Strong maximum principles for fractional Laplacians

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Abstract

We give a unified approach to strong maximum principles for a large class of nonlocal operators of order $s \in (0,1)$, that includes the Dirichlet, the Neumann Restricted (or Regional) and the Neumann Semirestricted Laplacians.

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1 Introduction

In this paper we prove strong maximum principles for a large class of fractional Laplacians of order $s \in (0,1)$, including the *Dirichlet Laplacian*

$$(-\Delta)^s u(x) = C_{n,s} \cdot \text{P.V.} \int\limits_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy,$$

the *Restricted Neumann Laplacian*

$$(-\Delta_\Omega^N)_{\rm R}^s u(x) = C_{n,s} \cdot \text{P.V.} \int\limits_{\Omega} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy$$

(also called *Regional Laplacian*), and intermediate operators, such as the *Semirestricted Neumann Laplacian*

$$(-\Delta_\Omega^N)_{\rm Sr}^s u = \chi_\Omega \cdot (-\Delta)^s u + \chi_{\Omega^c} \cdot (-\Delta_\Omega^N)_{\rm R}^s u \, . \tag{1.1}$$

Here Ω is a domain in $\mathbb{R}^n$, $n \geq 1$, $\Omega^{\mathsf{c}} = \mathbb{R}^n \setminus \Omega$, χ_V is the characteristic function of the set $V \subset \mathbb{R}^n$, $C_{n,s} = \frac{s 2^{2s} \Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}} \Gamma(1-s)}$ and "P.V." means "principal value".

In recent years a lot of effort has been made to develop efficient technical tools suitable to handle nonlocal operators. Differently from the standard Laplacian, that acts by pointwise differentiation, nonlocal operators can be used to model phenomena in presence of long-range interactions (see for instance [8] for a long list of applications). Despite the abundant available literature, several basic questions about fractional Laplacians are still open, including strong maximum principles. It is needless to mention that, besides their independent interest, maximum principles play a fundamental role in existence, uniqueness, comparison and regularity issues.

The monograph [21] furnishes the basic knowledge about the popular operator $(-\Delta)^s$. The related available literature is quite large; we limit ourselves to cite the recent surveys [8, 15] and references therein. The fractional Laplacian $(-\Delta)^s$ appears, in particular, as the generator of the symmetric $(2s)$-stable process in $\mathbb{R}^n$ (the classical Brownian motion is recovered in the limit case $s = 1$), and can be used to describe the motion of a particle jumping from any point $x \in \mathbb{R}^n$ to any $y \in \mathbb{R}^n$ with a probability density proportional to $\frac{1}{|x-y|^{n+2s}}$.

In the so-called *censored processes* the particle obeys the same power law decay but can only jump between points that belong to a given (bounded, smooth) domain Ω; in a quite naive interpretation, we could say that the region Ω has a "reflecting boundary". To study these processes one is lead to restrict the kernel $\frac{1}{|x-y|^{n+2s}}$ to Ω, so that the resulting fractional operator is $(-\Delta_\Omega^N)^s_{\mathrm{R}}$. A vaste literature about the operator $(-\Delta_\Omega^N)^s_{\mathrm{R}}$ is available as well; its relevance with regard to censored processes has been pointed out in [3, 10, 12, 13]. We cite also [22, 23, 24] where Neumann, Robin and mixed boundary value problems for $(-\Delta_\Omega^N)^s_{\mathrm{R}}$ on not necessarily regular domains Ω are studied.

Different homogenous (for instance) Neumann boundary conditions in a fractional setting may be introduced by surrounding the domain Ω by a fat "collar" $U \supset \Omega$ (here we adopt the terminology from [10]). Roughly speaking, one could imagine a random movement of a particle that can only jump between points $x, y \in U$; moreover, the region $U \setminus \overline{\Omega}$ has a "reflecting effect", so that points $x \in U, y \in \Omega$ interact with probability density proportional to $\frac{1}{|x-y|^{n+2s}}$, while jumps between points $x, y \in U \setminus \overline{\Omega}$ are not allowed. The resulting fractional Laplacian clearly depends on the collar U.

To obtain the Semirestricted Neumann Laplacian $(-\Delta_\Omega^N)^s_{\mathrm{Sr}}$ one takes $U = \mathbb{R}^n$. The fractional Laplacian $(-\Delta_\Omega^N)^s_{\mathrm{Sr}}$ has been proposed in [9] to set up an alternative approach to Neumann problems; it also can be used to study non-homogeneous Dirichlet problems for $(-\Delta)^s$, see for instance the survey [17].

In this paper we propose a unifying approach to handle all the above-mentioned fractional Laplacians. More precisely, for any domain $\Omega \subseteq \mathbb{R}^n$ we consider collars that are given by the union of two open sets $U_1, U_2 \subseteq \mathbb{R}^n$ such that $\Omega \subseteq U_1 \cap U_2$. We allow only jumps from $x \in U_1$ to $y \in U_2$, and vice versa. Thus the symmetric difference $(U_1 \setminus U_2) \cup (U_2 \setminus U_1)$ has a "reflecting property" in the sense that, for instance, a point $x \in U_1 \setminus U_2$ can only interact with points $y \in U_2$. Next we put

$$Z = \big(U_1 \times U_2\big) \cup \big(U_2 \times U_1\big) \subseteq \mathbb{R}^n \times \mathbb{R}^n ,$$

so that $\Omega \times \Omega \subseteq Z$. For $s \in (0,1)$ we introduce the space

$$X^s(\Omega; Z) = \Big\{ u : U_1 \cup U_2 \to \mathbb{R} \text{ measurable} \ \Big| \ \frac{u(x)}{1 + |x|^{n+2s}} \in L^1(U_1 \cup U_2), \ u \in H^s_{\mathrm{loc}}(\Omega) \ \Big\}.$$

Notice that, in particular, $X^s(\Omega; Z)$ contains functions $u \in L^1_{\mathrm{loc}}(U_1 \cup U_2)$ such that

$$\mathcal{E}_s(u; Z) := \frac{C_{n,s}}{2} \iint\limits_{Z} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}}\, dxdy \tag{1.2}$$

is finite. For $u \in X^s(\Omega; Z)$ we introduce the distribution $\mathfrak{L}^s_Z u \in \mathcal{D}'(\Omega)$ defined via

$$\langle \mathfrak{L}^s_Z u, \varphi \rangle = \frac{C_{n,s}}{2} \iint\limits_{Z} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x-y|^{n+2s}}\, dxdy \ , \quad \varphi \in \mathcal{C}^\infty_0(\Omega) \, ,$$

see Lemma 2.1.

We understand the inequality $\mathfrak{L}^s_Z u \geq 0$ in Ω in distributional sense, that is,

$$\langle \mathfrak{L}^s_Z u, \varphi \rangle \geq 0, \qquad \text{if} \quad \varphi \in \mathcal{C}^\infty_0(\Omega), \ \varphi \geq 0.$$

Clearly, $(-\Delta)^s$ is recovered by choosing $U_1 = U_2 = \mathbb{R}^n$, the Restricted Laplacian is obtained by taking $U_1 = U_2 = \Omega$, while the intermediate case $(-\Delta^N_\Omega)^s_{\mathrm{Sr}}$ is given by core $U_1 = \mathbb{R}^n, U_2 = \Omega$.

In our main result, see Theorem 4.1, we provide a strong maximum principle for solutions to $\mathfrak{L}^s_Z u \geq 0$ in Ω, with no assumptions on Ω. To prove Theorem 4.1 we follow the outlines of the arguments in [14, Theorems 2.4 and 2.5], that cover the case $\mathfrak{L}^s_Z = (-\Delta)^s$, Ω bounded and smooth, $n \geq 2$, $u \in H^s(\mathbb{R}^n)$ and $u \geq 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. We cite also [6, Theorem 1.2] for a related result involving the fractional Dirichlet p-Laplacian.

