, dx \, ds + C \int_0^t \int_\Omega |\theta_\varepsilon|^2 \theta_\varepsilon^{\alpha-1} \, dx \, ds, \tag{5.17}$$

1426 where the constant  $C$  subsumes the norm  $|\mathbb{B}|$  as well. Combining (5.15)–(5.17) and then  
1427 arguing exactly in the same way as in the proof of Proposition 3.4, we end up with the  
1428 analogue of (3.36), i.e.,

$$\begin{aligned} 1429 \quad & \frac{1}{\varepsilon^\beta} \int_0^t \int_\Omega \mathbb{K}(z_\varepsilon, \theta_\varepsilon) \nabla(\theta_\varepsilon^{\alpha/2}) \cdot \nabla(\theta_\varepsilon^{\alpha/2}) \, dx \, ds + \int_\Omega \frac{\varepsilon}{\alpha} (\theta_\varepsilon^0)^\alpha \, dx \leq \int_\Omega \frac{\varepsilon}{\alpha} (\theta_\varepsilon(t))^\alpha \, dx \\ 1430 \quad & + C \int_0^t \int_\Omega \theta_\varepsilon^{\alpha+1}(s) \, dx \, ds, \end{aligned}$$

1431 whence  $\frac{1}{\varepsilon^\beta} \int_0^T \int_\Omega \mathbb{K}(z_\varepsilon, \theta_\varepsilon) \nabla(\theta_\varepsilon^{\alpha/2}) \cdot \nabla(\theta_\varepsilon^{\alpha/2}) \, dx \, dt \leq C$ . From this, with the same arguments  
1432 as in the third step of the proof of Proposition 3.4, cf. (3.41), we infer that

$$1433 \quad \int_0^T \int_\Omega |\nabla \theta_\varepsilon|^2 \, dx \, dt \leq C \varepsilon^\beta,$$

1434 i.e. (5.14g). Then, (5.14h) follows from (5.14g) and (5.14f), via the Poincaré inequality.  
1435 Finally, (5.14i) ensues by interpolation, as in the proof of Proposition 3.4.  $\square$

1436 Observe that in the proof of Proposition 5.2 we have not been able to repeat the calculations  
1437 in the *Fourth and Fifth estimates*, cf. the proof of Proposition 3.4. In particular, from the  
1438 mechanical energy equality (5.8) we have not been able to deduce an estimate for  $\varepsilon^{1/2} e(\dot{u}_\varepsilon)$   
1439 in  $L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))$ , since we cannot bound the term  $\int_0^t \int_\Omega \theta_\varepsilon : e(\dot{u}_\varepsilon) \, dx \, ds$  on the  
1440 right-hand side of (5.8). Therefore, in the proof of our convergence result for vanishing  
1441 viscosity and inertia, Theorem 5.3 below, we shall have to resort to careful arguments in  
1442 order to handle the terms containing  $e(\dot{u}_\varepsilon)$ , in the passage to the limit in the momentum  
1443 equation and mechanical energy equality, cf. (5.30)–(5.33). In particular, differently from  
1444 Proposition 3.4, for a vanishing sequence  $(\varepsilon_n)_n$  the convergences

$$\begin{aligned} & \varepsilon_n e(\dot{u}_{\varepsilon_n}) \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d})) \quad \text{and} \quad \int_0^t \int_\Omega \theta_{\varepsilon_n} : e(\dot{u}_{\varepsilon_n}) \, dx \, ds \rightarrow 0, \\ & \theta_\varepsilon \rightarrow \Theta \text{ strongly in } L^2(0, T) \times \Omega \end{aligned} \tag{5.18}$$

1445 will now be extracted from the weak heat equation (5.9), using integration by parts and the  
1446 information that  $\Theta$  is constant in space. It is in this connection that we need to further assume  
1447 homogeneous Dirichlet boundary conditions for the displacement on the whole boundary  
1448  $\partial\Omega$ , cf. (5.1).

1451 **5.3 Convergence to Local Solutions of the Rate-Independent Limit System**

1452 Let us mention in advance that in Theorem 5.3 we will prove that, up to a subsequence, the  
 1453 functions  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)$  converge to a limit triple  $(u, z, \Theta)$  such that  $\Theta$  is spatially constant.  
 1454 As we will see, the pair  $(u, z)$  fulfills the (pointwise-in-time) *static* momentum balance (i.e.  
 1455 without viscosity and inertia), a semistability condition with respect to the energy  $\mathcal{E}$  arising  
 1456 from  $\mathcal{E}_\varepsilon$  (5.6) in the limit  $\varepsilon \downarrow 0$ , and an energy inequality, where the viscous, the inertial,  
 1457 and the thermal expansion contributions are no longer present. This inequality holds on  $[0, t]$   
 1458 for every  $t \in [0, T]$  in the general case, and on  $[s, t]$  for all  $t \in [0, T]$  and almost every  
 1459  $s \in (0, t)$ , under a further condition on the gradient term in the energy  $\mathcal{E}$ , i.e. that  $q > d$ .  
 1460 Indeed, the three properties (momentum balance, semistability, energy inequality) constitute  
 1461 the notion of *local solution* [41, 58, 65] to the rate-independent system driven by  $\mathcal{R}_1$  and  $\mathcal{E}$ .  
 1462 Observe that, in fact, the spatially constant  $\Theta$  does not appear in these relations, because it  
 1463 contributes with a zero term to the momentum balance.

