Scaling in fracture mechanics by BaŽant law: From finite to linearized elasticity

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Scaling in fracture mechanics by Bažant's law: from finite to linearized elasticity.

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Abstract. We consider crack propagation in brittle non-linear elastic materials in the context of quasi-static evolutions of energetic type. Given a sequence of self-similar domains $n\Omega$ on which the imposed boundary conditions scale according to Bažant's law, we show, in agreement with several experimental data, that the corresponding sequence of evolutions converges (for $n \to \infty$) to the evolution of a crack in a brittle linear-elastic material.

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1 Introduction

It is well known that in fracture mechanics the size effect plays an important role [4]. Scaling by small factors the size of a specimen is often enough to see a clear size dependence in the experimental data while scaling by large factors is sometimes necessary for real life applications.

Mathematical models, to be fully consistent with physics, should ideally predict the correct mechanical behaviour at every scale; in practice it is rarely so, since each model is applicable in a certain range of sizes. Making variations in the scale allows then to understand the relationship between different models which "live" in different ranges; this is indeed the spirit of our work and of several recent results in applied analysis, e.g. [11, 22, 18, 13] and many others.

In fracture mechanics, the literature offers several theories and laws to explain size effects, e.g. [5, 4], depending of course on the type of material (ductile, quasi-fragile or brittle) but also on the type of model. Here, following [24, 4], we will focus on the transition between finite and linearized elasticity in brittle fracture as the size of the domain becomes large, and ideally tends to infinity. More precisely, consider first a two dimensional domain Ω and a time depending boundary condition $u(t,\cdot) = g(t,\cdot)$ (for the displacement u) on $\partial_D\Omega \subset \partial\Omega$, assume that the material is brittle and hyper-elastic with a non-linear constitutive law (for the precise assumptions on the energy density we refer to section §2.2). Under these assumptions a quasi-static evolution is given by a crack $K(t)$ together with a displacement $u(t,\cdot)$ which satisfy (global) stability and energy balance, in the sense of [21] (for the rigorous definition, including the right spaces, see §2.3). Next, consider a family of domains $n\Omega$ for $n \in \mathbb{N}$ together with the boundary conditions $w(t,x) = n^{1/2} g(t, x/n)$. Let $H_n(t)$ and $w_n(t,\cdot)$ denote respectively the crack and the displacement of a quasi-static evolution in $n\Omega$. We are interested in understanding the behaviour of this system for n tending to ∞. In this picture the limit of the domains $n\Omega$ would be the infinite plane, usually identified with the complex plane $\mathbb{C}$ in the classical theories of fracture. In our asymptotic analysis it is instead technically more convenient to re-scale the domains $n\Omega$ back to the "reference" domain Ω. Accordingly we will re-scale the boundary conditions, the cracks and also the energy density. After performing this change of variable we are led to a sequence $K_n(t) = H_n(t)/n$ of cracks together with a sequence of rescaled displacements $u_n(t,x) = n^{-1/2} w_n(t, nx)$ which satisfy the boundary condition $u_n(t,\cdot) = g(t,\cdot)$ on $\partial_D \Omega$, (global) stability, and energy balance for the re-scaled energy. In this setting we pass to the limit with respect to $n \to \infty$. First, we prove that (up to subsequences) the sets $K_n(t)$ converge in the sense of Hausdorff to a set $K(t)$ while the displacements $u_n(t,\cdot)$ together with $Du_n(t,\cdot)$ converge strongly in L^2 to $u(t,\cdot)$ and $Du(t,\cdot)$ respectively (this is not a strong convergence in a Sobolev space since the domains $\Omega \setminus K_n(t)$ are changing with n). Second, we show that $K(t)$ and $u(t,\cdot)$ provide a quasi-static evolution which satisfies (global) stability and energy balance in the setting of LEFM, i.e. for a linear elastic energy density in the bulk together with Griffith's energy on the crack.

Let us comment on the scaling of the boundary conditions, which is crucial to recover the linear elastic energy in the limit. In fracture mechanics, considering two dimensional brittle solids, the natural scaling of the linear theory is exactly the one adopted here. With this choice the linear elastic energy and the dissipated energy both scale by a factor n and thus their balance, which drives the propagation of the crack, remains scale-invariant. When finite elasticity is employed it is less evident, on the theoretical level, that the "natural" scaling is still the same, since the elastic density may contain different terms which do not behave in the same way. On the other hand, experimental data suggest that the scaling is the same and, most importantly, that for large sizes the response of the non-linear system is very close to the response predicted by the linear theory. In the mechanical literature this is known as Bažant scaling law [4] and actually applies to a large number of materials with different mechanical properties, including for instance metals and concrete. This is precisely what is proved in [24] (for a pre-defined crack path) and here (for a wider class of admissible cracks). Note that our asymptotic analysis shows that the whole evolution converges while experimental results typically show convergence of the nominal strength [4].

Now, let us briefly discuss some technical aspects about the family of cracks and the evolution law. In this work, as in [11, 1], a crucial technical role is played by the so-called quantitative rigidity lemma of [18]; this result is the keystone for the boundedness in L^2 of the displacement gradients and applies in Lipschitz domains. First, note that our domains are of the type $\Omega \setminus K(t)$ and thus in general they are not Lipschitz, independently of the regularity of $K(t)$. However, if $\Omega \setminus K(t)$ can be decomposed into the union of a finite number of suitable Lipschitz sets then the rigidity estimate is still true. There is however a further difficulty: often in the proofs we deal with sequences of domains, say $\Omega \setminus K_n(t)$, for which it is crucial to have uniform constants in several inequalities, e.g. Poincaré, Korn and of course the quantitative rigidity. For this reason the family of admissible cracks should be crafted *ad hoc* and cannot be as general as that employed for instance in [12]. Despite the technical constraint, the family of admissible cracks is still quite general and includes the case of several unknown crack paths (its precise definition is quite lengthy and is given in §2.3). For the evolution law we employ quasi-static evolutions of energetic type [16, 21], i.e. those which satisfy (global) stability and energy balance; the advantage of this notion is its applicability under low regularity assumptions, and thus for general cracks, while the disadvantage could be its behaviour in time. Quasi-static evolutions of energetic type may indeed develop discontinuities in time for which the energy is not decreasing along any path connecting the right and left configuration. However, see e.g. [12], when the system evolves continuously in time the evolution actually satisfies Griffith's criterion and thus it seems natural to accept these evolutions at least until a discontinuity appears. Unfortunately, it is almost impossible (at least at the current stage) to provide estimates of the jump time. An analogous convergence result for quasi-static evolutions by critical points of the energy, instead of minimizers, can be found [24]; this type of evolutions is more demanding in terms of regularity of the cracks, indeed the analysis therein is restricted to the case of a straight crack.

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2 Mechanical setting

2.1 Reference configuration

The reference configuration of the elastic body is given by the closure $\overline{\Omega}$ of a bounded, connected open subset Ω of $\mathbb{R}^2$ with Lipschitz boundary. We fix a relatively open subset $\partial_D\Omega$ of $\partial\Omega$ on which the displacements are prescribed. Deformation and displacement (respectively) are denoted by $v, u : \Omega \to \mathbb{R}^2$; thus we have

$$v = \mathrm{id} + u\,, \qquad Dv = I + Du\,.$$

2.2 Energy densities

In the non-linear setting we will assume that the stored energy density $W : \mathbb{R}^{2\times2} \to [0,+\infty]$ is polyconvex, of class C^2 in $\mathbb{R}^{2\times2}_+$ (the subset of matrices with positive determinant) and satisfies the following conditions (cf. [24, 10])

$$W(F) = +\infty \quad \text{if } \det(F) \le 0, \tag{1}$$
$$W(F) = W(Q\,F) \quad \text{for every } Q \in SO(2), \tag{2}$$
$$W(F) = 0 \Leftrightarrow F \in SO(2), \tag{3}$$
$$W(F) \ge C|\sqrt{F^TF} - I|^2 = Cd^2(F, SO(2)), \tag{4}$$
$$|F|^2 + |\mathrm{det}F|^q \le C(W(F)+1) \quad \text{for } q > 1, \tag{5}$$
$$|F^T DW(F)| + |F^T D^2W(F)[FH]||H|^{-1} \le C(W(F)+1), \tag{6}$$

where (4) holds in a neighborhood of $SO(2)$. Let us recall that in the two dimensional setting the polyconvexity of W means that there exists a convex function G such that $W(F) = G(F, \det(F))$ for all $F \in \mathbb{R}^{2\times2}$. Moreover, (1) and (2) mean respectively that the energy is orientation preserving and frame invariant, (3) means that there is a unique well while (4) means, roughly speaking, that the energy "measures" the distance from rotations (at least in a neighborhood of the well). Finally, (5) and (6) are standard coercivity conditions, which will appear as natural bounds in some proofs.

In the linearized setting, we will denote by $\boldsymbol{\varepsilon}(u) = (Du^T + Du)/2$ the linear strain and by $\boldsymbol{\sigma}(u) = \mathbf{C}\boldsymbol{\varepsilon}(u)$ the linear stress, where $\mathbf{C} = D^2W(I)$ is the elasticity tensor.

We shall use in the sequel the estimates given in the following lemmas.

Lemma 2.1. *Under the above hypotheses*

$$|DW(A)\cdot BA| \le C|B|(W(A)+1)\,, \tag{7}$$

for some positive constant C, where the symbol $\cdot$ indicates the scalar product between matrices. Moreover there exists $\gamma \in (0,1)$ such that

$$W(AB) + W(BA) \le C(W(A)+1) \tag{8}$$

for every $B \in \mathbb{R}^{2\times 2}_+$ *with* $|B - I| \le \gamma$*. Finally, there exists* C*, such that*

$$|DW(A)A^T| \le C(W(A) + |A - I| + |A - I|^2)\,. \tag{9}$$

Proof. We have

$$|DW(A)\cdot BA| = |\mathrm{tr}(BA\,DW^T(A)| \le C|BA\,DW^T(A)| \le C|B|\,|A\,DW^T(A)| \le C|B|\,(W(A)+1)\,,$$

where the last inequality follows from (6). The second estimate follows from [3, Lemma 2.5], while the last one is a direct consequence of [24, Lemma 2.5]. □

Lemma 2.2. *Let* $W_n(E) = nW(I + n^{-1/2}E)$*. Then* $W_n(E) \to \frac{1}{2}\mathbf{C}E\cdot E = \frac{1}{2}\mathbf{C}E_{sym}\cdot E_{sym}$ *(where* $E_{sym} = (E^T + E)/2$ *denotes the symmetric part). Moreover, if* $|E| \le C$ *then* $W_n(E) \le C'$ *for* $n \gg 1$*.*

Proof. By Taylor expansion (for $n \gg 1$) for some $0 \le s \le 1$ it holds

$$W(I + n^{-1/2}E) = W(I) + n^{-1/2}DW(I)\cdot E + \tfrac{1}{2}n^{-1}D^2W(I + sn^{-1/2}E)E\cdot E.$$

By (3) and (4) for n sufficiently large

$$nW(I + n^{-1/2}E) = \tfrac{1}{2}D^2W(I + sn^{-1/2}E)E\cdot E.$$

As W is of class C^2 in a neighborhood of $SO(2)$ we get

$$W_n(E) = nW(I + n^{-1/2}E) = \tfrac{1}{2}D^2W(I + sn^{-1/2}E)E\cdot E \to \tfrac{1}{2}\mathbf{C}E\cdot E\,.$$

If $|E| \le C$ then for $n \gg 1$ we have $d(I + n^{-1/2}E, SO(2)) \ll 1$, therefore $W_n(E) \le C^*|E|^2$ and hence we get the boundedness of $W_n(E)$. □

2.3 Cracks

Let us introduce now the family $\mathcal{K}(\Omega)$ of admissible cracks. Given an initial crack K_0 we consider its evolution along regular paths inside Ω. More precisely, following [20], we assume that the initial crack K_0 is composed of M closed nondegenerate arcs of $C^{1,1}$-curves $K_0^1,\dots,K_0^M$, without self-intersections, and that each curve is contained in $\overline{\Omega}$. We assume further that $\partial_D\Omega \setminus K_0$ has positive Hausdorff measure and that it is the union of finitely many connected components. The curves $K_0^1,\dots,K_0^M$ are assumed to be pairwise disjoint unless their initial points coincide, and to have at least one end-point in Ω. Furthermore we assume that $\Omega \setminus K_0$ can be written as finite union of Lipschitz domains.

We consider as admissible cracks all possible extensions of K_0 along regular non interacting arcs, with a uniform control on the geometry made precise in the following definition, already employed in [20].

