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Locally compact groups and locally minimal group topologies

by

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Abstract. Minimal groups are Hausdorff topological groups G satisfying the open mapping theorem with respect to continuous isomorphisms, i.e., every continuous isomorphism $G \to H$, with H a Hausdorff topological group, is a topological isomorphism. A topological group (G,τ) is called *locally minimal* if there exists a neighbourhood V of the identity such that for every Hausdorff group topology $\sigma \leq \tau$ with $V \in \sigma$ one has $\sigma = \tau$. Minimal groups, as well as locally compact groups, are locally minimal. According to a well known theorem of Prodanov, every subgroup of an infinite compact abelian group K is minimal if and only if K is isomorphic to the group $\mathbb{Z}_p$ of p-adic integers for some prime p.

We find a remarkable connection of local minimality to Lie groups and p-adic numbers by means of the following results extending Prodanov's theorem: every subgroup of a locally compact abelian group K is locally minimal if and only if either K is a Lie group, or K has an open subgroup isomorphic to $\mathbb{Z}_p$ for some prime p. In the nonabelian case we prove that all subgroups of a connected locally compact group are locally minimal if and only if K is a Lie group, resolving Problem 7.49 from Dikranjan and Megrelishvili (2014) in the positive.

1. Introduction. Among the generalizations of compactness in the realm of topological groups (rather than topological spaces), the following one seems to be the most prominent:

Definition 1.1. A Hausdorff topological group (G,τ) is called *minimal* if for every Hausdorff group topology $\sigma \leq \tau$ on G one has $\sigma = \tau$.

The minimal groups were introduced simultaneously and independently in [26] and [18], where the first examples of noncompact minimal groups can be found. Answering a question of Choquet, Doïtchinov [18] showed that minimality (unlike compactness) is not preserved even under finite direct

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products. The surveys [5, 10, 11, 27] contain various information on minimal groups. The recent progress in the field of minimal groups is outlined in [12]; for minimal topological rings see [7]–[9].

In many of the proofs of the above results, as well as in some proofs of the present paper, the following useful criterion for minimality of a dense subgroup is used, where a subgroup H of a topological group G is said to be *essential* in G if H nontrivially meets each nontrivial closed normal subgroup N of G.

Fact 1.2 (Minimality criterion). *Let H be a dense subgroup of a topological group G. Then H is minimal iff G is minimal and H is essential in G.*

This criterion was first given by Stephenson [27] and by Prodanov [25] in the case of compact G (i.e., a dense subgroup H of a compact group G is minimal if and only if H is essential in G). The present general form was found later by Banaschewski [4].

Here we recall the celebrated theorem of Prodanov [25] characterizing the groups of p-adic integers that largely triggered this paper:

Theorem 1.3 ([25]). *An infinite compact abelian group K is isomorphic to $\mathbb{Z}_p$ for some prime p if and only if every subgroup H of K is minimal.*

While this theorem connects minimal groups with p-adic numbers, minimal topologies on torsion abelian groups led to a relevant connection with Lie groups in [14]. Nevertheless, a simultaneous connection of minimality with both issues in a single result has never been achieved so far.

Various counterparts of Theorem 1.3 concerning minimality in the class of topological rings were obtained in [7, 8, 9].

Regardless of its numerous nice features, the class of minimal groups has an important shortcoming, namely it does not contain the class of locally compact groups, which is the first basic generalization of compactness. Indeed, it was shown by Stephenson [27] that a locally compact abelian group is minimal if and only if it is compact. The following natural generalization of minimality was introduced by Morris and Pestov [23] with the aim to resolve this problem (local q-minimality was introduced later in [13]):

Definition 1.4 ([23], [13]). A topological group (G, τ) is called *locally minimal with respect to a neighbourhood V* of the identity of G if for every Hausdorff group topology $\sigma \leq \tau$ with $V \in \sigma$ one has $\sigma = \tau$. A locally minimal group G is called *locally q-minimal* whenever every Hausdorff quotient of G is still locally minimal.

The notion of local q-minimality here slightly differs from the original one in [13] (see [24] for a motivation for this choice). It was shown in [23] that locally compact groups are locally minimal. Locally minimal groups were studied in detail in [3, 13, 1, 2] (see also [22]). This suggests looking for a "local" version of Theorem 1.3 with "compact" replaced by "locally compact" and "minimal" by "locally minimal":

PROBLEM 1.5 ([12, Problem 7.47]). *Describe the locally compact groups G such that every subgroup of G is locally minimal.*

It turns out that at least in the case of Lie groups one can say even more. Let us first recall that locally compact groups are locally q-minimal [13].

LEMMA 1.6. *Every subgroup of a Lie group is locally q-minimal.*

We verify Lemma 1.6 in Section 2. Using this fact we prove in Section 2 our first main result which, beyond a complete solution of Problem 1.5 in the abelian case, naturally involves also Lie groups (as the above example suggests). A part of this result was anticipated without proof in [12, Theorem 7.48].

THEOREM 1.7. *For a locally compact abelian group K the following conditions are equivalent:*

(a) *every subgroup H of K is locally q-minimal;*
(b) *every subgroup H of K is locally minimal;*
(c) *either K is a Lie group, or K has an open subgroup isomorphic to $\mathbb{Z}_p$ for some prime p.*

The implication (a)⇒(b) is trivial, while (c)⇒(a) immediately follows from Lemmas 1.6 and 2.9. Therefore, the proof of Theorem 1.7 reduces to the proof of (b)⇒(c). The idea of the proof is this: property (b) is hereditary: if K satisfies (b), then all subgroups of K satisfy (b). Using this observation, we rule out many cases by showing that certain groups, like $\mathbb{Z}_p \times \mathbb{Z}_q$, $\mathbb{Z}_p \times \mathbb{T}$, $\mathbb{Z}_p \times \mathbb{R}$, $\mathbb{Z}(p)^{\mathbb{N}}$, etc., have subgroups that fail to be locally minimal, so these groups cannot appear as topological subgroups of a group K satisfying (b). From this we deduce that compact abelian groups satisfying (b) are finite-dimensional (see Lemma 3.9). This allows us to conclude the proof of Theorem 1.7 in Section 4.

