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An infinite-dimensional 2-generated primitive axial algebra of Monster type

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Abstract
Rehren proved in Axial algebras. Ph.D. thesis, University of Birmingham (2015), Trans Am Math Soc 369:6953–6986 (2017) that a primitive 2-generated axial algebra of Monster type (α, β), over a field of characteristic other than 2, has dimension at most 8 if $\alpha \notin \{2\beta, 4\beta\}$. In this note, we show that Rehren's bound does not hold in the case $\alpha = 4\beta$ by providing an example (essentially the unique one) of an infinite-dimensional 2-generated primitive axial algebra of Monster type $(2, \frac{1}{2})$ over an arbitrary field $\mathbb{F}$ of characteristic other than 2 and 3. We further determine its group of automorphisms and describe some of its relevant features.

Keywords Axial algebras · Finite simple groups · Monster group · Jordan algebras · Baric algebras

Mathematics Subject Classification 20D08 · 17C27 · 17Dxx

1 Introduction

This note is part of a project of the authors aimed at classifying all 2-generated primitive axial algebras of Monster type. Axial algebras were introduced by Hall, Rehren and Shpectorov [6, 7] in order to axiomatize some key features, that are relevant for

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the study of finite simple groups, of certain classes of algebras, such as the weight-2 components of OZ-type vertex operator algebras, Jordan algebras, Matsuo algebras, and Majorana algebras (see [8] and the introductions of [6] and [16]). In this paper, we assume that the underlying field $\mathbb{F}$ has characteristic different from 2, being the case of characteristic 2 somehow degenerate because the algebra turns out to be associative (Lemma 2.1). On the other hand, the case of characteristic other than 2 is of particular interest, since most of the finite simple groups can be faithfully represented as groups generated by special involutory automorphisms, called Miyamoto involutions [13], of such algebras. In particular, the Griess algebra (see [4]) is a real axial algebra of Monster type $(\frac{1}{4}, \frac{1}{32})$ and the Miyamoto involutions of this algebra (also called Majorana involutions) are precisely the involutions of type $2A$ in the Monster, i.e., those whose centralizer is the double cover of the Baby Monster.

The classification of 2-generated axial algebras has a fundamental rôle in the development of the theory of axial algebras and of the representations of the finite groups on them (see [2, 3, 9–12]). In a pioneering work [14], Norton classified subalgebras of the Griess algebra generated by two axes. Norton showed that there are exactly nine isomorphism classes of such subalgebras, corresponding to the nine conjugacy classes of dihedral subgroups of the Monster generated by two involutions of type $2A$. These algebras have been proven to be, up to isomorphisms, the only 2-generated primitive real axial algebras of Monster type $(\frac{1}{4}, \frac{1}{32})$ and are now known as Norton–Sakuma algebras [6, 9, 17]. In the minimal non-associative case of axial algebras of Jordan type η, the classification has been obtained by Hall, Rehren and Shpectorov in [7]. Note that axial algebras of Jordan type η are also axial algebras of Monster type (α, β) when $\eta \in \{\alpha, \beta\}$. In [15] and [16], Rehren started a systematic study of axial algebras of Monster type (α, β). In particular, he showed that, when $\alpha \notin \{2\beta, 4\beta\}$ every 2-generated primitive axial algebra of Monster type (α, β) has dimension at most 8.

In an unpublished work (arXiv:2101.10315, Jan. 25th 2021), the authors extend Rehren's result showing that, if $(\alpha, \beta) \neq (2, \frac{1}{2})$, then any symmetric 2-generated primitive axial algebra of Monster type (α, β) has dimension at most 8 (*symmetric* means that the algebra has an automorphism that swaps the generating axes). Still, the question of the existence of 2-generated axial algebras of infinite dimension has been around for some years. Here we give an elementary construction of such an example, proving that the case $(\alpha, \beta) = (2, \frac{1}{2})$ is indeed an exception.

Theorem 1.1 *For every field $\mathbb{F}$ of characteristic different from 2 and 3, there exists an infinite-dimensional 2-generated symmetric primitive axial algebra of Monster type $(2, \frac{1}{2})$ over $\mathbb{F}$.*

We would like to mention that a few weeks after this paper was posted in arXiv (arXiv:2007.02430, Jul. 5th 2020), another independent paper by Takahiro Yabe was submitted in that repository (arXiv:2008.01871, Aug. 4th 2020), where the same algebra appears (though with no details apart from the multiplication table) within a general result on 2-generated symmetric primitive axial algebras of Monster type. From Yabe's results, it would follow that in characteristic other than 5, any infinite-dimensional example of such algebras must be a quotient of this one.

2 The algebra $\mathcal{H}$

We first recall the definition of axial algebras of Monster type. Let $\mathbb{F}$ be a field and let $\mathcal{S}$ be a finite subset of $\mathbb{F}$ with $1 \in \mathcal{S}$. A *fusion law* on $\mathcal{S}$ is a map

$$\star : \mathcal{S} \times \mathcal{S} \to 2^{\mathcal{S}}.$$

An *axial algebra* over $\mathbb{F}$ with *spectrum* $\mathcal{S}$ and fusion law $\star$ is a commutative non-associative $\mathbb{F}$-algebra V generated by a set $\mathcal{A}$ of nonzero idempotents (called *axes*) such that, for each $a \in \mathcal{A}$,

(Ax1) $ad_a : v \mapsto av$ is a semisimple endomorphism of V with spectrum contained in $\mathcal{S}$;
(Ax2) for every $\lambda, \mu \in \mathcal{S}$, the product of a λ-eigenvector and a μ-eigenvector of ad_a is the sum of δ-eigenvectors, for $\delta \in \lambda \star \mu$.

Furthermore, V is called *primitive* if, for all $a \in \mathcal{A}$,

(Ax3) the 1-eigenspace of ad_a is $\langle a \rangle$.

An axial algebra over $\mathbb{F}$ is said to be *of Monster type* (α, β) if it satisfies the fusion law $\mathcal{M}(\alpha, \beta)$ given in Table 1, with $\alpha, \beta \in \mathbb{F} \setminus \{0, 1\}$, with $\alpha \neq \beta$.

