Università degli studi di Udine

PhD Course in Computer Science, Mathematics and Physics

A journey through computability, topology and analysis

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Abstract

This thesis is devoted to the exploration of the complexity of some mathematical problems using the framework of computable analysis and descriptive set theory. We will especially focus on Weihrauch reducibility as a means to compare the uniform computational strength of problems. After a short introduction of the relevant background notions, we investigate the uniform computational content of the open and clopen Ramsey theorems. In particular, since there is not a canonical way to phrase these theorems as multi-valued functions, we identify 8 different multi-valued functions (5 corresponding to the open Ramsey theorem and 3 corresponding to the clopen Ramsey theorem) and study their degree from the point of view of Weihrauch, strong Weihrauch and arithmetic Weihrauch reducibility. We then discuss some new operators on multi-valued functions and study their algebraic properties and the relations with other previously studied operators on problems. These notions turn out to be extremely relevant when exploring the Weihrauch degree of the problem DS of computing descending sequences in ill-founded linear orders. They allow us to show that DS, and the Weihrauch equivalent problem BS of finding bad sequences through non-well quasi-orders, while being very "hard" to solve, are rather weak in terms of uniform computational strength. We then generalize DS and BS by considering $\boldsymbol{\Gamma}$-presented orders, where $\boldsymbol{\Gamma}$ is a Borel pointclass or $\boldsymbol{\Delta}^1_1$, $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$. We study the obtained DS-hierarchy and BS-hierarchy of problems in comparison with the (effective) Baire hierarchy and show that they do not collapse at any finite level. Finally, we focus on the characterization, from the point of view of descriptive set theory, of some conditions involving the notions of Hausdorff/Fourier dimension and Salem sets. We first work in the hyperspace $\mathbf{K}([0,1])$ of compact subsets of $[0,1]$ and show that the closed Salem sets form a $\boldsymbol{\Pi}^0_3$-complete family. This is done by characterizing the complexity of the family of sets having sufficiently large Hausdorff or Fourier dimension. We also show that the complexity does not change if we increase the dimension of the ambient space and work in $\mathbf{K}([0,1]^d)$. We also generalize the results by relaxing the compactness of the ambient space and show that the closed Salem sets are still $\boldsymbol{\Pi}^0_3$-complete when we endow $\mathbf{F}(\mathbb{R}^d)$ with the Fell topology. A similar result holds also for the Vietoris topology. We conclude by showing how these results can be used to characterize the Weihrauch degree of the functions computing the Hausdorff and Fourier dimensions.

Preface

Three years have gone by since the beginning of my Ph.D. (well, three and a half, thanks to the pandemic), and as I'm here, a couple of days before the submission deadline, fiddling around with LaTeX[1] and fixing a few things here and there, I would like to take a few moments to thank all the people that made this possible. But this raises a question: who made this possible? If there is free will, this does not seem an easy problem[2]. We happily work in ZFC here, so it is natural to assume that people have a choice[3].

The first one that comes to mind is certainly my supervisor Alberto Marcone. Without his close guidance, I would have most certainly got lost over some triviality. He never lost his patience, no matter how blatantly wrong my claims were. The results in this thesis and their presentation highly benefited from his careful and constant support. I would have never got to this point without him.

But then, following backwards the cause-effect chain, I guess should thank Alonzo Church, and Poisson, Lagrange, Laplace, Euler before him, and many others[4]. Going back in the list of my mathematical ancestors does not sound like a feasible option. I will just try to think through the path I followed, and thank the people that changed its course, without hoping to be exhaustive.

An important role in my journey was played by Steffen Lempp: thanks to his support, my visit to Madison was flawless in every aspect, and I am very thankful for his warm hospitality and great availability. I also thank Joe Miller, Mariya Soskova, and the whole logic group in UW-Madison, for letting me feel part of the group. Many thanks in particular to Jun Le Goh: despite he graduated not long before I visited Madison, he already had a deep understanding of many very advanced topics. He welcomed me to Madison as a friend, and working with him was extremely smooth and pleasant.

Special thanks to Arno Pauly, for the many enlightening conversations. He was always ready to provide hints and pointers, when not fully detailed answers (24/7 support, literally), and his suggestions really improved the quality of my work, even before being a coauthor.

Sincere thanks to all the people in the mathematical community for many inspiring conversations and their generous support. The list of the people that shared some thoughts with me is definitely too long to be entirely written here, but I mention in particular Vasco Brattka, Riccardo Camerlo, Raphaël Carroy, Damir Dzhafarov, Guido Gherardi, Kyle Hambrook, Jan Reimann, Luca San Mauro, Matthias Schröder, Paul Shafer, Ted Slaman, Betsy Stovall, and Linda Brown Westrick. Thanks to my office mates at UW-Madison, and especially to Geoffrey Bentsen and Polly Yu, sharing the room with you has been super fun. Among the mathematicians in Udine, special thanks go to Giovanna D'Agostino (sorry for all the shouting from the other side of the wall), Vincenzo Dimonte, Roberta Musina, Giovanni Panti, and Fabio Zanolin.

My path from high school to the writing of this thesis was not linear, and not even affine! In fact, I started off as an engineer. I remember, back in the days of my B.Sc., during a walk with my friend Gabriele Pergola, I was arguing in favor of being a programmer, while he was doubtful on that, stressing the importance of a theoretical understanding of the matter. At that time, I

[1] It is a huge lie that you don't have to worry about layout when using LaTeX.

[2] Without free will, I guess the only one to thank is the prime mover if there is one, or nobody otherwise.

[3] No constructivists were harmed during the writing of this thesis.

[4] `https://www.mathgenealogy.org/`.

would have never imagined finding myself talking about things that calling "non-computable" is like saying that the ocean is "some water". When it rains, it pours, I guess!

Two people that had a fundamental impact on my journey, and yet I believe they did not receive enough credit for this, are my M.Sc. supervisors Claudio Agostinelli and Marcus Hutter. The former put a lot of unexpected and undeserved trust in me, giving me the maximum freedom on the topics I explored in my M.Sc. thesis. The latter promptly welcomed me in Canberra, was extremely kind and available, and gave me a huge help in the development of the core part of that work. It is not an overstatement to say that, without their support, I would not be studying computability now.

I ended up in Udine after a long search (and I thank Andrea Sorbi for his help in this), and over the years my path crossed the one of many other people. In particular, I thank Marta Fiori Carones for welcoming me to Udine, Eleonora Pippia for all the times she merciless proved my conjectures wrong, and the several discussions on the canons of beauty, Nicola Gigante Ph.D. for trapping me in becoming a Ph.D. representative, for the many lunch and pre-coffee talks, and for keeping my cat each time I was away for a conference. Special thanks to Dario "Johnny" Della Monica, for the many breaks we spent together, formally defining everyday-life notions[5]. Many thanks also to Giovanni Soldà for the several conferences we attended together and all the evenings spent playing cards. We may be enrolled in different Universities, but in my mind, we are part of the same research group. Best wishes to Davide Castelnovo, Vittorio Cipriani, and Martina Iannella, who joined the Udine logic group after me. I hope the pandemic will not hinder (too much) your Ph.D. experience.

A different and very special shout-out goes to all my friends, which are now spread all over the world. I am lucky enough that I cannot name you all one by one, or I'll miss the deadline. Thanks in particular to some of the best cryptographers in Italy[6], and to their weak link, for always keeping the level of non-sense very high. Many thanks to Eugenia Franco, for all the hours spent with me carefully checking cumbersome computations. Thanks also to Mike Pashos, one of the best housemates one can possibly dream to find, for his patience with all the questions about English.

Thanks to my family, and in particular to my aunt Aurelia, for taking care of me and setting up her larder according to my tastes. Thanks to my sister Anna Chiara, who only has one brother and there is not much she can do about it. Thanks to my girlfriend Paola who, against all odds[7], is still with me today. Thanks to my cat Nova, who was not asked who she wanted to live with, and still never misses a high five.

Thanks to my parents, who are not here to see this through.

I deeply apologize to all the people that deserve an acknowledgment but were not mentioned. I have a good memory, but it's short.

[5] In joint work with Dario and Nicola, we defined the *crazy cat lady coefficient* of a person p as $\sum_{c \in C} p_c$, where C is the set of all the cats in the world and p_c is the percentage of ownership of c by p. You are a *crazy cat lady* if your coefficient is greater than 1. There is nothing wrong with being a crazy cat lady, but if you are you have to accept it. Luckily, my coefficient is only 1.

[6] Among the best 15, to be precise.

[7] If you know how hard it is to stay with me for more than a few hours, think about doing it for several years straight.

Contents

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Introduction

The main theme of this work is the study of the complexity of mathematical problems. This is, of course, a very vague statement, and it has to be made precise. There is not a single universal definition of complexity that captures all the different aspects of complexity we may be interested in: for example, a problem can be "complicated" because it deals with "complicated" objects, or because the process required to obtain a solution for a given input is "hard".

On the one hand, we can explore the "absolute" complexity of problems, for example classifying them according to the complexity of their definitions. This can be done with the tools provided by (effective) descriptive set theory, which is a branch of mathematics that studies "definable sets" in Polish spaces (i.e. separable and completely metrizable spaces). These sets are organized in hierarchies, e.g. the Borel hierarchy, and its effective counterpart, the lightface hierarchy. The structure of these hierarchies and of the sets that inhabit their various levels is the matter of study of descriptive set theory. These notions turned out to be very fruitful, having applications in analysis, algebra, and other areas of mathematics.

On the other hand, the study of the complexity of problems can be done with the tools of computability theory: this is a subfield of mathematical logic that flourished starting with the work of Turing and the formal definition of "algorithm". Such a notion is extremely robust[8] and leads to a very rich theory, having applications in algebra, computer science, and combinatorics. Intuitively, computability theory suggests a way to compare problems by means of their "relative" computational strength: can we solve problem P, if we are able to solve problem Q? We expect that a strategy to solve the "more difficult" problem Q yields a technique to solve the "easier" problem P.

In other words, this corresponds to studying how the problems behave under a fixed notion of reducibility. This includes a local analysis (e.g. characterize the degree of a specific, concrete problem) as well as a global one (e.g. explore the algebraic properties of the degree structure induced by the fixed notion of reducibility).

Computable analysis is a generalization of (classical) computability, where we take into account the influence the representation of a mathematical object has on our capability to compute with it. In particular, the idea that mathematical objects have to be symbolically represented before we can do any computation is already present in classical computability theory, even if it rarely makes any (significant) difference. In the context of computable analysis, representations (by infinite sequences of natural numbers) are used to induce a notion of computability on *represented spaces*, i.e. spaces whose elements can be represented via sequences of naturals, like e.g. $\mathbb{R}$ or the set of continuous functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

Mathematical problems are formalized using multi-valued functions (which, roughly speaking, are functions that can have multiple outputs, see Definition 1.3): indeed, they can be seen as sets of instance-solution pairs, where a single instance can have multiple solutions (this is in line with Kolmogorov's idea of a "calculus of problems"). This underlines the inherently interdisciplinary scope of computable analysis: any problem that can be phrased as a multi-valued function can be studied in this framework.

This also draws a connection with reverse mathematics, which is a foundational research program aimed at determining the set-existence axioms that are necessary to prove theorems from

[8] Many alternative approaches captured the same intuitive idea, see [85].

"ordinary" mathematics (see Section 1.3 for a more detailed presentation). In fact, reverse mathematics provides yet another criterion of complexity that can be used to compare problems, namely their demonstrative strength (over a fixed weak set of axioms).

Notice that, since many theorems can be phrased in the form "for all X, if $\varphi(X)$ then there exists Y s.t. $\psi(X, Y)$", they have a natural interpretation as problems: the instances are the objects X that satisfy φ, and the solutions for X are the objects Y that satisfy $\psi(X, Y)$. We will say a bit more on the connections between reverse mathematics and computable analysis in Section 2.1.2.

When comparing functions between represented spaces, a (uniform) counterpart of Turing reducibility is provided by *Weihrauch reducibility*: intuitively, we say that f is Weihrauch-reducible to g if we can uniformly translate f-instances into g-instances, and then g-solutions into f-solutions (possibly accessing the original f-instance). This notion induces a degree structure on problems, analogously to how Turing reducibility induces a quasi-order structure on the subsets of $\mathbb{N}$.

The connection between reverse mathematics and Weihrauch reducibility has been first underlined in [41], and ever since researchers investigated the notion from both the reverse mathematics and the computable analysis perspective. Moreover, there is a significant interplay between (effective) descriptive set theory and computable analysis and often results and techniques in one field can shed light on problems in the other.

There are other related notions of reducibility (computable reducibility, generalized Weihrauch reducibility, polynomial-time Weihrauch reducibility, and so on) with different constraints (e.g. uniformity, resource-sensitiveness, etc). Together with Medvedev and Muchnik reducibilities, we have a quite rich toolbox of reducibilities, underlining different computational aspects of the problems under investigation.

Structure of the thesis

This thesis presents a series of results in computable analysis, Weihrauch reducibility, and descriptive set theory. While these topics are within the realm of mathematical logic, our results have direct connections and applications to combinatorics, topology, and harmonic analysis.

Here we briefly describe the structure of the thesis and the content of each chapter. Chapters 1 and 2 are devoted to the presentation of the background notions and the main tools that will be used in the following chapters, while Chapters from 3 to 6 contain the original contributions.

In Chapter 1, after fixing some general notation, we introduce Type-2 Theory of Effectivity (Section 1.1), as a means to extend classical computability to the Baire space first (Section 1.1.1), and then to arbitrary represented spaces (Section 1.1.2). In Section 1.2, we define the main tools used in descriptive set theory, mentioning also their effective counterparts (Section 1.2.1), and the connections with the theory of represented spaces (Section 1.2.2). In Section 1.3, we briefly introduce the framework of reverse mathematics, which will provide a motivational background for the topics presented in Chapters 3 and 5.

In Chapter 2, we formally introduce the notion of Weihrauch reducibility (Section 2.1), presenting some common operations on multi-valued functions (Section 2.1.1), and some computational problems that scaffold the Weihrauch lattice (Section 2.1.2). We also introduce the arithmetic Weihrauch reducibility in Section 2.2.

In Chapter 3, we study the Weihrauch degree of some multi-valued functions arising from the open and clopen Ramsey theorems. In particular, in Section 3.1 we recall the precise statement for the open and clopen Ramsey theorems and prove some useful lemmas. In Section 3.2 we define the multi-valued functions corresponding to the open and clopen Ramsey theorems (Section 3.2.1) and study their degrees. We divide the analysis into: functions that are reducible to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Section 3.2.2), functions that are reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ but not to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Section 3.2.3), and functions that are not reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Section 3.2.4). In Section 3.2.5 we characterize the strength of these functions from the point of view of strong Weihrauch reducibility. Finally, in Section 3.3 we focus on the behavior of these functions under arithmetic Weihrauch reducibility, and in Section 3.4 we draw some conclusions and list some open problems. The results of this chapter appear in [78].

In Chapter 4, we introduce and discuss the algebraic properties of some new operators on multi-valued functions. In particular, in Section 4.1 we study the union of two problems, showing how it can be applied to the choice principles (Section 4.1.1) and the Ramsey principles (Section 4.1.2). Some extra results are discussed in (Section 4.1.3). These results were obtained while exploring the open and clopen Ramsey theorems (Chapter 3).

In Section 4.2 we introduce the first-order part of a problem, a notion recently defined by Dzhafarov, Solomon, and Yokoyama [31]. In joint work with Giovanni Soldà, we characterized the first-order part of a parallelized problem (Section 4.2.1) and explored the interaction between the first-order part and other operations on problems (Section 4.2.2).

In Section 4.3, we define the deterministic part of a multi-valued function. We show how this operation is influenced by the choice of the codomain space (Section 4.3.1), we explore its interactions with the first-order part (Section 4.3.2) and with other operators on problems (Section 4.3.3). We also mention how the deterministic part of a problem was (implicitly) used in the literature (Section 4.3.4). The contents of Section 4.3 are, together with the results of Chapter 5, joint work with Jun Le Goh and Arno Pauly, and are included in [46].

In Chapter 5, we study the uniform computational strength of the problem DS of finding an infinite descending sequence through a given ill-founded linear order, which (a fortiori) is equivalent to the problem BS of finding a bad sequence through a given non-well quasi-order. In particular, after showing that the lower cone of DS misses many arithmetical problems (Section 5.1.1), we compare its strength with other combinatorial principles on linear orders (Section 5.1.2) and with the Ramsey principles (Section 5.1.3).

We then study how the presentation of a linear/quasi order can influence the uniform computational strength of the problems DS and BS (Section 5.2) and introduce the problems $\boldsymbol{\Gamma}$-DS and $\boldsymbol{\Gamma}$-BS. We consider the cases where $\boldsymbol{\Gamma}$ is $\boldsymbol{\Sigma}^0_n$, $\boldsymbol{\Pi}^0_n$ or $\boldsymbol{\Delta}^0_n$ (Section 5.2.1), and then turn our attention to the cases where $\boldsymbol{\Gamma}$ is $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$ or $\boldsymbol{\Delta}^1_1$ (Section 5.2.2). We finally draw the conclusions and highlight some open questions in Section 5.3.

In Chapter 6, we study the descriptive complexity of the family of closed Salem subsets of the Euclidean space, and some related conditions on the Hausdorff and Fourier dimension. After a brief introduction on the relevant background notions (Section 6.1), we explore the descriptive complexity of the family of closed Salem subset of $[0,1]$ (Section 6.2), of $[0,1]^d$ (Section 6.3), and of $\mathbb{R}^d$ (Section 6.4). In particular, the results of Section 6.2 answer a question raised by Ted Slaman during the IMS Summer School in Logic in Singapore (2018). Finally, in Section 6.5, we present a few additional results on the topic. Most of the results of this chapter have been collected in [77]. The results of Section 6.2 are joint work with Ted Slaman and Jan Reimann.

In Chapter 7, we study the effective counterparts of the results presented in Chapter 6. In particular, after presenting and proving some useful results (Section 7.1), we briefly discuss the hyperspaces of closed and compact sets as represented spaces (Section 7.2), and, in Section 7.3, we characterize the lightface complexity of the previously studied conditions. Finally, in Section 7.4, we characterize the Weihrauch degrees of the maps computing the Hausdorff and Fourier dimension (answering, in particular, a question by Fouché and Pauly).

1
Background

We now introduce the notation and the basic concepts we will use in the rest of the work. This introduction is not meant to be self-contained, and I am assuming the reader is familiar with the basic notions in (classical) computability theory, for which he/she is referred to [85, 107].

We write $\mathbb{N} := \{0, 1, \dots\}$ for the set of natural numbers, 0 included. We denote with $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the standard sets of the integer, rational, real, and complex numbers respectively. We also fix an effective enumeration $(q_i)_{i\in\mathbb{N}}$ of $\mathbb{Q}$. We use the symbol $\wp(X)$ for the powerset of X.

Let $2^{\mathbb{N}}$ be the standard Cantor set, i.e. the set of (total) functions $\mathbb{N} \to 2$ (which can be canonically identified with the set of subsets of $\mathbb{N}$). We denote with 2^n the set of functions $n \to 2$ (or, equivalently, the n-fold cartesian product of $\{0, 1\}$ with itself), and we let $2^{<\mathbb{N}} := \bigcup_{n\in\mathbb{N}} 2^n$. We will often think of the elements of $2^{<\mathbb{N}}$ as finite strings of $0, 1$, while the elements of $2^{\mathbb{N}}$ are infinite strings. We write i^ω for the constantly i sequence. We sometimes describe a string by a list of its elements. E.g. we write

$$(n_0, n_1, \dots, n_k),$$

for the string $\sigma := i \mapsto n_i$. Similarly, we can describe an infinite string by $(n_0, n_1, \dots)$, when it is clear from the context how to continue the sequence. We write $()$ for the empty sequence.

We denote with $\mathbb{N}^{\mathbb{N}}$ the Baire space, i.e. the space of functions $\mathbb{N} \to \mathbb{N}$, and with $\mathbb{N}^{<\mathbb{N}}$ the set of finite strings of natural numbers. In Chapter 3, a central role is played by the *Ramsey space* $[\mathbb{N}]^{\mathbb{N}}$ of total functions $\mathbb{N} \to \mathbb{N}$ that are strictly increasing (i.e. if $n < m$ then $f(n) < f(m)$). We write $[\mathbb{N}]^{<\mathbb{N}}$ for finite, strictly increasing strings.

For $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ we write:

- $\sqsubseteq$ for the prefix relation;
- $|\sigma|$ for the length of σ;
- $\sigma^\frown\tau$ for the concatenation of σ and τ;
- $\le_{lex}$ for lexicographical order;
- $\trianglelefteq$ for the domination relation, i.e. $\sigma \trianglelefteq \tau$ iff $|\sigma| = |\tau|$ and $(\forall i < |\sigma|)(\sigma(i) \le \tau(i))$.

The prefix relation can naturally be extended to the pairs $(\sigma, f) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. Similarly, we can consider $\sigma^\frown f$, for some finite string σ and some infinite string f. The lexicographical order and the domination relation also apply to infinite strings. Moreover, for every $x \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$ we write $x[m]$ for the prefix of x of length m.

If $f, g \in \mathbb{N}^{\mathbb{N}}$ we denote the composition $f \circ g$ by fg. Moreover, we write $f \preceq g$ if f is a subsequence of g, i.e. if there exists $h \in [\mathbb{N}]^{\mathbb{N}}$ s.t. $f = gh$. In particular for every $h \in [\mathbb{N}]^{\mathbb{N}}$ we have $gh \preceq g$. Similarly, we write $\sigma \preceq^* f$ if σ is a finite subsequence of f.

We will use the symbol $\langle \cdot \rangle$ to denote a fixed pairing function $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ (see e.g. [107, Notation I.3.6]). In particular, $\langle \cdot \rangle$ is a computable bijection with computable inverse. It is often convenient to write $\langle n_0, \dots, n_k \rangle$ in place of $\langle (n_0, \dots, n_k) \rangle$. In the literature, the symbol $\langle \cdot \rangle$ is often used to denote also the *join* between two (of the same length, finite or infinite) strings. With a (relatively) small abuse of notation, if x,y are two strings of the same length we will write[1]

$$\langle x, y \rangle(i) := \langle x(i), y(i) \rangle \ ;$$
$$\langle x_0, x_1, \dots \rangle(\langle i, j \rangle) := x_i(j) \ .$$

A set $T \subset \mathbb{N}^{<\mathbb{N}}$ is called *tree* if it is closed under the prefix relation, i.e. $\sigma \in T$ and $\tau \sqsubseteq \sigma$ implies $\tau \in T$. The *body* of a tree T, denoted $[T]$, is the set $\{x \in \mathbb{N}^{\mathbb{N}} : (\forall i)(x[i] \in T)\}$. For $\rho \in \mathbb{N}^{<\mathbb{N}}$ and T, S subtrees of $\mathbb{N}^{<\mathbb{N}}$ we define

- $\rho^\frown T := \{\rho^\frown \sigma : \sigma \in T\}$;
- $T \times S := \{\langle \sigma, \tau \rangle : \sigma \in T \text{ and } \tau \in S \text{ and } |\sigma| = |\tau|\}$.

We will use the symbol id_X to indicate the identity function on the space X. If $X = \mathbb{N}^{\mathbb{N}}$ we omit the subscript and just write id.

1.1 Computable analysis

Classical computability theory introduces a notion of computability for functions $f :\subseteq \mathbb{N} \to \mathbb{N}$. Intuitively speaking, we say that a Turing machine computes a function $f :\subseteq \mathbb{N} \to \mathbb{N}$ if, whenever executed with n represented on the input tape, it prints $f(n)$ on the output tape. The exact details of the definition of the Turing machine model are, often[2], unimportant. However, this idea contains the roots of the theory of represented spaces: formally, Turing machines transform finite strings into finite strings, and the burden of translating numbers into strings (and vice versa) is on the reader. In fact, the definition of computability for multivariate functions goes through some effective coding $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

In classical computability theory, however, objects are represented via finite strings on a finite or countable alphabet. However, this approach is not sufficient to deal with objects that require an infinite string to be represented (e.g. the reals).

Type-2 Theory of Effectivity (TTE) extends classical computability theory, introducing a notion of computability on the Baire space $\mathbb{N}^{\mathbb{N}}$ first, and then to more general represented spaces (Section 1.1.2). We stress the fact that, in TTE, the underlying model of computation is still the

[1] The exact details of the definition of the join are often not relevant. E.g. a common way to define the join of $x, y \in \mathbb{N}^{\mathbb{N}}$ is letting $\langle x, y \rangle(2n) := x(n)$ and $\langle x, y \rangle(2n+1) := y(n)$. However, this definition does not work well in the context of Chapter 3.

[2] The details may specify the number of tapes, the number of heads, the allowed alphabets on each tape, etc. Most of the time, the exact details yield the same computability notion, and therefore the authors are free to choose the one that makes the presentation simpler. This also includes the choice of the way numbers are represented on the input and output tapes, as well as the particular pairing function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. All of the "natural" choices are equally valid, see [85].

one of (classical) Turing machines. There are alternative approaches to hypercomputation (e.g. the Blum-Shub-Smale model [8], the Infinite Time Turing Machine [49], or the Ordinal Time Machine [68]), but the underlying model of computation is intrinsically stronger than the classical Turing machine.

There is a slight shift in the perspective though: classically, objects are coded via a finite string, and the computation should stop in finite time, producing therefore a finite string. In the context of TTE, however, objects are coded with infinite strings, hence Turing machines that stop after a finite amount of time are only able to produce a finite string, which would not be a valid representation for the output. We instead let the Turing machine run forever and produce longer and longer prefixes of a representation of the output. Therefore, a converging run is one that never halts! This will be made precise in the following section. We notice that this is, intuitively, what happens in the applications: we cannot store the exact value of π, but we can obtain arbitrarily precise approximations in finite time.

1.1.1 Computability in the Baire space

As already mentioned, the effective pairing function $\langle\cdot\rangle$ induces a notion of computability on functions $f :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$. We use this notion to define a notion of computability on functions $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ as follows:

Definition 1.1: A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is *computable* if there is a computable function $f :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ s.t.

1. f is *monotone*, i.e. $\sigma \sqsubseteq \tau \Rightarrow f(\sigma) \sqsubseteq f(\tau)$;
2. f is an *approximating function* for F, i.e. $F(x) = y$ iff

$$(\forall n)(\exists m \geq n)(y[n] \sqsubseteq f(x[m])).$$

A point $p \in \mathbb{N}^{\mathbb{N}}$ is *computable* if the constant map $q \mapsto p$ is computable.

Equivalently, the computable partial functions on the Baire space are exactly those that can be computed by a Type-2 Turing machine, i.e. a Turing machine with a read-only input tape, a write-only output tape, and finitely many work tapes. A Type-2 machine behaves just like a standard Turing machine, except that it is allowed to run with an infinite string on the input tape. We say that a Type-2 machine computes F if, whenever executed with p on the input tape, it runs forever and, in the limit, writes $F(p)$ on the output tape (without mind changes). The induced notion of computability is equivalent to the one introduced in Definition 1.1 ([112, Lem. 2.1.11]).

Computable functions enjoy several closure properties: they are closed under composition ([112, Thm. 2.1.12]), primitive recursion ([112, Thm. 2.1.14]) and map computable points to computable points ([112, Thm. 2.1.13]).

An important property of computable functions is that they are continuous. Indeed, since the Baire is canonically endowed with the product topology (i.e. the basic (cl)open sets are those of the form $\sigma^\frown\mathbb{N}^{\mathbb{N}}$ for $\sigma \in \mathbb{N}^{<\mathbb{N}}$), a function $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous iff it admits a monotone approximating function $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$. This idea can be used to prove an analog of the utm-theorem in the context of functions $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.

Theorem 1.2 ([112, Thm. 2.3.13]):
There is a computable function $\mathsf{U} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$*, called universal computable function, s.t. for every continuous* $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *there exists* $p \in \mathbb{N}^\mathbb{N}$ *s.t.*

$$\forall q \in \mathrm{dom}(F) \quad \mathsf{U}(\langle p, q\rangle) = F(q) \ .$$

This can be proved using the fact that a function $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is identified by its graph

$$\mathcal{G}_f := \{(\sigma, \tau) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \, : \, f(\sigma) = \tau\},$$

which in turn can be coded as a single $p \in \mathbb{N}^\mathbb{N}$ by enumerating the strings $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(f)$, and letting $p(n) := \langle\langle\sigma_n\rangle, f(\sigma_n)\rangle$. Notice that $f :\subseteq \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is computable iff $\mathcal{G}_f$ is c.e.. In particular, this implies that computable (partial) functions $\mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ are exactly those of the form $\mathsf{U}(\langle p, \cdot\rangle)$, for some computable $p \in \mathbb{N}^\mathbb{N}$.

This result can be interpreted as saying that a function is continuous iff it is computable w.r.t. some oracle $p \in \mathbb{N}^\mathbb{N}$.

1.1.2 Computability on problems and theory of represented spaces

We would now like to extend the notion of computability to the context of multi-valued functions, i.e., intuitively speaking, functions that can assign multiple values to the same input. Formally, we can introduce[3] them as follows:

Definition 1.3: A *(partial) multi-valued function* $f :\subseteq X \rightrightarrows Y$ is a function $f\colon X \to \wp(Y)$. We define the *domain* of f as $\mathrm{dom}(f) := \{x \in X \, : \, f(x) \neq \emptyset\}$ and the *codomain* or *range* of f as Y.

Whenever $f(x) = \{y\}$, we just write $f(x) = y$. If, for every $x \in X$, $f(x)$ is a singleton, then we identify f with the (partial) function that maps each $x \in \mathrm{dom}(f)$ to the unique y s.t. $y \in f(x)$.

In other words we can think of a (partial) multi-valued function as a relation $f \subset X \times Y$. The difference between multi-valued functions and relations rests on the way the composition between multi-valued functions is defined (see e.g. [19]):

Definition 1.4: Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Y \rightrightarrows Z$ be multi-valued functions. We define the *composition* $g \circ f$ between f and g as

$$g \circ f :\subseteq X \rightrightarrows Z := x \mapsto \{z \in Z \, : \, (\exists y \in Y)(y \in f(x) \text{ and } z \in g(y))\},$$

with $\mathrm{dom}(g \circ f) := \{x \in \mathrm{dom}(f) \, : \, f(x) \subset \mathrm{dom}(g)\}$.

Notice that we do not have this restriction on the domain of the composition of two relations, i.e. if R and S are relations, $(x, y) \in R$ and $(y, z) \in S$ then $(x, z) \in S \circ R$.

[3] The exact formal definition of a multi-valued function is not relevant. It can be defined simply as a relation, or in more elaborated ways (e.g. [112, Sec. 1.4, p. 11]).

This restriction is motivated by the association of multi-valued functions with *computational problems*: we think of an input for f as an *instance* of a problem, and of $f(a)$ as the set of possible *solutions*. The definition of composition between multi-valued functions ensures that we can apply g to any solution of $f(x)$. In particular, it implies that $g \circ f$ is still a multi-valued function. In the following, we will often use the term "problem" as a synonym of partial multi-valued function.

The connection between computational problems and multi-valued functions can be made precise using the notion of representation: intuitively, we use elements of the Baire space to "name" elements of an arbitrary set X. Then, we use the notion of computability define on $\mathbb{N}^\mathbb{N}$ to define a notion of computability between arbitrary *represented spaces*.

> **Definition 1.5 ([112, Def. 2.3.1]):** A *represented space* is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ is a surjective partial function called *representation map*. If $p \in \mathbb{N}^\mathbb{N}$ and $\delta_X(p) = x$ we say that p is a *name* or a *code* for x.

Whenever there is no ambiguity, we will not write explicitly the representation map and just say that X is a represented space. Notice that the representation map is not required to be injective (a single element of the space can have multiple names) nor to be total (not every string is necessarily a name for some element of the space).

Using the Baire space as a "name space", we can transform the problem of computing a f-solution y for a f-instance x into the problem of computing a name of a solution from a name of the instance.

> **Definition 1.6:** Let X, Y be represented spaces and let $f :\subseteq X \rightrightarrows Y$ be a (partial) multi-valued function. A partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is called a *realizer* of f (we write $F \vdash f$) iff
> $$\forall p \in \operatorname{dom}(f \circ \delta_X) \quad \delta_Y(F(p)) \in f(\delta_X(p)) \ .$$

This is often visualized by saying that the diagram

$$\begin{tikzcd} \mathbb{N}^\mathbb{N} \arrow[r, "F"] \arrow[d, "\delta_X"] & \mathbb{N}^\mathbb{N} \arrow[d, "\delta_Y"] \\ X \arrow[r, Rightarrow, "f"] & Y \end{tikzcd}$$

commutes for all $p \in \operatorname{dom}(f \circ \delta_X)$. The notion of realizer was already present in [112], even if the term is not explicitly used.

Notice that a realizer F for f is a choice function for the family

$$\{\delta_Y^{-1} f \delta_X(p) \,:\, p \in \operatorname{dom}(f \circ \delta_X)\}.$$

Indeed, the fact that every (partial multi-valued) function admits a realizer is equivalent to the following choice principle:

$$(\forall \mathcal{F} \subset \wp(\mathbb{N}^\mathbb{N}) \setminus \{\emptyset\})((|\mathcal{F}| = |\mathbb{N}^\mathbb{N}|) \to (\exists \operatorname{ch}\colon \mathcal{F} \to \cup\mathcal{F})(\forall F \in \mathcal{F})(\operatorname{ch}(F) \in F)).$$

In the following, we will not deal with the difficulties of developing the theory with restricted choice principles, and we will freely use the axiom of choice whenever needed.

Using the notion of realizer, we can induce a notion of computability on multi-valued functions between represented spaces as follows:

Definition 1.7 ([112, Def. 3.1.3]): Let (X, δ_X) and (Y, δ_Y) be represented spaces. A point $x \in X$ is called δ_X-*computable* if it has a δ_X-computable name. A partial multi-valued function $f :\subseteq X \rightrightarrows Y$ is called (δ_X, δ_Y)-*realizer-continuous* (resp. (δ_X, δ_Y)-*computable*) if it has a continuous (resp. computable) realizer.

We will omit the explicit dependency from the representation maps whenever they are clear from the context. In particular, we will say that a point is computable if it has a computable name, and that a function is realizer-continuous (resp. computable) if it has a continuous (resp. computable) realizer.

If X and Y are represented spaces there is a canonical way to induce a representation on the spaces $X \times Y$, $X \sqcup Y$, $X^* := \bigcup_{n\in\mathbb{N}}(\{n\} \times X^n)$, and $X^\mathbb{N}$. In particular, we define

$\delta_{X\times Y} :\subseteq \mathbb{N}^\mathbb{N} \to X \times Y := \langle p, q\rangle \mapsto (\delta_X(p), \delta_Y(q))$;

$\delta_{X\sqcup Y} :\subseteq \mathbb{N}^\mathbb{N} \to X \sqcup Y$, $\delta_{X\sqcup Y}((0)^\frown p) := \delta_X(p)$ and $\delta_{X\sqcup Y}((1)^\frown p) := \delta_Y(p)$;

$\delta_{X^*} :\subseteq \mathbb{N}^\mathbb{N} \to X^* := (n)^\frown\langle p_1, \dots, p_n\rangle \mapsto (n, \delta_X(p_1), \dots, \delta_X(p_n))$;

$\delta_{X^\mathbb{N}} :\subseteq \mathbb{N}^\mathbb{N} \to X^\mathbb{N} := \langle p_1, p_2, \dots\rangle \mapsto (\delta_X(p_1), \delta_X(p_2), \dots)$.

By the universal function theorem for $\mathbb{N}^\mathbb{N}$ (Theorem 1.2), we can exploit a universal computable function $\mathsf{U} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ to induce a representation on the space of realizer-continuous partial functions $X \to Y$. Indeed we can define

$\delta(p) := f :\iff \mathsf{U}(\langle p, \cdot\rangle) \vdash f$;

see also [112, Sec. 2.3]. Moreover, if $A \subset X$ then the representation on X induces a representation on A defined as

$\delta_A :\subseteq \mathbb{N}^\mathbb{N} \to A := \delta_X|_A$.

We also introduce the following notion:

Definition 1.8: We define the *jump of the represented space* (X, δ_X) as the represented space $X' = (X, \delta_{X'})$, where a $\delta_{X'}$-name for x is a string $\langle p_0, p_1, \dots\rangle$ s.t. $(p_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{N}^\mathbb{N}$ and

$$\delta_X\left(\lim_{n\to\infty} p_n\right) = x.$$

Such a notion was introduced in [115, Def. 2.1] (albeit Ziegler only focused on representation maps $2^\mathbb{N} \to 2^\mathbb{N}$), and it is the main ingredient to define the jump of a multi-valued function (Definition 2.9).

Trivial examples of represented spaces are $(\mathbb{N}^\mathbb{N}, \mathrm{id})$, $(2^\mathbb{N}, \mathrm{id}|_{2^\mathbb{N}})$ and $(\mathbb{N}, p \mapsto p(0))$. A more interesting example is the set of real numbers. The most common representation of a real number is the *Cauchy representation*: every $x \in \mathbb{R}$ can be represented via a *rapidly converging* sequence of rationals, i.e. a sequence $(q_n)_{n\in\mathbb{N}}$ s.t.

$$\forall n \quad |q_n - x| \le 2^{-n}.$$

The idea of representing the elements of a space via rapidly converging Cauchy sequences does not apply only to the real numbers.

Definition 1.9 ([112, Def. 8.1.2]): Let $X = (X, d, \alpha)$ be a separable metric space, where $d\colon X \times X \to \mathbb{R}$ is the distance function and $\alpha\colon \mathbb{N} \to X$ is an enumeration of a dense subset of X. We define the *Cauchy representation* on X as the map $\delta_X :\subseteq \mathbb{N}^\mathbb{N} \to X$ defined as

$$\delta_X(p) = x :\Longleftrightarrow \lim_{n\to\infty} \alpha(p(n)) = x,$$

where $\operatorname{dom}(\delta_X) := \{p \in \mathbb{N}^\mathbb{N} : (\forall n)(\forall m > n)(|\alpha(p(n)) - \alpha(p(m))| \le 2^{-n})\}$.

We say that X is a *computable metric space* if the set

$$\{(i, j, n, m) \in \mathbb{N}^4 : q_i < d(\alpha(n), \alpha(m)) < q_j\}$$

is computably enumerable.

We can always assume that α is an injective map (i.e. every element of the dense subset of X has a unique index). In other words, for every computable metric space (X, d, α) there is an injective subsequence β of α s.t. the spaces (X, d, α) and (X, d, β) are computably homeomorphic (i.e. there is a computable bijection with computable inverse) [47, Thm. 2.9].

Notice that, since $\mathbb{N}$ is a represented space, the notion of computability defined via Definition 1.7 coincides with the classical notion of computability.

Similarly, the notion of computability for real numbers induced by the Cauchy representation on $\mathbb{R}$ coincides with the "classical" notion of computability for a real number (see e.g. [93, Ch. 1, Def. 3]). Moreover, it is easy to see that the definition of computable metric space could be equivalently given by asking that the restriction of the distance map d to $\operatorname{ran}(\alpha)^2$ is computable.

Notice also that, for functions $\mathbb{R} \to \mathbb{R}$, the notions of continuity and realizer-continuity agree. It is important to stress that, in general, if X and Y are topological spaces then the two notions need not agree.

To show that realizer-continuity does not imply continuity, we can consider the two topological spaces $X = Y := \{0, 1\}$ endowed respectively with the trivial and the discrete topology. We represent both X and Y via the map $p \mapsto \min\{p(0), 1\}$. It is straightforward to see that the identity $\mathrm{id}_{\{0,1\}}$ is realizer-continuous but not continuous. On the other hand, a simple counter-example for the other direction can be presented anticipating some notions that will be introduced formally in Section 1.2. Let us consider the following two representation maps on X: we let δ_Π be s.t. $\delta_\Pi^{-1}(1)$ is a Π^1_1-complete subset of $\mathbb{N}^\mathbb{N}$, and $\delta_\Pi^{-1}(0) := \mathbb{N}^\mathbb{N} \setminus \delta_\Pi^{-1}(1)$. We also define δ_Σ so that $\delta_\Sigma^{-1}(i) = \delta_\Pi^{-1}(1 - i)$. The identity $\mathrm{id}_X\colon (X, \delta_\Pi) \to (X, \delta_\Sigma)$ is trivially continuous, but cannot be realizer-continuous (any continuous realizer would contradict the Π^1_1-completeness of $\delta_\Pi^{-1}(1)$). A more concrete counter-example is given by [112, Ex. 9.a]: if we endow $\mathbb{R}$ with the map δ_b, representing a real number via its binary expansion, then the map $h_3 := x \mapsto 3x$ is continuous (trivially) but not computable (and, in particular, not realizer-continuous). See also [100, Ex. 2.9].

For a list of "common" represented spaces, we refer the reader to Section 2.1.2.

Admissible representations

Often, spaces are naturally endowed with some canonical topology, and it would be desirable that the topological structure agrees with the computational one, i.e. that the notions of continuity and realizer-continuity agree. We will consider (and mainly focus our attention on) the so-called *admissible representations*, which intuitively are those that satisfy this requirement.

Definition 1.10 ([112, Def. 2.3.2]): Let X, Y be represented spaces with $X \subset Y$. We say that a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *translates* δ_X to δ_Y iff $(\forall p \in \operatorname{dom}(\delta_X))(\delta_X(p) = \delta_Y F(p))$. We define the following relations:

$$\begin{aligned} \delta_X \le \delta_Y :\iff & \text{ there exists a computable translation from } \delta_X \text{ to } \delta_Y; \\ \delta_X \le_t \delta_Y :\iff & \text{ there exists a continuous translation from } \delta_X \text{ to } \delta_Y. \end{aligned}$$

In the first case we say that δ_X is *reducible* to δ_Y, while in the second case we say that δ_X is *continuously* or *topologically reducible* to δ_Y. Two representation maps are called *equivalent* (resp. *continuously equivalent*) iff $\delta_X \le \delta_Y$ and $\delta_Y \le \delta_X$ (resp. $\delta_X \le_t \delta_Y$ and $\delta_Y \le_t \delta_X$).

A simple observation is that, for every two represented spaces X, Y with $X \subset Y$, and every (partial) function $f :\subseteq X \to Y$ we have

- $f \circ \delta_X \le \delta_Y \iff f$ is computable;
- $f \circ \delta_X \le_t \delta_Y \iff f$ is realizer-continuous.

In particular, letting $\iota \colon X \hookrightarrow Y$ be the inclusion map, we have that $\delta_X \le \delta_Y$ (resp. $\delta_X \le_t \delta_Y$) iff ι is computable (resp. realizer-continuous).

Definition 1.11 ([98, Def. 1]): Let (X, τ_X) be a topological space. A representation map δ_X of X is called *admissible w.r.t.* τ_X if it is continuous and, for every other continuous representation map[4] δ on X, we have $\delta \le_t \delta_X$.

In other words, an admissible representation of X is $\le_t$-maximal among the continuous representations of X. We will just say that a representation is admissible if there is no ambiguity on the topology.

To make the connection between admissible representation and continuity explicit, we introduce the following notion.

Definition 1.12 ([98, Sec. 3.1]): Let (X, τ) be a topological space. A family $\mathcal{B} \subset \wp(X)$ is called *pseudobase* iff for every open set $U \subset X$, every $x \in U$ and every sequence $(y_n)_{n\in\mathbb{N}}$ converging to x,

$$(\exists B \in \mathcal{B})(\exists n_0 \in \mathbb{N})(\{x\} \cup \{y_n : n \ge n_0\} \subset B \subset U).$$

Theorem 1.13 ([98, Thm. 13, p. 530]):
A topological space (X, τ_X) admits an admissible representation δ_X iff it is T_0 and admits a countable pseudobase.

[4] To be precise, the definition used by Schröder requires that every continuous map $\phi :\subseteq \mathbb{N}^\mathbb{N} \to X$ is continuously reducible to δ_X, i.e. there is a continuous function $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ s.t. for every $p \in \operatorname{dom}(\phi)$, $\phi(p) = \delta_X G(p)$. The two definitions are readily seen to be equivalent: indeed, a representation map is admissible (in the sense of Definition 1.11) iff for every continuous function $\phi :\subseteq \mathbb{N}^\mathbb{N} \to X$, there is continuous $G :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ s.t. $\delta_X = \phi \circ G$ ([9, Ex. 3.26]).

We briefly outline the main steps needed to prove the previous theorem, as they highlight some interesting facts.

The fact that the existence of an admissible representation implies that the space is T_0 follows by cardinality reasons.

If δ_X is admissible for X, then the family $\mathcal{B} := \{\delta_X(\sigma^\frown\mathbb{N}^\mathbb{N}) : \sigma \in \mathbb{N}^{<\mathbb{N}}\}$ is a countable pseudobase for X ([98, Lemma 11]).

On the other hand, if (X, τ_X) is T_0 and $(\beta_n)_{n\in\mathbb{N}}$ is an enumeration of a countable pseudobase for X, then the map

$$\begin{aligned}\delta_X(p) := x :\iff & (\forall n \in \mathbb{N})(p(n) > 0 \to x \in \beta_{p(n)-1}) \text{ and} \\ & (\forall U \in \tau_X : x \in U)(\exists n \in \mathbb{N})(p(n) > 0 \text{ and } \beta_{p(n)-1} \subset U)\end{aligned}$$

is an admissible representation for X ([98, Thm. 12]). The condition "$p(n) > 0$" in the above definition is just a technical trick that allows the name of x to "give no information at step n".

In particular, since every base is a pseudobase (trivial from the definition), for every T_1 second-countable space, a δ_X-name of $x \in X$ is an enumeration of a family of open neighborhoods $(U_n)_{n\in\mathbb{N}}$ of x s.t. $\{x\} = \bigcap_n U_n$ and $(U_n)_{n\in\mathbb{N}}$ is coinitial (w.r.t. set inclusion) for the family of all open neighborhoods of x.

Besides, using the fact that if a pseudobase is only made of open sets then it is actually a base, it is easy to prove that being a second-countable T_0 space is equivalent to admitting an open and admissible representation map ([9, Ex. 3.34]).

Notice that the definition of admissibility we are using does not agree (in general) with the definition given in [112, Def. 3.2.8]. Indeed, the notion introduced in [112] requires the space to be T_0 and second-countable. However, there are T_0 spaces with a countable pseudobase that are not second-countable (see [98, Ex. 3]). The two notions agree for second-countable T_0 spaces.

The importance of admissible representation lies in the following result:

Theorem 1.14 ([98, Thm. 4, pp. 524-525]):
Let $(X, \delta_X, \tau_X), (Y, \delta_Y, \tau_Y)$ be represented topological spaces with continuous representation maps. For every $f :\subseteq X \to Y$, we have

1. *f sequentially-continuous $\land\, \delta_Y$ admissible $\Rightarrow f$ realizer-continuous*
2. *f realizer-continuous $\land\, \delta_X$ admissible $\Rightarrow f$ sequentially-continuous*

Notice that, for admissibly represented spaces, the notions of first and second countability are equivalent: indeed, if X is first-countable then the family $\{\mathrm{Int}(\delta_X(\sigma^\frown\mathbb{N}^\mathbb{N})) : \sigma \in \mathbb{N}^{<\mathbb{N}}\}$, where $\mathrm{Int}(\cdot)$ denotes the interior part, is a countable base for τ_X ([9, Ex. 3.29]).

In particular, since every first-countable space is sequential, we obtain the following corollary:

Corollary 1.15 ([112, Thm. 3.2.11]):
Let (X, δ_X, τ_X), (Y, δ_Y, τ_Y) be admissibly represented second-countable T_0 spaces. For every $f :\subseteq X \to Y$,

$$f \text{ is continuous} \iff f \text{ is realizer-continuous.}$$

The family of admissibly represented spaces enjoys many natural closure properties. In particular, if X is an admissibly represented space then so is every $A \subset X$ with the relative topology. If $\{X_i\}_{i\in\mathbb{N}}$ is a family of admissibly represented spaces then the cartesian product $\prod_{i\in\mathbb{N}} X_i$, endowed with the product topology, is admissible. For every set Y, let $\{f_i\}_{i\in\mathbb{N}}$ is a family of functions with $f_i :\subseteq Y \to X_i$. If the weak topology on Y induced by $\{f_i\}_{i\in\mathbb{N}}$ is T_0 then it is admissible. If X and Y are admissibly represented spaces and X is sequential then the space of continuous functions $\mathcal{C}(X, Y)$ is admissibly represented when endowed with the topology induced by the prebase

$$\{\{f : f(\{x\} \cup \{y_n\}_{n\in\mathbb{N}}) \subset U\} : (y_n)_{n\in\mathbb{N}} \text{ converges to } x \text{ and } U \in \tau_X\}.$$

For details the reader is referred to [98, Sec. 4].

It is important to underline that admissible representations are only topologically equivalent, but not necessarily computably so. In particular, different admissible representations may induce different notions of computability on a represented space. For example, for a T_0 space (X, τ_X) with a countable pseudobase $(\beta_n)_{n\in\mathbb{N}}$, Theorem 1.13 provides the admissible representation δ_X. An alternative (topologically equivalent) representation is the map δ that names a point x by a list of all the n s.t. $x \in \beta_n$. The reduction $\delta_X \leq_t \delta$ is p-computable, where $p \in \mathbb{N}^\mathbb{N}$ is a list of all the pairs (n, m) s.t. $\beta_n \subset \beta_m$. An explicit example of two admissible but not equivalent representations is given in Section 1.2.2.

Final topology on represented spaces

We notice that every represented space can be naturally endowed with a topology induced by the representation map, namely the *final topology*.

Definition 1.16: Let (X, δ_X) be a represented space. The *final topology* on X, denoted with $\mathscr{O}(X)$, is the finest topology that makes the representation map δ_X continuous. It can be defined as

$$\mathscr{O}(X) := \{U \subset X : (\exists V \subset \mathbb{N}^\mathbb{N})(V \text{ is open and } \delta_X^{-1}(U) = V \cap \operatorname{dom}(\delta_X))\}.$$

Clearly, the representation map is a quotient map $\operatorname{dom}(\delta_X) \to X$. In other words, the space $(X, \mathscr{O}(X))$ is homeomorphic to the quotient space obtained identifying the elements of $\operatorname{dom}(\delta_X)$ that have the same image via δ_X. In particular, this implies that it is sequential (as quotient space of a metric space).

The importance of the final topology lies in the fact that there is a very close connection between an admissible representation on X and the final topology on X.

To make this connection precise, let us denote with $\operatorname{seq}(\tau)$ the smallest sequential topology that contains τ (equivalently, the intersection of all sequential topologies that contain τ). The topology $\operatorname{seq}(\tau)$ is called *sequentialization* or *sequential coreflection* of τ.

Theorem 1.17 ([100, Prop. 3.9]):
Let $X = (X, \tau)$ be a topological space. If δ_X is an admissible representation for X then:

1. *δ_X lifts to convergent sequences, i.e. for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x, there is a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ that converges to p s.t. $\delta_X(p) = x$ and, for every n, $\delta_X(p_n) = x_n$;*

2. *δ_X is a quotient map for τ iff τ is sequential;*

> 3. δ_X *is admissible for* $\operatorname{seq}(\tau)$*;*
>
> 4. $\operatorname{seq}(\tau)$ *is the final topology w.r.t.* δ_X *.*

In particular, δ_X is admissible w.r.t. τ iff it is admissible w.r.t. $\operatorname{seq}(\tau)$. Moreover, if δ_X is admissible then $\mathscr{O}(X) = \operatorname{seq}(\tau)$, see also [98, Thm. 7].

1.2 Descriptive set theory

Descriptive set theory is a branch of mathematical logic that studies the "definable sets" in topological spaces, with a special focus on *Polish spaces* (i.e. separable and completely metrizable spaces). It explores the relation between the structural properties of sets and the complexity of their definitions, underlining how "pathological" examples can be avoided by restricting our attention to sets that have a "simple" definition. The ideas and techniques introduced with descriptive set theory are, however, very powerful, and have been used to prove results in analysis for which there was no known solution before.

It is not realistic to give an exhaustive presentation of the results in descriptive set theory in an introductory chapter of a thesis. We will just recall the main notions and the definitions that we will use in the rest of the work.

Although, originally, most of the focus was on the structural properties of (the subsets of) $\mathbb{R}$, it turned out that many results could be extended to Polish spaces. In particular, a central role is played by the Cantor space $2^{\mathbb{N}}$ and the Baire space $\mathbb{N}^{\mathbb{N}}$. This is motivated by the following facts:

- For every non-empty perfect Polish space X there is an embedding of $2^{\mathbb{N}}$ into X [62, Thm. 6.2].
- The Baire space $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a $\mathbf{G}_\delta$ subspace of $2^{\mathbb{N}}$ (see below for the definition of $\mathbf{G}_\delta$).
- For every Polish space Y there is a closed set $F \subset \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f \colon F \to Y$. If Y non-empty then f extends to a continuous surjection $\mathbb{N}^{\mathbb{N}} \to Y$ [62, Thm. 7.9].

We say that a topological space X is *zero-dimensional* if it is Hausdorff and has a basis consisting of clopen sets ([62, Sec. 7.A]). The $2^{\mathbb{N}}$ and the $\mathbb{N}^{\mathbb{N}}$ space are especially pivotal among the zero-dimensional spaces:

- The Cantor space $2^{\mathbb{N}}$ is the unique, up to homeomorphism, perfect, non-empty, compact, metrizable, zero-dimensional space [62, Thm. 7.4].
- The Baire space $\mathbb{N}^{\mathbb{N}}$ is the unique, up to homeomorphism, non-empty, Polish, zero-dimensional space, for which all compact sets have empty interior [62, Thm. 7.7].
- Every zero-dimensional separable metrizable space can be embedded into $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$. Moreover, every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$ (and hence to a $\mathbf{G}_\delta$ subset of $2^{\mathbb{N}}$) [62, Thm. 7.8].

There is an extremely useful connection between the spaces of (infinite) sequences on a discrete space A (e.g. $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$) and the trees of finite strings on A. This fact will be used extensively in the following sections.

Theorem 1.18 ([62, Prop. 2.4]):
A set $F \subset A^{\mathbb{N}}$ is closed iff there is a tree $T \subset A^{<\mathbb{N}}$ s.t. $F = [T]$. The map $T \mapsto [T]$ is a bijection between pruned trees and closed subsets of $A^{\mathbb{N}}$.

The Borel hierarchy

For every topological space (X, τ), the family $\mathbf{B}(X)$ of *Borel subsets of* X is the smallest σ-algebra containing the open sets. This is a fundamental notion in topology and analysis. The family of Borel subsets of X can be stratified in a hierarchy, called the *Borel hierarchy*.

Let ω_1 be the first uncountable ordinal. The levels of the Borel hierarchy are defined by transfinite recursion on $1 \le \xi < \omega_1$. The classical definition (e.g. [62, Sec. 11.B]) is usually given in the context of Polish (or, more generally, Hausdorff) spaces. However, there is a small modification that allows us to give the definition for a generic topological space, while being equivalent to the classical one whenever for every Hausdorff space[5]. We therefore give the definition in the more general setting (see e.g. [23, Sec. 2.1.1]): we start from the families $\mathbf{\Sigma}^0_1(X)$ and $\mathbf{\Pi}^0_1(X)$ of the open and the closed subsets of X respectively. Then, for every $\xi > 1$ we define:

$$\mathbf{\Sigma}^0_\xi(X) := \left\{\bigcup_n A_n \setminus B_n \,:\, A_n, B_n \in \mathbf{\Sigma}^0_{\xi_n}(X),\, \xi_n < \xi,\, n \in \mathbb{N}\right\},$$

$$\mathbf{\Pi}^0_\xi(X) := \{X \setminus A \,:\, A \in \mathbf{\Sigma}^0_\xi(X)\}.$$

Moreover, for every ξ, we define $\mathbf{\Delta}^0_\xi(X) := \mathbf{\Sigma}^0_\xi(X) \cap \mathbf{\Pi}^0_\xi(X)$. In particular $\mathbf{\Delta}^0_1(X)$ is the family of clopen subsets of X. If X is a metric space, we can always assume $A_n = X$ in the definition of $\mathbf{\Sigma}^0_\xi(X)$, i.e. $\mathbf{\Sigma}^0_\xi(X)$ sets can be written as $\bigcup_n B_n$ sets, for $B_n \in \mathbf{\Pi}^0_{\xi_n}(X)$ with $\xi_n < \xi$.

The families $\mathbf{\Sigma}^0_2(X)$ and $\mathbf{\Pi}^0_2(X)$ are often written resp. $\boldsymbol{F}_\sigma(X)$ and $\boldsymbol{G}_\delta(X)$. It is known that

$$\mathbf{B}(X) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi(X) = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi(X) = \bigcup_{\xi<\omega_1} \mathbf{\Delta}^0_\xi(X)\ ,$$

i.e. every Borel set is obtained in less than ω_1 steps from the open sets, iterating the operations of complement, countable union and countable intersection. Whenever there is no ambiguity we will drop the dependency from the space X, and simply write $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$. If X is Polish and uncountable, then the hierarchy does not collapse at any level $\xi < \omega_1$ [62, Thm. 22.4].

Proposition 1.19 ([62, Prop. 22.1]):
For each $\xi \ge 1$, the classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ are closed under finite intersections, finite unions and continuous preimages. Moreover,

$\mathbf{\Sigma}^0_\xi$ *is closed under countable unions;*

$\mathbf{\Pi}^0_\xi$ *is closed under countable intersections;*

$\mathbf{\Delta}^0_\xi$ *is closed under complements.*

[5]For non-Hausdorff spaces, a open set may not be the union of closed sets.

For every level ξ of the Borel hierarchy, the pointclass $\mathbf{\Delta}^0_{\xi+1}$ can be further stratified in the so-called *difference hierarchy*. This is again defined by transfinite induction as follows:

$\mathbf{D}_1(\mathbf{\Sigma}^0_\xi(X)) := \mathbf{\Sigma}^0_\xi(X)$;

$A \in \mathbf{D}_{\alpha+1}(\mathbf{\Sigma}^0_\xi(X)) :\Longleftrightarrow A = U \setminus B$ for some $U \in \mathbf{\Sigma}^0_\xi(X)$ and $B \in \mathbf{D}_\alpha(\mathbf{\Sigma}^0_\xi(X))$;

if λ is a limit ordinal, then $A \in \mathbf{D}_\lambda(\mathbf{\Sigma}^0_\xi(X)) :\Longleftrightarrow A = \bigcup_{\alpha<\lambda, \alpha \text{ even}} B_{\alpha+1} \setminus B_\alpha$ for some growing sequence $(B_\alpha)_{\alpha<\lambda}$ of sets in $\mathbf{\Sigma}^0_\xi(X)$,

where even ordinals are those in the form $\lambda + n$, where λ is a limit ordinal (or 0) and $n < \omega$ is even.

In Polish spaces, the Hausdorff-Kuratowski theorem states that the difference hierarchy exhausts the $\mathbf{\Delta}^0_{\xi+1}$ sets [62, Thm. 22.27]. In second-countable spaces, the Hausdorff-Kuratowski theorem holds iff there is no $\mathbf{\Delta}^0_2$-complete set ([23, Cor. 3.9], the definition of $\mathbf{\Gamma}$-complete set is given in Definition 1.23 below). Notice that, using the difference hierarchy, we can rewrite the pointclass $\mathbf{\Sigma}^0_\xi$ as the family of countable unions of sets in $\mathbf{D}_2(\mathbf{\Sigma}^0_{\xi_n})$ with $\xi_n < \xi$.

The projective hierarchy

While the Borel classes $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ and $\mathbf{\Delta}^0_\xi$ are closed under continuous preimages, they are not (in general) closed under continuous *images*[6].

Definition 1.20 ([62, Def. 14.1 and Sec. 32.A]): Let X be a Polish space. A set $A \subset X$ is called *analytic* if there is a Polish space Y and a continuous function $f: Y \to X$ s.t. $\operatorname{ran}(f) = A$. Analytic sets are denoted by $\mathbf{\Sigma}^1_1(X)$.

The complement of an analytic set is called *co-analytic*, and the family of co-analytic sets is denoted by $\mathbf{\Pi}^1_1(X)$.

Analytic sets can be equivalently defined as projections of Borel sets. Formally:

Proposition 1.21 ([62, Ex. 14.3]):
Let X be Polish and let $A \subset X$. The following are equivalent:

1. *A is analytic;*
2. *there is a Polish space Y and $B \in \mathcal{B}(X \times Y)$ s.t. $A = \operatorname{proj}_X(B)$;*
3. *there is a closed $F \subset X \times \mathbb{N}^\mathbb{N}$ s.t. $A = \operatorname{proj}_X(F)$;*
4. *there is a $\mathbf{G}_\delta$ set $G \subset X \times 2^\mathbb{N}$ s.t. $A = \operatorname{proj}_X(G)$.*

This, in turn, induces a characterization of the co-analytic sets (as co-projections of Borel sets).

The analytic and co-analytic sets, as the symbols suggest, are the first levels of a higher-order hierarchy, called *projective hierarchy*. In particular, for every Polish space X and every $n \in \mathbb{N}$ we define

[6] In particular, an old mistake by Lebesgue was thinking that Borel sets are closed under projection.

$$\mathbf{\Sigma}^1_{n+1}(X) := \left\{\operatorname{proj}_X(A) \,:\, A \in \mathbf{\Pi}^1_n(X \times \mathbb{N}^{\mathbb{N}})\right\};$$

$$\mathbf{\Pi}^1_{n+1}(X) := \{X \setminus A \,:\, A \in \mathbf{\Sigma}^1_{n+1}(X)\};$$

$$\mathbf{\Delta}^1_{n+1}(X) := \mathbf{\Sigma}^1_{n+1}(X) \cap \mathbf{\Pi}^1_{n+1}(X).$$

By Souslin theorem ([62, Thm. 14.11 and cor. 26.2]), for every uncountable Polish space X we have

$$\mathbf{B}(X) = \mathbf{\Delta}^1_1(X) \subsetneq \mathbf{\Sigma}^1_1(X).$$

This is not necessarily true for arbitrary separable metric spaces (see the remarks in [62, p. 282]).

Proposition 1.22 ([62, Prop. 37.1]):
For every $n \geq 1$, the classes $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$ are closed under countable intersections, countable unions and continuous preimages. Moreover,

$\mathbf{\Sigma}^1_n$ *is closed under continuous images (in particular, projections);*

$\mathbf{\Pi}^1_n$ *is closed under co-projections (universal quantification over Polish spaces);*

$\mathbf{\Delta}^1_n$ *is closed under complements.*

In the following we will mostly focus on the classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$.

Wadge reducibility

The Borel classes and the projective classes are often called *pointclasses*, while, for every pointclass $\mathbf{\Gamma}$, a set $A \in \mathbf{\Gamma}$ is called *pointset*[7].

The fact that the classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$, and all the Borel classes, are closed under continuous preimages, suggests the following definition:

Definition 1.23 ([62, Def. 21.13 and Def. 22.9]): Let X and Y be topological spaces and $A \subset X$, $B \subset Y$. We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function $f\colon X \to Y$ s.t.

$$x \in A \iff f(x) \in B\ .$$

Let $\mathbf{\Gamma}$ be a Borel or projective class. Assume that X and Y are Polish and X is zero-dimensional. We say that $B \subset Y$ is $\mathbf{\Gamma}$-*hard* if $A \leq_W B$ for every $A \in \mathbf{\Gamma}(X)$. If B is $\mathbf{\Gamma}$-hard and $B \in \mathbf{\Gamma}(Y)$ then we say that B is $\mathbf{\Gamma}$-*complete*.

The notion of Wadge-reducibility induces a quasi-order on the subsets of topological spaces, and the equivalence classes are called *Wadge degrees*.

Notice that, while the definition of Wadge reducibility makes sense for every topological space X and Y, in the definition of $\mathbf{\Gamma}$-hardness and $\mathbf{\Gamma}$-completeness we restrict our attention to Polish spaces and, in particular, we require that the domain of the map witnessing the reduction is zero-dimensional.

[7]The terms pointset and pointclass can be used in a more general context, and need not be one of the Borel or projective ones, see [82, Sec. 1.B].