Definition 2.3. *Given* $\eta > 0$*,* $\mathcal{K}(\Omega)$ *is the class of sets* $K = K^1 \cup \dots \cup K^M$*, where each* K^m *is a closed arc of curve of class* $C^{1,1}$*, such that*

(a) $K^m \supset K_0^m$*,* $K^m \setminus K_0^m \subset\subset \Omega$*, and* $K^m \cap K^h = K_0^m \cap K_0^h$ *for every* $h \ne m$*;*

(b) for every point $x \in K^m \setminus K_0^m$ *there exist two open balls* $C_1, C_2 \subset \Omega$ *of radius* η*, such that* $(C_1 \cup C_2) \cap (K^m \cup \partial\Omega) = \emptyset$ *and* $\overline{C}_1 \cap \overline{C}_2 = \{x\}$*;*

(c) for every point $x \in K^m \setminus K_0^m$ *the open ball* C_3 *of radius* $2\,\eta$ *centered at* x *satisfies* $C_3 \cap K^h = \emptyset$ *for every* $h \ne m$*.*

We fix the value of the parameter η in such a way that for every $m = 1,\dots,M$ the curvature of K_0^m is controlled from above by $\frac{1}{\eta}$ at a.e. point, and the class $\mathcal{K}(\Omega)$ is not empty.

This class was introduced in [20] to which we refer for the proofs of the properties of $\mathcal{K}(\Omega)$ that we will need in the sequel. First let us remark that under these hypotheses, it is easy to see that there exist two constants $L, D > 0$ depending only on η, Ω, and K_0, such that $\mathcal{H}^1(K) \le L$ and $\mathrm{dist}(K \setminus K_0, \partial\Omega) \ge D$ for every $K \in \mathcal{K}(\Omega)$.

The next compactness result was proved in [20, Proposition 2.9]. We recall it here for reader's convenience.

Proposition 2.4. *For every sequence $\{K_n\} \subset \mathcal{K}(\Omega)$ there exist a subsequence (not relabelled) and a set $K \in \mathcal{K}(\Omega)$ such that K_n^m converges to K^m with respect to the Hausdorff distance and $\mathcal{H}^1(K_n^m) \to \mathcal{H}^1(K^m)$ for every m.*

We just recall briefly the main idea of the proof. Given a sequence in $\mathcal{K}(\Omega)$ the arc-length parametrizations of each set K^m converge to the parametrization of a $C^{1,1}$ curve, thanks to the uniform bound on the curvatures. By properties (b) and (c) above, the M limit curves have no self-intersections and none of them intersects the other ones, (except possibly at some of the initial points of the components of K_0); therefore, the limit crack belongs to $\mathcal{K}(\Omega)$. Note also that the convergence of the lengths is a consequence of the regularity of the curves.

Let us also recall that for $K \in \mathcal{K}(\Omega)$, the space $W^{1,\infty}(\Omega \setminus K; \mathbb{R}^2)$ is dense in $H^1(\Omega \setminus K; \mathbb{R}^2)$.

The next Lemma provides a useful way to represent the set $\Omega \setminus K$ as union of finitely many Lipschitz subsets.

Lemma 2.5. *For every $K \in \mathcal{K}(\Omega)$ there exist open connected Lipschitz sets $\Omega_i \subset \Omega$, for $i = 0, \dots, N$, with $\Omega \setminus K = \cup_{i=0}^N \Omega_i$ such that $\mathcal{H}^1(\partial\Omega_0 \cap (\partial_D\Omega \setminus K)) > 0$ and $|\Omega_0 \cap \Omega_i| > 0$ for each $i = 1, \dots, N$.*

Proof. Let us first illustrate the idea of the proof in the simpler case in which $K \subset \Omega$ can be written as $K = K^1 \cup \cdots \cup K^M$ with K^m pairwise disjoint arcs of $C^{1,1}$ curves satisfying conditions (b) and (c) in Definition 2.3. Then, for $r \ll \eta$ the one-sided r-neighborhoods

$$\Omega_{m,r} = \{x + s\nu(x) : x \in K^m,\ s \in [0, r]\}$$

are disjoint, well contained in Ω and Lipschitz continuous.

We define $\Omega_m = \Omega_{m,r}$ (for r sufficiently small) and let $\Omega_0 = \Omega \setminus \bigcup_{m=1}^M \Omega_{m,r/2}$. Then Ω_0 is still a Lipschitz set, $\Omega \setminus K = \bigcup_{m=0}^M \Omega_m$, $\mathcal{H}^1(\partial\Omega_0 \cap (\partial_D\Omega \setminus K)) > 0$ and $|\Omega_0 \cap \Omega_m| > 0$, as required.

In the general case, the initial cracks K_0^m can intersect the boundary $\partial\Omega$ and other branches K_0^j for $j \neq m$, therefore the one-sided neighborhoods used above could be either not contained in Ω or not disjoint. Anyway, since $\Omega \setminus K_0$ can be written as finite union of Lipschitz sets, we can still define the sets Ω_m. In this case their definition is formally more involved but essentially not different from the previous one, we just give an idea of the proof. For each K^m it is possible to find a suitable couple of one-sided neighborhoods in Ω, say $\Omega'_m \subset \Omega_m$, in such a way that Ω_m and $\Omega_0 = \Omega \setminus \cup_{m=1}^M \Omega'_m$ are connected, Lipschitz sets. Clearly the union of the Ω_m gives $\Omega \setminus K$ and $\Omega_0 \cap \Omega_m = \Omega_m \setminus \Omega'_m$ can be choosen of positive Lebesgue measure. If the sets Ω'_m are sufficiently small then we will also have that $\mathcal{H}^1(\partial\Omega_0 \cap (\partial_D\Omega \setminus K)) > 0$. $\square$

For the sake of simplicity in the next sections we will present the proofs in the case $M = 1$, i.e. the crack is represented by a single arc of curve. By standard localization arguments they can be extended to the general case of finitely many arcs.

2.4 Boundary conditions and admissible deformations

For technical reasons, related to the admissible variations in non-linear elasticity, it is convenient to use the multiplicative decomposition of the deformation v employed in [17] and [10]. To this end, we assume that the prescribed boundary deformation $\Psi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is of class C^2 and satisfies the following properties: for each $t \in [0, T]$ the map $x \mapsto \Psi(t, x)$, in the sequel denoted by Ψ_t, is invertible and the function $(t, y) \mapsto (\Psi_t)^{-1}(y)$ is of class C^2; moreover, the functions $D\Psi_t$, $D\Psi_t^{-1}$, $\dot{\Psi}_t$, $D\dot{\Psi}_t$, and $D\dot{\Psi}_t^{-1}$ are bounded uniformly with respect to $t \in [0, T]$ and $x \in \mathbb{R}^2$. Under these assumptions, the set of admissible deformations associated to a crack $K \in \mathcal{K}(\Omega)$ is given by

$$\mathcal{A}(t, K) = \{v \in H^1(\Omega \setminus K; \mathbb{R}^2) : v = \Psi_t \text{ on } \partial_D\Omega \setminus K\}.$$

Note that, being Ψ_t Lipschitz continuous, we can write the bound $|\Psi_t(z)| \leq C(|z| + 1)$ for $C > 0$ independent of time. Finally we impose a uniform bound on the elastic energy of the prescribed boundary deformation, assuming that there exists a constant $C > 0$ such that $\int_\Omega W(D\Psi_t(x))\, dx \leq C$ for every $t \in [0, T]$. This, together with the regularity of Ψ, implies in particular that $\det D\Psi_t(x) > 0$ for every $x \in \Omega$.

Throughout the paper, given a function $v \in H^1(\Omega \setminus K; \mathbb{R}^2)$ for some $K \in \mathcal{K}(\Omega)$, we always extend Dv to Ω by setting $Dv = 0$ a.e. on K. Note that, however, Dv is the distributional Jacobian matrix of v only in $\Omega \setminus K$, and, in general, it does not coincide in Ω with the distributional Jacobian matrix of an extension of v.

3 Energetic evolution for non-linear elasticity

Let $b \in W^{1,\infty}([0,T]; L^2(\Omega; \mathbb{R}^2))$ represent the applied body forces. Then the bulk energy $\mathcal{E} : [0,T] \times L^2(\Omega; \mathbb{R}^2) \times \mathcal{K}(\Omega) \to [0, +\infty]$ is defined as

$$\mathcal{E}(t, v, K) = \begin{cases} \displaystyle\int_\Omega W(Dv)\,dx + \int_\Omega b(t) v\,dx & \text{if } v \in \mathcal{A}(t, K), \\ +\infty & \text{otherwise.} \end{cases}$$

The dissipation distance $\mathcal{D} : \mathcal{K}(\Omega) \times \mathcal{K}(\Omega) \to [0, +\infty]$ is

$$\mathcal{D}(K_2, K_1) = \begin{cases} \mathcal{H}^1(K_2 \setminus K_1) & \text{if } K_2 \supseteq K_1, \\ +\infty & \text{otherwise.} \end{cases}$$

We consider here quasistatic evolutions of global minimizers, i.e., energetic evolutions in the sense of Mielke [21].

Definition 3.1. *We say that $t \mapsto (v, K)(t)$ is an energetic evolution if it satisfies the following two conditions:*

- *(global) stability: for every $t \in [0,T]$*

$$\mathcal{E}(t, v(t), K(t)) \le \mathcal{E}(t, v, K) + \mathcal{D}(K, K(t)) \tag{10}$$

  for every $K \in \mathcal{K}(\Omega)$ and every $v \in \mathcal{A}(t, K)$,

- *energy balance: for every $t \in [0,T]$*

$$\mathcal{E}(t, v(t), K(t)) + \mathcal{D}(K(t), K(0)) = \mathcal{E}(0, v(0), K(0)) + \int_0^t \mathcal{P}(\tau, v(\tau), K(\tau))\,d\tau\,, \tag{11}$$

  where $\mathcal{P}(t, v, K) = \partial_t \mathcal{E}(t, v, K)$ is the power of external forces.

Under the above assumptions on Ψ_t the power of external forces takes the following form [10, Remark 2.16]

$$\mathcal{P}(t, v, K) = \int_\Omega DW(Dv)(Dv)^T : D(\dot\Psi_t \circ \Psi_t^{-1}) \circ v\,dx + \int_\Omega \dot b(t) v\,dx\,.$$

Note that, since the map $t \mapsto K(t)$ we will construct will be non-decreasing, the energy balance (11) will hold as well between two instants t_1 and t_2 just by taking the difference.

Theorem 3.2. *Assume that (v_0, K_0), with $K_0 \in \mathcal{K}(\Omega)$ and $v_0 \in \mathcal{A}(0, K_0)$, satisfy (10) for $t = 0$. Then there exists an energetic evolution $t \mapsto (v, K)(t)$ with $v(0) = v_0$ and $K(0) = K_0$.*

We will prove the existence of an energetic solution by the standard procedure of time discretization, which consists in solving incremental minimum problem and passing to the limit as the time step tends to zero. In the context of nonlinear elasticity this has been done in [10] in the weak formulation involving functions of bounded variation. Due to the regularity assumptions made on the crack set we can use here the functional setting of Sobolev functions (on varying domains).

To prove existence it is more convenient to transfer the time-dependence from the boundary data to the functional and to employ the multiplicative representation introduced in [17]. To this end, given $K \in \mathcal{K}(\Omega)$, let

$$\mathcal{Z}(K) := \{z \in H^1(\Omega \setminus K; \mathbb{R}^2) : z = \mathrm{id} \text{ on } \partial_D \Omega \setminus K\}\,.$$

Thanks to the invertibility and to the regularity of Ψ_t, every $v \in \mathcal{A}(t, K)$ can be written in the form $v = \Psi_t(z)$ where $z = \Psi_t^{-1}(v)$ belongs to $\mathcal{Z}(K)$. Since $Dv = D\Psi_t(z)Dz$, it is natural to introduce the energy density $V : [0,T] \times \mathbb{R}^2 \times \mathbb{M}^{2\times 2} \to \mathbb{R}$ defined by

$$V(t, z, F) := W(D\Psi_t(z)\, F)\,.$$

In this way, if $v = \Psi_t(z)$, we have

$$\int_\Omega W(Dv)\, dx = \int_\Omega V(t, z, Dz)\, dx\,.$$

The assumptions on W and on Ψ_t imply the following property of V (cf. [10]).

Lemma 3.3. *There exists $C > 0$ such that*

$$V(t_2, z, F) + 1 \le (V(t_1, z, F) + 1)\, e^{C|t_2 - t_1|} \tag{12}$$

holds for every $t_1, t_2 \in [0,T]$, $z \in \mathbb{R}^2$ and $F \in \mathbb{R}^{2\times 2}_+$.

