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Corresponding Author: Professor Giovanna D'Agostino,

Corresponding Author's Institution: Univeristà degli Studi di Udine

First Author: Giovanna D'Agostino

Order of Authors: Giovanna D'Agostino; Giacomo Lenzi, Professor

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The μ-Calculus Alternation Depth Hierarchy is infinite over finite planar graphs

Giovanna D'Agostino (corresponding author)
Department of Math., Comp. Science, and Physics,
University of Udine, Via delle Scienze 206, 33100 Udine, Italy
giovanna.dagostino@uniud.it
and
Giacomo Lenzi
Department of Mathematics, University of Salerno,
Via Giovanni Paolo II 132, 84084 Fisciano (Salerno), Italy.
gilenzi@unisa.it

Abstract

We prove that the Alternation Hierarchy of the Modal μ-Calculus is infinite over finite planar graphs

Keywords Modal μ-Calculus, Alternation Hierarchy, Parity Games, Planar graphs

1 Introduction

This work is about he μ-Calculus, an extension of modal logic with least and greatest fixpoints of definable monotone operators. This logic, introduced in [10], highly increases the expressiveness of modal logic and subsumes many temporal logics used in verification of computer systems, such as CTL, CTL^*, PDL, etc. Moreover, the expressiveness of μ-Calculus is, in a sense, maximal because the μ-Calculus coincides with the fragment of monadic second order logic invariant under bisimulation (see [9]).

Understanding formulas with many alternation of least and greatest fixed points is a difficult task. Luckily, many properties used in system verification can be translated in the μ-Calculus using few fixpoints. One greatest fixpoint is sufficient for safety, one least fixpoint is sufficient for liveness, and fairness can be expressed with two alternating fixpoints. This does not mean in general that a bounded number of alternations always suffices, because for any n there are properties of graphs that need at least n alternations to be expressed in the μ-Calculus ([3, 2]). Hence, the alternation hierarchy of the μ-Calculus is infinite on the class of graphs.

One may ask what happens if we restrict the class of graphs where the μ-Calculus is interpreted. The first remark is that the μ-Calculus enjoys the finite model property, so the hierarchy remains infinite if we restrict the semantics to finite graphs. Other restrictions may change the situation drastically. One fundamental result in this direction is the De Jongh-Sambin Theorem, which (in its semantical version) says that in the class GL of all transitive well-founded graphs, the μ-Calculus collapses to modal logic. Another well known collapse applies when the relation of the graph is an equivalence: here again, μ-Calculus adds nothing to modal logic. In [1] this result is generalized to graphs which are both symmetric and transitive (removing reflexivity). There are also intermediate situations, witnessed by relaxations of the notion of equivalence relation. In [1], [7], and [5] one can find different proofs that on transitive graphs, every formula of the μ-Calculus is equivalent to a formula without alternations of fixed point operators. This status is also valid over reflexive and transitive graphs. [1] also proves that, over reflexive graphs, the hierarchy is infinite, while [6] proves the same for the class of symmetric and reflexive graphs.

In this paper we consider the class of *planar graphs*. It is well known that many graph problems have simpler solutions when considered over planar graphs, e.g. the isomorphism problem, which is in NP on the class of all finite graphs, becomes $LOGSPACE$ when we restrict to finite planar graphs [4]. In the case of the alternation hierarchy, however, we shall see that planarity does not help: we will prove that the alternation hierarchy of the μ-Calculus is strict also on finite planar graphs. To prove this result we will use (restricted) parity games because, as it is well known, these games are in a sense equivalent to the μ-Calculus formulas, with priorities corresponding to maximal number of alternations.

The paper is structured as follows. In Section 2 we fix the notation and recall some basic results on planar graphs, μ-Calculus, and parity games. In Section 3 we give some intuition on the constructions used in the paper, which will be then reintroduced formally in Section 4 where we finally give the proof of the strictness of the alternation hierarchy on planar graphs.

2 Preliminaries

2.1 Syntax and semantics of the μ-Calculus

Given a finite set of *propositions* $\mathcal{P}$, and a countable set of variables $\mathcal{V}ar$, the formulas of the μ-Calculus in $\mathcal{P} \cup \mathcal{V}ar$ are given by the following grammar:

$$\phi ::= P \mid \neg P \mid X \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Box\phi \mid \Diamond\phi \mid \mu X.\phi \mid \nu X.\phi,$$

where $P \in \mathcal{P}$ and $X \in \mathcal{V}ar$. Given a μ-formula ϕ, its free variables are defined as usual (considering μ, ν as quantifiers). The semantics of a μ-formula ϕ is given on graphs, vertex labeled by the propositions of ϕ, that is, on tuples $G = (V, R, L)$ where V is a set, R is a binary relation on V, and $L : \mathcal{P} \to Pow(V)$. Given a graph $G = (V, R, L)$ and a valuation $s : \mathcal{V}ar \to Pow(V)$ of the variables in $\mathcal{V}ar$, the set $[\![\phi]\!]_{G,s}$ is defined as follows:

$$\begin{array}{lcll}
[\![P]\!]_{G,s} & := & L(P) & \text{for } P \in \mathcal{P}; \\
[\![\neg P]\!]_{G,s} & := & V \setminus L(P) & \text{for } P \in \mathcal{P}; \\
[\![X]\!]_{G,v} & := & s(X) & \text{for } X \in \mathcal{V}ar; \\
[\![\phi \vee \psi]\!]_{G,s} & := & [\![\phi]\!]_{G,s} \cup [\![\psi]\!]_{G,v,s}; & \\
[\![\phi \wedge \psi]\!]_{G,s} & := & [\![\phi]\!]_{G,s} \cap [\![\psi]\!]_{G,v,s}; & \\
[\![\Diamond\phi]\!]_{G,s} & := & \{v \in V : R(v) \cap [\![\phi]\!]_{G,s} \neq \emptyset\}; & \\
[\![\Box\phi]\!]_{G,s} & := & \{v \in V : R(v) \subseteq [\![\phi]\!]_{G,s}\}; & \\
[\![\mu X.\phi]\!]_{G,s} & := & \bigcap\{S \subseteq V \mid [\![\phi]\!]_{G,s[X:=S]} \subseteq S\}; & \\
[\![\nu X.\phi]\!]_{G,s} & := & \bigcup\{S \subseteq V \mid [\![\phi]\!]_{G,s[X:=S]} \supseteq S\}; &
\end{array}$$

where $R(v) = \{w \in V : vRw\}$, and $s[X := S]$ is equal to s except that $s(X) = S$. Note that $[\![\mu X.\phi]\!]_{G,s}$ is the least fixpoint of the monotone operator $S \mapsto [\![\phi]\!]_{G,s[X:=S]}$, and $[\![\nu X.\phi]\!]_{G,s[X:=S]}$ is the greatest fixpoint of the same operator. Moreover, the semantics $[\![\phi]\!]_{G,s}$ does not depend on the value of s over bound variables in ϕ.

In the following, we denote $v \in [\![\phi]\!]_{G,s}$ by $(G, s, v) \models H$. If ϕ is a sentence we write simply $(G, v) \models \phi$, and if the graph G is a tree T, we use $T \models \phi$ to denote $(T, r) \models \phi$ where r is the root of T.

To classify formulas we may use the alternation of its fixed points (see e.g. [2]):

Definition 2.1. *The fixpoint hierarchy Σ_n, Π_n is defined recursively as follows.*

- $\Pi_0 = \Sigma_0$ *are the class of formulas without fixpoints;*

- Π_{n+1} *is the closure of* $\Sigma_n \cup \Pi_n$ *with respect to greatest fixpoints and composition: if* $\phi(X), \psi \in \Pi_{n+1}$*, where any occurrence of* X *in* ϕ *is positive, then the formula* $\phi[X/\psi]$*, obtained from* ϕ *by substituting each occurrence of* X *with* ψ*, is also in* Π_{n+1}*, provided no free variable in* ψ *becomes bounded in the substitution;*

- Σ_{n+1} *is the closure of* $\Sigma_n \cup \Pi_n$ *with respect to least fixpoints and composition.*

The alternation depth of a formula ϕ, denoted by $ad(\phi)$, is the least k such that $\phi \in \Sigma_{k+1} \cap \Pi_{k+1}$.