Since the proof of Theorem 1.7 does not rely on Theorem 1.3, as a first application we obtain a new self-contained proof of Theorem 1.3. Moreover, Theorem 1.7 unifies in an obvious way Lie theory and p-adic numbers "under the same umbrella", namely *local minimality.*

Since the second case in Theorem 1.7(c) cannot occur when the group is connected, the theorem leaves open the following question, already raised in [12]:

QUESTION 1.8 ([12, Problem 7.49]). *If every subgroup of a connected locally compact group G is locally minimal, is G necessarily a Lie group?*

We offer an affirmative answer to this question in the following theorem providing, among other things, a characterization of the connected Lie groups in terms of local minimality:

THEOREM 1.9. *Every subgroup of a connected locally compact group K is locally minimal if and only if K is a Lie group.*

Another characterization of Lie groups (via their zero-dimensional subgroups) was recently obtained in [16].

Now we can unify both theorems in a single statement containing the main result of the paper. The missing implication (c)⇒(a) in the nonabelian case follows from Example 1.6.

THEOREM 1.10. *For a locally compact group K that is either abelian or connected the following conditions are equivalent:*

(a) *every subgroup H of K is locally q-minimal;*
(b) *every subgroup H of K is locally minimal;*
(c) *either K is a Lie group, or K has an open subgroup isomorphic to $\mathbb{Z}_p$ for some prime p.*

Note that the second alternative in (c) (entailing total disconnectedness) is effectively present only in the abelian case, since it is obviously ruled out for connected locally compact groups K.

Finally, one may ask whether the hypothesis "either abelian or connected" in this corollary can be completely removed, obtaining in this way a definite answer to the general Problem 1.5. We conjecture that *this is not possible even in the case of compact groups.* More specifically, we conjecture that there exists a compact nonabelian totally disconnected group K satisfying (b), but failing to satisfy (c) (see Conjecture 5.1).

Notation and terminology. We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of natural numbers and primes, respectively; by $\mathbb{Z}$ the integers, by $\mathbb{Q}$ the rationals, by $\mathbb{R}$ the reals, and by $\mathbb{T}$ the unit circle group which is identified with $\mathbb{R}/\mathbb{Z}$. The cyclic group of order $n > 1$ is denoted by $\mathbb{Z}(n)$. For a prime p the symbol $\mathbb{Z}(p^\infty)$ stands for the quasicyclic p-group (the Prüfer group) and $\mathbb{Z}_p$ stands for the p-adic integers.

The subgroup generated by a subset X of a group G is denoted by $\langle X\rangle$, and $\langle x\rangle$ is the cyclic subgroup of G generated by an element $x \in G$. The abbreviation $K \leq G$ is used to denote a subgroup K of G.

Abelian groups will be written additively. The *torsion part* $t(G)$ of an abelian group G is the set $\{g \in G : ng = 0 \text{ for some } n \in \mathbb{N}\}$. Clearly, $t(G)$ is a subgroup of G when G is abelian. For a prime p, the *p-primary component* $t_p(G)$ of G is the subgroup of G that consists of all $x \in G$ satisfying $p^n x = 0$ for some positive $n \in \mathbb{N}$ (so, $\mathbb{Z}(p^\infty) = t_p(\mathbb{T})$). The group G is said to be *divisible* if $nG = G$ for every positive $n \in \mathbb{N}$. We denote by $r_0(G)$ the free rank of G.

Throughout the paper all topological groups are assumed to be Hausdorff. For a topological group G we denote by $c(G)$ the connected component of the identity in G. The group G is called *hereditarily disconnected* when $c(G)$ is trivial. The Pontryagin dual of a topological abelian group G will be denoted by $\widehat{G}$.

All unexplained terms related to general topology can be found in [19]. For background on abelian groups, see [20].

2. Background on local minimality. In this section we prepare the necessary background for the proofs of Theorems 1.7 and 1.9.

The group $\mathbb{Z}$ endowed with group topologies with a local base at 0 formed by open subgroups will play a relevant role in our proofs. That is why we start with an example that will be used in the proofs of Section 3.

EXAMPLE 2.1. For a natural number $m > 1$ the m-adic topology τ_m of $\mathbb{Z}$ has as basic neighborhoods of 0 the family of subgroups $\{m^n\mathbb{Z} : n \in \mathbb{N}\}$. If $m = pq$ with distinct primes p, q, then $(\mathbb{Z}, \tau_m)$ is not locally minimal.

Indeed, let $V = m^{n_0}\mathbb{Z} = p^{n_0}q^{n_0}\mathbb{Z}$, $n_0 \in \mathbb{Z}$ nonnegative, be a basic neighborhood of 0. Then the family $\mathfrak{V} = \{p^n q^{n_0}\mathbb{Z} : n \geq n_0\}$ is a base of 0 for a Hausdorff group topology σ. Moreover, $V \in \sigma \leq \tau_m$, as $\mathfrak{V} \subseteq \tau_m$. On the other hand, $p^{n_0+1}q^{n_0+1}\mathbb{Z} \in \tau_m$ and $p^{n_0+1}q^{n_0+1}\mathbb{Z} \notin \sigma$, so $\sigma \neq \tau_m$. Therefore the m-adic topology of $\mathbb{Z}$ is not locally minimal with respect to V.

This example shows that the m-adic topology of $\mathbb{Z}$ is locally minimal only if $m = p$ is a prime. On the other hand, it is well known that the p-adic topologies on $\mathbb{Z}$ are precisely the minimal topologies on $\mathbb{Z}$. Hence, for the family of m-adic topologies of $\mathbb{Z}$, local minimality is equivalent to minimality.