Proof. We have

$$\begin{aligned} V(t_2, z, F) - V(t_1, z, F) &= \int_{t_1}^{t_2} D_t V(t, z, F)\, dt = \\ &= \int_{t_1}^{t_2} D_t(W(D\Psi_t(z)\, F))\, dt = \int_{t_1}^{t_2} DW(D\Psi_t(z)\, F) \cdot D\dot\Psi_t(z) F\, dt\,. \end{aligned}$$

By (7) applied with $A = D\Psi_t(z)\, F$ and $B = D\dot\Psi_t(z)(D\Psi_t(z))^{-1}$ we get

$$|DW(D\Psi_t(z)\, F) \cdot D\dot\Psi_t(z)F| \le C\, |D\dot\Psi_t(z)(D\Psi_t(z))^{-1}|(W(D\Psi_t(z)\, F) + 1)\,.$$

Since $D\dot\Psi_t$ and $D\Psi_t^{-1}$ are uniformly bounded we conclude that

$$|V(t_1, z, F) - V(t_2, z, F)| \le C\left|\int_{t_1}^{t_2} (V(t, z, F) + 1)\, dt\right|\,. \tag{13}$$

By Gronwall's Lemma we deduce that (12) holds. $\square$

For $z \in \mathcal{Z}(K)$ let $\mathcal{V}(t)(z)$ be given by:

$$\mathcal{V}(t)(z) = \int_\Omega V(t, z, Dz)\, dx + \int_\Omega b(t)\Psi_t(z)\, dx = \int_\Omega W(D\Psi_t(z)\, Dz)\, dx + \int_\Omega b(t)\Psi_t(z)\, dx\,.$$

If the function z is such that $\mathcal{V}(t_0)(z) < +\infty$ for some t_0 (and hence for every t), then $t \mapsto \mathcal{V}(t)(z)$ is of class C^1 on $[0,T]$ with derivative $\dot{\mathcal{V}}(t)(z)$ given by

$$\dot{\mathcal{V}}(t)(z) = \int_\Omega DW(D\Psi_t(z)\, Dz) \cdot D\dot\Psi_t(z)\, Dz\, dx + \int_\Omega \dot b(t)\Psi_t(z) + b(t)\dot\Psi_t(z)\, dx\,.$$

Now we can define the discrete evolution. Fix $k \in \mathbb{N}$. Let $\{t_k^i\}_{0\le i\le k}$ be a time discretization of $[0,T]$ with $\max_i |t_k^{i+1} - t_k^i| \to 0$. We will use the following notation: $\Psi_k^i = \Psi_{t_k^i}$, $b_k^i = b(t_k^i)$ and

$$\mathcal{V}_k^i(z) := \mathcal{V}(t_k^i)(z) = \int_\Omega W(D\Psi_k^i(z)\, Dz)\, dx + \int_\Omega b_k^i \Psi_k^i(z)\, dx\,.$$

Let $K_0 \in \mathcal{K}(\Omega)$ and $z_0 \in \mathcal{Z}(K_0)$ satisfy

$$\mathcal{V}_0(z_0) + \mathcal{H}^1(K_0) \le \mathcal{V}_0(z) + \mathcal{H}^1(K) \quad \text{for every } K \in \mathcal{K}(\Omega) \text{ with } K \supseteq K_0 \text{ and for every } z \in \mathcal{Z}(K)\,.$$

We set $(z_k^0, K_k^0) := (z_0, K_0)$. For $i = 1, \dots, k$, let (z_k^i, K_k^i) be a solution of

$$\min\{\mathcal{V}_k^i(z) + \mathcal{H}^1(K \setminus K_k^{i-1}) : K \in \mathcal{K}(\Omega)\,,\ K \supseteq K_k^{i-1}\,,\ z \in \mathcal{Z}(K)\}\,. \tag{14}$$

Lemma 3.4. *There exists a solution of the minimum problem* (14).

Proof. Let (z_m, K_m) be a minimizing sequence. By the compactness of $\mathcal{K}(\Omega)$ there exists a subsequence (not relabelled) such that $K_m \to K$ in the Hausdorff metric. By coercivity (5)

$$\int_\Omega |D\Psi_k^i(z_m)Dz_m|^2\,dx + \int_\Omega b_k^i \Psi_k^i(z_m)\,dx \le C(\mathcal{V}_k^i(z_m)+1) < \infty.$$

By the assumptions on $D\Psi_k^i$, see §2.4, we can write

$$\int_\Omega |Dz_m|^2\,dx = \int_\Omega |(D\Psi_k^i)^{-1}(z_m)D\Psi_k^i(z_m)Dz_m|^2\,dx \le \|(D\Psi_k^i)^{-1}\|_\infty^2 \int_\Omega |D\Psi_k^i(z_m)Dz_m|^2\,dx.$$

Moreover, remembering that $|\Psi_t(z)| \le C(|z|+1)$, we have

$$\Big|\int_\Omega b_k^i \Psi_k^i(z_m)\,dx\Big| \le C'(\|z_m\|_2+1) \le C''(\|Dz_m\|_2+1) \tag{15}$$

where the second inequality follows from Proposition A.2 and thus C'' is uniform with respect to n. Therefore the bound on the energy gives

$$-C''(\|Dz_m\|_2+1) + \|(D\Psi_k^i)^{-1}\|_\infty^{-2}\,\|Dz_m\|_2^2 \le C(\mathcal{V}_k^i(z_m)+1) < \infty$$

from which it follows that $\|Dz_m\|_2$ and then $\|z_m\|_2$ are bounded. Thus there exists a subsequence (again not relabelled) such that $z_m \rightharpoonup z$ in $L^2(\Omega;\mathbb{R}^2)$ and $Dz_m \rightharpoonup Dz$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$. By Mosco convergence [8] (see Theorem A.6) it follows that $z \in \mathcal{Z}(K)$.

The lower semi-continuity of the functional $\mathcal{V}_k^i$ is established in [10, Theorem 3.1] while the lower semi-continuity of the dissipated energy $\mathcal{H}^1(K \setminus K_k^{i-1})$ is known as Golab's Theorem. By the direct method of the Calculus of Variations the existence of a minimizer follows. □

Let $\tau_k(t)$, $(z_k(t), K_k(t))$, $\mathcal{V}_k(t)$ be the piecewise constant interpolations given by

$$\tau_k(t) := t_k^i\,,\quad (z_k(t),K_k(t)) := (z_k^i, K_k^i)\,,\quad \mathcal{V}_k(t) = \mathcal{V}_k^i \qquad \text{for } t_k^i \le t < t_k^{i+1} \quad \text{and } i = 0,\dots,k-1\,.$$

Lemma 3.5. *Let $t \mapsto (z_k(t), K_k(t))$ be the sequence of approximate solutions defined just above. Then, $\|Dz_k(t)\|_2$, $\|z_k(t)\|_2$, $\dot{\mathcal{V}}(t)(z_k(t))$ and $\mathcal{H}^1(K_k(t))$ are bounded uniformly in k and t. Moreover*

$$\mathcal{V}_k(t)(z_k(t)) + \mathcal{H}^1(K_k(t)) \le \mathcal{V}_k(0)(z_k(0)) + \mathcal{H}^1(K_k(0)) + \int_0^{\tau_k(t)} \dot{\mathcal{V}}(s)(z_k(s))\,. \tag{16}$$

Proof. We follow closely the proof of [10, Proposition 3.10]. To show the uniform bound on the bulk functionals it is enough to notice that the identity map and the crack K_k^{j-1} are admissible competitors, indeed, by the boundedness of Ψ_t, $D\Psi_t$ e b we have

$$\mathcal{V}_k^j(z_k^j) + \mathcal{H}^1(K_k^j \setminus K_k^{j-1}) \le \mathcal{V}_k^j(\mathrm{id}) = \int_\Omega W(D\Psi_k^j)\,dx + \int_\Omega b_k^j\Psi_k^j\,dx < C$$

for some positive constant C independent of k and j. Arguing as in the proof of Lemma 3.4, we get the uniform bound on $\|Dz_k^i\|_2$ and $\|z_k^i\|_2$, and thus on $\|Dz_k(t)\|_2$ and $\|z_k(t)\|_2$.

Remember that

$$\dot{\mathcal{V}}(t)(z_k^i) = \int_\Omega DW(D\Psi_t(z_k^i)\,Dz_k^i)\cdot D\dot\Psi_t(z_k^i)\,Dz_k^i\,dt + \int_\Omega \dot b(t)\Psi_t(z_k^i) + b(t)\dot\Psi_t(z_k^i)\,.$$

By (7) with $A = D\Psi_t(z_k^i)Dz_k^i$ and $B = D\dot\Psi_k^i(z)(D\Psi_k^i(z))^{-1}$ we get

$$\begin{aligned}
|DW(D\Psi_t(z_k^i)\,Dz_k^i)\cdot D\dot\Psi_t(z_k^i)\,Dz_k^i| &\le C|D\dot\Psi_k^i(z)(D\Psi_k^i(z))^{-1}|(W(D\Psi_t(z_k^i)Dz_k^i)+1)\\
&\le C\|D\dot\Psi_k^i\|_\infty\,\|(D\Psi_k^i)^{-1}\|_\infty(W(D\Psi_t(z_k^i)Dz_k^i)+1)\,.\\
&\le C'e^{\hat C|t_k^i - t|}\,(W(D\Psi_k^i(z_k^i)Dz_k^i)+1)
\end{aligned}$$

where the last bound follows from Lemma 3.3. Therefore,

$$\begin{aligned}\int_\Omega DW(D\Psi_t(z_k^i)\,Dz_k^i)\cdot D\dot\Psi_t(z_k^i)\,Dz_k^i\,dt &\le C''\int_\Omega (W(D\Psi_k^i(z_k^i)Dz_k^i)+1)\,dx\\ &\le C''(\mathcal{V}_k^i(z_k^i)+\|b_k^i\|_2\|\Psi_k^i(z_k^i)\|_2+|\Omega|)<\infty,\end{aligned}$$

where the boundedness of the L^2-norms follows as in (15) and C'' depends on $\max_i|t_k^{i+1}-t_k^i|$. Moreover, being $b\in C^1([0,T],L^2(\Omega,\mathbb{R}^2))$ and by the boundedness of Ψ_t and $\dot\Psi_t$ we have

$$\int_\Omega (\dot b(t)\Psi_t(z_k^i)+b(t)\dot\Psi_t(z_k^i))\,dx \le \|\dot b(t)\|_2\|\Psi_t(z_k^i)\|_2+\|b(t)\|_2\|\dot\Psi_t(z_k^i)\|_2<\infty\,.$$

It follows that $|\dot{\mathcal{V}}(t)(z_k^i)|$ is uniformly bounded.

For $j=1,\dots,k$ take (z_k^{j-1},K_k^{j-1}) as a competitor in the minimum problem solved by (z_k^j,K_k^j). (Here the multiplicative decomposition of the boundary data proves to be useful.) Then

$$\mathcal{V}_k^j(z_k^j)+\mathcal{H}^1(K_k^j\setminus K_k^{j-1})\le \mathcal{V}_k^j(z_k^{j-1})\,. \tag{17}$$

Recalling that on $[t_k^{j-1},t_k^j)$ it is $z_k(s)=z_k^{j-1}$, we can write

$$\mathcal{V}_k^j(z_k^{j-1})=\mathcal{V}_k^{j-1}(z_k^{j-1})+\int_{t_k^{j-1}}^{t_k^j}\dot{\mathcal{V}}(s)(z_k(s))\,ds\,.$$

Hence

$$\mathcal{V}_k^j(z_k^j)-\mathcal{V}_k^{j-1}(z_k^{j-1})+\mathcal{H}^1(K_k^j)-\mathcal{H}^1(K_k^{j-1})\le\int_{t_k^{j-1}}^{t_k^j}\dot{\mathcal{V}}(s)(z_k(s))\,ds.$$

Summing up for $j=1$ to $\tau_k(t)$ we obtain (16). The uniform bound on the Hausdorff measure of the cracks is for free in the family $\mathcal{K}(\Omega)$. □

Since we have a uniform bound on $\mathcal{H}^1(K_k(t))$ we can apply Helly's Theorem [12, Theorem 6.3] and conclude that there exists a subsequence, still denoted K_k, and an increasing function $t\mapsto K(t)$ such that, for every t, $K_k(t)$ converges to $K(t)$ in the Hausdorff distance and, by Proposition 2.4, $K(t)\in\mathcal{K}(\Omega)$. For every time t, both $\|z_k(t)\|_2$ and $\|Dz_k(t)\|_2$ are bounded, by the previous Lemma, thus (up to subsequences) $z_k(t)\to z(t)$ in $L^2(\Omega;\mathbb{R}^2)$ and $Dz_k(t)\rightharpoonup Dz(t)$ in $L^2(\Omega;\mathbb{R}^{2\times2})$, where $z(t)\in\mathcal{Z}(K(t))$ by Mosco convergence (Theorem A.6). This argument defines for every t a couple $z(t),K(t)$ which in turn will be an energetic evolution.

Theorem 3.6. *The function $t\mapsto(z(t),K(t))$ obtained as limit of the discretization procedure described above is an energetic evolution, i.e., for every $t\in[0,T]$ it satisfies the global stability condition:*

$$\mathcal{V}(t)(z(t))+\mathcal{H}^1(K(t))\le\mathcal{V}(t)(z)+\mathcal{H}^1(K) \tag{18}$$

for every $K\in\mathcal{K}(\Omega)$ with $K\supseteq K(t)$ and every $z\in\mathcal{Z}(K)$, and the energy balance

$$\mathcal{V}(t)(z(t))+\mathcal{H}^1(K(t))=\mathcal{V}(0)(z(0))+\mathcal{H}^1(K(0))+\int_0^t\dot{\mathcal{V}}(s)(z(s))\,ds\,. \tag{19}$$

In the proof we will use the following result, which corresponds in our setting to the jump transfer lemma [15].