2.2 Parity games and μ-Calculus

We use the following notation for parity games.

Definition 2.2. *A parity game* (G, v) *of index* n *is given by a graph* (V, R) *(the* arena*), a starting vertex* $v \in V$*, and a coloring of the set of vertices* V *with colors in the set*

$$A_n = \{E_i, O_i : i = 1 \dots n\}.$$

Vertices colored by E_i *(*O_i*) have* priority i *and are called* E*-vertices (*O*-vertices, respectively).* E*-vertices are positions in which player Even has to move, while in* O*-vertices it is player Odd turn. The set of possible moves for a player in a vertex* v *is* $\{w : vRw\}$*. If one player cannot move, the other wins. Otherwise, an infinite sequence of vertices is generated, and the play is won by Even if the maximal priority seen infinitely often in the play is even. Otherwise, player Odd wins.*

Alternatively, we shall describe a parity game by giving the arena, the initial vertex, the positions for Even, the ones for Odd, and their priorities.

Parity games are connected to the μ-Calculus as follows. First of all, the model checking problem $(G, v) \models \phi$, for a μ-formula ϕ (in negative normal form) over a graph $G = (V, R)$, can be presented as a parity game, the *model checking game* of ϕ over (G, v). In this game we call the two players Verifier (for Even) and Falsifier (for Odd), and the arena is given by the following graph:

- the set of vertices is $V \times FL(\phi)$, where $FL(\phi)$ is the Fisher Ladner closure of the formula ϕ (see [10]);

- (w, α) is a position for Verifier if α is a disjunction, a diamond, a fixed point, or a literal with $w \not\models \alpha$;

- (w, α) is a position for Falsifier if α is a conjunction, a box, or a literal with $w \models \alpha$;
- if α is a literal, (w, α) is a terminal position;
- from $(w, \alpha \wedge \beta)$ there is an edge to (w, α) and an edge to (w, β), and the same holds from $(w, \alpha \vee \beta)$;
- from $(w, \Box\alpha)$ there is an edge to (w', α) for every w' with wRw', and the same holds from $(w, \Diamond\alpha)$;
- from $(w, \sigma x \alpha(x))$, where $\sigma \in \{\mu, \nu\}$ there is an edge to $(w, \alpha(\sigma x \alpha(x)))$;
- a position (w, α) has priority k if $k = ad(\phi)$ is the alternation depth of the formula α.

Lemma 2.1.

$$(G, v) \models \phi$$

$$\Updownarrow$$

Verifier has a winning strategy in the model checking game of ϕ over (G, v).

On the opposite direction, moving from parity games to formulas, we have:

Lemma 2.2. *([13]) If (G, v) is a parity game of index n, then there is a Σ_n formula W_n in the alphabet A_n, called the Walukiewicz formula of index n, which expresses the fact that player Even has a winning strategy in the parity game associated to the graph G:*

$$(G, v) \models W_n \Leftrightarrow \text{ Even has a winning strategy in the parity game } (G, v)$$

Walukiewicz formulas also witness the strictness of the alternation hierarchy over the class of all graphs:

Theorem 2.1 ([3])**.** *The formula W_n is a Σ_n-formula which is not equivalent to any Π_n-formula over the class of all graphs. It follows that the alternation hierarchy is strict over the class of all graphs.*

2.3 Planar graphs

For technical reasons, we define planar graphs using the concept of broken line (but the definition is equivalent to the usual one), where a *broken line* is a union of a finite number of segments of the form $\overline{x_i x_{i+1}}$ for $i = 1, \ldots, n-1$, where $x_1, \ldots, x_n$ are points in the plane.

Definition 2.3. *A finite undirected graph is* planar *if it can be drawn in the plane in such a way that edges correspond to broken lines that cross only in the vertices of the graph.*
A finite directed graph is planar *if the undirected graph obtained from G by forgetting the orientation of the edges is planar.*

The first question we must consider when dealing with the hierarchy of the μ-Calculus on the class of finite directed planar graphs is whether there exists a satisfiable μ-formula which is unsatisfiable in finite directed planar graphs: otherwise the strictness of the hierarchy on finite directed planar graphs would easily follow from the strictness of the hierarchy on finite graphs. To find such a formula we shall use the fact that any undirected planar graph contains a vertex of degree less than 5. This can be proved by starting from the following well known results:

Lemma 2.3. *(Handshake lemma) Let $G = (V, E)$ be a finite undirected graph with $|E| = e$. Then*

$$2e = \Sigma_{w \in V} deg(w).$$

Theorem 2.2. *(Euler's formula) Let $G = (V, E)$ be an undirected planar graph, with $|E| = e$, $|V| = v$, and f faces. Then $v - e + f = 2$.*

The previous two results imply:

Lemma 2.4. *In every finite undirected planar graph there is a vertex of degree at most* 5.

Proof. Suppose for an absurdity that $G = (V, E)$ is a finite undirected planar graph where every vertex has at least degree 6. Let v, e, f be the number of vertices, the number of edges, and the number of faces, respectively. Then by the handshake lemma we have

$$2e \geq 6v$$

hence

$$e \geq 3v.$$

Moreover from the Euler formula we have

$$3v - 3e + 3f = 6$$

Let us consider the dual graph G^d of G, whose vertices are the faces of G and where there is an edge between every pair of adjacent faces. Since every face in G has at least three edges in its border, by applying the handshake lemma to the dual graph of G we obtain

$$2e \geq 3f$$

so using the previous equation we get

$$6 - 3v + 3e = 3f \leq 2e.$$

By simplifying the inequality we have

$$e \leq 3v - 6$$

contradicting $3v \leq e$. This concludes the proof. □

Using the previous lemma it is now easy to prove:

Theorem 2.3. *There exists a formula which is satisfiable on the class of finite graphs but is unsatisfiable on the class of finite directed planar graphs.*

Proof. Consider the set of proposition $P_1, \ldots, P_7$ and use them to color the 7 vertices of the complete symmetric graph, obtaining

$$K_7 = (V, R, P_1, \ldots, P_7)$$

where $V = \{1, \ldots 7\}$, $R = V \times V$, and $P_i = \{i\}$, for all $i = 1, \ldots, 7$.
Let ϕ_7 be the μ-formula characterizing $(K_7, 1)$ modulo bisimulation; then for all graph G and $v \in G$ it holds:

$$(G, v) \models \phi_7 \Leftrightarrow (G, v) \text{ is bisimilar to } (K_7, 1)$$

It follows that if $(G, v) \models \phi_7$ then all vertices in G reachable from v must have out degree at least 6. By Lemma 2.4 it follows that ϕ_7 has no finite planar models. □

The remaining of this section is about drawings of graphs in the plane, respecting particular conditions. We prove some results that will be useful during the proof of the strictness of the μ-Calculus hierarchy on finite directed planar graph (Theorem 4.1 and Corollary 4.1).

We first show that, for any fixed partition S, U of the edges of a graph, we can always draw the graph in the plane in such a way to avoid “mixed” crossing.

Lemma 2.5. *If G is a finite undirected graph and $\{S, U\}$ is a bipartition of the edges in G, then it is always possible to draw G in the plane in such a way that*

- *edges are replaced by broken lines consisting of two segments;*
- *three segments never cross, and crossings are always between two edges belonging to the same component of the partition (either both in S or both in U).*

Proof. Suppose $S = \{e_1, \ldots, e_s\}$ with $e_i = (v_i, w_i)$. First we put the vertices of G on the horizontal axis. Starting from $k = 1$, we draw e_k as the broken line $\overline{v_k z_k}\ \overline{z_k w_k}$ in the upper halfplane, by choosing z_k is such a way that:

- z_k does not belong to any line $v_i z_i$ or $z_i w_i$ for $i < k$;
- $v_k z_k w_k$ contains no z_i, for all $i < k$, and none of the finitely many crossings between edges which we have already drawn.