A common technique to show that a set $B \subset X$ is $\mathbf{\Gamma}$-hard is to show that there is a Wadge reduction $A \leq_W B$, for some A which is already known to be $\mathbf{\Gamma}$-complete. Standard examples of $\mathbf{\Gamma}$-complete sets are the following (see [62, Sec. 23.A, sec. 27.A and Ex. 33.1]):

$$\begin{array}{ll}
Q_2 := \{x \in 2^{\mathbb{N}} \,:\, (\forall^\infty m)(x(m) = 0)\} & \mathbf{\Sigma}^0_2\text{-complete,} \\
N_2 := \{x \in 2^{\mathbb{N}} \,:\, (\exists^\infty m)(x(m) = 0)\} & \mathbf{\Pi}^0_2\text{-complete,} \\
S_3 := \{x \in 2^{\mathbb{N}\times\mathbb{N}} \,:\, (\exists k)(\exists^\infty m)(x(k,m) = 0)\} & \mathbf{\Sigma}^0_3\text{-complete,} \\
P_3 := \{x \in 2^{\mathbb{N}\times\mathbb{N}} \,:\, (\forall k)(\forall^\infty m)(x(k,m) = 0)\} & \mathbf{\Pi}^0_3\text{-complete,} \\
\mathrm{IF} := \{T \subset \mathbb{N}^{<\mathbb{N}} \,:\, \text{ T is a tree and } [T] \neq \emptyset\} & \mathbf{\Sigma}^1_1\text{-complete,} \\
\mathrm{UB} := \{T \subset \mathbb{N}^{<\mathbb{N}} \,:\, \text{ T is a tree and } |[T]| = 1\} & \mathbf{\Pi}^1_1\text{-complete,}
\end{array}$$

where $(\exists^\infty m)$ and $(\forall^\infty m)$ mean respectively $(\forall n \in \mathbb{N})(\exists m \geq n)$ and $(\exists n \in \mathbb{N})(\forall m \geq n)$.

Proposition 1.24 ([62, Ex. 22.11 and ex. 24.20]):
Let X be a Polish space. For every $\xi \geq 1$ and every $A \subset X$,

$$A \in \mathbf{\Sigma}^0_\xi \setminus \mathbf{\Pi}^0_\xi \iff A \text{ is } \mathbf{\Sigma}^0_\xi\text{-complete.}$$

The statement is true also interchanging $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

The above theorem cannot be extended to $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$. Indeed, the statement

$$A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Pi}^1_1 \iff A \text{ is } \mathbf{\Sigma}^1_1\text{-complete,}$$

is equivalent, over ZFC, to $\mathbf{\Sigma}^1_1$-*determinacy*, which is the principle asserting that every infinite game on $\mathbb{N}$ with payoff $W \subset \mathbf{\Sigma}^1_1(\mathbb{N}^\mathbb{N})$, is determined (see [62, Sec. 26.B and thm. 26.4]).

1.2.1 Effective descriptive set theory

We now consider the effective counterpart of the notions introduced in the previous section. In other words, we introduce a hierarchy of subsets of a topological space X where sets are classified according to their computability properties.

Classically, the focus is mainly on separable metric spaces (as in [82]). However, the theory can be developed in a more general context. An *effective second-countable space*[8] is a pair $(X, (B_n)_{n\in\mathbb{N}})$, where X is a second-countable space and $(B_n)_{n\in\mathbb{N}}$ is an enumeration of a basis for the topology of X s.t. there is a computable function $\psi\colon \mathbb{N}^3 \to \mathbb{N}$ s.t., for all n, m

$$B_n \cap B_m = \bigcup_{k\in\mathbb{N}} B_{\psi(n,m,k)} \ .$$

Equivalently, the above condition can be stated for finite intersections, requiring that there is a computable function $\varphi\colon \mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ s.t.

$$\bigcap_{i<|\sigma|} B^X_{\sigma(i)} = \bigcup_{k\in\mathbb{N}} B^X_{\varphi(\sigma,k)} \ .$$

[8] Effective second-countable spaces are often called *countably based* spaces in the literature (see e.g. [23, Sec. 2]). They are called *basic spaces* in [71, Sec. 2.3.1]. This conflicts with the notation used in [82], as the author calls “basic” every perfect Polish space (or $\mathbb{N}$) that is deemed relevant, and then develops the theory abstracting from the particular choices.

If $(B_n)_{n\in\mathbb{N}}$ satisfies the above condition we say that it is an *effective basis*. If $(X, (B_n^X)_{n\in\mathbb{N}})$ is an effective second-countable space and $Y \subset X$ comes with the induced topology then Y can be canonically endowed with an effective basis defining $B_n^Y := B_n^X \cap Y$ ([71, Sec. 2.3.2]). If $(X_i, (B_n^i)_{n\in\mathbb{N}})$, with $i < k$, are effective second-countable spaces then their product $\prod_{i<k} X_i$ can be canonically endowed with an effective basis defining $B_\sigma^X := \prod_{i<k} B_{\sigma(i)}^i$ for every $\sigma \in \mathbb{N}^k$. Similarly, this can be done for countable products of effective second-countable spaces (see e.g. [71, Sec. 2.3.3]).

Notice that, in a computable metric space (X, d, α), there is a canonical choice for an effective basis, namely $B_{\langle n,m\rangle} := B\,(\alpha(i), q_j)$. As an historical remark, the effective descriptive set theory is often developed in the context of *recursively presented metric spaces*: let (X, d, α) be a separable metric space, where d is a distance function and $\alpha\colon \mathbb{N} \to X$ is a dense sequence in X. We say that α is a *recursive presentation* of X if the conditions

$$
\begin{aligned}
P^{d,X}(i,j,k) &:\iff d(\alpha(i), \alpha(j)) \le q_k,\\
Q^{d,X}(i,j,k) &:\iff d(\alpha(i), \alpha(j)) < q_k
\end{aligned}
$$

are computable [82, Sec. 3B]. We underline that, in general, being a recursively presented metric space is strictly stronger than being a computable metric space ([47, Obs. 2.4 and Ex. 2.5]). However, for every computable metric space $X = (X, d, \alpha)$ there is a computable real $\beta \le 1$ s.t. the computable metric space $X' = (X, \beta d, \alpha)$ is a recursively presented metric space and there is a computable bijection $X \to X'$ with computable inverse ([47, Thm. 2.10]).

For every effective second-countable space $(X, (B_n)_{n\in\mathbb{N}})$, we say that $A \subset X$ is *effectively open* if $A = \bigcup_{n\in\mathbb{N}} B_{\varphi(n)}$ for some computable function $\varphi\colon \mathbb{N} \to \mathbb{N}$. The set of effectively open subsets of X, denoted by $\Sigma_1^0(X)$ is called *effective topology*[9]. In other words, an effective open set is a computable union of basic open sets. The complement of an effectively open set is called *effectively closed* and the family of all effectively closed subsets of X is denoted by $\Pi_1^0(X)$.

Notice that $\Sigma_1^0(X)$ sets can be indexed using the code for a computable function defining them. In other words, there is a canonical indexing $(A_i)_{i\in\mathbb{N}}$ of the $\Sigma_1^0(X)$ sets. This allows us to define

$$
\begin{aligned}
&\Sigma_2^0(X) := \{A \subset X \,:\, A = \textstyle\bigcup_{n\in\mathbb{N}} A_{\varphi(2n+1)} \setminus A_{\varphi(2n)}, \text{ for some computable } \varphi\};\\
&\Pi_2^0(X) := \{X \setminus A \,:\, A \in \Sigma_2^0(X)\}.
\end{aligned}
$$

We can inductively define the *(Kleene's) arithmetical hierarchy*, also called *lightface hierarchy*, by letting $(A_i^n)_{i\in\mathbb{N}}$ be an effective indexing of the $\Sigma_n^0(X)$ sets and defining

$$
\begin{aligned}
&\Sigma_{n+1}^0(X) := \{A \subset X \,:\, A = \textstyle\bigcup_{i\in\mathbb{N}} A^n_{\varphi(2n+1)} \setminus A^n_{\varphi(2n)}, \text{ for some computable } \varphi\};\\
&\Pi_{n+1}^0(X) := \{X \setminus A \,:\, A \in \Sigma_{n+1}^0(X)\}.
\end{aligned}
$$

Alternatively, the same classes can be defined by letting Σ_{n+1}^0 be the set of effective unions of Boolean combinations of Σ_n^0 sets ([102, Sec. 3]).

We can define the effective difference hierarchy $(\mathrm{D}_\xi(\Sigma_n^0(X)))_{\xi<\omega_1^{\mathrm{CK}}}$ where ω_1^{CK} is the first non-recursive ordinal (see [96, Sec. I.2.3]) by replacing $\boldsymbol{\Sigma}_n^0$ with Σ_n^0 in the definition of $\mathbf{D}_\xi(\boldsymbol{\Sigma}_n^0(X))$ and letting $\mathrm{D}_\lambda(\Sigma_n^0(X))$, for $\lambda < \omega_1^{\mathrm{CK}}$ limit ordinal, be the family of computable unions of difference of sets in $\Sigma_n^0(X)$ (see also [102, Sec. 3]). With this definition, the $\Sigma_{n+1}^0(X)$ sets are obtained as computable unions of $\mathrm{D}_2(\Sigma_n^0(X))$ sets.

If X is a computable metric space, the $\Sigma_{n+1}^0(X)$ classes can be equivalently defined by letting

$$\Sigma_{n+1}^0(X) := \{A \subset X \,:\, (\exists B \in \Pi_n^0(X \times \mathbb{N}))(A = \mathrm{proj}_X B)\},$$

[9]The terminology may be misleading, as (in general) $\Sigma_1^0(X)$ is not a topology on X.

as in e.g. [82, Sec. 3E]. As in the boldface case, the difference in the definition is due to the fact that, in non-Hausdorff spaces, an effectively open set may not be the effective union of Π^0_1 sets (as, e.g., for the Sierpiński space, see the following Section 1.2.2).

The arithmetical hierarchy can be extended, by means of Borel codes, to every $\xi < \omega_1^{\mathrm{CK}}$. The resulting hierarchy is called *hyperarithmetical hierarchy* (see[10] [82, Sec. 7B]).

For a computable metric space, the effective counterpart of the projective hierarchy, called *(Kleene's) analytical hierarchy*[11], is defined iteratively as follows:

$$\Sigma^1_1(X) := \left\{A \subset X \,:\, (\exists C \in \Pi^0_1(\mathbb{N}^\mathbb{N} \times X))(A = \mathrm{proj}_X\, C)\right\},$$
$$\Pi^1_1(X) := \{X \setminus A \,:\, A \in \Sigma^1_1(X)\},$$
$$\Sigma^1_{n+1}(X) := \left\{A \subset X \,:\, (\exists C \in \Pi^1_n(\mathbb{N}^\mathbb{N} \times X))(A = \mathrm{proj}_X\, C)\right\},$$
$$\Pi^1_{n+1}(X) := \{X \setminus A \,:\, A \in \Sigma^1_{n+1}(X)\}.$$

Moreover, for every n, we define $\Delta^1_n(X) := \Sigma^1_n(X) \cap \Pi^1_n(X)$. The $\Delta^1_1(\mathbb{N})$ pointclass coincides with the hyperarithmetic sets (see [96, Ch. II]). In general, the pointclass $\Delta^1_1(X)$ is the set of Borel subsets of X with recursive Borel code [82, Ex. 7B.6].

The lightface hierarchy can be relativized in a straightforward manner, by defining

$$\Sigma^{0,z}_1(X) := \left\{A \subset X \,:\, A = \bigcup_{n\in\mathbb{N}} B_{f(n)} \text{ for some } z\text{-computable function } f\right\},$$

and then, define the classes $\Pi^{i,z}_n$, $\Sigma^{i,z}_{n+1}$, $\Delta^{i,z}_n$, for $i < 2$, accordingly. It is important to mention that the lightface classes are universal for their corresponding boldface ones. Formally, if Γ is a lightface class among Σ^0_n, Π^0_n or Σ^1_n, Π^1_n and $\boldsymbol{\Gamma}$ is the corresponding boldface pointclass, then

$$P \in \boldsymbol{\Gamma}(X) \iff (\exists z \in \mathbb{N}^\mathbb{N})(P \in \Gamma^z(X)),$$

see e.g. [82, Thm. 3E.4].

Theorem 1.25:
All the arithmetical and analytical classes are closed under finite union and intersection. Moreover,

Σ^0_ξ *is closed under computable union;*

Π^0_ξ *is closed under computable intersection;*

Σ^1_n *is closed under projection over* $\mathbb{N}^\mathbb{N}$*;*

Π^1_n *is closed under co-projection over* $\mathbb{N}^\mathbb{N}$*;*

Δ^0_ξ *and* Δ^1_n *are closed under complements;*

see e.g. [82, Cor. 3E.2].

All the above classes are closed under computable preimages [82, Thm. 3G.2]. The analytical classes are closed under Δ^1_1*-preimages [82, Thm. 3E.5].*

[10] While Moschovakis develops the theory only for separable metric spaces, the same ideas can be applied to a generic effective second-countable space.

[11] Notice the different suffix between *analytic*, i.e. $\boldsymbol{\Sigma}^1_1$ and *analytical*, i.e. belonging to $\bigcup_n \Sigma^1_n$. Maybe not the most unambiguous choice of words, but it is widespread.

The lightface hierarchy can be used to induce a notion of computability on any effective second-countable space. We say that a partial function $f :\subseteq X \to Y$ between two effective second-countable spaces $(X, (B^X_n)_{n\in\mathbb{N}})$ and $(Y, (B^Y_n)_{n\in\mathbb{N}})$ is Γ-*recursive on its domain* iff there is $P \in \Gamma(X \times \mathbb{N})$ s.t., for all $x \in X$,

$$x \in \mathrm{dom}(f) \Rightarrow ((x, n) \in P \iff f(x) \in B^Y_n) \ .$$

A (partial) function is called *recursive on its domain* if it is Σ^0_1-recursive on it (see [82, Sec. 3G]). If f is total, being Γ-recursive corresponds to

$$G_f := \{(x, n) \,:\, f(x) \in B^Y_n\} \in \Gamma(X \times \mathbb{N}) \ .$$

We say that $A \subset X$ is *effectively Wadge reducible* to $B \subset Y$, and write $A \le_m B$, if there is a recursive functional $f : X \to Y$ s.t. $x \in A$ iff $f(x) \in B$.

Fix a lightface pointclass Γ as above. Assume Y is an effective Polish space and $B \subset Y$. We say that B is Γ-*hard* if $A \le_m B$ for every $A \in \Gamma(2^{\mathbb{N}})$. If B is Γ-hard and $B \in \Gamma(Y)$ then we say that B is Γ-*complete*.

1.2.2 Descriptive set theory and represented spaces

There is a close connection between descriptive set theory and the theory of represented spaces. As already mentioned, every separable metric space (X, d, α) can be endowed with the Cauchy representation. Moreover, for every $k \ge 1$, we can define the represented spaces $(\boldsymbol{\Sigma}^0_k(X), \delta_{\boldsymbol{\Sigma}^0_k(X)})$, $(\boldsymbol{\Pi}^0_k(X), \delta_{\boldsymbol{\Pi}^0_k(X)})$, $(\boldsymbol{\Delta}^0_k(X), \delta_{\boldsymbol{\Delta}^0_k(X)})$ inductively by:

- $\delta_{\boldsymbol{\Sigma}^0_1(X)}(p) := \bigcup_{\langle i,j\rangle \in \mathrm{ran}(p)} B\,(\alpha(i), q_j)$;
- $\delta_{\boldsymbol{\Pi}^0_k(X)}(p) := X \setminus \delta_{\boldsymbol{\Sigma}^0_k(X)}(p)$;
- $\delta_{\boldsymbol{\Sigma}^0_{k+1}(X)}(\langle p_0, p_1, \ldots\rangle) := \bigcup_{i\in\mathbb{N}} \delta_{\boldsymbol{\Pi}^0_k(X)}(p_i)$;
- $\delta_{\boldsymbol{\Delta}^0_k(X)}(\langle p, q\rangle) := \delta_{\boldsymbol{\Sigma}^0_k(X)}(p)$, iff $p, q \in \mathrm{dom}(\delta_{\boldsymbol{\Sigma}^0_k(X)})$ and $\delta_{\boldsymbol{\Sigma}^0_k(X)}(p) = X \setminus \delta_{\boldsymbol{\Sigma}^0_k(X)}(q)$,

where $B\,(x, r)$ denotes the ball with center x and radius r.

The set $\boldsymbol{\Sigma}^1_1(X)$ of analytic subsets of X can be seen as a represented space defining a name for S to be a name for a closed set $A \subset X \times \mathbb{N}^{\mathbb{N}}$ s.t. $S = \mathrm{proj}_X(A)$. Moreover, we can define a name for a coanalytic set $R \in \boldsymbol{\Pi}^1_1(X)$ to be a name for its complement.

Analogously, if $(Y, (B^Y_n)_{n\in\mathbb{N}})$ is an effective second-countable space, we can define a name for a $\boldsymbol{\Sigma}^0_1(Y)$ set U to be any $p \in \mathbb{N}^{\mathbb{N}}$ s.t. $U = \bigcup_{i\in\mathbb{N}} B^Y_{p(i)}$. A name p for a $\boldsymbol{\Sigma}^0_2(Y)$ set A is (the join of) a sequence $(p_n)_{n\in\mathbb{N}}$ of names of $\boldsymbol{\Sigma}^0_1(Y)$ sets $(A_n)_{n\in\mathbb{N}}$ s.t. $A = \bigcup_{n\in\mathbb{N}} A_{p_{2n+1}} \setminus A_{p_{2n}}$. We can then define a representation map for every Borel and analytic pointclass accordingly. With this in mind, we can prove the following:

Proposition 1.26:
Let $(Y, (B^Y_n)_{n\in\mathbb{N}})$ be an effective second-countable space. For every $A \subset Y$,

$$A \in \Sigma^0_k(Y) \iff A \in \boldsymbol{\Sigma}^0_k(Y) \text{ and } A \text{ has a computable } \delta_{\boldsymbol{\Sigma}^0_k(Y)}\text{-name.}$$

PROOF: By induction on k, see also [10, Sec. 3]. ■

Notice that a name for a closed subset of X is, in fact, a name for its complement. In other words, a closed set F is represented via a list of open balls whose union is the complement of F. This is the so-called *negative information representation* for the closed sets. The space $(\mathbf{\Pi}^0_1(X), \delta_{\mathbf{\Pi}^0_1(X)})$ is often denoted $(\mathcal{A}(X), \psi_-)$ in the literature. On the other hand, we may consider the so-called *positive information representation* ψ_+ for the closed sets: a name for F consists in an enumeration of all the basic open sets that *intersect* F. The space of the closed sets, represented via the positive information representation is often denoted $(\mathcal{V}(X), \psi_+)$, and its elements are called *closed overt sets* (see [27]). We can also consider the *full information representation* ψ, where a name for a closed set F is a string $\langle p, q\rangle$ s.t. $F = \psi_-(p) = \psi_+(q)$. The complexity of various topological operations, according to the different representations, has been explored in [12].

A natural question is how the notion of recursiveness compares with the notion of computability defined in the context of TTE by means of representation maps. Here is a small proposition to fill the gap. To avoid ambiguity, we use the word "computable" in the sense of TTE, while "recursive" is used in the sense of effective descriptive set theory.

Proposition 1.27:
Let $(X, \delta_X, (B^X_n)_{n\in\mathbb{N}})$, $(Y, \delta_Y, (B^Y_n)_{n\in\mathbb{N}})$ be two represented effective second-countable spaces, where the representation δ_X (resp. δ_Y) is the admissible representation that names a point $x \in X$ (resp. $y \in Y$) by a list of all the n s.t. $x \in B^X_n$ (resp. $y \in B^Y_n$). A (partial) function $f :\subseteq X \to Y$ is computable iff it is recursive on its domain.

PROOF: Assume f is computable and let F be a computable realizer for f. Since X is an effective space, there is a computable function φ_X s.t. for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$

$$\bigcap_{i<|\sigma|} B^X_{\sigma(i)} = \bigcup_{k\in\mathbb{N}} B^X_{\varphi_X(\sigma,k)} .$$

With a small abuse of notation[12], we define the following Σ^0_1 sets

$$G := \{(\varphi_X(\sigma,k), \tau(j)) \,:\, F(\sigma) = \tau \text{ and } j < |\tau|\};$$
$$P := \bigcup_{(i,j)\in G} B^X_i \times \{j\} .$$

Notice that, if $F(\sigma) = \tau$ then

$$f\left(\operatorname{dom}(f) \cap \bigcap_{i<|\sigma|} B^X_{\sigma(i)}\right) \subset \bigcap_{j<|\tau|} B^Y_{\tau(j)} .$$

In other words, if $F(\sigma) = \tau$ then every $x \in \operatorname{dom}(f)$ that has a name that begins with σ must be mapped, via f, to some $y \in \bigcap_{j<|\tau|} B^Y_{\tau(j)}$. This shows that, if $x \in \operatorname{dom}(f)$ and $(x, n) \in P$ then $f(x) \in B^Y_n$. On the other hand, recall that the functional F maps a name p for some $x \in \operatorname{dom}(f)$

to a name q of $f(x)$, i.e. to a list of all the indexes n s.t. $f(x) \in B^Y_n$. If $f(x) \in B^Y_n$ then, for every name p of x, there is a prefix σ of p s.t. $F(\sigma)(j) = n$, for some j. In particular, since $x \in \bigcap_{i<|\sigma|} B^X_{\sigma(i)}$, there is k s.t. $x \in B^X_{\varphi_X(\sigma,k)}$, and therefore $(x, n) \in B^X_{\varphi_X(\sigma,k)} \times B^Y_n$. This shows that $(x, n) \in P$.

Let us now assume that f is recursive on its domain with witness P. Let also $\varphi_P : \mathbb{N} \to \mathbb{N}^2$ be a computable function s.t. $P = \bigcup_{n\in\mathbb{N}} B^X_{\varphi_P(n)_0} \times \{\varphi_P(n)_1\}$. It is straightforward to show that f has a computable realizer: indeed, given a name p for $x \in \operatorname{dom}(f)$, we can computably produce a name for $f(x)$ by enumerating all the n s.t. $(p(i), n) \in \operatorname{ran}(\varphi_P)$ for some $i \in \mathbb{N}$. ■

Notice that the choice of the representation maps for X and Y played a crucial role in the previous proposition: if the representations were not admissible, we may not be able to use the realizer to retrieve the list of all the indexes of basic open sets containing $f(x)$. As a simple example, recall that multiplication by 3 is not a computable operation $\mathbb{R} \to \mathbb{R}$, if we represent the real numbers by their binary expansion (which is not an admissible representation).

Notice also that the fact that being an effective (second-countable) space is a property of the pair $(X, (B_n)_{n\in\mathbb{N}})$ and not just of the space X. In general, the lightface structure is not a topological property (as it is not invariant under homeomorphisms), but it depends on the choice of the basis and of its enumeration. In fact, every second-countable space has an effective basis (just expand a fixed basis with all the finite intersections, see below for an explicit example).

As already mentioned, two admissible representations on the same space may induce different notions of computability. Consider the following example: let $\mathbb{R}$ be represented with the Cauchy representation, and let $(B_n)_{n\in\mathbb{N}}$ be the effective basis for the Euclidean topology on $\mathbb{R}$ where $B_{\langle i,j\rangle}$ is the open ball with center q_i and radius q_j. It is straightforward to see that the Cauchy representation for $\mathbb{R}$ is equivalent to the representation that names a point with a list of all the B_n that contain it. We can now define a new effective basis $(C_n)_{n\in\mathbb{N}}$ as follows: fix a non-c.e. set $S \subset \mathbb{N}$ and define

- $C_{2n} := B_n$;
- $C_1 := \bigcup_{n\in S} \left(n - \frac{1}{4}, n + \frac{1}{4}\right)$;
- for every non-empty $\sigma \in \mathbb{N}^{<\mathbb{N}}$, $C_{2\langle\sigma\rangle+1} := C_1 \cap \bigcap_{i<|\sigma|} B_{\sigma(i)}$.

Clearly $(C_n)_{n\in\mathbb{N}}$ is a basis for the Euclidean topology on $\mathbb{R}$. It is actually an effective basis: given two indexes i and j, $C_i \cap C_j$ belongs to the basis and its index is computable from i and j. Indeed, if i and j are even then it is trivial. Moreover, $C_1 \cap C_{2n} = C_{2\langle(n)\rangle+1}$; if $\sigma \neq ()$, $C_{2\langle\sigma\rangle+1} \cap C_{2n} = C_{2\langle\sigma^\frown(n)\rangle+1}$, and finally, for every $\tau \in \mathbb{N}^{<\mathbb{N}}$, $C_{2\langle\sigma\rangle+1} \cap C_{2\langle\tau\rangle+1} = C_{2\langle\sigma^\frown\tau\rangle+1}$.

The two effective bases, however, do not induce the same notion of computability on $\mathbb{R}$. In other words, the two corresponding admissible representation maps are not equivalent (they are only topologically equivalent). To see this it is enough to consider the inclusion map $\mathbb{N} \hookrightarrow \mathbb{R}$: such a map is $(B_n)_{n\in\mathbb{N}}$-recursive, but not $(C_n)_{n\in\mathbb{N}}$-recursive: if it were $(C_n)_{n\in\mathbb{N}}$-recursive then there would be a c.e. way to tell whether $n \in C_1$, i.e. whether $n \in S$, contradicting the fact that S is not c.e..

While, most often, there is a natural choice for an effective basis, when working with represented spaces we can exploit the representation map to induce a lightface structure in a canonical way.

[12] We are using the fact that a computable functional $\mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is identified by a computable function $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ (see Definition 1.1).

Let us introduce the *Sierpiński space* $\mathbb{S} := \{0, 1\}$ (sometimes its elements are denoted $\perp$ and $\top$). The space $\mathbb{S}$ is endowed with the topology $\{\emptyset, \{1\}, \mathbb{S}\}$. Notice that such a topology is not metrizable, and in fact not T_1 (as 1 belongs to every non-empty open set). This space is represented as follows: the name for 0 is 0^ω, while every string that is not constantly 0 is a name for 1.

We can notice that, if (X, δ_X) is a represented space and $\mathscr{O}(X)$ is the final topology on X induced by δ_X, then the open sets $U \in \mathscr{O}(X)$ are exactly the subsets of X s.t. the characteristic function $\chi_U : X \to \mathbb{S}$ is realizer-continuous[13]. In particular, this means that we can represent an open set $U \in \mathscr{O}(X)$ using a name for χ_U (recall that realizer-continuous partial functions are canonically endowed with a representation map, see Section 1.1.2). This, in turn, allows us to extend the negative information representation to arbitrary represented spaces, representing a closed set (in the final topology on X) via a name for its complement. These ideas are essentially those leading to the formalization of synthetic topology ([34, Sec. 3.2]). Using the jumps of the Sierpiński space, we can obtain an analogous characterization for the pointclasses $\boldsymbol{\Sigma}^0_\xi(X)$ ([91, Sec. III and prop. 30], see also [26]).

In other words, using the Sierpiński space, we can define a representation map for the sets $\boldsymbol{\Sigma}^0_k(X)$, $\boldsymbol{\Pi}^0_k(X)$, $\boldsymbol{\Delta}^0_k(X)$, $\boldsymbol{\Sigma}^1_1(X)$, $\boldsymbol{\Pi}^1_1(X)$, $\boldsymbol{\Delta}^1_1(X)$, for any represented space (X, δ_X). For separable metric spaces, the two representations are equivalent (see [89, 10]).

The same ideas allow us to induce a lightface structure on any represented space. Indeed, for a represented space (X, δ_X), we can define the effectively open sets as follows:

$$A \in \Sigma^0_1(X) :\iff \text{the characteristic function } \chi_A : X \to \mathbb{S} \text{ of } A \text{ is computable.}$$

Such a choice is motivated by the following result:

Proposition 1.28 ([17, Prop. 2.9]):
Let X be a computable metric space and let $A \subset X$. The following are equivalent:

1. $A \in \Pi^0_1(X)$*;*
2. $\chi_{X \setminus A} : X \to \mathbb{S}$ *is computable.*

In other words, the Sierpiński space is useful to obtain a notion of semi-decidability in represented spaces. As in the boldface case, we can define the higher levels of the lightface hierarchy by means of the jumps of the Sierpiński space, namely define $A \in \Sigma^0_n(X)$ iff the characteristic function $\chi_A : X \to \mathbb{S}^{(n)}$ of A is computable, where $\mathbb{S}^{(n)}$ is the n-th jump of $\mathbb{S}$. There are several reasons why the approach via representation maps looks preferable. For a more detailed discussion the reader is referred to [23, 91, 88].

There is a close connection between the descriptive complexity of a set A and the descriptive complexity of the set of names of points in A.

Theorem 1.29 ([25, Thm. 68]):
Let (X, δ_X) be an admissibly represented second-countable T_0 space. For any countable ordinals $\alpha, \xi > 0$ and $A \subset X$

$$A \in \mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\xi(X)) \iff \delta_X^{-1}(A) \in \mathbf{D}_\alpha(\boldsymbol{\Sigma}^0_\xi(\mathrm{dom}(\delta_X)))$$

[13] This is a simple exercise. See also [89, Sec. 4].

The effective counterpart of the previous theorem can be stated as follows:

Theorem 1.30 ([23, Thm. 4.1]):
Let (X, δ_X) be an effective second-countable T_0 space. For any $n, m \in \mathbb{N}$, the map $\mathbf{D}_m(\mathbf{\Sigma}^0_n(\mathrm{dom}(\delta_X))) \to \mathbf{D}_m(\mathbf{\Sigma}^0_n(X))$ mapping P to $\delta_X(P)$ is computable. In particular, for every $A \subset X$

$$A \in \mathrm{D}_m(\Sigma^0_n(X)) \iff \delta_X^{-1}(A) \in \mathrm{D}_m(\Sigma^0_n(\mathrm{dom}(\delta_X)))$$

In particular, this shows that, to study the (effective) descriptive complexity of a subset A of a represented space X, it is enough to study the (effective) descriptive complexity of the set of names of points in A.

1.3 Reverse mathematics

Reverse mathematics is a subfield of mathematical logic whose goal is to characterize the demonstrative strength of mathematical statements. It started with the work of Friedman [37], where he asks "What are the proper axioms to use in carrying out proofs of particular theorems, or bodies of theorems, in mathematics? What are those formal systems which isolate the essential principles needed to prove them?". In other words, the goal of reverse mathematics is to establish the set-existence axioms needed to prove theorems from "ordinary" mathematics (e.g. real and complex analysis, number theory, topology of complete separable metric spaces).

The typical reverse mathematical questions are of the form: working in a relatively weak system of axioms B, what are the weakest axioms R we need to add to our system to prove a given theorem T? Does $B + R$ prove T and $B + T$ prove every axiom in R?

The reverse mathematics investigations are usually carried out in the context of *second-order arithmetic*: the formal language L_2 used is a two-sorted extension of the language of Peano arithmetic, augmented with a relation symbol $\in$. The second sort variables are thought as *set variables*, and $\in$ is intended as the membership relation. A L_2-structure M is a tuple

$$(|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M),$$

where $\mathcal{S}_M$ is a set of subsets of $|M|$, $+_M$ and $\cdot_M$ are binary operations on $|M|$, $<_M$ is a binary relation on $|M|$ and 0_M, 1_M, are elements of $|M|$. We work with the Henkin semantics, and therefore $\mathcal{S}_M$ is not necessarily $\wp(M)$. An *ω-model* is a L_2-structure where $|M| = \omega$ is the set of natural numbers and the interpretation of the operations, the constants and the $<_M$ relation are the usual ones. A *β-model* M is an ω-model s.t., for every Σ^1_1 formula φ (possibly with parameters in M), $M \models \varphi$ iff φ is true in the standard model. The difference between ω-models and β-models will play an important role in Chapter 5.

The early studies in the field revealed that a large number of theorems are equivalent to one of five subsystems of second-order arithmetic: the so-called Big Five systems can be briefly introduced as follows:

RCA_0 (Recursive Comprehension Axiom) : this is usually assumed to be the base theory. It consists of the basic axioms of Peano arithmetic (asserting that the model is a commutative linearly-ordered semiring), plus the *set induction scheme*:

$$(\forall X)(((0 \in X \land (\forall n)(n \in X \to n+1 \in X)) \to (\forall n)(n \in X));$$

the $\mathbf{\Sigma}^0_1$-*induction scheme*: for every Σ^0_1 formula φ, possibly with parameters in the model,

$$\big(\varphi(0) \land (\forall n)(\varphi(n) \to \varphi(n+1))\big) \to (\forall n)(\varphi(n));$$

the $\mathbf{\Delta}^0_1$-*comprehension scheme*: for every Σ^0_1 formulas φ and ψ, possibly with parameters in the model,

$$(\forall n)(\varphi(n) \leftrightarrow \psi(n)) \to (\exists X)(\forall n)(n \in X \leftrightarrow \varphi(n)).$$

WKL_0 (Weak König's Lemma) : RCA_0 plus the statement "every infinite binary tree has a path";

ACA_0 (Arithmetic Comprehension Axiom) : WKL_0 plus the arithmetic comprehension scheme;

ATR_0 (Arithmetic Transfinite Recursion) : ACA_0 plus every arithmetic formula can be iterated along a well-order;

$\mathbf{\Pi}^1_1{-}\mathrm{CA}_0$ (Π^1_1 Comprehension Axiom) : ATR_0 plus the $\mathbf{\Pi}^1_1$ comprehension scheme.

A thorough presentation of each of the big five, as well as a comprehensive list of the theorems that are equivalent to each of them, is out of the scope of this thesis, and the reader is referred to [106]. We briefly mention that, despite the big five still occupy a central position in the picture, the perspective of reverse mathematics has shifted after the proof that Ramsey's theorem for pairs is not equivalent to any of the big five: it is, in fact, provable from ACA_0 but does not prove, nor it is provable by WKL_0. Ever since, a wide variety of "natural" problems that do not fit the "big five" picture has been discovered, yielding the so-called "reverse mathematics zoo" [52].

We notice that many theorems from "ordinary mathematics" are of the form

$$(\forall X)(\varphi(X) \to (\exists Y)(\psi(X,Y))),$$

and therefore have a natural interpretation as problems: the instances are the objects X that satisfy φ, and the solutions for X are the objects Y that satisfy $\psi(X,Y)$. In other words, there is a close connection between reverse mathematics and computable analysis, as theorems can be formalized as multi-valued functions on represented spaces. This connection was made explicit in [41], and ever since techniques and results on one field have been used to shed light on the other.

2
Computable reducibilities

Now that we have a notion of computability for functions $\mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, and, more in general, for multi-valued functions between represented spaces, the natural step is to compare their computational strength. In other words, the idea is to generalize the notion of Turing reducibility to higher-order objects.

Before formally introducing the *Weihrauch reducibility* (and its variants), we briefly mention the notions of *Medvedev and Muchnik reducibilities* (see [109, 94]).

> **Definition 2.1:** Let $A, B \subset \mathbb{N}^\mathbb{N}$. We say that A is *Medvedev reducible* to B, and we write $A \le_{\mathrm{M}} B$, if
> $$(\exists e \in \mathbb{N})(\forall y \in B)(\Phi_e(y) \in A),$$
> where Φ_e is the e-th Turing functional. The non-uniform version of Medvedev reducibility is called *Muchnik reducibility*: formally, we say that A is Muchnik-reducible to B, and we write $A \le_w B$ (where w stands for "weak"), if
> $$(\forall y \in B)(\exists e \in \mathbb{N})(\Phi_e(y) \in A).$$

The idea behind Medvedev/Muchnik reducibility is that a set $A \subset \mathbb{N}^\mathbb{N}$ (sometimes called *mass problem*) corresponds to the set of "solutions" to a particular problem. For example, the problem of enumerating a set $E \subset \mathbb{N}$ corresponds to $\{f \in \mathbb{N}^\mathbb{N} : \operatorname{ran}(f) = E\}$. In this context, the reduction $A \le_{\mathrm{M}} B$ can be interpreted as "given a solution for B we can uniformly compute a solution for A".

It is easy to see that the Medvedev (resp. Muchnik) reducibility is a reflexive and transitive relation, and therefore it induces a degree structure on the subsets of $\mathbb{N}^\mathbb{N}$, called *Medvedev* (resp. *Muchnik*) *degrees*. The two degree structures have been extensively explored in the literature (see e.g. [105]).

2.1 Weihrauch reducibility

While the Medvedev/Muchnik reducibilities deal with higher-order objects than Turing reducibility, they have the "downside" of considering each single instance of a problem as a separate mass problem. As an example, we can consider the problem of finding a path through an ill-founded tree $T \subset \mathbb{N}^{<\mathbb{N}}$ with a unique path. The corresponding mass problem is the singleton $[T] = \{x\}$. In particular, different trees (may) correspond to different mass problems. A Medvedev reduction $[T] \le_{\mathrm{M}} [S]$ does not give any information on the difficulty of obtaining the tree S from the tree T.

As an additional example, given a computable tree T s.t. $[T] = \{x\}$ for some non-arithmetic x (the existence of such a tree follows e.g. by [96, Thm. II.4.2]), we can consider a convergent sequence $p := (p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ s.t. $\lim_n p_n = x$. It is obvious that $[T] \equiv_{\mathrm{M}} \mathsf{lim}_p$, where lim_p is the mass problem that corresponds to finding the limit of the sequence p. However, there is no computable way to obtain p from T.

Compare this example with a Turing reduction with a single oracle call: if $D \le_T E$, with $D, E \subset \mathbb{N}$, to answer the question $n \in D$ we computably map n to some m. We then query the oracle E on m, and computably obtain an answer for "$n \in D$?" from an answer to "$m \in E$?".

In general, a Medvedev/Muchnik reduction does not give any information on the complexity of the "pre-processing phase". This leads us to the notion of *Weihrauch reducibility*. We give the definition in its full generality, as a notion of reducibility of partial multi-valued functions on represented spaces.

Definition 2.2: Let X, Y, Z, W be represented spaces and $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions. We say that f is *Weihrauch reducible* to g, and write[1] $f \le_{\mathrm{W}} g$, if

$$(\exists \text{ computable } \Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}})(\forall G \vdash g)(\Psi\langle \mathrm{id}, G\Phi\rangle \vdash f),$$

where $\Psi\langle \mathrm{id}, G\Phi\rangle := \langle p, q\rangle \mapsto \Psi(\langle p, G\Phi(q)\rangle)$.

We say that f is *strongly Weihrauch reducible* to g, and write $f \le_{\mathrm{sW}} g$, if

$$(\exists \text{ computable } \Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}})(\forall G \vdash g)(\Psi G \Phi \vdash f).$$

The notion of Weihrauch reducibility was originally introduced in [111, p. 5]. For a historical digression see [17, Sec. 3]. It yields a quasi-order on problems, and, in turn, a degree structure called *Weihrauch degrees*.

Intuitively, the maps Φ and Ψ play the roles of "pre-processing" and "post-processing" phases of the computation, while g plays the role of the oracle. Since we are dealing with computability on represented spaces, the oracle call to g is actually an oracle to an *arbitrary* realizer G of g. In other words, $f \le_{\mathrm{W}} g$ if there are two computable maps Φ and Ψ s.t.

- for every name p_x for some $x \in \mathrm{dom}(f)$, $\Phi(p_x)$ is a name for $z \in \mathrm{dom}(g)$;
- for every name p_w for some $w \in g(z)$, $\Psi(\langle p_x, p_w\rangle)$ is a name for $y \in f(x)$.

Such a computation can be represented via a block diagram as in Figure 2.1. With a small abuse of notation, we may consider $\Psi :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, and therefore write $\Psi(p_x, p_w)$ instead of $\Psi(\langle p_x, p_w\rangle)$. The difference between Weihrauch and strong Weihrauch reduction is that, in the latter, the map Ψ is not allowed to have access to the original input p_x. In particular, it follows that $f \le_{\mathrm{sW}} g$ implies $f \le_{\mathrm{W}} g$.

[1] There is a potential source of ambiguity: the symbol for the Weihrauch reducibility ($\le_{\mathrm{W}}$) is essentially the same as the symbol for the Wadge reducibility ($\le_W$). While the reader may pay attention to the different font styles of the letter "W", we hope that the context can solve any ambiguity.

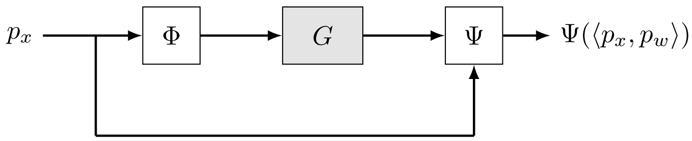

Figure 2.1: Block representation of the Weihrauch reduction

Alternatively, we can graphically represent the reduction $f \leq_W g$ as follows:

$$\begin{tikzcd} p_x \arrow[r, "\Phi(\cdot)"] \arrow[d, shift right, "f"] \arrow[d, shift left] & p_z \arrow[d, shift right] \arrow[d, shift left, "g"] \\ p_y & p_w \arrow[l, "\Psi(p_x,\cdot)"] \end{tikzcd}$$

This diagram motivates the following terminology: we say that the map Φ is the *forward functional* of the reduction, while Ψ is the *backward functional*. Unless otherwise mentioned, we will implicitly assume that Φ is the forward functional and Ψ is the backward one.

The non-uniform version of (strong) Weihrauch reducibility is called *(strong) computable reducibility* (see [52, Sec. 2.2]): the difference with (strong) Weihrauch reducibility is that the forward functional can depend on the particular instance of the problem, and the backward functional can depend on the particular solution. In other words, $f \leq_c g$ if every $x \in \mathrm{dom}(f)$ computes a $z \in \mathrm{dom}(g)$ and every solution $w \in g(z)$ is s.t. the join of x and w computes a solution $y \in f(x)$ (for the strong computable reduction, the solution y is w-computable).

Notice that, if Φ, Ψ witness the Weihrauch reduction $f \leq_W g$, then, for every name p_x for some $x \in \mathrm{dom}(f)$, the map Ψ witnesses the Medvedev reduction

$$\{\delta_Y^{-1}(y) \,:\, y \in f(x)\} \leq_M \{G(\Phi(p_x)) \,:\, G \vdash g\}.$$

If we assume that $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ then we can restate the previous reduction in a clearer form:

$$f(x) \leq_M g(\Phi(x)).$$

This highlights how Medvedev reduction is dealing with "only half" of the reduction of a problem f to a problem g.

As mentioned, many theorems can be written in a $\forall\exists$-form, and therefore have a natural interpretation as computational problems. In particular, we can use the framework of Weihrauch reducibility to study the (uniform) computational content of $\forall\exists$-statements. In contrast, Medvedev reducibility appears to be suited only for theorems in the $\exists$ form (i.e. finding a solution to a particular problem).

Intuitively, we can think of a Weihrauch reduction $f \leq_W g$ as a computation of f where g is used as an oracle, and is called *exactly* once[2]. The reason behind such a strong constraint on the number of oracle calls is that it allows a very fine-grained analysis of the computational strength

[2]Assuming that every problem has a realizer. Any problem g with no realizer is a top element w.r.t. Weihrauch reducibility. However, a reduction $f \leq_W g$ does not provide a way to solve f using g.

of problems. It is interesting to see that, in some cases, more than one oracle call is *needed*, and one does not suffice. In the next section, we introduce the operation and the formalism to allow for more than one, or even arbitrarily many, oracle calls.

We stress that the number of calls is exactly one, and cannot be zero. While this appears as a technicality, it is a small price to pay in order to develop a theory of computability on *arbitrary* represented spaces.

Definition 2.3: A problem $f :\subseteq X \rightrightarrows Y$ is called *pointed* if $\operatorname{dom}(f)$ has a computable point.

Almost every interesting problem is pointed, but non-pointed functions can be used to build particular counterexamples. In particular, no pointed function can be reduced to a non-pointed one. The same argument shows that

$$\mathrm{id} \equiv_{\mathrm{W}} f \iff f \text{ is pointed and computable.} \tag{$\star$}$$

Indeed, if f is computable then $f \le_{\mathrm{W}} \mathrm{id}$ (e.g. you can use either the forward functional to compute f), while if f is pointed then $\mathrm{id} \le_{\mathrm{W}} f$ (since Ψ has access to the original input, you only need to produce a computable input for f and then ignore the output). The same argument fails if f is not pointed (see also [13, Lem. 2.8]). This also shows that there is a (trivial) bottom Weihrauch degree: no problem can be Weihrauch reduced to a function with empty domain. On the other hand, no natural top degree is available (see [19])[3].

Observe that ($\star$) does not hold for the strong Weihrauch reducibility (id is not strong Weihrauch reducible to any constant function). Albeit this is a simple example, it shows that the Weihrauch and the strong Weihrauch reducibility are two very different notions (in some cases the two notions may agree, see Definition 2.5).

2.1.1 Operations on problems

There many natural operations that can be defined on problems, and that capture several intuitive ways of combining them (obtaining other problems). While they are actually operations on multi-valued functions (and so they could have been introduced in Section 1.1.2), we present them here are as they lift to Weihrauch degrees.

Definition 2.4: Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions. We define the following operations:

parallel product: $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ is defined as $(f \times g)(x, z) := f(x) \times g(z)$ with $\operatorname{dom}(f \times g) := \operatorname{dom}(f) \times \operatorname{dom}(g)$;

coproduct: $f \sqcup g :\subseteq X \sqcup Z \rightrightarrows Y \sqcup W$ with $\operatorname{dom}(f \sqcup g) := \operatorname{dom}(f) \sqcup \operatorname{dom}(g)$, defined as $(f \sqcup g)(0, x) := \{0\} \times f(x)$ and $(f \sqcup g)(1, z) := \{1\} \times g(z)$;

countable coproduct: the coproduct can be naturally extended to the countable case: if $\{f_i\}_{i\in\mathbb{N}}$ is a family of multi-valued functions with $f_i :\subseteq X_i \rightrightarrows Y_i$ then we define $\bigsqcup_{i\in\mathbb{N}} f_i :\subseteq \bigcup_{i\in\mathbb{N}} \{i\} \times X_i \rightrightarrows \bigcup_{i\in\mathbb{N}} \{i\} \times Y_i$ as

$$\left(\bigsqcup_{i\in\mathbb{N}} f_i \right)(i, x) := \{i\} \times f_i(x),$$

with $\operatorname{dom}(\bigsqcup_{i\in\mathbb{N}} f_i) := \bigcup_{i\in\mathbb{N}} \{i\} \times \operatorname{dom}(f_i)$;

[3] Recall that we are happily working in ZFC, hence, in particular, every problem has a realizer.

meet: $f \sqcap g :\subseteq X \times Z \rightrightarrows Y \sqcup W$ is defined as $(f \sqcap g)(x,z) := f(x) \sqcup g(z)$ with domain $\mathrm{dom}(f \sqcap g) := \mathrm{dom}(f) \times \mathrm{dom}(g)$;

finite parallelization: $f^* :\subseteq X^* \rightrightarrows Y^*$ is defined as

$$f(x_1, \dots, x_n) := \prod_{i=0}^{n} f(x_i),$$

with domain $\mathrm{dom}(f^*) := \mathrm{dom}(f)^* = \bigcup_{n\in\mathbb{N}} \mathrm{dom}(f)^n$;

(infinite) parallelization: $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ is defined as $f((x_n)_{n\in\mathbb{N}}) := \prod_{i\in\mathbb{N}} f(x_i)$ with domain $\mathrm{dom}(\widehat{f}) := \mathrm{dom}(f)^{\mathbb{N}}$.

All these operations are motivated by some intuition:

the product $f \times g$ corresponds to the problem of solving f and g in parallel. A solution of $f \times g$ is a solution for both f and g;

the coproduct $f \sqcup g$ corresponds to the problem of solving either f or g, but not both at the same time. Analogously for its countable analog (it is important to keep in mind that the countable coproduct is not degree theoretic);

the problem $f \sqcap g$ is similar to the problem $f \sqcup g$, in the sense that in both cases we obtain either a solution of f or g, but not both at the same time. The difference is that we are not able to specify whether we want to solve f or g but we only learn a posteriori whether the solution solves f or g;

the finite parallelization f^* correspond to solving finitely many instances of f (notice that the number of instances is part of the input). Similarly, its infinite analog $\widehat{f}$ corresponds to solving countably many instances of f in parallel.

Using the parallel product, we can introduce the following notion:

Definition 2.5 ([13, Def. 3.4]): A multi-valued function $f :\subseteq X \rightrightarrows Y$ is called a *cylinder* if $\mathrm{id} \times f \le_{\mathrm{sW}} f$.

Notice that the equivalence $f \equiv_{\mathrm{W}} \mathrm{id} \times f$ holds for every f (straightforward from the definition of Weihrauch reducibility), but in general we only have $f \le_{\mathrm{sW}} \mathrm{id} \times f$. A counterexample for the reverse reduction is e.g. a constant (total) function $\mathbb{N} \to \mathbb{N}$. Moreover, since id is a cylinder (straightforward from the fact that the pairing function $\langle\cdot\rangle$ is computable with computable inverse), we have that for every f, $\mathrm{id} \times f$ is a cylinder, and therefore every Weihrauch degree has a representative which is a cylinder.

Intuitively, we can think of a cylinder as a problem that “is able to use the output to (uniformly) reconstruct the (name for the) input”, i.e. we can compute x from $f(x)$, for every $x \in \mathrm{dom}(f)$. The notion of cylinder is very useful as it draws a simple connection between Weihrauch and strong Weihrauch reducibility.

Proposition 2.6 ([13, Cor. 3.6]):
A problem f is a cylinder iff for every problem g

$$g \le_{\mathrm{W}} f \iff g \le_{\mathrm{sW}} f.$$

This is especially useful to prove non-reductions, as if f is a cylinder then, to prove that $g \not\le_{\mathrm{W}} f$, it suffices to show that $g \not\le_{\mathrm{sW}} f$.

It is also possible to consider the composition between problems, as defined in Definition 1.4. Unfortunately, $f \circ g$ does not match very well the intuition of "applying g, and then applying f". The reason for this is that, for $f \circ g$ to be well-defined, we need that the codomain of g is contained in the domain of f. In many practical situations, however, we do not have this perfect match, but we are only able to use the output of g to *compute* a valid input for f. To capture this idea we introduce the following operation:

Definition 2.7 ([16, Def. 4.1]): For every multi-valued functions f and g, we define the *compositional product* $f * g$ as

$$f * g := \max_{\le_{\mathrm{W}}}\{f_0 \circ g_0 \,:\, f_0 \le_{\mathrm{W}} f \text{ and } g_0 \le_{\mathrm{W}} g\}.$$

We also write $f^{[n]}$ to denote the n-fold compositional product of f with itself, where $f^{[0]} := \mathrm{id}$ and $f^{[1]} := f$.

The fact that this operation is well-defined was proved in [19, Cor. 3.7]. Notice that the compositional product $f * g$ does not identify a single multi-valued function, but rather $*$ is an operator that maps two multi-valued functions to a Weihrauch degree. However, with a small abuse of notation, we will often write $h \le_{\mathrm{W}} f * g$ with the obvious meaning "h is Weihrauch-reducible to any problem in $f * g$".

Notice that if Φ is computable then $f \equiv_{\mathrm{W}} f \circ \Phi$ and $g \equiv_{\mathrm{W}} \Phi \circ g$ (whenever the compositions are well-defined), therefore $f * g$ matches the idea of being able to "do some computable operation to map the output of g to an input of f", and analogously for $f * g \le_{\mathrm{W}} h$.

Usually it is easier to prove that $h \le_{\mathrm{W}} f * g$ (it suffices to present a computable function Φ s.t. $h \le_{\mathrm{W}} f \circ \Phi \circ g$), rather than the opposite reduction $f * g \le_{\mathrm{W}} h$. A result that is extremely useful in practice is the so-called *cylindrical decomposition*:

Proposition 2.8 ([19, Lem. 3.10]):
For all f, g and all cylinders F, G with $F \equiv_{\mathrm{W}} f$ and $G \equiv_{\mathrm{W}} g$ there exists a computable K such that $f * g \equiv_{\mathrm{W}} F \circ K \circ G$.

In particular, knowing that for every function f we have that $f \equiv_{\mathrm{W}} \mathrm{id} \times f$ and that the latter is a cylinder, we can always take a representative of $f * g$ of the form $(\mathrm{id} \times f) \circ \Phi_e \circ (\mathrm{id} \times g)$ for some computable function Φ_e. In particular, we can always assume that $f * g$ is a cylinder.

Definition 2.9 ([16, Def. 5.1]): Recall that the jump of the represented space (X, δ_X) is the represented space $X' = (X, \delta_{X'})$, where a $\delta_{X'} := \delta_X \circ \lim$ (Definition 1.8). The *jump of* $f :\subseteq X \rightrightarrows Y$ is defined as $f' :\subseteq X' \rightrightarrows Y := x \mapsto f(x)$, where $\operatorname{dom}(f') := \operatorname{dom}(f)'$. We write $f^{(n)}$ to denote the result of applying the jump operation n times.

In other words, the difference between f and f' only rests on the representation of the input. Notice that, if $\langle p_0, p_1, \dots\rangle$ is a name for an input of f', we do not require that $p_i \in \operatorname{dom}(\delta_X)$.

It turns out that the name "jump" is an unfortunate one, as the jump does not behave on the Weihrauch degrees as the Turing jump behaves on Turing degrees. In fact, even if it is monotone (and hence degree-theoretic) w.r.t. the strong Weihrauch reducibility ($f \le_{\mathrm{sW}} g$ implies $f' \le_{\mathrm{sW}} g'$) and, for every f, $f \le_{\mathrm{sW}} f'$, trivial examples show that $f \equiv_{\mathrm{sW}} f'$ is possible (e.g. let f be any constant map). Moreover, it does not lift to Weihrauch degrees, and it is possible that $f \le_{\mathrm{W}} g$ while $g' <_{\mathrm{sW}} f'$ (take e.g. $f = \mathrm{id}$ and $g = \mathrm{id}_2$).

Let us introduce the problem $\mathsf{lim} :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ defined[4] as

$$\mathsf{lim}(\langle p_0, p_1, \dots\rangle) := \lim_{n\to\infty} p_n,$$

where $\operatorname{dom}(\mathsf{lim})$ consists of all converging sequences. Since an input for f' is a sequence converging to a name for an input $x \in \operatorname{dom}(f)$, it is clear that f and f' are connected via an application of lim.

Proposition 2.10 ([16, Cor. 5.16]):
For every problem f, $f' \le_{\mathrm{W}} f * \mathsf{lim}$. Moreover, if f is a cylinder then $f' \equiv_{\mathrm{W}} f * \mathsf{lim}$.

We conclude this section introducing the *totalization* of a function. This operation was introduced formally in [15], but it was already used in the literature (e.g. [84, Prop. 24], [64, Def. 8.1]).

Definition 2.11: For every $f :\subseteq X \rightrightarrows Y$, we define the *total continuation* or *totalization* of f, written $\mathsf{T}f$, as the total multi-valued function $\mathsf{T}f(x) \colon X \rightrightarrows Y$ defined as

$$\mathsf{T}f(x) := \begin{cases} f(x) & \text{if } x \in \operatorname{dom}(f) \\ Y & \text{otherwise.} \end{cases}$$

Clearly $\mathsf{T}f = f$ iff f is total. Notice that the definition of $\mathsf{T}f$ is sensitive to the particular definition of f as a multi-valued function between represented spaces. In particular, since $\operatorname{dom}(f)$ is a represented space (with the representation induced by δ_X), the restriction $f|_{\operatorname{dom}(f)} : \operatorname{dom}(f) \rightrightarrows Y$ is total, and trivially $f \equiv_{\mathrm{sW}} f|_{\operatorname{dom}(f)}$. Since there are examples of functions s.t. $f <_{\mathrm{W}} \mathsf{T}f$ (in particular this is the case for lim [15, Cor. 8.5]), this shows that $\mathsf{T}(\cdot)$ is not a degree-theoretic operation.

We omit a detailed presentation of how the above operations interact with each other. For a detailed compendium of such algebraic properties, the reader is referred to [19].

[4] We use a different font style to indicate the limit as a problem between represented spaces (lim), to distinguish it from lim, which denotes the classical topological limit.

2.1.2 An overview of the Weihrauch lattice

We now introduce a list of known represented spaces, and multi-valued functions on them, as they will be useful in the development of the work. We mention that every multi-valued function $f :\subseteq X \rightrightarrows Y$ has a strong Weihrauch equivalent version $f^r :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, sometimes called *realizer version* (see e.g. [17, Lem. 3.8]). This shows that, from the point of view of Weihrauch/strong Weihrauch degrees, it is enough to consider problems $\mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$. However, the algebraic properties of non-degree-theoretic operators may depend critically on the domain and codomain of the problem.

We already mentioned a few natural represented spaces, as well as a few canonical ways to induce a representation on products of represented spaces (see Section 1.1.2). In particular, $\mathbb{N}$, $\mathbb{S}$, $2^\mathbb{N}$, $\mathbb{N}^\mathbb{N}$, and $\mathbb{R}$ are represented spaces. Moreover, in Section 1.2.2 we defined the represented spaces $(\boldsymbol{\Sigma}^0_k(X), \delta_{\boldsymbol{\Sigma}^0_k(X)})$, $(\boldsymbol{\Pi}^0_k(X), \delta_{\boldsymbol{\Pi}^0_k(X)})$, $(\boldsymbol{\Delta}^0_k(X), \delta_{\boldsymbol{\Delta}^0_k(X)})$ for every represented space X and $k \geq 1$.

In Chapter 3, a central role is played by the Ramsey space $[\mathbb{N}]^\mathbb{N}$ of strictly increasing functions $\mathbb{N} \to \mathbb{N}$. This space is canonically endowed with the induced topology from the Baire space $\mathbb{N}^\mathbb{N}$, which makes it is computably isometric to $\mathbb{N}^\mathbb{N}$. There is actually a canonical choice for a computable bijection $\mathbb{N}^\mathbb{N} \to [\mathbb{N}]^\mathbb{N}$. Since $[\mathbb{N}]^\mathbb{N} \subset \mathbb{N}^\mathbb{N}$, a natural representation for $[\mathbb{N}]^\mathbb{N}$ is $\mathrm{id}|_{[\mathbb{N}]^\mathbb{N}}$.

We denote with $\mathbf{Tr}$ the space of trees on $\mathbb{N}$ represented via their characteristic function. Similarly, we denote with $\mathbf{Ti}$ the space of trees with strictly increasing strings, represented analogously. The function $[\cdot]\colon \mathbf{Tr} \to \boldsymbol{\Pi}^0_1(\mathbb{N}^\mathbb{N})$ that maps a tree to the set of its paths is computable with multi-valued computable inverse (this is a simple exercise). This implies that a closed set A of $\mathbb{N}^\mathbb{N}$ or $[\mathbb{N}]^\mathbb{N}$ can be equivalently represented via the characteristic function of a tree T s.t. $[T] = A$.

Similarly, an open set P of $\mathbb{N}^\mathbb{N}$ can be equivalently represented via an enumeration p of a prefix-free subset of $\mathbb{N}^{<\mathbb{N}}$ s.t. $P = \{f \in \mathbb{N}^\mathbb{N} : (\exists i)(p(i) \sqsubset f)\}$. With a small abuse of notation we may write $\tau \in p$ in place of $\tau \in \mathrm{ran}(p)$. The same considerations can be made for the space $[\mathbb{N}]^\mathbb{N}$.

We denote by $\mathrm{LO} = (\mathrm{LO}, \delta_{\mathrm{LO}})$ the represented space of linear orders on $\mathbb{N}$, where an order L is represented by the characteristic function of the set $\{\langle a, b\rangle \in \mathbb{N} : a \leq_L b\}$. Similarly, we denote by $\mathrm{WO} = (\mathrm{WO}, \delta_{\mathrm{WO}})$ and $\mathrm{QO} = (\mathrm{QO}, \delta_{\mathrm{QO}})$ respectively the represented spaces of well-orders and of countable quasi-orders on $\mathbb{N}$, both represented via the characteristic function of the relation. These represented spaces will be central in Chapter 5.

For every tree $T \subset \mathbb{N}^{<\mathbb{N}}$ we denote by $\mathrm{KB}(T)$ the Kleene-Brouwer order on T, defined as $\sigma \leq_{\mathrm{KB}(T)} \tau$ iff $\sigma, \tau \in T$ and $\tau \sqsubseteq \sigma$ or $\sigma \leq_{lex} \tau$. The map $T \mapsto \mathrm{KB}(T)$ from $\mathbf{Tr}$ to LO is computable. It is known that $\mathrm{KB}(T)$ is a well-order iff $[T] = \emptyset$ (see e.g. [106, Lem. V.1.3]).

Problems in the Weihrauch lattice

We now formally introduce a few problems that will be useful in the following sections to calibrate the computational strength of problems from the point of view of Weihrauch reducibility.

We have already introduced the problem lim of finding the limit of a convergent sequence in the Baire space. In general, we denote with lim_X the problem of finding the limit of a convergent sequence in a topological space X. The problem lim is closely related (in fact, strongly Weihrauch equivalent) to the problem $\mathsf{J}\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ consisting in computing the Turing jump of $p \in \mathbb{N}^\mathbb{N}$. Formally:

$$\mathsf{J}(p)(e) := \begin{cases} 1 & \text{if } \{e\}^p(e)\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

We mention that lim is a cylinder, and that for each n,

$$\mathsf{lim}^{(n)} <_{\mathrm{W}} \mathsf{lim}^{(n+1)} \equiv_{\mathrm{W}} \mathsf{lim}^{(n)} * \mathsf{lim}.$$

The problem $\mathsf{LPO}\colon \mathbb{N}^\mathbb{N} \to \{0,1\}$ (which stands for *Limited Principle of Omniscience*) is defined as $\mathsf{LPO}(p) := 1$ iff $(\exists n)(p(n) > 0)$. It is often convenient to think of LPO as the problem of finding a yes/no answer to a $\Sigma^{0,p}_1$ or $\Pi^{0,p}_1$ question. In particular, since asking whether $\mathsf{J}(p)(e) = 1$ is a $\Sigma^{0,p}_1$ question, it follows that $\mathsf{J} \le_{\mathrm{W}} \widehat{\mathsf{LPO}}$. The reduction $\widehat{\mathsf{LPO}} \le_{\mathrm{W}} \mathsf{J}$ follows from the Σ^0_1-completeness of the halting problem. The jump LPO' of LPO (and its iterated jumps $\mathsf{LPO}^{(k)}$) will play an important technical role. We notice that $\mathsf{lim}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}^{(n)}}$ (see e.g. [17, Thm. 6.7 and Prop. 6.10]).

An important family of problems is given by Ramsey's theorem for n-tuples and k colors: for every $A \subset \mathbb{N}$, let $[A]^n := \{B \subset A : |B| = n\}$ be the set of subsets of A with cardinality n. A map $c\colon [\mathbb{N}]^n \to k$ is called a *k-coloring* of $[\mathbb{N}]^n$, where $k \ge 2$. An infinite set H s.t. $c([H]^n) = \{i\}$ for some $i < k$ is called a *homogeneous solution* for c, or simply *homogeneous*.

The classical Ramsey theorem can be stated as follows:

Theorem 2.12 (Ramsey's theorem):
For every $n, k \ge 1$ and every coloring $c\colon [\mathbb{N}]^n \to k$ there is an infinite subset $H \subset \mathbb{N}$ that is homogeneous for c.

The set $\mathcal{C}_{n,k}$ of k-colorings of $[\mathbb{N}]^n$ can be seen as a represented space, where a name for a coloring c is the string $p \in \mathbb{N}^\mathbb{N}$ s.t. for each $(i_0, \dots, i_{n-1}) \in [\mathbb{N}]^n$, $p(\langle i_0, \dots, i_{n-1}\rangle) = c(i_0, \dots, i_{n-1})$.

We define $\mathsf{RT}^n_k\colon \mathcal{C}_{n,k} \rightrightarrows 2^\mathbb{N}$ as the total multivalued function that maps a coloring c to the set of all homogeneous sets for c. Similarly we define $\mathsf{RT}^n_\mathbb{N}\colon \bigcup_{k\ge 1} \mathcal{C}_{n,k} \rightrightarrows 2^\mathbb{N}$ as $\mathsf{RT}^n_\mathbb{N}(c) := \mathsf{RT}^n_k(c)$, where $k-1$ is the maximum of the range of c. Note that the input for $\mathsf{RT}^n_\mathbb{N}$ does not include information on which colors appear in the range of the coloring.

We also define $\mathsf{cRT}^n_k\colon \mathcal{C}_{n,k} \rightrightarrows k$ as the multivalued function that produces only the color of a homogeneous solution. We define $\mathsf{cRT}^n_\mathbb{N}$ analogously.