DEFINITION 2.2 ([2]). Let H be a subgroup of a topological group G. We say that H is *locally essential* in G if there exists a neighborhood V of 0 in G such that $H \setminus \{0\}$ meets each nontrivial closed normal subgroup N of G which is contained in V.

When necessary, we shall say H is *locally essential with respect to* V to emphasize the fact that V witnesses local essentiality. In such a case this also holds true for any smaller neighborhood of zero.

DEFINITION 2.3. A topological group G is said to have *no small subgroups* (or briefly, to be an *NSS group*) if G has a neighborhood of the identity element that contains no nontrivial subgroups.

Remark 2.4. Obviously, every subgroup of an NSS group is vacuously locally essential with respect to any neighborhood of the neutral element satisfying the NSS property.

The following criterion for local minimality was established in [2]:

Fact 2.5 (Local Minimality Criterion). *Let H be a dense subgroup of a topological group G. Then H is locally minimal if and only if G is locally minimal and H is locally essential in G.*

The next remark will turn out to be crucial for Lemma 1.6.

Remark 2.6. It easily follows from Fact 2.5 and Remark 2.4, that every dense subgroup of a locally minimal NSS group is again locally minimal.

Remark 2.7. We recall first that a topological group G is a Lie group if and only if G is a locally compact NSS group.

(a) According to the structure theory of LCA groups, every LCA group K has the form $K = \mathbb{R}^n \times G_0$, where G_0 contains an open compact subgroup C.
(b) One can deduce from the above equivalent form of the definition of Lie group and from (a) that an abelian topological group is a Lie group precisely when it is topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times D$, where D is a discrete group and $m, n \in \mathbb{N}$.

Proof of Lemma 1.6. Let L be a Lie group and suppose that H is a subgroup of L. Then $\overline{H}$ is locally compact as a closed subgroup of the locally compact Lie group L. Hence, $\overline{H}$ is locally minimal. As a subgroup of the NSS group L, the group $\overline{H}$ is NSS as well. By Remark 2.6, H is locally minimal.

In order to prove that H is also locally q-minimal we use the fact that $\overline{H}$ is a Lie group, being a closed subgroup of the Lie group L. Let N be a closed normal subgroup of H. Then $\overline{N}$ is a closed normal subgroup of $\overline{H}$. The quotient homomorphism $q : \overline{H} \to \overline{H}/\overline{N}$ is open and $\overline{N} \cap H = N$, as N is closed. This entails that $\overline{N} \cap H$ is dense in $\overline{N}$. According to [15, Lemma 4.3.2], this implies that also the restriction $q\restriction_H \colon H \to q(H)$ is open, when the group $q(H)$ carries the subgroup topology inherited by the quotient group $\overline{H}/\overline{N}$. The openness of $q\restriction_H$ implies that H/N is isomorphic to the dense topological subgroup $q(H)$ of $\overline{H}/\overline{N}$. By what we have proved above, H/N is locally minimal as $\overline{H}/\overline{N}$ is a Lie group. ∎

Lemma 2.8. *If an open subgroup N of a Hausdorff topological group G is locally minimal, then G itself is locally minimal.*

Proof. Denote by τ the topology of G and assume that V witnesses the locally minimality of $(N, \tau\restriction_N)$. Then V is a neighborhood of 0 in (G, τ), as N is τ-open in G. Now we will prove that V also witnesses the local minimality of (G,τ). Pick a Hausdorff group topology $\sigma \le \tau$ on G such that V is a σ-neighborhood of 0. Then $\sigma\restriction_N$ is a Hausdorff group topology on N with $\sigma\restriction_N \le \tau\restriction_N$ and $V \in \sigma\restriction_N$. By the choice of V, we have $\sigma\restriction_N = \tau\restriction_N$. Pick an arbitrary neighborhood U of 0 in (G,τ). Then $U_1 = U \cap N \in \tau\restriction_N = \sigma\restriction_N$. Hence, there exists a σ-neighborhood U_2 of 0 such that $U_1 = U_2 \cap N$. Since N is a σ-neighborhood of 0, also U_1, as well as U, are σ-neighborhoods of 0. Therefore $(G,\tau) = (G,\sigma)$. ∎

LEMMA 2.9. *If K is an abelian group with an open subgroup $N \cong \mathbb{Z}_p$ for some prime p, then every subgroup of K is locally q-minimal.*

Proof. Let H be a subgroup of K. By Prodanov's Theorem 1.3, the (open) subgroup $H \cap N$ of H is minimal, being also a subgroup of N. To prove that H is locally q-minimal take a closed subgroup L of H and consider the quotient homomorphism $q : H \to H/L$. Since q is an open map, $q(H\cap N)$ is an open subgroup of H/L. According to Lemma 2.8, in order to deduce that H/L is locally minimal, it suffices to check that the open subgroup $q(H \cap N)$ of H/L is minimal. This will be done by considering two cases.

If $L \cap N = \{0\}$, then q sends $H \cap N$ isomorphically onto $q(H \cap N)$. As $H\cap N$ is minimal, $q\restriction_{H\cap N} : H\cap N \to q(H\cap N)$ is a topological isomorphism, so $q(H \cap N)$ is minimal as well.

In case $L\cap N$ is not $\{0\}$, its closure N_1 in N is a closed nonzero subgroup of $N \cong \mathbb{Z}_p$, so $N_1 = p^n N$ for some nonnegative integer n. In particular, $N/N_1 \cong \mathbb{Z}(p^n)$ is finite. Then $q(H \cap N)$ is finite as well (being algebraically isomorphic to a subgroup of N/N_1). Therefore, $q(H \cap N)$ is minimal (being actually compact). ∎

The following lemma will be frequently used to provide dense non-locally-minimal subgroups.