The creation of the new edge $e_k = \overline{v_k z_k}\ \overline{z_k w_k}$ will possibly produce new crossings, but they will have multiplicity 2 and they will be finitely many, allowing the inductive procedure to continue. When $k = s$ a correct drawing of S will be obtained.

In this way we have considered only the edges belonging to S (see fig. 1); for the U edges we do the same, using the inferior half plane instead that the superior one. $\square$

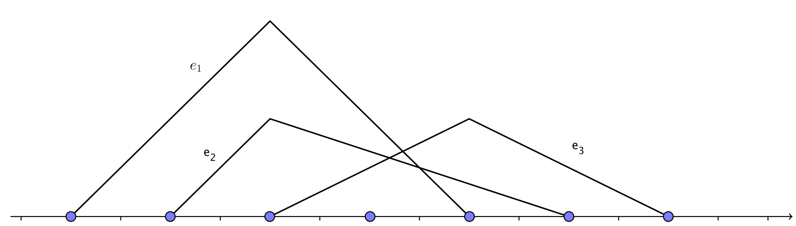


Figure 1: Dealing with edges in S

We next prove that if a graph G is planar and v is a vertex of G we can always draw G in the plane in such a way that edges intersect only in vertices and the vertex v is on the "border" of the graph. More precisely:

Lemma 2.6. *Let G be a planar undirected graph and v one of its vertices. Then it is possible to draw G on the plane in such a way that edges intersect only in vertices and there is a half line S with initial vertex in v such that $S \setminus \{v\}$ has no intersection with the edges of G.*

Proof. We start by drawing G on the plane in such a way that edges are unions of consecutive segments intersecting only in the vertices of G, and no such segment belongs to a line containing v. Then we define the *nestedness* of v in G as the minimum number n such that there exists a half line S with initial vertex in v such that $S \setminus \{v\}$ intersect the edges of G in n points. The proof goes by induction on the nestedness $n(v, G)$.
If $n(v, G) = 0$ we are done. If $n(v, G) = n > 0$, let S be a half line from v such that $S \setminus \{v\}$ has n intersections with the edges of G. Suppose without loss of generality that the only vertex of G in S is v, and (see the figure below) let x be the last intersection of the half line S with the edges of G (i.e. the intersection whose distance form v is maximal). Let e be the edge of G intersecting S in x and consider two points y, z in e such that:

1. the segment $\overline{yz}$ contains x;

2. $\overline{yz}$ is small enough so that it is possible to replace it by a union σ of consecutive segments starting in y and ending in z such that σ does not intersect neither S nor any edge in G (see figure 2);

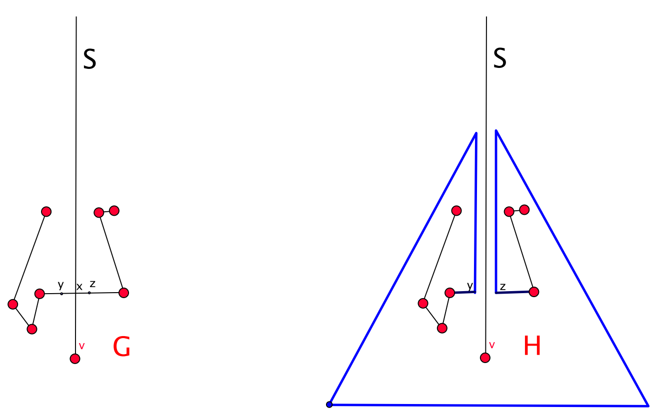


Figure 2: Eliminating an intersection with S

3. the closed curve $\overline{yz}\sigma$ contains all of G.

The graph H obtained in this way is isomorphic to G, but $n(v, H) < n(v, G)$. We can then apply the inductive hypothesis. $\square$

Definition 2.4. *Given two directed graphs $G = (V, R)$, $G' = (V', R')$, $v \in V$, and $v' \in V'$, the graph $G'' = (V'', R'')$ obtained by appending (G', v') to (G, v) is defined as follows:*

$$V'' = V \dot{\cup} V', \quad R'' = R \dot{\cup} R' \dot{\cup} \{(v, v')\}.$$

Corollary 2.1. *Given two planar directed graphs G, G', $v \in V$, and $v' \in V'$, the graph G'' obtained by appending (G', v') to (G, v) is planar.*

Proof. If we apply the previous lemma to the undirected version of (G, v) and (G', v'), we may draw these graphs in the plane in such a way that:

- the two half lines S, S', depicted in the previous lemma, which intersect the graphs G, G' only in v, v', respectively, belong to the same line r, with different orientation;

- the distance between v, v' is greater then the distance from v to any vertex of G and from v' to any vertex of G'.

Then the resulting graph is planar (see figure 3), and the same is true if we restore the orientation of the edges. $\square$

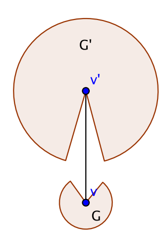


Figure 3: Merging two planar graphs

In the following, we only consider finite directed graphs (and omit the word directed).

3 Informal Description of the Main Proof

The key idea of our proof goes through parity games and consists in reducing a parity game over a finite non planar graph to a restricted form of parity game over a finite planar graph. To give an intuition of what we shall do, consider the graph of a parity game which, when drawn in the plane, admits a crossing between two edges (as in figure 4 on the left).

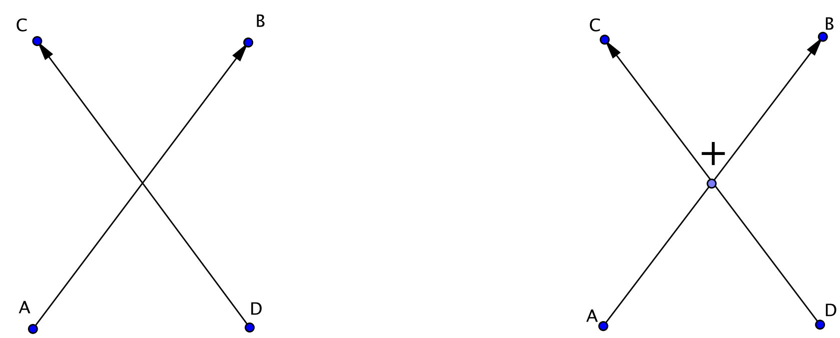


Figure 4: Adding a new vertex to avoid a crossing

To transform the graph into a planar graph, avoiding the crossing, we could add a new vertex marked with a new symbol $+$, as shown on the right of figure 4; however, this addition creates new paths: it is now possible to go from vertex A to vertex C, going through the vertex $+$, while this is not possible in the original graph.

To avoid the possibility for a player of the parity game to use this kind of new paths we shall "mark" the paths arriving and starting from the cross by adding new vertices colored by new letters a, b, as shown in figure 5; in this way we are able to recognize the old paths from the new ones, because the old paths always follow a pattern of type $a, +, b$ or $b, +, a$ and never a pattern of type $a, +, a$ or $b, +, b$.