Note that, given an axial algebra A of Monster type (α, β), every axis a induces an obvious $\mathbb{Z}_2$-grading $A = A_a^+ \oplus A_a^-$, where A_a^+ is the sum of the 0-, 1-, α-eigenspaces of ad_a and A_a^- is the β-eigenspace of ad_a. Furthermore, we have a $\mathbb{Z}_2$-grading $A_a^+ = A_a^{++} \oplus A_a^{+-}$ of A_a^+, where A_a^{++} is the sum of the 0- and 1-eigenspaces of ad_a and A_a^{+-} is the α-eigenspace of ad_a. An easy argument due to Mathias Stout (see p. 59 in *Modular representation theory and applications to decomposition algebras*, master thesis Gent University 2021, https://algebra.ugent.be/ ˜ tdemedts/research-students.php#), who kindly allowed us to include it here, shows that

Lemma 2.1 *If A is a primitive axial algebra of Monster type (α, β) over a field of characteristic 2, then A is associative.*

Proof Let b be an axis and assume $b = b_+ + b_-$, with $b_+ \in A_a^+$ and $b_- \in A_a^-$. Then, by the fusion law,

$$b = b^2 = (b_+ + b_-)^2 = b_+^2 + b_-^2 \in A_a^+.$$

Table 1 Fusion law $\mathcal{M}(\alpha, \beta)$

$\star$	1	0	α	β
1	1		α	β
0		0	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

So all axes lie in A_a^+, whence $A = A_a^+$. Now, the same argument on A_a^+ shows that also the α-eigenspace of ad_a is trivial. Thus, the adjoint of every axis has spectrum $\{1, 0\}$, which is equivalent to the algebra being associative (see [6]). □

Therefore, from now on we assume that $\mathbb{F}$ is a field of characteristic other than 2. A straightforward computation shows that the map that negates A_a^- and induces the identity on A_a^+ is an involutory algebra automorphism called *Miyamoto involution* (see [9, 13]).

Let $\mathcal{H}$ be the infinite-dimensional $\mathbb{F}$-vector space with basis $\{a_i, \sigma_j \mid i \in \mathbb{Z}, j \in \mathbb{Z}_+\}$,

$$\mathcal{H} := \bigoplus_{i\in\mathbb{Z}} \mathbb{F}a_i \oplus \bigoplus_{j\in\mathbb{Z}_+} \mathbb{F}\sigma_j.$$

Set $\sigma_0 = 0$. Define a commutative non-associative product on $\mathcal{H}$ by linearly extending the following values on the basis elements:

($\mathcal{H}$1) $a_i a_j := \frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}$;

($\mathcal{H}$2) $a_i \sigma_j := -\frac{3}{4}a_i + \frac{3}{8}(a_{i-j} + a_{i+j}) + \frac{3}{2}\sigma_j$;

($\mathcal{H}$3) $\sigma_i \sigma_j := \frac{3}{4}(\sigma_i + \sigma_j) - \frac{3}{8}(\sigma_{|i-j|} + \sigma_{i+j})$.

In particular, $a_i^2 = \frac{1}{2}(a_i + a_i) + \sigma_0 = a_i$, so each a_i is an idempotent.

We call $\mathcal{H}$ the *Highwater algebra* because it was discovered in Venice during the disastrous floods in November 2019. In what follows, double angular brackets denote algebra generation while single brackets denote linear span.

Theorem 2.2 *If* $\mathrm{char}(\mathbb{F}) \neq 3$ *then* $\mathcal{H} = \langle\langle a_0, a_1 \rangle\rangle$ *is a symmetric primitive axial algebra of Monster type* $(2, \frac{1}{2})$.

Manifestly, this result implies Theorem 1.1.

If $\mathrm{char}(\mathbb{F}) = 3$ then $2 = \frac{1}{2}$ and so the concept of an algebra of Monster type $(2, \frac{1}{2})$ is not defined. However, the four-term decomposition typical for algebras of Monster type still exists, and so $\mathcal{H}$ in characteristic 3 is an example of an axial decomposition algebra as defined in [1]. We also prove that $\mathcal{H}$ in this case is a Jordan algebra. Note, however, that because a lot of structure constants in $\mathcal{H}$ become zero in characteristic 3, $\mathcal{H}$ is no longer generated by a_0 and a_1. In fact, every pair of distinct axes a_i, a_j generates a 3-dimensional subalgebra, linearly spanned by a_i, a_j, and $\sigma_{|i-j|}$, and isomorphic to the 3-dimensional Jordan algebra denoted by $Cl^{00}(\mathbb{F}^2, b_2)$ in [7, Theorem (1.1)].

We start with a number of observations concerning the properties of $\mathcal{H}$. First of all, we show that it is not a simple algebra. Consider the linear map $\lambda : \mathcal{H} \to \mathbb{F}$ defined on the basis of $\mathcal{H}$ as follows: $\lambda(a_i) = 1$ for all $i \in \mathbb{Z}$ and $\lambda(\sigma_j) = 0$ for all $j \in \mathbb{Z}_+$.

Lemma 2.3 *The map* λ *is a homomorphism of algebras.*

Proof It suffices to show that λ is multiplicative, i.e., $\lambda(uv) = \lambda(u)\lambda(v)$ for all $u, v \in \mathcal{H}$. Since this equality is linear in both u and v, it suffices to check it for the elements of the basis. If $u = a_i$ and $v = a_j$ then $\lambda(a_i a_j) = \lambda(\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}) = \frac{1}{2} + \frac{1}{2} = 1 = 1 \cdot 1 = \lambda(a_i)\lambda(a_j)$. If $u = a_i$ and $v = \sigma_j$ then $\lambda(a_i\sigma_j) = \lambda(-\frac{3}{4}a_i + \frac{3}{8}(a_{i-j} + a_{i+j}) + \frac{3}{2}\sigma_j) = -\frac{3}{4} + \frac{3}{8} + \frac{3}{8} = 0 = 1 \cdot 0 = \lambda(a_i)\lambda(\sigma_j)$. Finally, if $u = \sigma_i$ and $v = \sigma_j$ then $\lambda(\sigma_i\sigma_j) = \lambda(\frac{3}{4}(\sigma_i + \sigma_j) - \frac{3}{8}(\sigma_{|i-j|} + \sigma_{i+j})) = 0 = \lambda(\sigma_i)\lambda(\sigma_j)$. So the equality holds in all cases and so λ is indeed an algebra homomorphism. □

An $\mathbb{F}$-algebra homomorphism from an $\mathbb{F}$-algebra A to $\mathbb{F}$ is called a *weight function* and an algebra admitting a weight function is called a *baric* (or *weighted*) algebra. Note that, if A is an axial algebra with a weight function ω, then the axes, being idempotents, have weights 0 or 1 and, since the axes generate the whole algebra, at least one axis has weight 1. We say that A is a *baric axial algebra* if all axes have weight 1. So $\mathcal{H}$ is a baric axial algebra with weight function λ. Clearly its kernel J is an ideal of codimension 1.