The problem $\mathsf{NON}\colon \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ maps a string p to the set $\{q \in \mathbb{N}^\mathbb{N} : q \text{ is not computable in } p\}$. Similarly, $\mathsf{NHA}\colon \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is defined as

$$\mathsf{NHA}(p) := \{q : q \text{ is not hyperarithmetic in } p\}.$$

When working with the represented space $(\boldsymbol{\Gamma}(X), \delta_{\boldsymbol{\Gamma}(X)})$, it is often awkward to construct $\delta_{\boldsymbol{\Gamma}(X)}$-names explicitly. If we want to construct a $\delta_{\boldsymbol{\Gamma}(X)}$-name for a set $A \subseteq X$, we typically only check that there is a $\boldsymbol{\Gamma}$-formula which defines A. By invoking computable closure properties, one can construct a computable map which takes a $\boldsymbol{\Gamma}$-formula ϕ and its parameter p to a $\delta_{\boldsymbol{\Gamma}(X)}$-name for the set defined by ϕ. Conversely, one can construct a computable map which takes a $\delta_{\boldsymbol{\Gamma}}$-name p for a set A to a $\boldsymbol{\Gamma}$-formula ϕ with parameter p which defines A.

We define the (single-valued) functions $\boldsymbol{\Gamma}\text{-}\mathsf{CA} :\subseteq \boldsymbol{\Gamma}(\mathbb{N}) \to 2^\mathbb{N}$ corresponding to comprehension principles: given a $\delta_{\boldsymbol{\Gamma}(\mathbb{N})}$-name p for a subset A of $\mathbb{N}$, produce its characteristic function. Notice that, for each k and each $A \in \boldsymbol{\Sigma}^0_{k+1}(\mathbb{N})$, we can use $\mathsf{LPO}^{(k)}$ to check whether $n \in A$ (intuitively, for every p we can use $\mathsf{LPO}^{(k)}$ to answer a $\Sigma^{0,p}_{k+1}$ question). This shows that, for each k,

$$\mathsf{lim}^{(k)} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}^{(k)}} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^0_{k+1}\text{-}\mathsf{CA},$$

as it is somewhat implicitly written in [10].

The problem $\chi_{\Pi^1_1}\colon \mathbb{N}^\mathbb{N} \to \{0,1\}$ is the characteristic function of the set of names for well-founded trees (which is a Π^1_1-complete set). The problem $\boldsymbol{\Pi}^1_1\text{-}\mathsf{CA}$ is Weihrauch equivalent to the parallelization of $\chi_{\Pi^1_1}$.

A central role is played by the choice problems: given a represented space X we define $\boldsymbol{\Gamma}\text{-}\mathsf{C}_X :\subseteq \boldsymbol{\Gamma}(X) \rightrightarrows X$ as the multi-valued function that chooses an element from a non-empty set $A \in \boldsymbol{\Gamma}(X)$. If $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^0_1$ we simply write C_X. We also write $\boldsymbol{\Gamma}\text{-}\mathsf{UC}_X$ if the choice is restricted to singletons. Different spaces X can lead to different Weihrauch degrees. In particular, letting C_k be the choice problem on $\{0, \dots, k-1\}$, we have

$$\begin{tikzcd}
 & \mathsf{C}_\mathbb{N} \arrow[dr] & & & \\
\mathsf{C}_k \arrow[ur] \arrow[dr] & & \mathsf{C}_\mathbb{R} \arrow[r] & \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \arrow[r] & \mathsf{C}_{\mathbb{N}^\mathbb{N}} \\
 & \mathsf{C}_{2^\mathbb{N}} \arrow[ur] & & &
\end{tikzcd}$$

where the arrows represent strict Weihrauch reduction in the direction of the arrow. Several variants of the choice problems have been explored in the literature: for example, we can consider the restriction of C_X to convex sets (XC_X), to sets with positive measure (PC_X), to cofinite sets ($\mathsf{C}_X^{\mathsf{cof}}$) and so on (see [17]). In particular, the choice on cofinite sets will play an important role in Chapter 5.

For each $\boldsymbol{\Gamma}$ we introduce the problem $\boldsymbol{\Gamma}{-}\mathsf{Bound} :\subseteq \boldsymbol{\Gamma}(\mathbb{N}) \rightrightarrows \mathbb{N}$, defined as the problem that takes as input a finite $\boldsymbol{\Gamma}$ subset of the natural numbers and returns a bound for it. Formally

$$\operatorname{dom}(\boldsymbol{\Gamma}{-}\mathsf{Bound}) := \{A \in \boldsymbol{\Gamma}(\mathbb{N}) \,:\, (\forall^\infty n)(A(n) = 0)\},$$
$$\boldsymbol{\Gamma}{-}\mathsf{Bound}(A) := \{n \in \mathbb{N} \,:\, (\forall m \geq n)(A(m) = 0)\}.$$

We will be especially interested in the principle $\boldsymbol{\Pi}^1_1{-}\mathsf{Bound}$. A simple observation is that $\boldsymbol{\Pi}^1_1{-}\mathsf{Bound} \equiv_{\mathrm{sW}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}$. Indeed, the reduction $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}} \leq_{\mathrm{sW}} \boldsymbol{\Pi}^1_1{-}\mathsf{Bound}$ is trivial. On the other hand, given a finite $\boldsymbol{\Pi}^1_1$ subset X of $\mathbb{N}$ we can consider the set

$$Y := \{n \in \mathbb{N} \,:\, (\exists m \geq n)(m \in X)\}.$$

Clearly Y is a $\boldsymbol{\Pi}^1_1$ initial segment of $\mathbb{N}$, and therefore $\mathbb{N} \setminus Y$ is a valid input for $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}$. Moreover a name for Y can be uniformly computed from a name of X and $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}(\mathbb{N} \setminus Y) = \boldsymbol{\Pi}^1_1{-}\mathsf{Bound}(X)$. This shows that $\boldsymbol{\Pi}^1_1{-}\mathsf{Bound} \leq_{\mathrm{sW}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}$ and hence the two problems are (strongly) Weihrauch equivalent. Moreover, this argument allows us to assume that an input for $\boldsymbol{\Pi}^1_1{-}\mathsf{Bound}$ is a sequence $(T_m)_{m\in\mathbb{N}}$ of trees s.t. there exists k s.t. $[T_i] = \emptyset$ iff $i < k$.

The problem $\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}}$ has been studied in [2] under the name $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{AC}_{\mathbb{N}^\mathbb{N}}^{\mathsf{cof}}$. Moreover, [64] (implicitly) uses $\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}}$ in the proof of Lemma 4.7 to separate $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{WKL}$ from $\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}}$. It is known that $\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}^{\mathsf{cof}}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ [2, Thm. 3.34]. We will show in Proposition 5.52 that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \widehat{\boldsymbol{\Pi}^1_1{-}\mathsf{Bound}}$.

As mentioned, the fact that Π^1_2 theorems can be phrased as multi-valued functions allows us to draw a parallel between reverse mathematics and Weihrauch reducibility, so that we can informally talk of "analogs" of the big five in the Weihrauch lattice:

RCA_0 roughly corresponds to constructive mathematics, and its analog is the identity id;

WKL_0 can be directly identified with the problem WKL that takes in input an ill-founded subtree of $2^{<\mathbb{N}}$ and produces a path through it. As mentioned, a closed set $A \subset 2^\mathbb{N}$ is closed iff there is a tree $T \subset 2^{<\mathbb{N}}$ s.t. $A = [T]$. This is essentially the proof of $\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_{2^\mathbb{N}}$;

ACA_0 is equivalent to the existence of the Turing jump of any set (in the model), hence it corresponds to lim and its iterations; “full” König’s lemma KL (given an infinite finitely-branching subtree of $\mathbb{N}^{<\mathbb{N}}$, produces a path) corresponds to WKL' and we have $\mathsf{lim} <_{\mathrm{W}} \mathsf{KL} <_{\mathrm{W}} \mathsf{lim}'$ (see [21, Fact 2.3]). We informally refer to the family of problems that are dominated by $\mathsf{lim}^{(n)}$ for some n as the “arithmetic part” of the Weihrauch lattice (see also Section 2.2);

ATR_0 corresponds approximately to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ (see below);

$\mathbf{\Pi}^1_1{-}\mathrm{CA}_0$ corresponds to $\mathbf{\Pi}^1_1$-CA, i.e. given a $\mathbf{\Pi}^1_1$ code for a set, produce its characteristic function.

The “higher levels” of the Weihrauch lattice have not been thoroughly explored so far. Recently, Marcone [18] raised the question “What do the Weihrauch hierarchies look like once we go to very high levels of reverse mathematics strength?”. While $\mathbf{\Pi}^1_1$-CA appears to be a natural choice for an analogue[5] of $\mathbf{\Pi}^1_1{-}\mathrm{CA}_0$, finding an analog for ATR_0 seems a harder task, and various alternatives have been explored in [64].

The system ATR_0 proves the existence of jump-hierarchies on any well-order (or, equivalently, hierarchies obtained by iterating an arithmetic formula over a well-order). This may suggest that the analogue of ATR_0 should be the function $\mathsf{ATR}\colon \mathrm{WO} \times 2^\mathbb{N} \times \mathbb{N} \to 2^\mathbb{N}$ mapping a countable well-order L, a set $A \subset \mathbb{N}$ and an arithmetic formula θ to the unique set $Y \subset \mathbb{N}$ obtained iterating θ along the well-order X with parameter A. The problem ATR is strong Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ ([64, Thm. 3.13]), and, in turn, to the problem $\mathsf{lim}^\dagger$, which corresponds to the iteration of lim over a countable ordinal ([90]).

Overall, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ appears as a perfect analog of ATR_0 in the Weihrauch lattice. However, there is an interesting phenomenon that makes the comparison between ATR_0 and ATR a bit harder: the existence of *pseudo-well-orders*. When working with different ω-models of second order arithmetic, the notion of well-order depends on the model. Pseudo-well-orders are ill-founded linear orders s.t. no descending sequence exists within the model itself (hence they are well-order “from the point of view of the model”). While the exploitation of pseudo-well-orders is a powerful tool from the point of view of reverse mathematics (see [106, Sec. V.4]), in the context of computable analysis we are *guaranteed* that the linear order in input to ATR is a well-order, and hence the hierarchy we build on it is a “true hierarchy”.

This led Jun Le Goh to introduce the problem

$$\mathsf{ATR}_2\colon \mathrm{LO} \times 2^\mathbb{N} \times \mathbb{N} \rightrightarrows \{0,1\} \times \mathbb{N}^\mathbb{N}$$

as the two sided version of ATR ([43, Def. 3.2 and prop. 3.11]). Formally it is defined as the following multi-valued function:

- inputs are triples (L, A, θ) s.t. L is a linear order on $\mathbb{N}$, A is the characteristic function of a subset of $\mathbb{N}$ and θ is an arithmetic formula whose only free variables are n, Y, and A;
- the output is a pair (i, Y) s.t. either $i = 0$ and Y is a $<_L$-infinite descending chain or $i = 1$ and Y is a (pseudo)hierarchy $\langle Y_a \rangle_{a \in L}$ s.t. for all $b \in L$, $Y_b = \{n : \theta(n, \bigoplus_{a <_L b} Y_a, A)\}$.

It is known that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{ATR}_2 <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([43, Cor. 3.5 and 3.7], see also [44]).

In general, the fact that many theorems do not have a unique phrasing in terms of multi-valued functions leads to several examples where the “one-sided” version of a theorem exhibit different

[5] Together with Vittorio Cipriani and Alberto Marcone we started the analysis, from the point of view of Weihrauch reducibility, of the Cantor-Bendixon theorem and the perfect kernel theorem, both known to be equivalent to $\mathbf{\Pi}^1_1{-}\mathrm{CA}_0$, see [106, Sec. VI.6].

uniform computational strength than its "two-sided" counterpart (see [64] and the following chapter 3). In particular, it turns out that the problems $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ are very relevant to calibrate the strength of multi-valued functions that corresponds to theorems around ATR_0.

There have been several works exploring the Weihrauch lattice around $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$: Kihara, Marcone, and Pauly [64] have studied several principles, like the (strong) comparability of well-orders, the perfect tree theorem, and the open determinacy theorem; Goh [44, 43, 45] analyzed the weak comparability of well-orders and the König duality theorem; Anglès D'Auriac and Kihara [2] dealt with the Σ^1_1 choice on $\mathbb{N}$ and variants thereof.

It is known that $\mathsf{lim}^{(n)} <_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ for every n (see [11, Sec. 6], [17, Prop. 7.50]). Moreover $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ are closed under compositional product ([11, Thm. 7.3]). In [64] it is proved that $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. The fact that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ follows from the fact that the element of a Σ^1_1 singleton is hyperarithmetic, but the hyperarithmetic functions are not a basis for the Π^0_1 predicates (see [96, Thm. I.1.6 and thm. III.1.1]). In particular we have

Theorem 2.13 ([64, Cor. 3.4]):
Let $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows X$ be a (partial) multi-valued function, for some represented space X. If $f \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ then, for every $x \in \mathrm{dom}(f)$, $f(x)$ contains some y hyperarithmetic relative to x.

The Weihrauch degree of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ has been explored in [64, Sec. 8]. It is known that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ ([64, Prop. 8.2(1)]) and $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$ is one of the strongest problem studied so far that is still considered among the "ATR_0 analogs".

We mention a simple proposition that will be very useful to prove many non-reducibilities.

Proposition 2.14:
Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions between represented spaces and let $A \subset \mathrm{dom}(g)$ be s.t.

$$\{z \in \mathrm{dom}(g) \,:\, (\forall w \in g(z))(w \textit{ is not hyperarithmetic in } z)\} \subset A.$$

If $f \times \mathsf{NHA} \le_{\mathrm{W}} g$ then $f \le_{\mathrm{W}} g|_A$.

PROOF: Assume $f \times \mathsf{NHA} \le_{\mathrm{W}} g$ and let the reduction be witnessed by the computable functions Φ, Ψ. For every p_x which is the name of some $x \in \mathrm{dom}(f)$, the pair (p_x, p_x) is mapped via Φ to a name p_z for some element $z \in \mathrm{dom}(g)$.

It suffices to show that $z \in A$. If this were not the case then, for some $x \in \mathrm{dom}(f)$, p_z is the name of some $z \notin A$. By hypothesis, there is a $w \in g(z)$ s.t. w has a name p_w which is hyperarithmetic in p_z. Let G be a realizer of g s.t. $p_w = G(p_z)$. Since p_w is hyperarithmetic in p_z, and hence in p_x, we have reached a contradiction with the fact that $\Psi(p_w, p_x, p_x)$ computes a solution for $\mathsf{NHA}(p_x)$. ■

This result will often be used in combination with Theorem 2.13. In fact if there is a computable $x \in \mathrm{dom}(f)$ s.t. $f(x)$ does not contain any hyperarithmetic element, then $f \not\le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

2.2 Arithmetic Weihrauch reducibility

When dealing with multivalued functions that are very high in the Weihrauch lattice it is often convenient to use a coarser notion of reducibility than Weihrauch reducibility. The notion of *arithmetic Weihrauch reducibility* was introduced in [43, Def. 1.4] (see also [45, Def. 2.2] and [2, Sec. 2.4]), and is obtained by relaxing the computability requirements on the forward and backward functionals.

Definition 2.15: Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be partial multivalued functions between represented spaces. We say that f is *arithmetically Weihrauch reducible* to g, and we write $f \le_{\mathrm{W}}^{a} g$, if

$$(\exists \text{ arithmetic } \Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}})(\forall G \vdash g)\ \Psi\langle \mathrm{id}, G\Phi\rangle \vdash f$$

where a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called *arithmetic* if there is $n \in \mathbb{N}$ s.t. $F \le_{\mathrm{W}} \mathsf{lim}^{(n)}$.

It is straightforward to see that $f \le_{\mathrm{W}} g \Rightarrow f \le_{\mathrm{W}}^{a} g$. Notice moreover that $f \le_{\mathrm{W}}^{a} g$ iff there exists n s.t. $f \le_{\mathrm{W}} \mathsf{lim}^{(n)} * g * \mathsf{lim}^{(n)}$ (this follows directly from the definition of the compositional product).

Proposition 2.16:
For every multivalued function f

$$(\exists n)(f \le_{\mathrm{W}} \mathsf{lim}^{(n)}) \iff f \le_{\mathrm{W}}^{a} \mathrm{id}\,.$$

Proof: The right-to-left implication follows from the definition. Assume there is a strong reduction $f \le_{\mathrm{sW}} \mathsf{lim}^{(n)}$ witnessed by the computable maps Φ_f, Ψ_f. It is easy to see that the maps $\Phi := \Psi_f \circ \mathsf{lim}^{(n)} \circ \Phi_f$ and $\Psi := \mathrm{id}$ witness the reduction $f \le_{\mathrm{W}}^{a} \mathrm{id}$. ■

Corollary 2.17:
$\mathrm{id} \equiv_{\mathrm{W}}^{a} \mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}}^{a} \mathsf{LPO} \equiv_{\mathrm{W}}^{a} \mathsf{lim}^{(n)}$.

Proof: Straightforward from Proposition 2.16 and the fact that id is Weihrauch reducible to $\mathsf{C}_{2^{\mathbb{N}}}$, LPO and $\mathsf{lim}^{(n)}$. ■

Proposition 2.18:
For every (partial) multivalued functions f, g, if $f \le_{\mathrm{W}}^{a} \mathrm{id}$ then $f * g \equiv_{\mathrm{W}}^{a} g * f \equiv_{\mathrm{W}}^{a} g$.

Proof: Let us first prove $f * g \equiv^a_{\mathrm{W}} g$, the other equivalence is analogous. We only need to prove that $f * g \le^a_{\mathrm{W}} g$ as the converse reduction is trivial. We can assume w.l.o.g. that f, g are (partial) multivalued functions $:\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ (see e.g. [17, Lem. 3.8]). By the cylindrical decomposition, we can write

$$f * g \equiv_{\mathrm{W}} (\mathrm{id} \times f) \circ \Phi_e \circ (\mathrm{id} \times g)$$

for some computable Φ_e. In particular

$$(\mathrm{id} \times f) \circ \Phi_e \circ (\mathrm{id} \times g)(\langle p_1, p_2 \rangle) = \langle \Phi_1(p_1, g(p_2)), f \circ \Phi_2(p_1, g(p_2)) \rangle$$

where Φ_1, Φ_2 are the computable functions s.t. $\Phi_e(p) = \langle \Phi_1(p), \Phi_2(p) \rangle$.

Let Φ_f, Ψ_f be two arithmetic maps witnessing the reduction $f \le^a_{\mathrm{W}} \mathrm{id}$. It is straightforward to see that the maps

$$\Phi := \langle p_1, p_2 \rangle \mapsto p_2,$$

$$\Psi := (\langle p_1, p_2 \rangle, q) \mapsto \langle \Phi_1(p_1, q), \Psi_f(\Phi_2(p_1, q), \Phi_f \Phi_2(p_1, q)) \rangle$$

witness the reduction $(\mathrm{id} \times f) \circ \Phi_e \circ (\mathrm{id} \times g) \le^a_{\mathrm{W}} g$. ■

We mention that an analog of Proposition 2.14 holds for arithmetic reducibility. We make it explicit, as it will be useful in the rest of the thesis.

Proposition 2.19:
Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be multi-valued functions between represented spaces and let $A \subset \mathrm{dom}(g)$ be s.t.

$$\{z \in \mathrm{dom}(g) \,:\, (\forall w \in g(z))(w \textit{ is not hyperarithmetic in } z)\} \subset A.$$

If $f \times \mathsf{NHA} \le^a_{\mathrm{W}} g$ then $f \le^a_{\mathrm{W}} g|_A$.

Proof: It is enough to follow the proof of Proposition 2.14, replacing "computable" with "arithmetic" and $\le_{\mathrm{W}}$ with $\le^a_{\mathrm{W}}$. ■

3 The open and clopen Ramsey theorems in the Weihrauch lattice

In this chapter, we explore the uniform computational strength of some infinite-dimensional generalizations of Ramsey's theorem. The results obtained in this work are joint work with Alberto Marcone, and have been collected in [78].

We already mentioned the classical (finite-dimensional) Ramsey theorem in Section 2.1.2. Our focus will be on the infinite generalization of the above result. In particular, we will focus on Nash-Williams' theorem, also called the open Ramsey theorem:

Theorem 3.1 (Nash-Williams [83]):
The open subsets of $[\mathbb{N}]^{\mathbb{N}}$ *admit infinite homogeneous sets.*

We will also consider the restriction of Nash-Williams' theorem to clopen subsets of $[\mathbb{N}]^{\mathbb{N}}$. With this work, we join the quest for an ATR_0 analog in the Weihrauch lattice (as discussed in Section 2.1.2), as both the open and the clopen Ramsey theorems are known to be equivalent to ATR_0 over RCA_0 (see [106, Sec. V.9]).

Notice that, as already occurred to other principles equivalent to ATR_0 ([64, 43]), there is not a single multi-valued function corresponding to the open Ramsey theorem. Actually, in our case, the situation is even more complex than for the open determinacy or the perfect tree theorem, as the two alternatives (homogeneous solution on the open side or homogeneous solution on the closed side) given by the open Ramsey theorem are not mutually exclusive. Therefore given an open set we can ask for a homogeneous solution on the open side, a homogeneous solution on the closed side, or a homogeneous solution on either side. Altogether we will define five different multi-valued functions corresponding to the open Ramsey theorem and three different functions corresponding to the clopen Ramsey theorem.

In Figure 3.1 we summarize the results we obtain both with respect to Weihrauch reducibility and arithmetic Weihrauch reducibility. Notice that the multi-valued function $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$ is stronger than any multi-valued function related to ATR_0 considered so far. In fact all these functions are strictly Weihrauch reducible to $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$, which, by Corollary 3.42, is strictly below $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$.

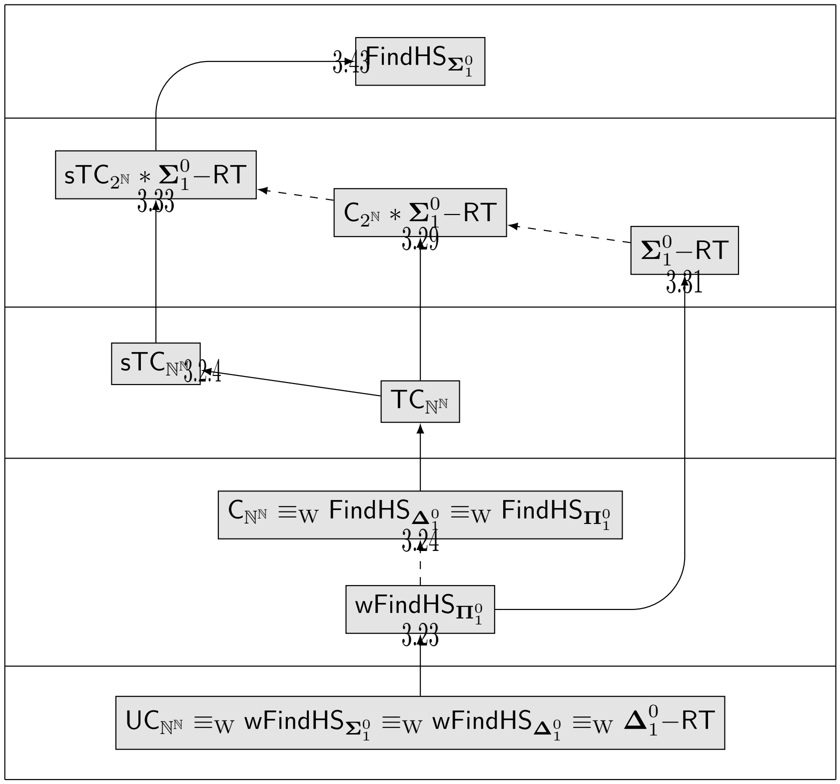

Figure 3.1: Multi-valued functions related to the open and clopen Ramsey theorems in the Weihrauch lattice. Dashed arrows represent Weihrauch reducibility in the direction of the arrow, solid arrows represent strict Weihrauch reducibility. The large rectangles indicate arithmetic Weihrauch equivalence classes. In particular, every function strictly arithmetically reduces to all the functions in rectangles above its own.

Notice also that, since $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is closed under parallel product (Proposition 3.39), it computes $\mathsf{sTC}^*_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^{\mathbb{N}}} \times \chi^*_{\Pi^1_1}$, which was suggested as an ATR_0 analogue in [64, Sec. 9].

In Section 3.1 we will recall the precise statement for the open and clopen Ramsey theorems and prove some lemmas that will be useful in proving the results on the Weihrauch degrees. The reader may skip these lemmas on the first read, and return to it as needed. In Section 3.2 we define the multi-valued functions corresponding to the open and clopen Ramsey theorems and study their degrees. In particular we divide the analysis into: functions that are reducible to $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ (Section 3.2.2), functions that are reducible to $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ (but not to $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, Section 3.2.3) and functions that are not reducible to $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ (Section 3.2.4). Moreover, in Section 3.2.5 we characterize the strength of these functions from the point of view of strong Weihrauch reducibility. Finally, in Section 3.3 we focus on the behavior of these functions under arithmetic Weihrauch reducibility, and in Section 3.4 we draw some conclusions and list some open problems.

3.1 Ramsey theorems

Recall that the space $[\mathbb{N}]^{\mathbb{N}}$, endowed with the induced topology from the Baire space $\mathbb{N}^{\mathbb{N}}$, is computably isometric to $\mathbb{N}^{\mathbb{N}}$. Moreover, there is a natural choice for a computable bijection $\mathbb{N}^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$.

Let $P \subset [\mathbb{N}]^{\mathbb{N}}$. We say that $f \in [\mathbb{N}]^{\mathbb{N}}$ is a *homogeneous solution for* P iff

$$(\forall g \in [\mathbb{N}]^{\mathbb{N}})(fg \in P) \vee (\forall g \in [\mathbb{N}]^{\mathbb{N}})(fg \notin P).$$

If f is homogeneous for P we say that f *lands in* P if the first disjunct of the above condition holds, i.e. if $(\forall g \in [\mathbb{N}]^{\mathbb{N}})(fg \in P)$. Vice versa, if $(\forall g \in [\mathbb{N}]^{\mathbb{N}})(fg \notin P)$ then we say that f *avoids* P. A set $P \subset [\mathbb{N}]^{\mathbb{N}}$ is called *Ramsey* (or we say that it has the *Ramsey property*) iff it has a homogeneous solution. We will denote the set of homogeneous solutions for P (which may either land in it or avoid it) with $\mathrm{HS}(P)$. Notice that, in general, a set can have both solutions that land in the set and solutions that avoid the set.

In the literature, the symbol $[\mathbb{N}]^{\mathbb{N}}$ is sometimes used to denote the family of all infinite subsets of $\mathbb{N}$. Also, if X is an infinite subset of $\mathbb{N}$, $[X]^{\mathbb{N}}$ denotes the family of all infinite subsets of X. It is easy to identify the Ramsey space $[\mathbb{N}]^{\mathbb{N}}$ with the space of infinite subsets of $\mathbb{N}$ (by identifying a function f with its range). With this in mind, we may write $[f]^{\mathbb{N}} := [\mathrm{ran}(f)]^{\mathbb{N}}$ to denote the set of all infinite subsequences of f. The definition of homogeneous solution can now be written as

$$[f]^{\mathbb{N}} \subset P \vee [f]^{\mathbb{N}} \cap P = \emptyset.$$

It is natural to ask which classes of subsets of $[\mathbb{N}]^{\mathbb{N}}$ have the Ramsey property. The problem is well studied and has an extensive literature. The Galvin-Prikry theorem ([39]) states that all Borel subsets of $[\mathbb{N}]^{\mathbb{N}}$ have the Ramsey property. This result can actually be extended to analytic sets ([103]). To go beyond the analytic sets we need axioms above ZFC (see e.g. [106, Rem. VI.7.6, p. 240], [60, pp. 1036–1037]). We will focus on Nash-Williams' theorem ([83]), which states that open sets have the Ramsey property. This is also known as the open Ramsey theorem. It, in turn, implies the clopen Ramsey theorem (which is the restriction of Nash-Williams' theorem to clopen sets). As already mentioned, the open and clopen Ramsey theorems are known to be equivalent to ATR_0 over RCA_0 (see [106, Thm. V.9.7]).

3.1.1 Some useful tools

Before formalizing the open and clopen Ramsey theorems in the context of Weihrauch reducibility as multi-valued functions, let us explicitly state some properties of the set of homogeneous solutions that will turn out to be useful in the rest of the paper. As a notational convenience we will use the letters $P, Q, \dots$ to denote open sets and $D, E, \dots$ to denote clopen sets.

We start by mentioning these simple properties of the representation maps:

Lemma 3.2:
The following maps are computable:

1. $\boldsymbol{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \hookrightarrow \boldsymbol{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) := D \mapsto D$;
2. $\boldsymbol{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \to \boldsymbol{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) := D \mapsto [\mathbb{N}]^{\mathbb{N}} \setminus D$;
3. $\cup\colon \boldsymbol{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \times \boldsymbol{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \to \boldsymbol{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) := (P, Q) \mapsto P \cup Q$.

PROOF:

1, 2 follow from the fact that a name for a clopen set is the join $\langle p, q\rangle$ of two names for open sets (one for the set and one for its complement);

3 see [10, Prop. 3.2(5)]. ∎

We notice that the open and clopen Ramsey theorems (and, in fact, the Galvin-Prikry theorem) can be applied to subspaces of $[\mathbb{N}]^{\mathbb{N}}$ as follows:

Proposition 3.3:
Let $\mathbf{\Gamma}$ be a definable (boldface) pointclass that is downward closed with respect to Wadge reducibility (i.e. it is a downward closed family of Wadge degrees), such as the families of open and of clopen sets. Assume that every $P \in \mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}})$ is Ramsey and that for every $f \in [\mathbb{N}]^{\mathbb{N}}$,

$$\mathbf{\Gamma}([f]^{\mathbb{N}}) = \{P \cap [f]^{\mathbb{N}} \,:\, P \in \mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}})\}.$$

Then for every $f \in [\mathbb{N}]^{\mathbb{N}}$, every $Q \in \mathbf{\Gamma}([f]^{\mathbb{N}})$ is Ramsey. Moreover if $P \in \mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}})$ and $f \in [\mathbb{N}]^{\mathbb{N}}$ there exists $h \in \mathrm{HS}(P)$ s.t. $h \preceq f$.

PROOF: It is easy to see that every $f \in [\mathbb{N}]^{\mathbb{N}}$ induces a f-computable homeomorphism $\varphi_f : [\mathbb{N}]^{\mathbb{N}} \to [f]^{\mathbb{N}}$ defined as

$$\varphi_f(p) := n \mapsto f(p(n)).$$

Notice also that

$$\varphi_f(h)g = fhg = \varphi_f(hg). \tag{$\star$}$$

In particular this homeomorphism preserves subsequences, i.e. for every $q \preceq p$ we have $\varphi_f(q) \preceq \varphi_f(p)$. Fix $f \in [\mathbb{N}]^{\mathbb{N}}$ and let $Q \in \mathbf{\Gamma}([f]^{\mathbb{N}})$. Since $\mathbf{\Gamma}$ is closed under Wadge reducibility we have that

$$P := \varphi_f^{-1}(Q) \in \mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}}).$$

Moreover, since every pointset in $\mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}})$ has the Ramsey property, there is $h \in \mathrm{HS}(P)$. Using ($\star$), it is straightforward to conclude that $\varphi_f(h) \in \mathrm{HS}(Q)$.

For the second part it suffices to apply the first part to $Q := [f]^{\mathbb{N}} \cap P$, which is in $\mathbf{\Gamma}([f]^{\mathbb{N}})$. ∎

The following proposition says that, under relatively mild conditions, the set of homogeneous solutions that land in $P \cup Q$ splits nicely in the set of homogeneous solutions for P that land in P and the set of homogeneous solutions for Q that land in Q.

Proposition 3.4:
Let $A, B \subset \mathbb{N}$ be disjoint. Let $P, Q \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ be s.t.

1. $(\forall f \in P)(f(0) \in A$ *and* $f(1) \in A)$;

2. *$(\forall g \in Q)(g(0) \in B$ and $g(1) \in B)$.*

If $R := P \cup Q$ then

$$\mathrm{HS}(R) \cap R = (\mathrm{HS}(P) \cap P) \cup (\mathrm{HS}(Q) \cap Q).$$

PROOF: The inclusion $(\mathrm{HS}(P) \cap P) \cup (\mathrm{HS}(Q) \cap Q) \subset \mathrm{HS}(R) \cap R$ is trivial and always holds, so we only need to prove the converse direction. Let $h \in \mathrm{HS}(R) \cap R$ and assume that $h \in P$. By induction we can easily show that $\mathrm{ran}(h) \subset A$. Indeed, by point 1, $h(0) \in A$ and $h(1) \in A$. Moreover, if $h(i) \in A$ then $h(i+1) \in A$ because h lands in R: indeed if not then the substring $(h(i), h(i+1), \dots)$ of h can neither be in P nor in Q (by the disjointness of A and B), hence it cannot be in R, contradicting the fact that h lands in R. This shows that $h \in P$ implies $h \in \mathrm{HS}(P) \cap P$. Notice also that, by the disjointness of A and B, $\mathrm{ran}(h) \subset A$ implies $\mathrm{ran}(h) \cap B = \emptyset$ and therefore no subsequence of h is in Q. Similarly we can show that $h \in \mathrm{HS}(R) \cap Q$ implies $h \in (\mathrm{HS}(Q) \cap Q) \setminus (\mathrm{HS}(P) \cap P)$ and therefore the claim follows. ■

The following construction was used by Avigad [4] in his proof of the open Ramsey theorem in ATR_0.

Definition 3.5: Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ and let $\langle P \rangle$ be a name for P. We can define the tree

$$T_{\langle P \rangle} := \{\sigma \in [\mathbb{N}]^{<\mathbb{N}} : (\forall \tau \preceq^* \sigma)(\tau \notin \langle P \rangle)\}.$$

Lemma 3.6:
Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$. For every name $\langle P \rangle$ of P and every $f \in [\mathbb{N}]^{\mathbb{N}}$ we have

$$f \in \mathrm{HS}(P) \setminus P \iff f \in [T_{\langle P \rangle}].$$

PROOF: If $f \notin [T_{\langle P \rangle}]$ then $(\exists n)(f[n] \notin T_{\langle P \rangle})$, i.e. $(\exists n)(\exists \tau \preceq^* f[n])(\tau \in \langle P \rangle)$. This implies that there exists a $g \preceq f$ s.t. $\tau \sqsubset g$. This shows that $g \in P$ and hence $f \notin \mathrm{HS}(P) \setminus P$.

Let $f \in [T_{\langle P \rangle}]$ and let $g \preceq f$. If $g \in P$ then $(\exists n)(g[n] \in \langle P \rangle)$, contradicting the fact that $f \in [T_{\langle P \rangle}]$ (by definition of $T_{\langle P \rangle}$). Therefore we have that $f \in \mathrm{HS}(P) \setminus P$. ■

Notice that the above lemma shows that $\mathrm{HS}(P) \setminus P$ is closed whenever P is open. In particular, if D is clopen then $\mathrm{HS}(D)$ is closed: indeed, letting $E := [\mathbb{N}]^{\mathbb{N}} \setminus D$, we have $\mathrm{HS}(D) = \mathrm{HS}(E)$ and

$$\mathrm{HS}(D) = (\mathrm{HS}(D) \cap D) \cup (\mathrm{HS}(D) \setminus D) = (\mathrm{HS}(E) \setminus E) \cup (\mathrm{HS}(D) \setminus D)$$

is the union of two closed sets.

On the other hand, the set of solutions for an open set P that lands in P can be Π^1_1-complete: let $(q_i)_{i\in\mathbb{N}}$ be an enumeration of the rationals. We can define

$$T := \{\sigma \in [\mathbb{N}]^{<\mathbb{N}} \,:\, (\forall i < |\sigma| - 1)(q_{\sigma(i+1)} <_{\mathbb{Q}} q_{\sigma(i)})\}.$$

A path through T is an infinite descending sequence in $\mathbb{Q}$. If we define $P := [\mathbb{N}]^{\mathbb{N}} \setminus [T]$ we have that $\mathrm{HS}(P) \cap P$ is the set of well-suborders of $\mathbb{Q}$ (every suborder of $\mathbb{Q}$ that is not a well-order contains an infinite descending sequence with increasing indexes, and therefore, a subsequence that lands in $[T]$) and hence is Π^1_1-complete.

This underlines a critical difference between the problem of finding a homogeneous solution that lands in P and finding one that avoids P.

The following construction will be used in the following to move open sets around while "preserving" homogeneous solutions.

Definition 3.7: For every $n > 1$ let $\mathrm{pow}_n := i \mapsto n^{i+1}$. We can define the map $\eta_n \colon ([\mathbb{N}]^{<\mathbb{N}} \cup [\mathbb{N}]^{\mathbb{N}}) \to ([\mathbb{N}]^{<\mathbb{N}} \cup [\mathbb{N}]^{\mathbb{N}})$ as

$$\eta_n(f) := \mathrm{pow}_n \circ f = i \mapsto n^{f(i)+1}.$$

It is clear that η_n is a computable injection with computable inverse.

Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ and let $\langle P\rangle$ be a name for P. We can define (with a small abuse of notation)

$$\eta_n(\langle P\rangle) := \bigcup_{\sigma\in\langle P\rangle} \{f \in [\mathbb{N}]^{\mathbb{N}} \,:\, \eta_n(\sigma) \sqsubset f\}.$$

We can naturally extend the definition to a multi-valued map $\mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ defining

$$\eta_n(P) := \{\eta_n(\langle P\rangle) \,:\, \langle P\rangle \text{ is a name of } P\}.$$

Lemma 3.8:
Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$. Fix $n > 1$ and let $Q := \eta_n(\langle P\rangle)$ for some name $\langle P\rangle$ of P. Then

$$f \in \mathrm{HS}(P) \cap P \Leftrightarrow \eta_n(f) \in \mathrm{HS}(Q) \cap Q.$$

Proof: It is straightforward to see that, for every $f \in [\mathbb{N}]^{\mathbb{N}}$, $f \in P$ iff $\eta_n(f) \in Q$. Moreover, if $g \in [\mathbb{N}]^{\mathbb{N}}$, then

$$\eta_n(f)g = i \mapsto n^{fg(i)+1} = \eta_n(fg).$$

If f is a homogeneous solution that lands in P then $\eta_n(f)g \in Q$ for every $g \in [\mathbb{N}]^{\mathbb{N}}$, i.e. $\eta_n(f) \in \mathrm{HS}(Q) \cap Q$. Vice versa, if $\eta_n(f) \in \mathrm{HS}(Q) \cap Q$ then for every $g \in [\mathbb{N}]^{\mathbb{N}}$ we have $\eta_n(f)g = \eta_n(fg) \in Q$, which implies $fg \in P$. ■

Definition 3.9: Let $\sigma, \tau \in [\mathbb{N}]^{<\mathbb{N}}$. We define $\sigma \boxtimes \tau$ to be the set of all strings of the form $(\langle \rho(i), \theta(i) \rangle \,:\, i < N)$ where $\rho, \theta \in [\mathbb{N}]^{<\mathbb{N}}$ and s.t.

$$\sigma \sqsubseteq \rho \land \tau \sqsubseteq \theta \land N = \max\{|\sigma|, |\tau|\}.$$

Clearly the map $\boxtimes$ can be extended to infinite strings by defining

$$f \boxtimes g := (\langle f(0), g(0) \rangle, \langle f(1), g(1) \rangle, \dots).$$

For the sake of readability, it is convenient to introduce the following notation: for $i = 1, 2$, we define

$$\pi_i : (\mathbb{N} \times \mathbb{N})^{<\mathbb{N}} \cup (\mathbb{N} \times \mathbb{N})^{\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^{\mathbb{N}}$$

as the map that, given in input a finite (resp. infinite) string of pairs, returns the finite (resp. infinite) string of the i-th elements of the pairs.

Let $\langle P \rangle, \langle Q \rangle$ be two names for two open subsets of $[\mathbb{N}]^{\mathbb{N}}$. We can define

$$\langle P \rangle \boxtimes \langle Q \rangle := \bigcup_{\sigma \in \langle P \rangle} \bigcup_{\tau \in \langle Q \rangle} \sigma \boxtimes \tau,$$

which is a name for a new open set.

This leads to a map $\mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \times \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ defined by

$$P \boxtimes Q := \{R \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \langle P \rangle \boxtimes \langle Q \rangle \text{ is a name for } R\}.$$

Notice that, in general, it is not true that if f lies in the open set with name $\langle P \rangle \boxtimes \langle Q \rangle$ then $\pi_i f \in [\mathbb{N}]^{\mathbb{N}}$.

Lemma 3.10:
Let $P_1, P_2 \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ and let $\langle P_1 \rangle, \langle P_2 \rangle$ be names for P_1, P_2 respectively s.t. every string in $\langle P_1 \rangle$ has length at least 2. Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}})$ be the open set with name $\langle P_1 \rangle \boxtimes \langle P_2 \rangle$. Then

$$\mathrm{HS}(P) \cap P = \{f \boxtimes g \,:\, f \in \mathrm{HS}(P_1) \cap P_1 \text{ and } g \in \mathrm{HS}(P_2) \cap P_2\}.$$

Proof: Notice first of all that $f \in \mathrm{HS}(P) \cap P$ implies that, for $i = 1, 2$,

$$\pi_i f \in [\mathbb{N}]^{\mathbb{N}}.$$

Indeed, fix $n \in \mathbb{N}$ and consider the substring $g := (f(n), f(n+1), \dots)$ of f. Since f is homogeneous we have $g \in P$. In particular, there is $\tau = (\langle \tau_1(i), \tau_2(i) \rangle \,:\, i < N) \in \langle P_1 \rangle \boxtimes \langle P_2 \rangle$ s.t. $\tau \sqsubset g$. Fix $\sigma_1 \in \langle P_1 \rangle$ and $\sigma_2 \in \langle P_2 \rangle$ s.t. $\tau \in \sigma_1 \boxtimes \sigma_2$. Since $|\sigma_1| \geq 2$ we have $|\tau| \geq 2$. Moreover

$$(\pi_i f)(n) = \tau_i(0) < \tau_i(1) = (\pi_i f)(n+1).$$

Let now $f \in \mathrm{HS}(P) \cap P$. For every $g \preceq f$ we have that $g \in \mathrm{HS}(P) \cap P$ and $\pi_1 g \in P_1$, $\pi_2 g \in P_2$ (indeed if $\pi_1 g \notin P_1$ or $\pi_2 g \notin P_2$ then $g \notin P$). Hence $\pi_i f \in \mathrm{HS}(P_i) \cap P_i$ for $i = 1, 2$. The reverse inclusion is straightforward as if $f_i \in \mathrm{HS}(P_i) \cap P_i$ then $f := f_1 \boxtimes f_2$ is a homogeneous solution for P. ■

The following generalizes the tree used by Solovay [108].

Definition 3.11: Let $\phi\colon \mathbb{N} \to \mathbb{N}$ be injective. For every tree $T \in \mathbf{Ti}$ we define the *Solovay open set* $W_\phi(T)$ as

$$W_\phi(T) := \{f \in [\mathbb{N}]^{\mathbb{N}} \,:\, (\exists k)(\forall \tau \trianglelefteq f[k])(\tau \notin T) \text{ and } (\exists n, m \in \mathbb{N})(f(0) = \phi(n) \text{ and } f(1) = \phi(m))\}.$$

If ϕ is the identity function we drop the subscript.

It is easy to see that $W_\phi(T)$ is an open set.

Lemma 3.12:
Let $T \in \mathbf{Ti}$ and let $W := W_\phi(T)$. Then

1. *If $[T] = \emptyset$ then $\mathrm{HS}(W) \cap W \neq \emptyset$. Moreover, if ϕ is surjective then $W = [\mathbb{N}]^{\mathbb{N}}$ and therefore $\mathrm{HS}(W) = \mathrm{HS}(W) \cap W = [\mathbb{N}]^{\mathbb{N}}$.*
2. *If $[T] \neq \emptyset$ then $\mathrm{HS}(W) = \mathrm{HS}(W) \setminus W$. Moreover every $f \in \mathrm{HS}(W)$ dominates a path through T.*

Proof: For each $f \in [\mathbb{N}]^{\mathbb{N}}$ define the tree

$$T_f := \{\sigma \in T \,:\, (\forall i < |\sigma|)(\sigma(i) \leq f(i))\}.$$

1 Notice that, for every $f \in [\mathbb{N}]^{\mathbb{N}}$, the set T_f is a finitely-branching well-founded subtree of T. By König's lemma, T_f must be finite and therefore there is a k s.t. every string in T_f has length $< k$. This implies that every $\tau \trianglelefteq f[k]$ is not in T. If $\mathrm{ran}(f) \subset \mathrm{ran}(\phi)$ (as is always the case if ϕ is surjective) then $f \in \mathrm{HS}(W) \cap W$.

2 Notice that $\mathrm{HS}(W) \cap W = \emptyset$ because for every path $x \in [T]$ and every $f \in [\mathbb{N}]^{\mathbb{N}}$, there exists $g \in [\mathbb{N}]^{\mathbb{N}}$ that grows sufficiently quickly s.t. $x \trianglelefteq fg$ (as proved in [108, p. 108]).

Moreover, if $f \in \mathrm{HS}(W) \setminus W$ then $(\forall k)(\exists \tau \trianglelefteq f[k])(\tau \in T)$. This implies that T_f is infinite and therefore, by König's lemma, $[T_f] \neq \emptyset$. This concludes the proof as $[T_f] \subset [T]$. ■

Definition 3.13: Let $\phi\colon \mathbb{N} \to \mathbb{N}$. For every tree $T \in \mathbf{Ti}$ we define the clopen set $D_\phi(T)$ as

$$D_\phi(T) := \{f \in [\mathbb{N}]^{\mathbb{N}} \,:\, (\exists \sigma_0, \sigma_1 \in T)(\, f(0) = \phi(\langle \sigma_0 \rangle) \text{ and } f(1) = \phi(\langle \sigma_1 \rangle) \text{ and } \sigma_0 \sqsubset \sigma_1)\}.$$

If ϕ is the identity function we just drop the subscript.

Lemma 3.14:
Let $T \in \mathbf{Ti}$ and let $D := D_\phi(T)$ for some computable strictly increasing function $\phi\colon \mathbb{N} \to \mathbb{N}$. If $[T] \neq \emptyset$ then $\mathrm{HS}(D) \cap D \neq \emptyset$ and there is a uniform computable surjection $\mathrm{HS}(D) \cap D \to [T]$. On the other hand, if $[T] = \emptyset$ then $\mathrm{HS}(D) \cap D = \emptyset$.

Proof: Let $y \in [T]$ and define $h \in D$ s.t. $h(i) = \phi(\langle y[i] \rangle)$. It is easy to see that h is an homogeneous solution for D landing in D, therefore $\mathrm{HS}(D) \cap D \neq \emptyset$. Moreover, given any f that lands in D we can compute a path x through T as

$$x := \bigcup_{k \in \mathbb{N}} \phi^{-1} f(k).$$

Indeed, since f lands in D we have that $\phi^{-1} f(k) \in T$ for every $k \in \mathbb{N}$. Moreover, for every k, $(f(k), f(k+1), \dots) \in D$ so that $\phi^{-1} f(k+1)$ is a finite string that properly extends $\phi^{-1} f(k)$. Therefore x is a well-defined function $\mathbb{N} \to \mathbb{N}$. Finally it is easy to see that $x \in [T]$: for every i, let $j > i$ s.t. there is a k s.t. $x[j] = \phi^{-1} f(k)$. By definition of D we have that $\phi^{-1} f(k) \in T$. Since T is a tree, we can conclude that $x[i] \in T$. Notice that ϕ^{-1} is computable, therefore x is computable from f. Notice also that, if y and h are as in the beginning of this proof, then $y = \bigcup_{k \in \mathbb{N}} \phi^{-1} h(k)$, which proves that the mapping is a surjection.

On the other hand, if $[T] = \emptyset$ then, for every $f \in D$, there is an i s.t. $\phi^{-1}(f(i)) \notin T$ or $\phi^{-1}(f(i)) \not\sqsubset \phi^{-1}(f(i+1))$, otherwise we could compute a path through T. In any case if $f \in D$ then it is not a homogeneous solution for D. ■

Lemma 3.15:
The following maps are computable:

1. $\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \to \mathbf{\Pi}^0_1([\mathbb{N}]^\mathbb{N}) := P \mapsto \mathrm{HS}(P) \setminus P$;
2. $\eta_n\colon \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \rightrightarrows \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) := P \mapsto \eta_n(P)$, *for every* $n \in \mathbb{N}$;
3. $\boxtimes\colon \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \times \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \rightrightarrows \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) := (P, Q) \mapsto P \boxtimes Q$;
4. $W_\phi\colon \mathbf{Ti} \to \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) := T \mapsto W_\phi(T)$, *for every injective map* $\phi\colon \mathbb{N} \to \mathbb{N}$ *with computable range;*
5. $D_\phi\colon \mathbf{Ti} \to \mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N}) := T \mapsto D_\phi(T)$, *for every invertible map* $\phi\colon \mathbb{N} \to \mathbb{N}$ *with computable inverse.*

Proof:

1 Let $P \subset [\mathbb{N}]^\mathbb{N}$ be open and let $\langle P \rangle$ be a name for P. The definition of $T_{\langle P \rangle}$ is computable in $\langle P \rangle$. Moreover, $x \in [T_{\langle P \rangle}]$ iff $x \in \mathrm{HS}(P) \setminus P$ (see Lemma 3.6). Since a name for $A \in \mathbf{\Pi}^0_1([\mathbb{N}]^\mathbb{N})$ can be a tree T s.t. $A = [T]$, the claim follows.

2, 3 Straightforward from the definition.

4 Let $T \in \mathbf{Ti}$ be represented by its characteristic function χ_T. We can define

$$\langle W_\phi(T)\rangle := \{\sigma \in [\mathbb{N}]^{<\mathbb{N}} \,:\, (\forall \tau \trianglelefteq \sigma)(\tau \notin T) \text{ and } (\exists n, m \in \mathbb{N})(\sigma(0) = \phi(n) \text{ and } \sigma(1) = \phi(m))\}.$$

Notice that the universal quantifier is bounded, while the formula in the scope of the existential quantifier is equivalent to requiring that $\sigma(0)$ and $\sigma(1)$ are in $\operatorname{ran}(\phi)$, which is computable by hypothesis. Therefore $\langle W_\phi(T)\rangle$ is computable in T.

5 Let $T \in \mathbf{Ti}$. By definition of $D_\phi(T)$, the basic clopen cone $\{f \in [\mathbb{N}]^{\mathbb{N}} \,:\, \tau \sqsubset f\}$ is a subset of $D_\phi(T)$ iff

$$\phi^{-1}\tau(0) \in T \text{ and } \phi^{-1}\tau(1) \in T \text{ and } \phi^{-1}\tau(0) \sqsubset \phi^{-1}\tau(1).$$

In particular, this shows that we can T-computably obtain open names for $D_\phi(T)$ and its complement. ■

3.2 Ramsey theorems in the Weihrauch lattice

3.2.1 Definitions

There are several ways to formalize the open Ramsey theorem as a multi-valued function.

Definition 3.16 (Open Ramsey Theorem): We define the *full version* of the open Ramsey theorem as the (total) multi-valued function

$$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}\colon \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}} := P \mapsto \mathrm{HS}(P).$$

We may modify the full version by adding the requirement on "which side" we want the solution to be in. In this case, however, we need to restrict the domain to the family of open sets that admit a solution. We can define the *strong versions* of the open Ramsey theorem as the multi-valued functions $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}, \mathsf{FindHS}_{\mathbf{\Pi}^0_1} :\subseteq \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ with domain respectively

$$\operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}) := \{P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(P) \cap P \neq \emptyset\},$$

$$\operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1}) := \{P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(P) \setminus P \neq \emptyset\}$$

and defined as $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(P) := \mathrm{HS}(P) \cap P$ and $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}(P) := \mathrm{HS}(P) \setminus P$. We may strengthen further the requirements, defining the *weak versions* of the open Ramsey theorem: namely we define $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ as the restriction of $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ to

$$\operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) := \{P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(P) \subset P\}.$$

Similarly we can define the weak version of $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ as the multi-valued function $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ obtained by restricting $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ to

$$\operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}) := \{P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(P) \cap P = \emptyset\}.$$

Recall that, in general, an open set can have both solutions that land in the set and solutions that avoid the set. The domain of $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ (resp. $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$) is therefore strictly smaller than the domain of $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (resp. $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$). As we will see the two versions exhibit very different behaviors. Notice also that the weak versions are restrictions of $\mathbf{\Sigma}^0_1\mathsf{-RT}$, while the strong versions are not (the set of solutions can be strictly smaller).

As in the case of the open Ramsey theorem, we can consider different multi-valued functions corresponding to the clopen Ramsey theorem.

Definition 3.17 (Clopen Ramsey Theorem): We define the full version of the clopen Ramsey theorem as the multi-valued function $\mathbf{\Delta}^0_1\mathsf{-RT}\colon \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}} := D \mapsto \mathrm{HS}(D)$.

The strong version of the clopen Ramsey theorem is the multi-valued function

$$\mathsf{FindHS}_{\mathbf{\Delta}^0_1} :\subseteq \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}} := D \mapsto \mathrm{HS}(D) \cap D$$

with domain

$$\mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Delta}^0_1}) := \{D \in \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(D) \cap D \neq \emptyset\}.$$

The weak version of the clopen Ramsey theorem is the problem $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1} :\subseteq \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined as the restriction of $\mathsf{FindHS}_{\mathbf{\Delta}^0_1}$ to

$$\mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}) := \{D \in \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \,:\, \mathrm{HS}(D) \subset D\}.$$

Notice that we defined only one strong and one weak version of the clopen Ramsey theorem. This is because, using Lemma 3.2.2, it is straightforward to see that the other two are (strongly) Weihrauch equivalent to the ones we defined.

3.2.2 Problems reducible to $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$

We show that $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$, $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ and $\mathbf{\Delta}^0_1\mathsf{-RT}$ are all Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$. None of these principles are strongly Weihrauch equivalent to $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, as we will show in Proposition 3.44.

Lemma 3.18:
$\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \leq_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$.

Proof: Since $\mathsf{ATR} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ [64, Thm. 3.13], it suffices to prove that $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \leq_{\mathrm{W}} \mathsf{ATR}$. The proof is the direct translation in the context of Weihrauch reducibility of the proof presented in [106, Lem. V.9.4]. More details on the construction can be found in the paper where the proof was first presented, i.e. [4, Sec. 3]. Let $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. The proof consists of four steps:

1. build the tree $T := T_{\langle P \rangle}$ of homogeneous solutions that avoid P;
2. build $\mathrm{KB}(T)$;
3. via arithmetic transfinite recursion along $\mathrm{KB}(T)$, obtain a sequence of infinite sets $(U_\sigma)_{\sigma \in T}$ and classify each $\sigma \in [\mathbb{N}]^{<\mathbb{N}}$ as "good" or "bad";

4. use this classification to build a solution f.

Notice that steps 1 and 2 are computable (using Lemma 3.15.1). For $\sigma \notin T$, we classify σ as good if the shortest prefix of σ which is not in T belongs to $\langle P \rangle$, and bad otherwise. For $\sigma \in T$, to define U_σ and classify σ as good or bad, we first define a set V_σ as follows:

- if σ is the minimum of $\mathrm{KB}(T)$ then $V_\sigma := \mathbb{N}$;
- if σ is the successor of τ in $\mathrm{KB}(T)$ then $V_\sigma := U_\tau$;
- if σ is a limit in $\mathrm{KB}(T)$ then we define V_σ by diagonal intersection: we computably and uniformly find a sequence τ_j cofinal in σ. Define

$$u_0 := \min U_{\tau_0};$$
$$u_{i+1} := \min \left\{ \bigcap_{j \le i} U_{\tau_j} \setminus \{u_j\} \right\};$$
$$V_\sigma := \{u_i \,:\, i \in \mathbb{N}\}.$$

It is easy to verify that V_σ is defined by an arithmetic formula. Let

$$V^1_\sigma := \{m \in V_\sigma \,:\, \sigma^\frown m \text{ is good}\},$$

and similarly $V^0_\sigma := \{m \in V_\sigma \,:\, \sigma^\frown m \text{ is bad}\} = V_\sigma \setminus V^1_\sigma$. Set

$$n \in U_\sigma :\Leftrightarrow (|V^1_\sigma| = \infty \text{ and } n \in V^1_\sigma) \text{ or } (|V^1_\sigma| < \infty \text{ and } n \in V^0_\sigma).$$

We now classify σ as good if V^1_σ is infinite, and bad otherwise.

We can obtain the information about $(U_\sigma)_{\sigma \in T}$ and the goodness (or badness) for each $\sigma \in T$ as a name for $Y \in \mathsf{ATR}(\mathrm{KB}(T), P, \theta)$, for an appropriate arithmetic formula θ.

As in [106, 4], one can show that () is good and compute a solution $f \in \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}(P)$ from Y. ∎

Lemma 3.19:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{wFindHS}_{\boldsymbol{\Delta}^0_1}$.

Proof: We follow the proof of the fact that the clopen Ramsey theorem implies ATR_0 over RCA_0 presented in [106, Lem. V.9.6]. We actually prove the reduction $\boldsymbol{\Sigma}^1_1\mathsf{-Sep} \le_{\mathrm{W}} \mathsf{wFindHS}_{\boldsymbol{\Delta}^0_1}$, as the equivalence $\boldsymbol{\Sigma}^1_1\mathsf{-Sep} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ has been proved in [64, Thm. 3.11].

Let $((T^0_k, T^1_k))_{k \in \mathbb{N}}$ be a sequence of pairs of trees s.t. for all k at most one of T^0_k and T^1_k has a path (i.e. the sequence is a valid input for $\boldsymbol{\Sigma}^1_1\mathsf{-Sep}$). Our goal is to find a set Z s.t. if T^0_k has a path then $k \in Z$ and if T^1_k has a path then $k \notin Z$.

Following [106], we use the sequence $((T^0_k, T^1_k))_{k \in \mathbb{N}}$ to uniformly compute a name for a clopen $D \in \mathrm{dom}(\mathsf{wFindHS}_{\boldsymbol{\Delta}^0_1})$ s.t. for every $f \in \mathsf{wFindHS}_{\boldsymbol{\Delta}^0_1}(D)$, f and $((T^0_k, T^1_k))_{k \in \mathbb{N}}$ uniformly compute some $Z \in \boldsymbol{\Sigma}^1_1\mathsf{-Sep}(((T^0_k, T^1_k))_{k \in \mathbb{N}})$. ∎

Theorem 3.20:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$.

PROOF: This follows from Lemma 3.18, Lemma 3.19 and the fact that $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ is the restriction of $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ to clopen sets. ■

Theorem 3.21:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_1{-}\mathsf{RT}$.

PROOF: By Theorem 3.20 it suffices to prove that $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_1{-}\mathsf{RT}$. The reduction $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1} \le_{\mathrm{W}} \mathbf{\Delta}^0_1{-}\mathsf{RT}$ is trivial (the former is a restriction of the latter).

Let us prove the reverse reduction. By Proposition 3.3, for every open $P \subset [\mathbb{N}]^\mathbb{N}$, if $f \notin \mathrm{HS}(P)$ then there is a $g \preceq f$ s.t. $g \in \mathrm{HS}(P)$. This implies that the set $[\mathbb{N}]^\mathbb{N} \setminus \mathrm{HS}(P)$ has no homogeneous solution that lies in itself.

Let $D \in \mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N})$ and fix a name $\langle p, q\rangle$ for D. Consider the set

$$E := \{f \in [\mathbb{N}]^\mathbb{N} \,:\, (\exists \sigma, \tau \in p)(\sigma^\frown\tau \sqsubset f) \lor (\exists \sigma, \tau \in q)(\sigma^\frown\tau \sqsubset f)\}.$$

It is clear that E is a clopen set and a name for E is computable from $\langle p, q\rangle$.

Notice also that $\mathrm{HS}(D) \subset E$. Indeed, let f be a homogeneous solution for D and assume first that $f \in D$. Since D is open there must be a $\sigma \in p$ s.t. $\sigma \sqsubset f$. Moreover, since f is a homogeneous solution, $g := (f(|\sigma|), f(|\sigma|+1), \dots)$ must again be in D, hence there must be a $\tau \in p$ s.t. $\tau \sqsubset g$. This implies that $\sigma^\frown\tau \sqsubset f$, i.e. $f \in E$. The case $f \in [\mathbb{N}]^\mathbb{N} \setminus D$ is analogous by replacing D with $[\mathbb{N}]^\mathbb{N} \setminus D$.

Moreover we can notice that there are no homogeneous solutions that land in $[\mathbb{N}]^\mathbb{N} \setminus E$, i.e. $E \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Delta}^0_1})$. Indeed assume that f avoids E and let $\sigma, \tau \in [\mathbb{N}]^{<\mathbb{N}}$ be s.t. $\sigma^\frown\tau \sqsubset f$ and $\sigma \in p$ and $\tau \in q$ (the case in which $\tau \in p$ and $\sigma \in q$ is analogous). Let also $g := (f(|\sigma|), f(|\sigma|+1), \dots)$. Since $g \preceq f$ and f is homogeneous there must be $\rho \in p$ s.t. $\tau^\frown\rho \sqsubset g$. We can now notice that the subsequence h of f defined as

$$h := (f(0), \dots, f(|\sigma|-1), f(|\sigma|+|\tau|), f(|\sigma|+|\tau|+1), \dots)$$

is s.t. $\sigma^\frown\rho \sqsubset h$, and therefore $h \in E$, contradicting the fact that f avoids E.

We now claim that $\mathrm{HS}(D) = \mathrm{HS}(E)$. Once the claim is proved, we can finish the reduction using $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}(E) = \mathbf{\Delta}^0_1{-}\mathsf{RT}(D)$.

It is straightforward to notice that $\mathrm{HS}(D) \subset E$ implies $\mathrm{HS}(D) \subset \mathrm{HS}(E)$, hence we only need to prove the inclusion $\mathrm{HS}(E) \subset \mathrm{HS}(D)$. Let $f \in \mathrm{HS}(E)$. Since $E \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Delta}^0_1})$ we must have $f \in E$. Assume w.l.o.g. that $f \in D$ and let $\sigma \sqsubset f$ be s.t. $\sigma \in p$. Fix $g \preceq f$ and let $\rho \sqsubset g$ be s.t. $\rho \in p \cup q$. Let $k \in \mathbb{N}$ be s.t. $\max \rho < f(k)$ and $\max \sigma < f(k)$, and consider $\tau \in p \cup q$ be s.t. $\tau \sqsubset (f(k), f(k+1), \dots)$. By the homogeneity of f we have that $\tau \in p$ (otherwise every substring of f that begins with $\sigma^\frown\tau$ would not be in E). Let $h \preceq f$ be s.t. $\rho^\frown\tau \sqsubset h$. Again, by the homogeneity of f we have that $\rho \in p$, hence $g \in D$. Since g was arbitrary, we have that $f \in \mathrm{HS}(D)$. ■

3.2.3 Problems reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$

Here we consider $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$, $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ and $\mathsf{FindHS}_{\mathbf{\Delta}^0_1}$.

Theorem 3.22:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$.

Proof: We first show that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \leq_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Given a name $\langle P \rangle$ for some open set $P \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$, by Lemma 3.15.1 we can compute a name for the closed set $\mathrm{HS}(P) \setminus P$, which by hypothesis is nonempty. Therefore we can use $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to pick a solution.

Since $\mathsf{FindHS}_{\mathbf{\Delta}^0_1} \leq_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ is trivial, it remains to show that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Delta}^0_1}$. Let $T \subset [\mathbb{N}]^{<\mathbb{N}}$ be s.t. $[T] \neq \emptyset$ and let $D := D(T)$, i.e.

$$D = \{f \in [\mathbb{N}^{<\mathbb{N}}]^\mathbb{N} : f(0) \in T \text{ and } f(1) \in T \text{ and } f(0) \sqsubseteq f(1)\}.$$

Recall that D is computable from T (see Lemma 3.15.5). Moreover, by Lemma 3.14, we have that $D \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Delta}^0_1})$ and that every $f \in \mathsf{FindHS}_{\mathbf{\Delta}^0_1}(D)$ uniformly computes a path through T. ■

Proposition 3.23:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

Proof: The reduction is straightforward knowing that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ (Theorem 3.20). The fact that the reduction is strict follows from [108, Sec. 3]. In particular, Solovay showed that there is an open set W with computable code s.t. every homogeneous solution avoids W (hence W is a valid input for $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$) and is neither Σ^1_1 nor Π^1_1 (in particular it is not hyperarithmetic), while every computable instance of $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ has an hyperarithmetic solution (Theorem 2.13). ■

Proposition 3.24:
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

Proof: The first reduction follows from Theorem 3.22 as $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ is the restriction of $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ to a smaller domain. The reduction $\mathsf{C}_{2^\mathbb{N}} * \mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is straightforward from $\mathsf{C}_{2^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is closed under compositional product.