1464 Moreover, testing the weak heat equation (5.9) with functions  $\eta$  that are constant in space  
 1465 (which is the property of the limit temperature  $\Theta$  by (5.14g)) and taking into account the  
 1466 bounds (5.11), (5.13), (5.14f), and convergence (5.18), we find in the limit relation  $0 = 0$ .  
 1467 This shows that the temporal evolution of  $\Theta$  is irrelevant in the rate-independent limit model.  
 1468 In fact, in order to gain insight into the time evolution of  $\Theta$ , we will perform the limit  
 1469 passage in the heat equation (5.9) rescaled by the factor  $1/\varepsilon$  and tested by  $\eta \in H^1(0, T)$ ,  
 1470 constant in space. In this way, the heat-transfer term involving  $\mathbb{K}_\varepsilon = \frac{1}{\varepsilon^\beta} \mathbb{K}$  will disappear.  
 1471 This will lead to an ODE for the limit function  $\Theta$ , cf. (5.26). Such an ODE involves a *defect*  
 1472 measure  $\mu$ , i.e. a Radon measure on  $[0, T]$  arising in the limit of the viscous dissipation term  
 1473  $\|\varepsilon \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\|_{L^1(\Omega)}$ , see (5.27) below.

1474 In the following proof, notice that Steps 0–3 can be proven for  $\beta > 0$ , while in Step 4  
 1475 we need  $\beta \geq 2$ . Furthermore, the condition that the tensor  $\mathbb{B}$  is constant in space will have a  
 1476 crucial role in handling the thermal expansion term  $\theta_\varepsilon \mathbb{B} : e(\dot{u}_\varepsilon)$  in the rescaled heat equation,  
 1477 cf. (5.32) ahead.

1478 **Theorem 5.3** Assume (2.1)–(2.4), (2.5), (2.8), and, in addition, let (5.1), (5.2) with  $\beta \geq 2$ ,  
 1479 (5.11), and (5.12) be satisfied. Let the initial data  $(u_\varepsilon^0, \dot{u}_\varepsilon^0, z_\varepsilon^0, \theta_\varepsilon^0)$  fulfill (2.7), (5.13),

$$1480 \quad \varepsilon \dot{u}_\varepsilon^0 \rightarrow 0 \quad \text{in } L^2(\Omega; \mathbb{R}^d), \tag{5.19}$$

1481 and suppose that there exist  $u_0 \in H_D^1(\Omega; \mathbb{R}^d)$  and  $z_0 \in \mathcal{Z}$  such that

$$1482 \quad u_\varepsilon^0 \rightharpoonup u_0 \text{ in } H_D^1(\Omega; \mathbb{R}^d), \quad z_\varepsilon^0 \rightharpoonup z_0 \text{ in } \mathcal{Z}, \quad \mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) \rightarrow \mathcal{E}(0, u_0, z_0) \quad \text{as } \varepsilon \downarrow 0, \tag{5.20}$$

1483 with  $\mathcal{E}_\varepsilon$  as in (5.6).

1484 Then, the functions  $(u_\varepsilon, z_\varepsilon, \theta_\varepsilon)_\varepsilon$  converge (up to subsequences) to a triple  $(u, z, \Theta)$  such  
 1485 that

$$1486 \quad u \in L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d)), \quad z \in L^\infty(0, T; W^{1,q}(\Omega)) \cap L^\infty((0, T) \times \Omega) \cap \text{BV}([0, T]; L^1(\Omega)),$$

$$1487 \quad \Theta \text{ is constant in space and } \Theta \in L^p(0, T) \text{ for any } p \in \begin{cases} [1, 8/3] & \text{if } d=3, \\ [1, 3] & \text{if } d=2. \end{cases} \tag{5.21}$$

1488 The pair  $(u, z)$  fulfills the unidirectionality as well as

- 1489 1. the semistability condition (2.12a) for all  $t \in [0, T]$ , with the mechanical energy  $\mathcal{E}$  defined as in (5.6) with  $f_\varepsilon$  replaced by the weak limit  $f$  of the sequence  $(f_\varepsilon)_\varepsilon$ , see (5.12);

1490 2. the weak momentum balance for all  $t \in [0, T]$

1491 
$$\int_{\Omega} \mathbb{C}(z(t))e(u(t)) : e(v) \, dx = \langle f(t), v \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \quad \text{for all } v \in H_D^1(\Omega; \mathbb{R}^d); \quad (5.22)$$

1492 3. the mechanical energy inequality for all  $t \in [0, T]$

1493 
$$\mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z(0) - z(t)) \, dx \leq \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_r \mathcal{E}(r, u(r), z(r)) \, dr; \quad (5.23)$$

1494 If in addition the function  $G$  fulfills the growth condition (2.5d) with  $q > d$ , then  $(u, z)$  also  
 1495 fulfill

1496 
$$\mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z(s) - z(t)) \, dx \leq \mathcal{E}(s, u(s), z(s)) + \int_s^t \partial_r \mathcal{E}(r, u(r), z(r)) \, dr \quad (5.24)$$

1497 for all  $t \in [0, T]$  and for almost all  $s \in (0, t)$ .