Lemma 3.7. *Let $K_h,K,H\in\mathcal{K}(\Omega)$ be such that $K_h\to K$ in the Hausdorff distance and $K\subset H$. Let $z\in\mathcal{Z}(H)$. Then there exist a sequence $\{H_h\}$ with $H_h\in\mathcal{K}(\Omega)$ and a sequence of functions $z_h\in\mathcal{Z}(H_h)$ such that $H_h\to H$ in the Hausdorff distance, $K_h\subset H_h$, $\mathcal{H}^1(H_h)\to\mathcal{H}^1(H)$, $z_h\to z$ strongly in $L^2(\Omega,\mathbb{R}^2)$, $Dz_h\to Dz$ strongly in $L^2(\Omega,\mathbb{R}^{2\times2})$ and $\mathcal{V}(t)(z_h)\to\mathcal{V}(t)(z)$.*

Proof. Step 1. Construction of the sets H_h. Let $\ell = \mathcal{H}^1(K)$ and $\gamma : [0,\ell] \to \mathbb{R}^2$ be an arc-length parametrization of K. Since $K \in \mathcal{K}(\Omega)$, we have that $\gamma \in W^{2,\infty}([0,\ell];\mathbb{R}^2)$. For the same reason we have an arc-length parametrization $\gamma_h : [0, \mathcal{H}^1(K_h)] \to \mathbb{R}^2$ of K_h with the same regularity.

We reparametrize K_h on the fixed interval $[0,\ell]$ by simply changing $s \to s\ell/\mathcal{H}^1(K_h)$ and still denote the (new) parametrization by γ_h. From the regularity of the curves and the Hausdorff convergence it follows that $\gamma_h \rightharpoonup \gamma$ weakly$*$ in $W^{2,\infty}([0,\ell];\mathbb{R}^2)$.

Assume for simplicity that H extends K from only one of its end-points. This surely happens if the other end-point belongs to $\partial\Omega$. Let $\hat\gamma : [0,L] \to \mathbb{R}^2$ be an arc-length parametrization of H, with $L = \mathcal{H}^1(H)$. We now define a $W^{2,\infty}$-extension $\hat\gamma_h$ of γ_h to the interval $[0,L]$. We set

$$\hat\gamma_h(s) := \begin{cases} \gamma_h(s) & \text{if } 0 \le s \le \ell \\ \hat\gamma(s) + \gamma_h(\ell) - \gamma(\ell) + (\gamma_h'(\ell) - \hat\gamma'(\ell))(s-\ell) & \text{if } s > \ell. \end{cases}$$

It is easy to check that, up to subsequences, $\hat\gamma_h \rightharpoonup \hat\gamma$ weakly$*$ in $W^{2,\infty}((0,L);\mathbb{R}^2)$. Let $H_h = \hat\gamma_h([0,L])$. Then $H_h \supset K_h$, and $H_h \to H$ in the Hausdorff distance. In order to have $H_h \in \mathcal{K}(\Omega)$ (at least for h sufficiently large) it is not sufficient to know that the curvature of the parametrization is uniformly bounded, since the conditions (b) and (c) in Definition 2.3 are global; these properties can be proved by contradiction (if there would be a subsequence of $H_h \notin \mathcal{K}(\Omega)$, by compactness of the circles C_i, we would get that $H \notin \mathcal{K}(\Omega)$).

Step 2. Construction of the functions z_h. By the regularity of H it is possible to choose $r > 0$ small enough ($r \ll \eta$) so that the projection Π_H on H is well-defined for all points in $\mathcal{I}_r(H) := \{x \in \mathbb{R}^2 : \mathrm{dist}(x,H) < r\}$ (in the terminology of [14] the set H has positive reach [14, Lemma 4.1]) and so that $x \mapsto \Pi_H(x)$ is a Lipschitz map in $\mathcal{I}_r(H)$ (see for instance [19, Corollary 4.4.9]). Further, for $x \in \mathcal{I}_r(H)$ let $s(x) \in [0,L]$ be such that $\hat\gamma(s(x)) = \Pi_H(x)$, since $\hat\gamma$ provides an arc length parametrization of H the map $x \mapsto s(x)$ is locally Lipschitz.

From the Hausdorff convergence of the curves we deduce that for h large enough all curves H_h are contained in the above set $\mathcal{I}_r(H)$. Let $d_h = \|\hat\gamma_h - \hat\gamma\|_{1,\infty}^{1/2}$. Clearly $d_h \to 0$ and $d_h < r$ for h large enough. Let $\lambda_h : \mathbb{R} \to \mathbb{R}$ be given by $\lambda_h(\rho) := (1 - \frac{|\rho|}{d_h})_+$, where $(\cdot)_+$ indicates the positive part. Define

$$\Lambda_h(x) := x + \lambda_h(|x - \Pi_H(x)|)(\hat\gamma_h(s(x)) - \hat\gamma(s(x)))\,. \tag{20}$$

Note that Λ_h maps H to H_h and that $\Lambda_h(x) = x$ for every $x \in \Omega \setminus \mathcal{I}_{d_h}(H)$. Moreover, the maps Λ_h are uniformly Lipschitz with $\|\Lambda_h - \mathrm{id}\|_{W^{1,\infty}(\Omega;\mathbb{R}^2)} \le C d_h$. Therefore Λ_h is globally invertible by Hadamard Theorem with $\|\Lambda_h^{-1} - \mathrm{id}\|_{W^{1,\infty}(\Omega;\mathbb{R}^2)} \to 0$ (e.g. [19, Theorem 6.2.3]).

Let $z_h := z \circ \Lambda_h^{-1}$. Then $z_h \in W^{1,2}(\Omega \setminus H_h, \mathbb{R}^2)$, $z_h = z$ on $\Omega \setminus \mathcal{I}_{d_h}(H)$; moreover, using the fact that $\hat\gamma_h \to \hat\gamma$ strongly in $W^{1,\infty}$ it follows that $z_h \to z$ in $L^2(\Omega,\mathbb{R}^2)$ and $Dz_h \to Dz$ strongly in $L^2(\Omega,\mathbb{R}^{2\times 2})$ as $h \to +\infty$.

It remains to show that $\mathcal{V}(t)(z_h) \to \mathcal{V}(t)(z)$. To this end, we have

$$\begin{aligned} \int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z_h)Dz_h)dx &= \int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z)DzD\Lambda_h^{-1})|\mathrm{det}D\Lambda_h|dx \\ &\le C\int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z)Dz)dx + C|\mathcal{I}_{d_h}(H)|\,, \end{aligned}$$

where the last estimate holds for h large enough since we have used (8). Therefore

$$\begin{aligned} \mathcal{V}(t)(z_h) - \mathcal{V}(t)(z) &= \int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z_h)Dz_h)dx - \int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z)Dz)dx \\ &\quad + \int_{\mathcal{I}_{d_h}(H)\cap\Omega} b(t)\big(\Psi_t(z_h) - \Psi_t(z)\big)\,dx \\ &\le (C-1)\int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z)Dz)dx + C|\mathcal{I}_{d_h}(H)| \\ &\quad + \|b(t)\|_2\,\|\Psi_t(z_h) - \Psi_t(z)\|_2\,. \end{aligned}$$

Since

$$\|\Psi_t(z_h) - \Psi_t(z)\|_2 \le \|D\Psi_t\|_\infty \, \|z_h - z\|_2$$

we can write

$$\mathcal{V}(t)(z_h) \le \mathcal{V}(t)(z) + (C-1)\int_{\mathcal{I}_{d_h}(H)\cap\Omega} W(D\Psi_t(z)Dz)dx + C|\mathcal{I}_{d_h}(H)| + C\|z_h - z\|_2\,,$$

while by the lower semi-continuity of $\mathcal{V}(t)$ we have

$$\mathcal{V}(t)(z) \le \liminf_h \mathcal{V}(t)(z_h)\,.$$

Since the upper bound holds for every h large enough, $d_h \to 0$ and $z_h \to z$ in $L^2(\Omega, \mathbb{R}^2)$, we have shown the convergence of the energies. □

Proof of Theorem 3.6. Global stability condition (18). Fix $t \in [0,T]$, $K \in \mathcal{K}(\Omega)$ with $K \supseteq K(t)$ and $z \in \mathcal{Z}(K)$; applying Lemma 3.7 to $K_k(t)$, $K(t)$, K, and z we obtain the existence of a sequence $\{H_k\} \subset \mathcal{K}(\Omega)$ with $K_k(t) \subseteq H_k$ such that H_k converges in the Hausdorff distance to K and $\mathcal{H}^1(H_k) \to \mathcal{H}^1(K)$, and of a sequence of functions $z_k \in \mathcal{Z}(H_k)$ such that $z_k \to z$ strongly in $L^2(\Omega;\mathbb{R}^2)$, $Dz_k \to Dz$ strongly in $L^2(\Omega;\mathbb{R}^{2\times 2})$ and

$$\mathcal{V}(t)(z_k) \to \mathcal{V}(t)(z)\,.$$

The pair (z_k, H_k) is an admissible competitor in the minimum problem solved by $(z_k(t), K_k(t))$, therefore

$$\mathcal{V}(\tau_k(t))(z_k(t)) + \mathcal{H}^1(K_k(t)) \le \mathcal{V}(\tau_k(t))(z_k) + \mathcal{H}^1(H_k)\,,$$

where $\tau_k(t)$ is the piecewise constant interpolation of the discrete time steps t_k^i. We now pass to the limit as $k \to +\infty$. For the left-hand side of the inequality we use the lower semicontinuity of the energy (with respect to Hausdorff convergence of the sets, weak convergence of the gradients in L^2 and strong convergence of the deformations in L^2) and the continuity in time. Let us now consider the right-hand side. By Lemma 3.3 we have

$$\begin{aligned}
|\mathcal{V}(\tau_k(t))(z_k) - \mathcal{V}(t)(z_k)| &\le \int_\Omega |V(\tau_k(t), z_k, Dz_k) - V(t, z_k, Dz_k)|\,dx + \\
&\quad + \int_\Omega |b(\tau_k(t))\Psi_{\tau_k(t)}(z_k) - b(t)\Psi_t(z_k)|\,dx \\
&\le \int_\Omega |V(\tau_k(t), z_k, Dz_k) + 1|\,(e^{C|\tau_k(t)-t|} - 1)\,dx + \\
&\quad + \|b(\tau_k(t))\|_2\,\|\Psi_{\tau_k(t)}(z_k) - \Psi_t(z_k)\|_2 + \|b(\tau_k(t)) - b(t)\|_2\,\|\Psi_t(z_k)\|_2 \\
&\le (e^{C|\tau_k(t)-t|} - 1)\int_\Omega |V(\tau_k(t), z_k, Dz_k) + 1|\,dx + \\
&\quad + C'\|\Psi_{\tau_k(t)}(z_k) - \Psi_t(z_k)\|_2 + C''|\tau_k(t) - t|.
\end{aligned}$$

Since

$$\Psi_{\tau_k(t)}(z_k) - \Psi_t(z_k) = \int_{\tau_k(t)}^t \dot{\Psi}_s(z_k)\,ds \le \|\dot{\Psi}\|_\infty |t - \tau_k(t)|$$

it follows that $\|\Psi_{\tau_k(t)}(z_k) - \Psi_t(z_k)\|_2 \to 0$. Therefore we have $|\mathcal{V}(\tau_k(t))(z_k) - \mathcal{V}(t)(z_k)| \to 0$. This together with the convergences of $\mathcal{V}(t)(z_k)$ to $\mathcal{V}(t)(z)$ provided by Lemma 3.7 allows us to conclude that (18) is satisfied.