We shall use this construction in a context in which, given a fixed formula ψ, we want to create a planar graph H starting from a non planar graph G in such a way that the two (pointed) graphs agree on ψ; in terms of parity games, we want that either Duplicator has a winning strategy in both the parity games corresponding to (H, ψ), (G, ψ) or Spoiler has one. Consequently, we shall consider a restricted form of parity game, where players are obliged to always follow patterns of type $a, +, b$ or $b, +, a$. Unfortunately, things are not that simple. Referring to the picture above, suppose that in G the move from A to B is successful for Duplicator in the ψ-model checking game, whereas the move from D to C is unsuccessful in the game: then we should not add a crossing between the two edges and try to draw the graph G on the plane whithout this crossing. This will be done using Lemma 2.5, where we proved that we can always draw a picture of G in the plane without mixed "successful-unsuccessful" crossing. Moreover, because of the existence of successful/unsuccessful crossing we shall have to use two different colors for the new vertices, the $(+)$ color for crossings between "successful" paths, and the $(-)$ color for crossings between "usuccessful" paths. In doing so we shall also need other colors a_+, b_+, a_-, b_- to distinguish the "original" paths in the crossing. This roughly explains the introduction of the pattern

$$\pi_8 := a_+, +, b_-, -, a_-, -, b_+, +$$

that, as we shall see, will prevent players of an octonary game to follow new dangerous paths.

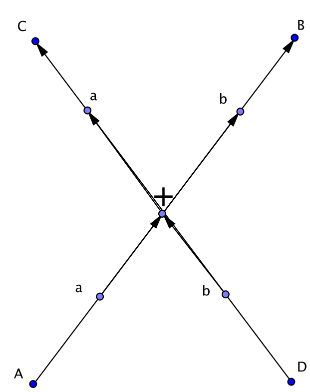


Figure 5: Adding letters to recognize original paths

4 Formal Proof

More formally, we give the following definitions. We consider a restricted version of parity games, where we add new letters as explained in the informal introduction and restrict the admissible paths. First, remember the alphabeth A_n used for (ordinary) parity games

$$A_n = \{E_i, O_i : i = 1 \dots n\},$$

and the pattern

$$\pi_8 := a_+, +, b_-, -, a_-, -, b_+, +.$$

Definition 4.1. *Let G be a graph vertex colored in the alphabet $B_n = A_n \cup \{-, +, a_-, b_-, a_+, b_+\}$. An* octonary path *in G is a path in the graph colored by a suffix of a word of the form*

$$Z_1(\pi_8)^* Z_2(\pi_8)^* Z_3(\pi_8)^* \dots$$

where $Z_i \in A_n$ and $(\pi_8)^*$ represent an arbitrary finite repetition of the octonary path defined above.
An octonary graph *is a B_n graph in which all possible paths are octonary.*

We now introduce a restricted form of parity games, the octonary games, that we shall use when we draw a octonary graph in the plane in such a way that crossing arises only in a $(+)$-vertex belonging to the pattern $a_+, +, b_-$ on one side, and to the pattern $b_+, +, a_+$ on the other side, or in a $(-)$-vertex belonging to the pattern $a_-, -, b_+$ on one side and to the pattern $b_-, -, a_-$ on the other side (see figure 6).

Definition 4.2. *A* well-pointed graph *is a pair (G, v) where G is a B_n-graph and v is a vertex in G colored in $B_n \setminus \{+, -\}$.*

An *octonary parity game* is given by a well-pointed graph (G, v), where we stipulate that that vertices colored by the new letters $\{-, +, a_-, b_-, a_+, b_+\}$ have priority 1. The play starts from the initial vertex v (which is colored in $B_n \setminus \{+, -\}$). We still have two players, Even and Odd; Even's positions are vertices colored by E_i or by $-$ andOdd's positions are vertices colored by O_i, $+$, a_-, b_-, a_+ or b_+.

Each round of a play starts from a vertex colored in $A_n \cup \{a_-, b_-, a_+, b_+\}$, and proceeds as follows:

- From a vertex colored in A_n, the player in charge has to move to a son colored a_+.

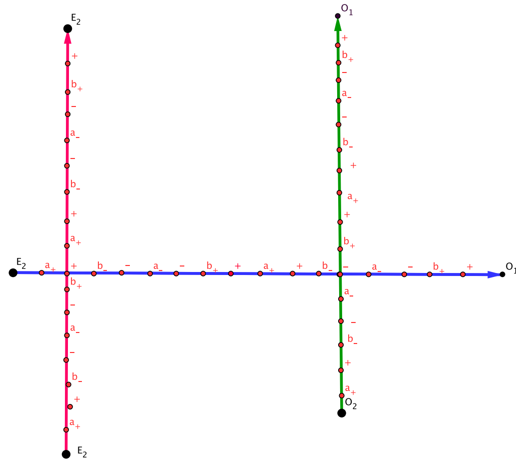


Figure 6: Crossing octonary paths

- From a vertex colored $a_\star$ with $\star \in \{+,-\}$ a subrun starts, with Odd choosing a son colored by $\star$; the next move is an Odd move if $\star = +$, and Odd has to move to a son colored by b_-; on the other hand, if $\star = -$, Even has to move to a son colored by b_+. Notice that at the end of the subrun either a pattern of type $a_+, +, b_-$, or a pattern of type $a_-, -, b_+$ has been overtaken.

- From a vertex colored $b_\star$ with $\star \in \{+,-\}$ a subrun starts, with Odd choosing a son colored by $\star$; in the next move, if $\star = +$, Odd has to move to a son colored by a_+, or to a son colored by $Z \in A_n$ (indicating that we are leaving a finite repetition of the pattern π_8), while if $\star = -$ Even has to move to a son colored by a_-. Notice that at the end of the subrun either a pattern of type $b_+, +, Z$, with $Z \in A_n \cup \{a_+\}$, or a pattern of type $b_-, -, a_-$ has been overtaken.

If a player cannot move, it loses. Otherwise an infinite play is generated, and Even wins if the maximal priority appearing infinitely often on the play is even; otherwise Odd wins. Notice that, in order not to lose, Odd cannot persist in choosing the pattern $b_+, +, a_+$ but sooner or later he has to choose a pattern of type $b_+, +, Z$, with $Z \in A_n$, leaving in this way a finite repetition of the pattern π_8.

Winning strategies for this restricted form of parity games may be defined by μ-formulas, as in the case of standard parity games:

Lemma 4.1. *There exists a formula W_n^8 such that for every well-pointed graph (G, v) it holds:*

$(G, v) \models W_n^8 \Leftrightarrow$ *Even has a winning strategy in the octonary game* (G, v)

Proof. If n is even we have:

$$W_n^8 = \nu X_n.\mu X_{n-1} \ldots \nu X_2.\mu X_1.$$
$$F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5 \wedge F_6$$

where

$$\begin{array}{lll} F_1 & := & (a_+ \to \Box(+ \to \Box(b_- \to X_1)) \\ F_2 & := & (b_- \to \Box(- \to \langle\rangle(a_- \wedge X_1)) \\ F_3 & := & (a_- \to \Box(- \to \langle\rangle(b_+ \wedge X_1)) \\ F_4 & := & (b_+ \to \Box(+ \to [\Box(a_+ \to X_1) \wedge \Box(\vee_i(E_i \vee O_i) \to X_i)))] \\ F_5 & := & \wedge_i(E_i \to \langle\rangle(a_+ \wedge X_1)) \\ F_6 & := & \wedge_i(O_i \to \Box(a_+ \to X_1)) \end{array}$$

The formula for n odd is similar but starts with μ instead of ν. $\Box$

Since octonary games only allow plays along octonary path, we easily obtain:

Lemma 4.2. *If $(G, v), (G', v)$ are well-pointed graphs over the same set of vertices having the same octonary paths then*

$$(G, v) \models W_n^8 \Leftrightarrow (G', v) \models W_n^8$$

The main result of this paper is the following theorem and its immediate corollary.

Theorem 4.1. *The formula W_n^8 is not equivalent over B_n-pointed, finite planar graphs to any μ-formula of alternation depth strictly smaller than n.*

Corollary 4.1. *The alternation hierarchy is strict over finite planar graphs.*

The proof is done in Section 4.7 and goes by contradiction, supposing there is a formula ψ of alternation depth smaller than n which is equivalent to W_n^8 over well-pointed finite planar graphs. In the next paragraph we shall start considering some definitions and constructions that will be used in the proof, depending on the formula ψ.

