On fractional Laplacians - 3

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Abstract

We investigate the role of the noncompact group of dilations in $\mathbb{R}^n$ on the difference of the quadratic forms associated to the fractional Dirichlet and Navier Laplacians. Then we apply our results to study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving the Navier-Laplacian .

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1 Introduction

The Sobolev space $H^m(\mathbb{R}^n) = W_2^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_m^2 = \int\limits_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^m |\mathcal{F}u(\xi)|^2 \, d\xi,$$

see for instance Section 2.3.3 of the monograph [19]. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx.$$

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For arbitrary $m \in \mathbb{R}$ we define fractional Laplacian on $\mathbb{R}^n$ by the quadratic form

$$Q_m[u] = ((-\Delta)^m u, u) := \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}u(\xi)|^2 d\xi,$$

with domain

$$\mathrm{Dom}(Q_m) = \{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, Q_m[u] < \infty\}.$$

Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. We introduce the "Dirichlet" fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)^m_D$) as the restriction of $(-\Delta)^m$. The domain of its quadratic form is

$$\mathrm{Dom}(Q^D_{m,\Omega}) = \{u \in \mathrm{Dom}(Q_m) \,:\, \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

Also we define the "Navier" fractional Laplacian as the m-th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q^N_{m,\Omega}[u] = ((-\Delta_\Omega)^m_N u, u) := \sum\nolimits_j \lambda_j^m \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively, and $\mathrm{Dom}(Q^N_{m,\Omega})$ consists of distributions in Ω such that $Q^N_{m,\Omega}[u] < \infty$.

For $m = 1$ these operators evidently coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)^m_N$, the operator $(-\Delta_\Omega)^m_D$ is not the m-th power of the Dirichlet Laplacian for $m \neq 1$.

It is well known that for $m > 0$ quadratic forms $Q^D_{m,\Omega}$ and $Q^N_{m,\Omega}$ generate Hilbert structures on their domains, and

$$\mathrm{Dom}(Q^D_{m,\Omega}) = \widetilde{H}^m(\Omega) \subseteq \mathrm{Dom}(Q^N_{m,\Omega}),$$

where

$$\widetilde{H}^m(\Omega) = \{u \in H^m(\mathbb{R}^n) \,:\, \mathrm{supp}\, u \subset \overline{\Omega}\}\,.$$

It is also easy to see that for $m \in \mathbb{N}$, $u \in \widetilde{H}^m(\Omega)$

$$Q^D_{m,\Omega}[u] = Q^N_{m,\Omega}[u].$$

In [12] and [14] we compared the operators $(-\Delta_\Omega)^m_D$ and $(-\Delta_\Omega)^m_N$ for non-integer m. It turned out that the difference between their quadratic forms is positive or negative depending on the fact whether $\lfloor m \rfloor$ is odd or even. However, roughly speaking, this difference disappears as $\Omega \to \mathbb{R}^n$.

Namely, denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. For any $u \in \mathrm{Dom}(Q^D_{m,\Omega})$ the form $Q^D_{m,\Omega'}[u]$ does not depend on $\Omega' \in F(\Omega)$ while the form $Q^N_{m,\Omega'}[u]$ does depend on $\Omega' \supset \Omega$, and the following relations hold.

Proposition 1 *([14, Theorem 2]). Let $m > -1$, $m \notin \mathbb{N}_0$. If $u \in \mathrm{Dom}(Q^D_{m,\Omega})$, then*

$$Q^D_{m,\Omega}[u] = \inf_{\Omega' \in F(\Omega)} Q^N_{m,\Omega'}[u], \quad \text{if} \quad 2k < m < 2k+1, \quad k \in \mathbb{N}_0; \tag{1.1}$$

$$Q^D_{m,\Omega}[u] = \sup_{\Omega' \in F(\Omega)} Q^N_{m,\Omega'}[u], \quad \text{if} \quad 2k-1 < m < 2k, \quad k \in \mathbb{N}_0. \tag{1.2}$$

The main result of our paper is a quantitative version of Proposition 1.