1498 Moreover, assume in addition that there exists  $\tilde{H} \in L^1(0, T)$  such that

1499 
$$\frac{1}{\varepsilon} (\|H_\varepsilon\|_{L^1(\Omega)} + \|h_\varepsilon\|_{L^1(\partial\Omega)}) \rightharpoonup \tilde{H} \quad \text{in } L^1(0, T). \quad (5.25)$$

1500 Then,  $\Theta$  fulfills

1501 
$$\begin{aligned} \eta(t) \int_{\Omega} \Theta(t) \, dx - \int_0^t \dot{\eta} \int_{\Omega} \Theta \, dx \, ds - \eta(0) \int_{\Omega} \Theta(0) \, dx \\ = \int_0^t \eta \, d\mu(s) + \int_0^t \eta \int_{\Omega} |\dot{z}| \, dx \, ds + \int_0^t \tilde{H} \eta \, ds \end{aligned} \quad (5.26)$$

1503 for a.a.  $t \in (0, T)$  and for every  $\eta \in H^1(0, T)$  constant in space, with the defect measure  $\mu$   
 1504 given by

1505 
$$\|\varepsilon \mathbb{D}(z_\varepsilon, \theta_\varepsilon) e(\dot{u}_\varepsilon) : e(\dot{u}_\varepsilon)\|_{L^1(\Omega)} \rightarrow \mu \quad \text{in the sense of Radon measures in } [0, T]. \quad (5.27)$$

1506 *Proof Step 0, compactness:* It follows from Proposition 5.2 that for every vanishing sequence  
 1507  $(\varepsilon_n)_n$  there exist a (not relabeled) subsequence and a triple  $(u, z, \Theta)$  as in (5.21) such that  
 1508 the following convergences hold

1509 
$$u_{\varepsilon_n} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d)), \quad (5.28a)$$

1510 
$$\varepsilon_n u_{\varepsilon_n} \overset{*}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(0, T; L^2(\Omega; \mathbb{R}^d)), \quad (5.28b)$$

1511 
$$z_{\varepsilon_n} \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty(0, T; W^{1,q}(\Omega)) \cap L^\infty((0, T) \times \Omega), \quad (5.28c)$$

1512 
$$z_{\varepsilon_n}(t) \rightarrow z(t) \quad \text{in } W^{1,q}(\Omega) \quad \text{for all } t \in [0, T], \quad (5.28d)$$

1513 
$$z_{\varepsilon_n}(t) \rightarrow z(t) \quad \text{in } L^r(\Omega) \quad \text{for all } 1 \leq r < \infty \text{ and for all } t \in [0, T], \quad (5.28e)$$

1514 
$$\theta_{\varepsilon_n} \rightarrow \Theta \quad \text{in } L^2(0, T; H^1(\Omega)) \cap L^p((0, T) \times \Omega) \text{ for all } p \text{ as in (5.14i)}. \quad (5.28f)$$

1516 Indeed, (5.28a) ensues from (5.14a), and it gives, in particular, that  $\varepsilon_n u_{\varepsilon_n} \rightarrow 0$  in  
 1517  $L^\infty(0, T; H_D^1(\Omega; \mathbb{R}^d))$ . Then, convergence (5.28b) directly follows from estimate (5.14b).  
 1518 Convergences (5.28c)–(5.28e) ensue from the very same compactness arguments as in the  
 1519 proof of Proposition 4.1, also using the Helly Theorem. Furthermore, (5.28f) follows from  
 1520 estimates (5.14h)–(5.14i) by weak compactness. Observe that in view of (5.14g) we have  
 1521 that

1522 
$$\nabla \theta_{\varepsilon_n} \rightarrow 0 \quad \text{in } L^2(0, T; L^2(\Omega; \mathbb{R}^d)). \quad (5.29)$$

1523 Therefore, we conclude that  $\nabla\Theta = 0$  a.e. in  $(0, T) \times \Omega$ . Since  $\Theta$  is spatially constant,  
 1524 hereafter we will write it as a function of the sole variable  $t$ .

1525 We now prove the enhanced convergence

$$1526 \theta_{\varepsilon_n} \rightarrow \Theta \text{ in } L^2(0, T; L^2(\Omega)). \tag{5.30}$$

1527 In fact, we use the Poincaré inequality

$$1528 \begin{aligned} & \|\theta_{\varepsilon_n} - \Theta\|_{L^2(0, T; L^2(\Omega))} \\ 1529 & \leq \|\nabla(\theta_{\varepsilon_n} - \Theta)\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^d))} \\ 1530 & + C(\Omega, T) \left| \int_0^T \int_{\Omega} (\theta_{\varepsilon_n} - \Theta) \, dx \, ds \right| \rightarrow 0, \end{aligned}$$

1531 where the gradient term tends to 0 by (5.29), and the convergence of the second term follows  
 1532 from (5.28f).