Our approach can be easily generalized for a wider class of kernels $\frac{A(x,y)}{|x-y|^{n+2s}}$ with A measurable, symmetric, bounded and bounded away from zero.

The paper is organized as follows. In Section 2 we prove some auxiliary statements. Section 3 is devoted to Caccioppoli-type estimates and to De Giorgi-type maximum estimates for (sub)solutions. In Section 4 we state and prove Theorem 4.1, and formulate corresponding results for the Dirichlet, Restricted and Semirestricted Neumann Laplacians.

In the Appendix we collect some more strong maximum principles for nonlocal Laplacians. First, we formulate a strong maximum principle for $(-\Delta)^s$ that is essentially contained in the remarkable paper [19] by Silvestre, who extended the classical theory of superhamonic functions to the case of fractional Laplacian. Then we discuss strong maximum principles for spectral fractional Laplacians. The *Spectral Dirichlet Laplacian* $(-\Delta_\Omega)^s_{\mathrm{Sp}}$ (also called *the Navier Laplacian*) is widely studied. Notice that for $\Omega = \mathbb{R}^n$ we have $(-\Delta_\Omega)^s_{\mathrm{Sp}} = (-\Delta)^s$, for other Ω these operators differ, see [16] for some integral and pointwise inequalities between them. The *Spectral Neumann Laplacian* $(-\Delta^N_\Omega)^s_{\mathrm{Sp}}$ is less investigated; we limit ourselves to cite [1, 4, 11] and references therein.

Notation. Here we recall some basic notions taken from [21]. For $Z \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and u measurable, let $\mathcal{E}_s(u; Z)$ be the quadratic form in (1.2). We put

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) \mid \mathcal{E}_s(u; \mathbb{R}^n \times \mathbb{R}^n) < \infty \right\},$$

that is an Hilbert space with respect to the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \mathcal{E}_s(u; \mathbb{R}^n \times \mathbb{R}^n) + \|u\|^2_{L^2(\mathbb{R}^n)}.$$

For any domain $G \subset \mathbb{R}^n$, we introduce the following closed subspace of $H^s(\mathbb{R}^n)$:

$$\widetilde{H}^s(G) = \left\{ u \in H^s(\mathbb{R}^n) \mid u = 0 \ \text{ on } \mathbb{R}^n \setminus \overline{G} \ \right\},$$

and its dual space $\widetilde{H}^s(G)'$.

We write $u \in H^s_{\mathrm{loc}}(\Omega)$ if for any $G \Subset \Omega$, the function u is the restriction to G of some $v \in H^s(\mathbb{R}^n)$, and we put

$$\|u\|_{H^s(G)} := \inf \left\{ \|v\|_{H^s(\mathbb{R}^n)} \ \mid \ v = u \ \text{ on } G \ \right\}.$$

It is well known that $u \in H^s_{\mathrm{loc}}(\Omega)$ if and only if $\eta u \in \widetilde{H}^s(\Omega)$ for any $\eta \in \mathcal{C}^\infty_0(\Omega)$, see for instance [21, Subsection 4.4.2].

We adopt the following standard notation:

$B_r(x)$ is the Euclidean ball of radius r centered at x, and $B_r = B_r(0)$;

$u^\pm = \max\{\pm u, 0\}$; $\sup\limits_G u$ and $\inf\limits_G u$ stand for essential supremum/infimum of the measurable function u on the measurable set G;

Through the paper, all constants depending only on n and s are denoted by c. To indicate that a constant depends on other quantities we list them in parentheses: $c(\cdots)$.

2 Preliminaries

For any function φ on $\mathbb{R}^n$ we put

$$\Psi_\varphi(x,y) = \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{n+2s}}. \tag{2.1}$$

LEMMA 2.1 *Let $u \in X^s(\Omega;Z)$. Then $\mathfrak{L}^s_Z u$ is a well defined distribution in Ω. Moreover, for any Lipschitz domain $G \Subset \Omega$, we have $\mathfrak{L}^s_Z u \in \widetilde{H}^s(G)'$ and*

$$\iint\limits_{G\times G} |u(x)u(y)|\Psi_\varphi(x,y)\,dxdy < \infty \qquad \textit{for any } \varphi \in \mathcal{C}^\infty_0(G).$$

Proof. Let $\varphi \in \mathcal{C}^\infty_0(\Omega)$. In order to have that $\mathfrak{L}^s_Z u$ is well defined we need to show that

$$g(x,y) := \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}} \in L^1(Z).$$

Take two Lipschitz domains $G, \widetilde{G}$, such that $\mathrm{supp}(\varphi) \subset G \Subset \widetilde{G} \Subset \Omega$. From $u \in H^s(\widetilde{G})$, we have

$$\iint\limits_{\widetilde{G}\times\widetilde{G}} |g(x,y)|\,dxdy \le \|u\|_{H^s(\widetilde{G})}\|\varphi\|_{H^s(\mathbb{R}^n)}.$$

Next, since φ vanishes outside G, and since

$$\big[Z \setminus (\widetilde{G}\times\widetilde{G})\big] \setminus (G^{\mathsf{c}} \times G^{\mathsf{c}}) = \Big[G \times \big((U_1\cup U_2)\setminus \widetilde{G}\big)\Big] \ \cup\ \Big[\big((U_1\cup U_2)\setminus\widetilde{G}\big)\times G\Big], \tag{2.2}$$

it is enough to prove that $g \in L^1(G\times\big((U_1\cup U_2)\setminus\widetilde{G}\big))$. We have

$$\iint\limits_{G\times((U_1\cup U_2)\setminus\widetilde{G})} |g(x,y)|\,dxdy \le \int\limits_G |\varphi(x)|\Big[\int\limits_{(U_1\cup U_2)\setminus\widetilde{G}} \frac{|u(x)|+|u(y)|}{|x-y|^{n+2s}}\,dy\Big]dx$$

$$\le c(\mathrm{dist}(G,\partial\widetilde{G}))\Big(\|\varphi\|_{L^2(G)}\|u\|_{L^2(G)} + \|\varphi\|_{L^1(G)}\int\limits_{U_1\cup U_2}\frac{|u(y)|}{1+|y|^{n+2s}}\,dy\Big) \le c(G,\widetilde{G},u)\|\varphi\|_{\widetilde{H}^s(G)}.$$

We proved that $\mathfrak{L}^s_Z u \in \mathcal{D}'(\Omega)$ and actually $\mathfrak{L}^s_Z u \in \widetilde{H}^s(G)'$, by the density of $\mathcal{C}^\infty_0(G)$ in $\widetilde{H}^s(G)$.

Further, take again $\varphi \in \mathcal{C}^\infty_0(G)$ and notice that $\Psi_\varphi(x,\ \cdot\ ) \in L^1(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$, because

$$\Psi_\varphi(x,y) \le c(\varphi)\Big(\frac{\chi_{\{|x-y|<1\}}}{|x-y|^{n-2(1-s)}} + \frac{\chi_{\{|x-y|>1\}}}{|x-y|^{n+2s}}\Big).$$

Actually $\int\limits_{\mathbb{R}^n} \Psi_\varphi(x,y)\,dy \le c(\varphi)$, and by the Cauchy-Bunyakovsky-Schwarz inequality we infer

$$\iint\limits_{G\times G} |u(x)u(y)|\Psi_\varphi\,dxdy \le \iint\limits_{G\times G} |u(x)|^2\Psi_\varphi\,dxdy \le c(\varphi)\int\limits_G |u(x)|^2\,dx \le c(u,\varphi,G) < \infty\,.$$

The lemma is proved. □

Next, for any domain $G \subseteq U_1 \cap U_2$ we introduce the **relative killing measure** $M_G^Z \in L^\infty_{\rm loc}(G)$,

$$M_G^Z(x) = C_{n,s} \int\limits_{(U_1\cup U_2)\setminus G} \frac{dy}{|x-y|^{n+2s}}\,, \quad x \in G\,.$$

When $U_1 \cup U_2 = \mathbb{R}^n$, that happens for instance in the Dirichlet and in the Semirestricted cases, see Section 4, the weight M_G^Z coincides with so-called *killing measure* of the set G:

$$M_G(x) := M_G^{\mathbb{R}^n\times\mathbb{R}^n}(x) = C_{n,s}\int\limits_{\mathbb{R}^n\setminus G} \frac{dy}{|x-y|^{n+2s}}\,.$$

In the Restricted case we have $U_1 \cup U_2 = \Omega$ and $M_G^Z = M_G^{\Omega\times\Omega}$; if $G \subset \Omega$ then $M_G^{\Omega\times\Omega}$ is the difference between the killing measures of the sets G and Ω.