LEMMA 2.10. *Let K be a topological abelian group and let H be a dense subgroup of K. If $H \cap N = \{0\}$ for a closed subgroup N of K that is not NSS, then H is not locally minimal.*

Proof. Assume for a contradiction that H is locally minimal. By Fact 2.5, there exists a neighborhood V of 0 in K witnessing local essentiality of H. Pick a neighborhood U of 0 in K such that $U + U \subseteq V$, so $\overline{U} \subseteq V$. As N is not NSS, the intersection $W = N \cap U$ contains a nontrivial subgroup N_1 of N. Then $\overline{N_1}$ is a closed subgroup of K contained in V, so $\overline{N_1} \cap H \neq \{0\}$. Since N is closed, $\overline{N_1} \le N$. This yields $N \cap H \neq \{0\}$, a contradiction. ∎

3. Compact abelian groups with a non-locally-minimal subgroup. In this section we present a number of compact abelian groups which have a non-locally-minimal subgroup.

Lemma 3.1. *If p is a prime number, then the group $\mathbb{Z}_p^2$ has a dense subgroup that fails to be locally minimal.*

Proof. Let $H = \mathbb{Z}\times\mathbb{Z}$. Obviously, H is a dense subgroup of $K = \mathbb{Z}_p^2$. Pick $\kappa \in \mathbb{Z}_p$ such that $\langle\kappa\rangle\cap\mathbb{Z} = \{0\}$. Then the subgroup $N = \{(\xi\kappa, \xi) : \xi \in \mathbb{Z}_p\}$ of K is closed and non-NSS, being obviously isomorphic to $\mathbb{Z}_p$ via $\xi \mapsto (\xi\kappa, \xi)$. By the choice of κ, one has $N \cap H = \{0\}$. Now Lemma 2.10 implies that H is not locally minimal. ∎

Lemma 3.2. *For any pair p, q of primes the group $K = \mathbb{Z}_p \times \mathbb{Z}_q$ has a dense subgroup that fails to be locally minimal.*

Proof. For $p = q$ this is Lemma 3.1, so we can assume that $p \neq q$. Let us check that the diagonal subgroup $D = \{(n, n) : n \in \mathbb{Z}\}$ of K is not locally minimal. To this end it suffices to note that D is topologically isomorphic to $\mathbb{Z}$ equipped with the pq-adic topology and apply Example 2.1.

An alternative way to see that D is not locally minimal is to notice that $D \cap (\mathbb{Z}_p \times 0) = \{0\}$ and apply Lemma 2.10. ∎

Lemma 3.3. *For every prime p the group $K = \mathbb{Z}(p)^{\mathbb{N}}$ has a dense subgroup that fails to be locally minimal.*

Proof. Let $H = \mathbb{Z}(p)^{(\mathbb{N})}$ be the direct sum, considered as a subgroup of K. Consider any partition

$$\mathbb{N} = \bigcup_{i=1}^{\infty} N_i \tag{3.1}$$

into infinite sets. For each $i \in \mathbb{N}$ let D_i be the diagonal subgroup of $G_i = \mathbb{Z}(p)^{N_i}$ (i.e., the subgroup consisting of elements of the form $(c, c, \ldots)$, $c \in \mathbb{Z}(p)$). Then $N = \prod_{i\in\mathbb{N}} D_i$. is a closed subgroup of $K = \prod_{i\in\mathbb{N}} \mathbb{Z}(p)^{N_i}$ that is not NSS and trivially meets H. By Lemma 2.10, H is not locally minimal. ∎

Lemma 3.4. *For any infinite sequence $p_1 < p_2 < \cdots$ of primes the group $K = \prod_n \mathbb{Z}(p_n)$ has a dense subgroup that fails to be locally minimal.*

Proof. Let c_k be a generator of $\mathbb{Z}(p_k)$ and $x = (c_1, c_2, \ldots) \in K$. Then the cyclic subgroup $H = \langle x\rangle$ of K is dense. We check that H is not locally minimal. According to Lemma 2.10 it suffices to note that H trivially meets the closed subgroup $K_1 := \prod_{n=1}^{\infty} \mathbb{Z}(p_{2n})$ of K (as K_1 is not NSS). ∎

Corollary 3.5. *If all subgroups of a compact abelian totally disconnected group H are locally minimal, then H is either finite or isomorphic to $\mathbb{Z}_p \times F$ with a finite group F.*

Proof. Let $H = \prod_{p\in\mathbb{P}} H_p$, where for each prime p the subgroup H_p is a pro-p-group. For every $p \in \mathbb{P}$, the subgroup $H_p[p] = \{x \in H_p : px = 0\}$ of H_p is a compact abelian group of exponent p, so it is isomorphic to $\mathbb{Z}(p)^{\kappa}$ for some cardinal κ. By Lemma 3.3, κ must be finite. So, each H_p with $p \in \mathbb{P}$

has finite p-rank. Since H_p is residually finite, H_p is reduced, so $r_p(H_p) < \infty$ implies that $t(H_p)$ is finite for every prime p [20]. By Lemma 3.4, $t(H_p) \neq 0$ only for finitely many primes p. In case all subgroups H_p are torsion, this proves that H is finite.

Assume that H is infinite, so H_p is nontorsion for some prime p, so H_p contains a copy of $\mathbb{Z}_p$. This means that at most one of the subgroups H_p may be nontorsion, by Lemma 3.2. Fix this p and note that all other subgroups H_q are torsion, so finite by what we have observed above. Hence the group $F_0 := \prod_{q\in\mathbb{P}\setminus\{p\}} H_q$ is finite. As $t(H_p)$ is finite, there exists $n \in \mathbb{N}$ such that $p^n H_p$ is torsion-free. By Lemma 3.2, H_p cannot contain copies of $\mathbb{Z}_p^2$, so $p^n H_p \cong \mathbb{Z}_p$, since an abelian torsion-free pro-p-group is isomorphic to $\mathbb{Z}_p^\kappa$ for some cardinal κ. Therefore $H_p \cong \mathbb{Z}_p \times F_p$, where $F_p = t(H_p)$ is a finite p-group. This proves our assertion $H \cong \mathbb{Z}_p \times F$ with $F = F_0 \times F_p$. ∎

In what follows, $\dim K$ denotes the covering dimension of a locally compact group G (according to a well known theorem of Pasynkov, all three principal dimension functions dim, ind and Ind coincide on locally compact groups). When K is compact abelian, $\dim K$ is equal to the free rank $r_0(\widehat{K})$ of $\widehat{K}$ [21]. Therefore, the dimension is monotone under taking closed subgroups and quotients of compact abelian groups.