Finally the reduction $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ is suggested by the proof of the corollary in [108, Sec. 3]. In particular, given an ill-founded tree $T \subset [\mathbb{N}]^\mathbb{N}$ we can computably define the open set $W := W(T)$ (Definition 3.11). By Lemma 3.12, $W \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$ and every solution $f \in \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}(W)$ dominates a path through T. Let X be the subtree of $\mathbb{N}^{<\mathbb{N}}$ of the strings that are dominated by f and let $T_f := T \cap X$. Since $\emptyset \neq [T_f] \subset [T]$, we can use $\mathsf{C}_{[X]}([T_f])$ to compute a path through T. To conclude the proof it is enough to notice that $\mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{[X]}$ ([17, Thm. 7.23]). ■

Notice that, by the choice-elimination principle ([17, Thm. 7.25]), if Y is a computable metric space, $f :\subseteq X \to Y$ is a single-valued function and $f \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ then $f \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

Since we are not able to show the equivalence of $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ with any known principle, it is worth studying its properties.

Proposition 3.25:
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{sW}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \times \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

Proof: Notice that if $P, Q \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$ then $P \cup Q \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$. Indeed, for every $f \in [\mathbb{N}]^\mathbb{N}$, by the open Ramsey theorem applied to $[f]^\mathbb{N} \cap P$ (see Proposition 3.3), there is a $g \preceq f$ s.t. $[g]^\mathbb{N} \subset [f]^\mathbb{N} \cap P$ or $[g]^\mathbb{N} \cap [f]^\mathbb{N} \cap P = \emptyset$. In the first case we would have a contradiction as $[g]^\mathbb{N} \subset [f]^\mathbb{N} \cap P$ implies $[g]^\mathbb{N} \subset P$, i.e. $\mathrm{HS}(P) \cap P \neq \emptyset$, against $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$. Therefore we have $[g]^\mathbb{N} \cap [f]^\mathbb{N} \cap P = [g]^\mathbb{N} \cap P = \emptyset$, i.e. $g \in \mathrm{HS}(P) \setminus P$. With a similar argument, we can now apply the open Ramsey theorem to $[g]^\mathbb{N} \cap Q$ and conclude that there is a $h \preceq g$ s.t. $h \in \mathrm{HS}(Q) \setminus Q$. In particular $h \notin Q$ and $h \notin P$ (as $h \preceq g$ and g avoids P). Therefore h is a subsequence of f that is not in $P \cup Q$. Since f was arbitrary, we have that $\mathrm{HS}(P \cup Q) \cap (P \cup Q) = \emptyset$. This shows that every homogeneous solution $f \in \mathrm{HS}(P \cup Q)$ avoids $P \cup Q$, and, in particular, avoids both P and Q. Since the union is computable (see Lemma 3.2.3) we can compute a solution for $(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \times \mathsf{wFindHS}_{\mathbf{\Pi}^0_1})(P, Q)$ by computing $f \in \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}(P \cup Q)$ and returning two copies of f. ■

Recall that, in Section 2.1.2, we introduced the problem $\mathbf{\Pi}^1_1{-}\mathsf{Bound}$, whose input can be assumed being a sequence $(T_m)_{m\in\mathbb{N}}$ of trees s.t. there exists k s.t. $[T_i] = \emptyset$ iff $i < k$. We will show in Proposition 5.52 that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \widehat{\mathbf{\Pi}^1_1{-}\mathsf{Bound}}$. We notice the following:

Theorem 3.26:
$\widehat{\mathbf{\Pi}^1_1{-}\mathsf{Bound}} \le_{\mathrm{sW}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

PROOF: We will use strings σ which are prefixes of an infinite string f obtained by joining countably many strings g_i; we write $\sigma = \operatorname{dvt}(\tau_0, \dots, \tau_n)$ if τ_i is the prefix of g_i contained in σ. Formally $\sigma = \operatorname{dvt}(\tau_0, \dots, \tau_n)$ iff

- $n = \max\{i : \langle i, 0\rangle < |\sigma|\}$,
- for each i, $|\tau_i| = \max\{j : \langle i, j\rangle < |\sigma|\} + 1$,
- for each $\langle i, j\rangle < |\sigma|$, $\tau_i(j) = \sigma(\langle i, j\rangle)$.

Let $(T_{n,m})_{n,m\in\mathbb{N}}$ be a double sequence of trees s.t. for every n there is k_n s.t. $[T_{n,m}] = \emptyset$ iff $m < k_n$. For every n we can define

$$T_n := () \cup \bigcup_{m\in\mathbb{N}} (m)^\frown T_{n,m}.$$

Notice that, by hypothesis, for every n we have $[T_n] \neq \emptyset$. Moreover, if $f \in [T_n]$ then $f(0) \in \mathbf{\Pi}^1_1{-}\mathsf{Bound}((T_{n,m})_{m\in\mathbb{N}})$. Define also

$$T := \{\sigma \in \mathbb{N}^{<\mathbb{N}} : \sigma = \operatorname{dvt}(\tau_0, \dots, \tau_k) \land (\forall i \leq k)(\tau_i \in T_i)\}.$$

Notice that if $f_n \in [T_n]$ then $\langle f_0, f_1, \dots\rangle \in [T]$, hence $[T] \neq \emptyset$. Moreover, if $f \in [T]$ and $i \leq j$ then, letting $f[i] = \operatorname{dvt}(\tau_0, \dots, \tau_k)$ and $f[j] = \operatorname{dvt}(\rho_0, \dots, \rho_h)$, for every $n \leq k$ we have $\tau_n \sqsubseteq \rho_n$. Therefore

$$f \in [T] \iff f = \langle f_0, f_1, \dots\rangle \text{ and } (\forall n \in \mathbb{N})(f_n \in [T_n]).$$

Let $W := W(T)$ be the Solovay open set for T. By Lemma 3.12, $W \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$ and every $h \in \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}(W)$ dominates a path through T.

To conclude the proof we notice that, if $f = \langle f_0, f_1, \dots\rangle \in [T]$ and h dominates f then, for every n,

$$h(\langle n, 0\rangle) \geq f_n(0) \in \mathbf{\Pi}^1_1{-}\mathsf{Bound}((T_{n,m})_{m\in\mathbb{N}}).$$

In particular $h(\langle n, 0\rangle) \in \mathbf{\Pi}^1_1{-}\mathsf{Bound}((T_{n,m})_{m\in\mathbb{N}})$. ■

In particular, this implies that $\widehat{\mathbf{\Pi}^1_1{-}\mathsf{Bound}}$ is not a cylinder, as $\mathrm{id}_2 \not\leq_{\mathrm{sW}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ (see the following Proposition 3.44).

Recall that $\mathsf{ATR}_2 \colon \mathrm{LO} \times 2^{\mathbb{N}} \times \mathbb{N} \rightrightarrows \{0,1\} \times \mathbb{N}^{\mathbb{N}}$ is the two sided version of ATR (see Section 2.1.2). Jun Le Goh (personal communication) observed the following corollary:

Corollary 3.27:
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \not\leq_{\mathrm{W}} \mathsf{ATR}_2$.

PROOF: The claim follows from the fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ (Proposition 3.24) while $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathsf{ATR}_2$ ([44, Cor. 8.5]). ■

Let us denote with $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}$ the total continuation of $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$, i.e. the (total) multi-valued function with domain $\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$ defined as

$$\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P) := \begin{cases} \mathrm{HS}(P) & \text{if } P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}); \\ [\mathbb{N}]^\mathbb{N} & \text{otherwise.} \end{cases}$$

The following proposition underlines the gap between $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ and $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ (recall that $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} <_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ by Lemma 3.18 and Proposition 3.23).

Proposition 3.28:
$\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{ATR}_2$, *and hence* $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \not\le_{\mathrm{W}} \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}$.

PROOF: Let $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$ be an input for $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}$ and consider the tree $T_{\langle P\rangle}$. We can computably build the linear order $\mathrm{KB}(T_{\langle P\rangle})$. Notice that it is not necessarily a well-order, as we are not assuming $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ (i.e. there may be solutions that avoid P). Let θ be the arithmetic formula defined in the proof of Lemma 3.18 and let $(i, Y) \in \mathsf{ATR}_2(\mathrm{KB}(T_{\langle P\rangle}), P, \theta)$. If $i = 0$ then Y is a $<_{\mathrm{KB}(T_{\langle P\rangle})}$-infinite descending sequence and $P \notin \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$; therefore any $f \in [\mathbb{N}]^\mathbb{N}$ is a valid output for $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P)$. Suppose now that $i = 1$, so that Y is a (pseudo)hierarchy. By construction, Y yields a labeling of each $\sigma \in [\mathbb{N}]^{<\mathbb{N}}$ as "good" or "bad" (see the proof of [106, Lem. V.9.4]). The classical proof shows that if $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then $()$ is good. In particular if $()$ is bad then we can immediately conclude that $P \notin \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ (and, again, any $f \in [\mathbb{N}]^\mathbb{N}$ is a valid solution for the original problem). On the other hand, if $()$ is good then we can follow the construction described in the classical proof and compute $f \in [\mathbb{N}]^\mathbb{N}$. This follows from the definition of the sets U_σ, which have to be infinite for every σ. Notice that if $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then $f \in \mathrm{HS}(P) = \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P)$. On the other hand, if $P \notin \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then $f \in \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P)$ (trivially).

The second part follows from Corollary 3.27. ■

3.2.4 Problems not reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$

Let us turn our attention to the last two remaining problems, namely $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ and $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Proposition 3.29:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ *and* $\chi_{\Pi^1_1} <_{\mathrm{W}} \mathsf{LPO} * \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$.

PROOF: We first show $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$. Let $T \subset [\mathbb{N}]^\mathbb{N}$ be a tree and let $W := W(T)$ be the Solovay open set of T. Let also $f \in \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}(W)$. By Lemma 3.12, if $[T] \neq \emptyset$ then $\mathrm{HS}(W) \cap W = \emptyset$ and f is a bound for some $x \in [T]$. On the other hand, if $[T] = \emptyset$ then $W = [\mathbb{N}]^\mathbb{N}$ and f is just an arbitrary infinite string.

Let X be the subtree of $\mathbb{N}^{<\mathbb{N}}$ of the strings that are dominated by f. Notice that $\mathsf{TC}_{[X]} \equiv_{\mathrm{W}} \mathsf{C}_{[X]}$. Indeed, to show that $\mathsf{TC}_{[X]} \le_{\mathrm{W}} \mathsf{C}_{[X]}$ we can notice that, given a tree $S \subset X$, we can computably define an ill-founded tree R as follows: for each level n we check whether S has no nodes at level n. If this happens for some n, we can (computably) extend S to an ill-founded tree R. If this never happens then $R = S$. It is straightforward to see that $\mathsf{C}_{[X]}([R]) \subset \mathsf{TC}_{[X]}([S])$.

Let $T_f := T \cap X$. By [17, Thm. 7.23], $\mathsf{C}_{2^\mathbb{N}} \equiv_\mathrm{W} \mathsf{C}_{[X]} \equiv_\mathrm{W} \mathsf{TC}_{[X]}$, therefore we can use $\mathsf{C}_{2^\mathbb{N}}$ to compute a solution $h \in \mathsf{TC}_{[X]}([T_f])$. Notice that $h \in \mathsf{TC}_{\mathbb{N}^\mathbb{N}}([T])$. Indeed, if $[T] \neq \emptyset$ then $[T_f] \neq \emptyset$ and h is a path through T.

A simple modification of the above argument shows that $\chi_{\Pi^1_1} \le_\mathrm{W} \mathsf{LPO} * \mathbf{\Sigma}^0_1{-}\mathsf{RT}$. In fact, we can see the tree T as an input for $\chi_{\Pi^1_1}$. If $f \in \mathbf{\Sigma}^0_1{-}\mathsf{RT}(W(T))$ then T_f is a finitely branching tree. Thus whether T_f is finite is a Σ^0_1 question in T_f. We can therefore use LPO to check if T_f is infinite and hence establish whether it is well-founded or not (by König's lemma, a finitely-branching tree is infinite iff it has a path).

The reduction is trivially strict as $\chi_{\Pi^1_1}$ always has a computable output. ■

It follows from Theorem 3.35 that the reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_\mathrm{W} \mathsf{C}_{2^\mathbb{N}} * \mathbf{\Sigma}^0_1{-}\mathsf{RT}$ is actually strict.

Corollary 3.30:
$\mathbf{\Sigma}^0_1{-}\mathsf{RT} \not\le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: If $\mathbf{\Sigma}^0_1{-}\mathsf{RT} \le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ then

$$\chi_{\Pi^1_1} \le_\mathrm{W} \mathsf{LPO} * \mathbf{\Sigma}^0_1{-}\mathsf{RT} \le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}} * \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}},$$

contradicting the fact that $\chi_{\Pi^1_1} \not\le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see [64, Sec. 7]). ■

Corollary 3.31:
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} <_\mathrm{W} \mathbf{\Sigma}^0_1{-}\mathsf{RT}$.

PROOF: The fact that $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \le_\mathrm{W} \mathbf{\Sigma}^0_1{-}\mathsf{RT}$ is trivial since $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ is a restriction of $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$ to a smaller domain. The reduction is strict because $\mathbf{\Sigma}^0_1{-}\mathsf{RT} \not\le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Corollary 3.30) but $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \le_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Proposition 3.24). ■

Definition 3.32: For every represented space X, we define the *strong total continuation of* C_X to be the multi-valued function $\mathsf{sTC}_X \colon \mathbf{\Pi}^0_1(X) \rightrightarrows 2 \times X$ defined as

$$\mathsf{sTC}_X(A) := \{(b, x) \in 2 \times X \,:\, (b = 0 \to A = \emptyset) \land (b = 1 \to x \in A)\}.$$

In particular, for $X = \mathbb{N}^\mathbb{N}$ (and analogously for $X = 2^\mathbb{N}$) we can think of $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$ as the total multi-valued function that, given in input a tree, returns a string $(b)^\frown x$ s.t. b codes whether the tree is well-founded or not and, if it is ill-founded, then x is a path through T.

It is clear that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$ (the fact that the reduction is strict follows from $\chi_{\Pi^1_1} \not\le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ [64, Cor. 8.6], while obviously $\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$). We can also notice the following:

Corollary 3.33:
$\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{2^\mathbb{N}} * \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$.

Proof: It suffices to repeat the proof of the first statement of Proposition 3.29, using $\mathsf{sTC}_{2^\mathbb{N}}$ in place of $\mathsf{C}_{2^\mathbb{N}}$. ■

We will prove in Corollary 3.53 that the above reduction is actually strict.

Proposition 3.34:
$\widehat{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}} \,|_{\mathrm{W}}\, \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$.

Proof: The fact that $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \widehat{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}}$ follows from the obvious observation that $\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$, while $\chi_{\Pi^1_1} \not\le_{\mathrm{W}} \widehat{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}}$ (see [64, Cor. 8.6]).

On the other hand, if $\widehat{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$ then, in particular, $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$. Since $\mathsf{NHA} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see e.g. [64, Cor. 3.6]), by Proposition 2.14, this implies that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}|_A$, where A is the set of non-empty closed sets of $\mathbb{N}^\mathbb{N}$ with no hyperarithmetic member (notice that $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}(\emptyset)$ has computable solutions). In particular, this implies that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, contradicting [64, Prop. 8.2.1]. ■

We will now show that $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \not\le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. We actually prove a stronger result that will be useful in Section 3.3.

Theorem 3.35:
For every $n \in \mathbb{N}$, $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \not\le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)}$.

Proof: Let (X, δ_X) be the represented space of computably open subsets of $[\mathbb{N}]^\mathbb{N}$ with no arithmetic homogeneous solution, where δ_X is the restriction of $\delta_{\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})}$ to computable names. Let us define $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}_X \colon X \rightrightarrows [\mathbb{N}]^\mathbb{N}$ as $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}_X(P) := \mathrm{HS}(P)$.

The reduction $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}_X \le_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ holds trivially, hence it is enough to prove that

$$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}_X \not\le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)}.$$

Assume by contradiction that there is a reduction. Since $\mathsf{lim}^{(n)}$ is a cylinder, we can assume that the reduction is a strong Weihrauch reduction. Let Φ_1, Φ_2, Ψ be the maps witnessing the strong reduction, with Φ_1 producing an input for $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$ and Φ_2 producing an input for $\mathsf{lim}^{(n)}$. Assume that there is an $P \in X$ s.t. $\Phi_1(\langle P \rangle)$ is a name for the empty set, for some name $\langle P \rangle$ of P. By definition, 0^ω is a valid output of $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}(\emptyset)$. Let $q := \mathsf{lim}^{(n)}(\Phi_2(\langle P \rangle))$. Notice that q is arithmetic, as $\langle P \rangle$ is computable by definition of X. We have now reached a contradiction as $\Psi(0^\omega, q)$ is arithmetic, against the fact that P has no arithmetic solution.

This implies that, for every $P \in X$ and every name $\langle P \rangle$ of P, $\Phi_1(\langle P \rangle)$ is a name for a non-empty closed set, hence we have a reduction $\mathbf{\Sigma}^0_1{-}\mathsf{RT}_X \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

We now claim that $\mathbf{\Sigma}^0_1{-}\mathsf{RT}_X \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, concluding the proof. We will in fact show that $c\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{LPO} * \mathbf{\Sigma}^0_1{-}\mathsf{RT}_X$, where $c\chi_{\Pi^1_1}$ is the restriction of $\chi_{\Pi^1_1}$ to computable trees. The claim then follows from the fact that $c\chi_{\Pi^1_1} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (as $c\chi_{\Pi^1_1}$ is not effectively Borel measurable, see [11, Thm. 7.7]) and the fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is closed under compositional product.

Let Φ_D be the forward functional witnessing $\mathbf{\Sigma}^1_1\text{-}\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ (recall that $\mathbf{\Sigma}^1_1\text{-}\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ [64, Thm. 3.11], while $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ has been proved in Lemma 3.19). Let also T_{NAR} be a computable input for $\mathbf{\Sigma}^1_1\text{-}\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ with a single non-arithmetic solution (recall that, by [96, Thm. II.4.2] every H-set is a Π^0_2 singleton).

Let T be an input for $c\chi_{\Pi^1_1}$. We can assume w.l.o.g. that T has no hyperarithmetic path: indeed if S is a computable ill-founded tree with no hyperarithmetic path then

$$T \times S = \{(\langle \sigma(0), \tau(0) \rangle, \dots, \langle \sigma(n-1), \tau(n-1) \rangle) : \sigma \in T \text{ and } \tau \in S\}$$

is ill-founded iff T is, and $T \times S$ has no hyperarithmetic path.

Let $W := W(T)$ be the Solovay open set for T and let Q be the clopen set with name $\Phi_D(T_{NAR})$. Notice that, since $Q \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Delta}^0_1})$, for every f we can computably find a subsequence $g \preceq f$ s.t. $g \in Q$.

We can computably define $P := W \cap Q$ (see [10, Prop. 3.2.4]). Since W and Q are computable then so is P. Let us show that P does not have any arithmetic solution, which implies $P \in X$. We distinguish two cases:

1. $[T] = \emptyset$: by Lemma 3.12 we have that $W = [\mathbb{N}]^\mathbb{N}$, hence $P = Q$ and $\mathrm{HS}(P) = \mathrm{HS}(Q)$. Since every solution for Q computes the non-arithmetic solution for T_{NAR}, P does not have arithmetic solutions.

2. $[T] \neq \emptyset$: notice first of all that $P \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$ as $P \subset W$ and $W \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Pi}^0_1})$ (see Lemma 3.12).

   Given $f \in \mathrm{HS}(P)$ then, by the above observation, we can computably find a subsolution $g \in \mathrm{HS}(P)$ s.t. $g \in Q$, thus $g \notin W$. By König's lemma such a g is a bound for a path through T (see the proof of Lemma 3.12). This also implies that every $f \in \mathrm{HS}(P)$ is not (hyper)arithmetic (as, by hypothesis, T does not have hyperarithmetic paths).

Given $f \in \mathbf{\Sigma}^0_1{-}\mathsf{RT}_X(P)$ we can computably find $g \preceq f$ s.t. $g \in Q$. Let T_g be the subtree of T bounded by g. Notice that g is a bound for a path through T iff T_g is ill-founded iff T is ill-founded (as shown in case 2 above). Since T_g is a finitely-branching tree, by König's lemma T_g is ill-founded iff it is infinite. Moreover, the problem of checking whether T_g is finite is a $\Sigma^{0,g}_1$ question, hence we can use LPO to solve the problem (as in the proof of Proposition 3.29). ■

Proposition 3.36:
For every (partial) multi-valued function f, if $f \times \mathsf{NHA} \leq_{\mathrm{W}} \mathbf{\Sigma}^0_1{-}\mathsf{RT}$ then $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: Assume that $f \times \mathsf{NHA} \leq_{\mathrm{W}} \mathbf{\Sigma}^0_1{-}\mathsf{RT}$ and let $B := \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. By definition $\mathbf{\Sigma}^0_1{-}\mathsf{RT}|_B = \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$, hence, since $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.20), the restriction of $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$ to B always has a solution that is hyperarithmetic relative to the input (Theorem 2.13). Since $f \times \mathsf{NHA} \leq_{\mathrm{W}} \mathbf{\Sigma}^0_1{-}\mathsf{RT}$, by Proposition 2.14, we have that f is reducible to the restriction of $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$ to

$$A := \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \setminus B = \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1}).$$

This implies that $f \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$, as for each $P \in A$ we have $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}(P) \subset \mathbf{\Sigma}^0_1{-}\mathsf{RT}(P)$ (and therefore every realizer for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ is also a realizer for $\mathbf{\Sigma}^0_1{-}\mathsf{RT}|_A$). The claim follows from the fact that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.22). ■

Corollary 3.37:
$\mathbf{\Sigma}^0_1{-}\mathsf{RT} <_{\mathrm{W}} \mathsf{NHA} \times \mathbf{\Sigma}^0_1{-}\mathsf{RT} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Sigma}^0_1{-}\mathsf{RT}$.

PROOF: The first reduction is straightforward and the second one follows from the fact that $\mathsf{NHA} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see [64, Cor. 3.6]). The fact that the first reduction is strict follows from Proposition 3.36 and the fact that $\mathbf{\Sigma}^0_1{-}\mathsf{RT} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Corollary 3.30). ■

To have a better understanding of the uniform strength of $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$, we now show that, even with parallel access to some hyperarithmetic computational power, $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$ does not reach the level of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Thus $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$ is not at the level of $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$, which is one of the strongest principles considered in [64] to be still at the level of ATR_0.

Proposition 3.38:
If $f :\subseteq X \rightrightarrows Y$ always has an hyperarithmetic solution relative to the input and $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ then $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq_{\mathrm{W}} f \times \mathbf{\Sigma}^0_1{-}\mathsf{RT}$.

PROOF: Notice that, if we define $B := \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$, then

$$(f \times \mathbf{\Sigma}^0_1{-}\mathsf{RT})|_{X \times B} = f \times \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}.$$

Since $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.20) we have that $(f \times \mathbf{\Sigma}^0_1{-}\mathsf{RT})|_{X \times B}$ always has a solution that is hyperarithmetic relative to the input (Theorem 2.13). Assume by contradiction that the reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_W f \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ holds. By Proposition 2.14 we have that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is reducible to the restriction of $f \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ to

$$A := X \times (\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \setminus B) = X \times \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1}).$$

In particular, this implies that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_W f \times \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ (see also the proof of Proposition 3.36). We have therefore reached a contradiction as we would have

$$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_W f \times \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}.$$ ■

In particular, Proposition 3.38 implies $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\le_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$.

Let us now turn our attention to $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$. We first notice the following useful property:

Proposition 3.39:
$\mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \le_{sW} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Proof: Let $\langle P_1\rangle, \langle P_2\rangle$ be names for two open sets $P_1, P_2 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$. Assume w.l.o.g. that every string $\sigma \in \langle P_1\rangle$ has length at least 2 (there is no loss of generality as we can computably modify the code of P_1 by replacing a string with length 1 with all its extensions of length 2).

Let P be the open set with name $\langle P_1\rangle \boxtimes \langle P_2\rangle$. Recall that P is computable from P_1 and P_2 (see Lemma 3.15). Moreover, by Lemma 3.10

$$\operatorname{HS}(P) \cap P = \{f \boxtimes g \,:\, f \in \operatorname{HS}(P_1) \cap P_1 \text{ and } g \in \operatorname{HS}(P_2) \cap P_2\}.$$

Since the projections π_i are computable, it is clear that, from every solution of $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(P)$, we obtain two homogeneous solutions that land in P_1 and P_2 respectively. ■

Corollary 3.40:
$\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ *is a cylinder.*

Proof: This follows from Proposition 3.39 and the fact that $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \le_{sW} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$, as it then follows that
$$\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \le_{sW} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \equiv_{sW} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}.$$

To prove that $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \le_{sW} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ we proceed as follows: let $p \in \mathbb{N}^\mathbb{N}$ and assume w.l.o.g. that $p \in [\mathbb{N}]^\mathbb{N}$. Consider the tree $T := \{p[k] \,:\, k \in \mathbb{N}\}$ of prefixes of p and let $D := D(T)$. By Lemma 3.14 we have that $D \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$ and that every $f \in \operatorname{HS}(D) \cap D$ uniformly computes p. ■

The problem $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is much stronger than all of the other Ramsey-related problems we introduced. We will in fact show that $\mathbf{\Sigma}^0_1{-}\mathsf{RT} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (and this holds even if we consider arithmetic reductions, see Theorem 3.54).

Although we will prove much stronger results, it is worth it to sketch a short proof for the reduction $\mathbf{\Sigma}^0_1{-}\mathsf{RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$. Given a name $\langle P\rangle$ for an open set P build the open set

$$Q := \eta_2(\langle P\rangle) \cup D_{\psi_3}(T_{\langle P\rangle}),$$

where $\psi_3 := \sigma \mapsto 3^{\langle\sigma\rangle+1}$ and $\langle\sigma\rangle$ is the code of σ. Using Lemma 3.8 and Lemma 3.14 one can prove that $Q \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$ and that every $f \in \mathrm{HS}(Q) \cap Q$ computes a solution for P.

Proposition 3.41:
$\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ *and hence* $\chi_{\Pi^1_1} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Proof: Let $\psi_n : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N} := \sigma \mapsto n^{\langle\sigma\rangle+1}$, where $\langle\sigma\rangle$ is the code of σ.

Let $T \subset [\mathbb{N}]^{<\mathbb{N}}$ be a tree. We can define the open set $P \subset [\mathbb{N}]^\mathbb{N}$ as $P := P_1 \cup P_2$, where $P_1 := D_{\psi_2}(T)$ and $P_2 := W_{\psi_3}(T)$. Notice that, by Lemma 3.14 and Lemma 3.12 we have

$$[T] = \emptyset \iff \mathrm{HS}(P_1) \cap P_1 = \emptyset \iff \mathrm{HS}(P_2) \cap P_2 \neq \emptyset.$$

Moreover, by Proposition 3.4,

$$\mathrm{HS}(P) \cap P = (\mathrm{HS}(P_1) \cap P_1) \cup (\mathrm{HS}(P_2) \cap P_2).$$

This implies that

$$[T] \neq \emptyset \Rightarrow \mathrm{HS}(P) \cap P = \mathrm{HS}(P_1) \cap P_1,$$
$$[T] = \emptyset \Rightarrow \mathrm{HS}(P) \cap P = \mathrm{HS}(P_2) \cap P_2.$$

In particular, given a $f \in \mathrm{HS}(P) \cap P$ we can know whether $f \in \mathrm{HS}(P_1) \cap P_1$ or $f \in \mathrm{HS}(P_2) \cap P_2$ just by checking $f(0)$. If $f(0)$ is a power of 2 then $[T] \neq \emptyset$ and we can compute a path through T by considering the string $x \in [\mathbb{N}]^\mathbb{N}$ s.t.

$$x = \bigcup_{i\in\mathbb{N}} \psi_2^{-1}(f(i)).$$

In the other case $[T] = \emptyset$ hence we can just return $(0)^\frown f$.

The result about $\chi_{\Pi^1_1}$ follows from $\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \chi_{\Pi^1_1}$ as $\chi_{\Pi^1_1}$ always has computable output. ■

Corollary 3.42:
$\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

PROOF: The reduction follows from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (Proposition 3.41) and the fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is closed under product (Proposition 3.39). The fact the reduction is strict follows from the fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ computes $\chi_{\Pi^1_1}$ (Proposition 3.41), while $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$ does not ([64, Cor. 8.6]). ■

This shows that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is properly stronger than any multi-valued function arising from statements related to ATR_0 studied so far.

Theorem 3.43:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} * \mathbf{\Sigma}^0_1{-}\mathsf{RT} \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

PROOF: By the cylindrical decomposition we can write

$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} * \mathbf{\Sigma}^0_1{-}\mathsf{RT} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} \circ \Phi_e \circ (\mathrm{id} \times \mathbf{\Sigma}^0_1{-}\mathsf{RT})$$

for some computable function Φ_e. It is enough to show that

$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \circ \Phi_e \circ (\mathrm{id} \times \mathbf{\Sigma}^0_1{-}\mathsf{RT}) \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \chi_{\Pi^1_1}$$

and the claim will follow from $\chi_{\Pi^1_1} \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (Proposition 3.41) and the fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is closed under product (Proposition 3.39).

Let $\langle p_1, p_2\rangle$ be an input for $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \circ \Phi_e \circ (\mathrm{id} \times \mathbf{\Sigma}^0_1{-}\mathsf{RT})$ and let P be the open set with name p_2. We can consider the tree T_{p_2} of homogeneous solutions for P that avoid P. We can now compute a tree R s.t. for every $x, y \in \mathbb{N}^\mathbb{N}$,

$$x \in [T_{p_2}] \text{ and } y \in [\Phi_e(p_1, x)] \iff \langle x, y\rangle \in [R].$$

Using the canonical computable bijection between $\mathbb{N}^\mathbb{N}$ and $[\mathbb{N}]^\mathbb{N}$ it is easy to transform R into a tree $S \in \mathbf{Ti}$ so that from any path through S we can compute a path through R.

Recall that $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1} \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see Proposition 3.28). Since $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is closed under compositional product we have that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \circ \Phi_e \circ (\mathrm{id} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}) \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Let Φ_A, Ψ_A be two computable maps witnessing the reduction. In particular, $\Phi_A(\langle p_1, p_2\rangle)$ is an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$ and every path through $\Phi_A(\langle p_1, p_2\rangle)$ computes a solution for $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \circ \Phi_e \circ (\mathrm{id} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1})$ via Ψ_A. Let also ψ_n be the function that maps σ to $n^{\langle\sigma\rangle+1}$, where $\langle\sigma\rangle$ is the code of σ.

Let $D := D_{\psi_2}(S)$ and define

$$U := D \cup D_{\psi_3}(\Phi_A(\langle p_1, p_2\rangle));$$
$$V := D \cup W_{\psi_3}(S).$$

Let us first show that $U, V \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$. Notice that if $P \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ then $[T_{p_2}] \neq \emptyset$ and $[S] \neq \emptyset$. By Lemma 3.14 we have that $\mathrm{HS}(D) \cap D \neq \emptyset$ and therefore $U, V \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$. On the other hand, assume $P \notin \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$. Since $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}$ is total we have that $\Phi_A(\langle p_1, p_2\rangle)$ is ill-founded. This implies that $D_{\psi_3}(\Phi_A(\langle p_1, p_2\rangle))$ has solutions that land in itself (again by Lemma 3.14), and hence $U \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$. Moreover, since $P \notin \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ we have that $[S] = \emptyset$ and therefore $\mathrm{HS}(W_{\psi_3}(S)) \cap W_{\psi_3}(S) \neq \emptyset$, which shows that $V \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1})$.

Let $(f, g, b) \in (\mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \chi_{\Pi^1_1})(U, V, S)$. We distinguish 2 cases:

- if $b = 1$ then $[S] = \emptyset$, and hence $P \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. By Proposition 3.4 f lands in $D_{\psi_3}(\Phi_A(\langle p_1, p_2\rangle))$. In particular, f computes a path through $\Phi_A(\langle p_1, p_2\rangle)$ (Lemma 3.14). Moreover $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P) = \mathrm{HS}(P) = \mathbf{\Sigma}^0_1{-}\mathsf{RT}(P)$, so that f computes also a solution for the compositional product (by applying Ψ_A to the path).
- if $b = 0$ then $[S] \neq \emptyset$ and hence $P \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$. Moreover $\mathrm{HS}(V) \cap V = \mathrm{HS}(D) \cap D$. Indeed, by Proposition 3.4,

  $$\mathrm{HS}(V) \cap V = (\mathrm{HS}(D) \cap D) \cup (\mathrm{HS}(W_{\psi_3}(S)) \cap W_{\psi_3}(S))$$

  and $\mathrm{HS}(W_{\psi_3}(S)) \cap W_{\psi_3}(S) = \emptyset$ by Lemma 3.12. In this case, g computes a path through S, hence a path through R, and eventually a solution for the compositional product (by projecting the path through R).

The previous two points describe a way to compute a solution for the compositional product given a solution to $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \chi_{\Pi^1_1}$, and therefore conclude the proof. ■

Notice that if $P \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then we cannot (in general) use U to compute a solution for the compositional product. Indeed, it may be that $\mathrm{HS}(P) \cap P \neq \emptyset$ and the solution obtained from $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(U)$ lands in $D_{\psi_3}(\Phi_A(\langle p_1, p_2\rangle))$. However, since every string is a valid solution for $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}(P)$, the solution we obtain is not guaranteed to have any connection with the original problem.

Notice moreover that $\mathbf{\Sigma}^0_1{-}\mathsf{RT} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ as the former is not closed under product with $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Corollary 3.37) while the latter is closed under product (Proposition 3.39) and computes $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see Proposition 3.41). We will prove a stronger result in Theorem 3.54.

3.2.5 A $0-1$ law for strong Weihrauch reducibility

We now characterize the strength of the Ramsey-related multi-valued functions from the point of view of strong Weihrauch reducibility.

Proposition 3.44:
Let $\mathbf{\Gamma}$ be a definable (boldface) pointclass that is downward closed with respect to Wadge reducibility. Assume also that every $P \in \mathbf{\Gamma}([\mathbb{N}]^\mathbb{N})$ is Ramsey and that, for every $h \in [\mathbb{N}]^\mathbb{N}$,

$$\mathbf{\Gamma}([h]^\mathbb{N}) = \{P \cap [h]^\mathbb{N} \,:\, P \in \mathbf{\Gamma}([\mathbb{N}]^\mathbb{N})\}.$$

If $\mathsf{R} :\subseteq \mathbf{\Gamma}([\mathbb{N}]^\mathbb{N}) \rightrightarrows [\mathbb{N}]^\mathbb{N}$ is a multi-valued function, s.t. for every $x \in \operatorname{dom}(\mathsf{R})$, $\mathsf{R}(x) = \mathrm{HS}(x)$, then

$$\mathrm{id}_2 \not\leq_{\mathrm{sW}} \mathsf{R}.$$

In particular id_2 (and, a fortiori, $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$) is not strongly Weihrauch reducible to $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$, $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$, $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$, $\mathbf{\Sigma}^0_1{-}\mathsf{RT}$, $\mathbf{\Delta}^0_1{-}\mathsf{RT}$.

Proof: Assume there is a strong Weihrauch reduction witnessed by the computable maps Φ, Ψ. Let $p_i := \Phi(i)$ (with a small abuse of notation we are identifying i with its name) and let

$P_i := \delta_{\mathbf{\Gamma}([\mathbb{N}]^{\mathbb{N}})}(p_i)$. By definition of strong Weihrauch reducibility, for every $f \in \mathrm{HS}(P_i)$ we have $\Psi(f) = i$. Fix $f \in \mathrm{HS}(P_0)$ and consider the set $[f]^{\mathbb{N}} \cap P_1 \in \mathbf{\Gamma}([f]^{\mathbb{N}})$. By Proposition 3.3 we have that every pointset in $\mathbf{\Gamma}([f]^{\mathbb{N}})$ has the Ramsey property, therefore there is a $g \preceq f$ s.t. $[g]^{\mathbb{N}} \subset P_1 \cap [f]^{\mathbb{N}}$ or $[g]^{\mathbb{N}} \subset [f]^{\mathbb{N}} \setminus P_1$. In both cases $g \in \mathrm{HS}(P_1)$ and therefore $\Psi(g) = 1$. However $g \preceq f$, hence $g \in \mathrm{HS}(P_0)$ and so $\Psi(g) = 0$, which is a contradiction. ■

On the other hand, Theorem 3.22 shows that $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Delta}^0_1}$, which implies that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ and $\mathsf{FindHS}_{\mathbf{\Delta}^0_1}$ are cylinders. Since $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is also a cylinder (Corollary 3.40) we have that, for every g and every $f \in \{\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}, \mathsf{FindHS}_{\mathbf{\Pi}^0_1}, \mathsf{FindHS}_{\mathbf{\Delta}^0_1}\}$

$$g \le_{\mathrm{W}} f \iff g \le_{\mathrm{sW}} f.$$

This shows that, from the point of view of strong Weihrauch reducibility, the principles related to the open and clopen Ramsey theorems are either very weak (they do not strongly uniformly compute the identity on the 2-element space) or they are as strong as possible (the notions of Weihrauch reducibility and strong Weihrauch reducibility coincide).

3.3 Arithmetic Weihrauch reducibility

Let us explore the strength of the multi-valued functions related to the open and clopen Ramsey theorems from the point of view of arithmetic Weihrauch reducibility (introduced in Section 2.2).

A first straightforward result is the following:

Corollary 3.45:
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \equiv^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

Proof: This follows from $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} * \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ (Proposition 3.24), using Corollary 2.17 and Proposition 2.18. ■

Lemma 3.46:
Let g be a (partial multi-valued) function that computes every arithmetic function and is closed under compositional product. For every (partial) multi-valued function f

$$f \le^a_{\mathrm{W}} g \Rightarrow f \le_{\mathrm{W}} g.$$

Proof: It is enough to notice that $f \le^a_{\mathrm{W}} g$ implies that there exists n s.t.

$$f \le_{\mathrm{W}} \mathsf{lim}^{(n)} * g * \mathsf{lim}^{(n)}.$$

The hypotheses on g immediately yield the claim. ■

Corollary 3.47:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} <^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$.

PROOF: The fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is trivial as $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. The separation follows from Lemma 3.46 (recall that, for every n, $\mathsf{lim}^{(n)} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$) and the fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. ■

Theorem 3.48:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}$.

PROOF: This follows from Proposition 2.16, Proposition 2.18 and the fact that

$$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{LPO} * \mathsf{TC}_{\mathbb{N}^\mathbb{N}}.$$ ■

We will now prove the fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <^a_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{--}\mathsf{RT}$. To do so we will first need some additional results about compositional products of iterations of lim and $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$.

Lemma 3.49:
Let $D(X,Y,Z)$ be an arithmetic predicate with free variables among X,Y,Z and let $\Phi\colon \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbf{Tr}$ be computable. Define the Π^1_1 predicate $P(X,Y,Z)$ as

$$D(X,Y,Z) \wedge ([\Phi(X,Y)] \neq \emptyset \to Z \in [\Phi(X,Y)]).$$

There exists a Π^0_1 predicate $S(X,Y,Z,W)$ s.t. an index for S is computable from indices for D and Φ s.t.

$$(\exists W)(S(X,Y,Z,W)) \Rightarrow D(X,Y,Z),$$
$$[\Phi(X,Y)] \neq \emptyset \Rightarrow (\ P(X,Y,Z) \iff (\exists W)(S(X,Y,Z,W))\ ).$$

PROOF: By Kleene's normal form theorem (see e.g. [94, Thm. 16.IV]), there is a Π^0_1 predicate T s.t.

$$D(X,Y,Z) \iff (\exists W)(T(X,Y,Z,W)).$$

Define the Π^0_1 predicate $S(X,Y,Z,W) := T(X,Y,Z,W) \wedge Z \in [\Phi(X,Y)]$. It follows from Kleene's normal form theorem that an index for S is computable from indices for D and Φ. The first property of S is immediate. For the second notice that, if $[\Phi(X,Y)] \neq \emptyset$ then

$$\begin{aligned} P(X,Y,Z) &\iff D(X,Y,Z) \land Z \in [\Phi(X,Y)] \\ &\iff (\exists W)(T(X,Y,Z,W) \land Z \in [\Phi(X,Y)]) \\ &\iff (\exists W)(S(X,Y,Z,W)). \end{aligned}$$

■

The previous lemma can be interpreted as follows: the predicate P describes the compositional product (on both sides) of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ with an arithmetic problem f, while D says that Y is a solution for $f(X,Z)$. Notice that, if we are considering the composition $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} * f$ then f (and therefore D) will not depend on the output Z of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. On the other hand, if we consider $f * \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ then we need to keep track of Z. The lemma proves that there is a uniform way to build a tree (whose body is the set of solutions to S) s.t., by projecting its paths, we can obtain the solutions to the original problem P. Notice however that the lemma does not guarantee such a tree to be ill-founded. In other words, we can recover (some) solutions to the original problem only if the tree is ill-founded.

Obviously, if D depends only on X, Y and not on Z, then a solution for D can be (arithmetically) computed without first finding a path through the tree $\Phi(X,Y)$.

Lemma 3.50:
For every $n \in \mathbb{N}$, $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)}$.

Proof: Fix $n \in \mathbb{N}$. The reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)} \le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)}$ trivially follows from the algebraic rules of the operations (see [19, Prop. 4.4]).

To prove the converse reduction, by the cylindrical decomposition we can write

$$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)} \equiv_{\mathrm{W}} (\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \circ \Phi_e \circ \mathsf{lim}^{(n)},$$

for some computable function Φ_e. In particular

$$(\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \circ \Phi_e \circ \mathsf{lim}^{(n)}(p) = \langle \Phi_1(\mathsf{lim}^{(n)}(p)), \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \Phi_2(\mathsf{lim}^{(n)}(p)) \rangle$$

where Φ_1, Φ_2 are the computable functions s.t. $\Phi_e(p) = \langle \Phi_1(p), \Phi_2(p) \rangle$.

Let $D(X,Y)$ be the predicate that says

$$Y = \langle \Phi_1(\mathsf{lim}^{(n)}(X)), \Phi_2(\mathsf{lim}^{(n)}(X)) \rangle.$$

Notice that an index for D can be (uniformly) computed from an index of Φ_e. Define also the predicate $P(X,Y,Z)$ as

$$D(X,Y) \land ([\pi_2(Y)] \neq \emptyset \to Z \in [\pi_2(Y)]),$$

where $\pi_2 := \langle Y_1, Y_2 \rangle \mapsto Y_2$. Since $D(X,Y)$ is arithmetic, we can use Lemma 3.49 to define a computable tree S s.t.

$$\begin{gathered} (\exists W)((X,Y,Z,W) \in [S]) \Rightarrow D(X,Y), \\ [\pi_2(Y)] \neq \emptyset \Rightarrow (\ P(X,Y,Z) \iff (\exists W)((X,Y,Z,W) \in [S])\ ). \end{gathered}$$

For every fixed $p \in \operatorname{dom}((\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \circ \Phi_e \circ \mathsf{lim}^{(n)})$ we define $S_p := \{\sigma : \langle p[|\sigma|], \sigma\rangle \in S\}$. We now claim that, from an answer to $(\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)})([S_p], p)$ we can compute a solution to $(\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \circ \Phi_e \circ \mathsf{lim}^{(n)}(p)$.

Indeed $\Phi_1 \circ \mathsf{lim}^{(n)}(p)$ is trivially uniformly computed from $\mathsf{lim}^{(n)}(p)$. On the other hand, there is a unique q s.t. $D(p,q)$. Assume $[\pi_2(q)] \neq \emptyset$ and let $z_0 \in [\pi_2(q)]$. Since $P(p,q,z_0)$ holds, we have that $(\exists w)(\langle q, z_0, w\rangle \in [S_p])$, in particular $[S_p] \neq \emptyset$. Let $\langle y,z,w\rangle \in \mathsf{TC}_{\mathbb{N}^\mathbb{N}}([S_p])$. Notice that, since $\mathsf{lim}^{(n)}$ is single-valued, we have $y = q$. Hence we can conclude that $P(p,q,z)$ holds, and therefore, by projecting $\langle y,z,w\rangle$ on the second component we obtain a path through $[\pi_2(q)]$. If, on the other hand, $[\pi_2(q)] = \emptyset$, then any z belongs to $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}([\pi_2(q)])$. In both cases, by projecting the output of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}([S_p])$ we can compute a solution to $(\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \circ \Phi_2 \circ \mathsf{lim}^{(n)})(p)$ and this concludes the proof. ■

Lemma 3.51:
For every $n \in \mathbb{N}$, $\mathsf{lim}^{(n)} * \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(3n+5)}$.

PROOF: Fix $n \in \mathbb{N}$. By the cylindrical decomposition we can write

$$\mathsf{lim}^{(n)} * \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{lim}^{(n)} \circ \Phi_e \circ (\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}})$$

for some computable function Φ_e.

Let us define $F\colon \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ as $F := \mathsf{T}(\mathsf{lim}^{(n)} \circ \Phi_e)$. Recalling that $\mathsf{lim}^{(n)} = \mathsf{lim}^{[n+1]}$, it is immediate that being in the domain of $\mathsf{lim}^{(n)}$ is a Π^0_{2n+3} property. On the other hand, whether $\Phi_e(p,q)$ is defined is a Π^0_2 property. This implies that $F \leq_{\mathrm{W}} \mathsf{lim}^{[2n+5+n+1]} = \mathsf{lim}^{(3n+5)}$ and hence to prove the lemma it suffices to show that $\mathsf{lim}^{(n)} \circ \Phi_e \circ (\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \leq_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times F$.

Let $D(X,Y,Z)$ be the arithmetic predicate

$$Y = \mathsf{lim}^{(n)} \circ \Phi_e(\pi_1(X), Z).$$

Clearly an index for D is computable from an index of Φ_e. Let also $P(X,Y,Z)$ be the predicate

$$D(X,Y,Z) \land ([\pi_2(X)] \neq \emptyset \to Z \in [\pi_2(X)]).$$

Since $D(X,Y,Z)$ is arithmetic, we can use Lemma 3.49 to define a computable tree S s.t.

$$(\exists W)((X,Y,Z,W) \in [S]) \Rightarrow D(X,Y,Z),$$
$$[\pi_2(X)] \neq \emptyset \Rightarrow (\ P(X,Y,Z) \iff (\exists W)((X,Y,Z,W) \in [S])\ ).$$

For every fixed $p = \langle p_1, p_2\rangle \in \operatorname{dom}(\mathsf{lim}^{(n)} \circ \Phi_e \circ (\mathrm{id} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}))$ let $S_p := \{\sigma : \langle p[|\sigma|], \sigma\rangle \in S\}$. We define the forward Weihrauch functional as the map $\Phi := \langle p_1, p_2\rangle \mapsto ([S_p], (p_1, 0^\omega))$. Notice that, since F is total, $\Phi(\langle p_1, p_2\rangle)$ is a correct input for $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times F$.

Let $((b)^\frown\langle y,z,w\rangle, r) \in (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times F)([S_p], (p_1, 0^\omega))$. We claim that a valid solution for $\mathsf{lim}^{(n)}\ \Phi_e(p_1, \mathsf{TC}_{\mathbb{N}^\mathbb{N}}(p_2))$ is y if $b = 1$ or is r if $b = 0$.

Assume that $b = 1$, i.e. that $\langle y,z,w\rangle \in [S_p]$. In particular $D(\langle p_1, p_2\rangle, y, z)$ holds, i.e. $y = \mathsf{lim}^{(n)}(\Phi_e(p_1, z))$. Therefore it is enough to show that $z \in \mathsf{TC}_{\mathbb{N}^\mathbb{N}}([p_2])$. Assume that $[p_2] \neq \emptyset$

(the other case is trivial). Since $\langle y,z,w\rangle \in [S_p]$, we have that $P(\langle p_1,p_2\rangle,y,z)$ holds and therefore $z\in[p_2]$.

Assume now that $b=0$, i.e. for all y,z there is no w s.t. $\langle y,z,w\rangle\in[S_p]$. If $[p_2]\neq\emptyset$ then choose $z\in[p_2]$ and let $y=\mathsf{lim}^{(n)}\Phi_e(p_1,z)$. We then have that $D(\langle p_1,p_2\rangle,y,z)$ and $P(\langle p_1,p_2\rangle,y,z)$ hold. Therefore there exists w s.t. $\langle y,z,w\rangle\in[S_p]$, which is a contradiction. This implies that $[p_2]=\emptyset$ and therefore $0^\omega\in\mathsf{TC}_{\mathbb{N}^\mathbb{N}}([p_2])$ and $(p_1,0^\omega)\in\mathrm{dom}(\mathsf{lim}^{(n)}\Phi_e)$. Therefore $r=F(p_1,0^\omega)\in\mathsf{lim}^{(n)}\Phi_e(p_1,\mathsf{TC}_{\mathbb{N}^\mathbb{N}}([p_2]))$ and this concludes the proof. ■

We are now ready to prove the following characterization of arithmetic reducibility to $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$, conjectured by Arno Pauly during the BIRS-CMO 2019 workshop "Reverse Mathematics of Combinatorial Principles".

Theorem 3.52:
For every multi-valued function f,

$$f\le^a_{\mathrm{W}}\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \iff (\exists n)(f\le_{\mathrm{W}}\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}\times\mathsf{lim}^{(n)}).$$

Proof: The right-to-left implication follows from Corollary 2.17 and Proposition 2.18, as $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}\le_{\mathrm{W}}\mathsf{LPO}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. To prove the left-to-right implication, assume that there exists m s.t. $f\le_{\mathrm{W}}\mathsf{lim}^{(m)}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}}*\mathsf{lim}^{(m)}$. Notice that for every single-valued k and every g,h we have

$$(g\times h)*k\le_{\mathrm{W}}(g*k)\times(h*k). \tag{$\star$}$$

This fails for multi-valued k, as shown in [19, Prop. 4.9(19)]. We can therefore write

$$\begin{aligned}
\mathsf{lim}^{(m)}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}}*\mathsf{lim}^{(m)} &\le_{\mathrm{W}} (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}\times\mathsf{lim}^{[3m+6]})*\mathsf{lim}^{[m+1]} && \text{Lemma 3.51}\\
&\le_{\mathrm{W}} (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}}*\mathsf{lim}^{[m+1]})\times\mathsf{lim}^{[4m+7]} && (\star)\\
&\le_{\mathrm{W}} (\mathsf{LPO}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}}*\mathsf{lim}^{[m+1]})\times\mathsf{lim}^{[4m+7]} &&\\
&\equiv_{\mathrm{W}} (\mathsf{LPO}*(\mathsf{TC}_{\mathbb{N}^\mathbb{N}}\times\mathsf{lim}^{[m+1]}))\times\mathsf{lim}^{[4m+7]} && \text{Lemma 3.50}\\
&\le_{\mathrm{W}} (\mathsf{LPO}*\mathsf{lim}^{[m+1]}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}})\times\mathsf{lim}^{[4m+7]} &&\\
&\le_{\mathrm{W}} (\mathsf{lim}^{[m+2]}*\mathsf{TC}_{\mathbb{N}^\mathbb{N}})\times\mathsf{lim}^{[4m+7]} &&\\
&\le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}\times\mathsf{lim}^{[3m+9]}\times\mathsf{lim}^{[4m+7]} && \text{Lemma 3.51}\\
&\equiv_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}}\times\mathsf{lim}^{[n]}, &&
\end{aligned}$$

where $n=\max\{3m+9,4m+7\}$. ■

Corollary 3.53:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}}<^a_{\mathrm{W}}\boldsymbol{\Sigma}^0_1\text{–}\mathsf{RT}$.

Proof: The reduction follows from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} * \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ (Proposition 3.29) using Corollary 2.17 and Proposition 2.18. The fact that the reduction is strict follows from Theorem 3.52 as $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \not\le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(k)}$ for any k (Theorem 3.35). ■

Theorem 3.54:
$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} <^a_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Proof: It suffices to show that $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} <^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$, as $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ follows from $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (see Theorem 3.43), $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ (see Proposition 3.41) and the fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is closed under parallel product (Proposition 3.39).

The reduction is trivial, so we only need to prove the separation. Notice that Proposition 2.19 is the analogue of Proposition 2.14 for arithmetic Weihrauch reduction. This allows us to repeat the proof of Proposition 3.36, obtaining that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le^a_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ implies $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, which is a contradiction by Corollary 3.47 and Corollary 3.53. ■

Theorem 3.55:
$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}^* \equiv^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$.

Proof: The right-to-left reduction is a trivial consequence of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <^a_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$ (Corollary 3.53). To prove the left-to-right reduction we first notice that

$$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}\ .$$

Indeed, given an open set $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$, we can uniformly compute the input $([T_{\langle P\rangle}], P)$ for $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1}$. This is clearly a valid input as both the functions are total. Fix a pair $((b)^\frown x, f) \in (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1})([T_{\langle P\rangle}], P)$. If $b = 1$ then $x \in \mathrm{HS}(P) \setminus P$ (Lemma 3.6), and therefore $x \in \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}(P)$. If $b = 0$ then $[T_{\langle P\rangle}] = \emptyset$, which implies that $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ and hence $f \in \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}(P)$.

We then have

$$\mathbf{\Sigma}^0_1\text{–}\mathsf{RT} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}},$$

where $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ follows from $\mathsf{TwFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{ATR}_2$ (Proposition 3.28). From this $\mathbf{\Sigma}^0_1\text{–}\mathsf{RT}^* \le^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$ follows immediately. ■

3.4 Conclusions

Some problems resisted full characterization. In particular two questions remain open:

Question 3.56: $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$?

Question 3.57: $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_W \mathbf{\Sigma}^0_1\text{–}\mathsf{RT}$?

Observe that a positive answer to the first question automatically yields a positive answer to the second one by Corollary 3.31. We can expect that answering one of the two questions can shed light on the other.

As already observed in [64], there is not a single "analog" of ATR_0 in the context of Weihrauch reducibility, and theorems that are equivalent from the reverse mathematics point of view can exhibit very different behaviors when phrased as multi-valued functions.

Moreover, the classical proofs of the equivalences, over RCA_0, of ATR_0 and the open and clopen Ramsey theorems are useful only to obtain the equivalences $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_W \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$.

Finding a homogeneous solution that lands in an open set, when there are also solutions that avoid it, is a much harder problem. In particular, notice that a natural candidate for $\mathbf{\Pi}^1_1\text{–}\mathrm{CA}_0$ in the Weihrauch lattice is $\widehat{\chi_{\Pi^1_1}}$. The fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ computes $\chi_{\Pi^1_1}$ and is closed under parallel product implies that $\chi^*_{\Pi^1_1} \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$. This naturally leads to the following question:

Question 3.58: $\widehat{\mathsf{FindHS}}_{\mathbf{\Sigma}^0_1} \le_W \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$?

A positive answer to this question would locate $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ in the realm of $\mathbf{\Pi}^1_1\text{–}\mathrm{CA}_0$, in sharp contrast with what happens in reverse mathematics.

Let us now derive a few computability-theoretic corollaries.

Corollary 3.59 (Solovay's theorem [108, Thm. 1.8]):
If $P \subset [\mathbb{N}]^\mathbb{N}$ is open with computable code then either there is an hyperarithmetic homogeneous solution landing in P or there is a homogeneous solution avoiding P.

Proof: This follows from Theorem 2.13 and $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Lemma 3.18). ■

The following result is attributed to Solovay [108] (see [74, Thm. 1] for an explicit proof).

Corollary 3.60:
The set of homogeneous solutions for a clopen set with computable code always contains a hyperarithmetic element.

Proof: This follows from Theorem 2.13 and the equivalence $\mathbf{\Delta}^0_1-\mathsf{RT} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.21). ■

Corollary 3.61:
There is a clopen set $D \subset [\mathbb{N}]^{\mathbb{N}}$ with computable code s.t. every homogeneous solution that lands in D is not hyperarithmetic.

Proof: This follows from the fact that $\mathsf{FindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.22): if every computable clopen set had an hyperarithmetic solution landing in itself then every computable instance of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ would have a hyperarithmetic solution, contradicting $\mathsf{NHA} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([64, Cor. 3.6]). ■

Corollary 3.62:
Every open set $P \subset [\mathbb{N}]^{\mathbb{N}}$ with computable code has a homogeneous solution f that is strictly Turing reducible to Kleene's $\mathcal{O}$.

Proof: It follows from the proof of Gandy basis theorem (see [96, Chap. III, Thm. 1.4]) that $\{f : f <_T \mathcal{O}\}$ is a basis for the Σ^1_1 predicates. If $P \in \mathrm{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then, by Corollary 3.59, it has an hyperarithmetic solution. Otherwise $P \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ hence, by Lemma 3.6, a homogeneous solution for P can be computed from any element of $[T_{\langle P\rangle}]$ (the tree $T_{\langle P\rangle}$ is computable from $\langle P\rangle$, see Lemma 3.15). By the Gandy basis theorem the claim follows. ■

In particular Corollary 3.62 shows that the difference, in the (arithmetic) Weihrauch lattice, between $\mathbf{\Sigma}^0_1-\mathsf{RT}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ cannot be explained in terms of complexity of the solutions but rests entirely on the lack of uniformity.

4

Operators on multi-valued functions

In this chapter, we introduce and study some operators on multi-valued functions. The first-order part of a problem (Section 4.2) and the deterministic part of a problem (Section 4.3) have been formally introduced only recently (resp. in [31] and [46]), but the underlying ideas have been already exploited in the literature. The union operator (next section) reminds one of the if-then-else operator introduced in [64] (see Definition 4.9 below), but its definition and algebraic properties are very different.

4.1 The union of problems

We define a new operator on multi-valued functions, called *union*. The definition was motivated by some considerations on the uniform strength of $\boldsymbol{\Sigma}^0_1\text{−}\mathsf{RT}$ and, more generally, on the strength of problems whose behavior presents some form of non-computable disjunction between two different cases.

The results presented here are not conclusive[1], but can still reveal some interesting computational aspects of problems that lie at the level of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, like the perfect tree theorem and the determinacy of open games (see [64]).

Definition 4.1: Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be two partial multi-valued functions between represented spaces. We define the *union* of f and g as the problem

$$f \cup g :\subseteq X \times Z \rightrightarrows Y \cup W := (x, z) \mapsto f(x) \cup g(z)$$

It follows from the definition that

$$\mathrm{dom}(f \cup g) := \{(x, z) \,:\, x \in \mathrm{dom}(f) \vee z \in \mathrm{dom}(g)\}$$

as $x \notin \mathrm{dom}(f)$ means $f(x) = \emptyset$, hence if $x \notin \mathrm{dom}(f)$ and $z \notin \mathrm{dom}(g)$ then $f(x) \cup g(z) = \emptyset$, and therefore $(x, z) \notin \mathrm{dom}(f \cup g)$.

[1] The original hope was to apply these results to characterize the Weihrauch degree of $\boldsymbol{\Sigma}^0_1\text{−}\mathsf{RT}$. Unfortunately, that was not the case.

Intuitively $f \cup g$ is the problem that takes in input a pair of questions s.t. at least one of the two is correct, and produces a correct solution to one of the correct inputs. Trivially, the operation is symmetric. Notice that the solution does not come with an indication on "the side" it belongs to, i.e. given a solution $t \in (f \cup g)(x,z)$, we may not be able to tell whether $t \in f(x)$ or $t \in g(z)$.

Definition 4.2: A partial (multi-valued) function g is called *copointed* if there is a computable point $z \notin \operatorname{dom}(g)$.

A first immediate proposition is the following:

Proposition 4.3:
If g is copointed then $f \le_{\mathrm{sW}} f \cup g$.

Proof: Just consider the map $x \mapsto (x,z)$ for some fixed computable $z \notin \operatorname{dom}(g)$. ■

Albeit being not too strong, the hypothesis that g is copointed cannot be (easily) relaxed:

Proposition 4.4:
If f is total then $f \cup g \le_{\mathrm{sW}} f$. Moreover, if g is copointed then $f \cup g \equiv_{\mathrm{sW}} f$.

Proof: The reduction $f \cup g \le_{\mathrm{sW}} f$ is a trivial consequence of the fact that f is total. If, additionally, g is copointed, then the strong equivalence follows from Proposition 4.3. ■

This proposition can be used to show that $\cup$ does not lift to (strong) Weihrauch degrees. Consider indeed $\mathsf{C}_{\mathbb{N}}$ and lim. Since $\mathsf{C}_{\mathbb{N}}$ is copointed we have that $\mathsf{lim} \le_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}} \cup \mathsf{lim}$. However, $\mathsf{C}_{\mathbb{N}}$ is (strongly) equivalent to the problem $h :\subseteq \boldsymbol{\Pi}^0_1(\mathbb{N}) \setminus \{\emptyset\} \rightrightarrows \mathbb{N}$ defined as $h(A) := \mathsf{C}_{\mathbb{N}}(A)$. Since h is total and lim is copointed, we have $h \cup \mathsf{lim} \le_{\mathrm{sW}} h \equiv_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}}$.

4.1.1 Choice on $\mathbb{N}^{\mathbb{N}}$ and union

Let us explore how the problems $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, and $\mathsf{TC}_{\mathbb{N}^{\mathbb{N}}}$ interact with the union operator.

Theorem 4.5:
$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \cup \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

Proof: The left-to-right reduction is trivial. To prove the reduction $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \cup \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ it is enough to notice that, given a pair $(A_1, A_2) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \cup \mathsf{C}_{\mathbb{N}^{\mathbb{N}}})$, we have

$$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}(A_1 \cup A_2) = (\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \cup \mathsf{C}_{\mathbb{N}^{\mathbb{N}}})(A_1, A_2).$$ ■

Theorem 4.6:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$.

Proof: The reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ follows from the fact that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is copointed, while the reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ follows from the fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is total.

The equivalence $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ can be proved analogously (changing $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ with $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$).

Finally, the equivalence $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ trivially follows from the fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is total. ∎

Before proving the following Theorem 4.8, we introduce the following lemma:

Lemma 4.7:
Let f_0, f_1, g be multi-valued functions, and assume w.l.o.g. that $\mathrm{dom}(f_0)$ and $\mathrm{dom}(f_1)$ are subsets of $\mathbb{N}^\mathbb{N}$. Assume also that $\mathrm{dom}(f_0) \in \boldsymbol{\Gamma}(\mathbb{N}^\mathbb{N})$ and $\mathrm{dom}(f_1) \in \boldsymbol{\Gamma}'(\mathbb{N}^\mathbb{N})$, for some pointclasses $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}'$. If $f_0, f_1 \le_W g$ then, for every pointclass $\boldsymbol{\Lambda}$ s.t. $\boldsymbol{\Gamma} \cup \boldsymbol{\Gamma}' \subset \boldsymbol{\Lambda}$ and the inclusion maps $\boldsymbol{\Gamma} \hookrightarrow \boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}' \hookrightarrow \boldsymbol{\Lambda}$ are computable, we have

$$f_0 \cup f_1 \le_W g * \boldsymbol{\Lambda}\text{-}\mathsf{C}_2$$

Proof: It is enough to notice that, given an input (x_0, x_1) for $f_0 \cup f_1$, we can use $\boldsymbol{\Lambda}$-C_2 to obtain an index i s.t. $x_i \in \mathrm{dom}(f_i)$. Since $f_i \le_W g$, we can then use g to compute a solution $y \in f_i(x_i) \subset (f_0 \cup f_1)(x_0, x_1)$. ∎

Theorem 4.8:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof: It is enough to prove the right-to-left reduction. Since $\mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \in \boldsymbol{\Pi}^1_1(\mathbb{N}^\mathbb{N})$, by Lemma 4.7 we have that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} * \boldsymbol{\Pi}^1_1\text{-}\mathsf{C}_2$. Notice that $\widehat{\boldsymbol{\Pi}^1_1\text{-}\mathsf{C}_2} \equiv_W \boldsymbol{\Sigma}^1_1\text{-}\mathsf{Sep}$ and $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{Sep} \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ [64, Thm. 3.11]. Since $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is closed under compositional product the claim follows. ∎

Before studying the degree of $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, we define the if-then-else operator, first defined in [64, Def. 7.6]:

Definition 4.9: A *space of truth values* is just a represented space with underlying set $\{\bot, \top\}$. For every space of truth values $\mathbb{B}$ and every multi-valued functions $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$, we define the operator $[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ f\ \mathsf{else}\ g] :\subseteq X \times Z \rightrightarrows Y \times W$ as

$$[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ f\ \mathsf{else}\ g](b, x, z) := \begin{cases} f(x) \times W & \text{if } b = \top \\ Y \times g(z) & \text{if } b = \bot \end{cases}$$

with domain $\{\top\} \times \mathrm{dom}(f) \times Z \cup \{\bot\} \times X \times \mathrm{dom}(g)$.