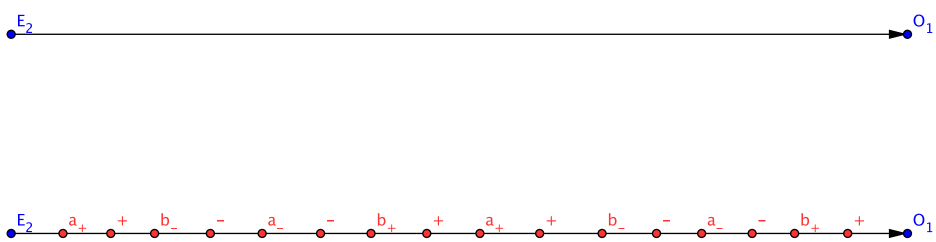
4.1 Subdivided graphs

Definition 4.3. *Let G be a A_n-graph. The k-*subdivision *of G is the B_n-graph $SUBD^k(G)$ obtained by substituting each edge (v, w) in G by a simple path starting in v and ending in w in such a way that:*

- *the colors of v, w are the same as in G;*
- *the inner path (excluding v and w) is labelled by $(\pi_8)^k$.*

Definition 4.4. *A path $v_0 \to v_1 \ldots \to v_n$ in a B_n graph G is a* macro-edge *if the vertex v_i have degree one, for all $1 \leq i < n$, and the inner path (excluding v_0 and v_n) is labelled by $(\pi_8)^k$, for some k.*

In short, $SUBD^k(G)$ is obtained from G by substituting edges with macro-edges, as shown in the following figure, where the original edge in a graph has been replaced, in a 2-subdivision, by a simple octonary path:



Lemma 4.3. *if G is an A_n-graph then for all k it holds*

$$(G, v) \models W_n \Leftrightarrow (SUBD^k(G), v) \models W_n^8.$$

In particular, all G-subdivisions agree on the formula W_n^8.
Moreover, if s, t are vertices colored in $B_n \setminus \{+, -\}$ and belonging to the same macro-edge of $SUBD^k(G)$ then

$$(SUBD^k(G), s) \models W_n^8 \Leftrightarrow (SUBD^k(G), t) \models W_n^8.$$

Proof. By Lemma 2.2 and Lemma 4.1 we know that

$$(G, v) \models W_n \Leftrightarrow \textit{ Even} \text{ has a winning strategy in the parity game } (G, v)$$

and

$$(SUBD^k(G), v) \models W_n^8$$

$$\Updownarrow$$

Even has a winning strategy in the octonary game $(SUBD^k(G), v)$.

Since winning strategies for *Even* in the parity game (G, v) correspond exactly to winning strategies for *Even* in the restricted game $(SUBD^k(G, v)$, the result follows. $\square$

Definition 4.5. *A macro-edge in $SUBD^k(G)$ is successful if there exists a vertex s in the macro-edge colored by $B_n \setminus \{+, -\}$ with $(SUBD^k(G), s) \models W_n^8$. Otherwise the macro-edge is unsuccessful.*

4.2 Decorations

While subdivisions are used to create new vertices which will represent crossings between edges in a non planar graph, *decorations* shall be used to neutralize unwanted turns in these crossings. To define decorations we need the notion of ψ equivalence, for a fixed B_n-formula ψ.

Definition 4.6. *If ψ is a μ-formula in the language B_n, the* ψ-equivalence relation *on well-pointed graphs is defined by stipulating that $(G, v), (G', v')$ are ψ-equivalent if and only if they verify the same μ-formulas of the Fischer-Ladner closure of ψ.*

ψ-equivalence is clearly an equivalence relation on the class of well-pointed graphs, and we may consider its equivalence classes.

Definition 4.7. *Fix a planar representative for each ψ-equivalence class of well-pointed graphs containing a finite planar graph. The ψ-decoration $DEC(G)$ of a B_n-graph G consists in the graph obtained from G by appending:*

- *all planar representatives of B_n-pointed, finite planar graphs satisfying ψ to each $(+)$-vertex;*
- *all planar representatives of B_n-pointed, finite planar graphs satisfying $\neg\psi$ to each $(-)$-vertex.*

The first result we need about decorations is that if ψ is equivalent to W_n^8 on finite well-pointed planar graphs then decorating a well-pointed graph G does not change the truth value of W_n^8.

Lemma 4.4. *Suppose ψ is equivalent to W_n^8 on B_n-pointed, finite planar graphs; then for any well-pointed graph (G, v) it holds:*

$$(G, v) \models W_n^8 \quad \Leftrightarrow \quad (DEC(G), v) \models W_n^8$$

Proof. Let ψ be equivalent to W_n^8 on B_n-pointed, finite planar graphs, and suppose first that $(G, v) \models W_n^8$. By Lemma 4.1 we know that Even has a winning strategy Σ in the octonary parity game (G, v). We now show how to modify Σ in order to obtain a winning strategy Σ' for Even in the octonary parity game $(DEC(G), v)$, from which $(DEC(G), v) \models W_n^8$ follows. Notice that the only problem that Even can face in order to apply the strategy Σ to win the parity game $(DEC(G), v)$ is that in this game player Odd could move from a $+$ vertex in (G, v) to a vertex outside (G, v) by going into a decoration. However, in this case we know that the play would enter in a representative (P, u) of a finite planar graph satisfying ψ, and, since ψ is equivalent to W_n^8 on finite planar graphs, we would have $(P, u) \models W_n^8$. Hence, if Odd moves outside (G, v), in the game over $DEC(G)$ Even can leave strategy Σ and follow a winning strategy for the octonary parity game on (P, u).

If, on the other hand, $(G, v) \not\models W_n^8$, then Odd has a winning strategy in the octonary parity game (G, v). As before, the only problem that Odd can face in order to apply this strategy to win the octonary parity game $(DEC(G), v)$ is that in this game player Even could move from a $-$ vertex in (G, v) to a vertex outside (G, v), by going into a decoration. However, in this case we know that the play would enter in a representative (P, u) of a B_n-pointed, finite planar graph satisfying $\neg\psi$, and since ψ is equivalent to W_n^8 on B_n-pointed, finite planar graphs, we would have $(P, u) \models \neg W_n^8$. Hence, if this is the case, Odd can always change strategy and start following a winning strategy for player Odd in the octonary parity game on (P, u). This proves that in this case we have $(DEC(G), v) \not\models W_n^8$. $\square$

The second result we need on decorations regards finite planar graphs.

Lemma 4.5. *If P is a finite planar graph then the graph $DEC(P)$ is still a finite planar graph.*

Proof. By applying repeatedly Corollary 2.1. $\square$

4.3 A Pumping Lemma

Since $+$ and $-$ vertices will be used to represent crossing among edges, we want to be able to have enough of such vertices. To this end we prove a sort of pumping lemma, allowing us to extend macro-edges as much as we like, provided that we start with macro-edges sufficiently long (where "sufficiently" only depending on ψ).

Lemma 4.6. *For any B_n-formula ψ there exists a number $N := N(\psi)$, depending only on ψ, with the following property: in any B_n-graph H, if we substitute any pattern $(\pi_8)^N$ in a macro-edge p with $(\pi_8)^{2N}$ then we obtain a graph H' such that $DEC(H)$ and $DEC(H')$ are both ψ-equivalent and W_n^8-equivalent.*

Proof. We start enumerating the union of the Fischer Ladner closure of ψ with the Fischer Ladner closure of W_n^8:

$$FL(\psi) \cup FL(W_n^8) = \{\chi_1, \dots, \chi_s\}.$$

For all $i = 1, \dots, s$, let $\mathcal{A}_i$ be a non deterministic parity automaton, with n_i states, which is equivalent to χ_i (see [8] for a definition of parity automata). Let

$$N(\psi) := (8 \cdot max\{n_i : i = 1, \dots, s\})!$$

For all $v \in H$ and $i = 1, \dots, s$ we shall prove that

$$(Dec(H), v) \models \chi_i \Rightarrow (Dec(H'), v) \models \chi_i,$$

from which the Lemma follows.