LEMMA 2.2 *Let $G \subseteq U_1 \cap U_2$ be a Lipschitz domain. If $u \in \widetilde{H}^s(G)$, then*

$$\mathcal{E}_s(u;Z) = \mathcal{E}_s(u;G\times G) + \int\limits_G M_G^Z(x)|u(x)|^2\,dx$$

and in particular u is square integrable on G with respect to the measure $M_G^Z(x)dx$.

Proof. Trivially $\mathcal{E}_s(u;Z) < \infty$, as $u \in \widetilde{H}^s(G) \hookrightarrow H^s(\mathbb{R}^n)$. Since u vanishes on $\overline{G}^{\mathsf{c}}$, using (2.2) with $\widetilde{G} = G$ we have

$$\begin{aligned}\mathcal{E}_s(u;Z) &= \mathcal{E}_s(u;G\times G) + 2\mathcal{E}_s(u;G\times\big[(U_1\cup U_2)\setminus G\big])\\ &= \mathcal{E}_s(u;G\times G) + C_{n,s}\int\limits_G |u(x)|^2\Big(\int\limits_{(U_1\cup U_2)\setminus G}\frac{dy}{|x-y|^{n+2s}}\Big)dx,\end{aligned}$$

and the lemma is proved. □

The next two elementary lemmata deal with certain quantities, depending on functions $u \in X^s(\Omega; Z)$, that will be involved in the crucial Caccioppoli-type inequality in the next section.

For $u \in X^s(\Omega; Z)$ and for any domain $G \subseteq \Omega$ we use Lemma 2.1 to introduce the distribution

$$\langle (-\Delta^N_{(U_1 \cup U_2)\setminus G})^s_{\mathrm{R}} u, \varphi \rangle := \frac{C_{n,s}}{2} \iint\limits_{G \times [(U_1 \cup U_2)\setminus G]} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}\, dxdy, \quad \varphi \in \mathcal{C}^\infty_0(G),$$

that is the restriction on G of the Regional Laplacian of u relative to the set $(U_1 \cup U_2) \setminus G$.

Lemma 2.3 *Let $G \subseteq \Omega$ be a domain, $u \in X^s(\Omega; Z)$. Then for any $\varphi \in \mathcal{C}^\infty_0(G)$*

$$\int\limits_G |u(x)||\varphi(x)|^2 \Big(\int\limits_{(U_1 \cup U_2)\setminus G} \frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\, dy \Big) dx < \infty. \tag{2.3}$$

In particular, $u \cdot (-\Delta^N_{(U_1 \cup U_2)\setminus G})^s_{\mathrm{R}} u \in L^1_{\mathrm{loc}}(G)$.

Proof. Similarly as in the proof of Lemma 2.1, we estimate the integral in (2.3) by

$$\int\limits_G |u(x)||\varphi(x)|^2 \Big[\int\limits_{(U_1 \cup U_2)\setminus G} \frac{|u(x)|+|u(y)|}{|x-y|^{n+2s}}\, dy \Big] dx$$

$$\le c(\mathrm{dist}(\mathrm{supp}(\varphi), \partial G)) \|\varphi\|^2_{L^\infty(G)} \Big(\|u\|^2_{L^2(\mathrm{supp}(\varphi))} + \|u\|_{L^1(\mathrm{supp}(\varphi))} \int\limits_{U_1 \cup U_2} \frac{|u(y)|}{1+|y|^{n+2s}}\, dy \Big) < \infty\,,$$

and the lemma follows. □

Lemma 2.4 *If $u \in X^s(\Omega; Z)$, then $u^\pm \in X^s(\Omega; Z)$; moreover for any $G \Subset \Omega$ we have*

$$\mathcal{E}(u^\pm; G \times G) < \mathcal{E}(u; G \times G)\ , \quad \mathcal{E}(|u|; G \times G) < \mathcal{E}(u; G \times G)$$

unless u has constant sign on G.

Proof. We compute

$$(u(x)-u(y))^2 - (u^+(x)-u^+(y))^2 = (u^-(x)-u^-(y))^2 + 2\Big(u^+(x)u^-(y) + u^-(x)u^+(y)\Big) \ge 0.$$

Thus $\mathcal{E}(u^+; G \times G) \le \mathcal{E}(u; G \times G) < \infty$ for any $G \Subset \Omega$. Therefore $u^+ \in H^s_{\mathrm{loc}}(\Omega)$, and $u^+ \in X^s(\Omega; Z)$ follows.

Next, assume that $\mathcal{E}(u; G \times G) = \mathcal{E}(u^+; G \times G)$ on some domain $G \Subset \Omega$. Then

$$(u^-(x)-u^-(y))^2 + 2\Big(u^+(x)u^-(y) + u^-(x)u^+(y)\Big) = 0$$

for a.e. $(x, y) \in G \times G$. We infer that u^- is constant a.e. on G. If $u^- = 0$ then $u \ge 0$ on G; if $u^- \ne 0$ we get $u^+ = 0$, that is, $u \le 0$ on G. The proofs for u^- and $|u|$ follow in a similar way. □

Remark 2.5 *If $u \in L^1_{\rm loc}(U_1 \cup U_2)$ and $\mathcal{E}_s(u;Z)$ is finite, then $\mathcal{E}_s(u^\pm;Z) < \mathcal{E}_s(u;Z)$ and $\mathcal{E}_s(|u|;Z) < \mathcal{E}_s(u;Z)$, unless u has constant sign on $U_1 \cup U_2$. The proof runs with no changes.*

Our proof of Theorem 4.1 requires the construction of a suitable barrier function. The next lemma slightly generalizes a result by Ros-Oton and Serra [18].

Lemma 2.6 *Let $B_R(x^0) \subset \Omega$. For any $r \in (0,R)$ there exists a constant $c = c(R/r) > 0$ and a function $\Phi \in H^s(\mathbb{R}^n)$ satisfying*

$$\begin{gathered} \mathfrak{L}^s_Z \Phi \le 0 \qquad \text{in } B_R(x^0) \setminus \overline{B}_r(x^0)\ ; \\ \Phi \equiv 1 \ \text{ in } B_r(x^0)\ , \quad \Phi \equiv 0 \ \text{ in } \mathbb{R}^n \setminus B_R(x^0)\ , \quad \Phi(x) \ge c(R-|x|)^s \ \text{ in } B_R(x^0)\ . \end{gathered} \tag{2.4}$$

Proof. Without loss of generality we can assume $x^0 = 0$. Lemma 3.2 in [18], see also [14, Lemma 2.2], provides the existence of $\Phi \in \widetilde{H}^s(B_R)$ satisfying (2.4) and $(-\Delta)^s\Phi \le 0$ in $B_R \setminus \overline{B}_r$. To conclude we claim that the distribution $(-\Delta)^s\Phi - \mathfrak{L}^s_Z\Phi$ is nonnegative in Ω. Indeed, take a nonnegative function $\eta \in \mathcal{C}^\infty_0(\Omega)$. Since both Φ and η vanish on $\mathbb{R}^n \setminus \Omega$, we have

$$\begin{aligned} \langle (-\Delta)^s\Phi - \mathfrak{L}^s_Z\Phi, \eta\rangle &= \frac{C_{n,s}}{2} \iint\limits_{\mathbb{R}^{2n}\setminus Z} \frac{(\Phi(x)-\Phi(y))(\eta(x)-\eta(y))}{|x-y|^{n+2s}}\, dxdy \\ &= C_{n,s} \int\limits_{\Omega} \Phi(x)\eta(x) \Big(\int\limits_{\mathbb{R}^n\setminus(U_1\cup U_2)} \frac{dy}{|x-y|^{n+2s}}\, dy \Big) dx, \end{aligned}$$

and the claim follows. In particular, $\mathfrak{L}^s_Z\Phi \le (-\Delta)^s\Phi \le 0$ in $B_R \setminus \overline{B}_r$, and we are done. □