LEMMA 3.6. *For every prime p the group $K = \mathbb{Z}_p \times \mathbb{T}$ has a dense subgroup that fails to be locally minimal.*

Proof. Pick a nontorsion element $c \in \mathbb{T}$, so that c generates a dense cyclic subgroup C of $\mathbb{T}$. Let $a = (1, c) \in K$, where $1 \in \mathbb{Z}$, considered as a subgroup of $\mathbb{Z}_p$. Our aim will be to prove that the subgroup H generated by a is not locally minimal. To this end we show first that H is dense in K. Let K_1 denote the closure of H in K. Then the projection $p_2 : K \to \mathbb{T}$ satisfies $p_2(K_1) = \mathbb{T}$, as $p_2(K_1)$ is a compact subgroup of $\mathbb{T}$ containing the dense subgroup C of $\mathbb{T}$. Since the continuous homomorphic image $p_2(K_1)$ of K is connected, we deduce that K_1 cannot be hereditarily disconnected since hereditarily disconnected compact groups are zero-dimensional [21], and zero-dimensionality of compact groups is preserved by taking quotients. As $c(K_1) \leq c(K) = \{0\} \times \mathbb{T}$ and as all proper subgroups of $\mathbb{T}$ are zero-dimensional, we deduce that $c(K_1) = \{0\} \times \mathbb{T}$. In particular, $K_1 \geq \{0\} \times \mathbb{T}$. Therefore, $K_1 = A \times \mathbb{T}$ with $A = p_1(K_1)$, where $p_1 : K \to \mathbb{Z}_p$ is the first projection. As $1 = p_1(a) \in A$ and A is a closed subgroup of $\mathbb{Z}_p$, we deduce that $A = \mathbb{Z}_p$, so that $K_1 = K$. This proves that H is dense in K.

At this point we note that the closed subgroup $N = \mathbb{Z}_p \times \{0\}$ of K is not NSS and $H \cap N = \{0\}$. With Lemma 2.10 we conclude that H is not locally minimal, a contradiction. ∎

Lemma 3.7. *For any prime p the group $K = \mathbb{Z}_p \times \mathbb{R}$ has a dense subgroup that fails to be locally minimal.*

Proof. Let $D = \langle(1,1)\rangle$ be the diagonal subgroup of $\mathbb{Z} \times \mathbb{Z}$ considered as a subgroup of K. Then D is a discrete, hence closed subgroup of K. The quotient K/D is compact. Indeed, let $q : K \to K/D$ denote the quotient homomorphism and $N := \mathbb{Z}_p \times \{0\}$. Then $M = q(N) \cong \mathbb{Z}_p$ is a closed compact subgroup of K/D and

$$(K/D)/M \cong K/(D+N) = K/(\mathbb{Z}_p \times \mathbb{Z}) \cong \mathbb{T}.$$

Since M and $(K/D)/M$ are compact, so is K/D.

Our next aim is to see that K/D is divisible. To this end we have to show that $q(K/D) = K/D$ for every prime q. Indeed, for $q \neq p$ this is obvious, as K itself is q-divisible for every prime $q \neq p$. On the other hand,

$$p(K/D) = (pK + D)/D = (p\mathbb{Z}_p \times \mathbb{R} + D)/D = K/D,$$

as $(p\mathbb{Z}_p \times \mathbb{R}) + D = K$.

As K/D is divisible, it is a connected compact metrizable group, so it is monothetic [15, 21]. Let $C = \langle c\rangle$ be a dense cyclic subgroup of K/D. There exists $y = (y_1, y_2) \in K$ with $q(y) = c$. Then the subgroup $H = D + C$ is a dense subgroup of K.

Next we check that $H \cap N = \{0\}$. According to Lemma 2.10, this will imply that H is not locally minimal, as N is not NSS.

Take $z = my + d \in N \cap H$ for some $m \in \mathbb{Z}$ and $d \in D$. If $m = 0$, then $z = d \in N \cap D = \{0\}$, and we are done. Assume that $m \neq 0$ and let $f : K/D \to (K/D)/M \cong \mathbb{T}$ be the natural projection. Then $q(z) = q(my) = mc$, so $f(mc) = mf(c) = 0$ in $(K/D)/M \cong \mathbb{T}$. Therefore, $f(C)$ is a finite (cyclic) subgroup of the infinite group $(K/D)/M$. On the other hand, $f(C)$ must be a dense subgroup of $(K/D)/M$, as C is a dense subgroup of K/D, a contradiction. ∎

Corollary 3.8. *If L is a nondiscrete abelian Lie group, then for any prime p the group $K = \mathbb{Z}_p \times L$ has a subgroup that fails to be locally minimal.*

Proof. If L is compact, then L contains a circle $\mathbb{T}$, so that Lemma 3.6 applies. If L is noncompact, then L contains a line $\mathbb{R}$, and Lemma 3.7 applies. ∎

Lemma 3.9. *If every subgroup of a compact abelian group K is locally minimal, then* $\dim K < \infty$.