Using λ, we can also define a bilinear form on $\mathcal{H}$. Namely, for $u, v \in \mathcal{H}$, we set $(u, v) := \lambda(u)\lambda(v)$. It is immediate that this is a bilinear form; furthermore, it associates with the algebra product. Indeed, for $u, v, w \in \mathcal{H}$, we have $(uv, w) = \lambda(uv)\lambda(w) = \lambda(u)\lambda(v)\lambda(w) = \lambda(u)\lambda(vw) = (u, vw)$. In the theory of axial algebras such forms are called *Frobenius forms*. The form $(\cdot,\cdot)$ further satisfies the property that $(a_i, a_i) = \lambda(a_i)\lambda(a_i) = 1 \cdot 1 = 1$, which is often required in the definition of a Frobenius form.

Lemma 2.4 *If* char($\mathbb{F}$) $\neq 3$, *the only nonzero finite-dimensional subalgebras of* $\mathcal{H}$ *are the* 1*-dimensional ones generated by axes.*

Proof Let $L \neq 0$ be a finite-dimensional subalgebra of $\mathcal{H}$. Every element e of L can be written uniquely as

$$e = \sum_{i\in\mathbb{Z}} r_i(e)a_i + \sum_{j\in\mathbb{Z}_+} s_j(e)\sigma_j$$

with the coefficients $r_i(e)$ and the $s_j(e)$ in $\mathbb{F}$ and only finitely many of them being nonzero. Let

$$R := \{i \in \mathbb{Z} \mid r_i(e) \neq 0 \text{ for some } e \in L\}$$

and

$$S := \{j \in \mathbb{Z}_+ \mid s_j(e) \neq 0 \text{ for some } e \in L\}.$$

Since L is finite dimensional, both R and S are finite sets. Also, since $L \neq 0$, at least one of the two sets is nonempty. If both R and S are nonempty, let $n := \max(R)$ and $m := \max(S)$. Select $e \in L$ such that $r_n(e) \neq 0$. If also $s_m(e) \neq 0$ then observe that, by ($\mathcal{H}$ 2), $r_{n+m}(e^2) = 2 \cdot \frac{3}{8} r_n(e)s_m(e) \neq 0$, a contradiction, since $n + m > n$. So $s_m(e) = 0$. Now select $e' \in L$ such that $s_m(e') \neq 0$. Again, by ($\mathcal{H}$2), $r_{n+m}(ee') = \frac{3}{8} r_n(e)s_m(e') \neq 0$. The contradiction shows that R and S cannot both be nonempty.

If R is empty and S is not, choose $e \in L$ with $s_m(e) \neq 0$, where again $m = \max(S)$. By ($\mathcal{H}$3), $s_{2m}(e^2) = -\frac{3}{8} s_m(e)^2 \neq 0$; which is a contradiction. Finally, if S is empty and R is not, let $n := \max(R)$ and $k := \min(R)$. If $n = k$, then L is 1-dimensional generated by a_n. So suppose that $k < n$. Again, select $e \in L$ such that $r_n(e) \neq 0$. If also $r_k(e) \neq 0$ then $s_{n-k}(e^2) = 2r_n(e)r_k(e) \neq 0$ by ($\mathcal{H}$1). This is a contradiction since $S = \emptyset$. Finally, if $r_k(e) = 0$, choose $e' \in L$ with $r_k(e') \neq 0$. Now $s_{n-k}(ee') = r_n(e)r_k(e') \neq 0$ by ($\mathcal{H}$1), and this is a contradiction in this final case. □

Corollary 2.5 *If* char($\mathbb{F}$) $\neq 3$, *the elements* $\{a_i \mid i \in \mathbb{Z}\}$ *are the only nontrivial idempotents in* $\mathcal{H}$.

The next observation to make is that $\mathcal{H}$ is quite symmetric. Let $D := \langle \tau, \pi \rangle$ be the infinite dihedral group acting naturally on $\mathbb{Z}$, where, for all $i \in \mathbb{Z}$, $\tau : i \mapsto -i$ and $\pi : i \mapsto -i+1$. For $\rho \in D$, let ϕ_ρ be the linear map that fixes all σ_j and sends a_i to a_{i^ρ}. Then ϕ_ρ is an automorphism of $\mathcal{H}$ and, since the map $\rho \mapsto \phi_\rho$ defines a faithful representation of D as an automorphism group of $\mathcal{H}$, with an abuse of notation, from now on we identify ϕ_ρ with ρ. In particular,

Proposition 2.6 *If* char($\mathbb{F}$) $\neq 3$, *then* $Aut(\mathcal{H}) \cong D$.

Proof Let $\varphi \in \mathrm{Aut}(\mathcal{H})$. By Corollary 2.5, φ induces a permutation on the set $\{a_i \mid i \in \mathbb{Z}\}$. Let $j \in \mathbb{Z}_+$ and set $a_h := a_0^\varphi$ and $a_k := a_j^\varphi$. Then, by ($\mathcal{H}$1),

$$\sigma_j^\varphi = \left(a_0 a_j - \frac{1}{2}(a_0 + a_j)\right)^\varphi = a_0^\varphi a_j^\varphi - \frac{1}{2}(a_0^\varphi + a_j^\varphi) = a_h a_k - \frac{1}{2}(a_h + a_k) = \sigma_{|h-k|},$$

whence φ induces a permutation also on the set $\{\sigma_j \mid j \in \mathbb{Z}_+\}$. Now observe that the action on the latter set has to be trivial. Indeed, for $j \in \mathbb{Z}_+$, define the graph Γ_j with vertices $\{a_i, \mid i \in \mathbb{Z}\}$, where a_i is adjacent to a_l if and only if $\sigma_j = a_i a_l - \frac{1}{2}(a_i + a_l)$. Thus, for all $h, k \in \mathbb{Z}_+$ such that $(\sigma_h)^\varphi = \sigma_k$, φ induces a graph isomorphism from Γ_h to Γ_k. On the other hand, by ($\mathcal{H}$1), a_i is adjacent to a_l in Γ_j if and only if $|i-l| = j$. Hence, Γ_j has exactly j distinct connected components, corresponding to the congruence classes of $\mathbb{Z}$ modulo j. In particular, the graph Γ_h is isomorphic to Γ_k if and only if $h = k$.