Remark 2.7 *It is worth to note that if $Z \subset Z'$ then for any nonnegative $\Phi \in \widetilde{H}^s(\Omega)$ the inequality $\mathfrak{L}^s_Z\Phi \le \mathfrak{L}^s_{Z'}\Phi$ holds in Ω. The proof runs without changes.*

We conclude this preliminary section by the following remark. We fix an exponent $\bar p > 2$; precisely we choose $\bar p = 4$ (for instance) if $n = 1 \le 2s$, and $\bar p = 2^*_s = \frac{2n}{n-2s}$ if $n > 2s$. Take any radius $r \in \big(1,2\big]$. The Sobolev embedding theorem implies $\mathcal{E}_s(u;\mathbb{R}^n\times\mathbb{R}^n) \ge c \Big(\int\limits_{B_r} |u|^{\bar p}\, dx\Big)^{2/\bar p}$ for any $u \in \widetilde{H}^s(B_r)$.

Now let $\rho \in \big(1,r\big)$. Since for $u \in \widetilde{H}^s(B_\rho)$ one has

$$\mathcal{E}_s(u;\mathbb{R}^n\times\mathbb{R}^n) = \mathcal{E}_s(u;B_r\times B_r) + C_{n,s} \int\limits_{B_r} |u(x)|^2 \Big(\int\limits_{\mathbb{R}^n\setminus B_r} \frac{dy}{|x-y|^{n+2s}} \Big) dx,$$

we plainly infer that

$$\mathcal{E}_s(u;B_r\times B_r)+\frac{1}{(r-\rho)^{2s}}\int\limits_{B_r}|u|^2\,dx\ge c\Big(\int\limits_{B_\rho}|u|^{\bar p}\,dx\Big)^{\frac{2}{\bar p}}\quad\text{for any } u\in\widetilde H^s(B_\rho). \tag{2.5}$$

3 Pointwise estimates for $\mathfrak{L}^s_Z$-subharmonic functions

First, we prove a Caccioppoli-type inequality. We use again the notation introduced in (2.1).

Lemma 3.1 *Let $G\subseteq\Omega$ be a Lipschitz domain, $w\in X^s(\Omega;Z)$ and $\varphi\in\mathcal{C}^\infty_0(G)$. Then*

$$\begin{aligned}\mathcal{E}_s(\varphi w^+;G\times G)\ &\le\ \langle\mathfrak{L}^s_Z w,\varphi^2w^+\rangle \\ &+\ \frac{C_{n,s}}{2}\iint\limits_{G\times G}w^+(x)w^+(y)\Psi_\varphi(x,y)\,dxdy\ -\int\limits_G w^+\varphi^2(-\Delta^N_{(U_1\cup U_2)\setminus G})^s_{\mathrm{R}}w^+dx\,.\end{aligned}\tag{3.1}$$

Proof. Note that all quantities in (3.1) are finite by Lemmata 2.1, 2.3 and 2.4. We compute

$$\begin{aligned}&(w(x)-w(y))((\varphi^2w^+)(x)-(\varphi^2w^+)(y))-((\varphi w^+)(x)-(\varphi w^+)(y))^2\\ &\qquad=-w^+(x)w^+(y)(\varphi(x)-\varphi(y))^2+\big(\varphi(y)^2w^-(x)w^+(y)+\varphi(x)^2w^+(x)w^-(y)\big)\\ &\qquad\ge-w^+(x)w^+(y)(\varphi(x)-\varphi(y))^2\end{aligned}$$

to infer

$$\begin{aligned}\langle\mathfrak{L}^s_Zw,\varphi^2w^+\rangle\ &=\ \frac{C_{n,s}}{2}\iint\limits_Z\frac{(w(x)-w(y))((\varphi^2w^+)(x)-(\varphi^2w^+)(y))^2}{|x-y|^{n+2s}}dxdy\\ &\ge\ \mathcal{E}_s(\varphi w^+;Z)-\frac{C_{n,s}}{2}\iint\limits_Z w^+(x)w^+(y)\Psi_\varphi dxdy\\ &=\ \mathcal{E}_s(\varphi w^+,G\times G)+\int\limits_G M^Z_G(x)|w^+\varphi|^2\,dx-\frac{C_{n,s}}{2}\iint\limits_Z w^+(x)w^+(y)\Psi_\varphi dxdy\end{aligned}$$

by Lemma 2.2. We compute

$$\begin{aligned}\int\limits_G M^Z_G(x)|w^+\varphi|^2\,dx&=C_{n,s}\int\limits_G w^+(x)|\varphi(x)|^2\Big(\int\limits_{(U_1\cup U_2)\setminus G}\frac{(w^+(x)-w^+(y)+w^+(y))}{|x-y|^{n+2s}}\,dy\Big)dx\\ &=\int\limits_G w^+|\varphi|^2(-\Delta^N_{(U_1\cup U_2)\setminus G})^s_{\mathrm{R}}w^+dx+C_{n,s}\int\limits_G w^+(x)|\varphi(x)|^2\Big(\int\limits_{(U_1\cup U_2)\setminus G}\frac{w^+(y)}{|x-y|^{n+2s}}\,dy\Big)dx\,.\end{aligned}$$

Since $\Psi_\varphi \equiv 0$ on $G^{\mathsf{c}} \times G^{\mathsf{c}}$ we have, by (2.2) with $\widetilde{G} = G$,

$$
\begin{aligned}
\frac{C_{n,s}}{2} \iint\limits_{Z} & w^+(x) w^+(y) \Psi_\varphi \, dxdy \\
&= \frac{C_{n,s}}{2} \iint\limits_{G\times G} w^+(x) w^+(y) \Psi_\varphi \, dxdy + C_{n,s} \int\limits_{G} w^+(x) \Big(\int\limits_{(U_1\cup U_2)\setminus G} w^+(y) \Psi_\varphi \, dy \Big) dx \\
&= \frac{C_{n,s}}{2} \iint\limits_{G\times G} w^+(x) w^+(y) \Psi_\varphi \, dxdy + C_{n,s} \int\limits_{G} w^+(x) \varphi(x)^2 \Big(\int\limits_{(U_1\cup U_2)\setminus G} \frac{w^+(y)}{|x-y|^{n+2s}} \, dy \Big) dx \, ,
\end{aligned}
$$

and the lemma follows. $\square$

Remark 3.2 *Inequality (3.1) was essentially proved in [7, Theorem 1.4], in a weaker form but in a non-Hilbertian setting and for more general kernels.*

The next De Giorgi-type result is obtained by suitably modifying the argument for [7, Theorem 1.1].

Lemma 3.3 *For any $u \in X^s(\Omega; Z)$ such that $\mathfrak{L}^s_Z u \le 0$ in Ω and for every ball $B_{2r}(x^0) \subseteq \Omega$, one has*

$$
\sup_{B_r(x^0)} u \le \Big(\frac{\widehat{c}}{r^n} \int\limits_{B_{2r}(x^0)} |u^+(x)|^2 \, dx \Big)^{\frac{1}{2}} + r^{2s} \int\limits_{(U_1\cup U_2)\setminus B_r(x^0)} \frac{|u^+(x)|}{|x-x^0|^{n+2s}} \, dx \, , \tag{3.2}
$$

where $\widehat{c} > 0$ depends only on n and s. In particular, u is locally bounded from above in Ω.

