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OPTIMAL STABILITY IN THE IDENTIFICATION OF A RIGID INCLUSION IN AN ISOTROPIC KIRCHHOFF–LOVE PLATE*

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Abstract. In this paper we consider the inverse problem of determining a rigid inclusion inside a thin plate by applying a couple field at the boundary and by measuring the induced transversal displacement and its normal derivative at the boundary of the plate. The plate is made by nonhomogeneous, linearly elastic, and isotropic material. Under suitable a priori regularity assumptions on the boundary of the inclusion, we prove a constructive stability estimate of log type. A key mathematical tool is a recently proved optimal three-spheres inequality at the boundary for solutions to the Kirchhoff–Love plate's equation.

Key words. inverse problems, elastic plates, unique continuation, stability estimates, rigid inclusion

AMS subject classifications. Primary, 35B60; Secondary, 35B30, 35Q74, 35R30

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1. Introduction. In this paper we consider the inverse problem of the stable determination of a rigid inclusion embedded in a thin elastic plate by measuring the transverse displacement and its normal derivative at the boundary induced by a couple field applied at the boundary of the plate. We prove that the stability estimate of log-log type found in [M-Ro-Ve2] can be improved to a single logarithm in the case in which the plate is made of isotropic linear elastic material.

From the point of view of applications, modern requirements of structural condition assessment demand the identification of defects using nondestructive methods, and, therefore, the present results can be useful in quality control of plates. We refer to, among other contributions, Bonnet and Constantinescu [Bo-Co] for a general overview of inverse problems arising in diagnostic analysis applied to linear elasticity and, in particular, to plate theory [Bo-Co, section 5.3], and to [K] for the identification of a stiff inclusion in a composite thin plate based on wavelet analysis of the eigenfunctions.

In order to describe our stability result, let us introduce the Kirchhoff–Love model of a thin, elastic isotropic plate under infinitesimal deformation; see, for example, [G]. Let the middle plane of the plate Ω be a bounded domain of $\mathbb{R}^2$ with regular boundary. The rigid inclusion D is modeled as a simply connected domain compactly contained in Ω. Under the assumptions of vanishing transversal forces in Ω, and for a given

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couple field $\widehat{M}$ acting on $\partial\Omega$, the transversal displacement $w \in H^2(\Omega)$ of the plate satisfies the mixed boundary value problem

$$\begin{cases} \mathrm{div}(\mathrm{div}(\mathbb{P}\nabla^2 w)) = 0 & \text{in } \Omega \setminus \overline{D}, & (1.1)\\ (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n & \text{on } \partial\Omega, & (1.2)\\ \mathrm{div}(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega, & (1.3)\\ w|_{\overline{D}} \in \mathcal{A} & \text{in } \overline{D}, & (1.4)\\ w^e{}_{,n} = w^i_n & \text{on } \partial D, & (1.5)\end{cases}$$

coupled with the *equilibrium conditions* for the rigid inclusion D

$$(1.6)\quad \int_{\partial D} \left(\mathrm{div}(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s}\right) g - ((\mathbb{P}\nabla^2 w)n\cdot n) g_{,n} = 0 \qquad \text{for every } g \in \mathcal{A},$$

where $\mathcal{A}$ denotes the space of affine functions. We recall that, from the physical point of view, the boundary conditions (1.4)–(1.5) correspond to ideal connection between the boundary of the rigid inclusion and the surrounding elastic material; see, for example, [O-Ri, section 10.10]. The unit vectors n and τ are the outer normal and the tangent vector to the boundary of $\Omega \setminus \overline{D}$, respectively. We denote by $w_{,s}$, $w_{,n}$ the derivatives of the function w with respect to the arclength s and to the normal direction, respectively. Moreover, we have defined $w^e \equiv w|_{\Omega\setminus\overline{D}}$ and $w^i \equiv w|_{\overline{D}}$. The functions $\widehat{M}_\tau$, $\widehat{M}_n$ are the twisting and bending component of the assigned couple field $\widehat{M}$, respectively. The plate tensor $\mathbb{P}$ is given by $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$, where h is the constant thickness of the plate and $\mathbb{C}$ is the nonhomogeneous Lamé elasticity tensor describing the response of the material.

The existence of a solution $w \in H^2(\Omega)$ of the problem (1.1)–(1.6) is ensured by general results, provided that $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, with $\int_{\partial\Omega} \widehat{M}_i = 0$, for $i = 1, 2$ (where $\widehat{M} = \widehat{M}_2 e_1 + \widehat{M}_1 e_2$ is the representation of $\widehat{M}$ in cartesian coordinates), and $\mathbb{P}$ is bounded and strongly convex. Let us notice that w is uniquely determined up to addition of an affine function.

Let us denote by w_i a solution to (1.1)–(1.6) for $D = D_i$, $i = 1, 2$. In order to deal with the stability issue, we found it convenient to replace each solution w_i with $v_i = w_i - g_i$, where g_i is the affine function which coincides with w_i on ∂D_i, $i = 1, 2$. By this approach, maintaining the same letter to denote the solution, the equilibrium problem (1.1)–(1.5) can be rephrased in terms of the following mixed boundary value problem in $\Omega \setminus \overline{D}$ with homogeneous Dirichlet conditions on the boundary of the rigid inclusion:

$$\begin{cases} \mathrm{div}(\mathrm{div}(\mathbb{P}\nabla^2 w)) = 0 & \text{in } \Omega \setminus \overline{D}, & (1.7)\\ (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n & \text{on } \partial\Omega, & (1.8)\\ \mathrm{div}(\mathbb{P}\nabla^2 w)\cdot n + ((\mathbb{P}\nabla^2 w)n\cdot\tau)_{,s} = (\widehat{M}_\tau)_{,s} & \text{on } \partial\Omega, & (1.9)\\ w = 0 & \text{on } \partial D, & (1.10)\\ w_{,n} = 0 & \text{on } \partial D, & (1.11)\end{cases}$$

for which there exists a unique solution $w \in H^2(\Omega \setminus \overline{D})$. The arbitrariness of this normalization, related to the fact that g_i is unknown, $i = 1, 2$, leads to the following formulation of the stability issue: *Given an open portion Σ of $\partial\Omega$, satisfying suitable regularity assumptions, and given two solutions w_i to (1.7)–(1.11) when $D = D_i$, $i = 1, 2$, satisfying, for some $\epsilon > 0$,*

$$\min_{g\in\mathcal{A}}\left\{\|w_1-w_2-g\|_{L^2(\Sigma)}+\|(w_1-w_2-g),_n\|_{L^2(\Sigma)}\right\}\leq\epsilon, \tag{1.12}$$

to evaluate the rate at which the Hausdorff distance $d_{\mathcal{H}}(\overline{D_1},\overline{D_2})$ between D_1 and D_2 tends to zero as ϵ tends to zero.