In particular, the entire group $\mathrm{Aut}(\mathcal{H})$ fixes σ_1, and so it acts on the infinite string graph Γ_1. Since this action is faithful, we conclude that $\mathrm{Aut}(\mathcal{H}) \cong D$. □

We are aiming to show that the a_i's are axes satisfying the fusion law $\mathcal{M}(2, \frac{1}{2})$. Since D is transitive on the a_i's, it suffices to check this for just one of them, say $a = a_0$. We start with the eigenvalues and eigenspaces of ad_a.

Select $j \in \mathbb{Z}_+$ and set $U = \langle a, a_{-j}, a_j, \sigma_j \rangle$. It is immediate to see that U is invariant under ad_a and, with respect to the basis $(a, a_{-j}, a_j, \sigma_j)$ the restriction of ad_a to U is represented by the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ -\frac{3}{4} & \frac{3}{8} & \frac{3}{8} & \frac{3}{2} \end{pmatrix}$$

This has characteristic polynomial $x^4 - \frac{7}{2}x^3 + \frac{7}{2}x^2 - x = (x-1)x(x-2)(x-\frac{1}{2})$ and eigenspaces $U_1 = \langle a \rangle$, $U_0 = \langle u_j \rangle$, $U_2 = \langle v_j \rangle$, $U_{\frac{1}{2}} = \langle w_j \rangle$, where

$$\begin{aligned} u_j &:= 6a - 3(a_{-j} + a_j) + 4\sigma_j \\ v_j &:= 2a - (a_{-j} + a_j) - 4\sigma_j \qquad (1) \\ w_j &:= a_{-j} - a_j. \end{aligned}$$

The only exception to this statement arises when char($\mathbb{F}$) = 3. In this case, $2 = \frac{1}{2}$ and so U_2 and $U_{\frac{1}{2}}$ merge into a single 2-dimensional eigenspace $U_2 = U_{\frac{1}{2}} = \langle v_j, w_j \rangle$.

In all cases, we can write that $U = \langle a, u_j, v_j, w_j \rangle$, that is, a_{-j}, a_j and σ_j can be expressed via these vectors. From this, we deduce the following.

Lemma 2.7 *The adjoint map* ad_a *is semisimple on* $\mathcal{H}$ *and*

(a) *if* char($\mathbb{F}$) $\neq 3$ *then the spectrum of* ad_a *is* $\{1, 0, 2, \frac{1}{2}\}$ *and the eigenspaces are* $\mathcal{H}_1 = \langle a \rangle$, $\mathcal{H}_0 = \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, $\mathcal{H}_2 = \langle v_j \mid j \in \mathbb{Z}_+ \rangle$, *and* $\mathcal{H}_{\frac{1}{2}} = \langle w_j \mid j \in \mathbb{Z}_+ \rangle$;
(b) *if* char($\mathbb{F}$) = 3 *then the spectrum is* $\{1, 0, \frac{1}{2}\}$ *and the eigenspaces are* $\mathcal{H}_1 = \langle a \rangle$, $\mathcal{H}_0 = \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, *and* $\mathcal{H}_{\frac{1}{2}} = \langle v_j, w_j \mid j \in \mathbb{Z}_+ \rangle$.

In order to avoid the complication arising in characteristic 3, we will use the notation $\mathcal{H}_u := \langle u_j \mid j \in \mathbb{Z}_+ \rangle$, $\mathcal{H}_v := \langle v_j \mid j \in \mathbb{Z}_+ \rangle$, and $\mathcal{H}_w = \langle w_j \mid j \in \mathbb{Z}_+ \rangle$ calling these subspaces the u-, v-, and w-parts of $\mathcal{H}$, respectively. A similar terminology will be used for sums of these subspaces. Thus, in all characteristics, we have the decomposition

$$\mathcal{H} = \langle a \rangle \oplus \mathcal{H}_u \oplus \mathcal{H}_v \oplus \mathcal{H}_w.$$

Let us relate this decomposition to the ideal J.

Lemma 2.8 $J = \mathcal{H}_u \oplus \mathcal{H}_v \oplus \mathcal{H}_w$.

Proof By inspection, the weight function is zero on $\mathcal{H}_u \oplus \mathcal{H}_v \oplus \mathcal{H}_w$, so this sum is contained in J. Now, the equality is forced because both J and $\mathcal{H}_u \oplus \mathcal{H}_v \oplus \mathcal{H}_w$ have codimension 1 in $\mathcal{H}$. □

The stabilizer D_a of a in D has order 2, and it is generated by the involution τ sending every a_i to a_{-i} (and fixing every σ_j). From this and Eq. (1), it follows that

Lemma 2.9 *The involution* τ *acts as identity on* $\langle a \rangle \oplus \mathcal{H}_u \oplus \mathcal{H}_v$ *and as minus identity on* $\mathcal{H}_w$. *In particular,* $\mathcal{H}_\tau := \langle a \rangle \oplus \mathcal{H}_u \oplus \mathcal{H}_v$ *is the fixed subalgebra of* τ.

Clearly, τ will turn out to be the Miyamoto involution associated to the axis a.

We find a further subalgebra by intersecting $\mathcal{H}_\tau$ with the ideal J. Namely, we set $V := \mathcal{H}_\tau \cap J = \mathcal{H}_u \oplus \mathcal{H}_v$, the uv-part of $\mathcal{H}$. Clearly, V is an ideal of $\mathcal{H}_\tau$.

We want now to determine the fusion law in the subalgebra V with respect to its basis consisting of the vectors u_j and v_j. To this aim, it is convenient to use a second basis, given by the elements σ_j and

$$c_j := 2a - (a_{-j} + a_j), \tag{2}$$

for all $j \in \mathbb{Z}_+$. From the definitions, we immediately get that, for all $j \in \mathbb{Z}_+$,

$$u_j = 3c_j + 4\sigma_j \quad \text{and} \quad v_j = c_j - 4\sigma_j \tag{3}$$

and

$$c_j = \frac{1}{4}(u_j + v_j) \quad \text{and} \quad \sigma_j = \frac{1}{16}u_j - \frac{3}{16}v_j. \tag{4}$$

In particular, this means that the set of all vectors c_j and σ_j is a basis of V. In the following lemma, we compute the products for this basis. Note that we only need to compute the products $c_i c_j$ and $c_i\sigma_j$, because $\sigma_i\sigma_j$ are given in ($\mathcal{H}3$). It will be convenient to use the following notation: we set $c_0 := 0$, and, for $i,j \in \mathbb{Z}_+$, we let $c_{i,j} := -2c_i - 2c_j + c_{|i-j|} + c_{i+j}$ and $\sigma_{i,j} := -2\sigma_i - 2\sigma_j + \sigma_{|i-j|} + \sigma_{i+j}$. For example, ($\mathcal{H}3$) can now be restated as follows: $\sigma_i\sigma_j = -\frac{3}{8}\sigma_{i,j}$.