Proof. First of all let us recall that for $U_1 \cup U_2 = \mathbb{R}^n$ the last term in (3.2) is called *nonlocal tail*. For $Z \neq \mathbb{R}^n \times \mathbb{R}^n$ we call this term **relative nonlocal tail** and denote it by $\mathrm{Tail}_Z(u^+; x^0, r)$.

By rescaling we can assume without loss of generality that $r = 1$ and $x^0 = 0$. We introduce a parameter $\tilde{k} > 0$ satisfying

$$
\tilde{k} \ge \mathrm{Tail}_Z(u^+; 0, 1) \tag{3.3}
$$

(its value will be chosen later). For any integer $j \ge 0$ we put

$$
\begin{aligned}
& r_j = 1 + 2^{-j} \, , && k_j = \tilde{k}(1 - 2^{-j}) \, , && B_j = B_{r_j} \, ; \\
& \tilde{r}_j = \frac{r_j + r_{j+1}}{2} \, , && \tilde{k}_j = \frac{k_j + k_{j+1}}{2}, && \widetilde{B}_j = B_{\tilde{r}_j} \, ; \\
& w_j = (u - k_j)^+ \, , && \widetilde{w}_j = (u - \tilde{k}_j)^+ \, , && \alpha_j = \Big(\int\limits_{B_j} |w_j|^2 \, dx \Big)^{\frac{1}{2}}.
\end{aligned}
$$

The following relations are obvious:

$$r_j \searrow 1\ , \quad r_{j+1} < \tilde r_j < r_j\ ; \qquad k_j \nearrow \tilde k\ , \quad k_j < \tilde k_j < k_{j+1};$$

$$\widetilde w_j \le w_j\ , \qquad\qquad \widetilde w_j \le \frac{w_j^2}{\tilde k_j - k_j} = \frac{2^{j+2}}{\tilde k}\, w_j^2\ ; \tag{3.4}$$

$$\alpha_0^2 = \int\limits_{B_2} |u^+|^2\,dx\ , \qquad\qquad \alpha_j^2 \to \int\limits_{B_1} |(u-\tilde k)^+|^2\,dx \quad \text{as} \quad j\to\infty. \tag{3.5}$$

In addition, we have

$$w_{j+1}^2\Big(\frac{\tilde k}{2^{j+2}}\Big)^{\bar p-2} = w_{j+1}^2(k_{j+1}-\tilde k_j)^{\bar p-2} \le \widetilde w_j^{\bar p}\ , \tag{3.6}$$

where the exponent $\bar p > 2$ was introduced at the end of Section 2.

Next, for any integer $j\ge 0$ we fix a cut-off function φ_j satisfying

$$\varphi_j \in \mathcal{C}^\infty_0(\widetilde B_j)\ , \quad 0\le\varphi_j\le 1\ , \quad \varphi\equiv 1 \ \text{ on } B_{j+1}, \quad \|\nabla\varphi\|_\infty \le 2^{j+3}.$$

Since $\widetilde w_j\varphi_j \in \widetilde H^s(B_j)$ and $1<\tilde r_j<r_j<2$, by (2.5) with $\rho=\tilde r_j$ and $r=r_j$, we have

$$c\Big(\int\limits_{B_j} |\widetilde w_j\varphi_j|^{\bar p}\Big)^{\frac{2}{\bar p}} \le \mathcal{E}_s(\widetilde w_j\varphi_j; B_j\times B_j) + 2^{2s(j+2)}\int\limits_{B_j}|\widetilde w_j\varphi_j|^2\,dx. \tag{3.7}$$

Notice that $\langle \mathfrak{L}^s_Z(u-\tilde k_j), \widetilde w_j\varphi_j^2\rangle \le 0$, since $\mathfrak{L}^s_Z(u-\tilde k_j) = \mathfrak{L}^s_Z u \le 0$ in B_j and $\widetilde w_j\varphi_j^2 \in \widetilde H^s(B_j)$ is nonnegative. Using Lemma 3.1 with $w = u-\tilde k_j$, we infer

$$\mathcal{E}_s(\widetilde w_j\varphi_j; B_j\times B_j) \le c \iint\limits_{B_j\times B_j} \widetilde w_j(x)\widetilde w_j(y)\Psi_{\varphi_j}(x,y)\,dxdy - \int\limits_{B_j}\widetilde w_j\varphi_j^2(-\Delta^N_{(U_1\cup U_2)\setminus B_j})^s_{\mathrm{R}}\widetilde w_j\,dx,$$

so that

$$\Big(\int\limits_{B_j}|\widetilde w_j\varphi_j|^{\bar p}\,dx\Big)^{\frac{2}{\bar p}} \le c\Big(J_1 - J_2 + \tilde k^2\, 2^{2sj}\,\Big(\frac{\alpha_j}{\tilde k}\Big)^2\Big) \tag{3.8}$$

by (3.7), where

$$J_1 = \iint\limits_{B_j\times B_j}\widetilde w_j(x)\widetilde w_j(y)\Psi_{\varphi_j}(x,y)\,dxdy\ , \quad J_2 = \int\limits_{B_j}\widetilde w_j\varphi_j^2(-\Delta^N_{(U_1\cup U_2)\setminus B_j})^s_{\mathrm{R}}\widetilde w_j)\,dx.$$

We estimate from below the left-hand side of (3.8) via (3.6):

$$\int\limits_{B_j}|\widetilde w_j\varphi_j|^{\bar p}\,dx \ge \int\limits_{B_{j+1}}|\widetilde w_j|^{\bar p}\,dx \ge c\,\Big(\frac{\tilde k}{2^j}\Big)^{\bar p-2}\int\limits_{B_{j+1}}|w_{j+1}|^2\,dx = c\,\tilde k^{\bar p}\,2^{j(2-\bar p)}\,\Big(\frac{\alpha_{j+1}}{\tilde k}\Big)^2. \tag{3.9}$$

We estimate J_1 by using

$$\Psi_{\varphi_j}(x,y) \le \|\nabla\varphi_j\|_\infty^2\, |x-y|^{-(n+2s-2)} \le c2^{2j}\, |x-y|^{-(n+2s-2)}$$

and the Cauchy-Bunyakovsky-Schwarz inequality, to obtain

$$\begin{aligned} J_1 &\le c\,2^{2j} \iint\limits_{B_j\times B_j} \frac{\widetilde{w}_j(x)}{|x-y|^{\frac{n+2s-2}{2}}}\, \frac{\widetilde{w}_j(y)}{|x-y|^{\frac{n+2s-2}{2}}} dxdy \le c\,2^{2j} \iint\limits_{B_j\times B_j} \frac{|\widetilde{w}_j(x)|^2}{|x-y|^{n+2s-2}} dxdy \\ &= c\,2^{2j} \int\limits_{B_j} |\widetilde{w}_j(x)|^2 \Big(\int\limits_{B_j} \frac{dy}{|x-y|^{n+2s-2}} \Big) dx \le c r_j^{2-2s}\, 2^{2j} \alpha_j^2 \le c\, \tilde{k}^2\, 2^{2j} \Big(\frac{\alpha_j}{\tilde{k}}\Big)^2. \end{aligned} \tag{3.10}$$

We handle J_2 as follows. For $x \in \operatorname{supp}(\varphi_j) \subset \widetilde{B}_j$ and $y \in \Omega \setminus B_j$ we have

$$\frac{|y|}{|x-y|} \le 1 + \frac{|x|}{|x-y|} \le 1 + \frac{r_j}{r_j - \tilde{r}_j} \le c\,2^j.$$