This operator has been used in [64] to study the uniform computational strength of "two-sided" versions of problems arising from the perfect tree theorem and the open determinacy. In particular, the authors considered the space of truth values $\mathbb{S}_{\mathbf{\Sigma}^1_1}$, where $\delta_{\mathbb{S}_{\mathbf{\Sigma}^1_1}}$ is s.t. a name for $\top$ is a name for an ill-founded tree (and $\bot$ is named with codes of well-founded trees), and they used the if-then-else operator in combination with $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, showing that ([64, Cor. 7.8 and prop. 8.2(5, 6)])

$$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} [\mathsf{if}\ \mathbb{S}_{\mathbf{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}] <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} .$$

Notice that, as the union operator, the if-then-else does not lift to Weihrauch degrees ([64, Footnote 7]).

Lemma 4.10:
$\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof: Given $T \subset \mathbb{N}^{<\mathbb{N}}$, consider the pair $(2^\frown T, 0^\omega \cup 1^\frown T)$. Notice that

- If $[T] = \emptyset$ then $[2^\frown T] \notin \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$, while $[0^\omega \cup 1^\frown T] = \{0^\omega\} \in \mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$.
- If $[T] \neq \emptyset$ then $[2^\frown T] \in \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$, while $|[0^\omega \cup 1^\frown T]| > 1$. In particular, this implies that $[0^\omega \cup 1^\frown T] \notin \mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$.

This shows that $([2^\frown T], [0^\omega \cup 1^\frown T]) \in \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})$. Moreover, if $[T] = \emptyset$ then every solution begins with 0. Conversely, if $[T] \neq \emptyset$ then every solution begins with 2. In other words, given a solution $x \in (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})([2^\frown T], [0^\omega \cup 1^\frown T])$, we have $[T] = \emptyset$ iff $x(0) = 0$. ■

Theorem 4.11:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} [\mathsf{if}\ \mathbb{S}_{\mathbf{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}] <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof: The reduction $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} [\mathsf{if}\ \mathbb{S}_{\mathbf{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}]$ was proved in [64, Cor. 7.8 and Prop. 8.2(5)]. To prove the reduction $[\mathsf{if}\ \mathbb{S}_{\mathbf{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}] \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, let (T, T_1, T_2) be an input for $[\mathsf{if}\ \mathbb{S}_{\mathbf{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}]$. Consider the pair (S_1, S_2) defined as $S_1 := T \times T_1$ and $S_2 := 0^\frown T \cup 1^\frown T \cup 2^\frown T_2)$. Notice that

- If $[T] = \emptyset$ then $[S_1] \notin \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$, while $[S_2] = [2^\frown T_2] \in \operatorname{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$, as $[T] = \emptyset$ implies that $|[T_2]| = 1$.
- If $[T] \neq \emptyset$ then $[T_1] \neq \emptyset$, and therefore $[S_1] \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$. On the other hand, $|[S_2]| > 1$ and therefore $[S_2] \notin \operatorname{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$.

This shows that $([S_1], [S_2]) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})$. It is easy to see that, for every x in $(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})([S_1], [S_2])$, letting $y := n \mapsto x(n+1)$ we have

$$(y, y) \in [\mathsf{if}\ \mathbb{S}_{\boldsymbol{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}](T, T_1, T_2).$$

To show that the reduction is strict it is enough to notice that $\chi_{\Pi^1_1} \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Lemma 4.10), while $\chi_{\Pi^1_1} \not\le_W [\mathsf{if}\ \mathbb{S}_{\boldsymbol{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}]$ ([64, Prop. 8.2(6) and Cor. 8.6]). ■

Notice therefore that, while $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ does not increase its computational power when combined with itself, $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is much stronger (this is another example of the non-monotonicity of the union operator).

Notice also that $\operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \in \boldsymbol{\Sigma}^1_1(\mathbb{N}^\mathbb{N})$, while $\operatorname{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \in \boldsymbol{\Pi}^1_1(\mathbb{N}^\mathbb{N})$. Applying Lemma 4.7 we have

$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} * \mathbf{D}_2\text{-}\mathsf{C}_2 \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} * \boldsymbol{\Delta}^1_2\text{-}\mathsf{C}_2,$$

where $\mathbf{D}_2 = \{A \cap B : A \in \boldsymbol{\Sigma}^1_1 \text{ and } B \in \boldsymbol{\Pi}^1_1\}$. In particular, as a corollary of Theorem 4.11, we have $\mathbf{D}_2\text{-}\mathsf{C}_2 \not\le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Remark 4.12: While it may seem that the union is stronger than the if-then-else operator, this is actually not (always) the case. The use of the space of truth values $\mathbb{B}$ plays a crucial role: indeed the union $f \cup g$ does not allow to control "which problem to solve" in case it receives in input a pair (x, z) s.t. both x and z are in the domain of the respective function.

Indeed, the reduction

$$[\mathsf{if}\ \mathbb{S}_{\boldsymbol{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}] \le_W [\mathsf{if}\ \mathbb{S}_{\boldsymbol{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}]$$

is straightforward from the definition. In particular, this implies that

$$[\mathsf{if}\ \mathbb{S}_{\boldsymbol{\Sigma}^1_1}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}] \not\le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ .$$

Theorem 4.13:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ *is a cylinder.*

Proof: It is enough to notice that, given a pair of trees (T_1, T_2) s.t. $([T_1], [T_2]) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})$ and a string $p \in \mathbb{N}^\mathbb{N}$, we can uniformly compute the pair $(p \times T_1, p \times T_2)$, where $p \times T$ is a shorthand for $\{p[n] : n \in \mathbb{N}\} \times T$. It is straightforward to see that, for every tree T and every string p, $[p \times T] = \emptyset$ iff $[T] = \emptyset$, and that, from every path $x \in [p \times T]$, we can (uniformly) compute p and a path through $[T]$ by projection. ■

Theorem 4.14:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \mid_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: Let us first prove the incomparability. The fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ follows from the fact that $\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Lemma 4.10) while $\chi_{\Pi^1_1} \not\le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([64, Cor. 8.6]).

The fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ follows by Proposition 2.14. Indeed, if there is a reduction then $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})|_A$, where

$$A := \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \times \boldsymbol{\Pi}^0_1(\mathbb{N}^\mathbb{N}) \setminus \mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}).$$

In particular, since $(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})|_A \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, this would contradict the fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([64, Prop. 8.2(1)]).

The fact that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ follows trivially from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.11). The reduction is strict simply because $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \mid_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$. ■

Theorem 4.15:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} <^a_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: The reduction is a trivial corollary of Theorem 4.14. We now prove that the reduction is strict even if w.r.t. the arithmetic Weihrauch reducibility. This follows from Proposition 2.19 (see also the proof of Theorem 3.54). Indeed, if $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ then $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is arithmetically Weihrauch reducible to the restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ to the set

$$A := \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \times \boldsymbol{\Pi}^0_1(\mathbb{N}^\mathbb{N}) \setminus \mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}).$$

In particular, this implies that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, against Corollary 3.47. ■

Theorem 4.16:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: The left-to-right reduction follows trivially from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.14). To prove the right-to-left reduction, it is enough to show that

$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}},$$

where the arithmetic equivalence follows from the fact that $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. Since $\mathsf{TUC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (see the proof of [64, Prop. 8.2(6)]), we map every input (T_1, T_2) for $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$

to (T_1, S), where S is the (ill-founded) tree obtained applying the forward Turing functional of the reduction $\mathsf{TUC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to T_2.

Let $((b)^\frown x, y) \in (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}})([T_1],[S])$. If $b = 1$ then $x \in [T_1] \subset (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})([T_1],[T_2])$. Otherwise $[T_1] = \emptyset$ and therefore y uniformly computes the unique path through T_2. ■

Corollary 4.17:
$(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})^* \equiv^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$.

Proof: It is enough to notice that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.11) and that in the proof of Theorem 4.16 we showed that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. ■

In the same spirit of Theorem 3.52, we now characterize the multi-valued functions that are arithmetically Weihrauch reducible to $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. We first need the following lemma:

Lemma 4.18:
For every n, $\mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Proof: The right-to-left reduction is straightforward:

$$\begin{aligned}\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} &\le_{\mathrm{W}} (\mathsf{LPO} * \mathsf{TC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \\ &\le_{\mathrm{W}} \mathsf{LPO} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \le_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}).\end{aligned}$$

Let us now prove the left-to-right reduction. Since $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (and hence $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$) is a cylinder, using the cylindrical decomposition we can write

$$\mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{lim}^{(n)} \circ \Phi_e \circ (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}),$$

for some Turing functional Φ_e.

Consider the following predicate $D_1(X, Y_1, Y_2, Z)$:

$$Y_1 \in [\pi_1(X)] \wedge Y_2 \in [\pi_2(X)] \wedge Z = \mathsf{lim}^{(n)} \circ \Phi_e(Y_1, Y_2).$$

Intuitively, D_1 describes the compositional product $\mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}})$: the variable X plays the role of the join of the inputs X_1 and X_2 resp. of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, Y_1 and Y_2 are two solutions respectively of $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}(X_1)$ and of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}(X_2)$, and Z is the output of the compositional product. Notice that, if $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ receives in input the empty set, then there is no choice of the input variables that makes D_1 true.

Consider also the predicate $D_2(X, Y_2, Z)$ defined as

$$Y_2 \in [\pi_2(X)] \wedge Z = \mathsf{lim}^{(n)} \circ \Phi_e(0^\omega, Y_2).$$

Notice that $(0^\omega, Y_2) \in \operatorname{dom}(\mathsf{lim}^{(n)} \circ \Phi_e)$, as the pair $(\emptyset, \{Y_2\})$ is a valid input for $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. However, if $X_1 \neq \emptyset$ then 0^ω may not be a valid solution for $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}(X_1)$, and hence Z gives no information on the solution.

Since both D_1 and D_2 are arithmetic (and hence Σ^1_1), by the Kleene normal form theorem there are two Π^0_1 predicates $S_1(X, Y_1, Y_2, Z, W)$, $S_2(X, Y_2, Z, W)$ s.t.

- $(\exists W)(S_1(X, Y_1, Y_2, Z, W)) \iff D_1(X, Y_1, Y_2, Z)$;
- $(\exists W)(S_2(X, Y_2, Z, W)) \iff D_1(X, Y_2, Z)$.

This shows that we can uniformly build two trees R_1, R_2 s.t. the solutions for S_i can be obtained by projecting the path(s) of R_i. We apply $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to $([R_1], [R_2])$. Let $((b)^\frown x, y) \in (\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}})([R_1], [R_2])$.

Notice that if $b = 1$ then R_1 is ill-founded. In particular, this implies that a solution for the original compositional product can be uniformly computed from x (by projection). On the other hand, if $b = 0$ then $[R_1] = \emptyset$. This implies that the input for $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ is empty, and therefore $(0, y) \in \operatorname{dom}(\Phi_e)$. In this case, a solution for the compositional product can be obtained from y (again by projection). ■

Theorem 4.19:
For every multi-valued function f,

$$f \le^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \iff f \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}.$$

Proof: The right-to-left implication follows trivially from the fact that $\mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. To prove the converse implication, assume there is n s.t. $f \le_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) * \mathsf{lim}^{(n)}$. Recall that, for every single-valued k and every g, h we have $(g \times h) * k \le_{\mathrm{W}} (g * k) \times (h * k)$.

Using Lemma 3.50 and Lemma 4.18 we obtain:

$$\begin{aligned}
\mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) * \mathsf{lim}^{(n)} &\le_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)}) \\
&\equiv_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} * \mathsf{lim}^{(n)} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \\
&\equiv_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{lim}^{(n)} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \\
&\equiv_{\mathrm{W}} \mathsf{lim}^{(n)} * (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}) \\
&\equiv_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}
\end{aligned}$$

which concludes the proof. ■

This raises the following question:

Question 4.20: $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{sTC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$?

4.1.2 Open Ramsey theorem and union

We now explore how the union operator interacts with the multi-valued functions arising from the open Ramsey theorem. A first immediate observation is that, whenever f is a problem whose domain can be partitioned in two disjoint sets A_0, A_1, if there are two multi-valued functions g_0, g_1 s.t. $\operatorname{dom}(g_i) = A_i$ and, for every $x_i \in A_i$, $g_i(x_i) \subset f(x_i)$, then $f \le_{\mathrm{W}} g_0 \cup g_1$.

In particular, applying this reasoning to the open Ramsey theorem we obtain

$$\boldsymbol{\Sigma}^0_1\text{−}\mathsf{RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}.$$

Clearly, we could also write $\boldsymbol{\Sigma}^0_1\text{−}\mathsf{RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Pi}^0_1}$. In the following, however, we will focus on $\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$. Indeed, since $\mathsf{wFindHS}_{\boldsymbol{\Pi}^0_1}$ is copointed, we immediately have $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Pi}^0_1}$. The following result shows that the problem $\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ is a tighter upper bound for $\boldsymbol{\Sigma}^0_1\text{−}\mathsf{RT}$.

Proposition 4.21:
$\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$

Proof: We show that

$$\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \times \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \times \chi_{\Pi^1_1}$$

and the reduction will follow from the fact that $\chi_{\Pi^1_1} \le_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$ (Proposition 3.41) and that $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$ is closed under product (Proposition 3.39).

Let (P_1, P_2) be an input for $\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ and assume w.l.o.g. that every string $\sigma \in \langle P_2 \rangle$ has length at least 2. The idea is to produce two different open sets (one for each $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$ in the product) and use $\chi_{\Pi^1_1}$ to check whether $P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})$. According to the answer of $\chi_{\Pi^1_1}$ the correct answer will be the one provided by the first or by the second instance of $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}$.

Let $T_i := T_{\langle P_i \rangle}$ be the tree of homogeneous solutions for P_i that avoid P_i (with $i = 1, 2$). Let ψ_n be the function that maps σ to $n^{\langle \sigma \rangle + 1}$, where $\langle \sigma \rangle$ is the code of σ.

Define the set $Q := Q_1 \cup Q_2$ as

$$Q_1 := D_{\psi_2}(T_1);$$
$$Q_2 := \eta_3(\langle P_2 \rangle),$$

where η_n was introduced in Definition 3.7. Define also the (open) set $R := R_1 \cup R_2$, where $R_1 := Q_1$ and $R_2 := W_{\psi_3}(T_1)$.

Let us show that Q and R are in $\operatorname{dom}(\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1})$. If $P_1 \in \operatorname{dom}(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1})$ then so does $Q_1 = R_1$ (Lemma 3.14). On the other hand, if $P_1 \in \operatorname{dom}(\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})$ then so do P_2. In particular this implies that both Q_2 and R_2 are in $\operatorname{dom}(\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1})$ (the former because of Lemma 3.8 and the latter because of Lemma 3.12.1). This shows that both Q and R have a subset which belongs to $\operatorname{dom}(\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1})$, and hence they are in $\operatorname{dom}(\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1})$ as well.

Notice also that, by Proposition 3.4,

$$\operatorname{HS}(Q) \cap Q = (\operatorname{HS}(Q_1) \cap Q_1) \cup (\operatorname{HS}(Q_2) \cap Q_2),$$
$$\operatorname{HS}(R) \cap R = (\operatorname{HS}(R_1) \cap R_1) \cup (\operatorname{HS}(R_2) \cap R_2).$$

We consider the input (Q, R, T_2) for $\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \times \mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1} \times \chi_{\Pi^1_1}$. Notice that:

if $P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then every solution $f \in \mathrm{HS}(Q) \cap Q$ computes a solution for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$. Indeed, we can computably tell whether $f \in \mathrm{HS}(Q_1)$ or $f \in \mathrm{HS}(Q_2)$ (just by checking the first symbol) and we can use f to compute a solution for $(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(P_1, P_2)$ accordingly.

if $P_2 \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then we cannot (in general) use Q to compute a solution for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$. Indeed, it may be that $\mathrm{HS}(P_2) \cap P_2 \neq \emptyset$ and the solution obtained from $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(Q)$ computes a solution for P_2 (which would be incorrect). However, if $P_2 \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then $P_1 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ and $\mathrm{HS}(R) \cap R = \mathrm{HS}(R_1) \cap R_1$ (Lemma 3.12.2). In this case every solution f for $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(R)$ computes a solution $g \in \mathrm{HS}(P_1) \setminus P_1$, which is a valid output for $(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(P_1, P_2)$.

Given a solution $(f, g, b) \in (\mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \mathsf{FindHS}_{\mathbf{\Sigma}^0_1} \times \chi_{\Pi^1_1})(Q, R, T_2)$, we use b to determine whether $P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$, and then use f or g (accordingly) to compute a solution for $(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(P_1, P_2)$ (as described above). ■

We can easily prove that the reduction is strict. Indeed, it is straightforward to apply Proposition 2.14 and obtain the following result:

Proposition 4.22:
$\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} <_{\mathrm{W}} (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$

Proof: Recall that, since $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Lemma 3.18), by Theorem 2.13 every input (P, Q) for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ s.t. $Q \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ has a hyperarithmetic solution (relative to the input). Recall also that

$$\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N}) \setminus \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) = \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$$

and let $A := \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1}) \times \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$.

If $(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ then, by Proposition 2.14,

$$\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \leq_{\mathrm{W}} (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})|_A \equiv_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}},$$

against the fact that $\mathbf{\Sigma}^0_1\text{--}\mathsf{RT} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Corollary 3.30). ■

Corollary 4.23:
$\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$

Proof: The reduction was proved in Proposition 4.21. The separation follows from the fact that $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is closed under product (Proposition 3.39), while $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ is not (Proposition 4.22). ■

The problem $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ belongs to a relatively poorly explored region of the Weihrauch lattice, and it is hard to characterize exactly its degree.

Theorem 4.24:
$\mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$.

PROOF: Given a tree $T \subset [\mathbb{N}]^{<\mathbb{N}}$, consider the two sets $P_1, P_2 \subset [\mathbb{N}]^{<\mathbb{N}}$ defined as $P_1 := D(T)$ (Definition 3.13) and $P_2 := W(T)$ (Definition 3.11).

Notice that, by Lemma 3.14 and Lemma 3.12, at most one between P_1 and P_2 has a homogeneous solution that lies in itself. Indeed if $[T] = \emptyset$ then $\mathrm{HS}(P_1) \cap P_1 = \emptyset$ and $P_2 = [\mathbb{N}]^\mathbb{N}$. On the other hand, if $[T] \neq \emptyset$ then $\mathrm{HS}(P_1) \cap P_1 \neq \emptyset$ and $\mathrm{HS}(P_2) \cap P_2 = \emptyset$.

This shows that the pair (P_1, P_2) is a valid input for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$. Given a solution $f \in (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(P_1, P_2)$ we compute a solution for $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}([T])$ as follows: let Ψ be the Turing functional described in Lemma 3.14 witnessing the surjection $\mathrm{HS}(P_1) \cap P_1 \to [T]$ (when $[T] \neq \emptyset$). Notice that the condition $f \notin \mathrm{dom}(\Psi)$ is a $\Sigma^{0,f}_1$ problem: indeed it is equivalent to

$$(\exists i)(f(i) \notin T \text{ or } f(i) \not\sqsubset f(i+1)).$$

We run Ψ with input f. If $f \in \mathrm{dom}(\Psi)$ then we return $\Psi(f)$. Otherwise, there is a finite stage s in which we will recognize that $f \notin \mathrm{dom}(\Psi)$. In this case we just computably extend the (finite) string σ (obtained by stage s) as $\sigma^\frown(\sigma(n)+1, \sigma(n)+2, \ldots)$, where $n := |\sigma| - 1$.

Notice that the procedure described above always produces an element of $[\mathbb{N}]^\mathbb{N}$. Moreover, if $[T] \neq \emptyset$ then f describes a path through T and therefore $\Psi(f) \in [T]$.

The fact that the reduction is strict follows from $\mathbf{\Sigma}^0_1\mathsf{-RT} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$. In fact, Corollary 3.53 implies that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \not\le^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. ■

We now compare the uniform strength of $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ with the one of $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$.

Theorem 4.25:
$\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

PROOF: Let $(P, Q) \in \mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. Let $T_{\langle Q\rangle}$ be the tree of homogeneous solutions that avoid Q. Let also θ be the arithmetic formula defined in Lemma 3.18.

Let Φ, Ψ be two Turing functionals witnessing the (strong) reduction $\mathsf{ATR}_2 \le_{\mathrm{sW}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. We can use the forward functional Φ with input $(\mathrm{KB}(T_{\langle Q\rangle}), Q, \theta)$ to compute a (ill-founded) tree $S \subset \mathbb{N}^{<\mathbb{N}}$ s.t. for every path $x \in [S]$, $\Psi(x) = (i, Y)$, where if $i = 0$ then Y is an infinite descending sequence through $\mathrm{KB}(T_{\langle Q\rangle})$, while if $i = 1$ then Y is a (pseudo)-hierarchy obtained by iterating θ on $\mathrm{KB}(T_{\langle Q\rangle})$ with parameter Q. We can assume that, if $T_{\langle Q\rangle}$ is well-founded then $|[S]| = 1$. This follows from the fact that, if $T_{\langle Q\rangle}$ is well-founded then there are no infinite descending sequences through $\mathrm{KB}(T_{\langle Q\rangle})$, hence there is a unique hierarchy obtained iterating θ on $\mathrm{KB}(T_{\langle Q\rangle})$ with parameter Q. By (carefully) applying the Kleene normal form theorem, we obtain a tree S as required (see [106, Lem. V.5.4], [43, Lem. 1.18]).

We consider the input $(0^\frown T_{\langle P\rangle}, 1^\frown T_{\langle Q\rangle} \cup 2^\frown S)$ as input for $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$. Indeed:

- $P \in \mathrm{dom}(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1})$ iff $[0^\frown T_{\langle P\rangle}] \neq \emptyset$ (Lemma 3.6);
- $Q \in \mathrm{dom}(\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})$ iff $|[1^\frown T_{\langle Q\rangle} \cup 2^\frown S]| = 1$. Indeed, if $Q \in \mathrm{dom}(\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})$ then $[T_{\langle Q\rangle}] = \emptyset$. Moreover $\mathrm{KB}(T_{\langle Q\rangle})$ is a well-order, hence, in particular S has a unique path which encodes the hierarchy. On the other hand, if $Q \notin \mathrm{dom}(\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})$ then $T_{\langle Q\rangle}$ has uncountably many paths, and therefore $1^\frown T_{\langle Q\rangle} \cup 2^\frown S \notin \mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$.

This shows that $(0^\frown T_{\langle P\rangle}, 1^\frown T_{\langle Q\rangle} \cup 2^\frown S) \in \mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})$.

Moreover, for every $x \in (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})(0^\frown T_{\langle P\rangle}, 1^\frown T_{\langle Q\rangle} \cup 2^\frown S)$, we can computably tell whether $x \in \mathsf{C}_{\mathbb{N}^\mathbb{N}}(0^\frown T_{\langle P\rangle})$ or $x \in \mathsf{UC}_{\mathbb{N}^\mathbb{N}}(1^\frown T_{\langle Q\rangle} \cup 2^\frown S)$. If we define $y := n \mapsto x(n+1)$ then, in the first case, we have $y \in \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1}(P)$. In the second case, we reason as in Lemma 3.18 to obtain a solution for $\mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}(Q)$ from y. ■

Theorem 4.26:
$(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq^a_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$

Proof: Let us first prove the (arithmetic) equivalence. The right-to-left reduction follows trivially from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ (Theorem 4.24). The left-to-right one follows from $\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.25) and the fact that $(\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.16).

To prove that $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \not\leq^a_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ we use the analogue of Proposition 2.14 for arithmetic Weihrauch reduction. In particular, if $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \leq^a_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ then $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \leq^a_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1}$, against Corollary 3.47 (see also the proof of Theorem 4.15). ■

Theorem 4.27:
$(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})^* \equiv^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$.

Proof: This comes as a corollary of Theorem 4.26. Indeed the right-to-left reduction follows from $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{W}} \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}$ (Theorem 4.24), while the left-to-right one follows from

$$(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})^* \leq^a_{\mathrm{W}} (\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}})^* \equiv_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}.$$ ■

In particular, since $\boldsymbol{\Sigma}^0_1\text{–}\mathsf{RT}^* \equiv^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.55), we obtain also

$$(\mathsf{FindHS}_{\boldsymbol{\Pi}^0_1} \cup \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1})^* \equiv^a_{\mathrm{W}} \boldsymbol{\Sigma}^0_1\text{–}\mathsf{RT}^* .$$

We now show that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ is closed under parallel product with "hyperarithmetic problems".

Proposition 4.28:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \times (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \equiv_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$

Proof: The right-to-left reduction is trivial, so we just need to prove the left-to-right one. Since $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 3.20), it is enough to prove that

$$\mathsf{wFindHS}_{\mathbf{\Delta}^0_1} \times (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}.$$

Let $D \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Delta}^0_1})$ and assume w.l.o.g. that every string in $\langle D \rangle$ has length at least 2. Since D is clopen we can consider the tree $T = T_{\langle [\mathbb{N}]^\mathbb{N} \setminus D\rangle}$ of homogeneous solutions that land in D.

Let $(P_1, P_2) \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. Define the tree $S \subset [\mathbb{N}]^{<\mathbb{N}}$ as $S := T \cap T_{\langle P_1\rangle}$, where $T_{\langle P_1 \rangle}$ is the tree of homogeneous solutions for P_1 that avoid P_1. Let Q_1 and Q_2 the open sets defined as $Q_1 := [\mathbb{N}]^\mathbb{N} \setminus [S]$ and $Q_2 := D \cap P_2$. We observe that:

$P_1 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ iff $Q_1 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$. Indeed, if $P_1 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ then, by Proposition 3.3, for every $f \in \mathrm{HS}(P_1) \setminus P_1$ there is $g \preceq f$ s.t. $g \in \mathrm{HS}(D) = \mathrm{HS}(D) \cap D$. In particular, this shows that $[S] \neq \emptyset$. Moreover, since $[S]$ is closed under subsequences (i.e. if $f \in [S]$ then $g \in [S]$ for every $g \preceq f$), we have $Q_1 \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$.

On the other hand, if $P \notin \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$ then $[S] \subset [T_{\langle P_1 \rangle}] = \emptyset$ and hence $Q_1 = [\mathbb{N}]^\mathbb{N}$. In particular, $Q_1 \notin \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$.

Notice that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}(Q_1) \subset \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}(D) \cap \mathsf{FindHS}_{\mathbf{\Pi}^0_1}(P_1)$.

$P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ iff $Q_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. Indeed, assume that $P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. Applying Proposition 3.3 twice, we can conclude that for every $f \in [\mathbb{N}]^\mathbb{N}$ there are $g, h \in [\mathbb{N}]^\mathbb{N}$ s.t. $h \preceq g \preceq f$, $g \in \mathrm{HS}(P_2) \cap P_2$ and $h \in \mathrm{HS}(Q_2) \cap Q_2$. In particular, $Q_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$.

On the other hand, if $P_2 \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then, for every $f \in \mathrm{HS}(P_2) \setminus P_2$ we have $f \in \mathrm{HS}(Q_2) \setminus Q_2$ and therefore $Q_2 \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$.

Notice that $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}(Q_1) \subset \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}(D) \cap \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}(P_2)$.

This shows that (Q_1, Q_2) is a valid input for $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ and that, for every $f \in (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(Q_1, Q_2)$,

$$f \in \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}(D) \cap (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(P_1, P_2),$$

which concludes the proof. ■

We conclude this section by noticing that $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ is not a cylinder. In particular, this shows that there cannot be any strong reduction $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \leq_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$.

Proposition 4.29:
$(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \times \mathrm{id}_2 \not\leq_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$

Proof: Assume towards a contradiction that there is a reduction witnessed by the maps Φ, Ψ. With a small abuse of notation, let us identify an open set with its name. For $i < 2$, fix some open sets P_i, Q_i, s.t. $\Phi(P_i, Q_i, i) = (R_i, S_i)$, where R_i and S_i are open sets and $S_i \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$.

If there are no such sets then there is $i < 2$ s.t., for every P, Q, $\Phi(P, Q, i) = (R, S)$ and $S \notin \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$. In particular, this implies that $R \in \operatorname{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1})$, and therefore the maps Φ, Ψ would yield a reduction $(\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \times \mathrm{id}_2 \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, which is a contradiction.

Fix a solution $f_0 \in \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}(S_0) \subset (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})(R_0, S_0)$. In particular, for every subsolution $g \preceq f$, $(\pi_2 \Psi(g)) = 0$, where π_2 denotes the projection on the second component. However, by Proposition 3.3, there is a subsolution $f_1 \preceq f_0$ s.t. $f_1 \in \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}(S_1)$. In particular, $(\pi_2 \Psi(f_1))(0) = 1$, reaching again a contradiction. ■

With a similar reasoning we can also prove the following:

Proposition 4.30:
$\chi_{\Pi^1_1} \not\leq_{\mathrm{sW}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$

Proof: Assume towards a contradiction that there are two Turing functional Φ, Ψ witnessing the strong Weihrauch reduction. For the sake of readability define $V_i := \Psi^{-1}(\{p \in \mathbb{N}^\mathbb{N} : p(0) = i\})$ for $i < 2$. Clearly both V_0 and V_1 are non-empty and disjoint.

Let $T \subset \mathbb{N}^\mathbb{N}$ be a tree and define $b := \chi_{\Pi^1_1}(T)$. Let also P_1, P_2 be the open sets given by $\Phi(T)$. If $P_2 \in \operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$ then $\mathrm{HS}(P_2) \subset V_b$. By Proposition 3.3, for every $f \in V_{1-b}$ there is $g \preceq f$ s.t. $g \in \mathrm{HS}(P_2)$. In particular, $g \in V_0 \cap V_1$, which is a contradiction.

This shows that P_2 cannot be in $\operatorname{dom}(\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})$, and therefore Φ and Ψ witness a reduction $\chi_{\Pi^1_1} \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, against the fact that $\chi_{\Pi^1_1} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. ■

We summarize the results in Figure 4.1. In particular, we draw the attention to the following question:

Question 4.31: $\chi_{\Pi^1_1} \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$?

Answering this question (in either way) would automatically yield several separation results.

We can also summarize the results from the point of view of arithmetic Weihrauch reducibility as follows:

- $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \leq^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \leq^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$;
- $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} <^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$;

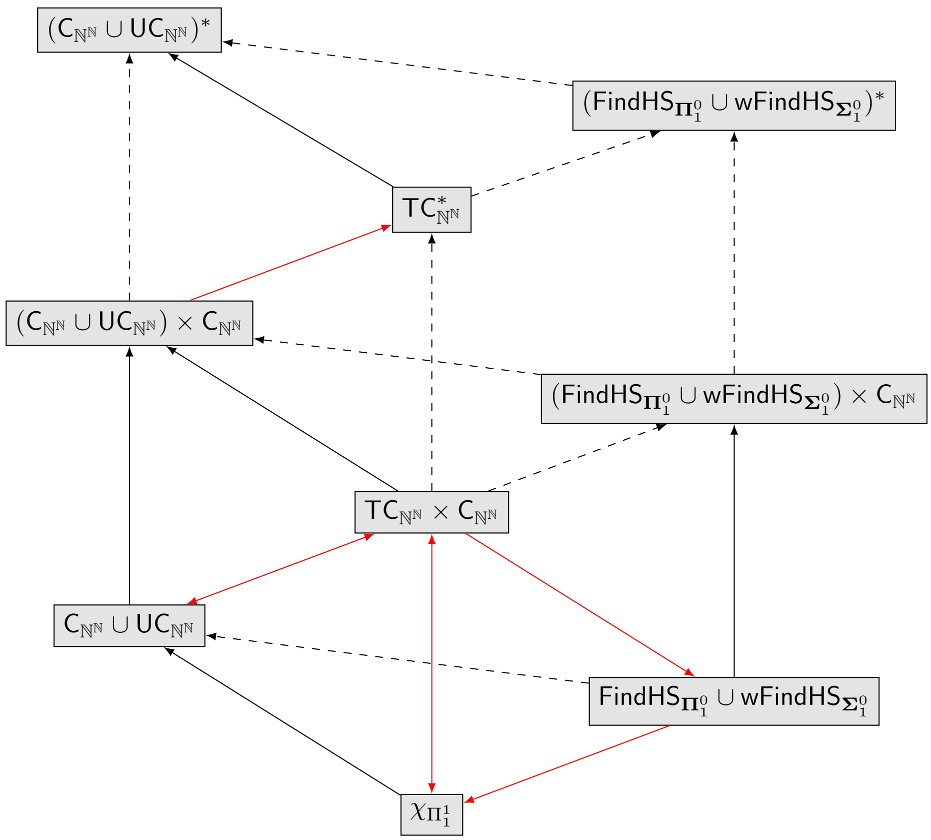

Figure 4.1: Dashed arrows represent Weihrauch reducibility in the direction of the arrow, solid arrows represent strict Weihrauch reducibility. Red arrows represent non-Weihrauch reduction in the direction of the arrow. The existence of every reduction that is not in the transitive closure of the diagram is open.

- $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}) \times \mathsf{C}_{\mathbb{N}^\mathbb{N}}$;
- $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le^a_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}} \equiv^a_{\mathrm{W}} (\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}})^* \equiv^a_{\mathrm{W}} (\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \cup \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1})^*$.

4.1.3 Further comments

While not being a degree-theoretic operator, the union of multi-valued functions can still provide interesting insights into the uniform strength of problems.

As already noticed, the fact that $\mathrm{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$ is $\mathbf{\Sigma}^1_1$-complete while $\mathrm{dom}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$ is $\mathbf{\Pi}^1_1$-complete is reflected by the fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, while $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \cup \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is much stronger.

It would be interesting to identify a (relatively mild) set of hypotheses under which the union operator is monotone (and, hence, degree-theoretic). Notice, for example, that while the if-then-else operator is not degree-theoretic, there is a simple condition that guarantees the monotonicity:

Proposition 4.32:
Let $\mathbb{B}$ be a space of truth values and let h be a multi-valued function. Let also $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ for some represented spaces X, Y, Z, W. If there is a reduction $f \le_{\mathrm{W}} g$ witnessed by two maps Φ, Ψ s.t.

- $\Phi|_{\operatorname{dom}(\delta_X)} : \operatorname{dom}(\delta_X) \to \operatorname{dom}(\delta_Z)$,
- $\Psi|_{\operatorname{dom}(\delta_X)\times\operatorname{dom}(\delta_W)} : \operatorname{dom}(\delta_X) \times \operatorname{dom}(\delta_W) \to \operatorname{dom}(\delta_Y)$,

then

$$[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ f\ \mathsf{else}\ h] \le_{\mathrm{W}} [\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ g\ \mathsf{else}\ h].$$

Proof: The proof is straightforward from the definition. Indeed, given some input (b, x, a) for $[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ f\ \mathsf{else}\ h]$, if p_x is a name for x then $p_z := \Phi(p_x)$ is a name for some element $z \in Z$. Notice that if $b = \top$ then $x \in \operatorname{dom}(f)$ and therefore $z \in \operatorname{dom}(g)$ (as Φ is the forward functional of a Weihrauch reduction). In particular, (b, z, a) is a valid input for $[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ g\ \mathsf{else}\ h]$.

Let (w, b) be a solution for $[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ g\ \mathsf{else}\ h](b, x, a)$, and let p_w be a name for w. We can compute the pair (y, b), where $y := \delta_Y(\Psi(p_x, p_w))$. In particular, if $b = \top$ then $w \in g(z)$, and therefore the backward functional Ψ computes a valid solution for $f(x)$.

If $b = \bot$, then x, z, w, and y are not relevant for the reduction, and we only need to ensure that they are elements of the respective represented spaces. This is guaranteed by the hypotheses on Φ and Ψ. ■

Intuitively, the requirements on Φ and Ψ in the previous proposition are some kinds of "totality" conditions. In particular, we ask that Φ can be run on any (name of an) element of X, and will always produce some (name for an) element of Z (and similarly for Ψ). In other words, this prevents the reduction $[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ f\ \mathsf{else}\ h] \le_{\mathrm{W}} [\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ g\ \mathsf{else}\ h]$ to get stuck due to some input that is, in fact, only garbage.

Notice that these conditions are naturally met in many cases: in particular this is trivially the case for the reductions $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$. This is also the case for the reductions $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ and $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. In particular we have

$$[\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ \mathsf{C}_{\mathbb{N}^\mathbb{N}}\ \mathsf{else}\ h] \equiv_{\mathrm{W}} [\mathsf{if}\ \mathbb{B}\ \mathsf{then}\ \mathsf{FindHS}_{\mathbf{\Pi}^0_1}\ \mathsf{else}\ h].$$

It is not clear whether Proposition 4.32 can be applied in case of $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ or $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$. Indeed, the backward functional defined in the proof of Lemma 3.18 as a witness of the reduction $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, and the forward functional defined in the proof of Lemma 3.19 as a witness of the reduction $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ do not satisfy the requirements of Proposition 4.32.

Let us now turn our attention to the union operator. We observe that, in this case, the hypotheses of Proposition 4.32 are too weak, and some extra care is needed. In particular:

1. if $f \le_{\mathrm{W}} g$ then the "totality" of the reduction maps is not sufficient to conclude that $f \cup h \le_{\mathrm{W}} g \cup h$ (take e.g. $f = \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $g = h = \mathsf{C}_{\mathbb{N}^\mathbb{N}}$);
2. if the backward functionals of the reductions $f_0 \le_{\mathrm{W}} g_0$ and $f_1 \le_{\mathrm{W}} g_1$ are different, then (in general), given a solution for $g_0 \cup g_1$ we may not be able to produce a correct output for $f_0 \cup f_1$.

To address the first point we need to take care that "no correct input is produced from an incorrect one". This can be guaranteed by asking that the forward functional witnesses a many-one (or effective Wadge) reduction between $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$.

Notice that, while the backward functional will always receive in input a correct solution (and therefore there is no need for a "totality" condition as in Proposition 4.32), we may not be able to tell whether a solution for $g \cup h$ is, in fact, a solution for g or a solution for h, and this brings us to the second point. This may not be a problem if the backward functional is the same in both cases (e.g. that was the case in Theorem 4.13). In general, however, asking that there is a unique functional playing the role of the backward functional of the reductions $f \le_{\mathrm{W}} g$ and $h \le_{\mathrm{W}} h$ (in some cases it may be useful to not use the identity) is a too strong condition.

We can notice that many multi-valued functions have a "recognizability" property: we say that f is recognizable if, for every $n, k \in \mathbb{N}$, there is a restriction $f_{n,k}$ of f s.t. $f_{n,k} \equiv_{\mathrm{W}} f$ and, for every $y \in \mathrm{dom}(f_{n,k})$, every name p for a solution of $f_{n,k}(y)$ is s.t. $p(n) = k$. In other words, we can uniformly modify every input x for f to an input y s.t. every name for a solution of $f(y)$ belongs to a fixed clopen set. This recognizability property can be useful when working with the union operator as, if g and h are recognizable[2], then we can uniformly tell whether a solution for $g \cup h$ is, in fact, a solution for g or a solution for h. Simple examples of recognizable problems are $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

We can therefore state the following proposition:

Proposition 4.33:
For every multi-valued functions f, g, h, if

1. *$f \le_{\mathrm{W}} g$ and the forward functional witnesses a many-one reduction* $\mathrm{dom}(f) \to \mathrm{dom}(g)$*,*
2. *g and h are recognizable and, for some $n \in \mathbb{N}$ and some $k_1 \neq k_2$, the forward map of the reductions $g \equiv_{\mathrm{W}} g_{n,k_1}$ (resp. $h \equiv_{\mathrm{W}} h_{n,k_2}$) witnesses a many-one reduction* $\mathrm{dom}(g) \to \mathrm{dom}(g_{n,k_1})$ *(resp.* $\mathrm{dom}(h) \to \mathrm{dom}(h_{n,k_2})$*),*

then $f \cup h \le_{\mathrm{W}} g \cup h$.

Proof (Sketch): The proof is straightforward: given an input (x, z) for $f \cup h$, we can use the reduction $f \le_{\mathrm{W}} g$ to obtain a pair $(y, z) \in \mathrm{dom}(g \cup h)$. Since both g and h are recognizable, we first obtain a pair $(y', z') \in \mathrm{dom}(g_{n,k_1} \cup h_{n,k_2})$. Then, from any solution w of $(g_{n,k_1} \cup h_{n,k_2})(y', z')$, if $w \in g_{n,k_1}(y')$ then we uniformly compute a solution for $g(y)$ and hence for $f(x)$, otherwise we just (uniformly) compute a solution for $h(z)$. ■

The fact that, in the previous proof, all the forward functionals of the reductions witness a many-one reduction guarantees that no correct input can be obtained from an invalid one.

We can notice that the previous proposition does not provide a set of necessary conditions for the union to be monotone. In fact, we could not find any set of necessary conditions that is (significantly) more informative than just the definition of Weihrauch reducibility. We therefore conclude this section by restating the original question:

[2]Slightly less then recognizability suffices: indeed, it is enough that both g and h are recognizable for a fixed n, i.e. that we can choose n and $k_1 \neq k_2$ s.t. $g \equiv_{\mathrm{W}} g_{n,k_1}$ and $h \equiv_{\mathrm{W}} h_{n,k_2}$.

Question 4.34: Is there a (relatively mild) set of hypotheses under which the union operator is monotone (and, hence, degree-theoretic)?

4.2 First-order part of a problem

Recently, Dzhafarov, Solomon, and Yokoyama [31], inspired by the first-order part of theories in reverse mathematics, introduced the following notion:

Definition 4.35: We say that a computational problem f is *first-order*, and write $f \in \mathcal{F}$, if the codomain of f is $\mathbb{N}$. For every problem $f :\subseteq Y \rightrightarrows Z$, the *first-order part of* f is the multi-valued function ${}^1 f :\subseteq \mathbb{N}^\mathbb{N} \times Y \rightrightarrows \mathbb{N}$ defined as follows:

- instances are pairs (w, y) s.t. $y \in \operatorname{dom}(f)$ and for every $z \in f(y)$ and every name p_z for z, $\mathsf{U}_w(p_z)(0)\downarrow$, where $\mathsf{U}_{(\cdot)}$ is a fixed universal Turing functional;
- a solution for (w, y) is any n s.t. there is a name p_z for a solution $z \in f(y)$ s.t. $\mathsf{U}_w(p_z)(0)\downarrow = n$.

Intuitively, we can think of $\mathsf{U}_{(\cdot)}$ as the universal Turing functional s.t., for every $w = (e)^\frown q$ and every input x, $\mathsf{U}_w(x)$ simulates the e-th computable Turing functional with oracle q and input x.

While the definition may look hard to digest at first, most of the complications come from the fact that we are considering a multi-valued function between arbitrary represented spaces.

Intuitively, the first-order part of f behaves "just like f, but stops at the first bit". The motivation for this notion comes from the following fact:

Proposition 4.36 ([31]):
For every problem f, ${}^1 f \equiv_{\mathrm{W}} \max_{\le_{\mathrm{W}}}\{g \in \mathcal{F} \,:\, g \le_{\mathrm{W}} f\}$.

We briefly give the idea of the proof, as it can guide the intuition when working with the first-order part of a problem (as well as with the deterministic part, which will be defined in the following section). Let g be a first-order problem that reduces to f via the functionals Φ, Ψ. Assume w.l.o.g. that $g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ and $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ (this makes the presentation easier, as we do not have to keep track of the representation maps). By definition of Weihrauch reducibility, for every input $p \in \operatorname{dom}(g)$, $\Phi(p) \in \operatorname{dom}(f)$ and, for every solution $q \in f\Phi(p)$, $\Psi(p, q)(0) \in g(p)$. We can computably find a string $r \in \mathbb{N}^\mathbb{N}$ s.t. for every $t \in \mathbb{N}^\mathbb{N}$,

$$\mathsf{U}_r(t) = \Psi(p, t).$$

It is easy to check that $(r, \Phi(p))$ is a valid input for ${}^1 f$, and that ${}^1 f(r, \Phi(p))(0) = \Psi(p, q)(0)$, for some solution $q \in f\Phi(p)$, i.e. it uniformly computes g.

Equivalently[3], we can define the first order part of f as the partial multi-valued function s.t.

- instances are triples $(p, e, i) \in \mathbb{N}^\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ s.t. $\delta_Y \Phi_e(p) =: y \in \operatorname{dom}(f)$ and for every $p_z \in \delta_Z^{-1}(f(y))$, $\Phi_i(p, p_z)\downarrow$;

- a solution for (p, e, i) is any n s.t. $\Phi_i(p, p_z)(0) \downarrow= n$, for some name p_z of a solution for $f\delta_Y\Phi_e(p)$.

The difference in the two definitions lies in the fact that, in the first case, we need to consider an input $w \in \mathbb{N}^\mathbb{N}$ that (intuitively) codes also the original input for the function we are reducing to f, while in the second case we need to specify two integer indexes, as the input will be automatically accessible (as part of the definition of Weihrauch reducibility).

It is easy to see that the first-order part is a degree theoretic operator, hence a common strategy to characterize the first-order part of a problem f is to show that a first-order function f_0 reduces to f and that, for every first-order g, if $g \le_W f$ then $g \le_W f_0$. A first example is the following:

Proposition 4.37:
${}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}$.

Proof: It is known that $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N} <_W \widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([2, Thm. 3.34]). On the other hand, if $f :\subseteq X \rightrightarrows \mathbb{N}$ is s.t. $f \le_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ via Φ, Ψ then, for every name p of some $x \in \operatorname{dom}(f)$, $\Phi(p)$ is the name of an ill-founded tree T_p and, for every $t \in [T_p]$ we have $\Psi(t)(0) \in f(x)$. This means that we can compute a solution choosing an element from

$$\{n \in \mathbb{N} : (\exists t \in \mathbb{N}^\mathbb{N})(t \in [T_p] \land \Psi(t)(0) = n)\},$$

which is a $\Sigma^{1,p}_1$ subset of $\mathbb{N}$. ■

The notion of first-order part will be applied in Chapter 5 to obtain several separation results. The following results on the first-order part are joint work with Giovanni Soldà.

4.2.1 First-order part and parallelization

We provide a characterization of the first-order part of a problem f, whenever $f \equiv_W \widehat{g}$ and g is a first-order problem.

Let us introduce the following "unbounded-*" operator. Intuitively it generalizes the finite parallelization *, by relaxing the requirement that the number of instances of the problem is specified a priori.

Definition 4.38: For every $f :\subseteq X \rightrightarrows Y$ define $f^{u*} :\subseteq \mathbb{N}^\mathbb{N} \times X^\mathbb{N} \rightrightarrows Y^{<\mathbb{N}}$ as follows:

- instances are pairs $(w, (x_n)_{n\in\mathbb{N}})$ s.t. $(x_n)_{n\in\mathbb{N}} \in \operatorname{dom}(\widehat{f})$ and for every $(y_n)_{n\in\mathbb{N}} \in \widehat{f}((x_n)_{n\in\mathbb{N}})$ there is $k \in \mathbb{N}$ s.t. for every $t \in \delta^{-1}_{Y^{<\mathbb{N}}}((y_n)_{n<k})$

$$\mathsf{U}_w(t)(0)\downarrow$$

where $\mathsf{U}_{(\cdot)}$ is a fixed universal Turing functional;

- a solution for $(w, (x_n)_{n\in\mathbb{N}})$ is every finite sequence $(y_n)_{n<k}$ s.t. $y_i \in f(x_i)$ and, for every $t \in \delta^{-1}_{Y^{<\mathbb{N}}}((y_n)_{n<k})$, $\mathsf{U}_w(t)(0)\downarrow$.

[3] This is actually the original definition used by Dzhafarov, Solomon, and Yokoyama.

Notice that if $Y = \mathbb{N}$ then we can equivalently think g^{u*} as being a multi-valued function $\mathbb{N}^\mathbb{N} \times X^\mathbb{N} \rightrightarrows \mathbb{N}$. In other words, if g is first-order then so is g^{u*}.

Proposition 4.39:
$g^{u*} \le_{\mathrm{W}} \widehat{g}$.

Proof: For every $(w, (x_n)_{n\in\mathbb{N}}) \in \operatorname{dom}(g^{u*})$, let $(y_n)_{n\in\mathbb{N}} \in \widehat{g}((x_n)_{n\in\mathbb{N}})$. For every k, let t_k be a name for $(y_n)_{n<k}$. We dove-tail all the computations $\mathsf{U}_w(t_k)$ for $k \in \mathbb{N}$ and return some $(y_n)_{n<k}$ s.t. $\mathsf{U}_w(t_k)(0)\downarrow$. ■

Theorem 4.40:
For every f and every $g :\subseteq Z \rightrightarrows \mathbb{N}$,

$$f \equiv_{\mathrm{W}} \widehat{g} \Rightarrow {}^1f \equiv_{\mathrm{W}} g^{u*}$$

Proof: The reduction $g^{u*} \le_{\mathrm{W}} {}^1f$ is straightforward using $g^{u*} \le_{\mathrm{W}} \widehat{g} \equiv_{\mathrm{W}} f$ and the fact that g^{u*} is first-order.

To prove the reduction ${}^1f \le_{\mathrm{W}} g^{u*}$, fix $f_0 :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ and let Φ, Ψ be two maps witnessing the reduction $f_0 \le_{\mathrm{W}} \widehat{g}$. In particular, for every $x \in \operatorname{dom}(f_0)$,

$$\Psi(x, \widehat{g}(\Phi(x)))(0) \in f_0(x)$$

For every given $x \in \operatorname{dom}(f_0)$, we can compute w s.t. for every $t \in \mathbb{N}^\mathbb{N}$,

$$\mathsf{U}_w(t) = \Psi(x, \sigma^\frown 0^\omega)$$

where $\sigma = \delta_{\mathbb{N}^{<\mathbb{N}}}(t)$.

By the continuity of Ψ, for every $y \in \widehat{g}(\Phi(x))$ there is k s.t. $\Psi(x, y[k]^\frown 0^\omega)(0)\downarrow$. This implies that $(w, \Phi(x)) \in \operatorname{dom}(g^{u*})$. Moreover, for every $\tau := (y(0), \dots, y(k-1)) \in g^{u*}(w, \Phi(x))$ we have $\Psi(x, \tau^\frown 0^\omega)(0) \in f_0(x)$, which shows that $f_0 \le_{\mathrm{W}} g^{u*}$. ■

We now consider the relation between f^{u*} and the diamond operator $f^\diamond$. The latter was introduced in [84, Def. 9] using generalized register machines, and intuitively corresponds to the possibility to call f as oracle an arbitrary but finite number of times during a computation. In [113, Def. 4], the author gives an alternative definition by means of a "higher-order" model of computation, and shows that the diamond operator corresponds to closure under compositional product for pointed problems ([113, Thm. 1]).

In the following, we will mostly use the game-theoretic definition introduced by [53, Def. 4.1 and def. 4.3].

Definition 4.41: Let $f, g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be two partial multi-valued functions. We define the *reduction game* $G(f \to g)$ as the following two-player game: on the first move, Player 1 plays $x_0 \in \operatorname{dom}(g)$, and Player 2 either plays an x_0-computable $y_0 \in g(x_0)$ and declares victory, or responds with an x_0-computable instance z_1 of f.

For $n > 1$, on the n-th move (if the game has not yet ended), Player 1 plays a solution x_{n-1} to the input $z_{n-1} \in \operatorname{dom}(f)$. Then Player 2 either plays a $\langle x_0, \dots, x_{n-1}\rangle$-computable solution to x_0 and declares victory, or plays a $\langle x_0, \dots, x_{n-1}\rangle$-computable instance z_n of f.

If at any point one of the players does not have a legal move, then the game ends with a victory for the other player. Player 2 wins if it ever declares victory (or if Player 1 has no legal move at some point in the game). Otherwise, Player 1 wins.

We say that *g is Weihrauch reducible to f in the generalized sense*, and write $g \le_{\mathrm{gW}} f$, if Player 2 has a computable winning strategy for the game $G(f \to g)$, i.e. there is a Turing functional Φ s.t. Player 2 always plays $\Phi(\langle x_0, \dots, x_{n-1}\rangle)$, and wins independently of the strategy of Player 1.

We described the game assuming that f, g have domain and codomain $\mathbb{N}^{\mathbb{N}}$. The definition can be extended to arbitrary multi-valued functions, and the moves of the players are *names* for the instances/solutions.

For every $f :\subseteq X \rightrightarrows Y$ we define $f^{\diamond} :\subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \rightrightarrows Y^{<\mathbb{N}}$ as the following problem:

- $\operatorname{dom}(f^{\diamond})$ is the set of pairs (e, p) s.t. Player 2 wins the game $G(f \to \mathrm{id})$ when Player 1 plays p as his first move, and Player 2 plays according to Φ_e;
- a solution is the list of moves of Player 1 for a run of the game (except for the first move).

Intuitively, in the reduction game $G(f \to g)$, Player 1 plays the role of the oracle f, while Player 2 plays the role of the algorithm trying to compute a solution for g, calling the oracle finitely many times. It is easy to see that $g \le_{\mathrm{W}} f^{\diamond}$ iff $g \le_{\mathrm{gW}} f$.

As described in [113], it is useful to think of a run of the game $G(f \to \mathrm{id})$ as the computation of a modified Type-2 Turing machine with a “Weihrauch problem plug-in”: fixed a computational problem f, on top of the standard operations, this machine can query the oracle f on (the element with name) $\Phi(t)$, where Φ is a computable functional (whose index can be obtained as part of the computation) and t is the content of the tapes. This results in a creation of a new tape that contains (a name for) an answer, or in an infinite loop if the operation was not allowed (i.e. if we try to apply a functional to a string that is not in its domain, or if $\Phi(t)$ is not a name for any element in $\operatorname{dom}(f)$).

Notice that, if f is first-order, then we can assume that $f^{\diamond}$ is first-order as well (via the canonical bijection $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$).

Proposition 4.42:
For every multi-valued function $f :\subseteq X \rightrightarrows Y$

$$f^{u*} \le_{\mathrm{sW}} f^{\diamond}.$$

If $f \colon \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is s.t. $\{(x, n) : n \in f(x)\} \in \Pi^0_1$ then $f^{u*} \equiv_{\mathrm{sW}} f^{\diamond}$.

PROOF: For the first claim, given $(w, (x_n)_{n\in\mathbb{N}}) \in \operatorname{dom}(f^{u*})$, consider the input (e, p) for $f^\diamond$, where $p = \langle p_0, p_1, \dots\rangle$ is a name for $(x_n)_{n\in\mathbb{N}}$ (i.e. $p_i \in \delta_X^{-1}(x_i)$) and e is an index of the Turing functional that works as follows: at stage s plays p_s and dove-tails all the computations $\mathsf{U}_w(\langle t_0, \dots, t_s\rangle)$ for s steps, where $t_i \in \delta_Y^{-1} f(x_i)$ is the i-th move played by Player 1 (i.e. it is a name for some $y_i \in f(x_i)$). The definition of f^{u*} guarantees us that there is k and a finite sequence $(t_n)_{n\le k}$ s.t. $\mathsf{U}_w(\langle t_0, \dots, t_k\rangle)(0)\downarrow$. In particular, this shows that we can use $f^\diamond$ to compute a solution $(y_n)_{n<k}$ for f^{u*}.

To prove the second part of the lemma, let f be a first-order problem that satisfies the hypotheses. Recall that an input for $f^\diamond$ is a pair (e, p) where $p \in \mathbb{N}^\mathbb{N}$ and e is an index for the computable strategy played by Player 2.

A solution for $f^\diamond(e, p)$ is (essentially) a run of the reduction game $G(f \to \mathrm{id})$, when Player 2 plays according to Φ_e. Equivalently, we can think of a solution as a branch of the tree $T \subset \mathbb{N}^{<\mathbb{N}}$ of possible answers obtained by Φ_e when calling the oracle f. In other words, $\sigma \in T$ iff $\sigma(i)$ is the answer to the i-th call made by Φ_e to the oracle f when executed with input p and when the answer to the j-th call, with $j < i$, is $\sigma(j)$. By definition, Player 2 wins the game $G(f \to \mathrm{id})$ when playing with Φ_e, or equivalently, T is well-founded.

We stress a subtle point that can easily be overlooked: the $(i+1)$-th oracle query depends on the *names* of the solutions to the first i calls. Since every natural number has (infinitely) many names, the $(i+1)$-th oracle query is not uniquely determined by $\sigma[i+1]$. In our context, we can avoid this problem by assuming that the unique name for $n \in \mathbb{N}$ is $(n)^\frown 0^\omega$, so that the sequence of the first answers uniquely determines the next oracle query.

Consider the tree $S \subset \mathbb{N}^{<\mathbb{N}}$ defined as follows: $\sigma \in S$ iff

- for every $s < |\sigma|$, $\sigma(s) = \langle n, i_1, \dots, i_n\rangle$, where n is the number of oracle calls made by Φ_e in s steps when executed with input p and when the answer to the j-th oracle call is i_j and
- Player 2 does not declare victory (i.e. the computation of Φ_e does not halt) in less than $|\sigma|$ steps and
- letting $\sigma(|\sigma| - 1) = \langle n, i_1, \dots, i_n\rangle$, we did not find a witness of the fact that i_j is not a correct answer to the $(j+1)$-th oracle call in less than $|\sigma|$ steps.

Intuitively, we build S by "guessing" a possible answer to every oracle call, and we stop extending σ if Player 2 declares victory or if we see that one of the oracle guesses was wrong. Notice that S is uniformly computable from p, e. Notice also that S is well-founded: indeed, if all the oracle guesses are correct then the claim follows from the fact that Φ_e is a winning strategy for Player 2. On the other hand, the fact that $\{(x, n) : n \in f(x)\}$ is Π^0_1 guarantees that every wrong guess is detected (and hence the branch is killed) in finite time.

The fact that S is computable does not immediately yield a computable enumeration of a list of inputs for f. In fact, in case some wrong oracle answer is guessed, the functional Φ_e may get stuck in an infinite loop while computing the next oracle query. This does not affect the computation of the tree S, as in that case we do not need to compute the oracle questions (we just detect the beginning of a question and then guess an answer).

To compute an input for f^{u*} we define a sequence $(q_\sigma)_{\sigma\in\mathbb{N}^{<\mathbb{N}}}$ as follows: if $\sigma \notin S$ or if $\sigma(i) = \langle 0\rangle$ for every $i < |\sigma|$ (i.e. if Φ_e did not commit to producing an oracle query by stage $|\sigma|$) we simply define $q_\sigma := 0^\omega$. Otherwise, let $s < |\sigma|$ be largest s.t. $\sigma(s) = \langle n, i_1, \dots, i_n\rangle$ and $\sigma(s+1) = \langle n+1, i_1, \dots, i_{n+1}\rangle$. At stage s, the functional Φ_e commits to producing the $(n+1)$-th oracle query. However, if the sequence $(i_1, \dots, i_n)$ is not a valid sequence of oracle answers (i.e. if some of the oracle guesses is wrong), then Φ_e is not guaranteed to produce a valid $(n+1)$-th query. By hypothesis, checking whether the solutions $(i_1, \dots, i_n)$ are correct is p-co-c.e.. We run Φ_e until we see that $(i_1, \dots, i_n)$ is not correct. If this never happens then Φ_e produces a valid input for f and we define q_σ to be the output of Φ_e. Otherwise, Φ_e only produces a finite string τ, and we define $q_\sigma := \tau^\frown 0^\omega$. Since f is total, the sequence $(q_\sigma)_{\sigma\in\mathbb{N}^{<\mathbb{N}}}$ is a valid input for $\widehat{f}$.

A solution $(y_\sigma)_{\sigma\in\mathbb{N}^{<\mathbb{N}}}$ for $\widehat{f}((q_\sigma)_{\sigma\in\mathbb{N}^{<\mathbb{N}}})$ contains enough information to compute a run of the game $G(f \to \mathrm{id})$. In fact, given (e,p), we can compute an index $w \in \mathbb{N}^\mathbb{N}$ for the functional that works as follows: it starts by simulating Φ_e on input p. We iteratively build a finite list $(\sigma_i)_i$ of finite strings and show that the list $(y_{\sigma_i})_i \in f^\diamond(e,p)$. Let σ_0 be the longest string in S s.t. for every s, $\sigma_0(s) = \langle 0 \rangle$. For every i, at stage $|\sigma_i|$ either Player 2 declares victory or it commits to producing the $(i+1)$-th oracle request. In the first case we are done, as that implies that Φ_e will never commit to another oracle call, and Φ_w returns a name for $(y_{\sigma_1}, \dots, y_{\sigma_i})$ (if $i = 0$ then the game ends in round 1 and Φ_w can return the empty string). In the second case, let σ_{i+1} be the longest string s.t. $\sigma_i \sqsubseteq \sigma_{i+1}$ and s.t. for every $j \in \{|\sigma_i|, \dots, |\sigma_{i+1}| - 1\}$, $\sigma_{i+1}(j) = \langle i+1, y_{\sigma_1}, \dots, y_{\sigma_{i+1}} \rangle$.

The fact that Player 2 wins the game $G(f \to \mathrm{id})$ when playing with Φ_e implies that the game ends after finitely many rounds, i.e. that the sequence $(y_{\sigma_i})_i$ is printed after finite time, and this concludes the proof. ■

To prove the second part of the claim, we could have proved that $f^\diamond \le_\mathrm{W} \widehat{f}$ and then $f^\diamond \le_\mathrm{W} f^{u*}$ would have followed by Theorem 4.40 as $f^\diamond$ is first-order. However, that would only yield a Weihrauch reduction and not necessarily a strong Weihrauch reduction. Notice that, in general, if f is not first-order then $f^{u*} \not\equiv_\mathrm{W} f^\diamond$. As a simple example, notice that $\mathsf{lim} \equiv_\mathrm{W} \mathsf{lim}^{u*} \equiv_\mathrm{W} \widehat{\mathsf{lim}}$, while $\mathsf{lim}^\diamond$ is not an arithmetic problem.

Corollary 4.43:
For every $f\colon \mathbb{N}^\mathbb{N} \rightrightarrows k$ s.t. $\{(x,n) : n \in f(x)\} \in \Pi^0_1$ we have $f^* \equiv_\mathrm{sW} f^\diamond$.

Proof: Notice that if f has codomain k, then the tree S defined in the proof of Proposition 4.42 is finitely branching and, for every $\sigma \in S$, we can (p,e)-uniformly compute a bound for the number i s.t. $\sigma^\frown(i) \in S$ from the input p. By König's lemma, the tree is finite, hence we can (p,e)-uniformly compute a level n s.t. no string of length n is in S. This gives an upper bound on the number of oracle calls needed to solve $f^\diamond$. In other words, in order to compute a run of the reduction game $G(f \to \mathrm{id})$ it suffices to apply f^* to $(q_\sigma)_{\langle\sigma\rangle<b}$ for some sufficiently large b. ■

Notice that, in the proof of Proposition 4.42, when simulating all possible runs of the reduction game, we "guess" a possible answer to the oracle calls. The answers to these calls will be used by the backward functional to compute a run of $G(f \to \mathrm{id})$. However, since we have no control over the behavior of the functional Φ_e when fed with incorrect answers, two problems may arise: either Φ_e will not produce an infinite string, or the string it produces is not a valid input for f.

To address the first case we required that $\{(x,n) : n \in f(x)\} \in \Pi^0_1$. In fact, if checking whether n is a wrong answer to $f(x)$ is x-c.e. then we can manually kill a branch whenever we see that some of the previously used guesses are incorrect. To address the second case we required that f is total with domain $\mathbb{N}^\mathbb{N}$, so that every infinite string is a valid input for f.

These hypotheses are sufficient but not necessary. Proposition 4.42 can be generalized using the notion of *completion*, introduced in [14]. For $p \in \mathbb{N}^\mathbb{N}$, we denote with $p-1$ the string $q \in \mathbb{N}^\mathbb{N} \cup \mathbb{N}^{<\mathbb{N}}$ with domain $|\{i : p(i) \neq 0\}|$ that maps n to $i_n - 1$, where i_n is the n-th non-zero element of p. Given a represented space (X, δ_X), we define its *completion* $(\overline{X}, \delta_{\overline{X}})$ as $\overline{X} := X \cup \{\bot\}$, where $\bot \notin X$, and

$$\delta_{\overline{X}}(p) := \begin{cases} \delta_X(p-1) & \text{if } p-1 \in \mathrm{dom}(\delta_X) \\ \bot & \text{otherwise.} \end{cases}$$

Intuitively, $\delta_{\overline{X}}$-names are obtained modifying δ_X-names, adding a "don't tell" symbol that postpones the information on the represented point. Every $\delta_{\overline{X}}$-name that is eventually 0 does not contain enough information to represent a point, hence it is assigned to $\bot$.

