Embedded loops in the hyperbolic plane with prescribed, almost constant curvature

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Abstract

Given a constant $k > 1$ and a real valued function K on the hyperbolic plane $\mathbb{H}^2$, we study the problem of finding, for any $\varepsilon \approx 0$, a closed and embedded curve u^ε in $\mathbb{H}^2$ having geodesic curvature $k + \varepsilon K(u^\varepsilon)$ at each point.

1 Introduction

Let Σ be an oriented Riemannian surface with empty boundary, Riemannian metric tensor g and Levi-Civita connection ∇^Σ. The geodesic curvature of a regular loop $u \in C^2(\mathbb{S}^1, \Sigma)$ is given by

$$K(u) = \frac{\langle \nabla^\Sigma_{u'} u', i_u u' \rangle_g}{|u'|_g^3}\,.$$

Here we denoted by $i_u : T_u\Sigma \to T_u\Sigma$ the isometry that rotates $T_u\Sigma$, in such a way that $\{\tau, i_u\tau\}$ is a positively oriented orthogonal basis of $T_u\Sigma$, for any $\tau \neq 0$.

Given a sufficiently smooth function $K : \Sigma \to \mathbb{R}$, the *K-loop problem* consists in finding regular curves $u \in C^2(\mathbb{S}^1, \Sigma)$ having geodesic curvature $K(u)$ at each point. This problem can be faced by studying the system of ordinary differential equations

$$\nabla^\Sigma_{u'} u' = L^\Sigma(u) K(u)\, i_u u'\,, \quad L^\Sigma(u) := \Big(\fint_{\mathbb{S}^1} |u'|_g^2\, dx \Big)^{\frac{1}{2}}\,. \tag{1.1}$$

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Indeed, every nonconstant solution $u \in C^2(\mathbb{S}^1, \Sigma)$ to (1.1) has constant speed $|u'|_g = L^\Sigma(u)$, use for instance the computations in [14, Chapter 4]. Therefore u is regular, and has curvature $K(u)$ at each point.

The K-loop problem has been largely studied since the seminal work [4] by Arnol'd. Most of the available existence results require compact target surfaces Σ; we limit ourselves to cite [9, 12, 13, 19, 20, 21, 22, 23] and references therein.

In the present paper we take Σ to be the (noncompact) hyperbolic plane $\mathbb{H}^2$. It turns out that the problem under consideration does not have solutions, in general (see Subsection 2.2). In particular, if $-1 \le K(q) \le 1$ for any $q \in \Sigma$, then no K-loop exists. If $K \equiv k > 1$ is constant (recall that changing the orientation of a curve changes the sign of its curvature), then any regular parameterization of an hyperbolic circle of radius

$$\rho_k = \operatorname{artanh}\frac{1}{k} = \frac{1}{2}\ln\frac{k+1}{k-1}$$

is a k-loop; conversely, any k-loop in $\mathbb{H}^2$ parameterizes some circle of radius ρ_k.

Our existence results involve curvatures that are small perturbations of a given constant $k > 1$. In Section 3 we carefully choose a reference parameterization ω of a circle of radius ρ_k. Then we take any point $z \in \mathbb{H}^2$ and compose ω with an hyperbolic translation to obtain a parameterization ω_z of $\partial D^{\mathbb{H}}_{\rho_k}(z)$. Next, given $K \in C^1(\mathbb{H}^2)$, we look for a point $z_0 \in \mathbb{H}^2$ and for embedded $(k+\varepsilon K)$-loops in $\mathbb{H}^2$ that suitably approach the circle ω_{z_0} as $\varepsilon \to 0$.

The center z_0 can not be arbitrarily prescribed. In fact, in Theorem 4.1 we prove that if there exists a sequence of $(k+\varepsilon_h K)$-loops u_h such that $\varepsilon_h \to 0$ and $u_h \to \omega_{z_0}$ suitably, then z_0 is a critical point for the Melnikov-type function

$$F^K_k(z) = \int\limits_{D^{\mathbb{H}}_{\rho_k}(z)} K(z)dV_{\mathbb{H}}\ , \qquad F^K_k : \mathbb{H}^2 \to \mathbb{R}\,. \tag{1.2}$$

One may wonder whether the existence of a critical point z_0 for F^K_k is sufficient to have the existence, for $\varepsilon \approx 0$, of an embedded $(k+\varepsilon K)$-loop $u_\varepsilon \approx \omega_{z_0}$. We can give a positive answer in case F^K_k has a *stable* critical point, accordingly with the next definition (see also [3, Chapter 2]).

Definition *Let $X \in C^1(\mathbb{H}^2)$ and let $A \Subset \mathbb{H}^2$ be an open set. We say that X has a stable critical point in A if there exists $r > 0$ such that any function $G \in C^1(\overline{A})$ satisfying $\|G - X\|_{C^1(\overline{A})} < r$ has a critical point in A.*

Sufficient conditions to have the existence of a stable critical point $z \in A$ for X are easily given via elementary calculus. For instance, one can assume that one of the following conditions holds:

i) $\nabla X(z) \neq 0$ for any $z \in \partial A$, and $\deg(\nabla \mathrm{X}, \mathrm{A}, 0) \neq 0$, where "deg" is Browder's topological degree;

ii) $\min_{\partial A} X > \min_{A} X$ or $\max_{\partial A} X < \max_{A} X$;

iii) X is of class C^2 on A, it has a critical point $z_0 \in A$, and the Hessian matrix of X at z_0 is invertible.

We are in position to state our main result.

Theorem 1.1 *Let $k > 1$ and $K \in C^1(\mathbb{H}^2)$ be given. Assume that F_k^K has a stable critical point in an open set $A \Subset \mathbb{H}^2$. Then for every $\varepsilon \in \mathbb{R}$ close enough to 0, there exists an embedded $(k + \varepsilon K)$-loop u^ε.*

Moreover, any sequence $\varepsilon_h \to 0$ has a subsequence ε_{h_j} such that $u^{\varepsilon_{h_j}} \to \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ as $j \to \infty$, where $z_0 \in \overline{A}$ is a critical point for F_k^K. In particular, if a point $z_0 \in A$ is the unique critical point for F_k^K in $\overline{A}$, then $u^\varepsilon \to \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ as $\varepsilon \to 0$.

Any stable critical point of the perturbation term K gives rise to a stable critical point for F_k^K, at least for k large enough. This is in essence the argument we use in Theorem 4.3 to obtain, via Theorem 1.1, the existence of $k + \varepsilon K$-loops whenever the perturbation curvature K admits stable critical points.

The proof of Theorem 1.1 is based on a Lyapunov-Schmidt reduction technique combined with variational arguments, as proposed in [1] (see also [3, Chapter 2]).

In fact, $(k + \varepsilon K)$-loops correspond to critical points of an energy functional $E_{k+\varepsilon K}(u) = E_{k+\varepsilon K}(u)$, where u runs in the class of nonconstant curves in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ (see Section 2.1 for details). In particular, critical points of the unperturbed functional E_k are circles of radius ρ_k. Let $\mathcal{S} = \{\omega_z \circ \xi\}$, where ξ is a rotation of $\mathbb{S}^1$, $z \in \mathbb{H}^2$, and ω_z is our reference parameterization of $\partial D^{\mathbb{H}}_{\rho_k}(z)$. Clearly $\mathcal{S}$ is a smooth three-dimensional manifold of solutions to the unperturbed problem $E_k'(u) = 0$.

The crucial and technically difficult nondegeneracy result is proved in Lemma 3.3, via an efficient functional change inspired by [17]. It states that for any $z \in \mathbb{H}^2$, the tangent space to $\mathcal{S}$ at ω_z coincides with the set of solutions to the linear problem

$E''_k(\omega_z)\varphi = 0$. In the last section we carry out the dimensional reduction argument and complete the proof of Theorem 1.1.

We conclude the paper with a short appendix about the much more easy problem of finding loops in $\mathbb{R}^2$ having prescribed, almost constant curvature.

The Lyapunov-Schmidt reduction argument has been successfully used to study related geometrical problems. We limit ourselves to cite the pioneering paper [24] by R. Ye, [2, 6, 7, 8, 10, 11, 16, 17] and references therein.

2 Notation and preliminaries

The Euclidean space $\mathbb{R}^2$ is endowed with the scalar product $p \cdot q$ and norm $|\cdot|$, so that the disk of radius R centered at $p \in \mathbb{R}^2$ is $D_R(p) = \{z \in \mathbb{R}^2 \mid |z-p| < R\}$. The canonical basis of $\mathbb{R}^2$ is $e_1 = (1,0), e_2 = (0,1)$.

Let $A, \Omega \subseteq \mathbb{R}^2$ be open sets. We write $A \Subset \Omega$ if $\overline{A}$ is a compact subset of Ω.

We will often use complex notation for points in $\mathbb{R}^2$. In particular we write $iz = (-z_2, z_1)$ and $z^2 = (z_1^2 - z_2^2, 2z_1z_2)$ for $z = (z_1, z_2) \in \mathbb{R}^2$.

Let $\mathbb{S}^1$ be the unit circle in the complex plane. Any $\xi \in \mathbb{S}^1$ is identified with the rotation $x \mapsto \xi x$.

The Poincaré half-plane model

We adopt as model for the two dimensional hyperbolic space the half-plane

$$\mathbb{H}^2 = \{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_2 > 0\}$$

endowed with the Riemannian metric $g_{lj}(z) = z_2^{-2}\delta_{lj}$. With some abuse of notation, we use the symbol $\mathbb{H}^2$ to denote the Euclidean upper half space as well.

The hyperbolic distance $d_{\mathbb{H}}(p,q)$ in $\mathbb{H}^2$ is related to the Euclidean one by

$$\cosh d_{\mathbb{H}}(p,q) = 1 + \frac{|p-q|^2}{2p_2q_2},$$

and the hyperbolic disk $D^{\mathbb{H}}_\rho(p)$ centered at $p = (p_1, p_2)$ is the Euclidean disk of center $(p_1, p_2 \cosh \rho)$ and radius $p_2 \sinh \rho$.