In this paper we prove the following quantitative stability estimate of log type for inclusions D of $C^{6,\alpha}$ class:

$$d_{\mathcal{H}}(\overline{D_1},\overline{D_2})\leq C|\log\epsilon|^{-\eta}, \tag{1.13}$$

where C, η, $C>0$ and $\eta>0$, are constants only depending on the a priori data; see Theorem 3.1 for a precise statement.

The above estimate is an improvement of the log-log type stability estimate found in [M-Ro-Ve2], although it must be said that the latter is not restricted to isotropic materials and also applies to less regular inclusions (e.g., D of $C^{3,1}$ class). On the other hand, in the present paper we remove the hypothesis that the support of the Neumann data $\widehat{M}$ is strictly contained in $\partial\Omega$, which was assumed in [M-Ro-Ve2, section 2.1].

It is worth noticing that a single logarithmic rate of convergence for the fourth order elliptic equation modeling the deflection of a Kirchhoff–Love plate is expected to be optimal, as it is in fact for the analogous inverse problem in the scalar elliptic case, which models the detection of perfectly conducting inclusions in an electric conductor in terms of measurements of potential and current taken on an accessible portion of the boundary of the body, as shown by the counterexamples due to Alessandrini [Al] and Alessandrini and Rondi [Al-R]; see also [Dc-R].

The methods used to prove (1.13) are inspired by the approach presented in the seminal paper [Al-Be-Ro-Ve], where, for the first time, it was shown how logarithmic stability estimates for the inverse problem of determining unknown boundaries can be derived by using quantitative estimates of strong unique continuation at the boundary (SUCB), which ensure a polynomial vanishing rate of the solutions satisfying homogeneous Dirichlet or Neumann conditions at the boundary. Precisely, in [Al-Be-Ro-Ve] the key tool was a doubling inequality at the boundary established by Adolfsson and Escauriaza in [A-E].

Following the direction traced in [Al-Be-Ro-Ve], other kinds of quantitative estimates of the SUCB turned out to be crucial properties to prove optimal stability estimates for inverse boundary value problems with unknown boundaries in different frameworks; see, for instance, [S], where the case of the Robin boundary condition is investigated. Let us recall, in the context of the case of thermic conductors involving parabolic equations, the three-cylinders inequality and the one-sphere two-cylinders inequality at the boundary [Ca-Ro-Ve1], [Ca-Ro-Ve2], [E-F-Ve], [E-Ve], [Ve1] and a similar estimate at the boundary for the case of wave equation with time independent coefficients [S-Ve], [Ve2], [Ve3].

In the present paper, the SUCB property used to improve the double logarithmic estimate found in [M-Ro-Ve2] takes the form of an optimal three-spheres inequality at the boundary. This latter result was recently proved in [Al-Ro-Ve] for isotropic elastic plates under homogeneous Dirichlet boundary conditions and leads to a finite vanishing rate at the boundary (Proposition 3.6).

Other main mathematical tools are quantitative estimates of strong unique continuation at the interior, essentially based on a three-spheres inequality at the interior obtained in [M-Ro-Ve1], which allows us to derive quantitative estimates of unique continuation from Cauchy data (Proposition 3.3), the finite vanishing rate at the interior (Proposition 3.5), and a Lipschitz estimate of propagation of smallness (Proposition 3.4) for the solutions to the plate equation.

Let us observe that estimate (1.13) is the first stability estimate with optimal rate of convergence in the framework of linear elasticity. Indeed, up to now, the analogous estimate for the determination, within isotropic elastic bodies, of rigid inclusions [M-Ro2], cavities [M-Ro1], or pressurized cavities [As-Be-Ro] shows a double logarithmic character, and the same convergence rate has been established by Lin, Nakamura, and Wang for star-shaped cavities inside anisotropic elastic bodies [L-N-W]. Finally, it is worth noticing that our approach could be extended to find a log-type stability estimate analogous to (1.13) for the determination of a cavity inside the plate, provided the SUCB property is available for homogeneous Neumann boundary conditions.

The plan of the paper is as follows. Main notation and a priori information are presented in section 2. In section 3, we first state our main result (Theorem 3.1). In the same section we also state some auxiliary propositions regarding the estimate of continuation from Cauchy data (Proposition 3.3) and from the interior (Proposition 3.4) and the determination of the finite vanishing rate of the solutions to the plate equation at the interior (Proposition 3.5) and at the Dirichlet boundary (Proposition 3.6). Finally, in the second part of section 3 we give a proof of Theorem 3.1.

2. Notation. Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius r and center P and by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes, of length $2a$ and $2b$, namely, $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a,\ |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

Given a bounded domain Ω in $\mathbb{R}^2$ we shall denote

$$\Omega_\rho = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) > \rho\}. \tag{2.1}$$

When representing locally a boundary as a graph, we use the following definition.

Definition 2.1 ($C^{k,\alpha}$ regularity). *Let Ω be a bounded domain in $\mathbb{R}^2$. Given k, α, with $k \in \mathbb{N}$, $0 < \alpha \le 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants r_0, $M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and*

$$\Omega \cap R_{r_0,2M_0r_0} = \{x \in R_{r_0,2M_0r_0} \quad | \quad x_2 > g(x_1)\},$$

where g is a $C^{k,\alpha}$ function on $[-r_0, r_0]$ satisfying

$$g(0) = g'(0) = 0,$$

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} \le M_0 r_0,$$

where

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} = \sum_{i=0}^{k} r_0^i \sup_{[-r_0,r_0]} |g^{(i)}| + r_0^{k+\alpha} |g|_{k,\alpha},$$

$$|g|_{k,\alpha} = \sup_{\substack{t,s \in [-r_0,r_0] \\ t \ne s}} \left\{ \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t-s|^\alpha} \right\}.$$

We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition when the dimensional parameter equals one. For instance,

$$\|w\|_{H^1(\Omega)} = r_0^{-1}\left(\int_\Omega w^2 + r_0^2\int_\Omega |\nabla w|^2\right)^{\frac{1}{2}},$$

and so on for boundary and trace norms.

Given a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is of class $C^{k,\alpha}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is,

$$\tau = e_3 \times n, \tag{2.2}$$

where $\times$ denotes the vector product in $\mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.

Throughout the paper, in order to simplify the notation, when we write $\int_{\partial\Omega} uv$ with $u \in H^{-1/2}(\partial\Omega, \mathbb{R}^2)$, $v \in H^{1/2}(\partial\Omega, \mathbb{R}^2)$, we mean the duality pairing $< u, v >_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$, and similarly for other trace norms.