For every $f :\subseteq X \rightrightarrows Y$, we define its *completion* $\overline{f} \colon \overline{X} \rightrightarrows \overline{Y}$ as

$$\overline{f}(x) := \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f) \\ \overline{Y} & \text{otherwise.} \end{cases}$$

For every computational problem f we have $f \le_{\mathrm{W}} \overline{f}$. If the converse reduction holds, i.e. $f \equiv_{\mathrm{W}} \overline{f}$, the problem is called *complete*.

Corollary 4.44:
For every complete problem $f :\subseteq X \rightrightarrows \mathbb{N}$

$$f^{u*} \equiv_{\mathrm{sW}} f^{\diamond}.$$

Proof: The proof follows the same strategy used in the proof of Proposition 4.42. When proving the reduction $f^\diamond \le_{\mathrm{W}} f^{u*}$, we modify the definition of the sequence $(q_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}}$ as follows: let Φ_*, Ψ_* be a pair of functionals witnessing the reduction $\overline{f} \le_{\mathrm{W}} f$. Whenever Φ_e commits to producing a new oracle query, instead of simulating Φ_e until we see that some of the previous guesses were wrong, we alternate the simulation of Φ_e with the simulation of the Turing functional that prints 0^ω. Moreover, we increase by 1 every output produced by Φ_e. We define r_σ as the output of this procedure. Clearly this is a valid name for some $x \in \overline{X}$. We then define $q_\sigma := \Phi_*(r_\sigma)$. Notice that if no oracle answer was guessed incorrectly, then we are guaranteed that Φ_e would produce a correct δ_X-name for an input for f. In this case, the procedure we described produces a $\delta_{\overline{X}}$-name for the same input.

Finally, to compute a solution for $f^\diamond$ from a solution $(y_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}}$ for $\widehat{f}((q_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}})$, we build a finite list $(\sigma_i)_i$ of finite strings as in the proof of Proposition 4.42, with the only difference that we use $\delta_{\overline{\mathbb{N}}}\Psi_*(r_{\sigma_i}, y_{\sigma_i})$ instead of y_{σ_i} as solution guess. ∎

Notice that if we do not assume that $\{(x,n) : n \in f(x)\}$ is Π^0_1 then we do not have any guarantee (in general) that the tree S of oracle guesses (defined in the proof of Proposition 4.42) is well-founded.

The results we obtained can be used to characterize the first-order part of many common problems. For example, knowing that $\mathsf{lim} \equiv_{\mathrm{W}} \widehat{\mathsf{LPO}}$ and that LPO is complete ([14, Prop. 5.8]) we obtain

$$^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{LPO}^{u*} \equiv_{\mathrm{W}} \mathsf{LPO}^{\diamond}.$$

Together with the fact that $\mathsf{LPO}^\diamond \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}$ ([84, Prop. 10]), we have

$$^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}^{u*} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}^\diamond.$$

In particular, in case of $\mathsf{C}_\mathbb{N}$ we also obtain that $\mathsf{C}_\mathbb{N} \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}^* \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}^{u*}$. This is not the case for LPO: indeed $\mathsf{LPO} <_{\mathrm{W}} \mathsf{LPO}^* <_{\mathrm{W}} \mathsf{LPO}^{u*}$. The fact that the reduction $\mathsf{LPO}^* <_{\mathrm{W}} \mathsf{LPO}^{u*}$ is strict follows from the fact that LPO^{u*} can compute the problem "given $A \in \mathbf{\Sigma}_1^0(\mathbb{N})$, say if A is empty and, if not, produce its minimum". The same problem cannot be solved by LPO^*.

We can use Theorem 4.40 also to characterize the first-order part of $\mathsf{WKL}^{(n)}$. Indeed, using [21, Fact 2.3 and cor. 4.18], we have

$$\mathsf{WKL}^{(n)} \equiv_{\mathrm{W}} \widehat{\mathsf{RT}_k^n} \equiv_{\mathrm{W}} \widehat{\mathsf{C}_2^{(n)}}\ ,$$

hence

$$^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{W}} {}^1\left(\widehat{\mathsf{RT}_k^n}\right) \equiv_{\mathrm{W}} (\mathsf{C}_2^{(n)})^{u*}\ .$$

Moreover, $\mathsf{C}_2 \equiv_{\mathrm{sW}} \mathsf{TC}_2$ (see e.g. [15, Prop. 6.3]) and TC_2 can be thought as being total with domain $\mathbb{N}^\mathbb{N}$, we can apply Corollary 4.43 to C_2 and conclude that

$$^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^* <_{\mathrm{W}} \mathsf{C}_\mathbb{N} \equiv_{\mathrm{W}} {}^1\mathsf{lim}.$$

4.2.2 First-order part and other operations

We now state a few results linking the first-order part and the other operators introduced in Section 2.1.1.

Proposition 4.45:

1. $^1(f \sqcup g) \equiv_{\mathrm{W}} {}^1f \sqcup {}^1g$
2. $^1(f \sqcap g) \equiv_{\mathrm{W}} {}^1f \sqcap {}^1g$
3. $^1f \times {}^1g \le_{\mathrm{W}} {}^1(f \times g)$, *the converse can fail*
4. $^1f * {}^1g \le_{\mathrm{W}} {}^1(f * g)$, *the converse can fail*

Proof:

1. This is straightforward from the definitions. Indeed an input for $f \sqcup g$ is of the type (i, x) where $x \in \mathrm{dom}(f)$ if $i = 0$ and $x \in \mathrm{dom}(g)$ if $i = 1$.

To prove the left-to-right reduction it suffices to map $(w, (i, x))$ to $(i, (w, x))$. To prove the right-to-left reduction it suffices to consider the inverse map $(i, (w, x)) \mapsto (w, (i, x))$.

2. To prove the left-to-right reduction notice that, by the monotonicity of $^1(\cdot)$, $^1(f \sqcap g) \le_{\mathrm{W}} {}^1f$ and $^1(f \sqcap g) \le_{\mathrm{W}} {}^1g$. Since $\sqcap$ is the meet in the Weihrauch lattice we have $^1(f \sqcap g) \le_{\mathrm{W}} {}^1f \sqcap {}^1g$. To prove the right-to-left reduction, recall that, by definition,

$$\begin{aligned}({}^1f \sqcap {}^1g)((w, x), (v, z)) &= {}^1f(w, x) \sqcup {}^1g(v, z)\\ &= \{0\} \times \mathsf{U}_w(f(x)) \cup \{1\} \times \mathsf{U}_v(g(z))\end{aligned}$$

where, with a small abuse of notation, we are identifying $f(x)$ and $g(z)$ with their names. We can uniformly compute $r \in \mathbb{N}^{\mathbb{N}}$ s.t.

$$\mathsf{U}_r((i,t)) = \begin{cases} \{0\} \times \mathsf{U}_w(t) & \text{if } i = 0 \\ \{1\} \times \mathsf{U}_v(t) & \text{if } i = 1 \end{cases}$$

It follows that every solution for ${}^1(f \sqcap g)(r,(x,z))$ is a solution for $({}^1f \sqcap {}^1g)((w,x),(v,z))$.

3. To show that the reduction holds it suffices to consider the map $((w,x),(v,z)) \mapsto (r,(x,z))$ where $\mathsf{U}_r(\langle y,t\rangle) = \langle \mathsf{U}_w(y), \mathsf{U}_v(t)\rangle$.

A counterexample for the converse reduction can be given considering the problem DS, which we will introduce and study in depth in Chapter 5. In particular, we will show that ${}^1\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}^1_1{-}\mathsf{Bound}$ (Theorem 5.9). It is easy to see that $\mathbf{\Pi}^1_1{-}\mathsf{Bound}$ is closed under product. Moreover $\mathsf{LPO}' \not\le_{\mathrm{W}} \mathsf{DS}$ (Corollary 5.16), while $\mathsf{LPO}' \le_{\mathrm{W}} \mathsf{DS} \times \mathsf{DS}$ (Theorem 5.17). These results show that

$$\mathbf{\Pi}^1_1{-}\mathsf{Bound} \equiv_{\mathrm{W}} {}^1\mathsf{DS} \times {}^1\mathsf{DS} <_{\mathrm{W}} {}^1(\mathsf{DS} \times \mathsf{DS})$$

4. Notice first of all that, by the monotonicity of ${}^1(\cdot)$, ${}^1f * {}^1g \le_{\mathrm{W}} f * g$. By the cylindrical decomposition there is a computable functional Φ_e s.t.

$${}^1f * {}^1g \equiv_{\mathrm{W}} (\mathrm{id} \times {}^1f) \circ \Phi_e \circ (\mathrm{id} \times {}^1g) \ .$$

In particular,

$$(\mathrm{id} \times {}^1f) \circ \Phi_e \circ (\mathrm{id} \times {}^1g)(\langle t_1, t_2\rangle) = \langle \Phi_1(t_1, {}^1g(t_2)), {}^1f(\Phi_2(t_1, {}^1g(t_2)))\rangle$$

where Φ_1 and Φ_2 are s.t. $\Phi_e(p) = \langle \Phi_1(p), \Phi_2(p)\rangle$. This shows that it is enough to solve ${}^1g(t_2)$ and ${}^1f(\Phi_2(t_1, {}^1g(t_2)))$. Since they are both first order problems we actually have ${}^1f * {}^1g \le_{\mathrm{W}} {}^1(f * g)$.

To prove that the converse reduction does not hold in general, consider $f = g = \mathsf{lim}$. We already showed that ${}^1\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{LPO}^{\diamond} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} {}^1f * {}^1g$, as $\mathsf{LPO}^{\diamond} * \mathsf{LPO}^{\diamond} \equiv_{\mathrm{W}} \mathsf{LPO}^{\diamond}$. On the other hand, using Theorem 4.40, ${}^1(\mathsf{lim} * \mathsf{lim}) \equiv_{\mathrm{W}} (\mathsf{LPO}')^{u*} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. ∎

To explore the connections between the first-order part and the jump in the Weihrauch lattice we introduce the following notion:

Definition 4.46: We say that a first-order function f is a *first-order cylinder* if, for every first-order g,

$$g \le_{\mathrm{W}} f \Rightarrow g \le_{\mathrm{sW}} f$$

Notice that no first-order function can be a (classical) cylinder, as every first-order function only has computable outputs (hence, in particular, it cannot strongly compute id).

Proposition 4.47:
If f is a cylinder then ${}^1f$ is a first-order cylinder.

PROOF: Assume $g \le_{\mathrm{W}} {}^1f$ via Φ, Ψ. For every name p_t for an input t of g, let (w, x) be the input for ${}^1f$ named by $\Phi(p_t)$. Intuitively, since f is a cylinder, we exploit the fact that $f \equiv_{\mathrm{sW}} f \times \mathrm{id}$, i.e. from $\Phi(p_t)$ we can uniformly compute an input y for f s.t. every name p_z for some $z \in f(y)$ uniformly computes $\langle w, r\rangle$ via Φ_e, where r is a name for a solution of $f(x)$. Let $q \in \mathbb{N}^{\mathbb{N}}$ be s.t.

$$\langle \sigma, n\rangle \sqsubseteq \mathsf{U}_q(p_z)$$

where $\Phi_e(p_z) = \langle w, r\rangle$, $n := \mathsf{U}_w(f(x))(0)$ and σ is a sufficiently long prefix of p_t s.t.

$$\Psi(\sigma^\frown 0^\omega, (n)^\frown 0^\omega)(0) \in g(t).$$

Such a σ exists because g is first-order. In other words, given p_t we uniformly compute an input (q, y) for ${}^1f$ s.t. $\Psi({}^1f(q, y))$ is a solution for $g(t)$. ■

As a trivial consequence, if f is a cylinder, g is a first-order cylinder and $g \equiv_{\mathrm{W}} {}^1f$, then $g \equiv_{\mathrm{sW}} {}^1f$ and hence $g' \equiv_{\mathrm{sW}} ({}^1f)'$ (as the jump lifts to the strong Weihrauch degrees).

Recall that, when working with the compositional product $*$, we identify $f * g$ with a representative of the degree that is also a cylinder. This allows us to consider $f * g$ when working with the strong Weihrauch reducibility.

Proposition 4.48:
For every multi-valued function f, ${}^1(f') \le_{\mathrm{sW}} ({}^1f)'$. Moreover, if f is a cylinder then ${}^1(f') \equiv_{\mathrm{sW}} ({}^1f)'$.

PROOF: The first statement is a trivial consequence of the definitions. Indeed, given an input $(w, (x_n)_{n\in\mathbb{N}})$ for ${}^1(f')$, where $(x_n)_{n\in\mathbb{N}}$ converges to $x \in \mathrm{dom}(f)$, it is enough to consider the input $((w_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}})$ for $({}^1f)'$, where $w_n := w$ is a constant sequence. Clearly

$$({}^1f)'((w_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) = \mathsf{U}_w(f(x))(0) = {}^1(f')(w, (x_n)_{n\in\mathbb{N}})$$

where, with a small abuse of notation, we identified $f(x)$ with its name.

Assume now that f is a cylinder. In particular we have that f' is a cylinder and $f * \mathsf{lim} \equiv_{\mathrm{sW}} f'$. This implies that

$$({}^1f)' \le_{\mathrm{sW}} {}^1f * \mathsf{lim} \le_{\mathrm{sW}} f * \mathsf{lim} \equiv_{\mathrm{sW}} f'$$

Since $({}^1f)'$ is first-order, the maximality of the first-order part implies that $({}^1f)' \le_{\mathrm{W}} {}^1(f')$. By Proposition 4.47, ${}^1(f')$ is a first-order cylinder, hence $({}^1f)' \le_{\mathrm{sW}} {}^1(f')$. ■

The reduction $({}^1f)' \le_{\mathrm{W}} {}^1(f')$ can fail if f is not a cylinder. To see this, fix a non-cylinder $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. By definition ${}^1f$ takes in input a pair (w, x) and produces $\mathsf{U}_w(f(x))(0)$. We can think of w as (e, X), where e is the index of a Turing machine and X is an oracle. By definition, $({}^1f)'$ takes in input a sequence $((w_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}})$ that converges to (w, x) and produces ${}^1f(w, x)$. Similarly ${}^1(f')$ takes in input $(v, (z_n)_{n\in\mathbb{N}})$ and produces $\mathsf{U}_v(f(z))$, where $z = \lim_n z_n$. If there is a reduction $({}^1f)' \le_{\mathrm{W}} {}^1(f')$ then the forward functional eventually has to commit to some $v(0)$, i.e. to some index for a Turing machine. You can fool the reduction by changing the sequence $(w_n)_{n\in\mathbb{N}}$

after that point so that the index of the Turing machine is different. You cannot do the same if f is a cylinder, as in that case $f' \equiv_{\mathrm{W}} f * \mathsf{lim}$, hence you can use lim to get the correct (w, x).

Lemma 4.49:
C_2^* is a first-order cylinder.

PROOF: Assume $g \leq_{\mathrm{W}} \mathsf{C}_2^*$ via Φ, Ψ. For every name z of some input for g, we can uniformly compute a bound k for the length of a prefix of z s.t. for every name y of a solution for $\mathsf{C}_2^*(\delta_{\mathbf{\Pi}^0_1(\mathbb{N})^{<\mathbb{N}}}(\Phi(z)))$,

$$\Psi(z, y)(0) = \Psi(z[k], y)(0),$$

see also the proof of **??**. Since $\mathrm{id}_{\mathbb{N}^{<\mathbb{N}}} \leq_{\mathrm{sW}} \mathsf{C}_2^*$ and $\mathsf{C}_2^* \times \mathsf{C}_2^* \equiv_{\mathrm{sW}} \mathsf{C}_2^*$ the claim follows. ■

Corollary 4.50:
For every $n \in \mathbb{N}$, ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n)}$

PROOF: By induction on $n \in \mathbb{N}$: for the base step we already know that ${}^1\mathsf{WKL} \equiv_{\mathrm{W}} \mathsf{C}_2^*$. Since both ${}^1\mathsf{WKL}$ and C_2^* are first-order cylinder (the former because of Proposition 4.47, as WKL is a cylinder, the latter because of Lemma 4.49) we have ${}^1\mathsf{WKL} \equiv_{\mathrm{sW}} \mathsf{C}_2^*$.

Assume the claim holds up to n. Recall that, for every n, $\mathsf{WKL}^{(n+1)}$ is a cylinder and $\mathsf{WKL}^{(n+1)} \equiv_{\mathrm{sW}} \mathsf{WKL}^{(n)} * \mathsf{lim}$. Proposition 4.48 implies that ${}^1(\mathsf{WKL}^{(n+1)}) \equiv_{\mathrm{sW}} ({}^1(\mathsf{WKL}^{(n)}))'$. By the inductive step we have ${}^1(\mathsf{WKL}^{(n)}) \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n)}$, therefore

$${}^1(\mathsf{WKL}^{(n+1)}) \equiv_{\mathrm{sW}} ((\mathsf{C}_2^*)^{(n)})' \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)^{(n+1)}.$$ ■

In particular, in case of $\mathsf{KL} \equiv_{\mathrm{sW}} \mathsf{WKL}'$ we obtain ${}^1\mathsf{KL} \equiv_{\mathrm{sW}} (\mathsf{C}_2^*)'$.

4.3 Deterministic part of a problem

The results of this section are joint work with Jun Le Goh and Arno Pauly. They have been obtained while studying the topics of Chapter 5, and are part of [46].

Definition 4.51: Let X be a represented space and $f :\subseteq Y \rightrightarrows Z$ be a multi-valued function. We define $\mathrm{Det}_X(f) :\subseteq \mathbb{N}^{\mathbb{N}} \times Y \to X$ by

$$\mathrm{Det}_X(f)(w, y) = x :\iff (\forall z \in \delta_Z^{-1}(f(y)))(\delta_X(\mathsf{U}_w(z)) = x),$$

where $\mathsf{U}_{(\cdot)}$ is a universal Turing functional. The domain of $\mathrm{Det}_X(f)$ is maximal for this to be well-defined. We just write $\mathrm{Det}(f)$ for $\mathrm{Det}_{\mathbb{N}^{\mathbb{N}}}(f)$.

Notice that $\mathrm{Det}(f)$ is always a cylinder. This is not true for all X (if $X = \mathbb{N}$ then $\mathrm{Det}_X(f)$ always has computable solutions, and therefore $\mathrm{id} \not\leq_{\mathrm{sW}} \mathrm{Det}_X(f)$).

Our interest in the principle $\mathrm{Det}_X(f)$ lies in the fact that it has the maximal Weihrauch degree of all (single-valued!) functions with codomain X that are Weihrauch below f:

Theorem 4.52:
$\mathrm{Det}_X(f) \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{g :\subseteq W \to X \,:\, g \leq_{\mathrm{W}} f\}$.

Proof: This argument is very similar to the one used to prove Proposition 4.36.

Clearly, $\mathrm{Det}_X(f)$ is itself present in the set on the right hand side. Assume $g :\subseteq W \to X$ satisfies $g \leq_{\mathrm{W}} f$ with reduction witnesses Φ and Ψ. Given a name q for an input to g, let $y = \delta_Y(\Phi(q))$ be the value f is called on, and let w be a name for the function $\Psi(q, \cdot)$. Then $\mathrm{Det}_X(f)(w, y) = g(\delta_W(q))$. ■

As in the case of the first-order part, we could equivalently define the deterministic part by requiring that the input specifies a pair of Turing functionals and a string in $\mathbb{N}^{\mathbb{N}}$.

In the same spirit, we can identify several other operators $\Lambda_{\mathcal{Y}}$ of the type

$$\Lambda_{\mathcal{Y}}(f) := \max_{\leq_{\mathrm{W}}}\{g \in \mathcal{Y} \,:\, g \leq_{\mathrm{W}} f\}.$$

In particular, the proof strategy used in Theorem 4.52 can be used to prove that $\Lambda_{\mathcal{U}_N}$ and $\Lambda_{\mathcal{V}_N}$ are total, where $\mathcal{U}_N$ is the set of first-order problems with codomain N, and $\mathcal{V}_N$ is the set of problems in $\mathcal{U}_N$ which are also single-valued. This will come into play in Theorem 5.29 and in Theorem 5.31.

Corollary 4.53:
$\mathrm{Det}_X(\cdot)$ *is an interior degree-theoretic operator on Weihrauch degrees, i.e.*

$$\mathrm{Det}_X(\mathrm{Det}_X(f)) \equiv_{\mathrm{W}} \mathrm{Det}_X(f) \leq_{\mathrm{W}} f \ ,$$
$$f \leq_{\mathrm{W}} g \Rightarrow \mathrm{Det}_X(f) \leq_{\mathrm{W}} \mathrm{Det}_X(g) \ .$$

4.3.1 Impact of the codomain space

We make some basic observations on how the space X impacts the degrees $\mathrm{Det}_X(f)$ for arbitrary f. Clearly, whenever Y computably embeds into X (i.e. there is a computable injection $Y \to X$ with computable inverse), then $\mathrm{Det}_Y(f) \leq_{\mathrm{W}} \mathrm{Det}_X(f)$. In general, we obtain many different operations. To see this, we consider the point degree spectrum of a represented space as introduced by Kihara and Pauly [66]. The point degree spectrum of (X, δ_X) is the set of Medvedev degrees of the form $\delta_X^{-1}(x)$ for $x \in X$.

The spectrum of Y is included in that of X iff Y can be decomposed into countably many parts each of which embeds into X ([66, Lem. 3.6]). If the spectrum of Y is not included in that of X, we can consider a constant function y witnessing this. Then $\mathrm{Det}_X(y) <_{\mathrm{W}} \mathrm{Det}_Y(y) \equiv_{\mathrm{W}} y$. We have thus seen that if $\mathrm{Det}_X(f) \equiv_{\mathrm{W}} \mathrm{Det}_Y(f)$ for all f, then X and Y must have the same point degree spectrum. Miller [81] has shown that the spectrum of $[0,1]^\omega$ is not contained in the Turing degrees (i.e. the spectrum of $2^{\mathbb{N}}$), which was extended in [66] to the result that the spectrum of a computable Polish space is contained in the Turing degrees relative to some oracle iff that space is countably dimensional. The spectra of further spaces have been explored in [65].

We can extend the separation arguments based on the spectrum by considering sequences rather than just constant functions[4]. Whenever we have a sequence $f_0 \colon \mathbb{N} \to X_0$ and a function $g_0 :\subseteq \mathbb{N}^{\mathbb{N}} \to X_1$ with $f_0 \equiv_{\mathrm{W}} g_0$, then there is a sequence $h \colon \mathbb{N} \to X_1$ with $f_0 \equiv_{\mathrm{W}} h$. A Weihrauch reduction $f \leq_{\mathrm{W}} g$ for $f \colon \mathbb{N} \to X$ and $g \colon \mathbb{N} \to Y$ gives rise to a computable partial function $F :\subseteq Y^{\mathbb{N}} \to X^{\mathbb{N}}$ with $F(g) = f$. It follows that it suffices to separate $Y^{\mathbb{N}}$ and $X^{\mathbb{N}}$ via their spectrum to conclude that $\mathrm{Det}_X(\cdot)$ and $\mathrm{Det}_Y(\cdot)$ are distinct operators. In particular, Miller's result implies that there is a function with codomain $\mathbb{R}$ that is not equivalent to any function with codomain $\mathbb{N}^{\mathbb{N}}$.

4.3.2 The deterministic part and the first-order part

Let us now explore the interplay between the deterministic part and the first-order part.

Proposition 4.54:
${}^1\mathrm{Det}(f) \equiv_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} \mathrm{Det}({}^1f)$.

Proof: By considering what the relevant maxima in the characterizations are taken about, it is clear that $\mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} {}^1\mathrm{Det}(f)$ and $\mathrm{Det}_{\mathbb{N}}(f) \leq_{\mathrm{W}} \mathrm{Det}({}^1f)$. To see that ${}^1\mathrm{Det}(f) \leq_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(f)$, we consider a function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and a multivalued function $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ with $g \leq_{\mathrm{W}} f$. But this reduction actually yields some choice function of g, showing that $g \leq_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(f)$. ■

Question 4.55: Is there some f with $\mathrm{Det}_{\mathbb{N}}(f) <_{\mathrm{W}} \mathrm{Det}({}^1f)$?

The question above asks whether whenever there is a countable cover making a partial function on Baire space piecewise computable, there also is a partition of the same or lower complexity that renders the function piecewise computable. The complexity here is not merely the complexity of the individual pieces, but the Weihrauch degree of the map that assigns the piece to any Baire space element.

Proposition 4.56:
$\mathrm{Det}(f) \leq_{\mathrm{W}} \widehat{\mathrm{Det}_{\mathbb{N}}}(f)$.

Proof: A function $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is reducible to the parallelization of its uncurried form $F :\subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ where $F(n,p) = f(p)(n)$. ■

[4]The ideas in this paragraph were pointed out to us by Mathieu Hoyrup.

Corollary 4.57:
$\mathrm{Det}(f) \le_{\mathrm{W}} \widehat{{}^{1}f}$.

4.3.3 Interaction with other operations on Weihrauch degrees

A first straightforward observation is that $\mathrm{Det}(f) \,\square\, \mathrm{Det}(g) \le_{\mathrm{W}} \mathrm{Det}(f \,\square\, g)$ whenever $\square$ is a degree-theoretic operator that preserves single-valuedness. We will look at the interaction with the usual well-studied operations on Weihrauch degrees.

It is imminent from the definition that $\mathrm{Det}(f) \sqcup \mathrm{Det}(g) \equiv_{\mathrm{W}} \mathrm{Det}(f \sqcup g)$.

Moreover, the reductions $\mathrm{Det}(f \sqcap g) \le_{\mathrm{W}} \mathrm{Det}(f)$ and $\mathrm{Det}(f \sqcap g) \le_{\mathrm{W}} \mathrm{Det}(g)$ hold by monotonicity, hence $\mathrm{Det}(f \sqcap g) \le_{\mathrm{W}} \mathrm{Det}(f) \sqcap \mathrm{Det}(g)$, as $\sqcap$ is the meet on Weihrauch degrees ([13, Prop. 3.11]). To see that the inequality can be strict, let $p, q \in 2^{\mathbb{N}}$ be a minimal pair of Turing degrees (which we identify with the constant functions returning these values). It follows that $\mathrm{Det}(p \sqcap q) \equiv_{\mathrm{W}} \mathrm{id} <_{\mathrm{W}} \mathrm{Det}(p) \sqcap \mathrm{Det}(q) \equiv_{\mathrm{W}} p \sqcap q$.

Our principle DS (to be defined) already witnesses that the deterministic part does not distribute over $\times$ and $*$, and does not commute with *, $^\diamond$ and $\widehat{\ }$: we will prove that $\mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$ (Theorem 5.15), while $\mathsf{LPO}' \le_{\mathrm{W}} \mathsf{DS} \times \mathsf{DS}$ (Theorem 5.17). Here we also give another example with a more computability-theoretic flavour:

Example 4.58: There is a Weihrauch degree f such that:

$$\mathrm{Det}(f) \equiv_{\mathrm{W}} \mathrm{id} <_{\mathrm{W}} f <_{\mathrm{W}} f \times f \equiv_{\mathrm{W}} f^\diamond \equiv_{\mathrm{W}} \widehat{f} \equiv_{\mathrm{W}} \mathrm{Det}(f \times f).$$

Indeed, consider the degrees of points in the spaces $\mathbb{R}_<$, $\mathbb{R}_>$ and $\mathbb{R}$ (see [66] for details). Let $x \in \mathbb{R}$ be neither left-c.e. nor right-c.e.; i.e. it lacks computable names in both $\mathbb{R}_<$ and $\mathbb{R}_>$. Then $x \in \mathbb{R}_<$ and $x \in \mathbb{R}_>$ have quasi-minimal degrees, that is do not compute any non-computable elements of Cantor space. We define $f \colon 2 \to \mathbb{R}_< + \mathbb{R}_>$ by $f(0) := x \in \mathbb{R}_<$ and $f(1) := x \in \mathbb{R}_>$. The quasi-minimality implies that $\mathrm{Det}(f) \equiv_{\mathrm{W}} \mathrm{id}$. However, $f \times f$ is equivalent to the constant function returning $x \in \mathbb{R}$, which is also equivalent to the constant function returning the decimal expansion of x. Thus, $f \times f \equiv_{\mathrm{W}} \mathrm{Det}(f \times f)$. Any of f^*, $f * f$, $f^\diamond$ and $\widehat{f}$ clearly share the same degree.

Theorem 4.59:
For every represented space X and all problems f, g,

$$\mathrm{Det}_X(f * g) \le_{\mathrm{W}} \mathrm{Det}_X(f) * g.$$

Proof: Fix a single-valued h with codomain X s.t. $h \le_{\mathrm{W}} f * g$ and assume w.l.o.g. that $\mathrm{dom}(h) \subset \mathbb{N}^{\mathbb{N}}$ (if h is single-valued then the map $p \mapsto h \circ \delta(p)$ is single-valued as well, where δ is the representation map for the domain of h). Assume also, for the sake of readability, that

f and g are cylinders (if not we can just replace f with $f \times \mathrm{id}$, as $\mathrm{Det}_X(\cdot)$ is a degree-theoretic operation).

By the cylindrical decomposition lemma, there is a computable function Φ_e s.t.

$$h \le_{\mathrm{sW}} f \circ \Phi_e \circ g.$$

Let Φ, Ψ be two maps witnessing this strong reduction. Define ϕ as the restriction of $\delta_X \circ \Psi \circ f \circ \Phi_e$ to $\mathrm{dom}(g \circ \Phi \circ h)$. The choice of the domain of ϕ guarantees that ϕ is single-valued: intuitively ϕ witnesses the "second part" of the reduction $h \le_{\mathrm{sW}} f \circ \Phi_e \circ g$, and the fact that h is single-valued implies that so is ϕ. In particular, $\phi \le_{\mathrm{W}} \mathrm{Det}_X(f)$ (as $\phi \le_{\mathrm{W}} f$ trivially). Since $h \le_{\mathrm{W}} \phi * g$ we have that $h \le_{\mathrm{W}} \mathrm{Det}_X(f) * g$. ■

Notice that this implies the choice elimination theorem [17, Thm. 7.25], as $\mathrm{Det}(\mathsf{C}_{2^\mathbb{N}}) \equiv_{\mathrm{W}} \mathrm{id}$ ([13, Cor. 8.8]).

Corollary 4.60:
If g is single-valued with codomain $\mathbb{N}^\mathbb{N}$ then $\mathrm{Det}(f * g) \equiv_{\mathrm{W}} \mathrm{Det}(f) * g$.

Proof: This follows from Theorem 4.59, as the reduction

$$\mathrm{Det}(f) * \mathrm{Det}(g) \le_{\mathrm{W}} \mathrm{Det}(f * g)$$

always holds and $\mathrm{Det}(g) \equiv_{\mathrm{W}} g$ as g is single-valued. ■

Notice however that it does not hold for every represented space, i.e. we cannot replace $\mathrm{Det}(\cdot)$ with $\mathrm{Det}_X(\cdot)$. A counterexample, suggested by Vasco Brattka, is obtained choosing $X = 2$ (with the standard representation) and $f = g = \mathsf{C}_\mathbb{N}$. Since $\mathrm{Det}_2(\mathsf{C}_\mathbb{N}) \equiv_{\mathrm{W}} \mathsf{lim}_2$ (as can be easily proved, see e.g. [16, Prop. 13.10]), we would obtain

$$\mathsf{lim}_3 \le_{\mathrm{W}} \mathsf{lim}_2 * \mathsf{lim}_2 \equiv_{\mathrm{W}} \mathrm{Det}_2(\mathsf{C}_\mathbb{N}) * \mathrm{Det}_2(\mathsf{C}_\mathbb{N}) \le_{\mathrm{W}} \mathrm{Det}_2(\mathsf{C}_\mathbb{N} * \mathsf{C}_\mathbb{N}) \le_{\mathrm{W}} \mathrm{Det}_2(\mathsf{C}_\mathbb{N}) \equiv_{\mathrm{W}} \mathsf{lim}_2,$$

which contradicts [16, Thm. 13.5].

This counterexample is based on the fact that there is no (computable) pairing $X \times X \to X$. However, even the reduction $\mathrm{Det}_X(f) * \mathrm{Det}_X(g) \le_{\mathrm{W}} \mathrm{Det}_{X \times X}(f * g)$ is not sound when generic represented spaces are involved. In fact, we cannot assume that $\mathrm{Det}_X(f) * \mathrm{Det}_X(g)$ is single-valued, as the computable functional connecting $\mathrm{Det}_X(f)$ and $\mathrm{Det}_X(g)$ may produce different instances for $\mathrm{Det}_X(f)$ depending on the name of the solution of $\mathrm{Det}_X(g)$. An explicit counterexample showing that it is possible to have $\mathrm{Det}_{X \times X}(f * g) <_{\mathrm{W}} \mathrm{Det}_X(f) * \mathrm{Det}_X(g)$ is the following (suggested by Arno Pauly): let $p_1, \dots, p_4 \in 2^\mathbb{N}$ be strongly Turing incomparable (i.e. for every i, p_i is not Turing reducible to $\bigoplus_{j \neq i} p_j$). Let $X := \mathbb{N}^\mathbb{N} \cup \{\bot\}$ be represented as follows: for every $p \in \mathbb{N}^\mathbb{N}$, $\delta_X^{-1}(p) := \{p\}$ and $\delta_X^{-1}(\bot) := \{p_1, p_2\}$. Let $f \colon \{p_1, p_2\} \to \{p_3, p_4\}$ be s.t. $f(p_i) := p_{i+2}$. Let also $g \colon \mathbb{N}^\mathbb{N} \to X$ be the constant map producing $\bot$.

It is easy to see that $\mathrm{Det}_X(f) \equiv_{\mathrm{W}} f$ and $\mathrm{Det}_X(g) \equiv_{\mathrm{W}} g$. In particular, $\mathrm{Det}_X(f) * \mathrm{Det}_X(g)$ computes the constant multi-valued function returning either p_3 or p_4. On the other hand, if h is a single-valued function with codomain $X \times X$ that is reducible to $f * g$ then every solution of h

must be computable from $(p_1 \wedge p_2) \oplus (p_3 \wedge p_4)$, where $\wedge$ denotes the meet in the Turing degrees. In particular, no h-solution can compute p_3 nor p_4.

Corollary 4.61:
For every cylinder f and every $k \in \mathbb{N}$

$$\mathrm{Det}(f)^{(k)} \equiv_{\mathrm{W}} \mathrm{Det}(f^{(k)})$$

Proof: The left-to-right reduction is straightforward as

$$\mathrm{Det}(f)^{(k)} \le_{\mathrm{W}} \mathrm{Det}(f) * \mathsf{lim}^{[k]} \le_{\mathrm{W}} f * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} f^{(k)},$$

where the last equality follows from the fact that f is a cylinder. Since $\mathrm{Det}(f)^{(k)}$ is single-valued, this implies $\mathrm{Det}(f)^{(k)} \le_{\mathrm{W}} \mathrm{Det}(f^{(k)})$.

The right-to-left reduction follows from Theorem 4.59 as

$$\mathrm{Det}(f^{(k)}) \equiv_{\mathrm{W}} \mathrm{Det}(f * \mathsf{lim}^{[k]}) \le_{\mathrm{W}} \mathrm{Det}(f) * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathrm{Det}(f)^{(k)},$$

where the last equality follows from the fact that $\mathrm{Det}(f)$ is a cylinder. ■

The previous corollary can be generalized in a straightforward way to any represented space X s.t. $\mathrm{Det}_X(f)$ is a cylinder. Notice that it is false (in general) if f is not a cylinder: take $f = \mathsf{C}_2$ and $k = 1$. Since $\mathsf{C}_2' \equiv_{\mathrm{W}} \mathsf{RT}^1_2$ (see e.g. [21, Fact 2.3 and Prop. 3.4]) we have $\mathrm{Det}(\mathsf{C}_2') \le_{\mathrm{W}} \mathsf{RT}^1_2$, hence in particular $\mathsf{lim} \not\le_{\mathrm{W}} \mathrm{Det}(\mathsf{C}_2')$. On the other hand $\mathsf{lim} \le_{\mathrm{W}} \mathrm{Det}(\mathsf{C}_2)'$ (as $\mathrm{Det}(\mathsf{C}_2)$ is a cylinder).

Definition 4.62: Given some $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ let $?f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be defined by $0^\omega \in ?f(0^\omega)$ and $0^n 1 p \in ?f(0^n 1 q)$ iff $p \in f(q)$.

It is easy to see that ? defines an operation on Weihrauch degrees, and represents the idea of being able to maybe ask a question to f – but never having to decide to forgo this (which would be the case for $1 \sqcup f$). Many well-studied principles are equivalent to their maybe-variants, this in particular holds for all pointed fractals. We introduce the operation here to be able to express how the deterministic part interacts with the notion of completion $\overline{(\cdot)}$ introduced by Brattka and Gherardi [14, 15].

Proposition 4.63:
$\mathrm{Det}(\overline{f}) \equiv_{\mathrm{W}} \mathrm{Det}(?f) \equiv_{\mathrm{W}} ?\mathrm{Det}(f)$.

Proof: To show that $\mathrm{Det}(\overline{f}) \le_{\mathrm{W}} \mathrm{Det}(?f)$, w.l.o.g. assume that $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ and consider a function $g :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with $g \le_{\mathrm{W}} \overline{f}$ witnessed by Φ, Ψ. Now if for some prefix w the computation of $\Psi(w, \cdot)$ outputs two different things depending on the second part of the input, then in order for g to be a function, we have the guarantee that all extensions of w in the domain of g will be mapped to inputs in the domain of f, i.e. we are actually calling f rather than making use of $\overline{f}$. On the other hand, if $\Psi(w, \cdot)$ would output the same thing regardless of the second argument, we can postpone actually calling f (which $?f$ lets us do) and go with that output for the time being. This reasoning establishes that $g \leq_W ?f$.

To see that $\mathrm{Det}(?f) \leq_W ?\mathrm{Det}(f)$, we just inspect the technical definition of $\mathrm{Det}(\cdot)$.

Finally, for $?\mathrm{Det}(f) \leq_W \mathrm{Det}(\overline{f})$ we observe that $?\mathrm{Det}(f)$ is single-valued with codomain $\mathbb{N}^\mathbb{N}$, thus it suffices to show $?\mathrm{Det}(f) \leq_W \overline{f}$. But already $?f \leq_W \overline{f}$ holds: $\overline{f}$ accepts an input that is completely void of information. We provide this as long as our $?f$ instance does not want to use f; if it ever does, we have the relevant f-instance which we can then feed into $\overline{f}$. Note that we do not get a strong reduction here, in general. ■

4.3.4 Previous appearances in the literature

While the deterministic part as such has not been introduced before, and in particular the observation that it is always well-defined is new, there are several results in the literature on Weihrauch degrees that implicitly use it. Already in the first paper introducing the modern definition of Weihrauch reducibility [41], it was shown that $\mathrm{Det}(\mathsf{C}_{2^\mathbb{N}}) \equiv_W \mathrm{id}$. It was observed in [70] that the argument actually even establishes that $\mathrm{Det}_X(\mathsf{C}_{2^\mathbb{N}}) \equiv_W \mathrm{id}$ for any computably admissible space X.

In [64, Sec. 6], the authors introduce the principle $\mathsf{wList}_{2^\mathbb{N}, \leq\omega}$ which produces an enumeration of the elements of a countable closed subset of Cantor space, and [64, Prop. 6.14] states that $\mathrm{Det}(\mathsf{wList}_{2^\mathbb{N}, \leq\omega}) \equiv_W \mathsf{lim}$. The authors also proved the following result, which will be useful in Proposition 5.49:

Theorem 4.64 ([64, Thm. 8.5]):
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathrm{Det}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_W \mathrm{Det}(\widehat{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}})$.

This, in particular, shows that $\mathrm{Det}(\cdot)$ is not useful to separate principles that are between $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

5

Finding descending sequences in ill-founded linear orders

The results of this chapter are joint work with Jun Le Goh and Arno Pauly, and have been collected in [46].

We study the difficulty of the following two (Weihrauch equivalent) computational problems:

- Given an ill-founded countable linear order, find an infinite decreasing sequence in it (DS)
- Given a countable quasi-order which is not well, find a bad sequence in it (BS).

Motivation for the first stems from the treatment of ordinals in reverse mathematics. We already mentioned the natural arising of pseudo-well-orders, when working within submodels of second order arithmetic. As a classic example of a pseudo-well-order, consider Kleene's computable linear order with no hyperarithmetic descending sequence ([96, Lem. III.2.1]). Such a linear order is a well-order when seen within the ω-model HYP consisting exactly of the hyperarithmetic sets. Pseudo-well-orders were first studied in [51] and proved to be a powerful tool in reverse mathematics, especially when working at the level of ATR_0 (see [106, Sec. V.4]). Our first task can essentially be rephrased as being concerned with the difficulty of revealing a pseudo-ordinal as not actually being an ordinal.

Our second task can be seen as an abstraction of the computational content of theorems in well-quasi-order (wqo) theory. There are many famous theorems asserting that wqo's are closed under certain operations. Examples such as Kruskal's tree theorem, as well as Extended Kruskal's theorem and Higman's theorem, have been well-studied in proof theory via their proof-theoretic ordinals (see [104]). However, in their usual form, these results lack computational content. Indeed, these theorems state that a certain quasi-order $(Q, \preceq_Q)$ is a wqo. Phrasing a result of this kind in the classical Π^1_2-form would yield a statement of the type "given an infinite sequence $(q_n)_{n\in\mathbb{N}}$ in Q, find a pair of indexes $i < j$ s.t. $q_i \preceq_Q q_j$". Such a pair (i, j) would be a witness of the fact that the sequence $(q_n)_{n\in\mathbb{N}}$ is not bad. However, while proving that $(Q, \preceq_Q)$ is a wqo can be "hard" (in particular Extended Kruskal's theorem is not provable in $\mathbf{\Pi}^1_1{-}\mathrm{CA}_0$ [104]), producing a pair of witnesses for each infinite sequence is a $\preceq_Q$-computable problem (as it can be solved by an extensive search)!

These theorems are very extreme examples of a well-known difference between reverse mathematics and computable analysis: quoting [41], the computable analyst is allowed to conduct an unbounded search for an object that is guaranteed to exist by (nonconstructive) mathematical knowledge, whereas the reverse mathematician has the burden of an existence proof with limited means.

On the other hand, considering the contrapositives of the above theorems can reveal some (otherwise hidden) computational content. For example, to show that a given quasi-order is not a wqo it suffices to produce a bad sequence in it. Extended Kruskal's theorem or Higman's theorem can be stated in the form "given a bad sequence for the derived quasi-order, find a bad sequence for the original quasi-order". Our second problem trivially is an upper bound for all these statements, as we disregard any particular reason for why the given quasi-order is not a wqo, and just start with the promise that it is not. Our results thus lay the groundwork for future exploration of the computational content of individual theorems from wqo theory.

Summary of our results

The parallel between reverse mathematics and Weihrauch reducibility identifies a "region" of the Weihrauch lattice that has been widely explored in the literature. This consists of the "arithmetic problems" (i.e. those that are reducible to some jump of lim), and the problems $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$, which are scaffolding the "ATR_0 analogs".

We show that DS does not belong to this "explored" part of the lattice. To put it in a nutshell, our results show that it is difficult to solve DS, but that DS is rather weak in solving other problems. For example, DS has computable inputs without any hyperarithmetic solutions, yet DS cannot guarantee to compute any specific real not Turing reducible to the Halting problem. We provide a few characterizations that tell us what the greatest Weihrauch degree with representatives of particular types below DS is, and include some general observations on this approach. The diagram in Figure 5.1 shows the relations between DS and several other Weihrauch degrees. Dashed arrows represent Weihrauch reducibility in the direction of the arrow, solid arrows represent strict Weihrauch reducibility. Next, we generalize our results by exploring how different presentations of the same order can affect the uniform strength of the same computational task (finding descending sequences in it). We study the problems $\boldsymbol{\Gamma}$-DS and $\boldsymbol{\Gamma}$-BS, where the name of the input order carries "less accessible information" on the order itself (namely $a \le_L b$ is assumed to be a Γ-condition relative to the name of the order). We summarize the results in Figure 5.2.

5.1 Finding descending sequences

Let us formally define the problem of finding descending sequences in an ill-founded linear order as a multivalued function.

Definition 5.1: Let $\mathsf{DS} :\subseteq \mathrm{LO} \rightrightarrows \mathbb{N}^\mathbb{N}$ be the multivalued function defined as

$$\mathsf{DS}(L) := \{x \in \mathbb{N}^\mathbb{N} : (\forall i)(x(i+1) <_L x(i))\},$$

with $\mathrm{dom}(\mathsf{DS}) := \mathrm{LO} \setminus \mathrm{WO}$.

5.1.1 The uniform strength of DS

We can immediately notice the following:

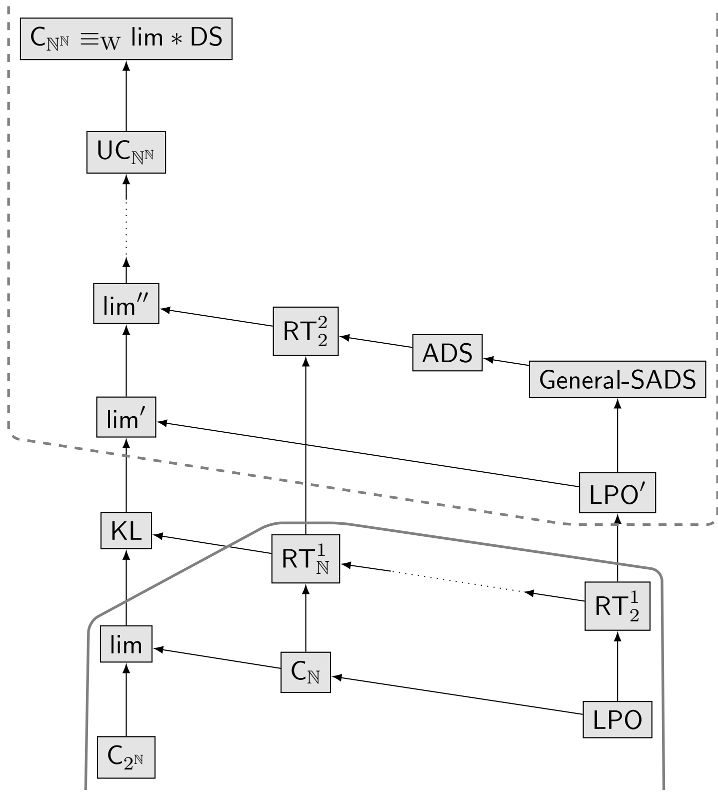

Figure 5.1: An overview of some parts of the Weihrauch lattice. The solid frame collects the degrees belonging to the lower cone of DS, while the dashed frame collects principles that are not Weihrauch reducible to DS. The only principle shown which is above DS is $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$. We do not know whether KL is reducible to DS.

Proposition 5.2:
$\mathsf{DS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *but* $\mathsf{DS} \not\le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof: To show that $\mathsf{DS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ it is enough to notice that being a descending sequence in a linear order L is a $\Pi^{0,L}_1$ property. In other words, we can obtain a descending sequence through L by choosing a path through the tree

$$\{\sigma \in \mathbb{N}^{<\mathbb{N}} : (\forall i < |\sigma| - 1)(\sigma(i+1) <_L \sigma(i))\}.$$

To show that $\mathsf{DS} \not\le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, recall that there is a computable linear order with no hyperarithmetic descending sequence (see e.g. [96, Lem. III.2.1]). A reduction $\mathsf{DS} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ would therefore contradict Theorem 2.13. ■

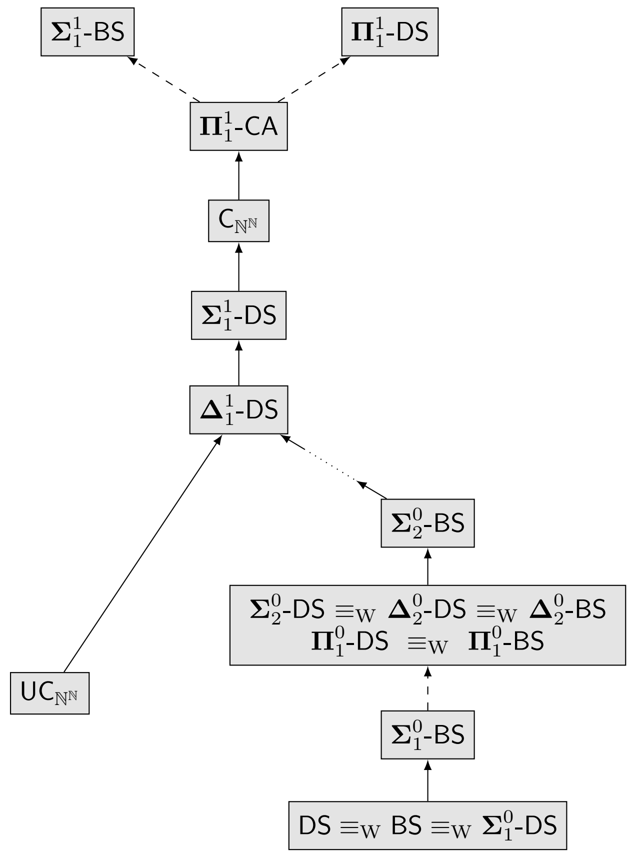

Figure 5.2: Diagram presenting the relations between the various generalizations of DS.

In particular, this shows that DS is not an arithmetic problem (i.e. $\mathsf{DS} \not\leq_{\mathrm{W}} \mathsf{lim}^{(n)}$, for any n).

Proposition 5.3:
$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{lim} * \mathsf{DS}$.

Proof: The reduction $\mathsf{lim} * \mathsf{DS} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ follows from the fact that both lim and DS are reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and that $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is closed under compositional product.

To prove the left-to-right reduction notice that, given a tree T, we can computably build the linear order $\mathrm{KB}(T)$. It is known that $[T] \neq \emptyset$ iff $\mathrm{KB}(T)$ is ill-founded (see e.g. [106, Lem. V.1.3]). Moreover, given a infinite descending sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathrm{KB}(T)$, the sequence $(\sigma_n{}^\frown 0^\omega)_{n\in\mathbb{N}}$ converges to some $x \in [T]$, and therefore the claim follows. ■

We can generalize the problem DS to the context of quasi-orders. It is easy to see that the problem of finding descending sequences in a quasi-order is Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Indeed, on the one hand, being a descending sequence in a quasi-order P is a $\Pi^{0,P}_1$ property. On the other hand, every tree, ordered by the prefix relation, is a partial order where the descending sequences provide arbitrarily long prefixes of a path.

When working with non-well quasi-orders, it is more natural to ask for bad sequences instead.

Definition 5.4: We define the multivalued function $\mathsf{BS} :\subseteq \mathrm{QO} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

$$\mathsf{BS}(P) := \{x \in \mathbb{N}^{\mathbb{N}} : (\forall i)(\forall j > i)(x(i) \not\preceq_P x(j))\},$$

where $\mathrm{dom}(\mathsf{BS})$ is the set of quasi-orders that are not well-quasi-orders.

It follows from the definition that every ill-founded linear order is a non-well quasi-order and that every bad sequence through an ill-founded linear order is indeed a descending sequence.

By expanding a bit on a classical argument we can prove that the two problems are uniformly equivalent.

Proposition 5.5:
$\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{BS}$.

Proof: The left-to-right reduction is trivial, so we only need to show that $\mathsf{BS} \le_{\mathrm{W}} \mathsf{DS}$. Let P be a non-well quasi-order. We will first compute an extension R of P s.t. every two elements of P are R-comparable, then we will computably pick an element from each R-equivalence class, so as to obtain a linear order.

We define R iteratively as follows: at every stage s s.t. $s \in P$, we define the R-relation between s and t, for every $t \in P$ s.t. $t < s$. If $t \,|_P\, s$ then we define $s \prec_R t$. Otherwise we define the R-relation between s and t so as to extend P.

It is easy to see that if $(p_i)_{i\in\mathbb{N}}$ is an $\prec_R$-descending sequence then it is a P-bad sequence. Indeed, for every i, j s.t. $i < j$, if $p_i \preceq_P p_j$ then $p_i \preceq_R p_j$ (as R extends P), contradicting the fact that $(p_i)_{i\in\mathbb{N}}$ is an $\prec_R$-descending sequence. Moreover R is ill-founded: indeed every $\prec_P$-descending sequence is also an $\prec_R$-descending sequence. On the other hand, every P-antichain $(q_i)_{i\in\mathbb{N}}$ has a subsequence $(q_{i_k})_{k\in\mathbb{N}}$ that is an $\prec_R$-descending sequence (define q_{i_k} inductively by letting i_k be the smallest integer s.t. $q_{i_k} > q_j$, for every $j < k$).

To conclude the proof it is enough to show that we can uniformly compute a linear order L by choosing an element from each R-equivalence class. We define L as the restriction of R to the set

$$\{p \in R : (\forall q < p)(p \not\equiv_R q)\}.$$

Clearly L is isomorphic to the quotient order induced by R on the set of R-equivalence classes, hence it is ill-founded. Moreover, every $<_L$-descending sequence is an $\prec_R$-descending sequence, and therefore $\mathsf{DS}(L) \subset \mathsf{BS}(P)$. ■

We will show that DS (and hence BS) is quite weak in terms of uniform computational strength (a fortiori $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \not\le_{\mathrm{W}} \mathsf{DS}$). Let us first underline the following useful proposition.

Proposition 5.6:
DS *is a cylinder.*

PROOF: Let $p \in \mathbb{N}^{\mathbb{N}}$ and L be an ill-founded linear order. Define

$$M := \{(p[n], n) \,:\, n \in L\},$$
$$(p[n], n) \leq_M (p[m], m) :\iff n \leq_L m.$$

It is easy to see that M is computably isomorphic to L, and hence it is a valid input for DS. In particular, letting $((p[n_i], n_i))_{i\in\mathbb{N}} \in \mathsf{DS}(M)$, we have that $(n_i)_{i\in\mathbb{N}}$ is a descending sequence in L and $p = \bigcup_{i\in\mathbb{N}} p[n_i]$. ■

In Section 2.1.2 we introduced the bounding problems $\mathbf{\Gamma}-\mathsf{Bound}$, and in particular, $\mathbf{\Pi}^1_1-\mathsf{Bound}$. Recall that we can assume that every instance of $\mathbf{\Pi}^1_1-\mathsf{Bound}$ is an initial segment of $\mathbb{N}$.

Proposition 5.7:
$\mathbf{\Pi}^1_1-\mathsf{Bound} <_{\mathrm{W}} \mathsf{DS}$.

PROOF: Let X be a $\mathbf{\Pi}^1_1$ initial segment of $\mathbb{N}$. By considering the Kleene-Brouwer ordering, we can think of a name for X as a sequence $(L_n)_{n\in\mathbb{N}}$ of linear orders s.t. $n \in X$ iff L_n is well-founded.

Define the linear order $L := \bigcup_n \{n\} \times L_n$, ordered lexicographically. Notice that L is ill-founded as X is not all of $\mathbb{N}$. Moreover, for every $<_L$-descending sequence $((n_i, a_i))_{i\in\mathbb{N}}$, we have that $n_0 \in \mathbf{\Pi}^1_1-\mathsf{Bound}(X)$. Indeed, for every $n \in X$ and every $a \in L_n$, the pair (n, a) lies in the well-founded part of L.

The fact that the reduction is strict follows from the fact that every solution to $\mathbf{\Pi}^1_1-\mathsf{Bound}$ is computable, whereas there is a computable input for DS with no hyperarithmetic solution. ■

We now show that ${}^1\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}^1_1-\mathsf{Bound}$. Let us first prove the following lemma, which will also be useful to prove Theorem 5.15.

Lemma 5.8:
Suppose that f is a problem which is Weihrauch reducible to DS via the computable maps Φ, Ψ. For every f-instance X, let $\leq^X$ be the linear order defined by Φ^X. We can uniformly compute a sequence $(F_s)_{s\in\mathbb{N}}$ of finite $<^X$-descending sequences s.t. (1) for every s, $\Psi^{X\oplus F_s}$ outputs some $j \in \mathbb{N}$; (2) for cofinitely many s, F_s extends to an infinite $<^X$-descending sequence.

Proof: Fix an f-instance X and run Φ^X for s steps. This produces a finite linear order $\leq_s^X$. Define

$$D_s := \{F \subseteq \leq_s^X \ : F \text{ is a } <_s^X\text{-descending sequence and } |F| \geq 1 \text{ and } \Psi^{X \oplus F} \text{ outputs some } j \in \mathbb{N} \text{ in } s \text{ steps}\}.$$

Note that D_s is finite and $t < s$ implies $D_t \subset D_s$. If $D_s \neq \emptyset$ we define F_s to be the $<_{\mathbb{N}}$-least element of D_s such that

$$(\forall F \in D_s) \left(\min_{<^X}(F) \leq_s^X \min_{<^X}(F_s) \right).$$

This ensures that if any $F \in D_s$ extends to an infinite $<^X$-descending sequence, then so does F_s. Observe that $(F_s)_s$ is uniformly computable from X. If $D_s = \emptyset$ we define $F_s := F_t$ where t is the first index greater than s s.t. $D_t \neq \emptyset$. (We will show below that such t exists, so we can computably search for it.)

Notice that for cofinitely many s, $D_s \neq \emptyset$. Indeed, let S be an infinite $<^X$-descending sequence (there must exist one because $<^X$ is a DS-instance). Since $\Psi^{X \oplus S}$ outputs some f-solution j of X, there is some finite nonempty initial segment F of S and some $t \in \mathbb{N}$ such that $\Psi^{X \oplus F}$ outputs j in t steps. Hence for all sufficiently large s, we have that $F \in D_s$. This shows that the sequence $(F_s)_{s \in \mathbb{N}}$ is well-defined. Moreover, as already observed, for every $t \geq s$, F_t extends to an infinite $<^X$-descending sequence.

The fact that, for every s, $\Psi^{X \oplus F_s}$ outputs some $j \in \mathbb{N}$ follows from the definition of D_s. ■

In particular, if f has codomain $\mathbb{N}$ the above lemma implies that, for cofinitely many s, $\Psi^{X \oplus F_s}$ outputs some f-solution for X.

Theorem 5.9:
${}^1\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}_1^1{-}\mathsf{Bound}$.

Proof: If $f \leq_{\mathrm{W}} \mathbf{\Pi}_1^1{-}\mathsf{Bound}$, then $f \leq_{\mathrm{W}} \mathsf{DS}$ by Proposition 5.7. Since $\mathbf{\Pi}_1^1{-}\mathsf{Bound}$ is first order, $f \leq_{\mathrm{W}} {}^1\mathsf{DS}$.

To prove the converse reduction, suppose that $f \leq_{\mathrm{W}} \mathsf{DS}$ as witnessed by the maps Φ and Ψ. Given an f-instance X, let $(F_s)_{s \in \mathbb{N}}$ be as in Lemma 5.8. Let $\leq^X$ denote the linear order represented by Φ^X. Define the following $\Pi_1^{1,X}$ set:

$$A := \{s \in \mathbb{N} \, : \, F_s \notin \mathrm{Ext}\},$$

where Ext denotes the set of finite sequences that extend to an infinite $<^X$-descending sequence.

Notice that A is finite as, for cofinitely many s, F_s is extendible. In particular A is a valid instance of $\mathbf{\Pi}_1^1{-}\mathsf{Bound}$ and, for every $b \in \mathbf{\Pi}_1^1{-}\mathsf{Bound}(A)$, F_b is extendible to an infinite $<^X$-descending sequence. By construction, $\Psi^{X \oplus F_b}$ commits to some $j \in \mathbb{N}$. The fact that F_b is extendible guarantees that j is a valid f-solution of X. ■

Corollary 5.10:
$\mathsf{DS} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Proof: If $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathsf{DS}$ then, by Proposition 4.37, $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N} \le_{\mathrm{W}} \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$. However, this would imply that $\widehat{\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} \le_{\mathrm{W}} \widehat{\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}}$, contradicting [2, Cor. 3.23]. ■

Definition 5.11: Let $f :\subseteq X \rightrightarrows \mathbb{N}$ be a multi-valued function. We say that f is *upwards-closed* if whenever $n \in f(x)$, then $m \in f(x)$ for all $m > n$.

It is straightforward from the definition that $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$ is upwards-closed.

Lemma 5.12:
If f is upwards-closed then $\mathrm{Det}_\mathbb{N}(f) \le_{\mathrm{W}} \mathsf{C}_\mathbb{N}$.

Proof: Let g be a single-valued function with codomain $\mathbb{N}$ and suppose that $g \le_{\mathrm{W}} f$ as witnessed by Φ, Ψ. Given a name p for a g-instance x, we use $\mathsf{C}_\mathbb{N}$ to guess some n, t such that $\Psi(p, n)$ converges to some k in at most t steps, and such that for no $m > n$ it ever happens that $\Psi(p, m)$ converges to anything but k. Since f is upwards-closed and g is single-valued, such n, t must exist. Moreover, the associated k is equal to $g(x)$. ■

Proposition 5.13:
$\mathrm{Det}_\mathbb{N}(\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}) \equiv_{\mathrm{W}} \mathrm{Det}_\mathbb{N}(\mathsf{C}_\mathbb{N}) \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}$, *and therefore* $\mathrm{Det}_\mathbb{N}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}$.

Proof: Let us first notice that $\mathsf{C}_\mathbb{N} \equiv_{\mathrm{W}} \mathsf{UC}_\mathbb{N}$ ([11, Prop. 6.2]) and therefore $\mathrm{Det}_\mathbb{N}(\mathsf{C}_\mathbb{N}) \equiv_{\mathrm{W}} \mathsf{C}_\mathbb{N}$. The fact that $\mathrm{Det}_\mathbb{N}(\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}) \le_{\mathrm{W}} \mathsf{C}_\mathbb{N}$ follows from Lemma 5.12. To prove the converse reduction it is enough to show that $\mathsf{UC}_\mathbb{N} \le_{\mathrm{W}} \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$.

Let $(n_i)_{i\in\mathbb{N}}$ be an enumeration of the complement of $\{x\} \subset \mathbb{N}$. Define

$$m(s) := \min\{j \in \mathbb{N} \,:\, (\forall i < s)(n_i \neq j)\},$$
$$A := \{s \in \mathbb{N} \,:\, (\exists t > s)(m(t) \neq m(s))\}.$$

Clearly $\lim_{s\to\infty} m(s) = x$, which implies that A is finite. Since m is computable (relative to $(n_i)_{i\in\mathbb{N}}$), A is a valid input for $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$. Moreover, for every $b \in \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}(A)$ we have $m(b) = x$.

This implies that $\mathsf{C}_{\mathbb{N}} \le_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(\mathsf{DS})$. To conclude the proof we notice that, for every single-valued g with codomain $\mathbb{N}$ we have

$$g \le_{\mathrm{W}} \mathsf{DS} \Rightarrow g \le_{\mathrm{W}} \mathbf{\Pi}^1_1\text{-}\mathsf{Bound} \Rightarrow g \le_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(\mathbf{\Pi}^1_1\text{-}\mathsf{Bound}) \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}. \qquad \blacksquare$$

Notice that $\mathbf{\Pi}^1_1\text{-}\mathsf{Bound} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$: indeed $\widehat{\mathsf{C}_{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{lim}$, while $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \widehat{\mathbf{\Pi}^1_1\text{-}\mathsf{Bound}}$ (see Proposition 5.52). This implies that $\mathrm{Det}_{\mathbb{N}}(\mathbf{\Pi}^1_1\text{-}\mathsf{Bound}) <_{\mathrm{W}} \mathbf{\Pi}^1_1\text{-}\mathsf{Bound}$. In this regard, we observe the following:

Proposition 5.14:
The Weihrauch degree of $\mathsf{C}_{\mathbb{N}}$ is the highest Weihrauch degree containing both of the following:

1. *a representative which is single-valued and has codomain $\mathbb{N}$;*
2. *a representative which is upwards-closed.*

Proof: To prove that $\mathsf{C}_{\mathbb{N}}$ satisfies point 1, consider $\mathsf{UC}_{\mathbb{N}}$, which is Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}}$ ([11, Prop. 6.2]). To prove that $\mathsf{C}_{\mathbb{N}}$ satisfies point 2, consider the problem $\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound}$ that produces a bound for a finite $\mathbf{\Sigma}^0_1$ subset of $\mathbb{N}$. Clearly $\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound}$ is upwards closed. The reduction $\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ follows from the fact that, for every $A \in \mathrm{dom}(\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound})$, the set

$$\{n \in \mathbb{N} : (\forall m \ge n)(m \notin A)\}$$

is a $\Pi^{0,A}_1$ subset of $\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound}(A)$. To prove the converse reduction, let p be a name for some $B \in \mathrm{dom}(\mathsf{C}_{\mathbb{N}})$. Define $m(s)$ to be the least number not enumerated in p by stage s. Clearly $\lim_{s\to\infty} m(s) = \min B$. In particular this implies that there are only finitely many stages s s.t. $m(s) \neq \min B$. Using $\mathbf{\Sigma}^0_1\text{-}\mathsf{Bound}$ we can obtain a stage b s.t. $m(b) = \min B$, hence solving $\mathsf{C}_{\mathbb{N}}$.