Energy balance (10). The proof of the energy balance is standard and follows exactly the proof of Theorem 2.14 in [10], which is based on the usual argument [9] of the approximation of a Lebesgue integral by suitable Riemann sums. □

4 Energetic evolution for linear elasticity

In this section we deal with the energetic evolution in linear elasticity. Let Ω, $\partial_D\Omega$ and $\mathcal{K}(\Omega)$ be as in §2. In analogy with Ψ, let $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be of class C^2; with a slight abuse of notation let $g(t)$ denote the field $g(t,\cdot)$ and assume that $g(t), \dot g(t)$ are bounded in $W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$ uniformly in time. As customary, we write the evolution in terms of the displacement u: for every $t \in [0,T]$ and $K \in \mathcal{K}(\Omega)$ the space of admissible displacements is given by

$$\mathcal{U}(t,K) = \{u \in H^1(\Omega \setminus K;\mathbb{R}^2) : u = g(t) \text{ on } \partial_D\Omega \setminus K\} .$$

Note that it would be possible, restricting to the linear setting, to employ much more general boundary conditions, however we need g to be regular since we will derive the linear energetic evolution from the non-linear one just by means of a scaling argument, which does not affect the regularity of the boundary datum. Then, the bulk energy $E : [0,T] \times L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega) \to [0,+\infty]$ is defined by

$$E(t,u,K) = \begin{cases} \frac{1}{2}\displaystyle\int_\Omega \boldsymbol{\sigma}(u)\cdot\boldsymbol{\varepsilon}(u)\,dx + \int_\Omega b(t)u\,dx & \text{if } u \in \mathcal{U}(t,K) \\ +\infty & \text{otherwise,} \end{cases}$$

where $\boldsymbol{\varepsilon}(u) = (Du + Du^T)/2$, $\boldsymbol{\sigma}(u) = \mathbf{C}\boldsymbol{\varepsilon}(u)$ and $b \in W^{1,\infty}([0,T];L^2(\Omega;\mathbb{R}^2))$. The dissipation distance $\mathcal{D} : \mathcal{K}(\Omega) \times \mathcal{K}(\Omega) \to [0,+\infty]$ is again

$$\mathcal{D}(K_2,K_1) = \begin{cases} \mathcal{H}^1(K_2 \setminus K_1) & \text{if } K_2 \supseteq K_1, \\ +\infty & \text{otherwise.} \end{cases}$$

An energetic evolution $t \mapsto (u,K)(t)$ is characterized by global stability and energy balance, respectively

$$E(t,u(t),K(t)) \le E(t,u,K) + \mathcal{D}(K,K(t))$$

for every $t \in [0,T]$ and every $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$, and

$$E(t,u(t),K(t)) + \mathcal{D}(K(t),K(0)) = E(0,u(0),K(0)) + \int_0^t P(\tau,u(\tau),K(\tau))\,d\tau$$

for every $t \in [0,T]$, where the power P is given by

$$P(\tau,u(\tau),K) = \int_\Omega \boldsymbol{\sigma}(u(\tau))\cdot\boldsymbol{\varepsilon}(\dot g(\tau))\,dx + \int_\Omega \dot b(t)u\,dx .$$

The existence of such an evolution will follow from §7.

5 Scaling

In this section we will see how to derive the evolution with linear elasticity from a sequence of evolutions with re-scaled non-linear energies. The mechanical meaning of the scaling law, which is actually the well known Bažant law [4], is explained in [24] in terms of scaling of domains.

Let Ω, $\partial_D\Omega$, $\mathcal{K}(\Omega)$, g, and $\mathcal{U}(t,K)$ be as in §4. For $n \in \mathbb{N}$ we define

$$W_n(F) = n\,W(I + n^{-1/2}F)$$

and consider the non-linear re-scaled bulk energy $E_n : [0,T] \times L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega) \to [0,+\infty]$ given by

$$E_n(t,u,K) = \begin{cases} \displaystyle\int_\Omega W_n(Du)\,dx + \int_\Omega b(t)u\,dx & \text{if } u \in \mathcal{U}(t,K) \\ +\infty & \text{otherwise.} \end{cases}$$

An energetic evolution $t \mapsto (u_n, K_n)(t)$ with $K_n(0) = K_0$ should then satisfy for every $t \in [0, T]$ (global) stability

$$E_n(t, u_n(t), K_n(t)) \leq E_n(t, u, K) + \mathcal{D}(K, K_n(t)) \tag{21}$$

for every $(u, K) \in L^2(\Omega; \mathbb{R}^2) \times \mathcal{K}(\Omega)$, and energy balance

$$E_n(t, u_n(t), K_n(t)) + \mathcal{D}(K_n(t), K_n(0)) = E_n(0, u_n(0), K_n(0)) + \int_0^t P_n(\tau, u_n(\tau), K_n(\tau))\, d\tau\,, \tag{22}$$

where $P_n(t, u, K) = \partial_t E_n(t, u, K)$ is the power of external forces (we will write later the explicit form of P_n in a convenient way). In the sequel such an evolution will be called a re-scaled non-linear energetic evolution.

In order to establish easily the existence of an energetic evolution which satisfies (21) and (22) it is convenient to introduce a change of variable in such a way that we can apply directly the existence result of §3 (which holds for the density W and not for the rescaled densities W_n). The change of variable will also shed some light on the choice of W_n itself.

For $n \in \mathbb{N}$ consider the scaled domains $\Omega_n = n\Omega$ with $\partial_D \Omega_n = n\partial_D \Omega$. Let $\mathcal{H}_n(\Omega_n) = n\mathcal{K}(\Omega)$ denote the family of admissible cracks. Given a crack $K \in \mathcal{K}(\Omega)$ and a displacement $u \in \mathcal{U}(t, K)$ let $H = nK$ and let $w(t, x) = n^{1/2} u(t, x/n)$. Clearly $H \in \mathcal{H}_n(\Omega_n)$, while w belongs to

$$\mathcal{W}_n(t, H) = \{w \in H^1(\Omega_n \setminus H; \mathbb{R}^2) : w(x) = n^{1/2} g(t, x/n) \text{ on } \partial_D \Omega_n \setminus H\},$$

which provides the set of admissible displacements for the scaled problem. In order to employ the existence result of §3 it is useful to introduce the deformation $v(t, x) = x + w(t, x) = x + n^{1/2} u(t, x/n)$ which will belong to the space

$$\mathcal{A}_n(t, H) = \{v \in H^1(\Omega_n \setminus H; \mathbb{R}^2) : v(x) = x + n^{1/2} g(t, x/n) \text{ on } \partial_D \Omega_n \setminus H\}\,.$$

Before proceeding let us prove the following lemma.

Lemma 5.1. *Let g be as in §4. Then there exists $n' > 0$ such that for every $n > n'$ the maps $\Phi_{n,t}(x) = x + n^{-1/2} g(t, x)$ are C^2 diffeomorphisms of $\mathbb{R}^2$. Moreover $D\Phi_{n,t}$, $D\Phi_{n,t}^{-1}$, $\dot{\Phi}_{n,t}$, $D\dot{\Phi}_{n,t}$ and $D\dot{\Phi}_{n,t}^{-1}$ belong to $L^\infty(\mathbb{R}^2; \mathbb{R}^{2\times 2})$. Finally, both $\Phi_{n,t} \to \mathrm{id}$ and $\Phi_{n,t}^{-1} \to \mathrm{id}$ in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$.*

Proof. Global invertibility is obtained by Hadamard Theorem (see e.g. [19, Theorem 6.2.3]). To this end it is enough to check that $\det D\Phi_{n,t} > 0$ and that $\lim_{|x| \to +\infty} |\Phi_{n,t}(x)| = +\infty$. We have

$$\det D\Phi_{n,t} = 1 + n^{-1/2} \mathrm{tr} Dg + n^{-1} \det Dg$$

which shows the first condition, while the second is true because $g \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$. The C^2 regularity of the inverse is instead obtained by the local regularity of the inverse (see e.g. [19, Theorem 6.2.4]).

Since $g(t) \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ and $\|\Phi_{n,t} - \mathrm{id}\|_{W^{1,\infty}} = n^{-1/2} \|g(t)\|_{W^{1,\infty}}$ we deduce that $\Phi_{n,t} \to \mathrm{id}$ in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$. By a simple argument we get $\Phi_{n,t}^{-1} \to \mathrm{id}$ in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ as well. $\square$

With the aid of the previous Lemma we can re-write the boundary condition by means of a global map $\Psi_{n,t}$ as follows

$$\Psi_{n,t}(x) = n\Phi_{n,t}(x/n) = n\big(x/n + n^{-1/2} g(t, x/n)\big) = x + n^{1/2} g(t, x/n) \tag{23}$$

where $\Phi_{n,t}$ has been defined in Lemma 5.1. Thanks to the regularity of $\Phi_{n,t}$, the map $\Psi_{n,t}$ satisfies all the regularity assumptions listed in §2.4.

Next, let us write the energy, the dissipation and the power for the non-linear scaled problems. The bulk energy $\mathcal{E}_n : [0, T] \times L^2(\Omega_n; \mathbb{R}^2) \times \mathcal{H}_n(\Omega_n) \to [0, +\infty]$ is defined as

$$\mathcal{E}_n(t, v, H) = \begin{cases} \displaystyle\int_{\Omega_n} W(Dv)\, dx + \int_{\Omega_n} b_n(t) v\, dx & \text{if } v \in \mathcal{A}_n(t, H) \\ +\infty & \text{otherwise,} \end{cases}$$

where $b_n(t,x) = n^{-3/2}b(t,x/n)$; the dissipation distance $\mathcal{D}_n : \mathcal{H}_n(\Omega_n) \times \mathcal{H}_n(\Omega_n) \to [0,+\infty]$ is still

$$\mathcal{D}_n(H_1,H_2) = \begin{cases} \mathcal{H}^1(H_2 \setminus H_1) & \text{if } H_2 \supseteq H_1, \\ +\infty & \text{otherwise.} \end{cases}$$

All the hypotheses needed for the existence of an energetic evolution (see Theorem 3.6) are fulfilled, therefore there exists $t \mapsto (v_n, H_n)(t)$ which enjoys (global) stability and energy conservation:

$$\mathcal{E}_n(t, v_n(t), H_n(t)) \le \mathcal{E}_n(t, v, H) + \mathcal{D}_n(H, H_n(t)) \tag{24}$$

for every $t \in [0,T]$ and every $(v,H) \in L^2(\Omega_n;\mathbb{R}^2) \times \mathcal{H}_n(\Omega_n)$, and

$$\mathcal{E}_n(t, v_n(t), H_n(t)) + \mathcal{D}_n(H_n(t), H_n(0)) = \mathcal{E}_n(0, v_n(0), H_n(0)) + \int_0^t \mathcal{P}_n(\tau, v_n(\tau), H_n(\tau))\, d\tau\,, \tag{25}$$

for every $t \in [0,T]$, where $\mathcal{P}_n(t,v,H) = \partial_t \mathcal{E}_n(t,v,H)$ denotes the power. As we will see in the sequel, the evolution $t \mapsto (u_n(t), K_n(t))$ obtained by the change of variables from $t \mapsto (v_n(t), H_n(t))$ will provide the evolution for the re-scaled problem in the reference domain Ω.

Let us denote

$$\beta_n(t) = \int_\Omega n^{1/2} b(t) \mathrm{id}\, dx.$$

With this notation, let us see that

$$\mathcal{E}_n(t,v,H) = nE_n(t,u,K) + n\beta_n(t)\,,$$

where $H = nK$ and $v(x) = x + w(x) = x + n^{1/2}u(x/n)$. Indeed, since $Dw(x) = n^{-1/2}Du(x/n)$, $W_n(F) = n\,W(I + n^{-1/2}F)$ and $b_n(t,x) = n^{-3/2}b(t,x/n)$, for $x' = x/n$ we can write

$$\begin{aligned} \mathcal{E}_n(t,v,K) &= \int_{\Omega_n} W(I + Dw(x))\, dx + \int_{\Omega_n} b_n(t,x)(x + w(x))\, dx \\ &= \int_\Omega W(I + n^{-1/2}Du(x'))n^2\, dx' + \int_\Omega n^{-3/2} b(t,x')(nx' + n^{1/2}u(x'))n^2\, dx' \\ &= n\int_\Omega W_n(Du)\, dx' + n\int_\Omega b(t)u\, dx' + n\beta_n(t) = nE_n(t,u,K) + n\beta_n(t)\,. \end{aligned}$$

As a consequence

$$\mathcal{P}_n(t,v,H) = \partial_t\mathcal{E}_n(t,v,H) = n\partial_t E_n(t,u,K) + n\dot\beta(t) = nP_n(t,u,K) + n\dot\beta(t)\,.$$

By a standard property of Hausdorff measures we have also, for $H_i = nK_i$,

$$\mathcal{D}(H_1,H_2) = n\mathcal{D}(K_1,K_2).$$

Note that, energy, power and dissipation, all scale by a factor n and thus the sequence $t \mapsto (u_n, K_n)(t)$ is an energetic evolution; indeed, dividing (25) by n we get

$$\begin{aligned} &E_n(t, u_n(t), K_n(t)) + \beta_n(t) + \mathcal{D}(K_n(t), K_n(0)) = \\ &\qquad = E_n(0, u_n(0), K_n(0)) + \beta_n(0) + \int_0^t (P_n(\tau, u_n(\tau), K_n(\tau)) + \dot\beta_n(\tau))\, d\tau \end{aligned} \tag{26}$$

for every $t \in [0,T]$, while in order to show the global stability it is enough to notice that given any $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$ its corresponding scaled pair $(\hat v_n, \hat H_n)$ is an admissible competitor for $(v_n(t), H_n(t))$; therefore

$$E_n(t, u_n(t), K_n(t)) + \beta_n(t) \le E_n(t,u,K) + \beta_n(t) + \mathcal{D}(K, K_n(t))$$

for every $t \in [0,T]$ and for every $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$. Then, simplifying the terms β_n we get respectively

$$E_n(t, u_n(t), K_n(t)) + \mathcal{D}(K_n(t), K_n(0)) = E_n(0, u_n(0), K_n(0)) + \int_0^t P_n(\tau, u_n(\tau), K_n(\tau))\, d\tau$$

for every $t \in [0,T]$ and

$$E_n(t, u_n(t), K_n(t)) \le E_n(t,u,K) + \mathcal{D}(K, K_n(t))$$

for every $t \in [0,T]$ and for every $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$.