Lemma 2.10 *For* $i,j \in \mathbb{Z}_+$, *we have* $c_i c_j = 2\sigma_{i,j}$ *and* $c_i\sigma_j = \frac{3}{8}c_{i,j}$.

Proof For $i \in \mathbb{Z}_+$, set $c_i^{\pm} := a - a_{\pm i}$. Then, by Eq. (2), $c_i = c_i^+ + c_i^-$ and, for $\delta, \varepsilon \in \{+,-\}$, using ($\mathcal{H}_1$), resp. ($\mathcal{H}_2$), we get

$$c_i^{\varepsilon} c_j^{\delta} = -\sigma_i - \sigma_j + \sigma_{|\varepsilon i - \delta j|}, \quad \text{resp.} \quad c_i^{\varepsilon}\sigma_j = -\frac{3}{4}c_i^{\varepsilon} + \frac{3}{8}(-c_j + c_{i+j}^{\varepsilon} + c_{|i-j|}^{\varepsilon}).$$

In particular, for all $i,j \in \mathbb{Z}_+$, $c_i^+ c_j^+ = c_i^- c_j^-$ and $c_i^+ c_j^- = c_j^+ c_i^-$. Thus

$$\begin{aligned} c_i c_j &= (c_i^+ + c_i^-)(c_j^+ + c_j^-) = 2(c_i^+ c_j^+ + c_i^+ c_j^-) \\ &= 2(-\sigma_i - \sigma_j + \sigma_{|i-j|} - \sigma_i - \sigma_j + \sigma_{i+j}) = 2\sigma_{i,j}. \end{aligned}$$

Similarly

$$\begin{aligned} c_i\sigma_j &= (c_i^+ + c_i^-)\sigma_j \\ &= -\frac{3}{4}c_i^+ + \frac{3}{8}(-c_j + c_{i+j}^+ + c_{|i-j|}^+) - \frac{3}{4}c_i^- + \frac{3}{8}(-c_j + c_{i+j}^- + c_{|i-j|}^-) \\ &= -\frac{3}{4}c_i - \frac{3}{4}c_j + \frac{3}{8}(c_{i+j} + c_{|i-j|}) = \frac{3}{8}c_{i,j}. \end{aligned}$$

□

Manifestly, all these products have a very uniform structure. Note that both $c_{i,j}$ and $\sigma_{i,j}$ are symmetric in i and j. In particular, it follows that $c_i\sigma_j = \frac{3}{8}c_{i,j} = \frac{3}{8}c_{j,i} = c_j\sigma_i$.

Similarly to the above, for $i,j \in \mathbb{Z}_+$, we introduce $u_{i,j} := -2u_i - 2u_j + u_{|i-j|} + u_{i+j}$ and $v_{i,j} := -2v_i - 2v_j + v_{|i-j|} + v_{i+j}$, where also $u_0 := 0, v_0 := 0$. By Eq. (3) we have

$$u_{i,j} = 3c_{i,j} + 4\sigma_{i,j} \ \text{and} \ v_{i,j} = c_{i,j} - 4\sigma_{i,j}. \tag{5}$$

Now we compute the products of the vectors u_j and v_j.

Lemma 2.11 *For all* $i,j \in \mathbb{Z}_+$, *we have that* $u_i u_j = 3u_{i,j}$, $u_i v_j = -3v_{i,j}$, *and* $v_i v_j = -u_{i,j}$.

Proof By Eq. (3),

$$u_i u_j = (3c_i + 4\sigma_i)(3c_j + 4\sigma_j) = 9c_i c_i + 12c_i\sigma_j + 12\sigma_i c_j + 16\sigma_i\sigma_j.$$

By Lemma 2.10, this becomes

$$9(2\sigma_{i,j}) + 12\left(\frac{3}{8}c_{i,j}\right) + 12\left(\frac{3}{8}c_{i,j}\right) + 16\left(-\frac{3}{8}\sigma_{i,j}\right) = 9c_{i,j} + 12\sigma_{i,j} = 3u_{i,j}.$$

Similarly,

$$\begin{aligned} u_i v_j &= (3c_i + 4\sigma_i)(c_j - 4\sigma_j) = 3c_ic_j + 4\sigma_ic_j - 12c_i\sigma_j - 16\sigma_i\sigma_j \\ &= 3(2\sigma_{i,j}) + 4\left(\frac{3}{8}c_{i,j}\right) - 12\left(\frac{3}{8}c_{i,j}\right) - 16\left(-\frac{3}{8}\sigma_{i,j}\right) = -3c_{i,j} + 12\sigma_{i,j} = -3v_{i,j} \end{aligned}$$

and

$$\begin{aligned} v_i v_j &= (c_i - 4\sigma_i)(c_j - 4\sigma_j) = c_ic_j - 4\sigma_ic_j - 4c_i\sigma_j + 16\sigma_i\sigma_j \\ &= 2\sigma_{i,j} - 4\left(\frac{3}{8}c_{i,j}\right) - 4\left(\frac{3}{8}c_{i,j}\right) + 16\left(-\frac{3}{8}\sigma_{i,j}\right) = -3c_{i,j} - 4\sigma_{i,j} = -u_{i,j}. \end{aligned}$$

□

We are now ready to establish the fusion law for the axis a.

*Proof of Theorem*2.2 Recall that, when $\mathrm{char}(\mathbb{F}) \neq 3$, we have $\langle a\rangle = \mathcal{H}_1$, $\mathcal{H}_u = \mathcal{H}_0$, $\mathcal{H}_v = \mathcal{H}_2$, and $\mathcal{H}_w = \mathcal{H}_{\frac{1}{2}}$ are the eigenspaces of ad_a. So we need to see how these subspaces behave under the product in $\mathcal{H}$.