Finally the maximality of $\mathsf{C}_{\mathbb{N}}$ follows from Lemma 5.12: indeed suppose $f\colon X \to \mathbb{N}$ is Weihrauch equivalent to some g which is upwards-closed. By Lemma 5.12, we have $\mathrm{Det}_{\mathbb{N}}(g) \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. By definition of $\mathrm{Det}(\cdot)$, we have $f \le_{\mathrm{W}} \mathrm{Det}_{\mathbb{N}}(g)$, hence $f \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$. $\blacksquare$

Let us now characterize the deterministic part of DS.

Theorem 5.15:
$\mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$.

Proof: Let us first prove that $\mathsf{lim} \le_{\mathrm{W}} \mathsf{DS}$. Let J be the Turing jump operator, i.e. $\mathsf{J}(p)(e) = 1$ iff $\varphi^p_e(e)$ halts, and recall that $\mathsf{J} \equiv_{\mathrm{sW}} \mathsf{lim}$. By relativizing the construction in [76, Lem. 4.2] we have that, for every p, we can p-computably build a linear order L of type $\omega + \omega^*$ s.t. every descending sequence through L computes $\mathsf{J}(p)$. This shows that $\mathsf{lim} \equiv_{\mathrm{W}} \mathsf{J} \le_{\mathrm{W}} \mathsf{DS}$.

To prove that $\mathrm{Det}(\mathsf{DS}) \le_{\mathrm{W}} \mathsf{lim}$, suppose that $f :\subseteq X \to \mathbb{N}^{\mathbb{N}}$ is single-valued and $f \le_{\mathrm{W}} \mathsf{DS}$ as witnessed by the maps Φ, Ψ. For every n, define f_n by $f_n(x) := f(x)(n)$. The maps Φ and Ψ witness that $f_n \le_{\mathrm{W}} \mathsf{DS}$ as well (modulo a trivial coding). Given an f-instance x, consider the sequences $(F_{s,n})_{s\in\mathbb{N}}$ obtained by applying Lemma 5.8 to each f_n. Define the sequence $(p_s)_{s\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ as $p_s(n) := \Psi^{x\oplus F_{s,n}}(0)$. Notice that, by Lemma 5.8, for every n, $\Psi^{x\oplus F_{s,n}}$ outputs some number, therefore $p_s(n)$ is well-defined and is uniformly computable from x. Moreover, since f_n is single-valued and, for cofinitely many s, $F_{s,n}$ is extendible, the sequence $(\Psi^{x\oplus F_{s,n}}(0))_{s\in\mathbb{N}}$ is eventually constant and equal to $f_n(x)$. In particular this shows that, letting $p := \lim_{s\to\infty} p_s$, for each n we have $p(n) = f_n(x)$, i.e. $p = f(x)$. ■

This result shows that, despite the fact that DS can have very complicated solutions, it is rather weak from the uniform point of view. In fact, its lower Weihrauch cone misses many arithmetic problems. In particular we have:

Corollary 5.16:
$\mathsf{DS} \,|_{\mathrm{W}}\, \mathsf{LPO}'$.

Proof: Since LPO is single-valued, so is LPO'. Since $\mathsf{LPO}' \not\le_{\mathrm{W}} \mathsf{lim}$ (see [16, Cor. 12.3 and Thm. 12.7]), it follows from Theorem 5.15 that $\mathsf{LPO}' \not\le_{\mathrm{W}} \mathsf{DS}$. On the other hand, $\mathsf{DS} \not\le_{\mathrm{W}} \mathsf{LPO}'$, as LPO' always has computable solutions. ■

Notice that Theorem 5.15 implies also that $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \not\le_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} * \mathsf{DS}$. Indeed, on the one hand, by Theorem 4.64 we have $\mathrm{Det}(\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}) \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, while, on the other hand, by Theorem 4.59 if f is single-valued and $f \le_{\mathrm{W}} \mathsf{C}_{2^{\mathbb{N}}} * \mathsf{DS}$ then $f \le_{\mathrm{W}} \mathsf{DS}$ (as $\mathrm{Det}(\mathsf{C}_{2^{\mathbb{N}}}) \equiv_{\mathrm{W}} \mathrm{id}$) and hence $\mathrm{Det}(\mathsf{C}_{2^{\mathbb{N}}} * \mathsf{DS}) \equiv_{\mathrm{W}} \mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$.

Using Corollary 5.16 we can prove that DS is not closed under (parallel) product:

Theorem 5.17:
$\mathsf{LPO}' \le_{\mathrm{W}} \mathsf{DS} \times \mathsf{lim}$ *and therefore* DS *is not closed under product.*

Proof: Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^{\mathbb{N}}$ converging to an instance p of LPO. For each s define

$$g(s) = \begin{cases} i+1 & \text{if } i \le s \land p_s(i) \neq 0 \land (\forall j < i)(p_s(j) = 0), \\ 0 & \text{otherwise.} \end{cases}$$

Let us define a linear order L inductively: at stage $s = 0$ we put 0 into L. At stage $s+1$ we do the following:

1. if $g(s) = g(s+1)$ we put $2(s+1)$ immediately below $2s$;
2. if $g(s) \neq g(s+1)$ and $g(s+1) = 0$ we put $2(s+1)$ at the bottom;

3. if $g(s) \neq g(s+1)$ and $g(s+1) > 0$ we put $2(s+1)$ at the top and we put $2s+1$ immediately above 0.

This construction produces a linear order on a computable subset of $\mathbb{N}$. It is clear that g and L are uniformly computable in $(p_n)_{n\in\mathbb{N}}$. Notice that if $\mathsf{LPO}(p) = 1$ then there is an s s.t. for every $t \geq s$, $g(t) = g(s)$ (this follows by definition of limit in the Baire space). In particular, L has order type $n + \omega^*$. On the other hand, if $\mathsf{LPO}(p) = 0$ we distinguish three cases: if $g(s)$ is eventually constantly 0 then L has order type ω^*. If there are infinitely many s s.t. $g(s) > 0$ then g is unbounded (because for each i, $\lim_s p_s(i) = p(i) = 0$ so g eventually stays above i). In particular, if there are infinitely many s and infinitely many t s.t. $g(s) = 0$ and $g(t) > 0$ then L has order type $\omega^* + \zeta$, where $\zeta := \omega^* + \omega$ is the order type of the integers. If instead $g(s) > 0$ for all sufficiently large s, then L has order type $n + \zeta$. In all cases, L is ill-founded.

We consider the input $(L, (p_n)_{n\in\mathbb{N}})$ for $\mathsf{DS} \times \mathsf{lim}$. Given an $<_L$-descending sequence $(q_n)_{n\in\mathbb{N}}$, we compute a solution for $\mathsf{LPO}'((p_n)_{n\in\mathbb{N}}) = \mathsf{LPO}(p)$ as follows: if q_0 is odd or $g(q_0/2) = 0$ then we return 0, otherwise we return $p(i)$ where i is s.t. $g(q_0/2) = i + 1$.

Notice that if $\mathsf{LPO}(p) = 1$ then the ω^* part of $\leq_L$ is the final segment of the even numbers that starts with the first index $2s$ s.t. for every $t \geq s$, $g(t) = i + 1$ and $p(i) = 1$. In particular every $<_L$-descending sequence starts with some even q_0 s.t. $g(q_0/2) > 0$. On the other hand, if $\mathsf{LPO}(p) = 0$ then, by definition of LPO, we have that $p = 0^{\mathbb{N}}$. In this case, the above procedure must return 0 so it produces the correct solution. This proves that $\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{DS} \times \mathsf{lim}$.

The fact that DS is not closed under product follows from the fact that $\mathsf{lim} \leq_{\mathrm{W}} \mathsf{DS}$ (Theorem 5.15) and Corollary 5.16. ■

5.1.2 Combinatorial principles on linear orders

We introduce the following notation to phrase many combinatorial principles from reverse mathematics as multi-valued functions.

Definition 5.18: Let $\mathsf{FindC}^X_Y :\subseteq \mathrm{LO} \rightrightarrows \mathrm{LO}$ be the partial multi-valued function defined as

$$\mathsf{FindC}^X_Y(L) := \{M \in \mathrm{LO} : M \subset L \text{ and } \mathrm{ordtype}(M) \in Y\},$$

with domain being the set of $L \in \mathrm{LO}$ s.t. $\mathrm{ordtype}(L) \in X$ and there is some $M \subset L$ s.t. $\mathrm{ordtype}(M) \in Y$.

Similarly we define $\mathsf{FindS}^X :\subseteq \mathrm{LO} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ to be the partial multi-valued function that takes as input a countable linear order L s.t. $\mathrm{ordtype}(L) \in X$ and produces a string $(b, x_0, x_1, \dots)$ s.t. $b \in \{0, 1\}$ and, for all i, if $b = 0$ then $x_i <_L x_{i+1}$ while if $b = 1$ then $x_{i+1} <_L x_i$.

If X or Y is not specified, we assume that it contains every countable order type.

There is an extensive literature that studies the "ascending/descending sequence principle" (ADS) and the "chain/antichain principle" (CAC) (see e.g. [54, 57]). These principles and, several of their variations, have been studied from the point of view of Weihrauch reducibility in [3].

Notice that, in particular, the problem ADS (given a linear order, produce an infinite ascending sequence or infinite descending sequence) corresponds to FindS. Similarly the problem $\mathsf{General\text{-}SADS}$ (given a stable — i.e. of order type $\omega + n, n + \omega^*$ or $\omega + \omega^*$ — linear order, produce an infinite ascending sequence or an infinite descending sequence), corresponds to FindS^X, where $X = \{\omega + n, n + \omega^*, \omega + \omega^*\}$.

Proposition 5.19:
$\mathsf{LPO}' \leq_{\mathrm{W}} \mathsf{FindS}^{\{\omega, n+\omega^*\}}$.

Proof: Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{N}^{\mathbb{N}}$ converging to an instance p of LPO. For every $s \in \mathbb{N}$ we define (as we did in the proof of Theorem 5.17)

$$g(s) = \begin{cases} i+1 & \text{if } i \leq s \wedge p_s(i) \neq 0 \wedge (\forall j < i)(p_s(j) = 0), \\ 0 & \text{otherwise.} \end{cases}$$

Let us define a linear order $\leq_L$ on $\mathbb{N}$ inductively: for each stage s we define a linear order on $\{0, \dots, s\}$. At stage $s = 0$ there are no decisions to make. At stage $s+1$ we do the following:

1. if $0 = g(s) = g(s+1)$ we put $s+1$ immediately above s;
2. if $0 < g(s) = g(s+1)$ we put $s+1$ immediately below s;
3. if $g(s) \neq g(s+1)$ we put $s+1$ at the top.

It is clear that g and $\leq_L$ are uniformly computable in $(p_n)_{n\in\mathbb{N}}$. Notice that if $\mathsf{LPO}(p) = 1$ then there is an s s.t. for every $t \geq s$, $g(t) = i+1$, where i is the smallest integer s.t. $p(i) = 1$ (this follows by definition of limit in the Baire space). In particular, $\leq_L$ has order type $n + \omega^*$. On the other hand, if $\mathsf{LPO}(p) = 0$ then g is either eventually constantly 0 or unbounded. In both cases the linear order $\leq_L$ has order type ω.

In other words $(\mathbb{N}, \leq_L)$ has order type ω iff $\mathsf{LPO}'((p_i)_{i\in\mathbb{N}}) = 0$. Since the output of $\mathsf{FindS}^{\{\omega, n+\omega^*\}}((\mathbb{N}, \leq_L))$ comes with an indication of the order type of the solution, this defines a reduction from LPO' to $\mathsf{FindS}^{\{\omega, n+\omega^*\}}$. ■

Corollary 5.20:
$\mathsf{FindS}^{\{\omega, n+\omega^*\}} \mid_{\mathrm{W}} \mathsf{DS}$, *and hence* $\mathsf{General\text{-}SADS} \mid_{\mathrm{W}} \mathsf{DS}$.

Proof: The fact that $\mathsf{FindS}^{\{\omega, n+\omega^*\}} \not\leq_{\mathrm{W}} \mathsf{DS}$ follows from Proposition 5.19 and the fact that $\mathsf{LPO}' \not\leq_{\mathrm{W}} \mathsf{DS}$ (Corollary 5.16). Moreover, since $\mathsf{FindS}^{\{\omega, n+\omega^*\}}$ is a restriction of $\mathsf{General\text{-}SADS}$, we have $\mathsf{General\text{-}SADS} \not\leq_{\mathrm{W}} \mathsf{DS}$.

To show that the converse reduction cannot hold it is enough to notice that $\mathsf{General\text{-}SADS}$ is an arithmetic problem, while $\mathsf{DS} \not\leq_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ (Proposition 5.2). ■

In particular, this implies that ADS, as well as the stable chain/antichain principle SCAC, and the weakly stable chain/antichain principle WSCAC, are Weihrauch incomparable with DS (as they are all arithmetic problems, and $\mathsf{General\text{-}SADS}$ is reducible to all of them, see [3]).

Proposition 5.21:
$\mathsf{FindC}_{\{\omega, n+\omega^*\}} \le_{\mathrm{W}} \mathsf{DS}$.

Proof: Given a linear order $(L, \le_L)$ we can computably build the linear order $Q := L + L^*$. Formally we define $(Q, \le_Q)$ as $Q := \{0\} \times L \cup \{1\} \times L$ and

$$(a,p) \le_Q (b,q) :\iff a < b \lor (a = b = 0 \land p \le_L q) \lor (a = b = 1 \land q \le_L p).$$

Notice that Q is always ill-founded, hence it is a valid input for DS. Given $(q_i)_{i\in\mathbb{N}} \in \mathsf{DS}(Q)$, we computably build the sequence $(x_i)_{i\in\mathbb{N}}$ defined by $x_i := \pi_1 q_i$ where $\pi_i := (a_0, a_1) \mapsto a_i$.

We distinguish 3 cases:

1. if $\pi_0 q_i = 0$ for every i then $(x_i)_{i\in\mathbb{N}}$ is an ω^*-sequence in L;
2. if $\pi_0 q_i = 1$ for every i then $(x_i)_{i\in\mathbb{N}}$ is an ω-sequence in L;
3. if there is a k s.t. for all $i < k$ we have $\pi_0 q_i = 1$ and for all $j \ge k$ we have $\pi_0 q_j = 0$ then, by point 1, $(x_j)_{j\ge k}$ is an ω^*-sequence in L, hence $(x_i)_{i\in\mathbb{N}}$ is of type $n + \omega^*$, with $n \le k$.

In any case the sequence $(x_i)_{i\in\mathbb{N}}$ is a valid solution for $\mathsf{FindC}_{\{\omega, n+\omega^*\}}$. ■

5.1.3 Relations with Ramsey theorems

We now explore the relations between DS and Ramsey's theorem for n-tuples and k colors. The basic definitions and notations have been introduced in Section 2.1.2.

Notice that $\mathsf{cRT}^n_k \equiv_{\mathrm{W}} \mathsf{RT}^n_k$ iff $n = 1$. Indeed the output of cRT^n_k is always computable, while for $n > 1$ there are computable k-colorings with no computable homogeneous solutions. Similarly $\mathsf{cRT}^n_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^n_{\mathbb{N}}$ iff $n = 1$. Moreover the equivalence cannot be lifted to a strong Weihrauch equivalence. Indeed RT^1_k and cRT^1_k are incomparable from the point of view of strong Weihrauch reducibility. The uniform computational content of Ramsey's theorems is well-studied (see e.g. [21, 28, 30, 87]).

In comparing RT^n_k with DS, we immediately notice that $\mathsf{RT}^2_2 \not\le_{\mathrm{W}} \mathsf{DS}$. This follows from the fact that $\mathsf{ADS} \le_{\mathrm{W}} \mathsf{RT}^2_2$ (see e.g. [54]), while $\mathsf{ADS} \not\le_{\mathrm{W}} \mathsf{DS}$ (see the remarks after Corollary 5.20). Hence $\mathsf{RT}^n_k \not\le_{\mathrm{W}} \mathsf{DS}$ for all $n, k \ge 2$.

Proposition 5.22:
$\mathsf{RT}^1_{\mathbb{N}} <_{\mathrm{W}} \mathbf{\Pi}^1_1\text{-}\mathsf{Bound}$, *and hence* $\mathsf{RT}^1_{\mathbb{N}} <_{\mathrm{W}} \mathsf{DS}$.

Proof: Given a coloring $c\colon \mathbb{N} \to k$, consider the $\Sigma^{0,c}_2$ set

$$X := \{n \in \mathbb{N} : (\forall^\infty j)(c(n) \ne c(j))\}.$$

It is easy to see that X is finite, as $\mathrm{ran}(c) \subset k$ and if there is no c-homogeneous set with color i then there are finitely many $j \in \mathbb{N}$ s.t. $c(j) = i$. In particular, given a bound b for X there is a homogeneous solution with color $c(b)$.

The separation follows from the fact that $\mathbf{\Pi}^1_1\mathsf{-Bound} \not\leq_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (as $\widehat{\mathbf{\Pi}^1_1\mathsf{-Bound}} \not\leq_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, see [2, Fact 3.25]), while $\mathsf{RT}^1_\mathbb{N} <_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (in particular $\mathsf{RT}^1_\mathbb{N} <_{\mathrm{W}} \mathsf{C}'_\mathbb{N}$, see [21, Prop. 7.2 and Cor. 7.6]). The fact that $\mathsf{RT}^1_\mathbb{N} <_{\mathrm{W}} \mathsf{DS}$ follows from $\mathbf{\Pi}^1_1\mathsf{-Bound} <_{\mathrm{W}} \mathsf{DS}$ (Proposition 5.7). ■

We now show that $\mathsf{RT}^1_\mathbb{N}$ is the strongest problem among those that are reducible to DS and whose instances always have finitely many solutions.

Definition 5.23: Let $f :\subseteq X \rightrightarrows \mathbb{N}$. We say that f is *pointwise finite* if, for each $x \in \mathrm{dom}(f)$, $|f(x)|$ is finite.

It is easy to see that cRT^1_k and $\mathsf{cRT}^1_\mathbb{N}$ are pointwise finite, as for each k-coloring c we have $|\mathsf{cRT}^1_k(c)| = |\mathsf{cRT}^1_\mathbb{N}(c)| \leq k$.

Lemma 5.24:
Let g be upwards-closed and let f be pointwise finite. If $f \leq_{\mathrm{W}} g$ then $f \leq_{\mathrm{W}} \mathsf{RT}^1_\mathbb{N}$.

Proof: Suppose that $f \leq_{\mathrm{W}} g$ as witnessed by Φ, Ψ. Let p be the name for the f-instance x we are given.

We define a coloring c as follows: we dove-tail all computations $\Psi(p, n)$ for $n \in \mathbb{N}$. Whenever some computation converges to some $j \in \mathbb{N}$, we define $c(i) := j$ where i is the first element on which c is not defined yet. Since g is upwards-closed, we know that for all but finitely many n, $\Psi(p, n)$ has to converge to some $j_n \in f(x)$. This implies that $\mathrm{ran}(c)$ contains only finitely many distinct elements. Moreover, any element repeating infinitely often is a correct solution to $f(x)$, therefore we can find a $y \in f(x)$ by applying $\mathsf{RT}^1_\mathbb{N}$ to c and returning the color of the solution. ■

Theorem 5.25:
If f is pointwise finite then $f \leq_{\mathrm{W}} \mathsf{DS}$ iff $f \leq_{\mathrm{W}} \mathsf{RT}^1_\mathbb{N}$.

Proof: The right-to-left implication always holds as $\mathsf{RT}^1_\mathbb{N} <_{\mathrm{W}} \mathsf{DS}$ (Proposition 5.22). On the other hand, if f is pointwise finite and $f \leq_{\mathrm{W}} \mathsf{DS}$ then, by Theorem 5.9 we have $f \leq_{\mathrm{W}} \mathbf{\Pi}^1_1\mathsf{-Bound}$. Since $\mathbf{\Pi}^1_1\mathsf{-Bound}$ is upwards-closed, by Lemma 5.24 we have $f \leq_{\mathrm{W}} \mathsf{RT}^1_\mathbb{N}$. ■

By Lemma 5.24 we also have the following:

Proposition 5.26:
The Weihrauch degree of $\mathsf{RT}^1_{\mathbb{N}}$ is the highest Weihrauch degree such that:

1. *it contains a representative which is pointwise finite;*
2. *it is Weihrauch reducible to some problem which is upwards-closed.*

Proof: Point 1 holds because $\mathsf{cRT}^1_{\mathbb{N}}$ is pointwise finite and $\mathsf{cRT}^1_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$. Point 2 holds because $\mathsf{RT}^1_{\mathbb{N}} <_{\mathrm{W}} \mathbf{\Pi}^1_1{-}\mathsf{Bound}$ (Proposition 5.22) and $\mathbf{\Pi}^1_1{-}\mathsf{Bound}$ is upwards-closed. Finally, the maximality follows from Lemma 5.24. ∎

Lemma 5.27:
If f is upwards-closed and $f \le_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$ then $f \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

Proof: Recall that $\mathsf{RT}^1_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}}$ and let Φ, Ψ be two computable maps witnessing $f \le_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}}$. Let p be a name for some $x \in \operatorname{dom}(f)$ and let c be the coloring represented by $\Phi(p)$. We define the following $\Pi^{0,p}_1$ set

$$\begin{aligned} A := \{\langle n, c_0, \dots, c_k, s\rangle \ :\ & (\forall i)(\exists j \le k)(c(i) = c_j) \text{ and} \\ & (\forall j \le k)(\exists i < s)(c(i) = c_j) \text{ and} \\ & (\forall j \le k)(\Psi(p, c_j)\downarrow \to \Psi(p, c_j) \le n)\}. \end{aligned}$$

Notice that, if $\langle n, c_0, \dots, c_k, s\rangle \in A$ then, by the first two conditions, there is a $j \le k$ s.t. c_j is a valid solution for $\mathsf{cRT}^1_{\mathbb{N}}(c)$. In particular $\Psi(p, c_j)\downarrow$ and is a correct solution for $f(x)$ (as Φ and Ψ witness that $f \le_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}}$). Since f is upwards-closed, every number greater than $\Psi(p, c_j)$ is a valid solution. In particular, the third condition implies that $n \ge \Psi(p, c_j)$ and therefore $n \in f(x)$. ∎

Notice that the previous lemma provides an alternative proof for $\mathbf{\Pi}^1_1{-}\mathsf{Bound} \not\le_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$, as $\mathbf{\Pi}^1_1{-}\mathsf{Bound} \not\le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

If we consider only bounded pointwise finite functions, we can improve Theorem 5.25 by replacing $\mathsf{RT}^1_{\mathbb{N}}$ with RT^1_k.

Lemma 5.28:
If f has codomain k, then $f \le_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$ iff $f \le_{\mathrm{W}} \mathsf{RT}^1_k$.

Proof: The right-to-left implication is trivial, so let us prove the left-to-right one. Since $\mathsf{RT}^1_{\mathbb{N}} \le_{\mathrm{W}} \mathbf{\Pi}^1_1{-}\mathsf{Bound}$ and $\mathbf{\Pi}^1_1{-}\mathsf{Bound}$ is upwards-closed, it suffices to show that if g is upwards-closed and $f \le_{\mathrm{W}} g$, then $f \le_{\mathrm{W}} \mathsf{RT}^1_k$. The proof closely follows the one of Lemma 5.24. Suppose that $f \le_{\mathrm{W}} g$ as witnessed by Φ, Ψ. Let p be the name for the f-instance x we are given. We define a k-coloring c as follows: we dove-tail all computations $\Psi(p,n)$ for $n \in \mathbb{N}$. Whenever some computation converges to some $j < k$, we define $c(i) := j$ where i is the first element on which c is not defined yet. Since g is upwards-closed, we know that for all but finitely many n, $\Psi(p,n)$ has to converge to some $j_n < k$ which lies in $f(x)$. Moreover, any element repeating infinitely often is a correct solution to $f(x)$, therefore we can find a $y \in f(x)$ by applying RT^1_k to c and returning the color of the solution. ■

Theorem 5.29:
If f has codomain k, then $f \le_{\mathrm{W}} \mathsf{DS}$ iff $f \le_{\mathrm{W}} \mathsf{RT}^1_k$.

Proof: The right-to-left implication always holds as $\mathsf{RT}^1_k \le_{\mathrm{W}} \mathsf{RT}^1_{\mathbb{N}}$ trivially and $\mathsf{RT}^1_{\mathbb{N}} <_{\mathrm{W}} \mathsf{DS}$ (Proposition 5.22). The left-to-right implication follows from Theorem 5.25 and Lemma 5.28. ■

To conclude the section we notice how we can improve the results if we restrict our attention to single-valued functions. Recall that $\mathsf{lim}_k :\subseteq k^{\mathbb{N}} \to k$ is the problem of computing the limit in the k-element space.

Lemma 5.30:
If f has codomain k and is single-valued, then $f \le_{\mathrm{W}} \mathsf{lim}_k$ iff $f \le_{\mathrm{W}} \mathsf{RT}^1_k$.

Proof: The left-to-right implication is trivial as $\mathsf{lim}_k \le_{\mathrm{W}} \mathsf{RT}^1_k$. To prove the converse direction recall that $\mathsf{RT}^1_k \equiv_{\mathrm{W}} \mathsf{cRT}^1_k$ and let the reduction $f \le_{\mathrm{W}} \mathsf{cRT}^1_k$ be witnessed by the maps Φ, Ψ. Let p be a name for some $x \in \mathrm{dom}(f)$ and let c be the coloring represented by $\Phi(p)$. Notice that, since f is single-valued, for every solution $j \in \mathsf{cRT}^1_k(c)$ we have $\Psi(p,j) = f(x)$. Furthermore, since the range of c is finite, there are only finitely many i such that $c(i)$ is not a solution. If we then define

$$n_i := \begin{cases} \Psi(p, c(i)) & \text{if } \Psi(p,c(i)) \text{ converges in } i \text{ steps and } \Psi(p,c(i)) < k, \\ 0 & \text{otherwise,} \end{cases}$$

we have that the sequence $(n_i)_{i\in\mathbb{N}} \in k^{\mathbb{N}}$ converges to $f(x)$. Therefore we can use lim_k to obtain $f(x)$. ■

Theorem 5.31:
If f has codomain k and is single-valued, then $f \le_{\mathrm{W}} \mathsf{lim}_k$ iff $f \le_{\mathrm{W}} \mathsf{DS}$.

Proof: The left-to-right implication follows from the fact that $\mathsf{lim} <_{\mathrm{W}} \mathsf{DS}$ (Theorem 5.15), while the other direction follows from Theorem 5.29 and Lemma 5.30. ■

5.2 Presentation of orders

In this section, we study how the presentation of a linear/quasi order can influence the uniform computational strength of the problems DS and BS.

Definition 5.32: For every $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_k, \mathbf{\Pi}^0_k, \mathbf{\Delta}^0_k, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1\}$, the represented spaces $\mathbf{\Gamma}(\mathrm{LO})$ and $\mathbf{\Gamma}(\mathrm{QO})$ are obtained by restricting the codomain of $\delta_{\mathbf{\Gamma}(\mathbb{N})}$ to the set of subsets of $\mathbb{N}$ which are characteristic functions of linear orders and quasi-orders respectively.

We define the problem $\mathbf{\Gamma}\text{-}\mathsf{DS} :\subseteq \mathbf{\Gamma}(\mathrm{LO}) \rightrightarrows \mathbb{N}^\mathbb{N}$ as $\mathbf{\Gamma}\text{-}\mathsf{DS}(L) := \mathsf{DS}(L)$. Similarly we define $\mathbf{\Gamma}\text{-}\mathsf{BS} :\subseteq \mathbf{\Gamma}(\mathrm{QO}) \rightrightarrows \mathbb{N}^\mathbb{N}$ as $\mathbf{\Gamma}\text{-}\mathsf{BS}(P) := \mathsf{BS}(P)$.

Despite the fact that $\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{BS}$ (Proposition 5.5), it is not the case that $\mathbf{\Gamma}\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Gamma}\text{-}\mathsf{BS}$ in general. In particular, we will show that $\mathbf{\Sigma}^0_k\text{-}\mathsf{BS} \not\le_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{DS}$ (Theorem 5.45) and $\mathbf{\Sigma}^1_1\text{-}\mathsf{BS} \not\le_{\mathrm{W}} \mathbf{\Sigma}^1_1\text{-}\mathsf{DS}$ (Corollary 5.55).

Furthermore, we strengthen Corollary 5.10 by showing that $\mathbf{\Sigma}^1_1\text{-}\mathsf{DS} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 5.53). In other words, even if we are allowed to feed DS a code for a $\mathbf{\Sigma}^1_1$ linear ordering, we still cannot compute $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$. On the other hand, we already showed that if we are allowed to perform a relatively small amount of post-processing (namely lim) on the output of DS, then we can compute $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Proposition 5.3). In particular, the use of lim absorbs any difference in uniform strength between DS and $\mathbf{\Sigma}^1_1\text{-}\mathsf{DS}$ and collapses the whole hierarchy (up to $\mathbf{\Sigma}^1_1\text{-}\mathsf{DS}$) to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Many of our separations are derived by analyzing the first-order part of the problems in question, or more generally by characterizing the problems satisfying certain properties (such as single-valuedness or having restricted codomain) which lie below the problems in question. On the contrary, we prove Theorem 5.53 using very different techniques due to Anglès d'Auriac and Kihara [2].

Before beginning our analysis, we record some preliminary observations. Note that $\mathsf{DS} = \mathbf{\Delta}^0_1\text{-}\mathsf{DS}$ and $\mathsf{BS} = \mathbf{\Delta}^0_1\text{-}\mathsf{BS}$. It is straightforward to see that, for every $\mathbf{\Gamma}$, $\mathbf{\Gamma}\text{-}\mathsf{DS} \le_{\mathrm{W}} \mathbf{\Gamma}\text{-}\mathsf{BS}$. Moreover, for every $\mathbf{\Gamma}, \mathbf{\Gamma}'$ s.t. $\mathbf{\Gamma}(X) \subset \mathbf{\Gamma}'(X)$ we have $\mathbf{\Gamma}\text{-}\mathsf{DS} \le_{\mathrm{W}} \mathbf{\Gamma}'\text{-}\mathsf{DS}$ and $\mathbf{\Gamma}\text{-}\mathsf{BS} \le_{\mathrm{W}} \mathbf{\Gamma}'\text{-}\mathsf{BS}$.

Notice also that the set of bad sequences through a $\mathbf{\Delta}^1_1$-quasi-order is $\mathbf{\Delta}^1_1$, hence it is straightforward to see that $\mathbf{\Delta}^1_1\text{-}\mathsf{BS} \le_{\mathrm{W}} \mathbf{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. This shows also that $\mathbf{\Gamma}\text{-}\mathsf{BS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ for every arithmetic $\mathbf{\Gamma}$.

Proposition 5.33:
For every $\mathbf{\Gamma} \in \{\mathbf{\Sigma}^0_k, \mathbf{\Pi}^0_k, \mathbf{\Delta}^0_k, \mathbf{\Sigma}^1_1, \mathbf{\Pi}^1_1, \mathbf{\Delta}^1_1\}$ the problems $\mathbf{\Gamma}\text{-}\mathsf{DS}$ and $\mathbf{\Gamma}\text{-}\mathsf{BS}$ are cylinders.

PROOF: The proof is a straightforward generalization of the proof of Proposition 5.6. ■

Theorem 5.34:
For every $k \in \mathbb{N}$ *and every* $\mathbf{\Gamma} \in \{\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}\}$

$$\mathbf{\Gamma}^0_{k+1}\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{DS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{DS}^{(k)},$$
$$\mathbf{\Gamma}^0_{k+1}\text{-}\mathsf{BS} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{BS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{BS}^{(k)}.$$

PROOF: Fix k and $\mathbf{\Gamma}$ as above. The reduction $\mathbf{\Gamma}^0_{k+1}\text{-}\mathsf{DS} \le_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{DS} * \mathsf{lim}^{[k]}$ follows from the fact that

$$\mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \mathbf{\Pi}^0_k\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}\text{-}\mathsf{CA},$$

hence we can use $\mathsf{lim}^{[k]}$ to compute a $\mathbf{\Gamma}^0_1$-name for the input linear order, and then apply $\mathbf{\Gamma}^0_1\text{-}\mathsf{DS}$ to get a descending sequence.

Let us now prove the converse reduction. Since both $\mathsf{lim}^{[k]}$ and $\mathbf{\Gamma}^0_1\text{-}\mathsf{DS}$ are cylinders, by the cylindrical decomposition there is an e s.t.

$$\mathbf{\Gamma}^0_1\text{-}\mathsf{DS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{DS} \circ \Phi_e \circ \mathsf{lim}^{[k]}.$$

Given any $p \in \operatorname{dom}(\mathbf{\Gamma}^0_1\text{-}\mathsf{DS} \circ \Phi_e \circ \mathsf{lim}^{[k]})$, the string $q := \Phi_e(\mathsf{lim}^{[k]}(p))$ is a $\mathbf{\Gamma}^0_1$-name for a linear order L_p. Since q is $\Delta^{0,p}_{k+1}$, the condition $a \le_{L_p} b$ is $\Gamma^{0,p}_{k+1}$ for every a, b. This shows that, given an input p we can uniformly compute a $\mathbf{\Gamma}^0_{k+1}$-name for the linear order L_p, and hence use $\mathbf{\Gamma}^0_{k+1}\text{-}\mathsf{DS}$ to compute an $<_{L_p}$-descending sequence.

The equivalence $\mathbf{\Gamma}^0_1\text{-}\mathsf{DS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{DS}^{(k)}$ follows from the fact that $\mathbf{\Gamma}^0_1\text{-}\mathsf{DS}$ is a cylinder.

The same reasoning works, mutatis mutandis, to show that

$$\mathbf{\Gamma}^0_{k+1}\text{-}\mathsf{BS} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{BS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Gamma}^0_1\text{-}\mathsf{BS}^{(k)}.$$ ■

Using this theorem, the relativized version of Proposition 5.5 can be proved explicitly as follows:

Corollary 5.35:
For every $k \ge 1$, $\mathbf{\Delta}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_k\text{-}\mathsf{BS}$.

PROOF: Using Proposition 5.5 and Theorem 5.34 we immediately have

$$\mathbf{\Delta}^0_{k+1}\text{-}\mathsf{BS} \equiv_{\mathrm{W}} \mathsf{BS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathsf{DS} * \mathsf{lim}^{[k]} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}\text{-}\mathsf{DS},$$

as $\mathsf{DS} = \mathbf{\Delta}^0_1\text{-}\mathsf{DS}$ and $\mathsf{BS} = \mathbf{\Delta}^0_1\text{-}\mathsf{BS}$. ■

This implies also that, for every k, $\mathbf{\Sigma}^0_k$-$\mathsf{BS} \leq_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}$-$\mathsf{DS}$ and $\mathbf{\Pi}^0_k$-$\mathsf{BS} \leq_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}$-$\mathsf{DS}$.

5.2.1 $\mathbf{\Gamma}^0_k$-DS and $\mathbf{\Gamma}^0_k$-BS

We will now show that the hierarchy of $\mathbf{\Gamma}$-DS problems does not collapse at any finite level. First we study the hierarchy of $\mathbf{\Delta}^0_k$-DS problems by characterizing their first-order parts (Theorem 5.36). Then we prove the analogues of Theorem 5.29 and Theorem 5.31 for $\mathbf{\Delta}^0_k$-DS (Proposition 5.39).

In the following, we will use the countable coproduct of multi-valued functions, defined in Section 2.1.1.

Theorem 5.36:
For every $k \geq 1$,

$$^1\mathbf{\Delta}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) * \mathbf{\Pi}^1_1\text{−}\mathsf{Bound}.$$

We split the proof into two lemmas.

Lemma 5.37:
For every $k \geq 1$, if $f :\subseteq X \rightrightarrows \mathbb{N}$ and $f \leq_{\mathrm{W}} \mathbf{\Delta}^0_k$-$\mathsf{DS}$ then

$$f \leq_{\mathrm{W}} \left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) * \mathbf{\Pi}^1_1\text{−}\mathsf{Bound}.$$

Proof: Fix Turing functionals Φ and Ψ which witness that $f \leq_{\mathrm{W}} \mathbf{\Delta}^0_k$-$\mathsf{DS}$. Given an f-instance with name x, Φ^x is a $\Delta^{0,x}_k$-code for the linear order $\leq^x$. Consider the $\Sigma^{0,x}_k$ set

$$\begin{aligned} D := \{F \in \mathbb{N} : \ & F \text{ codes a non-empty finite } <^x \text{-descending sequence and} \\ & \Psi^{x\oplus F} \text{ outputs some } j \in \mathbb{N}\}. \end{aligned}$$

We can uniformly express D as the increasing union over $s \in \mathbb{N}$ of finite sets $D_s \subseteq \{0, \dots, s\}$, which are uniformly $\Pi^{0,x}_{k-1}$.

We now define the set

$$A := \{s \in \mathbb{N} : (\forall F \in D_s)(F \notin \mathrm{Ext}_x)\},$$

where Ext_x is the set of finite sequences that extend to an infinite $<^x$-descending sequence. It is easy to see that A is $\Pi^{1,x}_1$, as being extendible in a Δ^0_k-linear order is a Σ^1_1 property.

We show that A is finite. Since $\leq^x$ is a $\mathbf{\Delta}^0_k$-DS-instance, we can fix an infinite $<^x$-descending sequence S. By definition of Weihrauch reducibility, $\Psi^{x\oplus S}$ outputs some f-solution $j \in \mathbb{N}$. By the continuity of Ψ, there is some finite non-empty initial segment F of S such that $\Psi^{x\oplus F}$ outputs j. Hence for all sufficiently large s, we have $F \in D_s$.

This shows that we can apply $\mathbf{\Pi}^1_1\mathsf{-Bound}$ to A to obtain some $b \in \mathbb{N}$ which bounds A. Note that D_b must be nonempty. We now define the following non-empty subset of D_b:

$$B := \left\{F \in D_b \,:\, (\forall G \in D_b)\left(\min_{<^x}(G) \le^x \min_{<^x}(F)\right)\right\}.$$

Notice that all the quantifications are bounded. In particular, B is a (non-empty) $\Delta^{0,x}_k$ subset of D_b because D_b is $\Pi^{0,x}_{k-1}$ and $\le^x$ is $\Delta^{0,x}_k$. Notice also that the definition of B ensures that each of its elements is extendible (as we know that there is some extendible element in D_b). In particular, this shows that, for every $F \in B$, it is enough to run $\Psi^{x\oplus F}$ to compute an f-solution for the original instance. We can find such $F \in B$ by applying $\left(\bigsqcup_s \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right)(b, B)$. ■

Notice that $(\bigsqcup_s \mathbf{\Delta}^0_1\text{-}\mathsf{C}_s)$ is computable, hence in case $k = 1$ we obtain Proposition 5.5.

Lemma 5.38:
For every $k \ge 1$,

$$\left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) * \mathbf{\Pi}^1_1\mathsf{-Bound} \le_{\mathrm{W}} \mathbf{\Delta}^0_k\text{-}\mathsf{DS}.$$

PROOF: Using the cylindrical decomposition we can write

$$\left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) * \mathbf{\Pi}^1_1\mathsf{-Bound} \equiv_{\mathrm{W}} \left(\left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) \times \mathrm{id}\right) \circ \Phi_e \circ (\mathbf{\Pi}^1_1\mathsf{-Bound} \times \mathrm{id})$$

for some computable map Φ_e. Let Φ_1, Φ_2 be computable maps s.t. $\Phi_e(p) = \langle \Phi_1(p), \Phi_2(p)\rangle$. Then we have

$$\left(\left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) \times \mathrm{id}\right) \circ \Phi_e \circ (\mathbf{\Pi}^1_1\mathsf{-Bound} \times \mathrm{id})(\langle p_1, p_2\rangle) =$$
$$\langle \left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s\right) \Phi_1(\mathbf{\Pi}^1_1\mathsf{-Bound}(p_1), p_2), \Phi_2(\mathbf{\Pi}^1_1\mathsf{-Bound}(p_1), p_2)\rangle.$$

Given an instance $\langle p_1, p_2\rangle$ of the above composition, we can think of p_1 as coding an input A to $\mathbf{\Pi}^1_1\mathsf{-Bound}$ via a tree T s.t. for each i, $i \in A$ iff the subtree $T_i := \{\sigma \in T \,:\, \sigma(0) = i\}$ of T is well-founded. For any $b \in \mathbf{\Pi}^1_1\mathsf{-Bound}(p_1)$, $\Phi_1(b, p_2)$ must be a name for an instance of $\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^0_k\text{-}\mathsf{C}_s$. Then $\pi_1\Phi_1(b, p_2)$ is a number s and $\pi_2\Phi_1(b, p_2)$ is a $\mathbf{\Delta}^0_k$-name for a non-empty subset A_s of $\{0, \dots, s-1\}$, where $\pi_i(\langle p_1, p_2\rangle) = p_i$ denotes the projection on the i-th component. Regardless of whether $b \in \mathbf{\Pi}^1_1\mathsf{-Bound}(p_1)$, we will interpret $\pi_1\Phi_1(b, p_2)$ and $\pi_2\Phi_1(b, p_2)$ as above.

We define a $\Delta^{0,\langle p_1,p_2\rangle}_k$ linear order as follows. First define

$$\begin{aligned} L := \{(\sigma, n) \,:\, & \sigma \in p_1 \text{ and} \\ & \pi_1\Phi_1(\sigma(0), p_2) \text{ outputs a number in less than } |\sigma| \text{ steps and} \\ & n \text{ lies in the set named by } \pi_2\Phi_1(\sigma(0), p_2)\}. \end{aligned}$$

We order the elements of L by

$$(\sigma, n) \leq_L (\tau, m) :\iff \sigma <_{\mathrm{KB}} \tau \vee (\sigma = \tau \wedge n \leq m).$$

It is easy to see that $(L, \leq_L)$ is $\Delta_k^{0,\langle p_1, p_2\rangle}$. Notice that it is a linear order, as the pairs are ordered lexicographically where the first components are ordered according to the Kleene-Brouwer order on $\mathbb{N}^{<\mathbb{N}}$ and the second components are ordered according to the order on $\mathbb{N}$.

Let $(q_i)_{i\in\mathbb{N}}$ be an $<_L$-descending sequence, with $q_i = (\sigma_i, n_i)$. Notice that for each i there is a $j > i$ s.t. $\sigma_j <_{\mathrm{KB}} \sigma_i$. Indeed, if there is an i s.t. for all $j > i$ we have $\sigma_j = \sigma_i$ then, by definition of $\leq_L$, the sequence $(n_j)_{j>i}$ would be a descending sequence in the natural numbers, which is impossible.

This implies that there is a subsequence $(q_{i_k})_{k\in\mathbb{N}}$ s.t. $(\sigma_{i_k})_{k\in\mathbb{N}}$ is a $<_{\mathrm{KB}}$-descending sequence. In particular, this implies that $T_{\sigma_0(0)}$ is ill-founded, i.e. $\sigma_0(0) \in \mathbf{\Pi}_1^1\!-\!\mathsf{Bound}(p_1)$. Moreover, by definition of L, this implies that n_0 lies in the set named by $\pi_2\Phi_1(\sigma_0(0), p_2)$.

In other words, given an $<_L$-descending sequence $(q_i)_{i\in\mathbb{N}}$ we have that

$$(\pi_1 q_0)(0) \in \mathbf{\Pi}_1^1\!-\!\mathsf{Bound}(p_1) \ ,$$

$$\pi_2 q_0 \in \left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}_k^0\text{-}\mathsf{C}_s\right) \Phi_1(\mathbf{\Pi}_1^1\!-\!\mathsf{Bound}(p_1), p_2) \ .$$

From this we can compute $\Phi_2(\pi_1 q_0, p_2)$ as well. This establishes the desired reduction. ∎

This completes the proof of Theorem 5.36.

With a small modification of the argument in the proof of Lemma 5.37 we can prove the following:

Proposition 5.39:
Fix $k \geq 1$. For every $f :\subseteq X \rightrightarrows \mathbb{N}$,

$$f \leq_{\mathrm{W}} \mathbf{\Delta}_k^0\text{-}\mathsf{DS} \iff f \leq_{\mathrm{W}} \mathbf{\Pi}_1^1\!-\!\mathsf{Bound} \times \mathsf{lim}^{[k-1]}.$$

If, in particular, f has codomain N for some $N \geq 1$ then

$$f \leq_{\mathrm{W}} \mathbf{\Delta}_k^0\text{-}\mathsf{DS} \iff f \leq_{\mathrm{W}} \mathsf{RT}_N^1 * \mathsf{lim}^{[k-1]}.$$

If, additionally, f is single-valued, then

$$f \leq_{\mathrm{W}} \mathbf{\Delta}_k^0\text{-}\mathsf{DS} \iff f \leq_{\mathrm{W}} \mathsf{lim}_N * \mathsf{lim}^{[k-1]}.$$

Proof: The right-to-left implication follows from Proposition 5.7 and Theorem 5.34:

$$\begin{aligned}\mathbf{\Pi}_1^1\!-\!\mathsf{Bound} \times \mathsf{lim}^{[k-1]} &\leq_{\mathrm{W}} \mathbf{\Pi}_1^1\!-\!\mathsf{Bound} * \mathsf{lim}^{[k-1]} \\ &\leq_{\mathrm{W}} \mathsf{DS} * \mathsf{lim}^{[k-1]} \equiv_{\mathrm{W}} \mathbf{\Delta}_k^0\text{-}\mathsf{DS}.\end{aligned}$$

To prove the left-to-right implication, fix a pair of Turing functionals Φ and Ψ witnessing the reduction $f \le_{\mathrm{W}} \boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}$. Fix an f-instance with name x and let $\le^x$ be the $\Delta^{0,x}_k$ linear order defined by Φ^x.

Define D, D_s and A as in the proof of Lemma 5.37. In that proof, we applied $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$ to A to obtain $b \in \mathbb{N}$. Then we restricted our attention to $B \subseteq D_b$. Here we will still apply $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}$ to A, but we will concurrently consider a subset B_s of each D_s. For each s, define

$$B_s := \left\{ F \in D_s \,:\, (\forall G \in D_s) \left(\min_{<^x}(G) \le^x \min_{<^x}(F) \right) \right\},$$
$$F_s := \min B_s,$$

where F_s is intended to be the empty sequence if B_s (and hence D_s) is empty.

Notice that $B_s \subset \{0, \dots, s-1\}$ is $\boldsymbol{\Delta}^0_k$ (as each D_s is $\boldsymbol{\Pi}^0_{k-1}$) and therefore F_s is $\boldsymbol{\Delta}^0_k$. Since $\mathsf{lim}^{[k-1]} \equiv_{\mathrm{W}} \boldsymbol{\Delta}^0_k\text{-}\mathsf{CA}$, it can determine which B_s is nonempty, and compute F_s if B_s is nonempty. Therefore the sequence $(F_s)_{s\in\mathbb{N}}$ can be computed using $\mathsf{lim}^{[k-1]}$. For every $b \in \boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound}(A)$ we have that F_b is extendible to an infinite $<^x$-descending sequence and that $\Psi^{x\oplus F_s}$ converges to some f-solution j (see also the proof of Lemma 5.37).

Assume now that f has codomain N for some $N \ge 1$. We can modify the above argument as follows: after computing the sequence $(F_s)_{s\in\mathbb{N}}$, we consider the RT^1_N-instance c defined as

$$c(s) := \begin{cases} 0 & \text{if } F_s = (), \\ \Psi^{x\oplus F_s}(0) & \text{otherwise.} \end{cases}$$

Since F_s is nonempty and extendible for cofinitely many s, if $c(s) = i$ for infinitely many s (i.e., c has an RT^1_N-solution of color i), then there is an extendible F_s s.t. $\Psi^{x\oplus F_s}(0) = i$, hence i is an f-solution.

If, additionally, f is single-valued, then there is only one possible i s.t. c has a homogeneous solution with color i. This shows that the sequence $(c(s))_{s\in\mathbb{N}}$ has a limit, and therefore it suffices to use lim_N to get the solution.

The fact that $\mathsf{RT}^1_N * \mathsf{lim}^{[k-1]}$ and $\mathsf{lim}_N * \mathsf{lim}^{[k-1]}$ are reducible to $\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}$ follows from the fact that the compositional product is a degree theoretic operation, as $\mathsf{RT}^1_N \le_{\mathrm{W}} \mathsf{DS}$ (Theorem 5.29), $\mathsf{lim}_N \le_{\mathrm{W}} \mathsf{DS}$ (Theorem 5.15) and $\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{DS} * \mathsf{lim}^{[k-1]}$ (Theorem 5.34). ■

Notice that $\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound} \times \mathsf{lim}^{[k-1]}$ is not a first-order problem, so the first statement in Proposition 5.39 is not an alternative characterization of ${}^1\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}$. It can be rephrased as

$${}^1\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} {}^1(\boldsymbol{\Pi}^1_1\text{−}\mathsf{Bound} \times \mathsf{lim}^{[k-1]}).$$

This concludes our discussion of the first-order problems that are Weihrauch reducible to $\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}$. As for the deterministic part of $\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}$:

Corollary 5.40:
For every $k \ge 1$, $\mathrm{Det}(\boldsymbol{\Delta}^0_k\text{-}\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}^{[k]}$.

Proof: This follows from $\mathrm{Det}(\mathsf{DS}) \equiv_{\mathrm{W}} \mathsf{lim}$ (Theorem 5.15) and the fact that, for cylinders, the jump commutes with the deterministic part (Corollary 4.61). ■

Theorem 5.41:
For every $k \geq 1$,

$$\mathbf{\Delta}^0_k\text{-}\mathsf{DS} <_{\mathrm{W}} \mathbf{\Delta}^0_{k+1}\text{-}\mathsf{DS}.$$

In particular this shows that the $\mathbf{\Gamma}$-DS-hierarchy does not collapse at any finite level.

Proof: This follows directly from Proposition 5.39 or, alternatively, from Corollary 5.40. Indeed it suffices to notice that, for every $k \geq 1$, $\mathsf{LPO}^{(k)} \leq_{\mathrm{W}} \mathsf{lim}^{[k+1]}$ but $\mathsf{LPO}^{(k)} \not\leq_{\mathrm{W}} \mathsf{lim}^{[k]}$, as $\mathsf{LPO}^{(k)}$ is the characteristic function of a Σ^0_{k+1}-complete set while $\mathsf{lim}^{[k]}$ is Σ^0_{k+1}-measurable. ■

Theorem 5.42:
For every $k \geq 1$, $\mathbf{\Delta}^0_{k+1}\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Pi}^0_k\text{-}\mathsf{DS}$.

Proof: The right-to-left reduction is trivial. To prove the left-to-right one it suffices to show that $\mathbf{\Delta}^0_1\text{-}\mathsf{DS}' \equiv_{\mathrm{W}} \mathbf{\Pi}^0_1\text{-}\mathsf{DS}$ and the proof will follow from Theorem 5.34 as

$$\mathbf{\Delta}^0_{k+1}\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_1\text{-}\mathsf{DS}' * \mathsf{lim}^{[k-1]} \equiv_{\mathrm{W}} \mathbf{\Pi}^0_1\text{-}\mathsf{DS} * \mathsf{lim}^{[k-1]} \equiv_{\mathrm{W}} \mathbf{\Pi}^0_k\text{-}\mathsf{DS}.$$

Let $p = (p_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^\mathbb{N}$ converging to the characteristic function of an ill-founded linear order L. In the following it is convenient to consider also the sequence $q = (q_n)_{n\in\mathbb{N}}$, where $q_n(i) := p_n(\langle i, i\rangle)$. Clearly q converges to the characteristic function of $\mathrm{dom}(L)$ and is uniformly computable from p.

For sake of readability, define the formula

$$\varphi((x_n)_{n\in\mathbb{N}}, \sigma) := (\forall i < |\sigma|)(x_{\sigma(i)}(i) \neq x_{\sigma(i)+1}(i) \land (\forall j > \sigma(i))(x_j(i) = x_{j+1}(i))).$$

Intuitively φ says that, for each $i < |\sigma|$, $\sigma(i)$ codes the positions in which the sequence $(x_n)_{n\in\mathbb{N}}$ changes for the last time in the i-th row. Let us also write $x_\sigma := |\sigma| - 1$. We define

$$\begin{aligned} M := \{(\sigma, \tau) \in \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}} \ : \ & \varphi(q, \sigma) \land \\ & q_{\sigma(x_\sigma)+1}(x_\sigma) = 1 \land \\ & \varphi(p, \tau) \land |\tau| = \langle x_\sigma, x_\sigma\rangle + 1\} \end{aligned}$$

Notice that the first two conditions imply that $x_\sigma \in L$. Intuitively x_σ is the $\leq_\mathbb{N}$-largest element that is witnessed by σ to enter in L. The last line says that τ is exactly as long as needed to witness all the relations between the elements of L that are $\leq_\mathbb{N} x_\sigma$.

We order the set M as follows:

$$(\sigma_0, \tau_0) \leq_M (\sigma_1, \tau_1) :\iff x_{\sigma_0} \leq_L x_{\sigma_1}$$

Notice that M is a $\Pi^{0,p}_1$ linear order as M is $\Pi^{0,p}_1$ and the order $\leq_M$ is p-computable: indeed, given two pairs $(\sigma_0, \tau_0), (\sigma_1, \tau_1) \in M$, we can use the longer string between τ_0 and τ_1 to p-compute whether $x_{\sigma_0} \leq_L x_{\sigma_1}$. Notice also that, for each l, there is exactly one string σ of length l witnessing $\varphi(q, \sigma)$ (by minimality). The third line in the definition of M implies that if σ satisfies the first two conditions then there is a unique τ s.t. $(\sigma, \tau) \in M$. The linearity of M follows by the linearity of L.

To conclude the proof it is enough to notice that if $((\sigma_i, \tau_i))_{i\in\mathbb{N}}$ is an $<_M$-descending sequence then $(x_{\sigma_i})_{i\in\mathbb{N}}$ is an $<_L$-descending sequence. ■

The following is essentially a classical result (see e.g. [29, Thm. 2.4]). The proof is simple enough that we can briefly sketch it.

Theorem 5.43:
For every $k \geq 1$, $\boldsymbol{\Sigma}^0_k$-$\mathsf{DS} \equiv_{\mathrm{W}} \boldsymbol{\Delta}^0_k$-$\mathsf{DS}$.

Proof: Given a $\boldsymbol{\Sigma}^0_k$ linear order L, we can uniformly consider a sequence $((L_s, \leq_s))_{s\in\mathbb{N}}$ of $\boldsymbol{\Delta}^0_k$ linear orders approximating L. We then define

$$M := \{(q, s) : q \in L_s \text{ and } (\forall t < s)(q \notin L_t)\},$$
$$(p, s) \leq_M (q, t) :\iff p \leq_L q.$$

Notice that $(p, s) \leq_M (q, t)$ can be written also as $p = q \lor (\forall i)(q \not\leq_i p)$, hence M is $\Delta^{0,L}_k$. Moreover, since for every $q \in L$ there is a unique s s.t. $(q, s) \in M$, it is easy to see that M is computably isomorphic to L. In particular, given an $<_M$-descending sequence we can obtain an $<_L$-descending sequence by projection. ■

Corollary 5.44:
For every $k \geq 1$, we have

$$\boldsymbol{\Pi}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \boldsymbol{\Pi}^0_k\text{-}\mathsf{BS} \equiv_{\mathrm{W}} \boldsymbol{\Delta}^0_{k+1}\text{-}\mathsf{BS} \equiv_{\mathrm{W}} \boldsymbol{\Delta}^0_{k+1}\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^0_{k+1}\text{-}\mathsf{DS}.$$

Proof: It is straightforward to see that $\boldsymbol{\Pi}^0_k$-$\mathsf{DS} \leq_{\mathrm{W}} \boldsymbol{\Pi}^0_k$-$\mathsf{BS} \leq_{\mathrm{W}} \boldsymbol{\Delta}^0_{k+1}$-$\mathsf{BS}$. By Corollary 5.35, $\boldsymbol{\Delta}^0_{k+1}$-$\mathsf{BS} \equiv_{\mathrm{W}} \boldsymbol{\Delta}^0_{k+1}$-$\mathsf{DS}$. It follows from Theorem 5.42 that the first four problems are equivalent. Finally, $\boldsymbol{\Delta}^0_{k+1}$-$\mathsf{DS} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^0_{k+1}$-$\mathsf{DS}$ by Theorem 5.43. ■

Theorem 5.45:
For every $k \geq 1$, $\mathsf{LPO}^{(k)} \leq_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{BS}$ and therefore $\mathbf{\Sigma}^0_k\text{-}\mathsf{BS} \not\leq_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{DS}$.

Proof: The second statement follows from the first because $\mathsf{LPO}^{(k)} \not\leq_{\mathrm{W}} \mathbf{\Delta}^0_k\text{-}\mathsf{DS}$ (proof of Theorem 5.41) and $\mathbf{\Delta}^0_k\text{-}\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{DS}$ (Theorem 5.43).

To prove the first statement, it is enough to show that $\mathsf{LPO}' \leq_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{-}\mathsf{BS}$, and the claim will follow by Theorem 5.34 as

$$\mathsf{LPO}^{(k)} \leq_{\mathrm{W}} \mathsf{LPO}' * \mathsf{lim}^{[k-1]} \leq_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{-}\mathsf{BS} * \mathsf{lim}^{[k-1]} \equiv_{\mathrm{W}} \mathbf{\Sigma}^0_k\text{-}\mathsf{BS}.$$

Let $(p_s)_{s\in\mathbb{N}}$ be a sequence in $\mathbb{N}^\mathbb{N}$ converging to an instance p of LPO. For every $s \in \mathbb{N}$ we define (as we did in the proofs of Theorem 5.17 and Proposition 5.19)

$$g(s) = \begin{cases} i+1 & \text{if } i \leq s \land p_s(i) \neq 0 \land (\forall j < i)(p_s(j) = 0), \\ 0 & \text{otherwise.} \end{cases}$$

Let us define a quasi-order Q inductively: at stage $s = 0$ we add $\langle g(0), 0\rangle$. At stage $s+1$ we do the following:

1. if $g(s) = g(s+1)$ we put $\langle g(s), s+1\rangle$ immediately below $\langle g(s), s\rangle$;
2. if $g(s) \neq g(s+1)$ we put $\langle g(s+1), s+1\rangle$ at the top and we put $\langle -1, s+1\rangle$ at the bottom. Moreover we collapse to a single equivalence class all the elements $\langle g, t\rangle$ with $t \leq s$ and $g \neq -1$.

This construction produces a quasi-order $(Q, \preceq_Q)$ which is computable in $(p_s)_{s\in\mathbb{N}}$.

Notice that if there is an s s.t. for every $t \geq s$, $g(t) = g(s)$ (in particular, this is the case if $\mathsf{LPO}(p) = 1$) then the equivalence classes of $\preceq_Q$ form a linear order of type $n+\omega^*$ and every $\preceq_Q$-bad sequence is a descending sequence of the form $(\langle g(s), s_n\rangle)_{n\in\mathbb{N}}$ for some strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$. On the other hand, if the sequence $(g(s))_{s\in\mathbb{N}}$ does not stabilize then the equivalence classes of $\preceq_Q$ are linearly ordered as ω^*, where all the elements $\langle g, s\rangle$ with $g \neq -1$ are equivalent and lie in the top equivalence class. This shows that the construction produces a non-well quasi-order.

For every $\preceq_Q$-bad sequence $(\langle g_n, s_n\rangle)_{n\in\mathbb{N}}$ produced by $\mathbf{\Sigma}^0_1\text{-}\mathsf{BS}(Q)$, we compute the solution for $\mathsf{LPO}'((p_s)_{s\in\mathbb{N}}) = \mathsf{LPO}(p)$ by returning 0 if $g_1 \leq 0$ and 1 otherwise. We consider two cases. If the sequence $(g(s))_{s\in\mathbb{N}}$ stabilizes, then the sequence $(g_n)_{n\in\mathbb{N}}$ is constant. Furthermore, its value is 0 if $\mathsf{LPO}(p) = 0$, otherwise its value is positive. On the other hand, if the sequence $(g(s))_{s\in\mathbb{N}}$ does not stabilize, then $\mathsf{LPO}(p) = 0$. Furthermore, for every $n > 0$, we have $g_n = -1 \leq 0$. (The first element $\langle g_0, s_0\rangle$ may lie in the top equivalence class, in which case g_0 may be positive. Hence we check g_1 instead of g_0). ■

5.2.2 $\mathbf{\Gamma}^1_1$-DS and $\mathbf{\Gamma}^1_1$-BS

We now turn our attention to the analytic classes. Notice first of all that being a descending sequence through a $\mathbf{\Sigma}^1_1$ linear order is a $\mathbf{\Sigma}^1_1$-property, hence $\mathbf{\Sigma}^1_1\text{-}\mathsf{DS} \leq_{\mathrm{W}} \mathbf{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. We will show that $\mathbf{\Sigma}^1_1$-DS is the strongest DS-principle that is still reducible to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (Theorem 5.56).

Proposition 5.46:
$\mathbf{\Delta}^1_1$-$\mathsf{DS} \equiv_{\mathrm{W}} \mathsf{DS} * \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathbf{\Delta}^1_1$-$\mathsf{BS} \equiv_{\mathrm{W}} \mathsf{BS} * \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

Proof: We will only prove the first statement. The proof of the second statement is similar.

To prove the left-to-right reduction, given a $\mathbf{\Delta}^1_1$ name for L we use $\mathbf{\Delta}^1_1$-CA (which is known to be equivalent to $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$, see [64, Thm. 3.11]) to compute a $\mathbf{\Delta}^0_1$ name for L. We can then apply DS to find a descending sequence through L.

To prove the converse reduction, using the cylindrical decomposition we can write

$$\mathsf{DS} * \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{DS} \circ \Phi_e \circ \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$$

for some computable function Φ_e. In particular, given $T \subset \mathbb{N}^{<\mathbb{N}}$ with a unique path x, $\Phi_e(x)$ is the characteristic function of a linear order L. Notice that x is $\Delta^{1,T}_1$-computable. Indeed,

$$\begin{aligned} x(n) = k &\iff (\exists \sigma \in T)(\sigma \in \mathrm{Ext} \wedge \sigma(n) = k) \\ &\iff (\forall \tau \in T)(\tau \in \mathrm{Ext} \rightarrow \tau(n) = k), \end{aligned}$$

where Ext is the set of finite strings that extend to a path through T ($\sigma \in \mathrm{Ext}$ is a $\Sigma^{1,T}_1$ property). We can therefore obtain a $\Delta^{1,T}_1$ name for L as

$$a \le_L b \iff \Phi_e(x)(\langle a, b\rangle) = 1,$$

and hence we use $\mathbf{\Delta}^1_1$-DS to find a descending sequence through L. ■

In particular, this implies that $\mathbf{\Delta}^1_1$ is the first level at which we can compute $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$. Indeed, for every k, we showed in the proof of Theorem 5.41 that $\mathsf{LPO}^{(k)} \not\le_{\mathrm{W}} \mathbf{\Delta}^0_k$-$\mathsf{DS}$, while $\mathsf{lim}^{[k]} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (see [11, Sec. 6]).

By adapting the proof of Corollary 5.35, we can relativize Proposition 5.5 and obtain the following:

Corollary 5.47:
$\mathbf{\Delta}^1_1$-$\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Delta}^1_1$-$\mathsf{BS}$.