Hence, using also (3.4) we can estimate

$$\widetilde{w}_j(x)|\varphi_j(x)|^2\, \frac{\widetilde{w}_j(y) - \widetilde{w}_j(x)}{|x-y|^{n+2s}} \le \frac{c}{\tilde{k}}\, |w_j(x)|^2\, 2^{j(n+2s+1)}\, \frac{w_j(y)}{|y|^{n+2s}}\,,$$

so that

$$\begin{aligned} -J_2 &= \int\limits_{B_j} \widetilde{w}_j(x)|\varphi_j(x)|^2 \Big(\int\limits_{(U_1\cup U_2)\setminus B_j} \frac{\widetilde{w}_j(y) - \widetilde{w}_j(x)}{|x-y|^{n+2s}} dy \Big)\, dx \\ &\le \frac{c}{\tilde{k}}\, 2^{j(n+2s+1)} \Big(\int\limits_{(U_1\cup U_2)\setminus B_j} \frac{w_j(y)}{|y|^{n+2s}} dy \Big) \int\limits_{B_j} |w_j|^2\, dx \\ &\le c\tilde{k}\, 2^{j(n+2s+1)}\ \mathrm{Tail}_Z(u^+;0,1) \Big(\frac{\alpha_j}{\tilde{k}}\Big)^2 \end{aligned}$$

because $B_j \supset B_1$ and $w_j \le u^+$. Comparing with (3.8), (3.9) and (3.10) we arrive at

$$2^{\frac{2(2-\bar{p})}{\bar{p}} j} \Big(\frac{\alpha_{j+1}}{\tilde{k}}\Big)^{\frac{4}{\bar{p}}} \le c\,2^{j(n+2s+1)} \Big(1 + \tilde{k}^{-1}\, \mathrm{Tail}_Z(u^+;0,1)\Big) \Big(\frac{\alpha_j}{\tilde{k}}\Big)^2.$$

Taking (3.3) into account, we can conclude that

$$\frac{\alpha_{j+1}}{\tilde{k}} \le (\widehat{c}^{\frac{\beta}{2}} \eta^{-\frac{1}{\beta}}) \eta^j \Big(\frac{\alpha_j}{\tilde{k}}\Big)^{\beta+1}, \tag{3.11}$$

where $\beta = \dfrac{\bar p}{2} - 1 > 0$, $\eta = 2^{\frac{\bar p}{4}(n+2s+1)+\beta} > 1$. Now we choose the free parameter $\tilde k$, namely

$$\tilde k = \mathrm{Tail}_Z(u^+;0,1) + \widehat{c}^{\frac{1}{2}}\alpha_0 = \mathrm{Tail}_Z(u^+;0,1) + \Big(\widehat{c}\int\limits_{B_1} |u^+|^2\,dx\Big)^{\frac{1}{2}},$$

compare with (3.2). The above choice of $\tilde k$ guarantees that

$$\widehat{c}^{\frac{1}{2}}\,\frac{\alpha_j}{\tilde k} \le \eta^{-\frac{j}{\beta}} \tag{3.12}$$

for $j = 0$. Using induction and (3.11) one easily gets that (3.12) holds for any $j \ge 0$. Thus $\alpha_j \to 0$ and hence $(u - \tilde k)^+ \equiv 0$ on B_1 by (3.5). The proof is complete. □

4 Main results

We are in position to state and prove a strong maximum principle for the nonlocal operator $\mathfrak{L}^s_Z$, that is the main result of the present paper.

THEOREM 4.1 *Let u be a nonconstant measurable function on $U_1 \cup U_2$ such that*

$$u \in H^s_{\rm loc}(\Omega)\ , \quad \int\limits_{U_1\cup U_2} \frac{|u(x)|}{1+|x|^{n+2s}}\,dx < \infty\ , \quad \mathfrak{L}^s_Z u \ge 0 \quad in\ \Omega.$$

Then u is lower semicontinuous on Ω, locally bounded from below on Ω and

$$u(x) > \inf_{U_1\cup U_2} u \qquad for\ every\ x \in \Omega.$$

Proof. First, local boundedness from below follows from Lemma 3.3.

To check the first claim it suffices to show that u has a representative that is lower semicontinuous on any fixed domain $G \Subset \Omega$. From

$$\frac{C_{n,s}}{2}\int\limits_G \Big(\int\limits_G \frac{(u(x)-u(y))^2}{|x-y|^{n+2s}}\,dy\Big)dx < \infty$$

we infer that

$$\int\limits_G \frac{(u(x)-u(y))^2}{|y-x|^{n+2s}}\,dy < \infty \tag{4.1}$$

for a.e. $x \in G$. Let G_0 be the set of Lebesgue points $x \in G$ for u that satisfy (4.1). We can assume that $u(x^0) = \liminf_{x \to x^0} u(x)$ for any $x^0 \in G \setminus G_0$, because $G \setminus G_0$ has null Lebesgue measure.

Our next goal is to show that $u(x^0) \le \liminf_{x \to x^0} u(x)$ for any $x^0 \in G_0$. We use Lemma 3.3 with u replaced by $u(x^0) - u$ to get

$$\inf_{B_r(x^0)} u \ge u(x^0) - \mathrm{Tail}_Z((u(x^0) - u)^+; x^0, r) - \Big(\frac{\widehat{c}}{r^n} \int\limits_{B_{2r}(x^0)} |(u(x^0) - u)^+|^2 \, dx \Big)^{\frac{1}{2}} \tag{4.2}$$

for any $r > 0$ small enough. First we split

$$\begin{aligned} \mathrm{Tail}_Z((u(x^0) - u)^+; x^0, r) &= r^{2s} \int\limits_{(U_1 \cup U_2) \setminus B_{2r}(x^0)} \frac{|u(x^0) - u(x)|}{|x - x^0|^{n+2s}} \, dx \\ &\le r^{2s} \int\limits_{(U_1 \cup U_2) \setminus G} \frac{|u(x^0)| + |u(x)|}{|x - x^0|^{n+2s}} \, dx \ + \ r^{2s} \int\limits_{G \setminus B_{2r}(x^0)} \frac{|u(x^0) - u(x)|}{|x - x^0|^{n+2s}} \, dx =: P_r + Q_r. \end{aligned}$$

We readily obtain

$$P_r \le c(\mathrm{dist}(x^0, \partial G)) \, r^{2s} \int\limits_{(U_1 \cup U_2)} \frac{|u(x^0)| + |u(x)|}{1 + |x|^{n+2s}} \, dx \to 0$$

as $r \to 0$. Next we use the Cauchy-Bunyakovsky-Schwarz inequality to estimate

$$Q_r \le r^{2s} \Big(\int\limits_G \frac{(u(x^0) - u(x))^2}{|x - x^0|^{n+2s}} \, dx \Big)^{\frac{1}{2}} \Big(\int\limits_{\mathbb{R}^n \setminus B_{2r}(x^0)} \frac{dx}{|x - x^0|^{n+2s}} \Big)^{\frac{1}{2}} = c r^s \Big(\int\limits_G \frac{(u(x^0) - u(x))^2}{|x - x^0|^{n+2s}} \, dx \Big)^{\frac{1}{2}}.$$

Since (4.1) is satisfied at $x = x^0$, we have that $Q_r \to 0$. Thus $\mathrm{Tail}_Z((u(x^0) - u)^+; x^0, r) \to 0$ as $r \to 0$. Further, the last term in (4.2) goes to zero as $r \to 0$ because x^0 is a Lebesgue point for u. Thus $\liminf_{x \to x^0} u(x) \ge u(x^0)$, and the first statement is proved.

Next, assume by contradiction that u is bounded from below and

$$\Omega_+ := \{ x \in \Omega \mid u(x) > m := \inf_{U_1 \cup U_2} u \ \}$$

is strictly contained in Ω. Since u is lower semicontinuous on Ω, the set Ω_+ is open and has a nonempty boundary in Ω.

Fix a point $\xi \in \Omega \cap \partial \Omega_+$, so that $u(\xi) = m$. Using again the lower-semicontinuity of u, we can find $R > r > 0$ and a point $x^0 \in \Omega_+$, such that $\xi \in B_R(x^0) \Subset \Omega$ and $u(x) \ge \frac{1}{2}(u(x^0) + m) > m$

for every $x \in B_r(x^0)$. We can assume that $x^0 = 0$ to simplify notations. Thus we have the following situation:

$$\xi \in B_R \subset \Omega\ , \quad u(\xi) = m\ , \quad \inf_{B_r} u(x) \ge m + \delta \tag{4.3}$$

for some $\delta > 0$. Let Φ be the function defined in Lemma 2.6. We claim that $u \ge m + \delta\Phi > m$ in $B_R \setminus \overline{B}_r$, that gives a contradiction with (4.3).