Theorem 1 *Assume that $m > 0$, $m \notin \mathbb{N}$. Let $u \in \widetilde{H}^m(\Omega)$, and let $\mathrm{supp}(u) \subset B_r \subset B_R \subset \Omega$. Then*

$$Q^N_{m,\Omega}[u] \le Q^D_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \quad \text{if} \quad \lfloor m \rfloor \,\vdots\, 2; \tag{1.3}$$

$$Q^D_{m,\Omega}[u] \le Q^N_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \quad \text{if} \quad \lfloor m \rfloor \,\not\vdots\, 2. \tag{1.4}$$

The proof of Theorem 1 is given in Section 2. In Section 3 we apply this result for studying the equations[1]

$$(-\Delta_\Omega)^m_N u = \lambda(-\Delta_\Omega)^s_N u + |u|^{2^*_m - 2} u \quad \text{in } \Omega, \tag{1.5}$$

$$(-\Delta_\Omega)^m_N u = \lambda |x|^{-2s} u + |u|^{2^*_m - 2} u \quad \text{in } \Omega, \tag{1.6}$$

where $0 \le s < m < \frac{n}{2}$ and $2^*_m = \frac{2n}{n-2m}$. By solution of (1.5) or (1.6) we mean a weak solution from $\mathrm{Dom}(Q^N_{m,\Omega})$.

In the basic paper [2] by Brezis and Nirenberg a remarkable phenomenon was discovered for the problem

$$-\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

which coincides with (1.5) and (1.6) with $n > 2$, $m = 1$, $s = 0$. Namely, the existence of a nontrivial solution for any small $\lambda > 0$ holds if $n \ge 4$; in contrast, for $n = 3$ non-existence phenomena for any sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical* for problem (1.7) (compare with [16], [8]).

[1] we assume that $0 \in \Omega$.

As was pointed out in [13], the Brezis–Nirenberg effect is a nonlinear analog of the so-called zero-energy resonance for the Schrödinger operators (see, e.g., [21] and [22, pp.287–288]).

After [2], a large number of papers have been focussed on studying the effect of lower order linear perturbations in noncompact variational problems, see for instance the list of references included in [8, Chapter 7] about the case $m \in \mathbb{N}$, $s = 0$, and the recent paper [13], where a survey of earlier results for the Dirichlet case was given. For the Navier case with non-integer m, the only papers we know consider $m \in (0, 1)$ and $s = 0$, see [18] and [1]. See also the recent paper [5] and references therein for nonlinear lower-order perturbations.

We consider the general case and prove the following result (see Section 3 for a more precise statement), that corresponds to [13, Theorem 4.2] .

Theorem 2 *Let $0 \le s < m < \frac{n}{2}$. If $s \ge 2m - \frac{n}{2}$ then n is not a critical dimension for the (1.5) and (1.6). This means that both these equations have ground state solutions for all sufficiently small $\lambda > 0$.*

Let us recall some notation. B_R is the ball with radius R centered at the origin, $\mathbb{S}_R$ is its boundary. We denote by c with indices all explicit constants while C without indices stand for all inessential positive constants. To indicate that C depends on some parameter a, we write $C(a)$.

2 Proof of Theorem 1

Notice that we can assume $u \in \mathcal{C}_0^\infty(\Omega)$, the general case is covered by approximation.

Proof of (1.3). Let $m = 2k + \sigma$, $k \in \mathbb{N}_0$, $\sigma \in (0, 1)$. Denote by $w^D(x, y)$, $x \in \mathbb{R}^n$, $y > 0$, the Caffarelli–Silvestre extension of $(-\Delta)^k u$ (see [4]), that is the solution of the boundary value problem

$$-\text{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad w\big|_{y=0} = (-\Delta)^k u,$$

given by the generalized Poisson formula

$$w^D(x, y) = c_1(n, \sigma) \int\limits_{\mathbb{R}^n} \frac{y^{2\sigma}\,(-\Delta)^k u(\xi)}{(|x - \xi|^2 + y^2)^{\frac{n+2\sigma}{2}}}\, d\xi. \tag{2.1}$$

In [4] it is also proved that

$$Q^D_{m,\Omega}[u] = Q^D_{\sigma,\Omega}[(-\Delta)^k u] = c_2(n, \sigma) \int\limits_0^\infty \int\limits_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w^D|^2\, dxdy. \tag{2.2}$$