1533 Finally, let us show that

$$1534 \varepsilon_n e(\dot{u}_{\varepsilon_n}) \rightarrow 0 \text{ strongly in } L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d})). \tag{5.31}$$

1535 Preliminarily, observe that, since  $\mathbb{B}$  and the limit function  $\Theta$  are constant in space, we have  
 1536 by integration by parts

$$1537 \int_0^t \int_{\Omega} \Theta \mathbb{B} : e(\dot{u}_{\varepsilon_n}) \, dx \, ds = \int_0^t \int_{\partial\Omega} \Theta \mathbb{B} \nu \cdot \dot{u}_{\varepsilon_n} \, d\mathcal{H}^{d-1}(x) \, ds - \int_0^t \int_{\Omega} \operatorname{div}(\Theta \mathbb{B}) \cdot \dot{u}_{\varepsilon_n} \, dx \, ds = 0, \tag{5.32}$$

1538 where we used  $\partial_D \Omega = \partial\Omega$ , hence  $\dot{u}_{\varepsilon_n} \in L^2(0, T; H_D^1(\Omega; \mathbb{R}^d))$  implies that  $\dot{u}_{\varepsilon_n} = 0$  a.e. in  
 1539  $(0, T) \times \partial\Omega$ . Using (5.32) in the weak heat equation (5.9) tested by 1 and applying Young’s  
 1540 inequality, we find

$$1541 \begin{aligned} & \varepsilon_n \left( \int_{\Omega} (\theta_{\varepsilon_n}(t) - \theta_{\varepsilon_n}^0) \, dx \right) \\ & \geq \int_0^t \int_{\Omega} [\varepsilon_n^2 \mathbb{D}(z_{\varepsilon_n}, \theta_{\varepsilon_n}) e(\dot{u}_{\varepsilon_n}) : e(\dot{u}_{\varepsilon_n}) - \varepsilon_n (\theta_{\varepsilon_n} - \Theta) \mathbb{B} : e(\dot{u}_{\varepsilon_n})] \, dx \, ds \\ & \geq \int_0^t \int_{\Omega} \varepsilon_n^2 \frac{C_{\mathbb{D}}}{2} |e(\dot{u}_{\varepsilon_n})|^2 \, dx \, ds - C \|\theta_{\varepsilon_n} - \Theta\|_{L^2(0, T; L^2(\Omega))}^2 \end{aligned} \tag{5.33}$$

1542 with  $C = |\mathbb{B}|/2$ . From this, taking into account that  $(\theta_{\varepsilon_n}^0)_n$  is bounded in  $L^1(\Omega)$  by (5.13),  
 1543 estimate (5.14f) for  $(\theta_{\varepsilon_n})_n$ , and convergence (5.30), we conclude that  $\lim_{\varepsilon_n \downarrow 0} \varepsilon_n \|e(\dot{u}_{\varepsilon_n})\|_{L^2(0, T; L^2(\Omega; \mathbb{R}^{d \times d}))} = 0$ , whence (5.31).

1544 In fact, by Korn’s inequality we conclude that

$$1546 \varepsilon_n u_{\varepsilon_n} \rightarrow 0 \text{ in } H^1(0, T; H_D^1(\Omega; \mathbb{R}^d)). \tag{5.34}$$

1547 *Step 1, passage to the limit in the momentum balance (5.7):* Convergence (5.34), joint  
 1548 with the boundedness (2.3e) of the tensor  $\mathbb{D}$ , ensures that the first and the second summands  
 1549 on the left-hand side of (5.7) tend to zero. Arguing as in the proof of Proposition 4.3, we show  
 1550 that for every test function  $v$  in (5.7),  $\mathbb{C}(z_{\varepsilon_n})e(v) \rightarrow \mathbb{C}(z)e(v)$  in  $L^2((0, T) \times \Omega; \mathbb{R}^{d \times d})$ . We  
 1551 combine this with (5.28a) and, also using (5.28f), we pass to the limit in the third term on  
 1552 the left-hand side of (5.7), recalling that the fourth summand converges to zero similarly to  
 1553 (5.32). As for the right-hand side, by (5.13) we have

$$1554 \varepsilon_n^2 \dot{u}_{\varepsilon_n}^0 \rightarrow 0 \text{ in } L^2(\Omega; \mathbb{R}^d), \tag{5.35}$$



1555 hence the first term converges to zero. The second one tends to zero for almost all  $t \in (0, T)$   
 1556 by (5.28b), which in particular gives

$$1557 \quad \varepsilon_n^2 \dot{u}_{\varepsilon_n} \rightarrow 0 \quad \text{in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^d)). \quad (5.36)$$