6 Γ-convergence and compactness

The following Lemma provides weak compactness of sequences equibounded in energy.

Lemma 6.1. *Let $\{K_n\} \subset \mathcal{K}(\Omega)$ with $K_n \to K$ in the Hausdorff metric. Then there exists a constant C (depending on the sequence $\{K_n\}$ and on its limit K) such that*

$$\int_\Omega |Du_n|^2\,dx \le C\int_\Omega W_n(Du_n)\,dx + C\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1\,, \tag{27}$$

for every sequence $u_n \in \mathcal{U}(t, K_n)$.

Proof. By Lemma 2.5 the domain $\Omega\setminus K$ can be decomposed into the union of a finite number of Lipschitz, connected subsets Ω_i, for $i = 0, ..., N$, in such a way that $\mathcal{H}^1(\partial\Omega_0 \cap (\partial_D\Omega\setminus K)) > 0$ and $|\Omega_0 \cap \Omega_i| > 0$.

As $\{K_n\} \subset \mathcal{K}(\Omega)$ with $K_n \to K$ in the Hausdorff metric, there exist bi-Lipschitz maps $\Lambda_n : \Omega \to \Omega$ (introduced in the proof of Lemma 3.7) such that $\Lambda_n(K) = K_n$. Consider now the subsets $\Omega_{i,n} = \Lambda_n(\Omega_i)$; by Lemma A.4 and by Theorem A.5 for each i and n there exists a rotation $R_{i,n}$ such that

$$\int_{\Omega_{i,n}} |Dv_n - R_{i,n}|^2\,dx \le Cn^{-1}\int_{\Omega_{i,n}} W_n(Du_n)\,dx \tag{28}$$

where $Dv_n = I + n^{-1/2}Du_n$. Note that by Theorem A.5 the constant C depends on Ω_i but it is uniform with respect to $n \in \mathbb{N}$. Being $i = 0, ..., N$ a finite index, the constant C can be choosen also independent of i. The next step follows closely the proof of [11, Proposition 3.4]. Let $\xi_{i,n}(x) = -R_{i,n}x$ and let $(v_n + \xi_{i,n})_{i,n}$ denote the average of $v_n + \xi_{i,n}$ in $\Omega_{i,n}$. Then, by Proposition A.2 we can write

$$\int_{\Omega_{i,n}} |v_n + \xi_{i,n} - (v_n + \xi_{i,n})_{i,n}|^2\,dx \le C\int_{\Omega_{i,n}} |Dv_n + D\xi_{i,n}|^2\,dx = C\int_{\Omega_{i,n}} |Dv_n - R_{i,n}|^2\,dx\,,$$

where the constant C is independent of i and n, since the sets $\Omega_{i,n}$ are all bi-Lipschitz equivalent to the sets Ω_i, for $i = 0, ..., N$. It follows that

$$\|v_n + \xi_{i,n} - (v_n + \xi_{i,n})_{i,n}\|_{H^1(\Omega_{i,n},\mathbb{R}^2)} \le Cn^{-1}\int_{\Omega_{i,n}} W_n(Du_n)\,dx\,.$$

The next step involves the boundary condition. Fix $i = 0$ and remember that, by Lemma 2.5, $\mathcal{H}^1(\partial\Omega_0\cap(\partial_D\Omega\setminus K)) > 0$. Moreover, by construction the bi-Lipschitz maps Λ_n are the identity in a neighborhhood $U(\partial\Omega)$ of the boundary $\partial\Omega$, therefore $\partial\Omega_{0,n}\cap(\partial_D\Omega\setminus K) = \partial\Omega_0\cap(\partial_D\Omega\setminus K_0)$ is actually independent of n. As a consequence, being $v_n(t) = \mathrm{id} + n^{-1/2}g(t)$ on $\partial_D\Omega\setminus K$ we have, by continuity of the trace operator in $\Omega_{0,n}\cap U(\partial\Omega) = \Omega_0 \cap U(\partial\Omega)$,

$$\int_{\partial\Omega_0\cap(\partial_D\Omega\setminus K)} |s - R_{0,n}s - (v_n+\xi_{0,n})_{0,n}|^2\,d\mathcal{H}^1(s) \le 2Cn^{-1}\int_{\Omega_{0,n}} W_n(Du_n)\,dx + 2n^{-1}\int_{\partial_D\Omega\setminus K} |g(t,s)|^2\,d\mathcal{H}^1(s)\,.$$

Note that the constant C is again independent of n since $\Omega_{0,n}\cap U(\partial\Omega) = \Omega_0\cap U(\partial\Omega)$ is independent of n. Then, following again [11, Lemma 3.3] we get

$$\begin{aligned}\int_{\Omega_{0,n}} |I - R_{0,n}|^2\,dx &\le C\int_{\partial\Omega_0\cap(\partial_D\Omega\setminus K)} |s - R_{0,n}s - (v_n + \xi_{0,n})_{0,n}|^2\,d\mathcal{H}^1(s)\\ &\le Cn^{-1}\int_{\Omega_{0,n}} W_n(Du_n)\,dx + Cn^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1,\end{aligned}$$

where C depends on $\partial\Omega_{0,n}\cap(\partial_D\Omega\setminus K) = \partial\Omega_0\cap(\partial_D\Omega\setminus K_0)$ and thus it is independent of n. Therefore

$$\int_{\Omega_{0,n}} |Dv_n - I|^2\,dx \le Cn^{-1}\int_{\Omega_{0,n}} W_n(Du_n)\,dx + Cn^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1.$$

In particular, if $\int_\Omega W_n(Du_n)\,dx$ is bounded, we have $Dv_n \to I$ in $L^2(\Omega_{0,n};\mathbb{R}^{2\times 2})$. Remembering that $Dv_n = I + n^{-1/2}Du_n$ we get

$$n^{-1}\int_{\Omega_{0,n}} |Du_n|^2\,dx \le Cn^{-1}\int_{\Omega_{0,n}} W_n(Du_n)\,dx + Cn^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1$$

which gives (27) in the set $\Omega_{0,n}$.

For each $i = 1, ..., N$ and each $n \in \mathbb{N}$, by (28) we have

$$\begin{aligned}\int_{\Omega_{0,n}\cap\Omega_{i,n}} |I - R_{i,n}|^2\,dx &\le 2Cn^{-1}\int_{\Omega_{i,n}} W_n(Du_n)\,dx + 2n^{-1}\int_{\Omega_{0,n}} |Du_n|^2\,dx\\ &\le 4Cn^{-1}\int_{\Omega_{0,n}\cup\Omega_{i,n}} W_n(Du_n)\,dx + 2Cn^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1.\end{aligned}$$

Thus

$$|\Omega_{0,n}\cap\Omega_{i,n}|\;|I - R_{i,n}|^2 \le 4Cn^{-1}\int_{\Omega_{0,n}\cup\Omega_{i,n}} W_n(Du_n)\,dx + 2Cn^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1.$$

Since $|\Omega_0\cap\Omega_i| = |\Lambda_n^{-1}(\Omega_{0,n}\cap\Omega_{i,n})| \le C|\Omega_{0,n}\cap\Omega_{i,n}|$ for some C independent of n, from the previous inequality and (28) we obtain

$$n^{-1}\int_{\Omega_{i,n}} |Du_n(x)|^2\,dx \le C'n^{-1}\int_{\Omega_{0,n}\cup\Omega_{i,n}} W_n(Du_n)\,dx + C'n^{-1}\int_{\partial_D\Omega\setminus K} |g(t)|^2\,d\mathcal{H}^1,$$

with C' independent of n. Taking the sum with respect to $i = 0, ..., N$ provides (27) in the domain Ω. $\square$

Lemma 6.2. *Let $\{K_n\} \subset \mathcal{K}(\Omega)$. If $K_n \to K$ in the Hausdorff metric then $E_n(t,\cdot,K_n)$ Γ-converge to $E(t,\cdot,K)$ in the strong topology of $L^2(\Omega;\mathbb{R}^2)$.*

Proof. Let us prove first the Γ-liminf inequality. Let $u_n \to u$ in $L^2(\Omega;\mathbb{R}^2)$; it is not restrictive to assume that $E_n(t,u_n,K_n)$ is bounded and thus that $u_n \in \mathcal{U}(t,K_n)$. Since

$$E_n(t,u_n,K_n) = \int_\Omega W_n(Du_n)\,dx + \int_\Omega b\,u_n\,dx$$

and

$$\int_\Omega bu_n\,dx \to \int_\Omega bu\,dx\,,$$

by the previous compactness lemma Du_n is bounded in $L^2(\Omega;\mathbb{R}^{2\times 2})$ and thus (upon extracting a subsequence) we can also assume that $Du_n \rightharpoonup Du$ in $L^2(\Omega;\mathbb{R}^{2\times 2})$. Then, by Theorem A.6 we have $u \in \mathcal{U}(t,K)$, while by [11, Proposition 4.4],

$$\tfrac{1}{2}\int_\Omega \boldsymbol{\sigma}(u)\cdot\boldsymbol{\varepsilon}(u) \le \liminf_{n\to\infty}\int_\Omega nW(I + n^{-1/2}Du_n)\,,$$

from which we get the Γ-liminf inequality.

We will prove the Γ-limsup inequality by a density argument. Let $E''(t,\cdot,K)$ be the Γ-limsup functional. Assume first that $u \in W^{1,\infty}(\Omega\setminus K;\mathbb{R}^2)$ and let $u_n = u\circ\Lambda_n^{-1}$, where Λ_n is the map employed in the previous Lemma. Then, u_n is unifomly bounded in $W^{1,\infty}(\Omega\setminus K_n;\mathbb{R}^2)$ and $u_n = u$ in $\Omega\setminus U_n$, where U_n is a neighborhood of K with $|U_n| \to 0$. In particular, by Lemma 2.2 we have that $W_n(Du_n) \to \frac{1}{2}\boldsymbol{\sigma}(u)\cdot\boldsymbol{\varepsilon}(u)$ pointwise in $\Omega\setminus K$ and $W_n(Du_n)$ is uniformly bounded. Therefore by dominated convergence $E_n(t,u_n,K_n) \to E(t,u,K)$ and hence

$$E''(t,u,K) \le E(t,u,K) \qquad \text{for } u \in W^{1,\infty}(\Omega\setminus K;\mathbb{R}^2)\,.$$

We can conclude the proof by the following density argument, classical in the theory of Γ-convergence [7]. Given $u \in W^{1,2}(\Omega\setminus K;\mathbb{R}^2)\setminus W^{1,\infty}(\Omega\setminus K;\mathbb{R}^2)$ there exists a sequence $u_k \in W^{1,\infty}(\Omega\setminus K;\mathbb{R}^2)$ such that

$u_k \to u$ strongly in $W^{1,2}(\Omega \setminus K; \mathbb{R}^2)$. As a consequence $E(t, u_k, K) \to E(t, u, K)$. Then, by the lower semi-continuity of $E''(t, \cdot, K)$ we can write

$$E''(t, u, K) \le \liminf_{k\to\infty} E''(t, u_k, K) \le \liminf_{k\to\infty} E(t, u_k, K) = E(t, u, K),$$

which ends the proof. □

Lemma 6.3. *Let $\{K_n\} \subset \mathcal{K}(\Omega)$. If $K_n \to K$ in the Hausdorff metric and $u_n \in \operatorname{argmin}\{E_n(t, \cdot, K_n) : u \in \mathcal{U}(t, K_n)\}$ then $u_n \to u$ strongly in $L^2(\Omega; \mathbb{R}^2)$ and $Du_n \to Du$ strongly in $L^2(\Omega; \mathbb{R}^{2\times 2})$, where $u \in \operatorname{argmin}\{E(t, \cdot, K) : u \in \mathcal{U}(t, K)\}$. In particular, $E_n(t, u_n, K_n) \to E(t, u, K)$.*

Proof. As $g(t)$ is a competitor for $u_n \in \mathcal{U}(t, K_n)$ it follows that $E_n(t, u_n, K_n)$ is uniformly bounded. Using inequality (27) yields

$$\|Du_n\|_2^2 \le C \int_\Omega W_n(Du_n)\, dx + C' \le E_n(t, u_n, K_n) + C''\|u_n\|_2 + C' \le \bar{C} + C''\|u_n\|_2.$$

Since $K_n \in \mathcal{K}(\Omega)$ and $K_n \to K$ in the Hausdorff metric, the sets $\Omega \setminus K_n$ are bi-Lipschitz equivalent to $\Omega \setminus K$, by means of the maps Λ_n introduced in the proof of Lemma 3.7. Thus, Proposition A.3 together with the boundedness of g provide the uniform Poincaré inequality

$$\|u_n\|_2^2 \le C\|Du_n\|^2 + C^*.$$

Joining the two previous inequalities implies first that u_n is bounded in $L^2(\Omega; \mathbb{R}^2)$ and second that Du_n is bounded in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Therefore, up to subsequences $Du_n \rightharpoonup Du$ weakly in $L^2(\Omega; \mathbb{R}^{2\times 2})$, with $u \in \mathcal{U}(t, K)$ by Theorem A.6. To pass to the strong convergence we can invoke [1, 25]. In our setting we have actually to deal with varying domains; however it is enough to pass to a fixed domain using the map Λ_n. Indeed, let $\tilde{u}_n := u_n \circ \Lambda_n \in H^1(\Omega \setminus K; \mathbb{R}^2)$ and

$$\int_\Omega W_n(Du_n)\, dx = \int_\Omega W_n(D\tilde{u}_n D\Lambda_n^{-1}) \det D\Lambda_n\, dx.$$

As $\Lambda_n \to \mathrm{id}$ in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ we can proceed step by step as in the proof of [1, Theorem 2.5] and conclude that $\boldsymbol{\varepsilon}(\tilde{u}_n) \to \boldsymbol{\varepsilon}(u)$ strongly in $L^2(\Omega; \mathbb{R}^{2\times 2})$. By Korn's inequality, in the set $\Omega \setminus K$, we deduce the strong convergence of $D\tilde{u}_n$ to Du and then by Poincaré inequality the strong convegence of $\tilde{u}_n$ to u in $L^2(\Omega; \mathbb{R}^2)$. As $\Lambda_n \to \mathrm{id}$ in $W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)$ we have also $Du_n \to Du$ strongly in $L^2(\Omega; \mathbb{R}^{2\times 2})$ and thus $u_n \to u$ stronlgy in $L^2(\Omega; \mathbb{R}^2)$.