Integrating by parts (2.1), we arrive at following estimates for $|x|>r$:

$$|w^D(x,y)| \le \frac{C(n,m)\,y^{2\sigma}\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+\sigma}{2}}}; \qquad |\nabla w^D(x,y)| \le \frac{C(n,m)\,y^{2\sigma-1}\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+\sigma}{2}}}. \tag{2.3}$$

Following [12, Theorem 3], we define, for $x \in \overline{B}_R$ and $y \ge 0$, the function

$$\widetilde{w}(x,y) = w^D(x,y) - \widetilde{\phi}(x,y),$$

where $\widetilde{\phi}(\cdot,y)$ is the harmonic extension of $w^D(\cdot,y)$ on B_R, that is,

$$-\Delta_x \widetilde{\phi}(\cdot,y) = 0 \quad \text{in } B_R; \qquad \widetilde{\phi}(\cdot,y) = w^D(\cdot,y) \quad \text{on } \mathbb{S}_R.$$

Clearly, $\widetilde{w}\big|_{y=0} = (-\Delta)^k u$ and $\widetilde{w}\big|_{x\in\mathbb{S}_R} = 0$. Further, we have

$$\begin{aligned}
\int_0^\infty \int_{B_R} y^{1-2\sigma} |\nabla \widetilde{w}|^2\,dxdy &= \int_0^\infty \int_{B_R} y^{1-2\sigma} (|\nabla w^D|^2 - 2\nabla w^D \cdot \nabla \widetilde{\phi} + |\nabla \widetilde{\phi}|^2)\,dxdy \\
&= \int_0^\infty \int_{B_R} y^{1-2\sigma} |\nabla w^D|^2\,dxdy - 2\int_0^\infty \int_{\mathbb{S}_R} y^{1-2\sigma} (\nabla w^D \cdot \mathbf{n})\,\widetilde{\phi}\,d\mathbb{S}_R(x)dy \\
&\qquad + \int_0^\infty \int_{B_R} y^{1-2\sigma} |\nabla \widetilde{\phi}(x,y)|^2\,dxdy.
\end{aligned} \tag{2.4}$$

Since $\widetilde{\phi}(\cdot,y) = w^D(\cdot,y)$ on $\mathbb{S}_R$, we can use (2.3) to get

$$\Big| \int_0^\infty \int_{\mathbb{S}_R} y^{1-2\sigma} (\nabla w^D \cdot \mathbf{n})\,\widetilde{\phi}\,d\mathbb{S}_R(x)dy \Big| \le \frac{C(n,m)\,R^{n-1}}{(R-r)^{2n+2m-1}} \cdot \|u\|^2_{L_1(\Omega)}.$$

Now we estimate the last integral in (2.4). It is easy to see that $|\nabla \widetilde{\phi}(\cdot,y)|^2$ is subharmonic in B_R and thus the function

$$\rho \mapsto \frac{1}{\rho^{n-1}} \int_{\mathbb{S}_\rho} |\nabla \widetilde{\phi}(x,y)|^2\,d\mathbb{S}_\rho(x)$$

is nondecreasing for $\rho \in (0,R)$. This implies

$$\begin{aligned}
\int_{B_R} |\nabla \widetilde{\phi}(x,y)|^2\,dx &= \int_0^R \int_{\mathbb{S}_\rho} |\nabla \widetilde{\phi}(x,y)|^2\,d\mathbb{S}_\rho(x)d\rho \\
&\le \frac{R}{n} \int_{\mathbb{S}_R} (|\nabla_x \widetilde{\phi}(x,y)|^2 + |\partial_y \widetilde{\phi}(x,y)|^2)\,d\mathbb{S}_R(x).
\end{aligned}$$

Using the fact that $\partial_y \widetilde{\phi}(x,y) = \partial_y w^D(x,y)$ for $x \in \mathbb{S}_R$ and the well known estimate

$$\int\limits_{\mathbb{S}_R} |\nabla_x \widetilde{\phi}(x,y)|^2 \, d\mathbb{S}_R(x) \le C(n) \int\limits_{\mathbb{S}_R} |\nabla_x w^D(x,y)|^2 \, d\mathbb{S}_R(x),$$

we can apply (2.3) and arrive at

$$\int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla \widetilde{\phi}(x,y)|^2 \, dxdy \le \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}.$$