Proof. Let $\kappa := \dim K$. To show that κ is finite we need

Fact 3.10 ([21, 8.15]). *If a compact abelian group K has* $\dim K = \kappa$, *then there exists a surjective continuous homomorphism $f : K \to \mathbb{T}^\kappa$ such that $N :=$* $\ker f$ *is a totally disconnected compact group.*

Coming back to the proof of Lemma 3.9, assume for a contradiction that κ is infinite. Then $\mathbb{T}^\kappa$ contains a copy of $\mathbb{T}^{\mathbb{N}}$. Since $\mathbb{Z}_p^2$, as a metric compact group, is isomorphic to a subgroup of $\mathbb{T}^{\mathbb{N}}$, we deduce that $\mathbb{T}^\kappa$ contains a subgroup L isomorphic to $\mathbb{Z}_p^2$. Let $G := f^{-1}(L)$. Then the compact group G has a quotient isomorphic to $L \cong \mathbb{Z}_p^2$. Since the continuous surjective homomorphism $f\restriction_L : G \to L$ induces an injective homomorphism

$$X = \widehat{L} \to Y = \widehat{G}$$

between the Pontryagin duals, and since $X \cong \mathbb{Z}(p^\infty)$ is divisible, we deduce that X splits as a subgroup of Y, hence $G \cong \mathbb{Z}_p^2 \times G_1$ splits as well. Consequently, our assumption that κ is infinite leads to the conclusion that K contains a subgroup isomorphic to $\mathbb{Z}_p^2$. This contradicts Lemma 3.1. ■

As the referee kindly pointed out, the argument of the above proof showing that an infinite-dimensional compact abelian group contains an isomorphic copy of $\mathbb{Z}_p^2$ gives the following more general fact:

Let $\mathbb{Z}^*$ denote the universal zero-dimensional compactification of $\mathbb{Z}$ (which is isomorphic to $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$). Then every infinite-dimensional compact abelian group K contains an isomorphic copy of $(\mathbb{Z}^*)^{\dim K}$.

Proposition 3.11. *If every subgroup of a compact abelian group K is locally minimal, then $n = \dim K < \infty$ and K has a closed subgroup $N \cong \mathbb{Z}_p^{e_p} \times F$, where p is a prime, $e_p \in \{0,1\}$, F is a finite group, and $K/N \cong \mathbb{T}^n$.*

Proof. The above lemma yields $n = \dim K < \infty$; here we use Fact 3.10 with $n = \dim K$. Hence, it remains to note that $N = \ker f$ (as in that fact) has the special form $N \cong \mathbb{Z}_p^{e_p} \times F$, as required in the proposition, by Corollary 3.5. ■

4. Proofs of Theorem 1.7 and Theorem 1.9. As a first corollary of Proposition 3.11 we obtain a proof of Theorem 1.7 in the compact case. Namely, if every subgroup of a compact abelian group K is locally minimal, then either $K \cong \mathbb{T}^n \times F$ is a Lie group, or $K \cong \mathbb{Z}_p \times F$, with F a finite group in both cases.

Indeed, G contains a closed subgroup $N \cong \mathbb{Z}_p^{e_p} \times F$, where p is a prime, $e_p \in \{0,1\}$, F is a finite group, and $K/N \cong \mathbb{T}^n$, according to Proposition 3.11. If $e_p = 0$, there is nothing to prove. Assume that $e_p = 1$, i.e., $N \cong \mathbb{Z}_p \times F$. We have to prove that $n = 0$. Assume for a contradiction that $n > 0$. Then $K/N \cong \mathbb{T}^n$, being a monothetic group, contains a dense cyclic subgroup $C = \langle c\rangle$. Let $q : K \to K/N$ be the quotient map. Pick $x \in K$ with $q(x) = c$. Let $H := \langle x\rangle$ and $K_1 = \overline{H}$. By the choice of c and x, one has $H \cap N = \{0\}$. Then, identifying N with $\mathbb{Z}_p \times F$ and letting $B = K_1 \cap \mathbb{Z}_p$, one also has

$$H \cap B = \{0\}. \tag{4.1}$$

Consider two cases:

CASE 1: $B \neq \{0\}$. Then B is a closed nontrivial subgroup of $\mathbb{Z}_p$, so $B \cong \mathbb{Z}_p$ is a non-NSS group. By (4.1) and Lemma 2.10, applied to K_1, H and B, we deduce that H is not locally minimal.

CASE 2: $B = \{0\}$. Now $F_1 := \ker q\restriction_{K_1} \leq N \cap K_1$ is finite, as $K_1 \cap \mathbb{Z}_p = \{0\}$ and $\mathbb{Z}_p \times \{0\}$ has finite index in N. Since $K_1/F_1 \cong \mathbb{T}^n$, we deduce that K_1 is a compact Lie group. Our hypothesis $K_1 \cap \mathbb{Z}_p = \{0\}$ implies that the sum $K_1 + \mathbb{Z}_p$ is a compact subgroup of K topologically isomorphic to the direct product $K_1 \times \mathbb{Z}_p$. Since K_1 is a Lie group, $K_1 \times \mathbb{Z}_p$ has a subgroup that fails to be locally minimal, by Corollary 3.8. This contradicts our assumption that every subgroup of K is locally minimal. Hence this case cannot occur.

This finishes the proof of Theorem 1.7 in the compact case. Now we can face the general case.

Proof of Theorem 1.7. Now assume that every subgroup of a locally compact abelian group K is locally minimal. By the structure theory of LCA groups, $K = \mathbb{R}^n \times G_0$, where G_0 contains an open compact subgroup C. As every subgroup of C is locally minimal, we deduce by the above argument that either C is a Lie group, or $C \cong \mathbb{Z}_p$ for some prime p. In the former case, K has an open subgroup that is a Lie group, so it is a Lie group itself. In the latter case, K contains an open subgroup isomorphic to $\mathbb{R}^n \times \mathbb{Z}_p$. According to Corollary 3.8, this is possible only if $n = 0$. This proves the theorem. ∎

Deduction of Theorem 1.3 from Theorem 1.7. Assume that all subgroups of an infinite compact abelian group K are minimal. By Theorem 1.7, K is either a Lie group or $K = \mathbb{Z}_p \times F$ for some prime p and a finite abelian group F. Since an infinite compact Lie group contains a torus $T \cong \mathbb{T}$, it suffices to note that $\mathbb{T}$ has nonminimal subgroups (e.g., every infinite cyclic subgroup, due to Fact 1.2). Therefore, $K = \mathbb{Z}_p \times F$. We now check that $F = \{0\}$.