A *loop* in the 2-dimensional hyperbolic space $\mathbb{H}^2$ is a curve $u : \mathbb{S}^1 \to \mathbb{H}^2$ of class C^2 having nonzero derivative at each point. We say that u is embedded if it is injective.

If $G : \mathbb{H}^2 \to \mathbb{R}$ is a differentiable function, then $\nabla^{\mathbb{H}} G(z) = z_2^2 \nabla G(z)$, where $\nabla^{\mathbb{H}}$, ∇ are the hyperbolic and the Euclidean gradients, respectively. In particular, $\nabla^{\mathbb{H}} G(z) = 0$ if and only if $\nabla G(z) = 0$.

The hyperbolic volume form $dV_{\mathbb{H}}$ is related to the Euclidean one by $dV_{\mathbb{H}} = z_2^{-2} dz$.

The Levi-Civita connection in $\mathbb{H}^2$ along a curve u in Σ is given by

$$\nabla^{\mathbb{H}}_{u'} u' = u'' - u_2^{-1} \Gamma(u') , \tag{2.1}$$

where, in complex notation, $\Gamma(z) = -iz^2$. In coordinates we have

$$\Gamma(z) := (2z_1 z_2, z_2^2 - z_1^2) = z_2 z - z_1 \, iz \ , \qquad \Gamma : \mathbb{H}^2 \to \mathbb{R}^2 . \tag{2.2}$$

For future convenience we compute the differential

$$\Gamma'(z)w = 2(w_2 z - w_1 \, iz) \ , \qquad z \in \mathbb{H}^2, \ w \in \mathbb{R}^2. \tag{2.3}$$

Isometries in $\mathbb{H}^2$

Hyperbolic translations are obtained by composing a horizontal (Euclidean) translation $w \mapsto w + se_1$, $s \in \mathbb{R}$ (sometimes called *parabolic isometry*), with an Euclidean homothety $w \mapsto tw$, $t > 0$ (in some literature, only homotheties are called hyperbolic translations). We obtain the two dimensional group of isometries $\mathbb{H}^2 \to \mathbb{H}^2$,

$$u \mapsto u_z := z_1 e_1 + z_2 u \ , \qquad z \in \mathbb{H}^2 .$$

Function spaces

Let $m \geq 0$, $n \geq 1$ be integer numbers. We endow $C^m(\mathbb{S}^1, \mathbb{R}^n)$ with the standard Banach space structure. If $f \in C^1(\mathbb{S}^1, \mathbb{R}^n)$, we identify $f'(x) \equiv f'(x)(ix)$, so that $f' : \mathbb{S}^1 \to \mathbb{R}^n$.

In $L^2 = L^2(\mathbb{S}^1, \mathbb{R}^2)$ we take the Hilbertian norm

$$\|u\|^2_{L^2} = \fint_{\mathbb{S}^1} |u(x)|^2 \, dx \ = \frac{1}{2\pi} \int_{\mathbb{S}^1} |u(x)|^2 \, dx \, .$$

If $T \subseteq C^0(\mathbb{S}^1, \mathbb{R}^2)$, the orthogonal to T with respect to the L^2 scalar product is

$$T^\perp = \{ \varphi \in C^0(\mathbb{S}^1, \mathbb{R}^2) \mid \fint_{\mathbb{S}^1} u \cdot \varphi \, dx = 0 \ \text{ for any } u \in T \ \}.$$

We look at $C^m(\mathbb{S}^1, \mathbb{H}^2)$ as an open subset of the Banach space $C^m(\mathbb{S}^1, \mathbb{R}^2)$, and identify $\mathbb{H}^2$ with the set of constant functions in $C^m(\mathbb{S}^1, \mathbb{H}^2)$. Thus $C^m(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ contains only nonconstant curves.

2.1 The variational approach

We put

$$L(u) := L_{\mathbb{H}^2}(u) = \Big(\fint_{\mathbb{S}^1} u_2^{-2} |u'|^2 \, dx \Big)^{\frac{1}{2}} \ , \qquad L : C^2(\mathbb{S}^1, \mathbb{H}^2) \to \mathbb{R} \, ,$$

that is a C^∞ functional, with Fréchet differential

$$L'(u)\varphi = \frac{1}{L(u)} \fint_{\mathbb{S}^1} u_2^{-2} \big(-u'' + u_2^{-1} \Gamma(u') \big) \cdot \varphi \, dx \, , \quad \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2) \, . \tag{2.4}$$

When $\Sigma = \mathbb{H}^2$, problem (1.1) reads

$$u'' - u_2^{-1} \Gamma(u') = L(u) K(u) \, i u' \, . \tag{$\mathcal{P}_K$}$$

The system ($\mathcal{P}_K$) admits a variational formulation. More precisely, its nonconstant solutions are critical points of the energy functional of the form

$$E_K(u) = L(u) + A_K(u) \ , \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2,$$

where $A_K(u)$ gives, roughly speaking, the signed area enclosed by the curve u with respect to the weight K (see Remark 2.2 below). More precisely, to introduce $A_K(u)$ we take any vectorfield $Q_K \in C^1(\mathbb{H}^2, \mathbb{R}^2)$ such that

$$\mathrm{div} Q_K(z) = z_2^{-2} K(z) \ , \qquad z \in \mathbb{H}^2$$

(here ″div″ is the usual Euclidean divergence). A possible choice is

$$Q_K(z_1, z_2) = \Big(\frac{1}{2} \, z_2^{-2} \int_0^{z_1} K(t, z_2) \, dt \Big) e_1 + \Big(\frac{1}{2} \int_1^{z_2} t^{-2} K(z_1, t) \, dt \Big) e_2 \, .$$

Then we define

$$A_K(u) = \fint_{\mathbb{S}^1} Q_K(u) \cdot i u' \, dx \ , \qquad A_K : C^2(\mathbb{S}^1, \mathbb{H}^2) \to \mathbb{R}.$$

By direct computations one gets that the functional A_K is Fréchet differentiable at any $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, with differential

$$A_K'(u)\varphi = \fint_{\mathbb{S}^1} u_2^{-2} \, K(u) \varphi \cdot \, i u' \, dx \, . \quad \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2) \, , \tag{2.5}$$

It follows that $A_K(u)$ does not depend on the choice of the vectorfield Q_K. Further, if $K \in C^1(\mathbb{H}^2)$ then the area functional A_K is of class C^2 on $C^2(\mathbb{S}^1, \mathbb{R}^2)$.

In conclusion, the following lemma holds.

Lemma 2.1 *Let $K \in C^1(\mathbb{H}^2)$. The functional $E_K(u) = L(u) + A_K(u)$ is of class C^2 on $C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$, and*

$$L(u)E'_K(u)\varphi = \fint_{\mathbb{S}^1} u_2^{-2}\big(- u'' + u_2^{-1}\Gamma(u') + L(u)K(u)\, iu'\big) \cdot \varphi \, dx$$

for any $u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2, \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2)$. In particular, if $u_0 \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ is a critical point for the functional $E_K(u)$, then u_0 solves $(\mathcal{P}_K)$, hence it is an hyperbolic K-loop.

Remark 2.2 *Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be an embedded loop. Then u is a regular parameterization of the boundary of an open set $\Omega_u \Subset \mathbb{H}^2$. Assume for instance that u is positively oriented, so that iu' gives the inner direction to Ω_u. Then*

$$A_K(u) = -\frac{1}{2\pi} \int_{\partial\Omega} Q_K(z) \cdot \nu \, ds = -\frac{1}{2\pi} \int_{\Omega} K(z) dV_{\mathbb{H}}$$

by the divergence theorem.

2.2 Nonexistence results

We start with a simple result that should be well known. We sketch its proof by adapting the argument in [15, p. 194].

Proposition 2.3 *Let $K \in C^0(\mathbb{H}^2)$. If $\|K\|_\infty \le 1$ then no K-loop exists.*

Proof. Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be a K-loop. We need to show that $|K| > 1$ somewhere in $\mathbb{H}^2$. Take the smallest closed disk $D_\rho = \overline{D^{\mathbb{H}}_\rho(z)}$ containing $u(\mathbb{S}^1)$. Then ∂D_ρ is tangent to $u(\mathbb{S}^1)$ at some point. At the contact point the absolute value of the curvature of u can not be smaller than the curvature $1/\tanh\rho$ of the circle ∂D_ρ, use a local comparison principle. The conclusion readily follows from $\tanh\rho < 1$. $\square$

Next, we point out few necessary conditions for the existence of K-loops.

Lemma 2.4 *Let $K \in C^1(\mathbb{H}^2)$ and let $\Omega \subset \mathbb{H}^2$ be a bounded open domain. Assume that $\partial\Omega$ is parameterized by a K-loop $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$. Then*

$$\int_{\Omega} \nabla K(z) \cdot e_1 \, dV_{\mathbb{H}^2} = \int_{\Omega} \nabla K(z) \cdot z \, dV_{\mathbb{H}^2} = \int_{\Omega} \nabla K(z) \cdot z^2 \, dV_{\mathbb{H}^2} = 0\,.$$

Proof. Direct computations based on integration by parts give

$$L'(u)e_1 = L'(u)u = L'(u)i(\Gamma u) = 0, \tag{2.6}$$

see (2.4) and (2.2). In addition, the curve u solves

$$-L(u)L'(u)\varphi = \fint_{\mathbb{S}^1} u_2^{-2}K(u)\varphi \cdot iu'\,dx \quad \text{for any } \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2).$$

Since $iu'(x) \neq 0$ is parallel to the outer normal ν to Ω at $u(x) \in \partial\Omega$, we infer that

$$\int_{\partial\Omega} z_2^{-2}K(z)e_1 \cdot \nu = \int_{\partial\Omega} z_2^{-2}K(z)z \cdot \nu = \int_{\partial\Omega} z_2^{-2}K(z)i\Gamma(z) \cdot \nu = 0.$$

Recall that we identify $i\Gamma(z) = z^2$, then use the divergence theorem to get

$$\int_{\Omega} \operatorname{div}\big(z_2^{-2}K(z)e_1\big)\,dz = \int_{\Omega} \operatorname{div}\big(z_2^{-2}K(z)z\big)\,dz = \int_{\Omega} \operatorname{div}\big(z_2^{-2}K(z)z^2\big)\,dz = 0\,.$$

The conclusion readily follows. □

Remark 2.5 *The identities in (2.6) hold indeed for any curve u, and are related to the group of isometries in $\mathbb{H}^2$. Notice indeed that $z \mapsto e_1, z \mapsto z, z \mapsto z^2$ are infinitesimal Killing vectorfields in $\mathbb{H}^2$.*

Lemma 2.4 readily implies the next nonexistence result.