In conclusion, from (2.4) we infer

$$\int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla \widetilde{w}|^2 \, dxdy \le \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla w^D|^2 \, dxdy + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}. \tag{2.5}$$

Now we use the Stinga–Torrea characterization of $Q^N_{\sigma,\Omega}$. Namely, a quite general result of [17] implies that

$$Q^N_{m,\Omega}[u] = Q^N_{\sigma,\Omega}[(-\Delta)^k u] = c_2(n,\sigma) \inf_{\substack{w|_{x\in\partial\Omega}=0 \\ w|_{y=0}=(-\Delta)^k u}} \int\limits_0^\infty \int\limits_\Omega y^{1-2\sigma} |\nabla w|^2 \, dxdy. \tag{2.6}$$

Relations (2.6), (2.5) and (2.2) give us

$$\begin{aligned} Q^N_{m,\Omega}[u] \le Q^N_{m,B_R}[u] &\le c_2(n,\sigma) \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla \widetilde{w}|^2 \, dxdy \\ &\le c_2(n,\sigma) \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla w^D|^2 \, dxdy + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)} \\ &\le Q^D_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \end{aligned}$$

and (1.3) follows.

Proof of (1.4). Let $m = 2k - \sigma$, $k \in \mathbb{N}$, $\sigma \in (0,1)$. Denote by $w^{-D}(x,y)$, $x \in \mathbb{R}^n$, $y > 0$, the "dual" Caffarelli–Silvestre extension of $(-\Delta)^k u$ (see [3] and [14]), that is the solution of the boundary value problem

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad y^{1-2\sigma}\partial_y w\big|_{y=0} = -(-\Delta)^k u,$$

given by the formula

$$w^{-D}(x,y) = c_3(n,\sigma)\int\limits_{\mathbb{R}^n} \frac{(-\Delta)^k u(\xi)}{(|x-\xi|^2+y^2)^{\frac{n-2\sigma}{2}}}\,d\xi. \tag{2.7}$$

Note that the representation (2.7) is true also for $n = 1 < 2\sigma$ while for $n = 1$, $\sigma = 1/2$ it should be rewritten as follows:

$$w^{-D}(x,y) = c_3(1,1/2)\int\limits_{\mathbb{R}^n} (-\Delta)^k u(\xi)\ln(|x-\xi|^2+y^2)\,d\xi.$$

It is also shown in [14] that

$$\begin{aligned} Q^D_{m,\Omega}[u] &= Q^D_{-\sigma,\Omega}[(-\Delta)^k u] \qquad (2.8)\\ &= \frac{1}{c_2(n,\sigma)}\bigg(2\int\limits_{\mathbb{R}^n}(-\Delta)^k u(x) w^{-D}(x,0)\,dx - \int\limits_0^\infty\int\limits_{\mathbb{R}^n} y^{1-2\sigma}|\nabla w^{-D}|^2\,dxdy\bigg). \end{aligned}$$

Integrating by parts (2.7), we arrive at following estimates for $|x| > r$:

$$|w^{-D}(x,y)| \le \frac{C(n,m)\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m-\sigma}{2}}}; \qquad |\nabla w^{-D}(x,y)| \le \frac{C(n,m)\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+1-\sigma}{2}}}. \tag{2.9}$$

Now we define, as in [14, Theorem 2],

$$\widehat{w}(x,y) = w^{-D}(x,y) - \widehat{\phi}(x,y), \qquad x \in \overline{B}_R,\ y \ge 0,$$

where

$$-\Delta_x \widehat{\phi}(\cdot,y) = 0 \quad \text{in } B_R; \qquad \widehat{\phi}(\cdot,y) = w^{-D}(\cdot,y) \quad \text{on } \mathbb{S}_R.$$

Clearly, $\widehat{w}\big|_{x\in\mathbb{S}_R} = 0$. Arguing as for (1.3) and using (2.9) instead of (2.3), we obtain

$$\int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla\widehat{w}|^2\,dxdy \le \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla w^{-D}|^2\,dxdy + \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)}. \tag{2.10}$$