First of all, clearly, $\langle a\rangle\mathcal{H}_u = 0$, $\langle a\rangle\mathcal{H}_v = \mathcal{H}_v$ and $\langle a\rangle\mathcal{H}_w = \mathcal{H}_w$. Now focus on the products of the u-, v-, and w-parts. These parts are all contained in J, and so their products are also contained in J, avoiding $\langle a\rangle$. Furthermore, the involution τ induces a grading on J, under which $V = \mathcal{H}_u \oplus \mathcal{H}_v$ is the even part and $\mathcal{H}_w$ is the odd part. Hence, $\mathcal{H}_u\mathcal{H}_w$ and $\mathcal{H}_v\mathcal{H}_w$ are contained in $\mathcal{H}_w$, while $\mathcal{H}_w\mathcal{H}_w \subseteq V = \mathcal{H}_u \oplus \mathcal{H}_v$. It remains to check the products within V. As can be seen from Lemma 2.11, $\mathcal{H}_u\mathcal{H}_u \subseteq \mathcal{H}_u$, $\mathcal{H}_u\mathcal{H}_v \subseteq \mathcal{H}_v$, and $\mathcal{H}_v\mathcal{H}_v \subseteq \mathcal{H}_u$. Hence, when $\mathrm{char}(\mathbb{F}) \neq 3$, a satisfies the fusion law of Table 2, which is slightly stricter than the fusion law $\mathcal{M}(2, \frac{1}{2})$.

Since D is transitive on the a_i, all of them are axes satisfying the law in Table 2. To complete the proof of Theorem 2.2, it remains to show that $\mathcal{H} = \langle\langle a_0, a_1\rangle\rangle$. Let $H := \langle\langle a_0, a_1\rangle\rangle$. Note that $\sigma_1 = a_0a_1 - \frac{1}{2}(a_0 + a_1) \in H$. Also, $\frac{3}{8}a_{-1} = a_0\sigma_1 + \frac{3}{4}a_0 - \frac{3}{8}a_1 - \frac{3}{2}\sigma_1 \in H$. Assuming that $\mathrm{char}(\mathbb{F}) \neq 3$, this gives us that $a_{-1} \in H$. Clearly, $\langle\langle a_0, a_1\rangle\rangle$ is invariant under the involution $\pi \in D$ switching a_0 and a_1. Now, we also see that $H = \langle\langle a_{-1}, a_0, a_1\rangle\rangle$ is invariant under the involution $\tau \in D_a$. Since $D = \langle \tau, \pi\rangle$, this makes H invariant under all of D, and so H contains all axes a_i. Clearly, this means that $H = \mathcal{H}$. This completes the proof of Theorem 2.2. □

Table 2 Fusion law for $\mathcal{H}$

$\star$	1	0	2	$\frac{1}{2}$
1	1		2	$\frac{1}{2}$
0		0	2	$\frac{1}{2}$
2	2	2	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$0, 2$

Note that, if $\mathrm{char}(\mathbb{F}) \neq 3$, then, for any two distinct i and j in $\mathbb{Z}$, the subalgebra $\langle\langle a_i, a_j\rangle\rangle$ is linearly spanned by the set $\{a_n, \sigma_n | n \text{ is a multiple of g.c.d. } (i,j)\}$. In particular $\langle\langle a_i, a_j\rangle\rangle$ is isomorphic to $\mathcal{H}$.

3 The characteristic 3 case

Let us discuss what happens when $\mathrm{char}(\mathbb{F}) = 3$. The fusion that we established for the decomposition $\mathcal{H} = \langle a\rangle \oplus \mathcal{H}_u \oplus \mathcal{H}_v \oplus \mathcal{H}_w$ remains true, but it becomes even stricter because so many coefficients are multiples of 3. For example, $\mathcal{H}_u\mathcal{H}_u = 0 = \mathcal{H}_u\mathcal{H}_v$. On the other hand, $2 = \frac{1}{2}$ in characteristic 3, so we need to merge $\mathcal{H}_v$ and $\mathcal{H}_w$ into a single eigenspace $\mathcal{H}_{\frac{1}{2}} = \mathcal{H}_v \oplus \mathcal{H}_w$. Let us see what fusion we get inside this eigenspace. We already know that $\mathcal{H}_v\mathcal{H}_v \subseteq \mathcal{H}_u$.

Lemma 3.1 *If* $\mathrm{char}(\mathbb{F}) = 3$ *then* $\mathcal{H}_v\mathcal{H}_w = 0$ *and* $\mathcal{H}_w\mathcal{H}_w \subseteq \mathcal{H}_u$.

Proof Indeed, for $i,j \in \mathbb{Z}_+$, $v_iw_j = (2a - (a_{-i} + a_i) - 4\sigma_i)(a_{-j} - a_j)$. Taking into account that $aw_j = \frac{1}{2}w_j$ and that $\sigma_iw_j = 0$ when $\mathrm{char}(\mathbb{F}) = 3$ [see ($\mathcal{H}2$)], we obtain that

$$\begin{aligned} v_iw_j &= (a_{-j} - a_j) - (a_{-i} + a_i)(a_{-j} - a_j) \\ &= (a_{-j} - a_j) - (\frac{1}{2}(a_{-i} + a_{-j}) + \sigma_{|i-j|}) - \left(\frac{1}{2}(a_i + a_{-j}) + \sigma_{i+j}\right) \\ &\quad + \left(\frac{1}{2}(a_{-i} + a_j) + \sigma_{i+j}\right) + \left(\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}\right) = 0. \end{aligned}$$

Also,

$$\begin{aligned} w_iw_j &= (a_{-i} - a_i)(a_{-j} - a_j) = \left(\frac{1}{2}(a_{-i} + a_{-j}) + \sigma_{|i-j|}\right) - \left(\frac{1}{2}(a_i + a_{-j}) + \sigma_{i+j}\right) \\ &\quad - \left(\frac{1}{2}(a_{-i} + a_j) + \sigma_{i+j}\right) + \left(\frac{1}{2}(a_i + a_j) + \sigma_{|i-j|}\right) = 2\sigma_{|i-j|} - 2\sigma_{i+j} \\ &= \frac{1}{2}u_{|i-j|} - \frac{1}{2}u_{i+j}. \end{aligned}$$

The last equality holds because $u_k = 4\sigma_k$ when $\mathrm{char}(\mathbb{F}) = 3$ [see Eq. (1)]. □

So we can see now that in characteristic 3 the axes a_i in $\mathcal{H}$ satisfy the fusion law in Table 3.

In particular, $\mathcal{H}$ is an algebra of Jordan type $\frac{1}{2}$. Let us prove that, in fact, $\mathcal{H}$ is a Jordan algebra when $\mathrm{char}(\mathbb{F}) = 3$. For this, we need to verify the Jordan identity $x(yx^2) = (xy)x^2$ for all $x, y \in \mathcal{H}$. We obtain this from the following more general result.