In what follows we shall denote by C constants which may change from line to line.

2.1. A priori information. (i) *A priori information on the domain.* Let us consider a thin plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ with middle surface represented by a bounded domain Ω in $\mathbb{R}^2$ and having uniform thickness h, $h << \text{diam}(\Omega)$.

We shall assume that, given r_0, $M_1 > 0$,

$$\text{diam}(\Omega) \leq M_1 r_0. \tag{2.3}$$

We shall also assume that Ω contains an open simply connected rigid inclusion D such that

$$\text{dist}(D, \partial\Omega) \geq r_0. \tag{2.4}$$

Moreover, we denote by Σ an open portion within $\partial\Omega$ representing the part of the boundary where measurements are taken.

Concerning the regularity of the boundaries, given $M_0 \geq \frac{1}{2}$ and α, $0 < \alpha \leq 1$, we assume that

$$\partial\Omega \text{ is of } \textit{class } C^{2,1} \textit{ with constants } r_0, M_0, \tag{2.5}$$

$$\Sigma \text{ is of } \textit{class } C^{3,1} \textit{ with constants } r_0, M_0, \tag{2.6}$$

$$\partial D \text{ is of } \textit{class } C^{6,\alpha} \textit{ with constants } r_0, M_0. \tag{2.7}$$

Let us notice that, without loss of generality, we have chosen $M_0 \geq \frac{1}{2}$ to ensure that $B_{r_0}(P) \subset R_{r_0, 2M_0 r_0}(P)$ for every $P \in \partial\Omega$.

Moreover, we shall assume that for some $P_0 \in \Sigma$

$$\partial\Omega \cap R_{r_0, 2M_0 r_0}(P_0) \subset \Sigma. \tag{2.8}$$

(ii) *Assumptions about the boundary data.* On the Neumann data $\widehat{M}$, $\widehat{M} = \widehat{M}_\tau n + \widehat{M}_n \tau$, we assume that

$$\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \quad (\widehat{M}_n, (\widehat{M}_\tau),_s) \not\equiv 0, \tag{2.9}$$

$$\mathrm{supp}(\widehat{M}) \subset\subset \Sigma, \tag{2.10}$$

the (obvious) compatibility condition on the ith component of $\widehat{M}$ in a given Cartesian coordinate system

$$\int_{\partial\Omega} \widehat{M}_i = 0, \quad i = 1, 2, \tag{2.11}$$

and that, for a given constant $F > 0$,

$$\frac{\|\widehat{M}\|_{L^2(\partial\Omega,\mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}} \leq F, \tag{2.12}$$

where we denote

$$\|\widehat{M}\|_{L^2(\partial\Omega,\mathbb{R}^2)} = \|\widehat{M}_n\|_{L^2(\partial\Omega)} + r_0 \|(\widehat{M}_\tau),_s\|_{H^{-1}(\partial\Omega)}, \tag{2.13}$$

$$\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)} = \|\widehat{M}_n\|_{H^{-\frac{1}{2}}(\partial\Omega)} + r_0 \|(\widehat{M}_\tau),_s\|_{H^{-\frac{3}{2}}(\partial\Omega)}, \tag{2.14}$$

and similarly for other trace norms.

(iii) *Assumptions about the elasticity tensor.* Let us assume that the plate is made by elastic isotropic material, the plate tensor $\mathbb{P}$ is defined by

$$\mathbb{P}A = B\left[(1-\nu)A^{sym} + \nu(\mathrm{tr}A)I_2\right] \tag{2.15}$$

for every 2×2 matrix A, where I_2 is the 2×2 identity matrix, and $\mathrm{tr}(A)$ denotes the trace of the matrix A. The *bending stiffness* (per unit length) of the plate is given by the function

$$B(x) = \frac{h^3}{12}\left(\frac{E(x)}{1-\nu^2(x)}\right), \tag{2.16}$$

where the *Young's modulus* E and the *Poisson's coefficient* ν can be written in terms of the Lamé moduli as follows:

$$E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \qquad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))}. \tag{2.17}$$

Hence, in this case, the displacement equation of equilibrium (1.1) is

$$\mathrm{div}\left(\mathrm{div}\left(B((1-\nu)\nabla^2 w + \nu \Delta w I_2)\right)\right) = 0 \qquad \text{in } \Omega. \tag{2.18}$$

We make the following strong convexity assumptions on the Lamé moduli:

$$\mu(x) \geq \alpha_0 > 0, \qquad 2\mu(x) + 3\lambda(x) \geq \gamma_0 > 0 \qquad \text{in } \overline{\Omega}, \tag{2.19}$$

where α_0, γ_0 are positive constants.

We assume that the Lamé moduli λ, μ satisfy the following regularity assumptions:

$$\|\lambda\|_{C^4(\overline{\Omega})}, \quad \|\mu\|_{C^4(\overline{\Omega})} \le \Lambda_0. \tag{2.20}$$

It should be noted that the regularity hypotheses required on the elastic coefficients and on the boundary of the inclusion are required to apply the arguments and techniques used in [Al-Ro-Ve] to derive the SUCB for isotropic elastic plates.

Under the above assumptions, the weak formulation of the problem (1.7)–(1.11) consists in finding $w \in H^2(\Omega \setminus \overline{D})$, with $w = 0$ and $w_{,n} = 0$ on ∂D, such that

$$\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 w \cdot \nabla^2 v = \int_{\partial\Omega} -\widehat{M}_{\tau,s} v - \widehat{M}_n v_{,n}\,, \tag{2.21}$$

for every $v \in H^2(\Omega \setminus \overline{D})$, with $v = 0$ and $v_{,n} = 0$ on ∂D. By standard variational arguments (see, for example, [Ag]), the above problem has a unique solution $w \in H^2(\Omega \setminus \overline{D})$ satisfying the stability estimate

$$\|w\|_{H^2(\Omega\setminus\overline{D})} \le C r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}, \tag{2.22}$$

where $C > 0$ only depends on α_0, γ_0, M_0, and M_1.

In what follows, we shall refer to the set of constants α_0, γ_0, Λ_0, α, M_0, M_1, and F as the a priori data.

3. Statement and proof of the main result. Here and in what follows we shall denote by G the connected component of $\Omega \setminus \overline{(D_1 \cup D_2)}$ such that $\Sigma \subset \partial G$.