We can use the "dual" Stinga–Torrea characterization of $Q^N_{-\sigma,\Omega}$. It was proved in [14] that

$$\begin{aligned} Q^N_{m,\Omega}[u] &= Q^N_{-\sigma,\Omega}[(-\Delta)^k u] \qquad (2.11)\\ &= \frac{1}{c_2(n,\sigma)}\sup_{w|_{x\in\partial\Omega}=0}\bigg(\int\limits_\Omega(-\Delta)^k u(x) w(x,0)\,dx - \int\limits_0^\infty\int\limits_\Omega y^{1-2\sigma}|\nabla w|^2\,dxdy\bigg). \end{aligned}$$

Relations (2.11), (2.10), (2.8) and the evident equality

$$\int_{B_R} (-\Delta)^k u(x)\widehat{\phi}(x,0)\,dx = 0\,,$$

give us

$$\begin{aligned}
Q^N_{m,\Omega}[u] \ge Q^N_{m,B_R}[u] &\ge \frac{1}{c_2(n,\sigma)}\left(2\int_{B_R}(-\Delta)^k u(x)\widehat{w}(x,0)\,dx - \int_0^\infty\int_{B_R} y^{1-2\sigma}|\nabla\widehat{w}|^2\,dxdy\right)\\
&\ge \frac{1}{c_2(n,\sigma)}\left(2\int_{B_R}(-\Delta)^k u(x)w^{-D}(x,0)\,dx - \int_0^\infty\int_{B_R} y^{1-2\sigma}|\nabla w^{-D}|^2\,dxdy\right)\\
&\quad - \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)} \le Q^D_{m,\Omega}[u] - \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)},
\end{aligned}$$

and (1.4) follows. The proof is complete. □

3 The Brezis–Nirenberg effect for Navier fractional Laplacians

We recall the Sobolev and Hardy inequalities

$$Q_m[u] \ \ge\ \mathcal{S}_m\left(\int_{\mathbb{R}^n}|u|^{2^*_m}\,dx\right)^{2/2^*_m} \tag{3.1}$$

$$Q_m[u] \ \ge\ \mathcal{H}_m\int_{\mathbb{R}^n}|x|^{-2m}|u|^2\,dx\,, \tag{3.2}$$

that hold for any $u \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ and $0 < m < \frac{n}{2}$. The *best Sobolev constant* $\mathcal{S}_m$ and the *best Hardy constant* $\mathcal{H}_m$ were explicitly computed in [6] and in [10], respectively.

It is well known that $\mathcal{H}_m$ is not attained, that is, there are no functions with finite left- and right-hand sides of (3.2) providing equality in (3.2). In contrast, it has been proved in [6] that $\mathcal{S}_m$ is attained by a unique family of functions, all of them being obtained from

$$\phi(x) = (1+|x|^2)^{\frac{2m-n}{2}} \tag{3.3}$$

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

A standard dilation argument implies that

$$\inf_{\substack{u\in \mathrm{Dom}(Q^D_{m,\Omega})\\ u\neq 0}} \frac{Q^D_{m,\Omega}[u]}{\Big(\int\limits_\Omega |u|^{2^*_m}\,dx\Big)^{2/2^*_m}} = \mathcal{S}_m.$$

The key fact used in further considerations is the equality

$$\inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega})\\ u\neq 0}} \frac{Q^N_{m,\Omega}[u]}{\Big(\int\limits_\Omega |u|^{2^*_m}\,dx\Big)^{2/2^*_m}} = \mathcal{S}_m\,, \tag{3.4}$$

that has been established in [15] (see also earlier results [9, 20] for $m = 2$, [8] for $m \in \mathbb{N}$ and [12] for $0 < m < 1$). Clearly, the Sobolev constant $\mathcal{S}_m$ is never achieved on $\mathrm{Dom}(Q^N_{m,\Omega})$.

The corresponding equality for the Hardy constant, that is,

$$\inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega})\\ u\neq 0}} \frac{Q^N_{m,\Omega}[u]}{\int\limits_\Omega |x|^{-2m}|u|^2\,dx} = \mathcal{H}_m\,, \tag{3.5}$$

was proved in [15] as well (see also [11] and [7] for $m \in \mathbb{N}$).