1558 For the third one, we use (5.12). We thus conclude that (5.22) holds at almost all  $t \in (0, T)$ .

1559 In order to check it at every  $t \in [0, T]$ , we observe that for every  $t \in [0, T]$  from  
 1560 the bounded sequence  $(u_{\varepsilon_n}(t))_n$  (along which convergences (5.28) hold) we can extract  
 1561 a subsequence, possibly depending on  $t$ , weakly converging to some  $\bar{u}(t)$  in  $H_D^1(\Omega; \mathbb{R}^d)$ .  
 1562 Relying on convergence (5.28e) for  $(z_{\varepsilon_n}(t))_n$  and on (5.12) for  $(f_{\varepsilon_n}(t))$ , with the same  
 1563 arguments as above we conclude that  $\int_\Omega \mathbb{C}(z(t))e(\bar{u}(t)) : e(v) \, dx = \langle f(t), v \rangle_{H_D^1(\Omega; \mathbb{R}^d)}$  for  
 1564 all  $v \in H_D^1(\Omega; \mathbb{R}^d)$ . Since this equation has a unique solution, we conclude that  $\bar{u}(t) = u(t)$   
 1565 for almost all  $t \in (0, T)$ , and that the whole sequence  $u_{\varepsilon_n}(t)$  weakly converges to  $\bar{u}(t)$  for  
 1566 every  $t \in [0, T]$ . In this way  $u$  extends to a function defined on  $[0, T]$ , such that

$$1567 \quad u_{\varepsilon_n}(t) \rightharpoonup u(t) \quad \text{in } H_D^1(\Omega; \mathbb{R}^d) \quad \text{for all } t \in [0, T], \quad (5.37)$$

1568 solving (5.22) at all  $t \in [0, T]$ .

1569 *Step 2, enhanced convergences for  $(u_{\varepsilon_n})_n$ :* As a by-product of this limit passage, we also  
 1570 extract convergences (5.39) and (5.38) below for  $(u_{\varepsilon_n})_n$ , which we will then use in the passage  
 1571 to the limit in the semistability and in the mechanical energy inequality. Indeed, we test (5.7)  
 1572 by  $u_{\varepsilon_n}$ , thus obtaining

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_0^t \int_\Omega (\mathbb{C}(z_{\varepsilon_n})e(u_{\varepsilon_n}) - \theta_{\varepsilon_n} \mathbb{B}) : e(u_{\varepsilon_n}) \, dx \, ds \\ & \leq \limsup_{n \rightarrow \infty} \varepsilon_n^2 \int_0^t \int_\Omega |\dot{u}_{\varepsilon_n}|^2 \, dx \, dt - \liminf_{n \rightarrow \infty} \int_0^t \int_\Omega \varepsilon_n \mathbb{D}(z_{\varepsilon_n}, \theta_{\varepsilon_n})e(\dot{u}_{\varepsilon_n}) : e(u_{\varepsilon_n}) \, dx \, ds \\ & \quad + \limsup_{n \rightarrow \infty} \varepsilon_n^2 \int_\Omega \dot{u}_{\varepsilon_n}^0 \cdot u_{\varepsilon_n}^0 \, dx - \liminf_{n \rightarrow \infty} \varepsilon_n^2 \int_\Omega \dot{u}_{\varepsilon_n}(t) \cdot u_{\varepsilon_n}(t) \, dx \\ & \quad + \limsup_{n \rightarrow \infty} \int_0^t \langle f_{\varepsilon_n}, u_{\varepsilon_n} \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds \\ & = 0 + 0 + 0 + 0 + \int_0^t \langle f, u \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \, ds = \int_0^t \int_\Omega \mathbb{C}(z)e(u) : e(u) \, dx \, ds \end{aligned}$$

1574 where the first term in the right-hand side converges to zero thanks to (5.34), the second  
 1575 one by the boundedness of  $\mathbb{D}$ , (5.28a), and (5.34), the third one by (5.35) combined with the  
 1576 boundedness of  $(u_{\varepsilon_n}^0)_n$ , the fourth one by (5.28a) and (5.36). The fifth term passes to the limit  
 1577 by (5.12) and (5.28a). The last identity follows from (5.22). Remark that the second term in  
 1578 the left-hand side converges to zero by (5.28a) and (5.28f), as done for (5.32).