Suppose $(Dec(H), v) \models \chi_i$ and let ρ be an accepting run of $\mathcal{A}_i$ on $(Dec(H), v)$. We show how to transform ρ in an accepting run ρ' of $\mathcal{A}_i$ on $(Dec(H'), v)$. If the run ρ never enters the macro-edge p we can simply define $\rho' := \rho$. If the run ρ enters the macro-edge p but then leaves p to enter a decoration, then ρ is an accepting run for $\mathcal{A}_i$ on $(Dec(H'), v)$, as well. On the other hand, if the run ρ completes the macro-edge p without entering in any decoration, then we can find two a_+ vertices v_i, v_j on the macro-edge which are at a distance $d \leq 8n_i$. Since d divides $N(\psi)$, say $N(\psi)/d = k$, we can substitute the run ρ over the interval (v_i, v_j) with $k+1$ copies of it, obtaining a new run ρ' which is now accepting over a macro-edge obtained from p by substituting the pattern $(\pi_8)^N$ with $(\pi_8)^{2N}$. In this way we can transform every accepting run of $\mathcal{A}_i$ on $(Dec(H), v)$ into an accepting run ρ' of $\mathcal{A}_i$ on $(Dec(H'), v)$. Since Fischer Ladner closures are closed under negation, the result follows. $\square$

By iterating this procedure, we obtain:

Corollary 4.2. *Fix $k \geq 1$. The two graphs*

$$(DEC(SUBD^{N(\psi)}(G)), v), \quad (DEC(SUBD^{kN(\psi)}(G)), v)$$

are ψ-equivalent and W_n^8-equivalent.

4.4 Planarization of a graph

Given an A_n graph G, let N_G be the smallest positive multiple of $N(\psi)$ which is greater then twice the number of edges in G [1] Using Lemma 2.5 we may draw $SUBD^{N_G}(G)$ in the plane in such a way that crossings are only between macro-edges belonging to the same element of the partition between "successful" and "unsuccesful" macro-edges (either both crossing macro-edges are successful, or they are both unsuccessful). Moreover, we can draw $SUBD^{N_G}(G)$ in the plane in such a way that, if the crossing is between successful macro-edges then it is represented by a (+)-vertex belonging to the pattern $a_+, +, b_-$ on one side and to the pattern $b_+, +, a_+$ on the other side, while if the crossing is is between unsuccessful macro-edges then it is represented by a $(-)$-vertex belonging to the pattern $a_-, -, b_+$ on one side and to the pattern $b_-, -, a_-$ on the other side. The resulting graph is called a *planarization* of $SUBD^{N_G}(G)$.

Definition 4.8. *(Planarization of a graph) Given an A_n graph G, we fix one planarization of $SUBD^{N_G}(G)$ and denote it by $PL(SUBD^{N_G}(G))$.*

We have:

Lemma 4.7. *If G is an A_n-graph and $v \in G$ then $(DEC(SUBD^{N_G}(G)), v)$ is a well-pointed graph and*

$$(DEC(SUBD^{N_G}(G)), v) \models W_n^8 \Leftrightarrow (DEC(PL(SUBD^{N_G}(G))), v) \models W_n^8$$

Proof. The result follows by Lemma 4.2, since the octonary paths of the two graphs are the same. □

4.5 The Key Lemma

We consider the following class of well-pointed finite graphs, parametric in the B_n-formula ψ:

$$\mathcal{G}_n^\psi = \{(DEC(SUBD^{N(\psi)}(G)), v) : G \text{ is a finite } A_n\text{-graph and } v \in G \ \}$$

Lemma 4.8. *If W_n^8 is equivalent to the formula ψ on well-pointed finite planar graphs then it is also equivalent to ψ over the class $\mathcal{G}_n^\psi$, that is, for any A_n-graph G and $v \in G$ it holds:*

$$(DEC(SUBD^{N(\psi)}(G)), v) \models W_n^8 \quad \Leftrightarrow \quad (DEC(SUBD^{N(\psi)}(G)), v) \models \psi.$$

[1] the factor 2 is needed because we shall apply Lemma 2.5 where an edge is substituted by a broken line composed by two segments.

Proof. We fix an A_n-graph G and a vertex $v \in G$. Consider the well-pointed graph $(DEC(SUBD^{N_G}(G)), v)$, where N_G is, as before, the smallest multiple of $N(\psi)$ which is greater then twice the number of edges in G. From Corollary 4.2 we know that

$$(DEC(SUBD^{N_G}(G)), v) \models W_n^8 \Leftrightarrow (DEC(SUBD^{N(\psi)}(G)), v) \models W_n^8 \quad (1)$$

and

$$(DEC(SUBD^{N_G}(G)), v) \models \psi \Leftrightarrow (DEC(SUBD^{N(\psi)}(G)), v) \models \psi. \quad (2)$$

Let P be the finite planar graph

$$P := DEC(PL(SUBD^{N_G}(G))).$$

By Lemma 4.7 we have

$$(DEC(SUBD^{N_G}(G)), v) \models W_n^8 \Leftrightarrow (P, v) \models W_n^8 \Leftrightarrow (P, v) \models \psi, \quad (3)$$

where the last equivalence holds because P is planar. Let

$$\theta(P, v) := \begin{cases} \psi, & \text{if } (P, v) \models \psi \\ \neg\psi, & \text{if } (P, v) \models \neg\psi. \end{cases}$$

By definition, $(P, v) \models \theta(P, v)$ and Verifier has a winning strategy Σ in the model checking game of $\theta(P, v)$ over (P, v). We use Σ to define a winning strategy Σ' for Verifier in the model checking game of $\theta(P, v)$ over the graph $(DEC(SUBD^{N_G}(G)), v)$. Define Σ' to be equal to Σ up to the first position $(\Diamond\chi, w)$, with $\chi \in FL(\theta(P, v)) = FL(\psi)$, after which Σ leaves the graph $DEC(SUBD^{N_G}(G))$ by taking a non octonary turn, and arriving to the position (χ, u). Notice that u is colored in $B_n \setminus \{+, -\}$ so that (P, u) is a B_n-pointed, finite planar graph. Since Σ is winning for Verifier, we have $(P, u) \models \chi$. If $Q := (P, u)$, let (Pl_Q, u') be the representative of the ψ-equivalence class of Q. Consider the following two possibilities:

1. w is a $(+)$-vertex. In this case, by construction of $PL(SUBD^{N_G}(G))$ we know that the crossing is among successful macro-edges, and hence $(P, u) \models W_n^8$; since P is planar we have $(P, u) \models \psi$, and in the graph $DEC(SUBD^{N_G}(G))$ we have the graph (Pl_Q, u') appended to w. Moreover, $(Pl_Q, u') \models \chi$, because (Pl_Q, u') is ψ-equivalent to $Q := (P, u)$. Let Σ'' be a winning strategy for Verifier in the model checking game of χ over (Pl_Q, u'). We define Σ' on the position $(\Diamond\chi, w)$ to be (χ, u') while from the position (χ, u') on the strategy coincides with Σ''.

2. w is a $(-)$-vertex. In this case, by the construction of $PL(SUBD^{N_G}(G))$ we know that the crossing is among unsuccessful macro-edges, and hence $(P, u) \models \neg W_n^8$; since P is planar we have $Q := (P, u) \models \neg\psi$ and in the graph $DEC(SUBD^{N(\psi)}(G))$ we have the graph (Pl_Q, u') appended to w. Moreover, $(Pl_Q, u') \models \chi$, because (Pl_Q, u') is ψ-equivalent to $Q := (P, u)$. Let Σ'' be a winning strategy for Verifier in the model checking game of χ over (Pl_Q, u'). We define Σ' on the position $(\Diamond\chi, w)$ to be (χ, u') while from the position (χ, u') on the strategy coincides with Σ''.