Indeed, define $v = u - \delta\Phi$, so that

$$v = u - \delta \ge m \ \text{ in } B_r, \quad v = u \ge m \ \text{ in } (U_1 \cup U_2) \setminus B_R.$$

Our goal is to show that $v \ge m$ also on $B_R \setminus \overline{B}_r$.

Clearly $v \in X^s(\Omega; Z)$ as $u, \Phi \in X^s(\Omega; Z)$. By Lemma 2.4 this implies $v^m_\pm := (v - m)^\pm \in X^s(\Omega; Z)$. Next, notice that $v^m_- = 0$ out of $\overline{B}_R \setminus B_r$. Therefore $v^m_- \in \widetilde{H}^s(B_R \setminus \overline{B}_r)$, and using Lemma 2.1 we obtain

$$\langle \mathfrak{L}^s_Z v, v^m_- \rangle = \langle \mathfrak{L}^s_Z u, v^m_- \rangle - \delta \langle \mathfrak{L}^s_Z \Phi, v^m_- \rangle \ge 0. \tag{4.4}$$

However,

$$\begin{aligned}
\langle \mathfrak{L}^s_Z v, v^m_- \rangle &= \frac{C_{n,s}}{2} \iint\limits_Z \frac{((v(x) - m) - (v(y) - m))(v^m_-(x) - v^m_-(y))}{|x - y|^{n+2s}}\, dxdy \\
&= -\frac{C_{n,s}}{2} \iint\limits_Z \frac{v^m_+(x) v^m_-(y) + v^m_+(y) v^m_-(x)}{|x - y|^{n+2s}}\, dxdy - \mathcal{E}_s(v^m_-; Z) \\
&\le -\mathcal{E}_s(v^m_-; B_R \times B_R),
\end{aligned}$$

so that (4.4) implies $\mathcal{E}_s(v^m_-; B_R \times B_R) = 0$, that together with $v^m_- \in \widetilde{H}^s(B_R \setminus \overline{B}_r)$ gives the conclusion. $\square$

By choosing $U_1 = U_2 = \mathbb{R}^n$ we have $Z = \mathbb{R}^n \times \mathbb{R}^n$. It is well known that $\mathfrak{L}^s_{\mathbb{R}^n \times \mathbb{R}^n} u = (-\Delta)^s u$ pointwise on $\mathbb{R}^n$ if $u \in C^2(\mathbb{R}^n)$. Thanks to Lemma 2.1, we can say that $\mathfrak{L}^s_{\mathbb{R}^n \times \mathbb{R}^n} u = (-\Delta)^s u$ in $\widetilde{H}^s(G)'$, for every $u \in X^s(\Omega; \mathbb{R}^n \times \mathbb{R}^n)$ and for any Lipschitz domain $G \Subset \Omega$. From Theorem 4.1 we immediately infer the next result.

Corollary 4.2 (Dirichlet Laplacian) *Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and let u be a non-constant measurable function on $\mathbb{R}^n$ such that*

$$u \in H^s_{\text{loc}}(\Omega)\ , \quad \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}}\, dx < \infty\ , \quad (-\Delta)^s u \ge 0 \ \text{ in } \Omega.$$

Then u is lower semicontinuous on Ω, locally bounded from below on Ω and $u(x) > \inf_{\mathbb{R}^n} u$ for every $x \in \Omega$.

The Restricted Laplacian is obtained by choosing $U_1 = U_2 = \Omega$. In fact, $\mathfrak{L}^s_{\Omega\times\Omega} u = (-\Delta^N_\Omega)^s_{\rm R} u$ on Ω if $u \in C^2(\Omega)$, and $\mathfrak{L}^s_{\Omega\times\Omega} u = (-\Delta^N_\Omega)^s_{\rm R} u$ in $\widetilde{H}^s(G)'$, for every $u \in X^s(\Omega; \mathbb{R}^n \times \mathbb{R}^n)$ and every Lipschitz domain $G \Subset \Omega$. From Theorem 4.1 we infer the next result.

Corollary 4.3 (Restricted Laplacian) *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let u be a nonconstant measurable function on Ω such that*

$$u \in H^s_{\rm loc}(\Omega) \ , \quad \int\limits_{\Omega} \frac{|u(x)|}{1 + |x|^{n+2s}}\, dx < \infty \ , \quad (-\Delta^N_\Omega)^s_{\rm R} u \ge 0 \quad in\ \Omega.$$

Then u is lower semicontinuous on Ω, locally bounded from below on Ω and $u(x) > \inf_{\Omega} u$ for every $x \in \Omega$.

Next, we choose $U_1 = \Omega, U_2 = \mathbb{R}^n$, so that $Z = \mathbb{R}^{2n} \setminus (\Omega^{\mathsf{c}})^2$. By [9, Lemma 3] we have that $\mathfrak{L}^s_{\mathbb{R}^{2n}\setminus(\Omega^{\mathsf{c}})^2} u = (-\Delta^N_\Omega)^s_{\rm Sr} u$ if $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, compare with (1.1). From the computations there and thanks to Lemma 2.1 we can identify the distributions $\mathfrak{L}^s_{\mathbb{R}^{2n}\setminus(\Omega^{\mathsf{c}})^2} u$ and $(-\Delta^N_\Omega)^s_{\rm Sr} u$ for functions $u \in X^s(\Omega; \mathbb{R}^{2n} \setminus (\Omega^{\mathsf{c}})^2)$ (see also [9, Definition 3.6]). Theorem 4.1 immediately implies the next corollary, see also [2, Theorem 1.1] for a related result.

Corollary 4.4 (Semirestricted Laplacian) *Let $\Omega \subset \mathbb{R}^n$ be a domain, and let u be a nonconstant measurable function on $\mathbb{R}^n$ such that*

$$u \in H^s_{\rm loc}(\Omega) \ , \quad \int\limits_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}}\, dx < \infty \ , \quad (-\Delta^N_\Omega)^s_{\rm Sr} u \ge 0 \quad in\ \Omega.$$

Then u is lower semicontinuous on Ω, locally bounded from below on Ω and $u(x) > \inf_{\mathbb{R}^n} u$ for every $x \in \Omega$.

We conclude by recalling that in the local case $s = 1$, the strong maximum principle states that every nonconstant superharmonic function u on Ω satisfies $u(x) > \inf_{\Omega} u$ for every $x \in \Omega$. Notice that in the non local, Neumann Restricted case we reached the same conclusion. In contrast, in the Dirichlet and in the Semirestricted cases a similar result can not hold, see the example in the next remark.

Remark 4.5 *Take any bounded domain* $\Omega \in \mathbb{R}^n$ *and two nonnegative functions* $u, \psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ *such that* $0 \le u \le 1$, $u \equiv 1$ *on* Ω, *supp* $\psi \subset \Omega$. *For any* $x \in \Omega$ *we have*

$$(-\Delta)^s u(x) = (-\Delta_\Omega^N)^s_{\mathrm{Sr}} u(x) = C_{n,s} \cdot \mathrm{P.V.} \int\limits_{\mathbb{R}^n} \frac{1-u(y)}{|x-y|^{n+2s}}\, dy > 0.$$

Since $(-\Delta)^s u, (-\Delta)^s \psi$ *are smooth functions, we have that* $(-\Delta)^s (u - \varepsilon\psi) \ge 0$ *in* Ω, *for some small* $\varepsilon > 0$. *Then* $u - \varepsilon\psi$ *satisfies the assumptions in Corollaries 4.2 and 4.4, but* $\inf\limits_{\Omega}(u - \varepsilon\psi)$ *is achieved in* Ω, *and actually* $u - \varepsilon\psi$ *has a strict local minimum in* Ω *if* ψ *has a strict local maximum. Clearly,* $\inf\limits_{\mathbb{R}^n}(u - \varepsilon\psi) = 0$ *is not achieved in* Ω.