Table 3 Fusion law for $\mathcal{H}$ in characteristic 3

$\star$	1	0	$\frac{1}{2}$
1	1		$\frac{1}{2}$
0			$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

Proposition 3.2 *Let B be a commutative baric algebra over a field of characteristic other than 2, with weight function ω. Suppose that*

(a) $B = A \oplus I$, *, where A is a subspace and I is an ideal,*
(b) $IB = 0$, *and*
(c) *for all $a, b \in A$, $ab - \frac{1}{2}(\omega(b)a + \omega(a)b) \in I$.*

Then, B is a Jordan algebra.

Proof Let $\mu : B \to A$ be the projection onto A with respect to the decomposition $B = A \oplus I$. Then, every $x \in B$ can be written in a unique way as $x = \mu(x) + x_I$, where $\mu(x) \in A$ and $x_I \in I$. Let $x, y \in B$. By hypothesis (b), $xy = \mu(x)\mu(y) = \mu(x)y$ and, by hypothesis (c), $\mu(x^2) = \omega(x)\mu(x)$. Hence, using the commutativity of B, we have

$$\begin{aligned} x(yx^2) &= \mu(x)(\mu(y)\mu(x^2)) = \mu(x)(\mu(y)\omega(x)\mu(x)) = \omega(x)\mu(x)(\mu(y)\mu(x)) \\ &= (\mu(x)\mu(y))\omega(x)\mu(x) = (\mu(x)\mu(y))\mu(x^2) = (xy)x^2. \end{aligned}$$

Therefore B is a Jordan algebra. □

Corollary 3.3 *Let A and I be vector spaces over a field $\mathbb{F}$ of characteristic other than 2, let $\omega : A \to \mathbb{F}$ be a linear map and $\sigma : A \times A \to I$ a symmetric bilinear map. Let $B := A \oplus I$ and define an algebra product on B by $BI = 0$ and, for $a, b \in A$, $ab := \frac{1}{2}(\omega(b)a + \omega(a)b) + \sigma(a, b)$. Then B is a baric Jordan algebra.*

Proof Define $\overline{\omega} : B \to \mathbb{F}$ by $\overline{\omega}(a + i) = \omega(a)$, for every $a \in A$, $i \in I$. Then, $\overline{\omega}$ is a weight function on B (note that it may not be the only choice for a weight function) and B satisfies the hypotheses of Proposition 3.2. The claim follows. □

Theorem 3.4 *If* $\mathrm{char}(\mathbb{F}) = 3$ *then $\mathcal{H}$ is a baric Jordan algebra.*

Proof It is enough to show that $\mathcal{H}$ satisfies the hypotheses of Proposition 3.2. We let $A := \langle a_i \mid i \in \mathbb{Z} \rangle$ and $I := \langle \sigma_j \mid j \in \mathbb{Z}_+ \rangle$. Then clearly $\mathcal{H}$ is the direct sum of A and I and, by ($\mathcal{H}$2) and ($\mathcal{H}$3), since $\mathrm{char}(\mathbb{F}) = 3$, we have $I\mathcal{H} = 0$ (in particular I is an ideal). Let $\mu : \mathcal{H} \to A$ be the projection onto A with respect to the decomposition $\mathcal{H} = A \oplus I$. Let $x, y \in A$ and write $x = \sum_{i \in \mathbb{Z}} r_i a_i$ and $y = \sum_{i \in \mathbb{Z}} t_i a_i$. Note that $\lambda(x) = \sum_{i \in \mathbb{Z}} r_i$ and $\lambda(y) = \sum_{i \in \mathbb{Z}} t_i$. By ($\mathcal{H}$1), we have

$$\begin{aligned} \mu(xy) &= \sum_{i,j \in \mathbb{Z}} r_i t_j \frac{1}{2}(a_i + a_j) \\ &= \sum_{i,j \in \mathbb{Z}} r_i t_j \frac{1}{2} a_i + \sum_{i,j \in \mathbb{Z}} r_i t_j \frac{1}{2} a_j \\ &= \frac{1}{2}\left(\sum_{j \in \mathbb{Z}} t_j\right)\left(\sum_{i \in \mathbb{Z}} r_i a_i\right) + \frac{1}{2}\left(\sum_{i \in \mathbb{Z}} r_i\right)\left(\sum_{j \in \mathbb{Z}} t_j a_j\right) \\ &= \frac{\lambda(y)}{2} x + \frac{\lambda(x)}{2} y. \end{aligned}$$

This proves that $xy - \frac{1}{2}(\lambda(y)x + \lambda(x)y)$ belongs to I. □

4 Final remarks

Now that we have proved our main results, let us tie the loose ends. In this section, $\mathbb{F}$ is a field of characteristic other than 2.

4.1 Frobenius form

We refer to [5] for a detailed discussion of the radical of an axial algebra, projection form and projection graph.

Proposition 4.1

(a) *The form $(\cdot,\cdot)$ is, up to a scalar factor, the only Frobenius form on $\mathcal{H}$.*
(b) *Every proper ideal of $\mathcal{H}$ is contained in J, which is the radical of $\mathcal{H}$.*

Proof The form $(\cdot,\cdot)$ that we introduced on $\mathcal{H}$ is a projection form, because it is a Frobenius form satisfying $(a_i, a_i) = 1$ for all $i \in \mathbb{Z}$. The projection graph has all a_i as vertices with an edge between a_i and a_j, $i \neq j$, whenever $(a_i, a_j) \neq 0$. In fact, since $(a_i, a_j) = 1$ for all $i,j \in \mathbb{Z}$, the projection graph of $\mathcal{H}$ is the complete graph. In particular, it is connected. It now follows from [5, Proposition 4.19] that $\mathcal{H}$ has only one Frobenius form up to a scalar factor. The same connectivity property implies, by [5, Corollary 4.15], that every proper ideal of $\mathcal{H}$ is contained in the radical of $\mathcal{H}$ (the largest ideal not containing axes). Finally, by [5, Corollary 4.11], the radical of $\mathcal{H}$ coincides with the radical of the projection form $(\cdot,\cdot)$, and that is J. □