1579 From the above chain of inequalities we thus obtain that

$$1580 \quad \limsup_{n \rightarrow \infty} \int_0^t \int_\Omega \mathbb{C}(z_{\varepsilon_n})e(u_{\varepsilon_n}) : e(u_{\varepsilon_n}) \, dx \, ds \leq \int_0^t \int_\Omega \mathbb{C}(z)e(u) : e(u) \, dx \, ds.$$

1581 Next, we may apply Lemma 4.7 to deduce that  $e(u_{\varepsilon_n})$  strongly converges to  $e(u)$  in  
 1582  $L^2((0, T) \times \Omega; \mathbb{R}^{d \times d})$ , see also Lemma 4.8. Hence, by Korn's inequality, we ultimately  
 1583 infer

$$1584 \quad u_{\varepsilon_n} \rightarrow u \quad \text{in } L^2(0, T; H_D^1(\Omega; \mathbb{R}^d)). \quad (5.38)$$

1585 For later convenience, we observe that, in particular, this yields

$$1586 \int_{\Omega} \mathbb{C}(z_{\varepsilon_n}(t))e(u_{\varepsilon_n}(t)) : e(u_{\varepsilon_n}(t)) \, dx \rightarrow \int_{\Omega} \mathbb{C}(z(t))e(u(t)) : e(u(t)) \, dx \quad \text{for a.a. } t \in (0, T). \tag{5.39}$$

1587  
 1588 *Step 3, passage to the limit in the semistability condition:* In view of the pointwise con-  
 1589 vergences (5.28d)–(5.28e) for  $z_{\varepsilon_n}$  and  $u_{\varepsilon_n}(t) \rightarrow u(t)$  in  $H_D^1(\Omega; \mathbb{R}^d)$  (by (5.38)) for all  
 1590  $t \in [0, T]$ , we may apply the mutual recovery sequence construction from Theorem 4.5 in  
 1591 order to pass to the limit as  $\varepsilon_n \downarrow 0$  in the semistability (5.5). Also taking into account con-  
 1592 vergence (5.12) for  $(f_{\varepsilon_n})_n$ , we conclude that  $(u, z)$  comply with the semistability condition  
 1593 (2.12a) for every  $t \in [0, T]$ .

1594 *Step 4, passage to the limit in the mechanical energy inequality on  $(0, t)$ :* By lower semi-  
 1595 continuity it follows from convergences (5.12), (5.37), (5.28d), and (5.28c) that

$$1596 \liminf_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) \geq \mathcal{E}(t, u(t), z(t)) \quad \text{for all } t \in [0, T]. \tag{5.40}$$

1597 Furthermore, combining (5.12) with (5.28a) we infer that

$$1598 \partial_t \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) = - \langle \dot{f}_{\varepsilon_n}(t), u_{\varepsilon_n} \rangle_{H_D^1(\Omega; \mathbb{R}^d)} \rightarrow - \langle \dot{f}(t), u \rangle_{H_D^1(\Omega; \mathbb{R}^d)} = \partial_t \mathcal{E}(t, u, z) \quad \text{in } L^2(0, T). \tag{5.41}$$

1599 We are now in a position to pass to the limit in the mechanical energy inequality (5.8). We  
 1600 notice that the first term on the left-hand side of (5.8) is positive. For the second one we use  
 1601 (5.40) and the third one converges to  $\int_{\Omega} (z(0) - z(t)) \, dx$  by (5.28e). The fourth one, given  
 1602 by

$$1603 \int_0^t \int_{\Omega} (\varepsilon \mathbb{D}(z_{\varepsilon}, \theta_{\varepsilon})e(\dot{u}_{\varepsilon}) - \theta_{\varepsilon} \mathbb{B}) : e(\dot{u}_{\varepsilon}) \, dx \, ds,$$

1604 is bounded from below by

$$1605 - \int_0^t \int_{\Omega} \theta_{\varepsilon_n} \mathbb{B} : e(\dot{u}_{\varepsilon_n}) \, dx \, ds.$$

1606 We can again argue as in (5.32)

$$1607 \begin{aligned} & \int_0^t \int_{\Omega} \theta_{\varepsilon_n} \mathbb{B} : e(\dot{u}_{\varepsilon_n}) \, dx \, ds \\ &= \int_0^t \int_{\partial\Omega} \theta_{\varepsilon_n} \mathbb{B} \nu \cdot \dot{u}_{\varepsilon_n} \, d\mathcal{H}^{d-1}(x) \, ds - \int_0^t \int_{\Omega} \operatorname{div}(\theta_{\varepsilon_n} \mathbb{B}) \cdot \dot{u}_{\varepsilon_n} \, dx \, ds \\ &= 0 - \int_0^t \int_{\Omega} \operatorname{div}(\theta_{\varepsilon_n} \mathbb{B}) \cdot \dot{u}_{\varepsilon_n} \, dx \, ds, \end{aligned} \tag{5.42}$$

1608 where we have used that  $\dot{u}_{\varepsilon_n}$  complies with homogeneous Dirichlet conditions on  $\partial_D \Omega = \partial \Omega$ ,  
 1609 and then observe that

$$1610 \|\operatorname{div}(\theta_{\varepsilon_n} \mathbb{B}) \cdot \dot{u}_{\varepsilon_n}\|_{L^1((0, T) \times \Omega)} = \|\varepsilon_n^{-1} \operatorname{div}(\theta_{\varepsilon_n} \mathbb{B}) \cdot \varepsilon_n \dot{u}_{\varepsilon_n}\|_{L^1((0, T) \times \Omega)} \leq C \|\varepsilon_n \dot{u}_{\varepsilon_n}\|_{L^2((0, T) \times \Omega)} \rightarrow 0, \tag{5.43}$$

1611 due to estimate (5.14g) and (5.34). Notice that here we have used the fact that  $\beta \geq 2$ ; this is  
 1612 the only point where we use such requirement. As for the right-hand side, we observe that the  
 1613 first term converges to zero by (5.19). The second term passes to the limit by the convergence  
 1614 (5.20) for the initial energies, and the third one by (5.41).