Assume for contradiction that $F \neq \{0\}$. Then F contains a nontrivial finite cyclic subgroup C. Denote by c its generator. Pick $\xi \in \mathbb{Z}_p$ such that ξ and 1 are independent and let H be the subgroup of $K_1 = \mathbb{Z}_p \times C$ generated by $\mathbb{Z} \times \{0\}$ and $x = (\xi, c)$. Let $m = |C|$, so that $mx = (m\xi, 0) \in \mathbb{Z}_p \times \{0\}$ and $mx \neq 0$ and 1 are independent. This means that $H \cong \mathbb{Z}^2$ is torsion-free. Since $\overline{H}$ contains $\mathbb{Z}_p \times \{0\}$, we deduce that $\overline{H} = \mathbb{Z}_p \times B$, where B is a subgroup of C. As the projection $p : K_1 \to C$ sends H onto C (as $c \in p(H)$), we deduce that $B = C$, i.e., H is a dense subgroup of K_1. Since the closed nontrivial subgroup $\{0\} \times C$ of K_1 trivially meets H, we deduce that H is not essential in K_1. By Fact 1.2, H is not minimal, a contradiction. ∎

Proof of Theorem 1.9. The sufficiency follows from Lemma 1.6. To prove the necessity suppose that every subgroup of a connected locally compact group K is locally minimal. We need to prove that K is a Lie group. We assume that K is nonabelian, since the abelian case was already covered by Theorem 1.7.

We consider first the case when K is compact and nonabelian. Using well known facts from the structure theory of compact connected groups [21], we have

$$K = K' \cdot Z(K), \tag{4.2}$$

implying $K/Z(K) \cong K'/(K' \cap Z(K))$ and an isomorphism $K'/(K' \cap Z(K)) \cong \prod\{L_i : i \in I\}$ for a family of connected compact algebraically simple Lie groups L_i. Moreover, if $\widetilde{L}_i$ denotes the universal covering group of each L_i ($i \in I$), then K' is isomorphic to a quotient of $L = \prod_{i\in I} \widetilde{L}_i$ by a closed totally disconnected subgroup N, contained in $Z(L)$. Using $\dim N = 0$, we are going to prove that I is finite, so L is a Lie group, and consequently K' is a Lie group as well.

Indeed, assume for a contradiction that I is infinite. Then picking in each $\widetilde{L}_i$ a one-dimensional torus $T_i \cong \mathbb{T}$ we obtain an infinite-dimensional subgroup $T = \prod_{i\in I} T_i \cong \mathbb{T}^I$ of L. Let $q : L \to K' \cong L/N$ be the canonical projection. Then its restriction $f = q\restriction_T : T \to q(T)$ induces an isomorphism $q(T) \cong T/(T \cap N)$. As $\dim(N \cap T) = \dim N = 0$, we conclude that $q(T)$ is infinite-dimensional as well.

By Lemma 3.9, $q(T)$ (so K' and K as well) has non-locally-minimal subgroups. This contradiction shows that I must be finite.

We have proved that K' is a Lie group. On the other hand, $Z(K)$ is a compact abelian group having all subgroups locally minimal. So either $Z(K)$ is a Lie group, or $Z(K) = N \times F$ with $N \cong \mathbb{Z}_p$ for some prime p and some finite group F. In the former case, we deduce from (4.2) that K is a Lie group and we are done. Let us see now that the latter case cannot occur. Indeed, in that case we pick a closed abelian subgroup L of K' isomorphic to $\mathbb{T}$. The subgroup N of K is central, hence the subgroup of K generated by N and L is abelian and coincides with $N \cdot L$. As all proper closed subgroups of $L \cong \mathbb{T}$ are finite and $N \cong \mathbb{Z}_p$ is totally disconnected (so cannot contain L), we deduce that $N \cap L$ is trivial, as N is torsion-free. Then $N \cdot L$ is isomorphic to $N \times L \cong \mathbb{Z}_p \times \mathbb{T}$, so that we can apply Lemma 3.6 to get a contradiction.

Finally, we can consider the general case of an arbitrary connected locally compact group K. By a theorem of Davis [6], K is homeomorphic to $C \times \mathbb{R}^n \times D$, where C is a compact subgroup of K, $n \in \mathbb{N}$ and D is a discrete space. Since K is connected, the subgroup D must be trivial and C is a compact connected subgroup of K. By the first part of the proof, C is a (compact) Lie group, as every subgroup of C, being also a subgroup of K, is locally minimal. In other words, C is a locally Euclidean space. Therefore, the whole group K, homeomorphic to $C \times \mathbb{R}^n$, is a locally Euclidean space, i.e., K is a Lie group. ∎

5. Final remarks and open questions. Let us note that in the alternative case of item (c) of Theorem 1.7 one has a one-dimensional p-adic Lie group. (Here dimension refers to the dimension of the group in question as a *p-adic manifold* [17].) This may suggest replacing in item (c) of Theorem 1.7 the current second case by the tempting "p-adic Lie group", thereby obtaining nice symmetry. Lemma 3.1 shows that the two-dimensional p-adic Lie group $\mathbb{Z}_p^2$ has non-locally-minimal subgroups, so higher dimensional p-adic Lie groups should be ruled out in the abelian case. Apparently this may change in the nonabelian case, as the following conjecture suggests:

CONJECTURE 5.1. *We conjecture that the linear p-adic group* $\mathrm{GL}_2(\mathbb{Q}_p)$ *has a locally compact subgroup, namely*

$$L = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p) : a, b \in \mathbb{Q}_p,\ a \neq 0 \right\} \cong (\mathbb{Q}_p, +) \rtimes (\mathbb{Q}_p \setminus \{0\}, \cdot),$$

such that every subgroup of L is locally minimal, so in particular also the compact group $(\mathbb{Z}_p, +) \rtimes (\mathbb{Z}_p \setminus p\mathbb{Z}_p, \cdot)$ must have the same property. In both semidirect products, the action is given by multiplication of p-adic numbers.