Corollary 2.6 *Let $K \in C^1(\mathbb{H}^2)$ be a given curvature function. Assume that one of the following conditions hold,*

i) *K is strictly monotone in the e_1 direction;*

ii) *K is radially strictly monotone, that is, $\nabla K(z) \cdot z$ never vanishes on $\mathbb{H}^2$;*

iii) *$\nabla K(z) \cdot z^2$ never vanishes on $\mathbb{H}^2$*

Then no embedded K-loop exists.

3 The unperturbed problem

In this section we take a constant $k > 1$ and study the system

$$u'' - u_2^{-1}\Gamma(u') = L(u)k\, iu' . \qquad (\mathcal{P}_k)$$

We start by introducing the radius

$$R_k := \sinh\rho_k = \frac{1}{k}\cosh\rho_k = \frac{1}{\sqrt{k^2-1}}$$

and the reference loop $\omega : \mathbb{S}^1 \to \mathbb{H}^2$,

$$\omega(x) = \frac{1}{k - x_2}\big(x_1 \,, \frac{1}{R_k} \big) \;, \quad x = x_1 + ix_2 \in \mathbb{S}^1 . \tag{3.1}$$

Notice that

$$|\omega - kR_k e_2| = R_k, \tag{3.2}$$

hence ω is a (positive) parametrization of the Euclidean circle $\partial D_{R_k}(kR_k e_2)$, that coincides with the hyperbolic circle $\partial D^{\mathbb{H}}_{\rho_k}(e_2)$. The next identities will be very useful:

$$\omega' = \omega_2\, i(\omega - kR_k e_2) \tag{3.3}$$

$$\omega_2^{-1}\Gamma(\omega') = (\omega_2 - kR_k)\, i\omega' + \omega_1\, \omega' \tag{3.4}$$

$$\omega_2^{-1}|\omega'| \equiv L(\omega) = R_k . \tag{3.5}$$

By differentiating (3.3) and using (3.5) one easily gets that ω solves $(\mathcal{P}_k)$. Next, for $z = (z_1, z_2) \in \mathbb{H}^2$ we parameterize $\partial D^{\mathbb{H}}_{\rho_k}(z)$ by the function

$$\omega_z = z_1 e_1 + z_2\omega .$$

Notice that $\omega = \omega_{e_2}$. It is easy to check that for any rotation $\xi \in \mathbb{S}^1$ and any point $z \in \mathbb{H}^2$, the circle $\omega_z \circ \xi$ solves $(\mathcal{P}_k)$ as well. Further, by Remark 2.2 we have

$$F_k^K(z) := \int\limits_{D^{\mathbb{H}}_{\rho_k}(z)} K(z)dV_{\mathbb{H}} = -2\pi A_K(\omega_z). \tag{3.6}$$

We know that any nonconstant solution u to $(\mathcal{P}_k)$ has constant curvature k, hence is a circle of hyperbolic radius ρ_k. Actually we need a sharper uniqueness result, that is, we have to classify solutions to $(\mathcal{P}_k)$.

Lemma 3.1 *Let $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$ be a nonconstant solution to $(\mathcal{P}_k)$. Then $\mu := L(u)/L(\omega)$ is an integer number, and there exist $\xi \in \mathbb{S}^1$, $z = (z_1, z_2) \in \mathbb{H}^2$ such that $u(x) = \omega_z \circ \xi$. In particular, u parameterizes $\partial D_{\rho_k}(z)$, and $L(u) = \mu L(\omega) = \mu R_k$.*

Proof. We have

$$\omega_2(-i) = e^{-\rho_k} = \min_{x\in\mathbb{S}^1} \omega_2(x) \ , \quad \omega(-i) = e^{-\rho_k} e_2 \ , \quad \omega'(-i) = e^{-\rho_k} L(\omega) e_1 \, .$$

Let $x_u \in \mathbb{S}^1$ such that

$$u_2(x_u) = m_u := \min_{x\in\mathbb{S}^1} u_2(x) \, .$$

Now we show that

$$u'(x_u) = m_u L(u) e_1. \tag{3.7}$$

Clearly $u_2'(x_u) = 0$ and $u_2''(x_u) \geq 0$. We first infer that $\Gamma(u'(x_u)) = -u_1'(x_u)\, iu'(x_u)$, compare with (2.2). Thus the system $(\mathcal{P}_k)$ for the second coordinate gives

$$\big(L(u)k - m_u^{-1} u_1'(x_u)\big)\, u_1'(x_u) = u_2''(x_u) \geq 0,$$

that implies $u_1'(x_u) \geq 0$. On the other hand, $u_2^{-1}|u'| \equiv L(u)$ on $\mathbb{S}^1$. Thus $u_1'(x_u) = |u'(x_u)| = m_u L(u)$, and (3.7) is proved.

In particular, u solves the Cauchy problem

$$v'' = v_2^{-1}\Gamma(v') + kL(u)\, iv' \ , \quad v(x_u) = u(x_u) \ , \quad v'(x_u) = m_u L(u) e_1. \tag{3.8}$$

It is easy to check that the function

$$\tilde{u}(x) := m_u e^{\rho_k}\, \omega\big(-ix_u^{-\mu} x^{\mu}\big) + u_1(x_u) e_1$$

solves (3.8) as well (use $f'(x) = i\mu x^\mu$ for $f(x) = x^\mu$, $f : \mathbb{S}^1 \to \mathbb{C}$). Thus $\tilde{u}(x) = u(x)$ for any $x \in \mathbb{S}^1$ and hence $u(x) = \omega_z \circ \xi$, where $z_1 = u_1(x_u)$, $z_2 = m_u e^{\rho_k}$, $\xi = -ix_u^{-\mu}$. Finally, μ is an integer number because u and ω are both well defined on $\mathbb{S}^1$. $\square$

The linearized problem

By Lemma 3.1, the 3-dimensional manifold

$$\mathcal{S} = \big\{\omega_z \circ \xi \mid \xi \in \mathbb{S}^1 \ , \ \ z \in \mathbb{H}^2 \ \big\} \subset C^2(\mathbb{S}^1, \mathbb{H}^2), \quad \omega_z = z_1 e_1 + z_2 \omega$$

is the set of embedded solutions to $(\mathcal{P}_k)$. The tangent space to $\mathcal{S}$ at ω_z is

$$T_{\omega_z}\mathcal{S} = T_\omega \mathcal{S} = \langle\ \omega', e_1, \omega\ \rangle.$$

Every loop in $\omega_z \circ \xi \in \mathcal{S}$ is a critical point for the energy functional

$$E_k(u) = L(u) + A_k(u) = \Big(\fint_{\mathbb{S}^1} u_2^{-2}|u'|^2\, dx\Big)^{\frac{1}{2}} - k \fint_{\mathbb{S}^1} u_2^{-1} u_1'\, dx$$

on $C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus H^2$, and $E_k(\omega_z \circ \xi) = E_k(\omega)$ is a constant. More generally one has

$$E_k(z_1 e_1 + z_2\, u \circ \xi) = E_k(u) \qquad \text{for any } \xi \in \mathbb{S}^1,\ z \in \mathbb{H}^2. \tag{3.9}$$

In order to handle the differential of E_k, it is convenient to introduce the function $J_0 : C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2 \to C^0(\mathbb{S}^1, \mathbb{H}^2)$ given by

$$\begin{aligned} J_0(u) &= -(u_2^{-2}u')' - u_2^{-3}|u'|^2 e_2 + L(u)k u_2^{-2}\, iu' \\ &= u_2^{-2}\big(-u'' + u_2^{-1}\Gamma(u') + L(u)k\, iu'\big)\,. \end{aligned} \tag{3.10}$$

By Lemma 2.1 we have

$$L(u)E_k'(u)\varphi = \fint_{\mathbb{S}^1} J_0(u)\cdot\varphi\, dx \quad \text{for any } \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \tag{3.11}$$

By differentiating (3.9) at $\xi = 1$, $z = e_2$ we readily get $E_k'(u)u' = E_k'(u)e_1 = E_k'(u)u = 0$ for any nonconstant curve $u \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, that is,

$$\fint_{\mathbb{S}^1} J_0(u)\cdot u'\, dx = 0\,, \quad \fint_{\mathbb{S}^1} J_0(u)\cdot e_1\, dx = 0\,, \quad \fint_{\mathbb{S}^1} J_0(u)\cdot u\, dx = 0\,. \tag{3.12}$$

Now we differentiate (3.11) with respect to u, at $u = \omega_z$. From $E_k'(\omega_z) = 0$ we get

$$L(\omega)E_k''(\omega_z)[\varphi, \tilde\varphi] = \fint_{\mathbb{S}^1} J_0'(\omega_z)\varphi\cdot\tilde\varphi\, dx \quad \text{for any } \varphi, \tilde\varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2).$$

Since E_k is of class C^2, then $J_0'(\omega_z)$ is self-adjoint in L^2, that means

$$\fint_{\mathbb{S}^1} J_0'(\omega_z)\varphi\cdot\tilde\varphi\, dx = \fint_{\mathbb{S}^1} J_0'(\omega_z)\tilde\varphi\cdot\varphi\, dx \quad \text{for any } \varphi, \tilde\varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \tag{3.13}$$

Finally, we differentiate $E_k'(\omega_z \circ \xi) = 0$ with respect to the variables $\xi \in \mathbb{S}^1, z \in \mathbb{H}^2$ to get $T_{\omega_z}\mathcal{S} \subseteq \ker J_0'(\omega_z)$. We shall see in the crucial Lemma 3.3 below that indeed $T_{\omega_z}\mathcal{S} = \ker J_0'(\omega_z)$.