By Γ-convergence the limit u is the unique minimizer of $E(t, \cdot, K)$ in $\mathcal{U}(t, K)$, thus it is independent of the subsequence and the above argument can be repetead for every subsequence. It follows that the convergence holds for the whole sequence u_n. □

7 Convergence of the rescaled evolutions

In this final section we prove that the (rescaled, non-linear elastic) evolution $t \mapsto (u_n, K_n)(t)$ converges to the (linear elastic) evolution $t \mapsto (u, K)(t)$. First of all let us recall the expression of the power

$$\mathcal{P}_n(t, v, H) = \int_{\Omega_n} DW(Dv)(Dv)^T \cdot D(\dot{\Psi}_{n,t} \circ \Psi_{n,t}^{-1}) \circ v\, dx + \int_{\Omega_n} \dot{b}_n v\, dx = nP_n(t, u, K) + n\dot{\beta}_n$$

in terms of $\Psi_{n,t}$, given by (23). Now we will derive a convenient expression for $P_n(t, u, K)$. By the regularity of g we have

$$\dot{\Psi}_{n,t}(x) = n^{1/2}\dot{g}(t, x/n)\,, \qquad D\dot{\Psi}_{n,t}(x) = n^{-1/2} D\dot{g}(t, x/n).$$

Since, by definition $\Psi_{n,t}(x) = n\Phi_{n,t}(x/n)$, we have $\Psi_{n,t}^{-1}(y) = n\Phi_{n,t}^{-1}(y/n)$ and

$$D\Psi_{n,t}^{-1}(y) = D\Phi_{n,t}^{-1}(y/n)\,.$$

Thus

$$D(\dot{\Psi}_{n,t}\circ\Psi_{n,t}^{-1})\circ v = n^{-1/2}D\dot{g}\big(t,(\Psi_{n,t}^{-1}\circ v)/n\big)D\Psi_{n,t}^{-1}\circ v$$
$$= n^{-1/2}D\dot{g}\big(t,\Phi_{n,t}^{-1}\circ(v/n)\big)D\Phi_{n,t}^{-1}\circ(v/n)\,.$$

Remember that $v(x)=x+w(x)$, where $w(x)=n^{1/2}u(x/n)=n^{1/2}u(x')$ and thus $Dv(x)=I+Dw(x)=I+n^{-1/2}Du(x')$. Then $(v/n)(x)=x/n+w(x)/n=x'+n^{-1/2}u(x')$. Setting, for convenience of notation, $\xi_n(x')=(v/n)(x)$, we have

$$D(\dot{\Psi}_{n,t}\circ\Psi_{n,t}^{-1})\circ v(x) = n^{-1/2}D\dot{g}\big(t,\Phi_{n,t}^{-1}\circ\xi_n(x')\big)D\Phi_{n,t}^{-1}\circ\xi_n(x')\,.$$

Thus the power P_n can be written as

$$\begin{aligned}
&P_n(t,v,H)=\\
&= n^{-1}\int_\Omega DW(I+n^{-1/2}Du(x'))(I+n^{-1/2}Du(x'))^T\cdot n^{-1/2}D\dot{g}\big(t,\Phi_{n,t}^{-1}\circ\xi_n(x')\big)D\Phi_{n,t}^{-1}\circ\xi_n(x')n^2\,dx'+\\
&\quad+\int_\Omega \dot{b}(t)u\,dx'\\
&= \int_\Omega n^{1/2}DW(I+n^{-1/2}Du(x'))(I+n^{-1/2}Du(x'))^T\cdot D\dot{g}\big(t,\Phi_{n,t}^{-1}\circ\xi_n(x')\big)D\Phi_{n,t}^{-1}\circ\xi_n(x')\,dx'+\\
&\quad+\int_\Omega \dot{b}(t)u\,dx'.
\end{aligned} \tag{29}$$

Lemma 7.1. *Let, for $n\in\mathbb{N}$, the map $t\mapsto(u_n,K_n)(t)$ be an energetic evolution for non-linear elasticity. Then there exist a subsequence (not relabelled), an increasing set function $K:[0,T]\to\mathcal{K}(\Omega)$, and a function $u:[0,T]\to L^2(\Omega;\mathbb{R}^2)$ such that for every $t\in[0,T]$ we have $K_n(t)\to K(t)$ in the Hausdorff distance, $u_n(t)\to u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$, $Du_n(t)\to Du(t)$ strongly in $L^2(\Omega;\mathbb{R}^{2\times2})$, and $u(t)\in H^1(\Omega\setminus K(t);\mathbb{R}^2)$.*

Proof. By [12, Theorem 6.3] there exists a subsequence (not relabelled) and an increasing function $t\mapsto K(t)$ such that $K_n(t)\to K(t)$ in the Hausdorff distance for every $t\in[0,T]$. By the closure of $\mathcal{K}(\Omega)$ (Proposition 2.4) we have $K(t)\in\mathcal{K}(\Omega)$. By Lemma 6.3 the convergence of the corresponding u_n follows. □

Lemma 7.2. *Let $t\mapsto(u_n,K_n)(t)$ and $t\mapsto(u,K)(t)$ be as in Lemma 7.1. Then*

$$P_n(t,u_n(t),K_n(t))\to P(t,u(t),K(t)) \text{ pointwise and in } L^1(0,T). \tag{30}$$

Proof. Let us fix $t\in(0,T)$ and see that

$$n^{1/2}DW(I+n^{-1/2}Du_n)(I+n^{-1/2}Du_n)^T\to\boldsymbol{\sigma}(u)\qquad\text{in } L^1(\Omega;\mathbb{R}^{2\times2}), \tag{31}$$

where for simplicity of notation we omitted the argument t. Remember that $u_n\to u$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and $Du_n\to Du$ strongly in $L^2(\Omega;\mathbb{R}^{2\times2})$. Up to extracting a subsequence (not relabelled) we can assume that $Du_n\to Du$ also a.e. in Ω. Thus $(I+n^{-1/2}Du_n)\to I$ a.e. in Ω and strongly in $L^2(\Omega;\mathbb{R}^{2\times2})$. Moreover, since $DW(I)=0$ we have $n^{1/2}DW(I+n^{-1/2}Du_n)\to D^2W(I)[Du]=\boldsymbol{\sigma}(u)$ a.e. in Ω. By (9)

$$\begin{aligned}
n^{1/2}|DW(I+n^{-1/2}Du_n)(I+n^{-1/2}Du_n)^T| &\le n^{1/2}C\big(W(I+n^{-1/2}Du_n)+|n^{-1/2}Du_n|+|n^{-1/2}Du_n|^2\big)\\
&\le C\big(n^{-1/2}W_n(Du_n)+|Du_n|+n^{-1/2}|Du_n|^2\big)\,.
\end{aligned} \tag{32}$$

The right hand side converges to $|Du|$ in $L^1(\Omega)$. Thus, by dominated convergence we deduce that (31) holds for the subsequence. Actually, as the limit is independent of the subsequence, we can conclude that (31) holds for the whole sequence.

Now, let us see that

$$D\dot{g}\big(t,\Phi_{n,t}^{-1}\circ\xi_n(x')\big)D\Phi_{n,t}^{-1}\circ\xi_n(x')\to D\dot{g}(t,x')\qquad\text{a.e. in }\Omega, \tag{33}$$

where $\xi_n(x') = x' + n^{-1/2}u_n(x')$. Notice that $\xi_n \to \mathrm{id}$ a.e. in Ω (and strongly in $L^2(\Omega;\mathbb{R}^2)$. Moreover $\Phi_{n,t} \to \mathrm{id}$ and $\Phi_{n,t}^{-1} \to \mathrm{id}$ in $W^{1,\infty}(\Omega,\mathbb{R}^2)$ by Lemma 5.1. Thus $\Phi_{n,t}^{-1} \circ \xi_n \to \mathrm{id}$ a.e. in Ω, indeed

$$|\Phi_{n,t}^{-1} \circ \xi_n(x') - x'| \le |\Phi_{n,t}^{-1} \circ \xi_n(x') - \xi_n(x')| + |\xi_n(x') - x'|\,;$$

the first term vanishes since $\Phi_{n,t}^{-1} \to \mathrm{id}$ uniformly, the second because $\xi_n \to \mathrm{id}$ a.e. in Ω. As a consequence, being $D\dot g$ continuous,

$$D\dot g\big(t, \Phi_{n,t}^{-1} \circ \xi_n(x')\big) \to D\dot g(t,x') \qquad \text{a.e. in } \Omega.$$

Moreover, $D\Phi_{n,t}^{-1} \circ \xi_n(x') \to I$ in $L^\infty(\Omega,\mathbb{R}^2)$ since $D\Phi_{n,t}^{-1} \to I$ in $L^\infty(\Omega;\mathbb{R}^2)$, so that (33) holds. Being $D\dot g(t, \Phi_{n,t}^{-1} \circ \xi_n(x'))D\Phi_{n,t}^{-1} \circ \xi_n(x')$ uniformly bounded, it follows by dominated convergence that

$$\int_\Omega n^{1/2} DW(I + n^{-1/2}Du_n(x'))(I + n^{-1/2}Du_n(x'))^T \cdot D\dot g\big(t, \Phi_{n,t}^{-1} \circ \xi_n(x')\big) D\Phi_{n,t}^{-1} \circ \xi_n(x')\, dx'$$

converges to

$$\int_\Omega \boldsymbol{\sigma}(u) \cdot D\dot g(t)\, dx = \int_\Omega \boldsymbol{\sigma}(u) \cdot \boldsymbol{\varepsilon}(\dot g(t))\, dx.$$

As $b \in W^{1,\infty}([0,T]; L^2(\Omega;\mathbb{R}^2))$, for a.e. $t \in (0,T)$

$$\int_\Omega \dot b(t) u_n\, dx \quad \to \quad \int_\Omega \dot b(t) u\, dx\,,$$

and we conclude that

$$P_n(t, u_n(t), K_n(t)) \to P(t, u(t), K(t)) \qquad \text{a.e. in } (0,T)\,.$$

To get the convergence in $L^1(0,T)$ it is enough to note that $P_n(t,u_n(t),K_n(t)) \le C$ for every t; indeed, by (32)

$$\begin{aligned} P_n(t,u_n(t),K_n(t)) &\le C\Big(n^{-1/2}\int_\Omega W_n(Du_n)\,dx + \int_\Omega |Du_n|\,dx + n^{-1/2}\int_\Omega |Du_n|^2\,dx + \|u_n\|_2\Big) \\ &\le C'\Big(n^{-1/2}\int_\Omega W_n(Dg(t))\,dx + \|Du_n\|_2^2 + \|u_n\|_2\Big). \end{aligned}$$

By Lemma 2.2 the energy term is uniformly bounded, while the H^1-norm is bounded by Lemma 7.1. This concludes the proof. □

Theorem 7.3. *Let $t \mapsto (u_n, K_n)(t)$ and $t \mapsto (u,K)(t)$ be as in Lemma 7.1. Then $t \mapsto (u,K)(t)$ is an energetic evolution for linearized elasticity, i.e. it satisfies*

$$E(t,u(t),K(t)) \le E(t,u,K) + \mathcal{D}(K,K(t))$$

for every $t \in [0,T]$ and every $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$, and

$$E(t,u(t),K(t)) + \mathcal{D}(K(t),K(0)) = E(0,u(0),K(0)) + \int_0^t P(\tau,u(\tau),K(\tau))\,d\tau$$

for every $t \in [0,T]$.