QUESTION 5.2. *Is it possible to transform the sufficient condition for non-local-minimality of dense subgroups from Lemma* 2.10 *into a criterion for non-local-minimality of dense subgroups of locally compact* (*abelian*) *groups?*

Lemmas 3.1, 3.2, 3.3, 3.4, 3.6 and 3.7 suggest the following:

QUESTION 5.3. *Is it possible to add in Theorem* 1.7 *also the following weaker condition:*

(b*) *every dense subgroup H of K is locally minimal?*

Is this possible for metrizable locally compact abelian groups G?

The metrizability restriction in the final part of the question is motivated as follows. Using the fact that the weight and the network weight of a locally minimal group coincide [1, Theorem 2.8], one can easily show that if all dense subgroups of a compact abelian group K are locally minimal, then K is metrizable.

Note added March, 2018: Recently, Conjecture 5.1 was answered positively: see [28].

Note added September, 2018: Question 5.3 has been answered negatively in [29] even for metrizable locally compact abelian groups. On the other hand, it is shown there that the answer is positive in the case of compact abelian groups.

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References

[1] L. Außenhofer, M. J. Chasco, D. Dikranjan and X. Domínguez, *Locally minimal topological groups 1*, J. Math. Anal. Appl. 370 (2010), 431–452.

[2] L. Außenhofer, M. J. Chasco, D. Dikranjan and X. Domínguez, *Locally minimal topological groups 2*, J. Math. Anal. Appl. 380 (2011), 552–570.

[3] T. Banakh, *Locally minimal topological groups and their embeddings into products of o-bounded groups*, Comment. Math. Univ. Carolin. 41 (2000), 811–815.

[4] B. Banaschewski, *Minimal topological algebras*, Math. Ann. 211 (1974), 107–114.

[5] W. W. Comfort and D. L. Grant, *Cardinal invariants, pseudocompactness and minimality: some recent advances in the topological theory of topological groups*, Topology Proc. 6 (1981), 227–265.

[6] H. F. Davis, *A note on Haar measure*, Proc. Amer. Math. Soc. 6 (1955), 318–321.

[7] D. Dikranjan, *Precompact minimal topologies on Noetherian domains*, C. R. Acad. Bulgare Sci. 29 (1976), 1561–1562.

[8] D. Dikranjan, *Minimal ring topologies and Krull dimension*, in: Topology (Budapest, 1978), Vol. II, Colloq. Math. Soc. Janos Bolyai 23, North-Holland, 1980, 357–366.

[9] D. Dikranjan, *Minimal topological rings*, Serdica 8 (1982), 139–165.

[10] D. Dikranjan, *Recent advances in minimal topological groups*, Topology Appl. 85 (1998), 53–91.

[11] D. Dikranjan, *Countably compact groups satisfying the open mapping theorem*, Topology Appl. 98 (1999), 81–129.

[12] D. Dikranjan and M. Megrelishvili, *Minimality conditions in topological groups*, in: Recent Progress in General Topology, III, Atlantis Press, Paris, 2014, 229–327.

[13] D. Dikranjan and S. Morris, *Subgroups of products of locally compact groups*, Topology Proc. 26 (2001–2002), 533–544.

[14] D. Dikranjan and I. Prodanov, *A class of compact abelian groups*, Annuaire Univ. Sofia Fac. Math. Méc. 70 (1975/76), 191–206.

[15] D. Dikranjan, I. Prodanov and L. Stoyanov, *Topological Groups: Characters, Dualities and Minimal Group Topologies*, Monogr. Textbooks Pure Appl. Math. 130, Dekker, New York, 1989.

[16] D. Dikranjan and D. Shakhmatov, *Characterizing Lie groups by controlling their zero-dimensional subgroups*, Forum Math. 30 (2018), 295–320.

[17] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, 2nd ed., Cambridge Stud. Adv. Math. 61, Cambridge Univ. Press, Cambridge, 1999.

[18] D. Doïtchinov, *Produits de groupes topologiques minimaux*, Bull. Sci. Math. (2) 97 (1972), 59–64.

[19] R. Engelking, *General Topology*, Sigma Ser. Pure Math. 6, Heldermann, Berlin, 1989.
[20] L. Fuchs, *Infinite Abelian Groups*, Vol. I, Academic Press, New York, 1970.
[21] K. H. Hofmann and S. Morris, *The Structure of Compact Groups. A Primer for the Student—a Handbook for the Expert*, de Gruyter Stud. Math. 25, de Gruyter, Berlin, 2013.
[22] F. Lin, *Locally minimal paratopological groups*, Topology Appl. 165 (2014), 69–83.
[23] S. Morris and V. Pestov, *On Lie groups in varieties of topological groups*, Colloq. Math. 78 (1998), 39–47.
[24] D. Peng, D. Dikranjan, W. He, Z. Xiao and W. Xi, *Products and quotients of locally minimal groups*, work in progress.
[25] I. Prodanov, *Precompact minimal group topologies and p-adic numbers*, Annuaire Univ. Sofia Fac. Math. Méc. 66 (1971/72), 249–266.
[26] R. M. Stephenson, Jr., *Minimal topological groups*, Math. Ann. 192 (1971), 193–195.
[27] R. M. Stephenson, Jr., *Some unsolved problems concerning P-minimal and P-closed spaces*, in: Topology (Memphis, TN, 1975), Lecture Notes in Pure Appl. Math. 24, Dekker, New York, 1976, 249–257.
[28] W. Xi, D. Dikranjan, M. Shlossberg and D. Toller, *Hereditarily minimal topological groups*, arXiv:1803.08033 (2018).
[29] W. Xi, D. Dikranjan, M. Shlossberg and D. Toller, *Densely locally minimal groups*, submitted.

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