Theorem 3.1 (stability result). *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying* (2.3) *and* (2.5). *Let D_i, $i = 1, 2$, be two simply connected open subsets of Ω satisfying* (2.4) *and* (2.7). *Moreover, let Σ be an open portion of $\partial\Omega$ satisfying* (2.6) *and* (2.8). *Let $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$ satisfy* (2.9)–(2.12) *and let the plate tensor $\mathbb{P}$ given by* (2.15) *with Lamé moduli satisfying the regularity assumptions* (2.20) *and the strong convexity condition* (2.19). *Let $w_i \in H^2(\Omega \setminus \overline{D_i})$ be the solution to* (1.7)–(1.11) *when $D = D_i$, $i = 1, 2$. If, given $\epsilon > 0$, we have*

$$\min_{g\in\mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + r_0 \left\|(w_1 - w_2 - g)_{,n}\right\|_{L^2(\Sigma)} \right\} \le \epsilon, \tag{3.1}$$

then we have

$$d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \le r_0 \omega \left(\frac{\epsilon}{r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}} \right), \tag{3.2}$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$\omega(t) \le C(|\log t|)^{-\eta} \quad \textit{for every } t,\ 0 < t < 1, \tag{3.3}$$

and C, η, $C > 0$, $\eta > 0$, are constants only depending on the a priori data.

Remark 3.2. Before presenting the proof of the theorem, it is appropriate to underline the optimality of the estimate (3.2) and the improvement it provides with respect to previous results. The presence of a logarithm in the stability estimate is expected and inevitable, since this is a consequence of the ill-posedness of the Cauchy problem (see Proposition 3.3). As mentioned in the introduction, a log-log type stability estimate was already derived in [M-Ro-Ve2]. In that paper, the additional logarithm was essentially the consequence of the application of a unique continuation result from the interior expressed in the form of the Lipschitz propagation of smallness for solutions to the plate equation (see also Proposition 3.5). In the proof of Theorem 3.1, instead, a different line of reasoning was followed, that is, inspired by the paper [Al-Be-Ro-Ve], we exploited the polynomial vanishing rate of the solution, both inside $\Omega \setminus D$ and close to the boundary of D. The former was in fact already available from the results in [M-Ro-Ve2], while the latter was only recently proved in [Al-Ro-Ve] for homogeneous Dirichlet boundary conditions on ∂D.

The proof of Theorem 3.1 is obtained from a sequence of propositions. The following proposition can be derived by merging Proposition 3.4 of [M-Ro-Ve2] and geometrical arguments contained in Proposition 3.6 of [Al-Be-Ro-Ve].

PROPOSITION 3.3 (stability estimate of continuation from Cauchy data [M-Ro-Ve2, Proposition 3.4]). *Let the hypotheses of Theorem* 3.1 *be satisfied. We have*

$$\int_{(\Omega\setminus\overline{G})\setminus\overline{D_1}} |\nabla^2 w_1|^2 \le r_0^2 \|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}\, \omega\left(\frac{\epsilon}{r_0^2\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{3.4}$$

$$\int_{(\Omega\setminus\overline{G})\setminus\overline{D_2}} |\nabla^2 w_2|^2 \le r_0^2 \|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}\, \omega\left(\frac{\epsilon}{r_0^2\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\right), \tag{3.5}$$

where ω is an increasing continuous function on $[0,\infty)$ which satisfies

$$\omega(t) \le C(\log|\log t|)^{-\frac{1}{2}} \qquad \text{for every } t < e^{-1} \tag{3.6}$$

with $C>0$ only depending on α_0, γ_0, Λ_0, M_0, and M_1.

Moreover, there exists $d_0 > 0$, with $\frac{d_0}{r_0}$ only depending on M_0, such that if $d_{\mathcal{H}}(\overline{\Omega\setminus D_1}, \overline{\Omega\setminus D_2}) \le d_0$, then (3.4)*–*(3.5) *hold with ω given by*

$$\omega(t) \le C|\log t|^{-\sigma} \qquad \text{for every } t < 1, \tag{3.7}$$

where $\sigma > 0$ and $C > 0$ only depend on α_0, γ_0, Λ_0, M_0, M_1, L, and $\frac{\tilde{r}_0}{r_0}$.

The next two propositions are quantitative versions of the SUCP property at the interior for solutions to the plate equilibrium problem. Precisely, Proposition 3.4 has global character and gives a lower bound of the strain energy density over any small disc compactly contained in $\Omega \setminus \overline{D}$ in terms of the Neumann boundary data. Instead, Proposition 3.5 establishes a polynomial order of vanishing for solutions to the plate problem at interior points of $\Omega \setminus \overline{D}$.

PROPOSITION 3.4 (Lipschitz propagation of smallness). *Let Ω be a bounded domain in $\mathbb{R}^2$ satisfying* (2.3) *and* (2.5)*. Let D be an open simply connected subset of Ω satisfying* (2.4), (2.7)*. Let $w \in H^2(\Omega\setminus\overline{D})$ be the solution to* (1.7)*–*(1.11)*, coupled with the equilibrium condition* (1.6)*, where the plate tensor $\mathbb{P}$ is given by* (2.15) *with Lamé moduli satisfying the regularity assumptions* (2.20) *and the strong convexity condition* (2.19) *and with $\widehat{M} \in L^2(\partial\Omega,\mathbb{R}^2)$ satisfying* (2.9)*–*(2.12)*.*

There exists $s > 1$, only depending on α_0, γ_0, Λ_0, and M_0, such that for every $\rho > 0$ and every $\bar{x} \in (\Omega\setminus\overline{D})_{s\rho}$, we have

$$\int_{B_\rho(\bar{x})} |\nabla^2 w|^2 \ge \frac{C r_0^2}{\exp\left[A\left(\frac{r_0}{\rho}\right)^B\right]} \|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}, \tag{3.8}$$

where $A>0$, $B>0$, and $C>0$ only depend on α_0, γ_0, Λ_0, M_0, M_1, and F.

Proof. The proof of this proposition, rather technical, is mainly based on the derivation of a lower bound of the strain energy density over the disc $B_\rho(\overline{x})$ in terms of the strain energy density over all the domain $\Omega \setminus \overline{D}$. This estimate requires a geometrical construction involving a number of iterated applications of the three-spheres inequality (3.8) which leads to an exponential dependence on the radius ρ.

The arguments follow the lines of the proof of Proposition 3.3 in [M-Ro-Ve2], the only difference consisting in estimating from below the strain energy $\int_{\Omega\setminus\overline{D}} |\nabla^2 w|^2$ in terms of the $H^{-\frac{1}{2}}$ norm of $\widehat{M}$ as defined in (2.14), that is, considering only the tangential derivative of the normal component $\widehat{M}_\tau$. This more natural choice allows us to remove the technical hypothesis that the support of $\widehat{M}$ is strictly contained in $\partial\Omega$, assumed in Lemma 4.6 in [M-Ro-Ve2]; see also [M-Ro-Ve1, Lemma 7.1] for details.