Proof. Since $K_n(t) \to K(t)$ for every $t \in [0,T]$, by Lemma 6.2 we have that $u(t) \in \operatorname{argmin}\{E(t,u,K(t)) : u \in \mathcal{U}(t,K(t))\}$ for every $t \in [0,T]$. We begin by showing that

$$E(t,u(t),K(t)) \le E(t,u,K) + \mathcal{D}(K,K(t)) \tag{34}$$

for every $(u,K) \in L^2(\Omega;\mathbb{R}^2) \times \mathcal{K}(\Omega)$.

Let $K \in \mathcal{K}(\Omega)$ with $K(t) \subseteq K$, and let u be the minimizer of $E(t,\cdot,K)$. By Lemma 3.7 there exists a sequence H_n with $H_n \in \mathcal{K}(\Omega)$ and $K_n(t) \subset H_n$ such that $H_n \to K$ in the Hausdorff distance. Recall that this gives also that $\mathcal{H}^1(H_n \setminus K_n(t)) \to \mathcal{H}^1(K \setminus K(t))$.

Thanks to Lemma 6.2 there exists a recovery sequence $\tilde{u}_n$ such that $\tilde{u}_n \to u$ strongly in $L^2(\Omega;\mathbb{R}^2)$ and $E_n(t,\tilde{u}_n,H_n) \to E(t,u,K)$. Since

$$E_n(t,u_n(t),K_n(t)) \le E_n(t,\tilde{u}_n,H_n) + \mathcal{H}^1(H_n \setminus K_n(t))\,.$$

passing to the limit we obtain (34).

By (22) for every $t \in [0,T]$

$$E_n(t,u_n(t),K_n(t)) + \mathcal{D}(K_n(t),K_n(0)) = E_n(0,u_n(0),K_n(0)) + \int_0^t P_n(\tau,u_n(\tau),K_n(\tau))\,d\tau.$$

First of all, we take the $\liminf_{n\to\infty}$ of the energy balance. By Lemma 6.3 we get convergence of the energy terms while by Lemma 7.2 we get the convergence of the work (integral of the power). By [12, Corollary 3.4] we get the following lower semi-continuity inequality

$$\mathcal{H}^1(K(t)\setminus K(s)) \le \liminf_{n\to\infty} \mathcal{H}^1(K_n(t)\setminus K_n(s))\,, \tag{35}$$

for every $s,t \in [0,T]$ with $s<t$.

By monotonicity of $K(\cdot)$ we have $\mathcal{H}^1(K(t)\setminus K(0)) = \mathcal{D}(K(t),K(0))$, similarly for K_n. Hence

$$\begin{aligned} E(t,u(t),K(t)) + \mathcal{D}(K(t),K(0)) &\le \liminf_{n\to\infty} E_n(t,u_n(t),K_n(t)) + \mathcal{D}(K_n(t),K_n(0)) \\ &= \liminf_{n\to\infty} E_n(0,u_n(0),K_n(0)) + \int_0^t P_n(\tau,u_n(\tau),K_n(\tau))\,d\tau \\ &= E(0,u(0),K(0)) + \int_0^t P(\tau,u(\tau),K(\tau))\,d\tau\,. \end{aligned} \tag{36}$$

To conclude the proof we will follow the same line of proof of [12, Lemma 7.9]. To this end, denote

$$J_n(t,u,K) = E_n(t,u,K) + \mathcal{D}(K,K_0)\,, \qquad J(t,u,K) = E(t,u,K) + \mathcal{D}(K,K_0)\,.$$

Again by (22) for every $0 \le s < t$ it holds

$$J_n(t,u_n(t),K_n(t)) - J_n(s,u_n(s),K_n(s)) = \int_s^t P_n(\tau,u_n(\tau),K_n(\tau))\,d\tau\,.$$

Passing to the limit by (35), Lemma 6.3 and Lemma 7.2 we get

$$J(t,u(t),K(t)) - J(s,u(s),K(s)) \le \int_s^t P(\tau,u(\tau),K(\tau))\,d\tau\,. \tag{37}$$

Moreover, writing (34) at time s in terms of J and choosing $K=K(t)$ as a competitor, yields $J(s,u(s),K(s)) \le J(s,u_{s,t},K(t))$ where $u_{s,t} \in \mathrm{argmin}\{E(s,u,K(t)) : u \in \mathcal{U}(s,K(t))\}$. Then

$$J(t,u(t),K(t)) - J(s,u(s),K(s)) \ge J(t,u(t),K(t)) - J(s,u_{s,t},K(t)) = \int_s^t P(\tau,u_{\tau,t},K(t))\,d\tau\,.$$

Hence

$$J(t,u(t),K(t)) - J(s,u(s),K(s)) \ge \int_s^t P(\tau,u_{\tau,t},K(t))\,d\tau\,. \tag{38}$$

Note that by the regularity in time of b and g it follows that $u_{\tau,t} \to u(t)$ strongly in $H^1(\Omega\setminus K(t);\mathbb{R}^2)$ as $\tau \to t$. Therefore, dividing (38) by $(t-s)$ and passing to the limit gives $\dot{J}(t,u(t),K(t)) = P(t,u(t),K(t))$ for a.e. t, from which the balance of energy follows. □

A Appendix

In this Appendix we collect, for the reader's convenience, few technical results which are fundamental to get uniform estimates in the case of varying domains.

Lemma A.1. *Let $\{K_n\} \subset \mathcal{K}(\Omega)$ with $K_n \to K$ in the Hausdorff distance. Then the sets $\Omega\setminus K_n$ are bi-Lipschitz equivalent to $\Omega\setminus K$ with uniformly controlled Lipschitz constants.*

Proof. The maps Λ_n given in the proof of Lemma 3.7 provide the bi-Lipschitz transforms. □

A.1 Uniform Poincaré inequalities

Proposition A.2. *Let Ω be a bounded, connected Lipschitz domain. For every $w \in H^1(\Omega)$ we have*

$$\int_\Omega |w - \bar{w}|^2 \, dx \le C \int_\Omega |Dw|^2 \, dx \,, \tag{39}$$

where $\bar{w}$ is the average of w and $C > 0$ depends only on Ω. The constant C appearing in (39) *can be chosen uniformly for a family of domains which are bi-Lipschitz equivalent to Ω with uniformly controlled Lipschitz constants. Further uniform estimates of this type may be found in [2].*

Proof. Consider a domain Ω' and let $\Psi : \Omega' \to \Omega$ be a Lipschitz map with Lipschitz inverse Ψ^{-1}. Let $w \in H^1(\Omega)$ and denote $w' = (w \circ \Psi) \in H^1(\Omega')$. Let us start with the L^2-norm.

$$\begin{aligned}
\int_\Omega |w(x) - \bar{w}|^2 \, dx &= \int_\Omega \left| w(x) - \fint_\Omega w(y) \, dy \right| dx = |\Omega|^{-1} \int_\Omega \left| \int_\Omega (w(x) - w(y)) \, dy \right| dx \\
&\le |\Omega|^{-1} \int_{\Omega'} \left| \int_{\Omega'} \big(w \circ \Psi(x') - w \circ \Psi(y') \big) \, |\mathrm{det} D\Psi(y')| dy' \right| \, |\mathrm{det} D\Psi(x')| dx' \\
&\le |\Omega|^{-1} |\Omega'| \, \|\mathrm{det} D\Psi\|_\infty^2 \int_{\Omega'} \left| w \circ \Psi(x') - \fint_{\Omega'} w \circ \Psi(y') dy' \right| \, dx' \\
&\le |\Omega|^{-1} |\Omega'| \, \|\mathrm{det} D\Psi\|_\infty^2 \int_{\Omega'} |w'(x') - \bar{w}'| dx'.
\end{aligned}$$

Let C' be the constant of (39) for the set Ω', thus

$$\int_{\Omega'} |w'(x') - \bar{w}'|^2 \, dx' \le C' \int_{\Omega'} |Dw'|^2 \, dx' \,.$$

Now, let us estimate the L^2-norm of the gradient.

$$\begin{aligned}
\int_{\Omega'} |Dw'|^2 dx' &= \int_\Omega |Dw' \circ \Psi^{-1}(x)|^2 \, |\mathrm{det} D\Psi^{-1}(x)| dx \\
&\le \|\mathrm{det} D\Psi^{-1}(x)\|_\infty \int_\Omega |Dw(x) D\Psi \circ \Psi^{-1}(x)|^2 dx \\
&\le \|\mathrm{det} D\Psi^{-1}(x)\|_\infty \, \|D\Psi\|_\infty^2 \int_\Omega |Dw(x)|^2 dx.
\end{aligned}$$

Joining all the inequalities proves the Corollary. □

Using Lemma 2.5 and the proof of Proposition A.2 it is not difficult to prove the following uniform Poincaré inequality.

Proposition A.3. *Let $K \in \mathcal{K}(\Omega)$ and $u \in H^1(\Omega \setminus K)$ with $u = 0$ on $\partial_D \Omega \setminus K$. There exists a constant $C > 0$ such that*

$$\int_\Omega |u|^2 \, dx \le C \int_\Omega |Du|^2 \, dx \,. \tag{40}$$

The constant C can be chosen uniformly for a family of domains which are bi-Lipschitz equivalent to $\Omega \setminus K$ with uniformly controlled Lipschitz constants.

A.2 Rigidity estimates

The next Lemma is proved in [11, Lemma 3.1].

Lemma A.4. *There exists a constant C (which depends only on the energy density W) such that*

$$\mathrm{dist}(F, SO(2))^2 \le C W(F) \,,$$

for every $F \in \mathbb{R}^{2\times 2}$.

The next result, known as quantitative rigidity estimate, is instead adapted from [18].

Theorem A.5. *Let Ω be a bounded, connected Lipschitz domain. There exists a constant C, depending on Ω, with the following property: For each $v \in H^1(\Omega;\mathbb{R}^2)$ there is an associated rotation $R \in SO(2)$ such that*

$$\|Dv - R\|_2 \leq C\|\mathrm{dist}(Dv, SO(2))\|_2 .$$

Moreover, the constant C can be chosen uniformly for a family of domains which are bi-Lipschitz equivalent to Ω with uniformly controlled Lipschitz constants.

A.3 Mosco convergence

We recall here the convergence in the sense of Mosco (cf. [23, Definition 1.1]) of the Sobolev spaces corresponding to a convergent sequence of cracked domains: we say that the spaces

$$\mathbb{H}_n := \{u \in H^1(\Omega \setminus K_n;\mathbb{R}^2)\,,\ u = g_n \text{ on } \partial_D\Omega \setminus K_n\}$$

converge to

$$\mathbb{H} := \{u \in H^1(\Omega \setminus K;\mathbb{R}^2)\ \ u = g \text{ on } \partial_D\Omega \setminus K\}$$

in the sense of Mosco, if the following two conditions hold:

(M_1) for every $u \in H^1(\Omega \setminus K;\mathbb{R}^2)$ with $u = g$ on $\partial_D\Omega \setminus K$ there exists a sequence $u_n \in H^1(\Omega \setminus K_n;\mathbb{R}^2)$, with $u_n = g_n$ on $\partial_D\Omega \setminus K_n$ such that u_n converges strongly to u in $L^2(\Omega;\mathbb{R}^2)$ and Du_n converges strongly to Du in $L^2(\Omega;\mathbb{R}^{2\times 2})$;

(M_2) if (h_n) is a sequence of indices that tends to $+\infty$, and u_n is a sequence with $u_n \in H^1(\Omega\setminus K_{h_n};\mathbb{R}^2)$ and $u_n = g_{h_n}$ on $\partial_D\Omega \setminus K_{h_n}$ for every n, such that u_n converges weakly in $L^2(\Omega;\mathbb{R}^2)$ to a function ϕ and Du_n converges weakly in $L^2(\Omega;\mathbb{R}^{2\times 2})$ to a function ψ, then there exists a function $u \in H^1(\Omega\setminus K;\mathbb{R}^2)$ with $u = g$ on $\partial_D\Omega \setminus K$ such that $\phi = u$ and $\psi = Du$.

The following result was proved in [8, Theorem 6.3] and employs some ideas of [6].

Theorem A.6. *Let $\{g_n\}$ be a sequence in $H^1(\Omega;\mathbb{R}^2)$ converging to $g \in H^1(\Omega;\mathbb{R}^2)$, and let $\{K_n\}$ be a sequence of compact subsets of $\overline{\Omega}$ converging to K in the Hausdorff metric. Assume that $\mathcal{H}^1(K_n)$ converges to $\mathcal{H}^1(K)$ and that K_n have a uniformly bounded number of connected components. Then $\mathbb{H}_n$ converges to $\mathbb{H}$ in the sense of Mosco.*

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