This will be done via a useful functional change.

A functional change and nondegeneracy

In order to avoid tricky computations, we use in $C^m(\mathbb{S}^1, \mathbb{R}^2)$, $m \ge 0$, the orthogonal frame $\omega', i\omega'$. We introduce the isomorphism

$$\Phi(g) = g_1\omega' + g_2 i\omega'\,, \qquad \Phi : C^m(\mathbb{S}^1, \mathbb{R}^2) \to C^m(\mathbb{S}^1, \mathbb{R}^2)$$

together with its inverse $\Phi^{-1}(\varphi) = R_k^{-2}\omega_2^{-2}(\,\varphi\cdot\omega'\, e_1 + \varphi\cdot i\omega'\, e_2)$ (recall that $|\omega'| = R_k\omega_2$) and the differential operator

$$Bg = -g'' - kR_k ig' + R_k^2\big(g_2 - k^2 \fint_{\mathbb{S}^1} g_2 dx\big)e_2\,, \qquad g \in C^2(\mathbb{S}^1, \mathbb{R}^2)\,. \tag{3.14}$$

Lemma 3.2 *Let z be any point in $\mathbb{H}^2$. The following facts hold.*

i) $J_0'(\omega_z)(\Phi(g)) = z_2^{-2}\omega_2^{-2}\Phi\big(Bg\big)$ *for any* $g \in C^2(\mathbb{S}^1, \mathbb{S}^2)$*;*

ii) $\displaystyle\fint_{\mathbb{S}^1} \omega_2^{-2}\Phi(g)\cdot\Phi(\tilde g)\,dx = R_k^2 \fint_{\mathbb{S}^1} g\cdot\tilde g\,dx$ *for any* $g, \tilde g \in C^2(\mathbb{S}^1, \mathbb{S}^2)$

Proof. Since $J_0(\omega_z) = z_2^{-1}J_0(\omega) = 0$ and $J_0'(\omega_z) = z_2^{-2}J_0'(\omega)$, it suffices to prove *i)* for $z = e_2$, that corresponds to $\omega_z = \omega$. We have to show that

$$\mathcal{J}(\varphi) := \omega_2^2 J_0'(\omega)\varphi = \Phi\big(Bg\big)\ , \quad \text{where } \ \varphi = g_1\omega' + g_2\, i\omega'\,. \tag{3.15}$$

To compute $\mathcal{J}(\varphi)$ it is convenient to recall (3.10) and to differentiate the identity

$$u_2^2\, J_0(u) = -u'' + u_2^{-1}\Gamma(u') + L(u)k\, iu'$$

at $u = \omega$. Since $J_0(\omega) = 0$ and $L(\omega) = R_k$, we get

$$\mathcal{J}(\varphi) = -\varphi'' + kR_k\, i\varphi' + \omega_2^{-1}\Gamma'(\omega')\varphi' - \omega_2^{-2}\varphi_2\Gamma(\omega') + k\big(L'(\omega)\varphi\big)i\omega'.$$

From (2.3) we find $\Gamma'(\omega')\varphi' = 2\varphi_2'\,\omega' - 2\varphi_1'\, i\omega'$. Taking also (3.4) into account, we obtain

$$\mathcal{J}(\varphi) \;\; = \;\; -\varphi'' + kR_k\, i\varphi' + A_1(\varphi)\,\omega' - \big(A_2(\varphi) - k\, L'(\omega)\varphi\,\big)\, i\omega'\,,$$

where

$$A_1(\varphi) = \big(2\varphi_2' - \varphi_2\omega_1\big)\omega_2^{-1}\ , \quad A_2(\varphi) = \big(2\varphi_1' + \varphi_2(\omega_2 - kR_k)\big)\omega_2^{-1}.$$

To compute the differential $L'(\omega)$ at φ we recall that ω solves $(\mathcal{P}_k)$. Thus (2.4) gives

$$L'(\omega)\varphi = -k\fint_{\mathbb{S}^1} \omega_2^{-2}\varphi\cdot i\omega'\,dx\,.$$

For the next computations we observe that the loop ω solves several useful differential systems. In particular, from $(\mathcal{P}_k)$, (3.4), (3.5) and (3.3) it follows that

$$\omega'' = \omega_1\omega' + \omega_2\, i\omega'\ , \quad \omega''' = (\omega_1^2 - 2\omega_2 + kR_k\omega_2)\,\omega' + 3\omega_1\omega_2\, i\omega'. \tag{3.16}$$

Now we take any $\psi \in C^2(\mathbb{S}^1, \mathbb{R})$ and we look for an explicit formula for $\mathcal{J}(\psi\omega')$. Clearly $L'(\omega)(\psi\omega') = 0$, as $\omega'\cdot i\omega' \equiv 0$. Direct computations based on (3.16) give

$$\begin{aligned} -(\psi\,\omega')'' + kR_k\, i(\psi\,\omega')' \;\; &= \;\; \big(-\psi'' - 2\omega_1\psi' - (\omega_1^2 - 2\omega_2^2 + 2kR_k\omega_2)\psi\big)\,\omega' \\ &\quad + \big((kR_k - 2\omega_2)\psi' + (kR_k - 2\omega_2)\omega_1\psi\big)\, i\omega' \end{aligned}$$

$$\begin{aligned} A_1(\psi\,\omega') \;\; &= \;\; 2\omega_1\psi' - (2\omega_2^2 - 2kR_k\omega_2 - \omega_1^2)\psi \\ A_2(\psi\,\omega') \;\; &= \;\; 2(kR_k - \omega_2)\psi' - (kR_k - 3\omega_2)\psi, \end{aligned}$$

and we find the formula

$$\mathcal{J}(\psi\omega') = -\psi''\omega' - kR_k\psi'\, i\omega'. \tag{3.17}$$

Now we handle $\mathcal{J}(\psi\, i\omega')$. From (3.5) we get

$$k\, L'(\omega)(\psi\, i\omega') = -k^2 \fint_{\mathbb{S}^1} \omega_2^{-2}|\omega'|^2\psi\, dx = -k^2R_k^2 \fint_{\mathbb{S}^1} \psi\, dx.$$

Then we use (3.2–3.5) and (3.16) to compute

$$\begin{aligned}-(\psi\, i\omega')'' + kR_k\, i(\psi\, i\omega')' &= \big((2\omega_2 - kR_k)\psi' + (3\omega_2 - kR_k)\omega_1\psi\big)\,\omega' \\ &\quad + \big(-\psi'' - 2\omega_1\psi' - (\omega_1^2 - 2\omega_2^2 + 2kR_k\omega_2)\psi\big)\, i\omega'\end{aligned}$$

$$\begin{aligned}A_1(\psi\, i\omega') &= -2(\omega_2 - kR_k)\psi' - (3\omega_2 - kR_k)\omega_1\psi \\ A_2(\psi\, i\omega') &= -2\omega_1\psi' + (\omega_2^2 - k^2R_k^2 - 2\omega_1^2)\psi.\end{aligned}$$

Since $R_k^2 = |\omega - kR_ke_2|^2 = |\omega|^2 - 2kR_k\,\omega_2 + k^2R_k^2$ by (3.2), we arrive at

$$\mathcal{J}(\psi\, i\omega') = kR_k\psi'\,\omega' + \big(-\psi'' + R_k^2\psi - k^2R_k^2 \fint_{\mathbb{S}^1} \psi\, dx\big)\, i\omega',$$

that together with (3.17) gives

$$\begin{aligned}\mathcal{J}(g_1\,\omega' + g_2\, i\omega') &= \big(-g_1'' - kR_kg_2'\big)\omega' \\ &\quad + \big(-g_2'' + kR_kg_1' + R_k^2g_2 - k^2R_k^2 \fint_{\mathbb{S}^1} g_2\, dx\big)i\omega'\end{aligned}$$

and concludes the proof of (3.15). The proof of *i)* is complete; the formula in *ii)* is immediate, because $\omega' \cdot i\omega' \equiv 0$ and $|\omega'| = R_k\omega_2$. □

We are in position to prove the main result of this section.

Lemma 3.3 (Nondegeneracy) *Let z be any point in $\mathbb{H}^2$. The following facts hold.*
i) $\ker J'(\omega_z) = T_\omega\mathcal{S}$;
ii) If $J_0'(\omega_z)\varphi \in T_\omega\mathcal{S}$, then $\varphi \in T_\omega\mathcal{S}$;
iii) For any $u \in T_\omega\mathcal{S}^\perp$ there exists a unique $\varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2) \cap T_\omega\mathcal{S}^\perp$ such that $J_0'(\omega_z)\varphi = u$.