Appendix

We start with a proposition in fact proved in [19]. It gives the same conclusion as in Corollary 4.2 under weaker summability assumptions on u. Notice however that $n > 2s$ is needed (this is a restriction only if $n = 1$), and that Silvestre's construction cannot be easily extended to more general operators such as the Restricted and Semirestricted ones.

Proposition A.1 *Assume* $n > 2s$ *and let* u *be a nonconstant measurable function on* $\mathbb{R}^n$ *such that* $\frac{u(x)}{1+|x|^{n+2s}} \in L^1(\mathbb{R}^n)$ *and* $(-\Delta)^s u \ge 0$ *in the distributional sense on* Ω, *that is,*

$$\langle (-\Delta)^s u, \varphi \rangle = \int\limits_{\mathbb{R}^n} u\, (-\Delta)^s \varphi\, dx \ge 0 \qquad \text{for any } \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n), \quad \varphi \ge 0.$$

Then u *is lower semicontinuous on* Ω *and* $u(x) > \inf\limits_{\mathbb{R}^n} u$ *for every* $x \in \Omega$.

Proof. First, notice that $(-\Delta)^s u$ is a well defined distribution, as $(1 + |x|^{n+2s})\, (-\Delta)^s \varphi$ is a bounded function on $\mathbb{R}^n$, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$. Proposition 2.2.6 in [19] gives the lower semicontinuity of u in Ω and the relations $u(x_0) \ge \int_{\mathbb{R}^n} u(x)\gamma_r^s(x_0 - x)\, dx > -\infty$ for any ball $B_r(x_0) \subset \Omega$, where γ_r^s is certain continuous and positive function on $\mathbb{R}^n$. If u is unbounded from below we are done; otherwise, we can assume $\inf\limits_{\mathbb{R}^n} u = 0$. Suppose that there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Take a ball $B_r(x_0) \subset \Omega$. Then $0 \ge \int_{\mathbb{R}^n} u(x)\gamma_r^s(x_0 - x)\, dx \ge 0$, that immediately implies $u \equiv 0$ in $\mathbb{R}^n$, a contradiction. $\square$

Now we recall that the Spectral Dirichlet/Neumann fractional Laplacian is the s-th power of standard Dirichlet/Neumann Laplacian in Ω in the sense of spectral theory.

A strong maximum principle for the Spectral Dirichlet Laplacian follows from [5, Lemma 2.6] and reads as follows.

Proposition A.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let a function $u \in \widetilde{H}^s(\Omega)$ be such that $(-\Delta_\Omega)^s_{\rm Sp} u \ge 0$ in Ω the sense of distributions. Then either $u \equiv 0$ or $\inf\limits_K u > 0$ for arbitrary compact set $K \subset \Omega$.*

A strong maximum principle for the Spectral Neumann Laplacian can be obtained from the results in [5].

Theorem A.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let a function $u \in H^s(\Omega)$ be such that $(-\Delta_\Omega^N)^s_{\rm Sp} u \ge 0$ in a subdomain $G \subset \Omega$ in the sense of distributions. Then either $u \equiv \mathrm{const}$ or $\inf\limits_K u > \inf\limits_\Omega u$ for arbitrary compact set $K \subset G$.*

Proof. It is well known, see [20], [4] and [1] for a general setting, that for any $u \in H^s(\Omega)$ the boundary value problem

$$-\mathrm{div}(y^{1-2s}\nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad w\big|_{y=0} = u; \qquad \partial_{\mathbf{n}} w\big|_{x\in\partial\Omega} = 0, \tag{A.1}$$

has a unique weak solution $w_s^N(x,y)$, and

$$(-\Delta_\Omega^N)^s_{\rm Sp} u(x) = -\,\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \cdot \lim_{y\to 0^+} y^{1-2s}\partial_y w_s^N(x,y)$$

(the limit is understood in the sense of distributions).

Without loss of generality we can assume that $\inf\limits_\Omega u = 0$. Then by the maximum principle for (A.1) we have $w \ge 0$. By [5, Lemma 2.6] the statement follows. $\square$

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References

[1] W. Arendt, A. F. M. ter Elst and M. Warma, Fractional powers of sectorial operators via the Dirichlet-to-Neumann operator, Comm. Partial Differential Equations, online first (2017), doi:10.1080/03605302.2017.1363229.

[2] B. Barrios and M. Medina, Strong maximum principles for fractional elliptic and parabolic problems with mixed boundary conditions, preprint arXiv:1607.01505v2 (2016).

[3] K. Bogdan, K. Burdzy and Z.-Q. Chen, Censored stable processes, Probab. Theory Related Fields **127** (2003), no. 1, 89–152.

[4] L. A. Caffarelli and P. R. Stinga, Fractional elliptic equations, Caccioppoli estimates and regularity, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 3, 767–807.

[5] A. Capella, J. Dávila, L. Dupaigne and Y. Sire, Regularity of radial extremal solutions for some non-local semilinear equations, Comm. Partial Differential Equations **36** (2011), no. 8, 1353–1384.

[6] L. M. Del Pezzo and A. Quaas, L. M. Del Pezzo and A. Quaas, A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian, J. Differential Equations **263** (2017), no. 1, 765–778.

[7] A. Di Castro, T. Kuusi and G. Palatucci, Local behavior of fractional p-minimizers, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 5, 1279–1299.

[8] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. **136** (2012), no. 5, 521–573.

[9] S. Dipierro, X. Ros-Oton and E. Valdinoci, Nonlocal problems with Neumann boundary conditions, Rev. Mat. Iberoam. **33** (2017), no. 2, 377–416. to appear.

[10] C. G. Gal and M. Warma, Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions, Evol. Equ. Control Theory **5** (2016), no. 1, 61–103.

[11] G. Grubb, Regularity of spectral fractional Dirichlet and Neumann problems, Math. Nachr. **289** (2016), no. 7, 831–844.

[12] Q. Y. Guan, Integration by parts formula for regional fractional Laplacian, Comm. Math. Phys. **266** (2006), no. 2, 289–329.

[13] Q. Y. Guan and Z. M. Ma, Boundary problems for fractional Laplacians, Stoch. Dyn., 5 (2005), 385–424.

[14] A. Iannizzotto, S. Mosconi and M. Squassina, H^s versus C^0-weighted minimizers, NoDEA Nonlinear Differential Equations Appl. **22** (2015), no. 3, 477–497.

[15] M. Kwaśnicki, Ten equivalent definitions of the fractional laplace operator, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51.

[16] R. Musina and A. I. Nazarov, On fractional Laplacians, Comm. Partial Differential Equations **39** (2014), no. 9, 1780–1790.

[17] X. Ros-Oton, Nonlocal elliptic equations in bounded domains: a survey, Publ. Mat. **60** (2016), no. 1, 3–26.

[18] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, J. Math. Pures Appl. (9) **101** (2014), no. 3, 275–302.

[19] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

[20] P. R. Stinga and J. L. Torrea, Extension problem and Harnack's inequality for some fractional operators, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122.

[21] H. Triebel, *Interpolation theory, function spaces, differential operators*, Deutscher Verlag Wissensch., Berlin, 1978.

[22] M. Warma, The p-Laplace operator with the nonlocal Robin boundary conditions on arbitrary open sets, Ann. Mat. Pura Appl. (4) **193** (2014), no. 1, 203–235.

[23] M. Warma, The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets, Potential Anal. **42** (2015), no. 2, 499–547.

[24] M. Warma, A fractional Dirichlet-to-Neumann operator on bounded Lipschitz domains, Commun. Pure Appl. Anal. **14** (2015), no. 5, 2043–2067.

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