It follows that Σ' is a winning strategy for Verifier in the model checking game of $\theta(P, v)$ over $(DEC(SUBD^{N_G}(G)), v)$. Hence,

$$(DEC(SUBD^{N_G}(G)), v) \models \theta(P, v).$$

By the definition of $\theta(P, v)$ we obtain

$$(P, u) \models \psi \quad \Leftrightarrow \quad (DEC(SUBD^{N_G}(G)), v) \models \psi,$$

and, from (3),

$$(DEC(SUBD^{N_G}(G)), v) \models \psi \quad \Leftrightarrow \quad (DEC(SUBD^{N_G}(G)), v) \models W_n^8;$$

finally, by Corollary 4.2 we obtain

$$(DEC(SUBD^{N(\psi)}(G)), v) \models \psi \quad \Leftrightarrow \quad (DEC(SUBD^{N(\psi)}(G)), v) \models W_n^8.$$

□

4.6 A class of trees

In the previous section we proved that if the B_n-formula ψ is equivalent to the formula W_n^8 over the class of finite well-pointed planar graphs, then the same holds over the class $\mathcal{G}_n^\psi$. However, in order to apply Arnold's technique (see [2]), which involves the existence of a fixed point over a complete metric space, we need to replace the class $\mathcal{G}_n^\psi$ by a class of *trees*. To this end, we first consider the decoration construction again, but this time we use decorations which are 3-unravelings.

Definition 4.9. *Given a graph G, and a vertex $v \in G$, the* 3*-unraveling of* (G, v) *is the tree consisting of the finite sequences:*

$$(v_0, i_1, v_1)(v_1, i_2, v_2) \dots (v_{n-1}, n, v_n)$$

such that $v_0 = v$, $i_j \in \{0,1,2\}$, *and* $v_i R v_{i+1}$ *in the graph* G*; the sons of a node* $(v_0, i_1, v_1)(v_1, i_2, v_2)\dots(v_{n-1}, n, v_n)$ *are all the sequences*

$$(v_0, i_1, v_1)(v_1, i_2, v_2)\dots(v_{n-1}, i_n, v_n)(v_n, i_{n+1}, v_{n+1})$$

in the tree. The root of the tree is the empty sequence.

The rationale for using 3-unravelings instead of simple unravelings in the following definition will become clear in Lemma 4.10.

Definition 4.10. *Let* ψ *be a* B_n*-formula. If* G *is a* B_n*-graph, then the graph* $DEC^{3\star}(G)$ *is obtained from* G *by appending all* 3-unravellings *of representatives of finite well-pointed planar graphs satisfying* ψ *to each* $(+)$*-vertex, and all* 3-unravellings *of representatives of finite well-pointed planar graphs satisfying* $\neg\psi$ *to each* $(-)$*-vertex.*

We still have a property corresponding to Lemma 4.4:

Lemma 4.9. *Suppose* ψ *is equivalent to* W_n^8 *on well-pointed finite planar graphs; then for any well-pointed graph* (G, v) *it holds:*

$$(G, v) \models W_n^8 \quad \Leftrightarrow \quad (DEC^{3\star}(G), v) \models W_n^8$$

Proof. Similar to the proof of Lemma 4.4. □

We are now able to define a class of trees:

$$\mathcal{T}_n^{\psi} = \{DEC^{3*}(SUBD^{N(\psi)}(T)) : T \text{ is a complete, } A_n\text{-colored, binary tree}\}$$

The class $\mathcal{T}_n^{\psi}$ satisfies the following lemma:

Lemma 4.10. *Suppose* T *is a complete,* A_n*-colored binary tree such that the corresponding* B_n*-tree* $DEC^{3*}(SUBD^{N(\psi)}(T))$ *has a finite number of subtrees, modulo isomorphism. Then* T *has a finite number of subtrees, modulo isomorphism, and* $DEC^{3*}(SUBD^{N(\psi)}(T))$ *is bisimilar to a graph in* $\mathcal{G}_n^{\psi}$*.*

Proof. Notice that the roots of decorations in $DEC^{3*}(SUBD^{N(\psi)}(T))$ are the only sons of nodes in $SUBD^{N(\psi)}(T)$ having outdegree 3. Hence, if $DEC^{3*}(SUBD^{N(\psi)}(T))$ is regular and s, t are nodes in $SUBD^{N(\psi)}(T)$ such that the $DEC^{3*}(SUBD^{N(\psi)}(T))$-subtrees rooted in s, t are isomorphic, then this isomorphism carries roots of decorations to roots of decorations and the restriction of this isomorphism to nodes in $SUBD^{N(\psi)}(T)$ is an isomorphism between the $SUBD^{N(\psi)}(T)$-subtrees rooted in s, t. Hence, if

$DEC^{3*}(SUBD^{N(\psi)}(T))$ has a finite number of subtrees, modulo isomorphism, the same is true for the tree $SUBD^{N(\psi)}(T)$. From this it easily follows that T has a finite number of subtrees, modulo isomorphism as well, and that T is bisimilar to a finite graph G in $\mathcal{G}_n^\psi$. Finally, it follows that $DEC^{3*}(SUBD^{N(\psi)}(T))$ is bisimilar to $DEC(SUBD^{N(\psi)}(G))$. $\square$

In order to prove that the hypothetical equivalence between ψ and W_n^8 carries over from the class $\mathcal{G}_n^\psi$ to the class of trees $\mathcal{T}_n^\psi$, we show the following finite model property:

Theorem 4.2. *Any μ-formula α which is satisfiable in $\mathcal{T}_n^\psi$ is satisfiable in $\mathcal{G}_n^\psi$.*

Proof. First, let us show that trees in $\mathcal{T}_n^\psi$ have bounded degree. Each element in the class is equal to a tree of the form $DEC^{3*}(SUBD^{N(\psi)}(T))$, where T is an A_n-colored binary tree; the vertices of the tree $SUBD^{N(\psi)}(T)$ have degree 1 or 2, and in the decoration DEC^{3*} we append unravellings of a finite fixed number of finite planar graphs to some vertices. It follows that there exists a number k such that the class $\mathcal{T}_n^\psi$ is contained in the class $\mathcal{D}_k$ of the trees with vertex degree less or equal to k. We claim that the class $\mathcal{T}_n^\psi$ is MSO-definable inside $\mathcal{D}_k$. Consider the $\mathcal{D}_k$ subclass

$$SUBD := \{SUBD^{N(\psi)}(T) : T \text{ is a complete } A_n\text{-colored binary tree}\}.$$

This class is definable, inside $\mathcal{D}_k$, by an MSO-formula θ_{SUBD} saying:

1. the vertices of the tree have degree smaller or equal to 2, and vertex colored in $\{+, -, a_-, b_-, a_+, b_+\}$ have degree 1;

2. the root of the tree satisfies $\bigvee_h (E_h \vee O_h)$;

3. in every maximal path starting from a vertex where $\bigvee_h (E_h \vee O_h)$ holds, either $\bigvee_h (E_h \vee O_h)$ holds infinitely often or the path is finite and ends in a vertex where $\bigvee_h (E_h \vee O_h)$ holds; moreover, the finite path between two successive occurrences of $\bigvee_h (E_h \vee O_h)$ is labeled by $\pi_8^{N(\psi)}$.

Using the formula θ_{SUBD} we can prove that the class $\mathcal{T}_n^\psi$ is MSO-definable inside $\mathcal{D}_k$ as follows. Let $\chi_1^+, \ldots, \chi_r^+$ be MSO-formulas characterizing the well-pointed finite planar representatives of the ψ-equivalence classes satisfying ψ modulo bisimulation; similarly, let $\chi_1^-, \ldots, \chi_s^-$ be MSO-formulas characterizing the well-pointed finite planar representatives of the ψ-equivalence classes satisfying $\neg\psi$ modulo bisimulation.