Proof. We start by studying the kernel of the operator B in (3.14). In coordinates, the linear problem $Bg = 0$ becomes

$$-g_1'' + kR_kg_2' = 0\ , \quad -g_2'' - kR_kg_1' + R_k^2\Big(g_2 - k^2 \fint_{\mathbb{S}^1} g_2dx\Big) = 0\,,$$

that is clearly equivalent to

$$\fint_{\mathbb{S}^1} g_2 dx = 0, \quad -g_1'' + kR_k g_2' = 0\ , \quad -g_2'' - kR_k g_1' + R_k^2 g_2 = 0 \tag{3.18}$$

because $k > 1$. The system (3.18) can be studied via elementary techniques. The conclusion is that $\ker B = \langle e_1, \gamma, \gamma' \rangle$, where $\gamma = \frac{1}{R_k}\big(\, kR_k x_1\, , -x_2\big)$. Since $\Phi(e_1) = \omega'$, $\Phi(\gamma) = \omega$ and $\Phi(\gamma') = e_1 - \omega'$, thanks to Lemma 3.2 we have

$$\ker J_0'(\omega_z) = \Phi(\ker\, B) = \Phi\big(\langle e_1, \gamma, \gamma' \rangle\big) = T_\omega \mathcal{S},$$

and the first claim is proved.

Now we prove *ii*). If $\tau := J_0'(\omega_z)\varphi \in T_\omega\mathcal{S} = \ker J'(\omega_z)$, then $J_0'(\omega_z)\tau = 0$. Taking (3.13) into account, we obtain

$$\fint_{\mathbb{S}^1} |J_0'(\omega_z)\varphi|^2\, dx = \fint_{\mathbb{S}^1} J_0'(\omega_z)\varphi \cdot \tau\, dx = \fint_{\mathbb{S}^1} J_0'(\omega_z)\tau \cdot \varphi\, dx = 0.$$

Thus $J_0'(\omega_z)\varphi = 0$, that means $\varphi \in T_\omega\mathcal{S}$.

It remains to prove *iii*). If $u \in T_\omega\mathcal{S}^\perp$, then $\Phi^{-1}(\omega_2^2 u)$ is orthogonal to $\ker B$ by *ii*) in Lemma 3.2. One can compute the Fourier coefficients of the unique solution $g_u \in \ker B^\perp$ of the system $Bg_u = \Phi^{-1}(\omega_2^2 u)$. Then $J_0'(\omega)(z_2^2\Phi(g_u)) = u$ by *i*) in Lemma 3.2. The function φ defined as the L^2-projection of $z_2^2\Phi(g_u)$ on $T_\omega\mathcal{S}^\perp$ solves $J_0'(\omega)\varphi = u$ as well, and is uniquely determined by u.

The lemma is completely proved. □

4 The perturbed problem

Let $k > 1$, $K \in C^1(\mathbb{H}^2)$ be given, and let $\varepsilon \in \mathbb{R}$ be a varying parameter. In this section we study the system

$$u'' - u_2^{-1}\Gamma(u') = L(u)(k + \varepsilon K(u))\, iu'\, . \tag{$\mathcal{P}_{k+\varepsilon K}$}$$

We start with a necessary condition for the existence of solutions to $(\mathcal{P}_{k+\varepsilon K})$ having some prescribed behavior as $\varepsilon \to 0$.

Theorem 4.1 *Let $k > 1$, $K \in C^1(\mathbb{H}^2)$, and $\varepsilon_h \to 0$ be given. For any integer h, let $u_h \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2$ be a solution to*

$$u_h'' = (u_h)_2^{-1}\Gamma(u_h') + L(u_h)(k + \varepsilon_h K(u_h))\, iu_h', \tag{$\mathcal{P}_{\varepsilon_h}$}$$

and assume that

$$L(u_h) \to L_\infty > 0, \qquad u_h \to U \text{ uniformly, for some } U \in C^0(\mathbb{S}^1, \mathbb{H}^2).$$

Then there exist $\mu \in \mathbb{N}$, $\xi \in \mathbb{S}^1$ and a critical point $z \in \mathbb{H}^2$ for F_k^K, such that $U(x) = \omega_z\big(\xi x^\mu\big)$.

Proof. We have $|u_h'| \equiv L(u_h)(u_h)_2$, thus the sequence $|u_h'|$ is uniformly bounded. It follows that u_h'' is uniformly bounded as well, because u_h solves $(\mathcal{P}_{\varepsilon_h})$. Thus, u_h' is bounded in $C^{0,s}$ for any $s \in (0,1)$ and using $(\mathcal{P}_{\varepsilon_h})$ again we infer that the sequence u_h converges in $C^{2,s}$ for any $s \in (0,1)$. In particular, $U \in C^2(\mathbb{S}^1, \mathbb{H}^2)$, $L_\infty = L(U)$ and U solves

$$U'' = U_2^{-1}\Gamma(U') + L(U)k\, iU'.$$

Lemma 3.1 applies and gives the existence of $\xi \in \mathbb{S}^1$, $z \in \mathbb{H}^2$, $\mu \in \mathbb{N}$ such that $U(x) = \omega_z(\xi x^\mu)$ and $L_\infty = L(U) = \mu L(\omega)$.

It remains to prove that z is a critical point for F_k^K. We rewrite $(\mathcal{P}_{\varepsilon_h})$ in the form

$$J_0(u_h) + \varepsilon_h L(u_h)(u_h)_2^{-2} K(u_h)\, iu_h' = 0, \tag{4.1}$$

see (3.10). Then we test (4.1) with the functions e_1 and u_h. Taking (3.12) into account, we find

$$\fint_{\mathbb{S}^1} (u_h)_2^{-2} K(u_h)\, e_1 \cdot iu_h'\, dx = 0\ , \quad \fint_{\mathbb{S}^1} (u_h)_2^{-2} K(u_h)\, u_h \cdot iu_h'\, dx = 0.$$

Since $u_h \to U(x) = \omega_z(\xi x^\mu)$, in the limit as $h \to \infty$ we obtain

$$\mu \fint_{\mathbb{S}^1} (\omega_z)_2^{-2} K(\omega_z)\, e_1 \cdot i\omega_z'\, dx = 0\ , \quad \mu \fint_{\mathbb{S}^1} (\omega_z)_2^{-2} K(\omega_z)\, \omega_z \cdot i\omega_z'\, dx = 0\,,$$

that is,

$$\partial_{z_1} A_K(\omega_z) = A_K'(\omega_z)e_1 = 0\ , \quad \partial_{z_2} A_K(\omega_z) = A_K'(\omega_z)\omega = 0.$$

Thus z is a critical point for F_k^K because of (3.6). □

4.1 Finite dimensional reduction

By Lemma 2.1, $k + \varepsilon K$-loops are the critical points of the functional

$$E_{k+\varepsilon K}(u) = E_k(u) + \varepsilon A_K(u) = L(u) + kA_1(u) + \varepsilon A_K(u)\ , \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2\,.$$

We introduce the C^1 function $J_\varepsilon : C^2(\mathbb{R}, \mathbb{H}^2) \setminus \mathbb{H}^2 \to C^0(\mathbb{R}, \mathbb{H}^2)$,

$$\begin{aligned} J_\varepsilon(u) &= J_0(u) + \varepsilon L(u) u_2^{-2} K(u)\, iu' \\ &= u_2^{-2}\big(-u'' + u_2^{-1}\Gamma(u') + L(u)(k + \varepsilon K(u))\, iu' \big)\,, \end{aligned}$$

compare with (3.10), so that

$$L(u)\, E_{k+\varepsilon K}'(u)\varphi = \fint_{\mathbb{S}^1} J_\varepsilon(u) \cdot \varphi\, dx\ , \quad u \in C^2(\mathbb{S}^1, \mathbb{H}^2),\ \varphi \in C^2(\mathbb{S}^1, \mathbb{R}^2). \tag{4.2}$$

We will look for critical points for $E_{k+\varepsilon K}$ by solving the problem $J_\varepsilon(u) = 0$.

First, we notice that $E_{k+\varepsilon K}(u\circ\xi) = E_{k+\varepsilon K}(u)$ for any $\xi\in\mathbb{S}^1$, that implies

$$\fint_{\mathbb{S}^1} J_\varepsilon(u)\cdot u'\,dx = 0 \qquad \text{for any } \varepsilon\in\mathbb{R},\ u\in C^2(\mathbb{S}^1,\mathbb{H}^2)\setminus\mathbb{H}^2. \tag{4.3}$$

In the next crucial lemma we carry out the Lyapunov-Schmidt procedure, in which we take advantage of the variational structure of problem $(\mathcal{P}_{k+\varepsilon K})$.

Lemma 4.2 *Let $\Omega\Subset\mathbb{H}^2$ be a given open set. There exist $\overline{\varepsilon}>0$ and a C^1 function*

$$[-\overline{\varepsilon},\overline{\varepsilon}]\times\overline{\Omega}\to C^2(\mathbb{S}^1,\mathbb{H}^2)\setminus\mathbb{H}^2\ , \quad (\varepsilon,z)\mapsto u_z^\varepsilon$$

such that the following facts hold.

i) *u_z^ε is an embedded loop and $u_z^0=\omega_z$;*

ii) *$u_z^\varepsilon-\omega_z\in T_\omega\mathcal{S}^\perp$;*

iii) *$J_\varepsilon(u_z^\varepsilon)\in T_\omega\mathcal{S}$. More precisely,*

$$\frac{1}{L(u_z^\varepsilon)}\,J_\varepsilon(u_z^\varepsilon) = \partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon))\,e_1 + \Big(\fint_{\mathbb{S}^1}|\omega|^2\,dx\Big)^{-1}\partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon))\,\omega\,; \tag{4.4}$$

iv) *As $\varepsilon\to0$, we have*

$$E_{k+\varepsilon K}(u_z^\varepsilon)-E_{k+\varepsilon K}(\omega_z)=o(\varepsilon) \tag{4.5}$$

uniformly on Ω, together with the derivatives with respect to the variable z.