We point out that the infima

$$\Lambda_1(m,s) := \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega})\\ u\neq 0}} \frac{Q^N_{m,\Omega}[u]}{Q^N_{s,\Omega}[u]}\,, \qquad \widetilde{\Lambda}_1(m,s) := \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega}[u])\\ u\neq 0}} \frac{Q^N_{m,\Omega}[u]}{\int\limits_\Omega |x|^{-2s}|u|^2\,dx} \tag{3.6}$$

are positive and achieved. Since $\mathrm{Dom}(Q^N_{m,\Omega})$ is compactly embedded into $\mathrm{Dom}(Q^N_{s,\Omega})$, this fact is well known for $\Lambda_1(m,s)$ and follows from (3.5) for $\widetilde{\Lambda}_1(m,s)$.

Weak solutions to (1.5), (1.6) can be obtained as suitably normalized critical points of the functionals

$$\mathcal{R}^\Omega_{\lambda,m,s}[u] = \frac{Q^N_{m,\Omega}[u] - \lambda Q^N_{s,\Omega}[u]}{\Big(\int\limits_\Omega |u|^{2^*_m}\,dx\Big)^{2/2^*_m}}\,, \tag{3.7}$$

$$\widetilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u] = \frac{Q^N_{m,\Omega}[u] - \lambda \int\limits_\Omega |x|^{-2s}|u|^2\,dx}{\Big(\int\limits_\Omega |u|^{2^*_m}\,dx\Big)^{2/2^*_m}}\,, \tag{3.8}$$

respectively. It is easy to see that both functionals are well defined on $\mathrm{Dom}(Q^N_{m,\Omega}) \setminus \{0\}$.

In fact, we prove the existence of ground states for functionals (3.7) and (3.8). We introduce the quantities

$$\mathcal{S}^\Omega_\lambda(m,s) = \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega}) \\ u\neq 0}} \mathcal{R}^\Omega_{\lambda,m,s}[u]; \qquad \widetilde{\mathcal{S}}^\Omega_\lambda(m,s) = \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega}) \\ u\neq 0}} \widetilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u]\,.$$

By standard arguments we have $\mathcal{S}^\Omega_\lambda(m,s) \le \mathcal{S}_m$. In addition, if $\lambda \le 0$ then $\mathcal{S}^\Omega_\lambda(m,s) = \mathcal{S}_m$ and it is not achieved. Similar statements hold for $\widetilde{\mathcal{S}}^\Omega_\lambda(m,s)$.

We are in position to prove our existence result that includes Theorem 2 in the introduction.

Theorem 3 *Assume* $s \ge 2m - \frac{n}{2}$.

i) For any $0 < \lambda < \Lambda_1(m,s)$ *the infimum* $\mathcal{S}^\Omega_\lambda(m,s)$ *is achieved and (1.5) has a nontrivial solution in* $\mathrm{Dom}(Q^N_{m,\Omega})$.

ii) For any $0 < \lambda < \widetilde{\Lambda}_1(m,s)$ *the infimum* $\widetilde{\mathcal{S}}^\Omega_\lambda(m,s)$ *is achieved and (1.6) has a nontrivial solution in* $\mathrm{Dom}(Q^N_{m,\Omega})$.

Proof. We prove *i*), the proof of the second statement is similar. Using the relation (3.4) and arguing for instance as in [13] one has that if $0 < \mathcal{S}^\Omega_\lambda(m,s) < \mathcal{S}_m$, then $\mathcal{S}^\Omega_\lambda(m,s)$ is achieved.

Since $0 < \lambda < \Lambda_1(m,s)$, then $\mathcal{S}^\Omega_\lambda(m,s) > 0$ by (3.6).

To obtain the strict inequality $\mathcal{S}^\Omega_\lambda(m,s) < \mathcal{S}_m$ we follow [2], and we take advantage of the computations in [13].

Let ϕ be the extremal of the Sobolev inequality (3.1) given by (3.3). In particular,

$$M := Q_m[\phi] = \mathcal{S}_m \bigg(\int\limits_{\mathbb{R}^n} |\phi|^{2^*_m}\, dx \bigg)^{2/2^*_m}. \tag{3.9}$$

Fix a cutoff function $\varphi \in \mathcal{C}^\infty_0(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside the ball $\{|x| < 2\delta\}$.