Precisely, we only need to prove the following trace-type inequality:

$$\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)} \leq C\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})}, \tag{3.9}$$

where C only depends on M_0, M_1, and Λ_0.

Let us first estimate the $H^{-\frac{1}{2}}(\partial\Omega)$ norm of $\widehat{M}_n$. Given any $g \in H^{\frac{1}{2}}(\partial\Omega)$, let $v \in H^2(\Omega \setminus \overline{D})$ such that $v = 0$, $v,_n = g$ on $\partial\Omega$, and $\|v\|_{H^2(\Omega\setminus\overline{D})} \leq Cr_0\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$, where C only depends on M_0 and M_1. By the weak formulation (2.21) of the equilibrium problem (1.7)–(1.11), we have

$$\begin{aligned}\int_{\partial\Omega} \widehat{M}_n g &= \int_{\partial\Omega} \widehat{M}_n v,_n + \int_{\partial\Omega} \widehat{M}_{\tau,s} v = -\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 w\cdot\nabla^2 v \\ &\leq \left(\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 w\cdot\nabla^2 w\right)^{\frac{1}{2}} \left(\int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 v\cdot\nabla^2 v\right)^{\frac{1}{2}} \\ &\leq C\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})}\|v\|_{H^2(\Omega\setminus\overline{D})} \leq Cr_0\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})}\end{aligned} \tag{3.10}$$

with C only depending on M_0, M_1, and Λ_0. Therefore

$$\|\widehat{M}_n\|_{H^{-\frac{1}{2}}(\partial\Omega)} = \sup_{\substack{g\in H^{1/2}(\partial\Omega)\\ \|g\|_{H^{1/2}(\partial\Omega)}=1}} \frac{1}{r_0}\int_{\partial\Omega} \widehat{M}_n g \leq C\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})} \tag{3.11}$$

with C only depending on M_0, M_1, and Λ_0.

Next, let us estimate the $H^{-\frac{3}{2}}(\partial\Omega)$ norm of $(\widehat{M}_\tau),_s$. Given any $g \in H^{\frac{3}{2}}(\partial\Omega)$, let $v \in H^2(\Omega \setminus \overline{D})$ such that $v = g$ on $\partial\Omega$, and $\|v\|_{H^2(\Omega\setminus\overline{D})} \leq C\|g\|_{H^{\frac{3}{2}}(\partial\Omega)}$, where C only depends on M_0 and M_1. By recalling (3.11), we can compute

$$\begin{aligned}-\int_{\partial\Omega} \widehat{M}_{\tau,s} g &= \int_{\partial\Omega} -\widehat{M}_{\tau,s} v - \widehat{M}_n v,_n + \int_{\partial\Omega} \widehat{M}_n v,_n \\ &= \int_{\Omega\setminus\overline{D}} \mathbb{P}\nabla^2 w\cdot\nabla^2 v + \int_{\partial\Omega} \widehat{M}_n v,_n \\ &\leq C\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})}\|v\|_{H^2(\Omega\setminus\overline{D})} + Cr_0\|\widehat{M}_n\|_{H^{-\frac{1}{2}}(\partial\Omega)}\|v,_n\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C\|v\|_{H^2(\Omega\setminus\overline{D})}\left(\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})} + \|\widehat{M}_n\|_{H^{-\frac{1}{2}}(\partial\Omega)}\right) \leq C\|g\|_{H^{\frac{3}{2}}(\partial\Omega)}\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})}\end{aligned} \tag{3.12}$$

with C only depending on M_0, M_1, and Λ_0. Therefore

$$\|\widehat{M}_{\tau,s}\|_{H^{-\frac{3}{2}}(\partial\Omega)} = \sup_{\substack{g\in H^{3/2}(\partial\Omega)\\ \|g\|_{H^{3/2}(\partial\Omega)}=1}} \frac{1}{r_0}\int_{\partial\Omega} \widehat{M}_{\tau,s} g \leq \frac{C}{r_0}\|\nabla^2 w\|_{L^2(\Omega\setminus\overline{D})} \tag{3.13}$$

with C only depending on M_0, M_1, and Λ_0. By (3.11) and (3.13), (3.9) follows. □

PROPOSITION 3.5 (finite vanishing rate at the interior). *Under the hypotheses of Proposition* 3.4, *there exist* $\widetilde{c}_0 < \frac{1}{2}$ *and* $C > 1$, *only depending on* α_0, γ_0, *and* Λ_0, *such that, for every* $\overline{r} \in (0, r_0)$ *and for every* $\overline{x} \in \Omega \setminus \overline{D}$ *such that* $B_{\overline{r}}(\overline{x}) \subset \Omega \setminus \overline{D}$, *and for every* $r_1 < \widetilde{c}_0\overline{r}$, *we have*

$$\int_{B_{r_1}(\overline{x})} |\nabla^2 w|^2 \geq C\left(\frac{r_1}{\overline{r}}\right)^{\tau_0} \int_{B_{\overline{r}}(\overline{x})} |\nabla^2 w|^2, \tag{3.14}$$

where $\tau_0 \geq 1$ *only depends on* α_0, γ_0, Λ_0, M_0, M_1, $\frac{r_0}{\overline{r}}$, *and* F.

Proof. The above estimate is based on the following three-spheres inequality at the interior, which was obtained in [M-Ro-Ve1, Theorem 6.3]: there exist c_0, $0 < c_0 < 1$, s_1, $0 < s_1 < 1$, and $C_1 > 1$ only depending on α_0, γ_0, and Λ_0, such that for every $\overline{r} > 0$, for every $\overline{x} \in (\Omega \setminus \overline{D})$ such that $B_{\overline{r}}(\overline{x}) \subset \Omega \setminus \overline{D}$, for every $r_1 < r_2 < c_0 r_3$, $r_3 < s_1\overline{r}$, and for any solution v to (2.18), we have

$$H(v; r_2) \leq \left(\frac{C_1\overline{r}}{r_2}\right)^{C_1} \left(H\left(v; \frac{r_1}{2}\right)\right)^{\vartheta_0} \left(H\left(v; \frac{r_3}{2}\right)\right)^{1-\vartheta_0}, \tag{3.15}$$

where

$$H(v;t) = \sum_{k=0}^{3} t^{2k} \int_{B_t(\overline{x})} \left|\nabla^k v\right|^2 \qquad \text{for every } t \in (0, \overline{r}] \tag{3.16}$$

and

$$\vartheta_0 = \frac{\log\left(\frac{c_0 r_3}{r_2}\right)}{2\log\left(\frac{r_3}{r_1}\right)}. \tag{3.17}$$

Let w be the solution to boundary value problem (1.7)–(1.11) and let us denote

$$v(x) = w(x) - a - \gamma\cdot(x - \overline{x}), \tag{3.18}$$

where

$$a = \frac{1}{|B_{r_1}(\overline{x})|}\int_{B_{r_1}(\overline{x})} w \qquad \text{and} \qquad \gamma = \frac{1}{|B_{r_1}(\overline{x})|}\int_{B_{r_1}(\overline{x})} \nabla w. \tag{3.19}$$

By a Caccioppoli-type inequality [M-Ro-Ve1, Proposition 6.2] and the Poincaré inequality we get

$$H\left(v; \frac{r_1}{2}\right) \leq C r_1^4 \int_{B_{r_1}(\overline{x})} \left|\nabla^2 w\right|^2, \tag{3.20}$$

where C depends on α_0, γ_0, and Λ_0 only.