Proof. In order to shorten formulae, for $r>0$, $m\in\{0,2\}$ and $\delta>0$ we write

$$\Omega_r=\{z\in\mathbb{R}^2\mid \operatorname{dist}(z,\Omega)<r\}\,,$$
$$C^m=C^m(\mathbb{S}^1,\mathbb{R}^2)\,,\quad \mathcal{U}_\delta:=\{\eta\in C^2\mid |\eta(x)|<\delta \ \text{ for any } x\in\mathbb{S}^1\ \}\,.$$

Take $r,\delta>0$ small enough, so that $\overline{\Omega}_{2r}\subset\mathbb{H}^2$ and $\omega_z+\eta\in C^2(\mathbb{S}^1,\mathbb{H}^2)\setminus\mathbb{H}^2$ for any $z\in\overline{\Omega}_{2r}$, $\eta\in\mathcal{U}_\delta$. Consider the differentiable function

$$\mathcal{F}:(\mathbb{R}\times\Omega_{2r})\times\mathcal{U}_\delta\times(\mathbb{R}\times\mathbb{R}^2)\to C^0\times(\mathbb{R}\times\mathbb{R}^2)\ ,\quad \mathcal{F}=\big(\mathcal{F}_1,\mathcal{F}_2\big),$$

whose coordinates

$$\mathcal{F}_1:(\mathbb{R}\times\Omega_{2r})\times\mathcal{U}_\delta\times(\mathbb{R}\times\mathbb{R}^2)\to C^0\,,\quad \mathcal{F}_2:(\mathbb{R}\times\Omega_{2r})\times\mathcal{U}_\delta\times(\mathbb{R}\times\mathbb{R}^2)\to\mathbb{R}\times\mathbb{R}^2$$

are given by

$$\begin{aligned}\mathcal{F}_1(\varepsilon,z;\eta;t,\vartheta) &= J_\varepsilon(\omega_z+\eta)-t\omega'-\vartheta_1e_1-\vartheta_2\omega,\\ \mathcal{F}_2(\varepsilon,z;\eta;t,\vartheta) &= \Big(\fint_{\mathbb{S}^1}\eta\cdot\omega'\,dx,\fint_{\mathbb{S}^1}\eta_1\,dx,\fint_{\mathbb{S}^1}\eta\cdot\omega\,dx\Big).\end{aligned}$$

Take $z \in \Omega_{2r}$ and notice that $\mathcal{F}(0, z; 0; 0, 0) = 0$ because $J_0(\omega_z) = 0$. The next goal is to solve the equation $\mathcal{F}(\varepsilon, z; \eta; t, \vartheta) = (0, 0)$ in a neighborhood of $(\varepsilon, z) = (0, z)$, $(\eta; t, \vartheta) = (0; 0, 0)$ via the implicit function theorem. Let

$$\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \to C^0 \times (\mathbb{R} \times \mathbb{R}^2)$$

be the differential of $\mathcal{F}(0, z; \cdot\,; \cdot\,, \cdot\,)$ computed at $(\eta; t, \vartheta) = (0; 0, 0) \in C^2 \times (\mathbb{R}\times\mathbb{R}^2)$. We need to prove that $\mathcal{L}$ is invertible. Explicitly, we have

$$\mathcal{L}_1 : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \to C^0, \qquad \mathcal{L}_1(\varphi; a, p) = J_0'(\omega_z)\varphi - a\omega' - p_1 e_1 - p_2\omega$$

$$\mathcal{L}_2 : C^2 \times (\mathbb{R} \times \mathbb{R}^2) \to \mathbb{R} \times \mathbb{R}^2, \quad \mathcal{L}_2(\varphi; a, p) = \Big(\fint_{\mathbb{S}^1} \varphi \cdot \omega'\, dx, \fint_{\mathbb{S}^1} \varphi_1\, dx, \fint_{\mathbb{S}^1} \varphi \cdot \omega\, dx \Big).$$

If $\mathcal{L}_1(\varphi; a, p) = 0$ then $J_0'(\omega_z)\varphi \in T_\omega\mathcal{S}$, hence $\varphi \in T_\omega\mathcal{S}$ by *ii)* in Lemma 3.3. If $\mathcal{L}_2(\varphi; a, p) = 0$ then $\varphi \in T_\omega\mathcal{S}^\perp$. Therefore, the operator $\mathcal{L}$ is injective.

To prove surjectivity take $u \in C^0, (b, q) \in \mathbb{R} \times \mathbb{R}^2$. We have to find $\varphi \in C^2$, $(a, p) \in \mathbb{R} \times \mathbb{R}^2$ satisfying $\mathcal{L}_1(\varphi; a, p) = u$ and $\mathcal{L}_2(\varphi; a, p) = (b, q_1, q_2)$, that is,

$$J_0'(\omega_z)\varphi = u + a\omega' + p_1 e_1 + p_2\omega \tag{4.6}$$

$$\fint_{\mathbb{S}^1} \varphi \cdot \omega'\, dx = b\ , \quad \fint_{\mathbb{S}^1} \varphi_1\, dx = q_1\ , \quad \fint_{\mathbb{S}^1} \varphi \cdot \omega\, dx = q_2. \tag{4.7}$$

By (3.13), for any $\varphi \in C^2$, $\tau \in T_\omega\mathcal{S} = \langle \omega', e_1, \omega \rangle = \ker J_0'(\omega_z)$ we have

$$\fint_{\mathbb{S}^1} J_0'(\omega_z)\varphi \cdot \tau\, dx = \fint_{\mathbb{S}^1} J_0'(\omega_z)\tau \cdot \varphi\, dx = 0.$$

Thus the unknowns $a \in \mathbb{R}$ and $p = (p_1, p_2) \in \mathbb{R}^2$ are determined by the condition

$$\fint_{\mathbb{S}^1} u \cdot \tau\, dx + a \fint_{\mathbb{S}^1} \omega' \cdot \tau\, dx + p_1 \fint_{\mathbb{S}^1} e_1 \cdot \tau\, dx + p_2 \fint_{\mathbb{S}^1} \omega \cdot \tau\, dx = 0 \quad \text{for any } \tau \in T_\omega\mathcal{S}. \tag{4.8}$$

Now we look for the L^2 projection of the unknown function φ on $T_\omega\mathcal{S}$ and its L^2 projection on $T_\omega\mathcal{S}^\perp$. The tangential component $\varphi^\top \in T_\omega\mathcal{S} = \langle \omega', e_1, \omega \rangle$ is uniquely determined by (4.7). Next, we notice that $u + a\omega' + p_1 e_1 + p_2\omega \in T_\omega\mathcal{S}^\perp$ by (4.8); then we use *iii)* in Lemma 3.3 to find $\varphi^\perp \in C^2 \cap T_\omega\mathcal{S}^\perp$ such that

$$J_0'(\omega_z)\varphi^\perp = u + a\omega' + p_1 e_1 + p_2\omega\,.$$

The function $\varphi = \varphi^\top + \varphi^\perp$ solves (4.6) because $J_0'(\omega_z)\varphi = J_0'(\omega_z)\varphi^\perp$, and surjectivity is proved.

We can now apply the implicit function theorem for any fixed $z \in \Omega_{2r}$. Actually, thanks a compactness argument we have that there exist $\varepsilon' > 0$ and (uniquely determined) C^1 functions

$$\begin{array}{lll} \eta : (-\varepsilon', \varepsilon') \times \Omega_r \to \mathcal{U}_\delta \subset C^2 & t : (-\varepsilon', \varepsilon') \times \Omega_r \to \mathbb{R} & \vartheta : (-\varepsilon', \varepsilon') \times \Omega_r \to \mathbb{R}^2 \\ \eta : (\varepsilon, z) \mapsto \eta^\varepsilon(z) & t : (\varepsilon, z) \mapsto t^\varepsilon(z), & \vartheta : (\varepsilon, z) \mapsto \vartheta^\varepsilon(z) \end{array}$$

such that

$$\eta^0(z) = 0 \ , \quad t^0(z) = 0 \ , \quad \vartheta^0(z) = 0 \ , \quad \mathcal{F}(\varepsilon, z; \eta^\varepsilon(z); t^\varepsilon(z), \vartheta^\varepsilon(z)) = 0.$$

We introduce the C^1 function

$$(-\varepsilon', \varepsilon') \times \Omega_r \to C^2(\mathbb{S}^1, \mathbb{H}^2) \setminus \mathbb{H}^2 \ , \quad (\varepsilon, z) \mapsto u_z^\varepsilon := \omega_z + \eta^\varepsilon(z) \, ,$$

that clearly satisfies $u_z^0 = \omega_z$. Since ω_z is embedded, then u_z^ε is embedded as well, provided that ε' is small enough. Moreover we have

$$J_\varepsilon(u_z^\varepsilon) = t^\varepsilon(z)\omega' + \vartheta_1^\varepsilon(z) e_1 + \vartheta_2^\varepsilon(z)\omega \in T_\omega \mathcal{S} \tag{4.9}$$

$$\fint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot \omega' \, dx = \fint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot e_1 \, dx = \fint_{\mathbb{S}^1} (u_z^\varepsilon - \omega_z) \cdot \omega \, dx = 0, \tag{4.10}$$

and (4.10) shows that $ii)$ is fulfilled.