Notice that a tree $S \in \mathcal{D}_k$ belongs to the class $\mathcal{T}_n^\psi$ if and only if S have a subtree S' satisfying θ_{SUBD} such that

1. for all $i = 1, \dots r$, every vertex $s \in S'$ labeled by $+$ has a son in $S \setminus S'$ satisfying χ_i^+;

2. for all $i = 1, \dots s$, every vertex $s \in S'$ labeled by $-$ has a son in $S \setminus S'$ satisfying χ_i^-;

3. every vertex s in $S \setminus S'$ is a descendant of a either a son of $+$ in $S \setminus S'$ satisfying $\vee_i \chi_i^+$, or a son of $-$ in $S \setminus S'$ satisfying $\vee_i \chi_i^-$.

From this the MSO-definability of $\mathcal{T}_n^\psi$ inside $\mathcal{D}_k$ follows, since all above properties are MSO-definable. We denote by τ_n^ψ the MSO-formula defining $\mathcal{T}_n^\psi$ inside $\mathcal{D}_k$.

Suppose now that α is satisfiable in $\mathcal{T}_n^\psi$; then $\alpha \wedge \tau_n^\psi$ is satisfiable in $\mathcal{D}_k$. Since MSO has the regular tree property over $\mathcal{D}_k$(see [11], [12]), we know that $\alpha \wedge \tau_n^\psi$ is satisfied by some regular tree in $\mathcal{D}_k$. Hence α is true in some regular tree in $\mathcal{T}_n^\psi$. Since a regular tree contains only a finite number of isomorphism classes modulo isomorphism, from Lemma 4.10 we know that every regular tree in $\mathcal{T}_n^\psi$ is bisimilar to an element in $\mathcal{G}_n^\psi$, and the thesis follows. $\square$

Corollary 4.3. *If W_n^8 is equivalent to a formula ψ over well-pointed finite planar graphs, then W_n^8 is equivalent to ψ in $\mathcal{T}_n^\psi$.*

Proof. By Lemma 4.8 we see that the equivalence carries over from finite planar graphs to the class $\mathcal{G}_n^\psi$, while Theorem 4.2 allow us to transfer the equivalence over the class $\mathcal{T}_n^\psi$. $\square$

4.7 The contraction

Throughout this final section, let us consider the B_n-formula ψ as a parameter. Since trees in $\mathcal{T}_n^\psi$ have bounded degree, we may suppose that trees in $\mathcal{T}_n^\psi$ are complete k-ary trees for some fixed number 2^k (by copying sons when necessary). If $T \in \mathcal{T}_n^\psi$ and t is a node in T, we fix an order $t_1, \dots, t_{2^k}$ of the sons of t and denote by (T, t_i) the subtrees with root t_i.

We now proceed as in [2]. Given an alternating automaton $\mathcal{A} = (Q, \delta, \Omega)$ in the B_n-alphabet (where δ is a function from $Q \times B_n$ to the positive modal formulas $Mod^+(Q)$ over Q, and $\Omega : Q \to \{1, \dots, n\}$), a tree $T \in \mathcal{T}_n^\psi$, and a state $q \in Q$, we shall define an A_n-binary tree $G_{(\mathcal{A},q)}(T)$ (hence, a parity game) such that

$$T \quad \text{is accepted by} \quad (\mathcal{A}, q) \quad \Leftrightarrow \quad G_{(\mathcal{A},q)}(T) \models W_n. \tag{4}$$

To define $G_{(\mathcal{A},q)}(T)$ we proceed as follows. First, for any formula $B \in Mod^+(Q)$ and $1 < i \leq n$ we define a finite binary tree $G_{(B,i)}(T)$ by recursion on B:

- if $B = q \in Q$ then $G_{(q,i)}(T)$ is just a root labelled by (q, r_T) where r_T is the root of T;
- If $B = B_1 \wedge B_2$ ($B = B_1 \vee B_2$) then $G_{(B,i)}(T)$ consists of a root labelled by O_i (E_i, respectively) and two sons to which we append the trees $G_{(B_1,i)}(T)$, $G_{(B_2,i)}(T)$;
- if $B = \Box B_1$, ($\Diamond B_1$) then $G_{(B,i)}(T)$ consists of a complete finite binary tree of height k where the root and internal nodes are labelled by O_i (E_i, respectively) and where the 2^k leaves are labelled by

  $$G_{(B_1,i)}(T, t_1), \ldots, G_{(B_1,i)}(T, t_{2^k}),$$

  where $t_1, \ldots, t_{2^k}$ are the sons of the root of T.

Finally, the tree $G_{(\mathcal{A},q)}(T)$ is the limit (in the natural metric over binary tree defined below) of the sequence of trees $(T_i)_{i \in \mathbb{N}}$ defined as follows:

$$T_0 := G_{(\delta(q_0, \lambda(r_T)), \Omega(q_0))}(T), \quad T_{i+1} = T_i[(q,t) \mapsto G_{(\delta(q,\lambda(t)),\Omega(q))}(T,t)]$$

where $\lambda(t) \in B_n$ is the color of t in T, and the tree

$$T_i[(q,t) \mapsto G_{(\delta(q,\lambda(t)),\Omega(q))}(T,t)]$$

is obtained from T_i by appending, for all $q \in Q$ and $t \in T$, the graph $G_{(\delta(q,\lambda(t)),\Omega(q))}(T,t)$ to any T_i leaf labelled by (q,t). Then, it is not difficult to check that the equivalence (4) holds.

Definition 4.11. *Given a B_n-formula $\theta \in \Sigma_n$, an alternating automaton $(\mathcal{A}, q)$ which is equivalent to θ, and a tree $T \in \mathcal{T}_n^{\psi}$, we define*

$$C_\theta^\psi(T) := DEC^{3*}(SUBD^{N(\psi)}(G_{(\mathcal{A},q)}(T)))$$

Recall that the complete k-ary trees form a complete metric space, where the distance of two different trees t, t' is $1/2^h$, and where h is the smallest level on which they differ. A contraction in a metric space is a map γ such that, for some real number $c < 1$, we have $d(\gamma(t), \gamma(t')) \leq c d(t,t')$ for every t, t' in the metric space.

Lemma 4.11. *For any B_n-formula $\theta \in \Sigma_n$, the function $C_\theta^\psi : \mathcal{T}_n^\psi \to \mathcal{T}_n^\psi$ is a contraction in the complete metric space $\mathcal{T}_n^\psi$.*

Proof. Similar to [2]. $\square$

We are now able to prove our main result, Theorem 4.1:

Theorem. *The formula W_n^8 is not equivalent over well-pointed finite planar graph to any Π_n-formula.*

Proof. Suppose by contradiction that W_n^8 is equivalent to a formula $\psi \in \Pi_n$ over finite planar graph; then, by Corollary 4.3 we see that the equivalence carries over to the the complete metric space $\mathcal{T}_n^\psi$. If we put $\theta := \neg\psi \in \Sigma_n$, we may consider the contraction C_θ^ψ, as defined in 4.11, and its fixed point T_0. Then, if $(\mathcal{A}, q)$ is the alternating automaton which is equivalent to θ, we have :

$$T_0 \models \neg W_n^8 \;\Leftrightarrow\; T_0 \models \theta \;\overset{\textit{equation } (4)}{\Leftrightarrow}\; G_{(\mathcal{A},q)}(T_0) \models W_n \;\Leftrightarrow$$

$$\overset{\textit{Lemma } (4.3)}{\Leftrightarrow}\; SUBD^{N(\psi)}(G_{(\mathcal{A},q)}(T_0)) \models W_n^8 \;\overset{\textit{Lemma}(4.9)}{\Leftrightarrow}\; C_\theta^\psi(T_0) \models W_n^8 \;\Leftrightarrow$$

$$\Leftrightarrow\; T_0 \models W_n^8$$

a contradiction (where the last equivalence holds because T_0 is a fixed point of the contraction C_θ^ψ). $\square$

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