Therefore we conclude that

$$\mathcal{E}(t, u(t), z(t)) + \int_{\Omega} (z(0) - z(t)) \, dx \leq \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t \mathcal{E}(s, u, z) \, ds.$$

Step 5, case  $q > d$ , enhanced convergence for  $(z_{\varepsilon_n})$  and energy convergence: We now prove that

$$\lim_{n \rightarrow \infty} \int_{\Omega} G(z_{\varepsilon_n}(t), \nabla z_{\varepsilon_n}(t)) \, dx = \int_{\Omega} G(z(t), \nabla z(t)) \, dx \quad \text{for a.a. } t \in (0, T), \quad (5.44)$$

which, combined with (5.12), (5.39) and (5.38) will yield the pointwise convergence of the energies

$$\lim_{n \rightarrow \infty} \mathcal{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), z_{\varepsilon_n}(t)) = \mathcal{E}(t, u(t), z(t)) \quad \text{for a.a. } t \in (0, T). \quad (5.45)$$

We obtain (5.44) testing semistability (5.5) by a suitable recovery sequence  $(\tilde{z}_{\varepsilon_n})_n$  for  $\tilde{z} = z(t)$ ; in the following lines, to avoid overburdening notation we will drop  $t$  when writing  $z_{\varepsilon_n}(t), z(t), u_{\varepsilon_n}(t)$ , and  $u(t)$ . Following [44, Lemma 3.9], where the recovery sequence right below has been introduced to deduce energy convergence, we set

$$\tilde{z}_{\varepsilon_n} := \max\{0, z - \|z_{\varepsilon_n} - z\|_{L^\infty(\Omega)}\}.$$

Now, for  $q > d$  the convergence  $z_{\varepsilon_n} \rightarrow z$  in  $W^{1,q}(\Omega)$ , see (5.28d), implies  $z_{\varepsilon_n} \rightarrow z$  in  $L^\infty(\Omega)$ . Thus, it can be checked that

$$\tilde{z}_{\varepsilon_n} \rightarrow z \text{ strongly in } W^{1,q}(\Omega). \quad (5.46)$$

Since  $\tilde{z}_{\varepsilon_n} \leq z_{\varepsilon_n}$ , we can choose it as a test function in (5.5). The term  $-(f_{\varepsilon_n}(t), u_{\varepsilon_n})_{H_D^1(\Omega; \mathbb{R}^d)}$  on both sides of the inequality cancels out and we deduce

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left( \int_{\Omega} \left( \frac{1}{2} \mathbb{C}(z_{\varepsilon_n}) e(u_{\varepsilon_n}) : e(u_{\varepsilon_n}) + G(z_{\varepsilon_n}, \nabla z_{\varepsilon_n}) \right) dx \right) \\ &= \limsup_{n \rightarrow \infty} \left( \int_{\Omega} \frac{1}{2} \mathbb{C}(\tilde{z}_n) e(u_{\varepsilon_n}) : e(u_{\varepsilon_n}) \, dx + \int_{\Omega} G(\tilde{z}_{\varepsilon_n}, \nabla \tilde{z}_{\varepsilon_n}) \, dx \right) \leq I_1 + I_2, \end{aligned} \quad (5.47)$$

where

$$I_1 := \lim_{n \rightarrow \infty} \int_{\Omega} \frac{1}{2} \mathbb{C}(\tilde{z}_n) e(u_{\varepsilon_n}) : e(u_{\varepsilon_n}) \, dx \leq \int_{\Omega} \frac{1}{2} \mathbb{C}(z) e(u) : e(u) \, dx,$$

combining (5.46) with (5.38) via the Lebesgue Theorem. It follows from (5.46), condition (2.5d) on the growth of  $G$  from above, and again the Lebesgue Theorem that

$$I_2 := \lim_{n \rightarrow \infty} \int_{\Omega} G(\tilde{z}_{\varepsilon_n}, \nabla \tilde{z}_{\varepsilon_n}) \, dx = \int_{\Omega} G(z, \nabla z) \, dx. \quad (5.48)$$

Taking into account the previously proven (5.39), from (5.47)–(5.48) we ultimately infer

$$\limsup_{n \rightarrow \infty} \int_{\Omega} G(z_{\varepsilon_n}, \nabla z_{\varepsilon_n}) \, dx \leq \int_{\Omega} G(z, \nabla z) \, dx,$$

whence (5.44).