Since integration by parts gives

$$\fint_{\mathbb{S}^1} \omega_z \cdot \omega' \, dx = 0 \ , \quad \fint_{\mathbb{S}^1} \omega_z \cdot e_1 \, dx = z_1 \ , \quad \fint_{\mathbb{S}^1} \omega_z \cdot \omega \, dx = z_2 \fint_{\mathbb{S}^1} |\omega|^2 \, dx \, ,$$

we can rewrite the orthogonality conditions (4.10) in the following, equivalent way:

$$\fint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega' \, dx = 0 \ , \quad \fint_{\mathbb{S}^1} u_z^\varepsilon \cdot e_1 \, dx = z_1 \ , \quad \fint_{\mathbb{S}^1} u_z^\varepsilon \cdot \omega \, dx = z_2 \fint_{\mathbb{S}^1} |\omega|^2 \, dx. \tag{4.11}$$

Our next aim is to show that $t^\varepsilon(z) = 0$ for any $z \in \overline{\Omega}$, provided that ε is small enough. We have that $\|(u_z^\varepsilon)' - \omega_z'\|_\infty = o(1)$ as $\varepsilon \to 0$, uniformly for $z \in \overline{\Omega}$. Thus

$$\fint_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' \, dx = \fint_{\mathbb{S}^1} \omega_z' \cdot \omega' \, dx + o(1) = z_2 \fint_{\mathbb{S}^1} |\omega'|^2 \, dx + o(1).$$

In particular, there exists $\overline{\varepsilon} \in (0, \varepsilon')$ such that $\int_{\mathbb{S}^1} (u_z^\varepsilon)' \cdot \omega' \, dx$ is bounded away from 0 if $(\varepsilon, z) \in [-\overline{\varepsilon}, \overline{\varepsilon}] \times \overline{\Omega}$. On the other hand, using (4.3), (4.9), integration by parts and (4.11), we have

$$
\begin{aligned}
0 &= \fint_{\mathbb{S}^1} J_\varepsilon(u_z^\varepsilon)\cdot (u_z^\varepsilon)'\,dx \\
&= t^\varepsilon(z)\fint_{\mathbb{S}^1} (u_z^\varepsilon)'\cdot\omega'\,dx + \vartheta_1^\varepsilon(z)\fint_{\mathbb{S}^1}(u_z^\varepsilon)'\cdot e_1\,dx + \vartheta_2^\varepsilon(z)\fint_{\mathbb{S}^1}(u_z^\varepsilon)'\cdot\omega\,dx \\
&= t^\varepsilon(z)\fint_{\mathbb{S}^1} (u_z^\varepsilon)'\cdot\omega'\,dx - \vartheta_2^\varepsilon(z)\fint_{\mathbb{S}^1} u_z^\varepsilon\cdot\omega'\,dx = t^\varepsilon(z)\fint_{\mathbb{S}^1}(u_z^\varepsilon)'\cdot\omega'\,dx.
\end{aligned}
$$

We see that $t^\varepsilon(z) = 0$ for any $(\varepsilon, z) \in [-\overline{\varepsilon}, \overline{\varepsilon}] \times \overline{\Omega}$, and therefore

$$
J_\varepsilon(u_z^\varepsilon) = \vartheta_1^\varepsilon(z)e_1 + \vartheta_2^\varepsilon(z)\omega\,. \tag{4.12}
$$

Now we compute the derivatives of the function $z \mapsto E_{k+\varepsilon K}(u_z^\varepsilon)$ via (4.2) and (4.12). For $j = 1, 2$ we obtain

$$
\begin{aligned}
L(u_z^\varepsilon)\partial_{z_j}(E_{k+\varepsilon K}(u_z^\varepsilon)) &= L(u_z^\varepsilon)E'_{k+\varepsilon K}(u_z^\varepsilon)\partial_{z_j}u_z^\varepsilon = \fint_{\mathbb{S}^1} J_\varepsilon(u_z^\varepsilon)\cdot\partial_{z_j}u_z^\varepsilon\,dx \\
&= \vartheta_1^\varepsilon(z)\fint_{\mathbb{S}^1}\partial_{z_j}u_z^\varepsilon\cdot e_1\,dx + \vartheta_2^\varepsilon(z)\fint_{\mathbb{S}^1}\partial_{z_j}u_z^\varepsilon\cdot\omega\,dx \\
&= \vartheta_1^\varepsilon(z)\partial_{z_j}\Big(\fint_{\mathbb{S}^1} u_z^\varepsilon\cdot e_1\,dx\Big) + \vartheta_2^\varepsilon(z)\partial_{z_j}\Big(\fint_{\mathbb{S}^1} u_z^\varepsilon\cdot\omega\,dx\Big).
\end{aligned}
$$

Then we use (4.11) to infer

$$
L(u_z^\varepsilon)\partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon)) = \vartheta_1^\varepsilon(z)\,, \quad L(u_z^\varepsilon)\partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon)) = \vartheta_2^\varepsilon(z)\Big(\fint_{\mathbb{S}^1}|\omega|^2\,dx\Big)\,,
$$

that compared with (4.12) give (4.4).

It remains to prove $iv)$. Take $z \in \overline{\Omega}$ and consider the function

$$
f_z(\varepsilon) = E_{k+\varepsilon K}(u_z^\varepsilon) = E_k(u_z^\varepsilon) + \varepsilon A_K(u_z^\varepsilon)\,, \quad f_z \in C^1(-\overline{\varepsilon}, \overline{\varepsilon}).
$$

Clearly $f_z(0) = E_k(\omega_z)$. To compute $f'_z(0)$ notice that $\partial_\varepsilon u_z^\varepsilon$ remains bounded in $C^2(\overline{\Omega})$ as $\varepsilon \to 0$, because the function $(\varepsilon, z) \mapsto u_z^\varepsilon$ is of class C^1. Thus $A'_K(u_z^\varepsilon)(\partial_\varepsilon u_z^\varepsilon)$ remains bounded as well. Further, $E'_k(u_z^\varepsilon) \to E'_k(\omega_z) = 0$ in the norm operator because $u_z^\varepsilon \to \omega_z$ in C^2 and since ω_z is a k-loop. We infer that

$$
f'_z(0) = E'_k(\omega_z)(\partial_\varepsilon u_z^\varepsilon) + A_K(u_z^\varepsilon) + o(1) = A_K(\omega_z) + o(1)
$$

uniformly on $\overline{\Omega}$. In fact we proved that

$$
f_z(\varepsilon) = E_{k+\varepsilon K}(u_z^\varepsilon) = E_k(\omega_z) + \varepsilon A_K(u_z^\varepsilon) + o(1)
$$

uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$. That is, (4.5) holds true "at the zero order".

To conclude the proof we have to handle $\partial_{z_j}\big(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)\big)$ for $j = 1, 2$. Since $J_\varepsilon(u) = J_0(u) + \varepsilon L(u) u_2^{-2} K(u)\, iu'$, we can rewrite (4.4) as follows,

$$\partial_{z_1}(E_{k+\varepsilon K}(u_z^\varepsilon))\, e_1 + \Big(\fint_{\mathbb{S}^1} |\omega|^2\, dx\Big)^{-1} \partial_{z_2}(E_{k+\varepsilon K}(u_z^\varepsilon))\, \omega$$
$$= \frac{1}{L(u_z^\varepsilon)} J_0(u_z^\varepsilon) + \varepsilon (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) i (u_z^\varepsilon)'. \tag{4.13}$$

Recall that $J_0(u_z^\varepsilon)$ is orthogonal to e_1 in L^2, see the second identity in (3.12). We test (4.13) with e_1 to obtain

$$\partial_{z_1}\big(E_{k+\varepsilon K}(u_z^\varepsilon)\big) = \varepsilon \fint_{\mathbb{S}^1} (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) e_1 \cdot i(u_z^\varepsilon)'\, dx = \varepsilon A_K'(u_z^\varepsilon) e_1 \tag{4.14}$$

by (2.5). Since $\partial_{z_1}\big(E_{k+\varepsilon K}(\omega_z)\big) = \partial_{z_1}\big(E_k(\omega) + \varepsilon A_K(\omega_z)\big) = \varepsilon A_K'(\omega_z) e_1$, we get

$$\partial_{z_1}\big(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)\big) = \varepsilon\big(A_K'(u_z^\varepsilon) e_1 - A_K'(\omega_z) e_1\big) = o(\varepsilon)$$

because of the continuity of $A_K'(\cdot)$ and since $u_z^\varepsilon \to \omega_z$.

To handle the derivative with respect to z_2 we test (4.13) with u_z^ε. Since $J_0(u_z^\varepsilon)$ is orthogonal to u_z^ε in L^2 by (3.12), using also (4.11) we obtain

$$z_1 \partial_{z_1}\big(E_{k+\varepsilon K}(u_z^\varepsilon)\big) + z_2 \partial_{z_2}\big(E_{k+\varepsilon K}(u_z^\varepsilon)\big) = \varepsilon \fint_{\mathbb{S}^1} (u_z^\varepsilon)_2^{-2} K(u_z^\varepsilon) u_z^\varepsilon \cdot i(u_z^\varepsilon)'\, dx = \varepsilon A_K'(u_z^\varepsilon) u_z^\varepsilon\,,$$

that compared with (4.14) gives

$$z_2 \partial_{z_2}\big(E_{k+\varepsilon K}(u_z^\varepsilon)\big) = \varepsilon A_K'(u_z^\varepsilon)(u_z^\varepsilon - z_1 e_1)\,.$$

From $z_2 \partial_{z_2}\big(E_{k+\varepsilon K}(\omega_z)\big) = z_2 \partial_{z_2}\big(E_k(\omega) + \varepsilon A_K(\omega_z)\big) = z_2 \varepsilon A_K'(\omega_z)\omega = \varepsilon A_K'(\omega_z)(\omega_z - z_1 e_1)$, we conclude that

$$z_2 \partial_{z_2}\big(E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)\big) = \varepsilon\big(A_K'(u_z^\varepsilon)(u_z^\varepsilon - z_1 e_1) - A_K'(\omega_z)(\omega_z - z_1 e_1)\big) = o(\varepsilon)\,.$$

The lemma is completely proved. □

4.2 Existence results

Proof of Theorem 1.1. We are assuming that there exists $r > 0$ such that any function $G \in C^1(\overline{A})$ satisfying $\|G + F_k^K\|_{C^1(\overline{A})} < r$ has a critical point in A. We recall also formula (3.6), that in particular gives

$$E_{k+\varepsilon K}(\omega_z) = E_k(\omega_z) + \varepsilon A_K(\omega_z) = E_k(\omega) - \frac{\varepsilon}{2\pi} F_k^K(z)\,. \tag{4.15}$$