If $\varepsilon > 0$ is small enough, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n}\varphi(x)\phi\left(\frac{x}{\varepsilon}\right) = \varphi(x)\left(\varepsilon^2 + |x|^2\right)^{\frac{2m-n}{2}}$$

has compact support in Ω.

From [13, Lemma 3.1] we conclude

$$
\begin{aligned}
\mathfrak{A}_m^\varepsilon := Q_{m,\Omega}^D[u_\varepsilon] \quad & \leq \quad \varepsilon^{2m-n}\left(M + C(\delta)\,\varepsilon^{n-2m}\right) \\
\mathcal{A}_s^\varepsilon := \int\limits_\Omega |x|^{-2s}|u_\varepsilon|^2\,dx \quad & \geq \quad \begin{cases} C(\delta)\,\varepsilon^{4m-n-2s} & \text{if} \quad s > 2m - \frac{n}{2} \\ \\ C(\delta)\,|\log\varepsilon| & \text{if} \quad s = 2m - \frac{n}{2} \end{cases} \\
\widetilde{\mathfrak{A}}_s^\varepsilon := Q_{s,\Omega}^N[u_\varepsilon] \quad & \geq \quad \mathcal{H}_s\,\mathcal{A}_s^\varepsilon \qquad\qquad [\text{ see (3.5) }] \\
\mathcal{B}^\varepsilon := \int\limits_\Omega |u_\varepsilon|^{2_m^*}\,dx \quad & \geq \quad \varepsilon^{-n}\left((M\mathcal{S}_m^{-1})^{2_m^*/2} - C(\delta)\,\varepsilon^n\right)\,.
\end{aligned}
$$

If m is an integer or if $\lfloor m\rfloor \not\vdots\, 2$, then by (1.2)

$$\widetilde{\mathfrak{A}}_m^\varepsilon := Q_{m,\Omega}^N[u_\varepsilon] \leq \mathfrak{A}_m^\varepsilon,$$

and we obtain

$$\mathcal{R}_{\lambda,m,s}^\Omega[u_\varepsilon] \leq \mathcal{S}_m\,\frac{1 + C(\delta)\,\varepsilon^{n-2m} - \lambda C(\delta)\,\varepsilon^{2m-2s}}{1 - C(\delta)\,\varepsilon^n}\,, \qquad \text{if} \quad s > 2m - \frac{n}{2} \tag{3.10}$$

$$\mathcal{R}_{\lambda,m,s}^\Omega[u_\varepsilon] \leq \mathcal{S}_m\,\frac{1 + C(\delta)\,\varepsilon^{n-2m} - \lambda C(\delta)\,\varepsilon^{n-2m}|\log\varepsilon|}{1 - C(\delta)\,\varepsilon^n}\,, \quad \text{if} \;\; s = 2m - \frac{n}{2}. \tag{3.11}$$

Thus, for any sufficiently small $\varepsilon > 0$ we have that $\mathcal{R}_{\lambda,m,s}^\Omega[u_\varepsilon] < \mathcal{S}_m$, and the statement follows.

It remains to consider the case $\lfloor m\rfloor \,\vdots\, 2$. Since $\|u_\varepsilon\|_{L_1(\Omega)} \leq C(\delta)$, the estimate (1.3) implies

$$\widetilde{\mathfrak{A}}_m^\varepsilon \leq \mathfrak{A}_m^\varepsilon + C(\delta) = \varepsilon^{2m-n}\left(M + C(\delta)\,\varepsilon^{n-2m}\right),$$

and we again arrive at (3.10), (3.11). □

For the case $s < 2m - \frac{n}{2}$ we limit ourselves to point out the next simple existence result, as in [13].

Theorem 4 *Assume* $s < 2m - \frac{n}{2}$.

i) There exists $\lambda^* \in [0, \Lambda_1(m,s))$ *such that for any* $\lambda \in (\lambda^*, \Lambda_1(m,s))$ *the infimum* $\mathcal{S}_\lambda^\Omega(m,s)$ *is attained, and hence (1.5) has a nontrivial solution.*

ii) There exists $\widetilde{\lambda}^* \in [0, \widetilde{\Lambda}_1(m,s))$ *such that for any* $\lambda \in (\widetilde{\lambda}^*, \widetilde{\Lambda}_1(m,s))$ *the infimum* $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ *is attained, and hence (1.6) has a nontrivial solution.*

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