Step 6, case  $q > d$ , passage to the limit in the mechanical energy inequality on  $(s, t)$ : We now pass to the limit in (5.8) written on an interval  $[s, t] \subset [0, T]$ , for every  $t \in [0, T]$  and almost all  $s \in (0, t)$ . Clearly, it is sufficient to discuss the limit passage on the right-hand side of (5.8), evaluated at  $s$ . The first summand tends to zero for almost all  $s$ , thanks to (5.34), which in particular ensures  $\varepsilon_n \dot{u}_{\varepsilon_n}(s) \rightarrow 0$  in  $L^2(\Omega; \mathbb{R}^d)$  for almost all  $s \in (0, T)$ . The second

1647 term passes to the limit by (5.45), while the third and the fourth ones can be dealt with by  
 1648 (5.42)–(5.43) and (5.41), respectively.

1649 *Step 7, limit passage in the rescaled heat equation and temporal evolution of  $\Theta$ :* We con-  
 1650 sider the heat equation (5.9) rescaled by the factor  $1/\varepsilon$  and tested by  $\eta \in H^1(0, T)$ , constant  
 1651 in space, which results in

$$\begin{aligned} & \eta(t) \int_{\Omega} \theta_{\varepsilon}(t) \, dx - \int_0^t \dot{\eta} \int_{\Omega} \theta_{\varepsilon} \, dx \, ds \\ &= \eta(0) \int_{\Omega} \theta_{\varepsilon}^0 \, dx + \int_0^t \eta \int_{\Omega} (\varepsilon \mathbb{D}(z_{\varepsilon}, \theta_{\varepsilon})e(\dot{u}_{\varepsilon}) - \theta_{\varepsilon} \mathbb{B}) : e(\dot{u}_{\varepsilon}) \, dx \, ds \\ & \quad + \int_0^t \eta \int_{\Omega} |\dot{z}_{\varepsilon}| \, dx \, ds + \frac{1}{\varepsilon} \int_0^t \eta \int_{\partial\Omega} h_{\varepsilon} \, d\mathcal{H}^{d-1}(x) \, ds + \frac{1}{\varepsilon} \int_0^t \eta \int_{\Omega} H_{\varepsilon} \, dx \, ds. \end{aligned} \tag{5.49}$$

1653 From the mechanical energy balance (5.8) we deduce by a comparison argument that

$$\begin{aligned} & \varepsilon \int_0^T \int_{\Omega} \mathbb{D}(z_{\varepsilon}, \theta_{\varepsilon})e(\dot{u}_{\varepsilon}) : e(\dot{u}_{\varepsilon}) \, dx \, ds \leq C, \text{ hence also} \\ & \varepsilon \int_0^T \eta \int_{\Omega} \mathbb{D}(z_{\varepsilon}, \theta_{\varepsilon})e(\dot{u}_{\varepsilon}) : e(\dot{u}_{\varepsilon}) \, dx \, ds \leq C\|\eta\|_{\infty} \end{aligned}$$

1656 for every  $\eta \in H^1(0, T)$ , taking into account (5.12), (5.13) as well as (5.18). This allows us to  
 1657 conclude that there exists a Radon measure  $\mu$  such that (5.27) holds. A comparison argument  
 1658 in (5.49) leads to

$$\left| \varepsilon \int_0^t \eta \int_{\Omega} \theta_{\varepsilon} \mathbb{B} : e(\dot{u}_{\varepsilon}) \, dx \, ds \right| \leq C\|\eta\|_{\infty},$$

1660 also in view of the bounds (5.11), (5.14i) and (5.14c). Since  $\eta$  is constant in space, inte-  
 1661 gration by parts and an argument along the lines of Step 4 yield that indeed  $\int_0^t \int_{\Omega} \eta \theta_{\varepsilon} \mathbb{B} :$   
 1662  $e(\dot{u}_{\varepsilon}) \, dx \, ds \rightarrow 0$ . Moreover, the third convergence in (5.18) implies that  $\theta_{\varepsilon}(t) \rightarrow \Theta(t)$  in  
 1663  $L^2(\Omega)$  for a.e.  $t \in (0, T)$ . Using (5.25), we finally pass to the limit in (5.49) and find that  $\Theta$   
 1664 satisfies (5.26). □

1665 **Acknowledgements** This work has been supported by the Italian Ministry of Education, University, and  
 1666 Research through the PRIN 2010-11 grant for the project *Calculus of Variations*, by the European Research  
 1667 Council through the two Advanced Grants *Quasistatic and Dynamic Evolution Problems in Plasticity and Frac-*  
 1668 *ture (290888)* and *Analysis of Multiscale Systems Driven by Functionals (267802)*, and by GNAMPA (Gruppo  
 1669 Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di  
 1670 Alta Matematica) through the project *Modelli variazionali per la propagazione di fratture, la delaminazione*  
 1671 *e il danneggiamento*. G.L. acknowledges also the support of the University of Würzburg, of the DFG grant  
 1672 SCHL 1706/2-1, of SISSA, of the University of Vienna, and of the FWF project P27052. This paper was  
 1673 submitted on October 24, 2014; the first referee report was received by the authors on December 6, 2017.

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Revised Proof