Take an open set $\Omega \in \mathbb{H}^2$ such that $A \Subset \Omega \Subset \mathbb{H}^2$, and let $(\varepsilon, z) \mapsto u_z^\varepsilon$, $(\varepsilon, z) \in [-\overline{\varepsilon}, \overline{\varepsilon}] \times \overline{\Omega}$ be the function given by Lemma 4.2. For $\varepsilon \neq 0$ consider the function

$$G^\varepsilon(z) = \frac{2\pi}{\varepsilon}(E_{k+\varepsilon K}(u_z^\varepsilon) - E_k(\omega))$$

and use (4.15) together with $iv)$ in Lemma 4.2 to get

$$\|G^\varepsilon + F_k^K\|_{C^1(\overline{A})} = \frac{2\pi}{|\varepsilon|}\big\|E_{k+\varepsilon K}(u_z^\varepsilon) - E_{k+\varepsilon K}(\omega_z)\big\| = o(1)$$

as $\varepsilon \to 0$. We see that for ε small enough the function G^ε has a critical point $z^\varepsilon \in A$. Since the derivatives of the function $z \mapsto E_{k+\varepsilon K}(u_z^\varepsilon)$ vanish at $z = z^\varepsilon$, then $J_\varepsilon(u_{z^\varepsilon}^\varepsilon) = 0$ by (4.4). That is, $u_{z^\varepsilon}^\varepsilon$ is and embedded $k + \varepsilon K$ loop.

The last conclusion in Theorem 1.1 follows via a simple compactness argument and thanks to Theorem 4.1. □

In the next result we apply Theorem 1.1 to obtain the existence of $k + \varepsilon K$-loops that shrink to a stable critical point for the curvature function K, as $k \to \infty$.

Theorem 4.3 *Let $K \in C^1(\mathbb{H}^2)$. Assume that K has a stable critical point in an open set $A \Subset \mathbb{H}^2$. There exists $k_0 > 1$ such that for any $k > k_0$ and for every ε close enough to 0, there exists an embedded $(k + \varepsilon K)$-loop.*

Moreover, let $k_h \to \infty, \varepsilon_h \to 0$ be given sequences. There exist subsequences $k_{h_j}, \varepsilon_{h_j}$, a point $z_\infty \in \overline{A}$ that is critical for K, and an embedded $(k_{h_j} + \varepsilon_{h_j} K)$-loop u^j such that u^j converges in $C^2(\mathbb{S}^1, \mathbb{H}^2)$ to the constant curve z_∞, as $h \to \infty$.

Proof. Recall that $R_k = (k^2 - 1)^{-1/2}$. In order to simplify notations we put

$$z^k := (z_1, kR_k z_2) = z + (kR_k - 1)z_2 e_2 \quad \text{for } z = (z_1, z_2) \in \mathbb{H}^2.$$

Since $D_{\rho_k}^{\mathbb{H}}(z) = D_{R_k z_2}(z^k)$ we have

$$F_k^K(z) = \int\limits_{D_{R_k z_2}(z^k)} p_2^{-2} K(p)\, dp = \int\limits_{D_{R_k}(0)} (q_2 + kR_k)^{-2} K(z_2 q + z^k)\, dq\,. \tag{4.16}$$

We put $\phi_K(q) = q_2^{-2} K(q)$ and rewrite (4.16) as follows:

$$\frac{1}{\pi R_k^2 z_2^2}\, F_k^K(z) = \fint\limits_{D_{R_k}(0)} \phi_K(z_2 q + z^k)\, dq\,.$$

Trivially $kR_k = k/\sqrt{k^2 - 1} \to 1$ and $|z^k - z| = (kR_k - 1)z_2 \to 0$ uniformly on $\overline{A}$, as $k \to \infty$. Since $\phi_K \in C^1(\mathbb{H}^2)$, it is easy to show that

$$\frac{1}{\pi R_k^2 z_2^2}\, F_k^K(z) \to \phi_K(z) = \frac{1}{z_2^2} K(z)$$

in $C^1(\overline{A})$. It follows that for k large enough, F_k^K has stable critical point in $A \Subset \mathbb{H}^2$. Theorem 1.1 applies and gives the conclusion of the proof. □

A Loops in the Euclidean plane

The argument we used to prove Theorem 1.1 applies also in the easier Euclidean case. It is well known that the only embedded loops in $\mathbb{R}^2$ having prescribed constant curvature $k > 0$ are circles of radius $1/k$. We take as a reference circle the loop

$$\omega(x) = \frac{1}{k}\, x \;, \qquad x \in \mathbb{S}^1 \subset \mathbb{R}^2,$$

that solves

$$u'' = L(u) k\, iu' \;, \quad \text{where} \quad L(u) := \Big(\,\fint_{\mathbb{S}^1} |u'|^2 \, dx \Big)^{\frac{1}{2}}$$

(in fact, $L(\omega)k = 1$ and $\omega'' = -\omega = i\omega'$).

Let $K \in C^1(\mathbb{R}^2)$ be given. If a nonconstant function $u \in C^2(\mathbb{S}^1, \mathbb{R}^2)$ solves

$$u'' = L(u)(k + \varepsilon K(u))\, iu' \,, \tag{A.1}$$

then $|u'| = L(u)$ is constant, and u parameterizes a loop in $\mathbb{R}^2$ having Euclidean curvature $k + \varepsilon K$ at each point. Further, problem (A.1) admits a variational structure, see [5], [18]. More precisely, its nonconstant solutions are critical points of the energy functional

$$E_{k+\varepsilon K}(u) = \Big(\,\fint_{\mathbb{S}^1} |u'|^2 \, dx \Big)^{\frac{1}{2}} + \varepsilon \fint_{\mathbb{S}^1} Q(u) \cdot iu' \;, \quad u \in C^2(\mathbb{S}^1, \mathbb{R}^2) \setminus \mathbb{R}^2,$$

where the vectorfield $Q \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies $\text{div}Q = K$.

Arguing as for Theorem 4.1 one can prove a necessary conditions for the existence of solutions to (A.1) for $\varepsilon = \varepsilon_h \to 0$.

Theorem A.1 *Let u_h be a $(k + \varepsilon_h K)$-loop solving (A.1) for $\varepsilon = \varepsilon_h$, and assume that*

$$L(u_h) \to L_\infty > 0, \qquad u_h \to U \text{ uniformly, for some } U \in C^0(\mathbb{S}^1, \mathbb{R}^2).$$

Then $U(x) = \omega\big(\xi x^\mu\big) + z$ for some $\mu \in \mathbb{N}$, $\xi \in \mathbb{S}^1$ and $z \in \mathbb{R}^2$, that is a critical point for the Melnikov function

$$F^K_k(z) = \int_{D_{\frac{1}{k}}(z)} K(q)\, dq \,, \quad F^K_k : \mathbb{R}^2 \to \mathbb{R} \,.$$

In the Euclidean case we have the following existence result.

Theorem A.2 *Let $k > 0$ and $K \in C^1(\mathbb{R}^2)$ be given. Assume that F^K_k has a stable critical point in an open set $A \Subset \mathbb{R}^2$. Then for every $\varepsilon \in \mathbb{R}$ close enough to 0, there exists an embedded $(k + \varepsilon K)$-loop $u^\varepsilon : \mathbb{S}^1 \to \mathbb{R}^2$.*

Moreover, any sequence $\varepsilon_h \to 0$ has a subsequence ε_{h_j} such that $u^{\varepsilon_{h_j}} \to \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ as $j \to \infty$, where $z_0 \in A$ is a critical point for F^K_k.

Sketch of the proof. We introduce the 3-dimensional space of embedded solutions to the unperturbed problem, namely

$$\mathcal{S} = \{ \omega \circ \xi + z \mid \xi \in \mathbb{S}^1 \ , \quad z \in \mathbb{R}^2 \ \},$$

and the functions $J_\varepsilon : C^2(\mathbb{R}, \mathbb{R}^2) \setminus \mathbb{R}^2 \to C^0(\mathbb{R}, \mathbb{R}^2)$, $\varepsilon \in \mathbb{R}$, given by

$$J_\varepsilon(u) = -u'' + L(u)(k + \varepsilon K(u))\, iu' = J_0(u) + L(u)K(u)\, iu'.$$

We have $\mathcal{S} \subset \{J_0 = 0\}$. Since $J_0'(\omega + z)\varphi = -\varphi'' + i\varphi' - k^2 \big(\displaystyle\fint_{\mathbb{S}^1} \varphi \cdot \omega \, dx \big)\omega$, it is quite easy to check that

$$T_{\omega+z}\mathcal{S} = \langle \omega', e_1, e_2 \rangle = \ker J_0'(\omega + z),$$

and that $J_0'(\omega + z) : T_{\omega+z}\mathcal{S}^\perp \to T_{\omega+z}\mathcal{S}^\perp$ is invertible. The remaining part of the proof runs with minor changes. $\square$

Theorem 4.3 has its Euclidean correspondent as well. We omit the proof of the next result.

Theorem A.3 *Let $K \in C^1(\mathbb{R}^2)$. Assume that K has a stable critical point in an open set $A \Subset \mathbb{R}^2$. Then there exists $k_0 > 1$ such that for any fixed $k > k_0$, and for every ε close enough to 0, there exists an embedded $(k + \varepsilon K)$-loop $u^{k,\varepsilon} : \mathbb{S}^1 \to \mathbb{R}^2$.*

Moreover, there exist sequences $k_h \to \infty$, $\varepsilon_h \to 0$ such that $u^{k_h, \varepsilon_{h_j}} \to \omega_{z_0}$ in $C^2(\mathbb{S}^1, \mathbb{R}^2)$ as $j \to \infty$, where $z_0 \in A$ is a critical point for K.

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