Now, we estimate from above $H\left(v; \frac{r_3}{2}\right)$. By the Caccioppoli-type inequality we have

$$H\left(v;\frac{r_3}{2}\right)\leq C\sum_{k=0}^{2}r_3^{2k}\int_{B_{\frac{2r_3}{3}}(\overline{x})}\left|\nabla^k v\right|^2, \tag{3.21}$$

where C depends on α_0, γ_0, and Λ_0 only. In addition, by (3.19) and Sobolev's inequality [Ag, Theorem 3.9] we have

$$|a|\leq\|w\|_{L^\infty\left(B_{2r_3/3}(\overline{x})\right)}\leq C\|w\|_{H^2\left(B_{2r_3/3}(\overline{x})\right)} \tag{3.22}$$

and

$$|\gamma|\leq\|\nabla w\|_{L^\infty\left(B_{2r_3/3}(\overline{x})\right)}\leq Cr_3^{-1}\|w\|_{H^3\left(B_{2r_3/3}(\overline{x})\right)}, \tag{3.23}$$

where C is an absolute constant. Hence, by (3.22), (3.23) and by the Caccioppoli-type inequality we have

$$\begin{aligned}\sum_{k=0}^{2}r_3^{2k}\int_{B_{\frac{2r_3}{3}}(\overline{x})}\left|\nabla^k v\right|^2&\leq C\sum_{k=0}^{3}r_3^{2k}\int_{B_{\frac{2r_3}{3}}(\overline{x})}\left|\nabla^k w\right|^2\\&\leq C'\sum_{k=0}^{2}r_3^{2k}\int_{B_{r_3}(\overline{x})}\left|\nabla^k w\right|^2,\end{aligned} \tag{3.24}$$

where C is an absolute constant and C' depends on α_0, γ_0, and Λ_0 only. In addition, by (2.22), (3.21), and (3.24) we obtain

$$H\left(v;\frac{r_3}{2}\right)\leq C_2r_0^6\|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}, \tag{3.25}$$

where C_2 depends on α_0, γ_0, Λ_0, M_0, and M_1 only.

By (3.15), (3.20), and (3.25) we have

(3.26)
$$r_2^4\int_{B_{r_2}(\overline{x})}\left|\nabla^2 w\right|^2\leq C_3\left(\frac{C_1\overline{r}}{r_2}\right)^{C_1}\left(r_1^4\int_{B_{r_1}(\overline{x})}\left|\nabla^2 w\right|^2\right)^{\vartheta_0}\left(C_2r_0^6\|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}\right)^{1-\vartheta_0},$$

where $C_3>1$ depends on α_0, γ_0, and Λ_0 only. Let us introduce the notation

$$g(r)=\frac{r^4\int_{B_r(\overline{x})}\left|\nabla^2 w\right|^2}{C_2r_0^6\|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)}}\qquad\text{for every } r\in\left(0,\frac{r_3}{2}\right] \tag{3.27}$$

and

$$K=C_3\left(\frac{C_1\overline{r}}{r_2}\right)^{C_1}, \tag{3.28}$$

so that (3.26) is equivalent to

$$g(r_1)\geq\left(\frac{g(r_2)}{K}\right)^{\vartheta_0^{-1}}. \tag{3.29}$$

We notice that

$$\vartheta_0^{-1}=\log\left(\frac{r_3}{r_1}\right)^m,$$

where $m = \frac{2}{\log(\frac{c_0 r_3}{r_2})}$. Recalling the elementary identity $\kappa^{\log\zeta} = \zeta^{\log\kappa}$, by (3.29) we have

$$g(r_1) \geq \left(\frac{r_1}{r_3}\right)^{m\log\left(\frac{K}{g(r_2)}\right)}. \tag{3.30}$$

Note that by (3.25), (3.27), and (3.28) we have $\frac{K}{g(r_2)} > 1$, so that

$$g(r_1) \geq \left(\frac{r_1}{\overline{r}}\right)^{m\log\left(\frac{K}{g(r_2)}\right)}. \tag{3.31}$$

Choosing $r_2 = \overline{c}_0\overline{r}$, where $\overline{c}_0 < \frac{1}{2}c_0$, by Proposition 3.4 we have

$$\frac{K}{g(r_2)} \leq C\exp\left[A\left(\frac{r_0}{\overline{c}_0\overline{r}}\right)^B\right], \tag{3.32}$$

where C depends on α_0, γ_0, Λ_0, M_0, M_1, F, and $\frac{r_0}{\overline{r}}$ only, and A, B are the same as in Proposition 3.4.

Finally, taking into account (2.22), by (3.30) and (3.32) the thesis follows. $\square$

As noted in the introduction, our key SUCB property is stated in the following proposition, which is the counterpart at the boundary ∂D of Proposition 3.5.

PROPOSITION 3.6 (finite vanishing rate at the boundary). *Under the hypotheses of Proposition* 3.4, *there exist* $\overline{c} < \frac{1}{2}$ *and* $C > 1$, *only depending on* α_0, γ_0, Λ_0, M_0, α, *such that, for every* $x \in \partial D$ *and for every* $r_1 < \overline{c}r_0$,

$$\int_{B_{r_1}(x)\cap(\Omega\setminus\overline{D})} w^2 \geq C\left(\frac{r_1}{r_0}\right)^\tau \int_{B_{r_0}(x)\cap(\Omega\setminus\overline{D})} w^2, \tag{3.33}$$

where $\tau \geq 1$ *only depends on* α_0, γ_0, Λ_0, M_0, α, M_1, *and* F.