4.2 Automorphisms of the subalgebra *V*

Recall that in Section 2, we introduced the subalgebra $V = \mathcal{H}_u \oplus \mathcal{H}_v$ and its two bases $\{u_i, v_j \mid i,j \in \mathbb{Z}_+\}$ and $\{c_i, \sigma_j \mid i,j \in \mathbb{Z}_+\}$. Let θ and ψ be the linear transformations $V \to V$ defined by

$$(u_j)^\theta = u_j, \quad (v_j)^\theta = -v_j$$

and

$$(c_j)^\psi = \frac{1}{2}v_j = \frac{1}{2}c_j - 2\sigma_j, \quad (\sigma_j)^\psi = -\frac{1}{8}u_j = -\frac{3}{8}c_j - \frac{1}{2}\sigma_j.$$

Lemma 4.2 *The linear transformations θ and ψ defined above are automorphisms of V.*

Proof This follows from the formulae in Lemmas 2.10 and 2.11. □

It is clear that θ has order 2. Note that ψ is also an involution, because the matrix

$$\begin{pmatrix} \frac{1}{2} & -2 \\ -\frac{3}{8} & -\frac{1}{2} \end{pmatrix}$$

squares to identity, and also that $\psi\rho$ is an element of order 3. This means that $T := \langle \theta, \psi \rangle \cong D_6$. It is interesting that the subalgebra V has symmetries independent of the entire algebra $\mathcal{H}$. Indeed, we claim that only the identity element from T extends to an element of $\mathrm{Aut}(\mathcal{H}) = D$. Define the support of an element

$$e = \sum_{i \in \mathbb{Z}} r_i(e) a_i + \sum_{j \in \mathbb{Z}_+} s_j(e) \sigma_j$$

as $\{i \in \mathbb{Z} \mid r_i(e) \neq 0\}$. The orbit under T of $c_1 = 2a - (a_{-1} + a_1)$ consists of the six elements $\pm c_1$, $\pm \frac{1}{2} v_1 = \pm(\frac{1}{2} c_1 - 2\sigma_1)$ and $\pm(\frac{1}{2} c_1 + 2\sigma_1)$. All these elements have support $\{-1, 0, 1\}$. Hence if $t \in T$ extends to an element of $d \in D$, then d, as an automorphism of $\mathbb{Z}$, must preserve the set $\{-1, 0, 1\}$. Manifestly, since the automorphisms of $\mathbb{Z}$ are either translations or reflections, the identity 1_D and τ are the only elements of D preserving this set. Both 1_D and τ act trivially on V, so $t = 1_T$, as claimed.

5 Discussion

In this final section, we pose some questions related to $\mathcal{H}$. We have to warn the reader that there is a lot of work in progress on this topic and many results quoted in this section have not yet been published nor undergone a blind refereeing process.

By Yabe's results, or by the preprint of the authors mentioned in the introduction, every symmetric 2-generated primitive algebra of Monster type (α, β) over a field of characteristic other than 2 is at most 8-dimensional unless $(\alpha, \beta) = (2, \frac{1}{2})$. In the non-symmetric case, such a bound has been proved only for $\alpha \neq 4\beta$ (see [16] and the paper by the authors posted in Mathematics arXiv:2101.10379).

Question 5.1 *Is $\mathcal{H}$ the only infinite-dimensional primitive algebra of Monster type? Is it the only such algebra having dimension greater than* 8?

In characteristic other than 5, by Yabe's result, for the symmetric case, the answer to this question depends on the knowledge of the ideals of $\mathcal{H}$. In characteristic 5, the first two authors have shown that the algebra $\mathcal{H}$ admits a proper cover which is still a symmetric primitive algebra of Monster type $(2, \frac{1}{2})$ (Mathematics arXiv:2101.09506).

Question 5.2 *Does $\mathcal{H}$ contain any nonzero ideal of infinite codimension? Is it possible to classify all ideals in $\mathcal{H}$?*

Note that, if $\mathrm{char}(\mathbb{F}) \neq 3$, J is not the only proper nonzero ideal of $\mathcal{H}$. For example, the ideal I generated by all elements σ_j has codimension 1 in J (and hence codimension 2 in $\mathcal{H}$). Indeed, from $(\mathcal{H}_2)$ we get that, for all $i \in \mathbb{Z}_+$,

$$a_0 \sigma_i = -\frac{3}{8} c_i + \frac{3}{2} \sigma_i,$$

whence $c_i \in I$, and $u_i, v_i \in I$. Further, again by $(\mathcal{H}_2)$, we have

$$w_1\sigma_1 = -\frac{3}{4}w_1 + \frac{3}{8}w_2 \text{ and, for } i > 1,\ w_i\sigma_1 = -\frac{3}{4}w_i + \frac{3}{8}(w_{i+1} - w_{i-1})$$

thus, recursively, we get that $w_i \in \langle w_1 \rangle + I$ for all $i \in \mathbb{Z}_+$. Hence I has codimension at most 1 in J. On the other hand, consider the 2-dimensional commutative algebra $\mathbb{F}\bar{a}_0 \oplus \mathbb{F}\bar{a}_1$ with multiplication defined by $\bar{a}_i\bar{a}_i = \bar{a}_i$, $i = 0, 1$ and $\bar{a}_0\bar{a}_1 = \frac{1}{2}(\bar{a}_0 + \bar{a}_1)$ and let φ be the linear map sending a_i to $i\bar{a}_1 - (i-1)\bar{a}_0$, for all $i \in \mathbb{Z}$, and σ_j to 0, for all $j \in \mathbb{Z}_+$. Then φ is a surjective algebra homomorphism with $I \leq \ker\varphi$, whence, by comparing the codimensions, $I = \ker\varphi$ has codimension 2 in $\mathcal{H}$.

In addition to finding all ideals of $\mathcal{H}$, it is also interesting to find all of its subalgebras and their automorphisms.

Question 5.3 *Which is the full automorphism group of $\mathcal{H}$ in characteristic 3? In any characteristic other than 2, which is the automorphism group of V? Which automorphisms of V extend to larger subalgebras of $\mathcal{H}$?*

Finally, the fact that $\mathcal{H}$ is a baric algebra looks exciting. A baric axial algebra of Monster type (α, β) satisfies the fusion law as in Table 2, but with 2 and $\frac{1}{2}$ substituted with arbitrary α and β. Indeed, for every $\eta \in \{0, \alpha, \beta\}$ and for every η-eigenvector v, the linearity of the weight function forces v to be in the kernel of the weight function and the claim follows since the kernel is an ideal.

Question 5.4 *Are there examples of baric algebras of Monster type (α, β) for other values of α and β? Is it possible to classify all such algebras?*

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