Proof. By Corollary 2.3 in [Al-Ro-Ve], there exist $c < 1$, only depending on M_0, α, and $C > 1$ only depending on α_0, γ_0, Λ_0, M_0, α, such that, for every $x \in \partial D$ and for every $r_1 < r_2 < cr_0$,

$$\int_{B_{r_1}(x)\cap(\Omega\setminus\overline{D})} w^2 \geq C\left(\frac{r_1}{r_0}\right)^{\frac{\log B}{\log\frac{cr_0}{r_2}}} \int_{B_{r_0}(x)\cap(\Omega\setminus\overline{D})} w^2, \tag{3.34}$$

where $B > 1$ is given by

$$B = C\left(\frac{r_0}{r_2}\right)^C \frac{\int_{B_{r_0}(x)\cap(\Omega\setminus\overline{D})} w^2}{\int_{B_{r_2}(x)\cap(\Omega\setminus\overline{D})} w^2}. \tag{3.35}$$

Let us choose in the above inequalities $r_2 = \overline{c}r_0$ with $\overline{c} = \frac{c}{2}$.

By (2.22) we have

$$\int_{B_{r_0}(x)\cap(\Omega\setminus\overline{D})} w^2 \leq Cr_0^6\|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)} \tag{3.36}$$

with C depending on α_0, γ_0, M_0, α, M_1. By interpolation estimates for solutions to elliptic equations (see, for instance, [Al-Ro-Ve, Lemma 4.7], stated for the case of hemidiscs, but which holds also in the present context), we have that

$$\int_{B_{r_2}(x)\cap(\Omega\setminus\overline{D})} w^2 \geq Cr_2^4 \int_{B_{\frac{r_2}{2}}(x)\cap(\Omega\setminus\overline{D})} |\nabla^2 w|^2,$$

with C depending on α_0, γ_0, and Λ_0. By Proposition 3.4 and recalling the definition of r_2, we derive

$$\int_{B_{r_2}(x)\cap(\Omega\setminus\overline{D})} w^2 \geq Cr_0^6 \|\widehat{M}\|^2_{H^{-\frac{1}{2}}(\partial\Omega,\mathbb{R}^2)} \tag{3.37}$$

with C depending on α_0, γ_0, Λ_0, M_0, α, M_1, and F.

By (3.36)–(3.37), we can estimate B from above, obtaining the thesis. □

Proof of Theorem 3.1. In order to estimate the Hausdorff distance between the inclusions,

$$\delta = d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}), \tag{3.38}$$

it is convenient to introduce the following auxiliary distances:

$$d = d_{\mathcal{H}}(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}), \tag{3.39}$$

$$d_m = \max\left\{\max_{x\in\partial D_1} \operatorname{dist}(x, \overline{\Omega\setminus D_2}), \max_{x\in\partial D_2} \operatorname{dist}(x, \overline{\Omega\setminus D_1})\right\}. \tag{3.40}$$

Let $\eta > 0$ such that

$$\max_{i=1,2} \int_{(\Omega\setminus\overline{G})\setminus\overline{D_i}} |\nabla^2 w_i|^2 \leq \eta. \tag{3.41}$$

Following the arguments presented in [Al-Be-Ro-Ve], the proof of Theorem 3.1 consists of four main steps. In Step 1, we control d_m in terms of η. Then, in Step 2 we use this estimate to control d in terms of η. The estimate of δ in terms of d is provided in Step 3. Finally, in Step 4 we use Proposition 3.3 in previous estimates to obtain the thesis.

Step 1. Let us start by proving the inequality

$$d_m \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right)^{\frac{1}{\tau}}, \tag{3.42}$$

where τ has been introduced in Proposition 3.6 and C is a positive constant only depending on the a priori data.

Let us assume, without loss of generality, that there exists $x_0 \in \partial D_1$ such that

$$\operatorname{dist}(x_0, \overline{\Omega\setminus D_2}) = d_m > 0. \tag{3.43}$$

Since $B_{d_m}(x_0) \subset D_2 \subset \Omega \setminus \overline{G}$, we have

$$B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1}) \subset (\Omega \setminus \overline{G}) \setminus \overline{D_1} \tag{3.44}$$

and then, by (3.41),

$$\int_{B_{d_m}(x_0)\cap(\Omega\setminus\overline{D_1})} |\nabla^2 w_1|^2 \leq \eta. \tag{3.45}$$

Let us assume that

$$d_m < \overline{c} r_0, \tag{3.46}$$

where $\overline{c}$ is the positive constant appearing in Proposition 3.6. Since $w_1 = 0$, $\nabla w_1 = 0$ on ∂D_1, by the Poincaré inequality (see, for instance, [Al-M-Ro, Example 4.4]) and noticing that $d_m \leq \mathrm{diam}(\Omega) \leq M_1 r_0$, we have

$$\eta \geq \frac{C}{r_0^4} \int_{B_{d_m}(x_0)\cap(\Omega\setminus\overline{D_1})} w_1^2, \tag{3.47}$$

where $C > 0$ is a positive constant only depending on α, M_0, M_1.

By Proposition 3.6, we have

$$\eta \geq \frac{C}{r_0^4} \left(\frac{d_m}{r_0}\right)^{\tau} \int_{B_{r_0}(x_0)\cap(\Omega\setminus\overline{D_1})} w_1^2, \tag{3.48}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0,α, M_0, M_1, and F.

By Lemma 4.7 in [Al-Ro-Ve], we have

$$\eta \geq C \left(\frac{d_m}{r_0}\right)^{\tau} \int_{B_{\frac{r_0}{2}}(x_0)\cap(\Omega\setminus\overline{D_1})} |\nabla^2 w_1|^2, \tag{3.49}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0,α, M_0, M_1.

By Proposition 3.4, we have

$$\eta \geq C \left(\frac{d_m}{r_0}\right)^{\tau} r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}, \tag{3.50}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0,α, M_0, M_1, F, from which we can estimate d_m,

$$d_m \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right)^{\frac{1}{\tau}}, \tag{3.51}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0,α, M_0, M_1, F.

Now, let us assume that

$$d_m \geq \overline{c} r_0. \tag{3.52}$$

By starting again from (3.45), and applying Proposition 3.4 and recalling $d_m \leq M_1 r_0$, we easily have

$$d_m \leq C r_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right), \tag{3.53}$$

where $C > 0$ is a positive constant only depending on α_0, γ_0, Λ_0, M_0, M_1, F. Assuming $\eta \leq r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}$, we obtain (3.42).

Step 2. Without loss of generality, let $y_0 \in \overline{\Omega \setminus D_1}$ such that

$$\mathrm{dist}(y_0, \overline{\Omega \setminus D_2}) = d. \tag{3.54}$$

It is significant to assume $d > 0$, so that $y_0 \in D_2 \setminus D_1$. Let us define

$$h = \mathrm{dist}(y_0, \partial D_1), \tag{3.55}$$

possibly $h = 0$.

There are three cases to consider:
(i) $h \leq \frac{d}{2}$;
(ii) $h > \frac{d}{2}$, $h \leq \frac{d_0}{2}$;
(iii) $h > \frac{d}{2}$, $h > \frac{d_0}{2}$.
Here the number d_0, $0 < d_0 < r_0$, is such that $\frac{d_0}{r_0}$ only depends on M_0, and it is the same constant appearing in Proposition 3.4. In particular, Proposition 3.6 in [Al-Be-Ro-Ve] shows that there exists an absolute constant $C > 0$ such that if $d \leq d_0$, then $d \leq Cd_m$.

Case (i). By definition, there exists $z_0 \in \partial D_1$ such that $|z_0 - y_0| = h$. By applying the triangle inequality, we get dist $(z_0, \overline{\Omega \setminus D_2}) \geq \frac{d}{2}$. Since, by definition, dist $(z_0, \overline{\Omega \setminus D_2}) \leq d_m$, we obtain $d \leq 2d_m$.

Case (ii). It turns out that $d < d_0$ and then, by the above recalled property, again we have that $d \leq Cd_m$, for an absolute constant C.

Case (iii). Let $\widetilde{h} = \min\{h, r_0\}$. We obviously have that $B_{\widetilde{h}}(y_0) \subset \Omega \setminus \overline{D_1}$ and $B_d(y_0) \subset D_2$. Let us set

$$d_1 = \min\left\{\frac{d}{2}, \frac{\widetilde{c_0}d_0}{4}\right\},$$

where $\widetilde{c_0}$ is the positive constant appearing in Proposition 3.5. Since $d_1 < d$ and $d_1 < \widetilde{h}$, we have that $B_{d_1}(y_0) \subset D_2 \setminus \overline{D_1}$ and therefore $\eta \geq \int_{B_{d_1}(y_0)} |\nabla^2 w_1|^2$.

Since $\frac{d_0}{2} < \widetilde{h}$, $B_{\frac{d_0}{2}}(y_0) \subset \Omega\setminus\overline{D_1}$ so that we can apply Proposition 3.5 with $r_1 = d_1$, $\overline{r} = \frac{d_0}{2}$, obtaining $\eta \geq C(\frac{2d_1}{d_0})^{\tau_0} \int_{B_{\frac{d_0}{2}}(y_0)} |\nabla^2 w_1|^2$, with $C > 0$ only depending on α_0, γ_0, Λ_0, M_0, M_1, and F. Next, by Proposition 3.4, recalling that $\frac{d_0}{r_0}$ only depends on M_0, we derive that

$$d_1 \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right)^{\frac{1}{\tau_0}},$$

where $C > 0$ only depends on α_0, γ_0, Λ_0, M_0, M_1, and F. For η small enough, $d_1 < \frac{\widetilde{c_0}d_0}{4}$, so that $d_1 = \frac{d}{2}$ and

$$d \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right)^{\frac{1}{\tau_0}}.$$

Collecting the three cases, we have

$$d \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|^2_{H^{-1/2}(\partial\Omega)}}\right)^{\frac{1}{\tau_1}} \tag{3.56}$$

with $\tau_1 = \max\{\tau, \tau_0\}$ and $C > 0$ only depends on α_0, γ_0, Λ_0, α, M_0, M_1, and F.

Step 3. In this step, we improve the results obtained in [Ca-Ro-Ve2, Proof of Theorem 1.1, step 2]. By (3.56), for η small enough, we have that

$$d < \frac{r_0}{4\sqrt{1 + M_0^2}}.$$

Let us notice that if a point y belongs to $\overline{D_1} \setminus D_2$, then $\mathrm{dist}(y, \partial D_1) \leq d$.

Without loss of generality let $\overline{x} \in \overline{D_1}$ such that $\mathrm{dist}(\overline{x}, \overline{D_2}) = \delta > 0$. Then $\overline{x} \in \overline{D_1} \setminus D_2$ and therefore $\mathrm{dist}(\overline{x}, \partial D_1) \leq d$.

Let $w \in \partial D_1$ such that $|w - \overline{x}| = \mathrm{dist}(\overline{x}, \partial D_1) \leq d$.

Letting n the outer unit normal to D_1 at w, we may write $\overline{x} = w - |w - \overline{x}|n$. By our regularity assumptions on D_1, the truncated cone $C(\overline{x}, -n, 2(\frac{\pi}{2} - \arctan M_0)) \cap B_{\frac{r_0}{4}}(\overline{x})$ having vertex $\overline{x}$, axis $-n$ and width $2(\frac{\pi}{2} - \arctan M_0)$, is contained in D_1.

On the other hand, by the definition of δ, $B_\delta(\overline{x}) \subset \Omega \setminus \overline{D_2}$, so that the truncated cone $C(\overline{x}, -n, 2(\frac{\pi}{2} - \arctan M_0)) \cap B_{\min\{\delta, r_0/4\}}(\overline{x})$ is contained in $D_1 \setminus \overline{D_2}$.

Let us see that $\delta < \frac{r_0}{4}$. In fact if, by contradiction, $\delta \geq \frac{r_0}{4}$, we can consider the point $z = \overline{x} - \frac{r_0}{4} n$. Since $z \in D_1 \setminus D_2$, as noticed above, $\mathrm{dist}(z, \partial D_1) \leq d$. On the other hand, by using the fact that $|z - w| \leq \frac{r_0}{2}$ and by the regularity of D_1, it is easy to compute that $\mathrm{dist}(z, \partial D_1) \geq \frac{r_0}{4\sqrt{1+M_0^2}}$, obtaining a contradiction.

Hence $\min\{\delta, \frac{r_0}{4}\} = \delta$ and, by defining $\overline{z} = \overline{x} - \delta n$ and by analogous calculations, we can conclude that $\delta \leq (\sqrt{1 + M_0^2})d$, which is the desired estimate of δ in terms of d.

Step 4. By Proposition 3.3,

$$d \leq C r_0 \left(\log \left| \log \left(\frac{\epsilon}{r_0^2 \| \widehat{M} \|^2_{H^{-1/2}(\partial\Omega)}} \right) \right| \right)^{-\frac{1}{2\tau_1}} \tag{3.57}$$

with $\tau_1 \geq 1$ and $C > 0$ only depends on α_0, γ_0, Λ_0, α, M_0, M_1, and F. By this first rough estimate, there exists $\epsilon_0 > 0$, only depending on on α_0, γ_0, Λ_0, α, M_0, M_1, and F, such that if $\epsilon \leq \epsilon_0$, then $d \leq d_0$. Therefore, the second part of Proposition 3.3 applies and the thesis follows. ∎

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