Similarly, the proofs of Theorem 5.36 and of Proposition 5.39 lead to the following equivalences:

Theorem 5.48:

$${}^1\mathbf{\Delta}^1_1\text{-}\mathsf{DS} \equiv_{\mathrm{W}} {}^1(\mathbf{\Pi}^1_1\text{–}\mathsf{Bound} \times \mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \left(\bigsqcup_{s\in\mathbb{N}} \mathbf{\Delta}^1_1\text{-}\mathsf{C}_s\right) * \mathbf{\Pi}^1_1\text{–}\mathsf{Bound}.$$

The deterministic part of $\boldsymbol{\Delta}^1_1$-DS and $\boldsymbol{\Sigma}^1_1$-DS can be easily characterized using Proposition 5.46, as the following proposition shows.

Proposition 5.49:
$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathrm{Det}(\boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}) \equiv_{\mathrm{W}} \mathrm{Det}(\boldsymbol{\Sigma}^1_1\text{-}\mathsf{DS})$.

Proof: The reductions $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \mathrm{Det}(\boldsymbol{\Delta}^1_1\text{-}\mathsf{DS}) \le_{\mathrm{W}} \mathrm{Det}(\boldsymbol{\Sigma}^1_1\text{-}\mathsf{DS})$ are straightforward from $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \boldsymbol{\Delta}^1_1$-$\mathsf{DS}$ (Proposition 5.46), $\boldsymbol{\Delta}^1_1$-$\mathsf{DS} \le_{\mathrm{W}} \boldsymbol{\Sigma}^1_1$-$\mathsf{DS}$ (trivial) and the fact that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is single-valued. To prove that $\mathrm{Det}(\boldsymbol{\Sigma}^1_1\text{-}\mathsf{DS}) \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ it is enough to notice that $\boldsymbol{\Sigma}^1_1$-$\mathsf{DS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, and therefore $\mathrm{Det}(\boldsymbol{\Sigma}^1_1\text{-}\mathsf{DS}) \le_{\mathrm{W}} \mathrm{Det}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ (Theorem 4.64). ■

In particular, the deterministic part does not help us separate $\boldsymbol{\Delta}^1_1$-DS and $\boldsymbol{\Sigma}^1_1$-DS. Instead, we separate them by considering their first-order parts. We characterized ${}^1\boldsymbol{\Delta}^1_1$-$\mathsf{DS}$ in Theorem 5.48. Notice that our proof (see the proof of Proposition 5.39) cannot be extended to establish the same result for $\boldsymbol{\Sigma}^1_1$-DS, because the definition of the corresponding $(F_s)_{s\in\mathbb{N}}$ would not be $\boldsymbol{\Sigma}^1_1$.

Proposition 5.50:
$\boldsymbol{\Sigma}^1_1$-$\mathsf{C}_\mathbb{N} \le_{\mathrm{W}} \boldsymbol{\Sigma}^1_1$-$\mathsf{DS}$.

Proof: Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of non-empty $\boldsymbol{\Sigma}^1_1$ subsets of $\mathbb{N}$. We define

$$L := \{(n,\sigma) \in \mathbb{N}\times\mathbb{N}^{<\mathbb{N}} \,:\, |\sigma| = n \land (\forall i < n)(\sigma(i) \in A_i)\},$$
$$(n,\sigma) \le_L (m,\tau) \iff n > m \lor (n = m \land \sigma \le_{lex} \tau).$$

It is easy to see that L is a $\boldsymbol{\Sigma}^1_1$ linear order (the linearity follows from the linearity of $\le$ and of $\le_{lex}$).

Let $((n_i,\sigma_i))_{i\in\mathbb{N}}$ be an $<_L$-descending sequence. Notice that, since each $A_i \subset \mathbb{N}$, for each n the set $\{\sigma \in \mathbb{N}^{<\mathbb{N}} : (n,\sigma) \in L\}$ is $\le_{lex}$-well-founded. Therefore there must be a subsequence $((n_{i_k},\sigma_{i_k}))_{k\in\mathbb{N}}$ s.t. the sequence $(n_{i_k})_{k\in\mathbb{N}}$ is strictly increasing.

This implies that, for each n, there is some m s.t. $|\sigma_m| \ge n$. In particular, by definition of L, $(\forall i < n)(\sigma_m(i) \in A_i)$ and the claim follows. ■

Proposition 5.50 implies that $\boldsymbol{\Sigma}^1_1$-$\mathsf{C}_\mathbb{N} \le_{\mathrm{W}} {}^1\boldsymbol{\Sigma}^1_1$-$\mathsf{DS}$. This, together with ${}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1$-$\mathsf{C}_\mathbb{N}$ (Proposition 4.37) and the observation that $\boldsymbol{\Sigma}^1_1$-$\mathsf{DS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, immediately yields the following:

Corollary 5.51:
${}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} {}^1\boldsymbol{\Sigma}^1_1$-$\mathsf{DS} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1$-$\mathsf{C}_\mathbb{N}$.

As a consequence, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ and $\boldsymbol{\Sigma}^1_1$-DS cannot be separated by means of their first-order part. But $\boldsymbol{\Delta}^1_1$-DS and $\boldsymbol{\Sigma}^1_1$-DS can, albeit somewhat indirectly:

Proposition 5.52:
$\mathbf{\Delta}^1_1$-$\mathsf{DS} <_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{DS}$.

PROOF: Notice first of all that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \widehat{\mathbf{\Pi}^1_1\text{−}\mathsf{Bound}}$. Indeed, given a tree $T \subset \mathbb{N}^{<\mathbb{N}}$ with a unique path, we can consider the following sequence of $\Pi^{1,T}_1$ sets:

$$A_n := \{k \in \mathbb{N} : (\forall \sigma \in T)((\exists x \in [T])(\sigma \sqsubset x) \to \sigma(n) \le k)\}.$$

Clearly each A_n is bounded by $x(n)$, where x is the unique path through T. Given $f \in \widehat{\mathbf{\Pi}^1_1\text{−}\mathsf{Bound}}((A_n)_{n\in\mathbb{N}})$, consider the space $X := \{\sigma \in \mathbb{N}^{<\mathbb{N}} : (\forall i < |\sigma|)(\sigma(i) \le f(i))\}$ and define $T_f := T \cap X$. Notice that $[T_f] = [T]$. In particular, since $[X]$ is f-computably compact, we can uniformly (in f) compute the unique path through $[T_f]$ (see [17, Thm. 7.23 and Cor. 7.26]). Notice that the reduction is actually strict, as $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \le_{\mathrm{W}} \widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_2}$ ([64, Lem. 4.4 and lem. 4.6]), while $\mathbf{\Pi}^1_1\text{−}\mathsf{Bound} \not\le_{\mathrm{W}} \widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_2}$ ([2, Cor. 3.23]).

If $\widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} \le_{\mathrm{W}} \mathbf{\Delta}^1_1$-$\mathsf{DS}$ then, by Theorem 5.48, $\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N} \le_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^\mathbb{N}} \times \mathbf{\Pi}^1_1\text{−}\mathsf{Bound}$ and therefore

$$\widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} \le_{\mathrm{W}} (\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \times \widehat{\mathbf{\Pi}^1_1\text{−}\mathsf{Bound}}) \equiv_{\mathrm{W}} \widehat{\mathbf{\Pi}^1_1\text{−}\mathsf{Bound}},$$

contradicting $\widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}} \not\le_{\mathrm{W}} \widehat{\mathbf{\Pi}^1_1\text{−}\mathsf{Bound}}$ ([2, Cor. 3.23]). ■

To separate $\mathbf{\Sigma}^1_1$-DS from $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ we generalize a technique based on inseparable Π^1_1 sets, first used in [2] to separate $\widehat{\mathbf{\Sigma}^1_1\text{-}\mathsf{C}_\mathbb{N}}$ from $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$. Our result is, in fact, slightly stronger (by Proposition 5.50).

Recall that, for every $A, B \subset \mathbb{N}^\mathbb{N}$, we say that A is Muchnik reducible to B, and write $A \le_w B$ if, for every $b \in B$ there is a Turing functional Φ_e s.t. $\Phi_e(b) \in A$.

Theorem 5.53:
$\mathsf{ATR}_2 \mid_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{DS}$, *and therefore* $\mathbf{\Sigma}^1_1$-$\mathsf{DS} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

PROOF: The fact that $\mathbf{\Sigma}^1_1$-$\mathsf{DS} \not\le_{\mathrm{W}} \mathsf{ATR}_2$ follows from the fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \lim * \mathbf{\Sigma}^1_1$-$\mathsf{DS}$ while $\lim * \mathsf{ATR}_2 <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ ([44, Cor. 8.5]).

Let us now prove that $\mathsf{ATR}_2 \not\le_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{DS}$. Assume towards a contradiction that there is a reduction witnessed by the maps Φ, Ψ. Let $(L_e)_{e\in\mathbb{N}}$ be an enumeration of the computable linear orders. Define the sets

$$S_e := \mathbf{\Sigma}^1_1\text{-}\mathsf{DS}(\Phi(L_e)),$$
$$DS_e := \{(x_n)_n \in \mathbb{N}^\mathbb{N} : (x_n)_n \text{ is an } <_{L_e} \text{-descending sequence}\},$$
$$JH_e := \{(y_n)_n \in \mathbb{N}^\mathbb{N} : (y_n)_n \text{ is a jump hierarchy on } L_e\}.$$

Notice that, for each e, S_e is Σ^1_1 (being a descending sequence through a Σ^1_1 linear order is a Σ^1_1 condition) while DS_e and JH_e are arithmetic.

Define now the sets

$$B := \{e \in \mathbb{N} : DS_e \not\leq_w S_e\},$$
$$C := \{e \in \mathbb{N} : JH_e \not\leq_w S_e\},$$

where $\leq_w$ represents Muchnik reducibility. In particular, if X is (hyper)arithmetic and Y is Σ^1_1 then $X \not\leq_w Y$ is a Σ^1_1 condition, and therefore $B, C \in \Sigma^1_1(\mathbb{N})$.

We now claim that $B \cap C = \emptyset$. Indeed, assume by contradiction that this is not the case and let $e \in B \cap C$. By definition of B and C this means that there are two descending sequences $(q_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ in $\Phi(L_e)$ s.t. $(q_n)_{n\in\mathbb{N}}$ does not compute any $<_{L_e}$-descending sequence and $(p_n)_{n\in\mathbb{N}}$ does not compute any jump hierarchy on L_e.

In particular, if we run the backward functional Ψ on $(q_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ then, by continuity, there is an n s.t. $\Psi((q_i)_{i<n})$ commits to producing a jump hierarchy on L_e and $\Psi((p_i)_{i<n})$ commits to producing an $<_{L_e}$-descending sequence. W.l.o.g. assume that $q_n \leq_{\Phi(L_e)} p_n$ (in the opposite case we just swap the roles of $(q_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$) and consider the sequence

$$r := (p_0, \dots, p_n, q_{n+1}, q_{n+2}, \dots).$$

Notice that $\Psi(r)$ must produce an $<_{L_e}$-descending sequence, contradicting the fact that $(q_n)_{n\in\mathbb{N}}$ does not compute any $<_{L_e}$-descending sequence.

Let $\mathbf{wf_{LO}}$ be the set of indexes for the computable well-orderings and let **hds** be the set of indexes for computable linear orderings with a hyperarithmetic descending sequence. Notice that $\mathbf{wf_{LO}} \subset B$, because for each e in $\mathbf{wf_{LO}}$, $DS_e = \emptyset \not\leq_w A$ for every non-empty set A. Likewise, $\mathbf{hds} \subset C$, as any ill-founded linear order which has a hyperarithmetic descending sequence cannot support a jump hierarchy (see[1] [38, Thm. 4]).

Since B, C are disjoint and Σ^1_1, by Σ^1_1-separation there must be a Δ^1_1 set separating them. Such a set would separate $\mathbf{wf_{LO}}$ and **hds** as well. This contradicts the fact that every Σ^1_1 set which separates $\mathbf{wf_{LO}}$ and **hds** must be Σ^1_1-complete [42]. ■

Finally we turn our attention to $\mathbf{\Sigma}^1_1$-BS and $\mathbf{\Pi}^1_1$-DS. We show below that these problems are much stronger in uniform computational strength than the problems considered so far. Indeed all the $\mathbf{\Gamma}$-DS problems, where $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1$ or below, are s.t.

$$\mathbf{\Gamma}\text{-}\mathsf{DS} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{lim} * \mathbf{\Gamma}\text{-}\mathsf{DS}.$$

In other words, $\mathbf{\Gamma}$-DS is arithmetically Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$, which is prominent among the "ATR_0 analogues".

On the other hand, a natural analog of $\mathbf{\Pi}^1_1\mathrm{-CA}_0$ in the Weihrauch lattice is $\mathbf{\Pi}^1_1$-CA (see Section 2.1.2).

We can notice that, using [75, Thm. 6.5], $\mathbf{\Pi}^1_1$-CA is equivalent to the problem of finding the leftmost path through an ill-founded tree. Using this fact we show that $\mathbf{\Sigma}^1_1$-BS and $\mathbf{\Pi}^1_1$-DS are in the realm of $\mathbf{\Pi}^1_1\mathrm{-CA}_0$.

[1] Friedman's result assumes that the linear order is adequate. We do not need this assumption because we choose to define jump hierarchies in a way such that each column (whether limit or successor) uniformly computes earlier columns, such as in [44, Def. 3.1]. This allows us to run Friedman's proof without assuming adequacy.

Theorem 5.54:
$\mathbf{\Pi}^1_1$-$\mathsf{CA} \le_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{BS}$.

Proof: Let $T \subset \mathbb{N}^{<\mathbb{N}}$ be an ill-founded tree. For each $\sigma \in T$, let $T_\sigma := \{\tau \in T : \tau \sqsubseteq \sigma \vee \sigma \sqsubseteq \tau\}$. We define a quasi-order on the extendible strings in T:

$$Q := \{\sigma \in T : [T_\sigma] \neq \emptyset\},$$
$$\sigma \preceq_Q \tau :\Longleftrightarrow (\exists \rho \in Q)(\rho <_{lex} \sigma) \vee \tau \sqsubseteq \sigma.$$

It is easy to see that $(Q, \preceq_Q)$ is $\Sigma^{1,T}_1$. Moreover, all the σ which are not prefixes of the leftmost path collapse in a bottom equivalence class. This shows that the equivalence classes of Q are linearly ordered as $1 + \omega^*$. To conclude the proof it is enough to notice that any $<_Q$-descending sequence gives longer and longer prefixes of the leftmost path, hence it computes $\mathbf{\Pi}^1_1$-CA. ■

Corollary 5.55:
$\mathbf{\Sigma}^1_1$-$\mathsf{DS} <_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{BS}$.

Proof: We have $\mathbf{\Sigma}^1_1$-$\mathsf{DS} \le_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathbf{\Pi}^1_1$-$\mathsf{CA} \le_{\mathrm{W}} \mathbf{\Sigma}^1_1$-$\mathsf{BS}$. ■

Theorem 5.56:
$\mathbf{\Pi}^1_1$-$\mathsf{CA} \le_{\mathrm{W}} \mathbf{\Pi}^1_1$-$\mathsf{DS}$.

Proof: Let $T \subset \mathbb{N}^{<\mathbb{N}}$ be an ill-founded tree. For each $\sigma \in T$, let $T_\sigma := \{\tau \in T : \tau \sqsubseteq \sigma \vee \sigma \sqsubseteq \tau\}$. We define a linear order

$$L := \{\sigma \in T : (\forall \tau \le_{lex} \sigma)([T_\tau] = \emptyset \vee \tau \sqsubseteq \sigma)\},$$
$$\le_L := \le_{\mathrm{KB}(T)} .$$

Clearly $(L, \le_L)$ is a $\Pi^{1,T}_1$ linear order. Notice that if $\sigma \in L$ and $[T_\sigma] \neq \emptyset$ then σ must be a prefix of the leftmost path. Moreover if ρ is strictly lexicographically above the leftmost path then $\rho \notin L$. In other words, L is the subset of T that is lexicographically below the leftmost path.

Moreover, every string that is not a prefix of the leftmost path lies in the well-founded part of L (by definition of KB). In particular every $<_L$-descending sequence computes arbitrarily long prefixes of the leftmost path. ■

5.3 Conclusions

In this chapter we explored the uniform computational content of the problem DS, and showed how it lies "on the side" w.r.t. the part of the Weihrauch lattice explored so far. We now draw the attention to some of the questions that did not receive an answer.

Among the lower bounds for DS, the most interesting question is probably:

Question 5.57: $\mathsf{KL} \le_{\mathrm{W}} \mathsf{DS}$?

We know that, if such a reduction exists, it must be strict (as KL is an arithmetic problem). On the other hand, none of the characterizations we used in Section 5.1 to describe the lower cone of DS can be used to prove a separation.

In Section 4.3.4 we introduced the problem $\mathsf{wList}_{2^{\mathbb{N}},\le\omega}$. Similarly to DS, this problem does not fit well within the effective Baire hierarchy: $\mathrm{Det}(\mathsf{wList}_{2^{\mathbb{N}},\le\omega}) \equiv_{\mathrm{W}} \mathsf{lim}$, but $\mathsf{wList}^{[3]}_{2^{\mathbb{N}},\le\omega} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$ ([64, Prop. 6.14 and Cor. 6.16]), hence in particular $\mathsf{wList}_{2^{\mathbb{N}},\le\omega}$ is not arithmetic.

Question 5.58: $\mathsf{wList}_{2^{\mathbb{N}},\le\omega} \le_{\mathrm{W}} \mathsf{DS}$?

Our results imply that $\mathsf{DS} \not\le_{\mathrm{W}} \mathsf{wList}_{2^{\mathbb{N}},\le\omega}$ (as $\mathsf{DS} * \mathsf{DS} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$), and hence a reduction would be strict.

In the context of $\mathbf{\Gamma}$-DS, there are a few problems that resisted full characterization. In particular:

Question 5.59: $\mathbf{\Delta}^0_2$-$\mathsf{DS} \le_{\mathrm{W}} \mathbf{\Sigma}^0_1$-$\mathsf{BS}$?

We expect that an answer to this question will yield a solution for every k (by relativization).

We notice that, in the statements involving $\mathbf{\Gamma}$-BS we proved slightly more than what claimed: indeed, in all the reductions, the quasi-order built is a *linear quasi-order*, i.e. a quasi-order whose equivalence classes are linearly ordered. Notice that every bad sequence through a non-well linear quasi-order is actually a descending sequence. If we introduce the problem $\mathbf{\Gamma}$-DS_{LQO} by restricting $\mathbf{\Gamma}$-BS to linear quasi-orders, our results imply that

$$\mathbf{\Delta}^0_k\text{-}\mathsf{DS} <_{\mathrm{W}} \mathbf{\Sigma}^0_{k+1}\text{-}\mathsf{DS}_{LQO} \le_{\mathrm{W}} \mathbf{\Sigma}^0_{k+1}\text{-}\mathsf{BS}.$$

A natural question is therefore

Question 5.60: $\mathbf{\Sigma}^0_{k+1}$-$\mathsf{BS} \le_{\mathrm{W}} \mathbf{\Sigma}^0_{k+1}$-$\mathsf{DS}_{LQO}$?

A negative answer would imply that the possibility of having infinite antichains provides extra uniform strength.

A very important structure that is left out of the picture is the one of partial orders. In the same spirit of the paper we can consider the problems $\mathbf{\Gamma}$-DS_{PO} and $\mathbf{\Gamma}$-BS_{PO}. The former is readily seen to be equivalent to $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ (see also the comment before Definition 5.4). Our results implicitly characterize $\mathbf{\Gamma}$-BS_{PO} for $\mathbf{\Gamma} \in \{\mathbf{\Delta}^0_k, \mathbf{\Pi}^0_k\}$ (by transitivity, as $\mathbf{\Gamma}$-$\mathsf{DS} \le_{\mathrm{W}} \mathbf{\Gamma}$-$\mathsf{BS}_{PO} \le_{\mathrm{W}} \mathbf{\Gamma}$-$\mathsf{BS}$).

Question 5.61: What is the relation between $\mathbf{\Sigma}^0_1\text{-}\mathsf{BS}_{PO}$ and the problems $\mathsf{DS} \equiv_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{-}\mathsf{DS}$, $\mathbf{\Sigma}^0_1\text{-}\mathsf{DS}_{LQO}$ and $\mathbf{\Sigma}^0_1\text{-}\mathsf{BS}$?

Answering these questions would yield very interesting insights on how the possibility to have equivalent non-equal elements can enhance the uniform computational strength.

6 On the descriptive complexity of Salem sets

In this chapter, I will present some results that lie at the intersection of (effective) descriptive set theory, geometric measure theory, and the theory of fractal dimensions. Most of the results have been collected in [77]. The results of Section 6.2 are joint work with Ted Slaman and Jan Reimann.

The notion of Salem set arises naturally in the context of geometric measure theory and the theory of fractal dimension. A set $A \subset \mathbb{R}^d$ is called *Salem* iff $\dim_{\mathcal{H}}(A) = \dim_{\mathrm{F}}(A)$, where $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$ denote the Hausdorff and the Fourier dimension respectively.

Hausdorff dimension is a fundamental notion in geometric measure theory and can be found in almost every textbook in the field. It describes the "size" of a set using the diameter of open sets covering it. When working with Borel subsets of $\mathbb{R}^d$, Frostman's lemma characterizes the Hausdorff dimension of a set via the existence of finite Radon measures supported on the set with certain regularity properties (see Section 6.1 for details).

This characterization establishes a close connection with the Fourier transform of a measure. Indeed, it can be shown that the decay of the Fourier transform of a (probability) measure supported on the set provides a lower bound for the Hausdorff dimension. This leads to the notion of Fourier dimension and hence to the one of Salem set. It is known that, for Borel subsets of $\mathbb{R}^d$, the Fourier dimension never exceeds the Hausdorff dimension.

This work is part of a long-term effort, involving many researchers, aimed at exploring the recursion-theoretic properties of the Fourier dimension. In recent work, J. and N. Lutz [72] proved a point-to-set principle linking the (classical) Hausdorff dimension of a set with the (relative) effective Hausdorff dimension of its points. If we restrict our attention to singletons, we can characterize the effective Hausdorff dimension of $\{\xi\}$ by means of the Kolmogorov complexity of ξ [73], which establishes a surprising connection between two (apparently) very distant notions.

While no point-to-set principles can hold for the Fourier dimension (in such a generality), analyzing the complexity of the Fourier dimension in simpler cases can shed light on the general behavior of the Fourier dimension itself (up to now, still not deeply understood).

The first non-trivial examples of Salem sets were based on random constructions ([97, 59]). Later Kahane [58] modified the original construction by Salem to produce an explicit Salem set of dimension α, for every $\alpha \in [0, 1]$. An important example of an explicit Salem set comes from the theory of Diophantine approximation of real numbers: Jarník [56] and Besicovitch [6] proved that, for $\alpha \geq 0$, the set $E(\alpha)$ of α-well approximable numbers is a fractal with Hausdorff dimension $2/(2 + \alpha)$. Kaufmann [61] improved the result by showing that there is a probability measure supported on a subset of $E(\alpha)$ witnessing the fact that $\dim_{\mathrm{F}}(E(\alpha)) \geq 2/(2+\alpha)$, which implies that $E(\alpha)$ is Salem (the reader is referred to [7] or [114] for detailed proofs of Kaufmann's theorem).

A classical example of a non-Salem set is Cantor middle-third set, which has Fourier dimension 0 and Hausdorff dimension $\log(2)/\log(3)$. Similarly, every symmetric Cantor set with dissection ratio $1/n$, with $n > 1$, is not Salem, as it has null Fourier dimension and Hausdorff dimension $\log(2)/\log(n)$ (see [79, Sec. 4.10] and [80, Thm. 8.1]). It can be proved that, for every $0 \leq x \leq y \leq 1$ there is a compact subset of $[0,1]$ with Fourier dimension x and Hausdorff dimension y ([69, Thm. 1.4]).

There are not many explicit (i.e. non-random) examples of subsets of $\mathbb{R}^d$ which are known to be Salem. As a corollary of a result of Gatesoupe [40], we know that if $A \subset \mathbb{R}$ is a Salem set of dimension α and has at least two points then the set $\{x \in \mathbb{R}^d : |x| \in A\}$ is Salem and has dimension $d-1+\alpha$. Recently, using a higher-dimensional analogue of $E(\alpha)$, some explicit examples of Salem subsets of $\mathbb{R}^2$ ([48]) and $\mathbb{R}^d$ ([36]) of arbitrary dimension have been constructed.

In this paper we study the complexity, from the point of view of descriptive set theory, of the family $\{A \in \mathbf{F}(X) : A \in \mathscr{S}(X)\}$, where $\mathbf{F}(X)$ is the hyperspace of closed subsets of X, $\mathscr{S}(X)$ is the family of Salem subsets of X, and X is either $[0,1]$, $[0,1]^d$ or $\mathbb{R}^d$. In other words, we study the complexity of the property "being a Salem set", when we restrict our attention to closed sets. For the sake of readability we write $\mathscr{S}_c(X) := \mathscr{S}(X) \cap \mathbf{F}(X)$ for the set of closed Salem subsets of X. We show that it is Borel and classify it in the Borel hierarchy.

We summarize our results for $X = [0,1]$ in the following table.

$p < 1$	$\{A \in \mathbf{K}([0,1]) : \dim_{\mathcal{H}}(A) > p\}$	$\mathbf{\Sigma}^0_2$-complete
$p > 0$	$\{A \in \mathbf{K}([0,1]) : \dim_{\mathcal{H}}(A) \geq p\}$	$\mathbf{\Pi}^0_3$-complete
$p < 1$	$\{A \in \mathbf{K}([0,1]) : \dim_{\mathrm{F}}(A) > p\}$	$\mathbf{\Sigma}^0_2$-complete
$p > 0$	$\{A \in \mathbf{K}([0,1]) : \dim_{\mathrm{F}}(A) \geq p\}$	$\mathbf{\Pi}^0_3$-complete
$\{A \in \mathbf{K}([0,1]) : A \in \mathscr{S}([0,1])\}$		$\mathbf{\Pi}^0_3$-complete

The complexities remain the same if we replace $[0,1]$ with any interval, with $[0,1]^d$ or $\mathbb{R}^d$. In particular, the fact that the family of closed Salem subsets of $[0,1]$ is $\mathbf{\Pi}^0_3$-complete answers a question asked by Slaman during the IMS Graduate Summer School in Logic, held in Singapore in 2018.

Our results can be used to obtain the classifications of the functions computing the dimensions of closed sets, both in the Baire hierarchy and in the effective hierarchy defined via Weihrauch reducibility, in particular answering a question raised by Fouché ([18]) and Pauly.

6.1 Background

For a general introduction to geometric measure theory the reader is referred to [35]. We briefly introduce the notions and notations that are used in this chapter.

Let X be a separable metric space and let $A \subset X$. Let also $\mathrm{diam}(A)$ denote the diameter of A. We say that a family $\{E_i\}_{i\in I}$ is a δ-cover of A if $A \subset \bigcup_{i\in I} E_i$ and $\mathrm{diam}(E_i) \leq \delta$ for each $i \in I$.

For every $s \geq 0$, $\delta \in (0, +\infty]$ we define

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i \in I} \operatorname{diam}(E_i)^s \,:\, \{E_i\}_{i\in I} \text{ is a } \delta\text{-cover of } A \right\},$$
$$\mathcal{H}^s(A) := \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

The function $\mathcal{H}^s$ is called *s-dimensional Hausdorff measure*. The *Hausdorff dimension* of A is defined as

$$\dim_{\mathcal{H}}(A) := \sup\{s \in [0, +\infty) \,:\, \mathcal{H}^s(A) > 0\}.$$

As a consequence of Frostman's lemma (see [79, Thm. 8.8]), for every Borel subset A of $\mathbb{R}^d$ (with the Euclidean norm), the Hausdorff dimension of A coincides with its capacitary dimension $\dim_c(A)$, defined as

$$\sup\{s \in [0, d] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R}^d)(\forall r > 0)\ (\mu(B\,(x, r)) \leq c r^s)\},$$

where $\mathbb{P}(A)$ is the set of Borel probability measures with support included in A and $B\,(x, r)$ denotes the ball with center x and radius r. We notice that the Hausdorff dimension is *countably stable* (i.e. for every family $\{A_i\}_{i\in\mathbb{N}}$ we have $\dim_{\mathcal{H}}(\bigcup_i A_i) = \sup_i \dim_{\mathcal{H}}(A_i)$, see [79, p. 59]) and, for every α-Hölder continuous map $f \colon \mathbb{R}^n \to \mathbb{R}^m$ we have $\dim_{\mathcal{H}}(f(A)) \leq \alpha^{-1} \dim_{\mathcal{H}}(A)$ (see [35, Prop. 3.3]). In particular every bi-Lipschitz map preserves the Hausdorff dimension.

For every probability measure μ on $\mathbb{R}^d$, we can define the *Fourier transform of* μ as the function $\widehat{\mu} \colon \mathbb{R}^d \to \mathbb{C}$ defined as

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{-i\,\xi\cdot x}\, d\mu(x)$$

where $\xi \cdot x$ denotes the scalar product. We define the *Fourier dimension* of $A \subset \mathbb{R}^d$ as

$$\dim_{\mathrm{F}}(A) := \sup\{s \in [0, d] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R}^d)\, (|\widehat{\mu}(x)| \leq c|x|^{-s/2})\}.$$

If we define $\dim_{\mathrm{F}}(\mu) := \sup\{s \in [0, d] \,:\, (\exists c > 0)(\forall x \in \mathbb{R}^d)\, (|\widehat{\mu}(x)| \leq c|x|^{-s/2})\}$ then we have $\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}(\mu) \,:\, \mu \in \mathbb{P}(A)\}$. For background notions on the Fourier transform the reader is referred to [110]. For its applications to geometric measure theory see [80].

The Fourier dimension is not as stable as the Hausdorff dimension. Some stability properties of the Fourier dimension have been investigated in [32]. We underline, however, that the definition of Fourier dimension given in [32] differs from the definition we use in this work (which agrees with the one that can be found in the literature [35, 79, 80, 114]). The "classical" definition of Fourier dimension agrees with the *compact Fourier dimension* $\dim_{\mathrm{FC}}$ of [32, Sec. 1.3] (this can be showed, e.g., using [32, Lem. 1]). The three notions agree if we restrict our attention to the dimension of closed sets. In general, requiring that the measure μ witnessing that $\dim_{\mathrm{F}}(A) > s$ gives full measure to A is strictly weaker[1] than requiring that μ is supported on A.

The fact that $\dim_{\mathrm{F}} = \dim_{\mathrm{FC}}$ implies that the Fourier dimension is inner regular for compact sets, i.e.

$$\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}(K) \,:\, K \subset A \text{ and } K \text{ is compact}\}.$$

On the other hand, the Fourier dimension is not finitely stable in general: the Bernstein set $B \subset \mathbb{R}$ (see [62, Example 8.24]) is s.t. every closed subset of B or $\mathbb{R} \setminus B$ is countable, and therefore

[1] In [32, Ex. 7], the authors show that there is a set X s.t. X is a countable union of compact sets and $\dim_{\mathrm{FC}}(X) = \dim_{\mathrm{F}}(X) = 0$. However, admitting measures giving full measure to the set would give X full dimension.

$\dim_{\mathrm{F}}(B) = \dim_{\mathrm{F}}(\mathbb{R} \setminus B) = 0$. On the other hand $\dim_{\mathrm{F}}(B \cup \mathbb{R} \setminus B) = \dim_{\mathrm{F}}(\mathbb{R}) = 1$ (see also [32, Sec. 1.3]).

We can recover countable stability if we restrict our attention to closed sets:

Theorem 6.1 ([32, Prop. 5]):
If $\{A_k\}_k$ is a finite or countable family of closed subsets of $\mathbb{R}^d$ then

$$\dim_{\mathrm{F}}\left(\bigcup A_k\right) = \sup_k \dim_{\mathrm{F}}(A_k).$$

It is also known that the Fourier dimension does not behave well under Hölder continuous maps: there is a Hölder continuous transformation that maps the Cantor middle-third set to the interval $[0,1]$, although they have Fourier dimension respectively 0 and 1 ([33, Sec. 8]). However, the following fact, which we will use repeatedly in the paper, can be proved using the properties of the Fourier transform (see also [33, Prop. 6]):

Theorem 6.2:
The Fourier dimension is invariant under affine invertible transformations.

As a consequence of Frostman's lemma, for every Borel subset A of $\mathbb{R}^d$, $\dim_{\mathrm{F}}(A) \le \dim_{\mathcal{H}}(A)$ (see [79, Chap. 12]). If $\dim_{\mathrm{F}}(A) = \dim_{\mathcal{H}}(A)$ then A is called *Salem set*. We denote the collection of Salem subsets of $X \subset \mathbb{R}^d$ with $\mathscr{S}(X)$.

For a topological space X, we denote by $\mathbf{F}(X)$ and $\mathbf{K}(X)$ respectively the hyperspaces of closed and compact subsets of X.

There is no canonical choice for the topology on $\mathbf{F}(X)$, and several alternatives have been explored in the literature [5, 67]. Let $\mathscr{U}$ be the collection of sets of the form

$$\{F \in \mathbf{F}(X) : F \cap C = \emptyset\},$$

where C ranges over all closed subsets of X. The topology having $\mathscr{U}$ as a prebase is called *upper topology* or *upper Vietoris topology* ([67, Def. 1.3.1]). In the same spirit, we can define $\mathscr{L}$ as the family of sets of the form

$$\{F \in \mathbf{F}(X) : F \cap U \neq \emptyset\},$$

where U ranges over the open subsets of X. The topology having $\mathscr{L}$ as a prebase is called *lower topology* or *lower Vietoris topology* ([67, Def. 1.3.2]). The *Vietoris topology* is the topology having as a prebase the family $\mathscr{L} \cup \mathscr{U}$.

The Vietoris topology is not always the preferred choice. As an alternative, we can consider the collection $\mathscr{U}_{\mathcal{K}}$ of sets of the form

$$\{F \in \mathbf{F}(X) : F \cap K = \emptyset\},$$

where K ranges over all compact subsets of X. The family $\mathscr{U}_{\mathcal{K}}$ is a prebase for a topology on $\mathbf{F}(X)$ called *upper Fell topology*. We can define the *Fell topology* on $\mathbf{F}(X)$ as the topology having as a prebase the set $\mathscr{U}_{\mathcal{K}} \cup \mathscr{L}$. For this reason, the lower Vietoris topology is often called *lower Fell topology*. In the following, the Fell topology will be our default choice. For the sake of readability, we will write $\mathbf{F}_U(X)$ (resp. $\mathbf{F}_L(X)$, $\mathbf{F}_{UV}(X)$, $\mathbf{V}(X)$) for the hyperspace of closed subsets of X endowed with the upper Fell topology (resp. lower Fell topology, upper Vietoris topology, Vietoris topology). Unless otherwise mentioned, $\mathbf{F}(X)$ will be endowed with the Fell topology.

Unlike the hyperspace $\mathbf{F}(X)$, there is a canonical choice for the topology for the hyperspace $\mathbf{K}(X)$ of compact subsets of X. In fact, $\mathbf{K}(X)$ is usually endowed with the topology induced from the Vietoris topology on $\mathbf{F}(X)$.

If X is a bounded metric space with distance d, we can define the *Hausdorff metric* $\mathrm{d}_{\mathcal{H}}$ on $\mathbf{K}(X)$ as follows:

$$\mathrm{d}_{\mathcal{H}}(K,L) := \begin{cases} 0 & \text{if } K = L = \emptyset \\ \operatorname{diam}(X) & \text{if exactly one between } K \text{ and } L \text{ is } \emptyset \\ \max\{\delta(K,L), \delta(L,K)\} & \text{otherwise} \end{cases}$$

where $\delta(K,L) := \max_{x\in K} d(x,L)$. It is known that the Hausdorff metric $\mathrm{d}_{\mathcal{H}}$ is compatible with the Vietoris topology on $\mathbf{K}(X)$ ([62, Ex. 4.21]) and that if X is Polish then so is $\mathbf{K}(X)$ ([62, Thm. 4.22]).

The choice of the Vietoris topology is, of course, not the only possible: any topology on $\mathbf{F}(X)$ induces a topology on $\mathbf{K}(X)$. For the sake of readability, we will write $\mathbf{K}_F(X)$ (resp. $\mathbf{K}_U(X)$, $\mathbf{K}_L(X)$) for the hyperspace of compact subsets of X, endowed with the Fell (resp. upper Fell, lower Fell) topology.

One of the main reasons why the Vietoris topology is not the canonical choice for $\mathbf{F}(X)$ is that it is not paracompact, and hence metrizable[2], if X is not compact ([63, Thm. 2]). On the other hand, if X Polish and locally compact then the Fell topology on $\mathbf{F}(X)$ gives rise to a Polish compact space and its Borel space is exactly the Effros-Borel space. The Fell and the Vietoris topologies coincide if X is compact ([62, Ex. 12.7]).

An important topological space is the space of Borel probability measures. If X is a separable metrizable space, we consider the space $\mathbb{P}(X)$ of Borel probability measures on X, endowed with the topology generated by the maps $\mu \mapsto \int f\, d\mu$, with $f \in \mathcal{C}_b(X)$ (i.e. $f\colon X \to \mathbb{R}$ is continuous and bounded, see [62, Sec. 17.E, p. 109]). A basis for the topology on $\mathbb{P}(X)$ is the family of sets of the form

$$U_{\mu,\varepsilon,f_0,\dots,f_n} := \left\{ \nu \in \mathbb{P}(X) \,:\, (\forall i \le n) \left(\left| \int_X f_i\, d\nu - \int_X f_i\, d\mu \right| < \varepsilon \right) \right\},$$

where $\mu \in \mathbb{P}(X)$, $\varepsilon > 0$, and $f_i \in \mathcal{C}_b(X)$ for every i. The space $\mathbb{P}(X)$ is separable metrizable iff so is X [86, Ch. II, thm. 6.2]. Moreover If X is compact metrizable (resp. Polish) then so is $\mathbb{P}(X)$ ([62, Thm. 17.22 and thm. 17.23]).

An important tool in descriptive set theory is Baire category. A set $A \subset X$ is called *nowhere dense* if its closure has empty interior, *meager* if it is the countable union of nowhere dense sets and *comeager* if its complement is meager. By the Baire category theorem (see [62, Thm. 8.4]), in every Polish space the intersection of countably many open dense sets is dense ([62, Prop. 8.1]). In particular every comeager set is dense (it follows from the definition that a set is comeager iff it contains a dense $\boldsymbol{G}_\delta$ set).

We conclude this section with the following lemma:

[2] Intuitively, the max in the definition of $\delta(K,L)$ is not guaranteed to exist, and two closed sets can be infinitely distant.

Lemma 6.3 ([1, Lem. 1.3]):
Let X be Polish and Y metrizable and $\mathbf{K}_\sigma$ (i.e. countable union of compact sets). If $F \subset X \times Y$ is $\mathbf{\Sigma}^0_2$ then $\operatorname{proj}_X(F)$ is also $\mathbf{\Sigma}^0_2$.

6.2 The complexity of closed Salem subsets of $[0,1]$

In this section, we characterize the complexity of the family of closed Salem subsets of $[0,1]$. We first obtain an upper bound for the complexity of the conditions $\dim_{\mathcal{H}}(A) > p$, $\dim_{\mathcal{H}}(A) \geq p$, $\dim_{\mathrm{F}}(A) > p$ and $\dim_{\mathrm{F}}(A) \geq p$. Since the upper Fell topology is coarser than the Vietoris topology, obtaining an upper bound for the above conditions when the hyperspace of compact subsets of $[0,1]$ is endowed with the upper Fell topology immediately yields an upper bound for the same conditions when the hyperspace is endowed with the Vietoris topology instead.

Lemma 6.4:
Let X be a closed subset of $\mathbb{R}^d$. The set

$$\{(\mu, K, x) \in \mathbb{P}(X) \times \mathbf{K}_U(X) \times \mathbb{R} \,:\, \mu(K) \geq x\}$$

is closed.

Proof: We prove that the complement is open. Let (μ, K, x) be s.t. $\mu(K) = x - \varepsilon < x$. By the outer regularity of μ, there are two open sets U, V s.t.

- $K \subset U \subset \overline{U} \subset V \subset X$,
- $\mu(V) < x - \varepsilon/2$.

Similarly, by the inner regularity of μ, there are two open sets W, Z s.t. $X \setminus W$ is compact and

- $W \subset \overline{W} \subset Z$,
- $\mu(Z) < \varepsilon/8$.

By Urysohn's lemma, there are two continuous functions $f, g\colon X \to [0,1]$ s.t. $f(\overline{U}) = 1 = g(\overline{W})$ and $f(X \setminus V) = 0 = g(X \setminus Z)$.

Recall that the set

$$U_{\mu,\varepsilon/16,f,g} = \left\{\nu \in \mathbb{P}(X) \,:\, \left|\int f\,d\nu - \int f\,d\mu\right| < \frac{\varepsilon}{16} \wedge \left|\int g\,d\nu - \int g\,d\mu\right| < \frac{\varepsilon}{16}\right\}$$

is a basic open set for $\mathbb{P}(X)$. Define the set

$$\mathcal{U} := \{H \in \mathbf{K}_U(X) \,:\, H \subset U \cup W\}.$$

Notice that $U \cup W$ has compact complement, hence $\mathcal{U}$ is a basic open subset of $\mathbf{K}_U(X)$.

We claim that for every $(\nu, H, y) \in U_{\mu,\varepsilon/16,f,g} \times \mathcal{U} \times B(x,\varepsilon/4)$ we have $\nu(H) < y$. Indeed

$$\begin{aligned}\nu(H) &\le \nu(U) + \nu(W) \le \int f\,d\nu + \int g\,d\nu \\ &\le \int f\,d\mu + \int g\,d\mu + \frac{\varepsilon}{8} \\ &\le \mu(V) + \mu(Z) + \frac{\varepsilon}{8} < x - \frac{\varepsilon}{4} < y.\end{aligned}$$ ■

Notice that the same set is not closed if we consider the lower Fell topology on $\mathbf{K}(X)$, essentially because X belongs to every non-empty open set $\mathcal{U}$ of $\mathbf{K}_L(X)$.

Proposition 6.5:

- $\{(A,p) \in \mathbf{K}_U([0,1]) \times [0,1] \,:\, \dim_{\mathcal{H}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
- $\{(A,p) \in \mathbf{K}_U([0,1]) \times [0,1] \,:\, \dim_{\mathcal{H}}(A) \ge p\}$ *is* $\mathbf{\Pi}^0_3$*.*

Proof: As noticed in the previous section, for Borel (in particular closed) $A \subset [0,1]$, the Hausdorff dimension $\dim_{\mathcal{H}}(A)$ coincides with the capacitary dimension $\dim_c(A)$. For ease of readability define

$$D(A) := \{s \in [0,1] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R})(\forall r > 0)(\mu(B(x,r)) \le c r^s)\}.$$

Notice that $D(A)$ is downward closed. Recall that $\dim_c(A) = \sup D(A)$. Clearly

$$\mu(B(x,r)) \le c r^s \iff \mu([0,1] \setminus B(x,r)) \ge 1 - c r^s.$$

Observe that the map $(x,r) \mapsto [0,1] \setminus B(x,r)$ is continuous when the codomain is endowed with the Vietoris topology. In particular, it is continuous as a function $\mathbb{R}^2 \to \mathbf{K}_U([0,1])$. By Lemma 6.4 the condition $\mu(B(x,r)) \le c r^s$ is closed, hence the set

$$C := \{(s,c,\mu) \,:\, (\forall x \in \mathbb{R})(\forall r > 0)(\mu(B(x,r)) \le c r^s)\}$$

is a closed subset of the product space $[0,1] \times [0,+\infty) \times \mathbb{P}(A)$. Notice also that

$$\mu \in \mathbb{P}(A) \iff \mu \in \mathbb{P}([0,1]) \text{ and } \mu(A) \ge 1.$$

Since the condition $\mu(A) \ge 1$ is closed (again, by Lemma 6.4) we have that, for each closed subset A of $[0,1]$, the set

$$Q := \{(s,\mu) \in [0,1] \times \mathbb{P}([0,1]) \,:\, (\exists c > 0)(\mu \in \mathbb{P}(A) \wedge (s,c,\mu) \in C)\}$$

is $\mathbf{\Sigma}^0_2$.

Recall that the space $\mathbb{P}([0,1])$ is metrizable and compact. Using Lemma 6.3 we can conclude that the set $D(A) = \operatorname{proj}_{[0,1]} Q$ is $\mathbf{\Sigma}^0_2$. To conclude the proof we notice that the conditions

$$\dim_c(A) > p \iff (\exists s \in \mathbb{Q})(s > p \wedge s \in D(A)),$$

$$\dim_c(A) \ge p \iff (\forall s \in \mathbb{Q})(s < p \to s \in D(A))$$

are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_3$ respectively. ■

Proposition 6.6:

- $\{(A,p) \in \mathbf{K}_U([0,1]) \times [0,1] \,:\, \dim_{\mathrm{F}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
- $\{(A,p) \in \mathbf{K}_U([0,1]) \times [0,1] \,:\, \dim_{\mathrm{F}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$.

Proof: For the sake of readability, let

$$D(A) := \{s \in [0,1] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R})(|\widehat{\mu}(x)| \leq c|x|^{-s/2})\}.$$

First of all we notice that the condition $|\widehat{\mu}(x)| > c|x|^{-s/2}$ is $\mathbf{\Sigma}^0_1$. To see this it is enough to show that the map $F\colon \mathbb{P}([0,1]) \times \mathbb{R} \to \mathbb{R}$ s.t. $F(\mu,x) = |\widehat{\mu}(x)|$ is continuous. Indeed, if that is the case, then the tuple (μ,x,s,c) satisfies the condition $|\widehat{\mu}(x)| > c|x|^{-s/2}$ iff it belongs to the preimage of $(0,+\infty)$ via the map $(\mu,x,s,c) \mapsto F(\mu,x) - c|x|^{-s/2}$, which is clearly continuous.

Recall that, for each finite Borel measure μ, the Fourier transform $\widehat{\mu}$ is a bounded uniformly continuous function.

Notice that the set

$$V_{\mu,\varepsilon,x} := \{\nu \in \mathbb{P}([0,1]) \,:\, |\widehat{\mu}(x) - \widehat{\nu}(x)| < \varepsilon\}$$

is open in the topology of $\mathbb{P}([0,1])$. Indeed, fix $\nu \in V_{\mu,\varepsilon,x}$ and let δ s.t. $|\widehat{\mu}(x) - \widehat{\nu}(x)| + \delta < \epsilon$. We claim that the basic open set $U_{\nu,\frac{\delta}{2},\cos(x\,\cdot),\sin(x\,\cdot)}$ is included in $V_{\mu,\varepsilon,x}$. In fact, for each $\eta \in U_{\nu,\frac{\delta}{2},\cos(x\,\cdot),\sin(x\,\cdot)}$ we have

$$\begin{aligned}|\widehat{\nu}(x) - \widehat{\eta}(x)| &= \left|\int e^{-i\,xt}\, d\nu(t) - \int e^{-i\,xt}\, d\eta(t)\right| \leq \\ &\leq \left|\int \cos(xt)\, d\nu(t) - \int \cos(xt)\, d\eta(t)\right| + \left|\int \sin(xt)\, d\nu(t) - \int \sin(xt)\, d\eta(t)\right| \leq \delta\end{aligned}$$

and therefore

$$|\widehat{\mu}(x) - \widehat{\eta}(x)| \leq |\widehat{\mu}(x) - \widehat{\nu}(x)| + |\widehat{\nu}(x) - \widehat{\eta}(x)| \leq |\widehat{\mu}(x) - \widehat{\nu}(x)| + \delta < \varepsilon.$$

To conclude the proof of the continuity we show that for each $\varepsilon > 0$ and each μ and x we can choose δ sufficiently small s.t. for every $(\nu,y) \in V_{\mu,\delta,x} \times B(x,\delta)$ we have $|\widehat{\mu}(x) - \widehat{\nu}(y)| < \varepsilon$. Indeed, by the triangle inequality

$$|\widehat{\mu}(x) - \widehat{\nu}(y)| \leq |\widehat{\mu}(x) - \widehat{\nu}(x)| + |\widehat{\nu}(x) - \widehat{\nu}(y)|.$$

The first term is bounded by δ by definition of $V_{\mu,\delta,x}$. Moreover

$$\begin{aligned}|\widehat{\nu}(x) - \widehat{\nu}(y)| &= \left|\int e^{-ixt} - e^{-iyt}\, d\nu(t)\right| \leq \\ &\leq \int |\cos(xt) - \cos(yt)|\, d\nu(t) + \int |\sin(xt) - \sin(yt)|\, d\nu(t)\,.\end{aligned}$$

By the sum-to-product formulas

$$\begin{aligned}\int |\cos(xt) - \cos(yt)|\, d\nu(t) &= \int 2\left|\sin\left(\frac{(x+y)t}{2}\right)\sin\left(\frac{(x-y)t}{2}\right)\right| d\nu(t) \leq \\ &\leq 2\sin\left(\frac{x-y}{2}\right)\end{aligned}$$

and similarly

$$\int |\sin(xt) - \sin(yt)|\, d\nu(t) \leq 2\sin\left(\frac{x-y}{2}\right).$$

hence the claim follows.

Since $\mu \in \mathbb{P}(A)$ is a closed condition (see the proof of Proposition 6.5), the set

$$\{(s,c,\mu) \in [0,1] \times [0,+\infty) \times \mathbb{P}([0,1]) \,:\, \mu \in \mathbb{P}(A) \wedge (\forall x \in \mathbb{R})(|\widehat{\mu}(x)| \leq c|x|^{-s/2})\}$$

is closed and, therefore, the set

$$Q := \{(s,\mu) \in [0,1] \times \mathbb{P}([0,1]) \,:\, (\exists c > 0)(\forall x \in \mathbb{R})(\mu \in \mathbb{P}(A) \wedge |\widehat{\mu}(x)| \leq c|x|^{-s/2})\}$$

is $\mathbf{\Sigma}^0_2$. As in the proof of Proposition 6.5, we can use Lemma 6.3 to conclude that the set $D(A) = \operatorname{proj}_{[0,1]} Q$ is $\mathbf{\Sigma}^0_2$ and hence the conditions

$$\dim_{\mathrm{F}}(A) > p \iff (\exists s \in \mathbb{Q})(s > p \wedge s \in D(A)),$$

$$\dim_{\mathrm{F}}(A) \geq p \iff (\forall s \in \mathbb{Q})(s < p \to s \in D(A))$$

are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_3$ respectively. ■

Theorem 6.7:
The set $\{A \in \mathbf{K}_U([0,1]) \,:\, A \in \mathscr{S}([0,1])\}$ *is* $\mathbf{\Pi}^0_3$.

Proof: To prove that $A \in \mathscr{S}([0,1])$ is a $\mathbf{\Pi}^0_3$ condition recall that, for Borel subsets of $\mathbb{R}^d$, $\dim_{\mathrm{F}}(A) \leq \dim_{\mathcal{H}}(A)$. For a closed subset A of $[0,1]$, the condition $\dim_{\mathcal{H}}(A) = \dim_{\mathrm{F}}(A)$ can be written as

$$(\forall r \in \mathbb{Q})(\dim_{\mathcal{H}}(A) > r \to \dim_{\mathrm{F}}(A) > r).$$

The claim follows from Proposition 6.5 and Proposition 6.6, as both $\dim_{\mathcal{H}}(A) > r$ and $\dim_{\mathrm{F}}(A) > r$ are $\mathbf{\Sigma}^0_2$ conditions. ■

We now show that the above conditions are complete for their respective classes (i.e. the upper bounds are tight) when the hyperspace of compact subsets of $[0,1]$ is endowed with the Vietoris topology. Since the Vietoris topology is finer than the upper Fell topology, the same lower bounds hold when the hyperspace of compact subsets of $[0,1]$ is endowed with the upper Fell topology.

The proof of the following Lemma 6.9 exploits the properties of the set $E(\alpha)$ of α-well approximable numbers.

Definition 6.8 ([35, Sec. 10.3]): For every $\alpha \geq 0$, we say that $x \in [0,1]$ is *α-well approximable* if there are infinitely many $n \in \mathbb{N}$ s.t.

$$\min_{m \in \mathbb{Z}} |nx - m| \leq n^{-1-\alpha}.$$

The set of α-well approximable numbers is denoted by $E(\alpha)$.

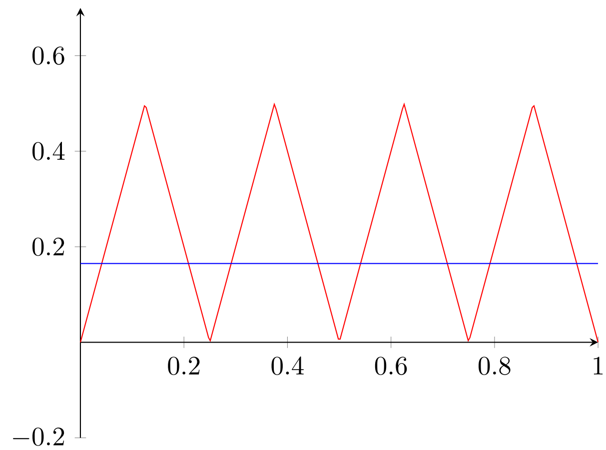

Figure 6.1: As an example, let $n = 4$ and $\alpha = 0.3$. The red line is the plot of the function $\min_{m\in\mathbb{Z}} |nx - m|$, while the blue line is the constant function $n^{-1-\alpha}$. The set $G_n(\alpha)$ is the set of points s.t. the red line lies below the blue line.

As mentioned in the introduction, $E(\alpha)$ is a Salem set of dimension $2/(2+\alpha)$. Notice that, by definition, the set $E(\alpha)$ is $\mathbf{\Pi}^0_3$, as it can be written in the form

$$E(\alpha) = \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} G_n,$$

where $G_n := \{x \in [0,1] \ : \ \min_{m\in\mathbb{Z}} |nx - m| \leq n^{-1-\alpha}\}$ is a closed set (it is a finite union of non-degenerate closed intervals), see Figure 6.1.

If $\alpha = 0$ then, by Dirichlet's theorem ([35, Ex. 10.8]), $E(\alpha) = [0,1]$. However, if $\alpha > 0$ then $E(\alpha)$ is not closed (because $E(\alpha)$ is dense in $[0,1]$ but does not have full dimension).

In the construction presented in [7], the author explicitly writes the support[3] $S(\alpha)$ of a measure witnessing that $\dim_{\mathrm{F}}(E(\alpha)) \geq 2/(2+\alpha)$. This, in particular, implies that $S(\alpha)$ itself is Salem with dimension $2/(2+\alpha)$. The set $S(\alpha)$ can be written as

$$S(\alpha) = \bigcap_{k\in\mathbb{N}} \bigcup_{k'\leq n\leq k''} G_n.$$

In other words, it is obtained from $E(\alpha)$ by making the inner union finite, where k' and k'' depend on k (and α) and are strictly increasing. Clearly $S(\alpha)$ is closed (as it is the infinite intersection of closed sets). We can rewrite $S(\alpha)$ as follows:

$$S(\alpha) = \bigcap_{k\in\mathbb{N}} S^{(k)}(\alpha)$$

where

$$S^{(k)}(\alpha) = \bigcup_{i\leq M_k} I_i(\alpha, k)$$

and, for each k, the $I_i(\alpha,k)$ are disjoint non-degenerate closed intervals.

[3]In [7] it is denoted with S_α.

We modify $S(\alpha)$ to obtain

$$R(\alpha) = \bigcap_{k\in\mathbb{N}} R^{(k)}(\alpha) = \bigcap_{k\in\mathbb{N}} \bigcup_{j\leq N_k} J_j(\alpha,k),$$

where each $J_j(\alpha,k)$ is a non-degenerate closed interval with the property that $R^{(k+1)}(\alpha) \subset R^{(k)}(\alpha)$, and, moreover, for every $i \leq N_k$ there exists $j \leq N_{k+1}$ s.t. $J_j(\alpha,k+1) \subset J_i(\alpha,k)$. To this end, define $R^{(k)}(\alpha)$ inductively as follows: $R^{(0)}(\alpha) := S^{(0)}(\alpha)$. At stage $k+1$, let

$$\tilde{R}^{(k+1)}(\alpha) := S^{(k+1)}(\alpha) \cup \bigcup_{n\in U_k} G_n,$$

where $U_k \subset \mathbb{N}$ is a finite set of indexes s.t. for every interval $j \leq N_k$,

$$\operatorname{Int}(J_j(\alpha,k)) \cap \tilde{R}^{(k+1)}(\alpha) \neq \emptyset,$$

where $\operatorname{Int}(\cdot)$ denotes the interior. Such a choice of U_k is always possible by the density of $E(\alpha)$. We obtain $R^{(k+1)}(\alpha)$ by considering the finitely many intervals whose union is $\tilde{R}^{(k+1)}(\alpha) \cap R^{(k)}(\alpha)$ and removing the degenerate ones.

Notice that, for every k, $S^{(k)}(\alpha) \setminus R^{(k)}(\alpha)$ is finite. This implies that $S(\alpha) \setminus R(\alpha)$ is countable and therefore, by Theorem 6.1, $\dim_{\mathrm{F}}(S(\alpha)) = \dim_{\mathrm{F}}(R(\alpha))$. Notice, moreover, that $R(\alpha) \subset E(\alpha)$, and therefore R is still a Salem set and $\dim(R(\alpha)) = 2/(2+\alpha)$.

Lemma 6.9:
For every $p \in [0,1]$ there exists a continuous map $f_p \colon 2^{\mathbb{N}} \to \mathbf{K}([0,1])$ s.t. for every x, $f_p(x)$ is Salem and

$$\dim(f_p(x)) = \begin{cases} p & \text{if } x \in Q_2 \\ 0 & \text{if } x \notin Q_2 \end{cases}$$

Proof: Recall that $Q_2 = \{x \in 2^{\mathbb{N}} : (\forall^\infty k)(x(k) = 0)\}$ is $\mathbf{\Sigma}^0_2$-complete.

The case $p = 0$ is trivial (just take the constant map $x \mapsto \emptyset$), so assume $p > 0$. Let $\alpha \geq 0$ s.t. $2/(2+\alpha) = p$ and consider the Salem set $S(\alpha)$ as defined above.

For each $x \in 2^{\mathbb{N}}$ we define a sequence $(F_x^{(k)})_{k\in\mathbb{N}}$ of nested closed sets s.t. each $F_x^{(k)}$ is a finite union of closed intervals. The idea is to follow the construction of $R(\alpha)$ until we find a k s.t. $x(k) = 1$. If this never happens then in the limit we obtain $R(\alpha)$, which is a Salem set of Fourier dimension p. On the other hand, each time we find a k s.t. $x(k) = 1$ we modify the next step of the construction by replacing each of the (finitely many) intervals $J_0, \dots, J_{N_k}$ whose union is the k-th level of the construction with sufficiently small subintervals $H_0, \dots, H_{N_k}$, and we reset the construction, starting again a (proportionally scaled down) construction of $R(\alpha)$ on each subinterval H_i. By carefully choosing the length of the subintervals H_i we can ensure that, if there are infinitely many k s.t. $x(k) = 1$ then F_x has null Hausdorff (and hence Fourier) dimension.

Formally, if $I = [a,b]$ is an interval then we define $R(\alpha, I)$ as the fractal obtained by scaling $R(\alpha)$ to the interval I. Notice that, by Theorem 6.2, $R(\alpha, I)$ is still a Salem set of dimension p.

We define $F_x^{(k)}$ recursively as

Stage $k = 0$: $F_x^{(0)} := [0,1]$;

Stage $k+1$: Let $J_0, \dots, J_{N_k}$ be the disjoint closed intervals s.t. $F_x^{(k)} = \bigcup_{i \le N_k} J_i$. If $x(k+1) = 1$ then choose, for each $i \le N_k$, a (non-degenerate) subinterval $H_i = [a_i, b_i] \subset J_i$ so that

$$\sum_{i \le N_k} \operatorname{diam}(H_i)^{2^{-k}} \le 2^{-k}.$$

Define then $F_x^{(k+1)} := \bigcup_{i \le N_k} H_i$.

If $x(k+1) = 0$ then let $s \le k$ be largest s.t. $x(s) = 1$ (or $s = 0$ if there is none) and let $I_0, \dots, I_{N_s}$ be the intervals of $F_x^{(s)}$. For each $i \le N_s$, apply the $(k+1-s)$-th step of the construction of $R(\alpha, I_i)$. Define $F_x^{(k+1)} := \bigcup_{i \le N_s} R^{(k+1-s)}(\alpha, I_i)$.

We define the map f_p as $f_p(x) := F_x = \bigcap_{k \in \mathbb{N}} F_x^{(k)}$. Clearly F_x is closed, as intersection of closed sets. To show that f_p is continuous, recall that the Vietoris topology is compatible with the Hausdorff metric $\mathrm{d}_{\mathcal{H}}$. Fix $x \in 2^{\mathbb{N}}$. For each $\varepsilon > 0$ we can choose k large enough so that all the intervals $J_0, \dots, J_{N_k}$ of $F_x^{(k)}$ have length $\le \varepsilon$. By construction, for every $y \in 2^{\mathbb{N}}$ that extends $x[k]$ we have $F_y \cap J_i \neq \emptyset$ (i.e. none of the intervals is ever removed completely) and $F_y \subset J_0 \cup \dots \cup J_{N_k}$ (i.e. nothing is ever added outside of $F_x^{(k)}$). This implies that

$$\mathrm{d}_{\mathcal{H}}(F_x, F_y) \le \max\{\operatorname{diam}(J_i) \,:\, i \le N_k\} \le \varepsilon,$$

which proves the continuity.

If $x \in Q_2$ then x is eventually null (i.e. there are finitely many 1s in x). Letting s be the largest index s.t. $x(s) = 1$ (or $s = 0$ if there is none) then $F_x = \bigcup_{i \le N_s} R(\alpha, J_i)$. Each set $R(\alpha, J_i)$ is a Salem set of dimension p (as we fixed α accordingly). Since the intervals J_i are closed and disjoint, using Theorem 6.1, we can conclude that F_x is a Salem set of dimension p.

On the other hand, if $x \notin Q_2$ then we want to show that $\dim_{\mathcal{H}}(F_x) = 0$. We will show that for each $s > 0$ and $\varepsilon > 0$ there is a cover $(A_n)_{n \in \mathbb{N}}$ of F_x s.t. $\sum_{n \in \mathbb{N}} \operatorname{diam}(A_n)^s \le \varepsilon$, i.e. for each $s > 0$, $\mathcal{H}^s(F_x) = 0$.

For fixed s and ε we can pick k large enough s.t. $2^{-k} \le s$, $2^{-k} \le \varepsilon$ and $x(k+1) = 1$. Notice that the intervals $(H_i)_{i \le N_k}$ (as defined in the construction of F_x) form a cover of F_x s.t.

$$\sum_{i \le N_k} \operatorname{diam}(H_i)^s \le \sum_{i \le N_k} \operatorname{diam}(H_i)^{2^{-k}} \le 2^{-k} \le \varepsilon,$$

as desired. ■

Proposition 6.10:
For every $p < 1$ *the sets*

$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are $\mathbf{\Sigma}_2^0$*-complete.*

PROOF: The hardness is a straightforward corollary of Lemma 6.9: fix q s.t. $p < q < 1$ and the $\mathbf{\Sigma}^0_2$-complete subset Q_2 of $2^{\mathbb{N}}$. We can consider the map $f_q\colon 2^{\mathbb{N}} \to \mathscr{S}_c([0,1])$ as in Lemma 6.9 and notice that

$$\dim(f_q(x)) > p \iff x \in Q_2.$$

The completeness follows from Proposition 6.5 and Proposition 6.6. ∎

Theorem 6.11:
For every $p \in (0,1]$ *there exists a continuous map* $F\colon 2^{\mathbb{N}\times\mathbb{N}} \to \mathbf{K}([0,1])$ *s.t. for every* $x \in 2^{\mathbb{N}\times\mathbb{N}}$, $F(x)$ *is a Salem set and* $\dim(F(x)) \ge p$ *iff* $x \in P_3$*. Letting*

$$X_1 := \{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathcal{H}}(A) \ge p\},$$
$$X_2 := \{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathrm{F}}(A) \ge p\}$$

we have that every set X *s.t.* $X_2 \subset X \subset X_1$ *is* $\mathbf{\Pi}^0_3$*-hard. In particular,* X_1 *and* X_2 *are* $\mathbf{\Pi}^0_3$*-complete.*

PROOF: The last statement follows from the first one using Proposition 6.5 and Proposition 6.6.

Consider the map $\Phi\colon 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ defined as

$$\Phi(x)(m,n) := \max_{i\le m} x(i,n).$$

It is easy to see that Φ is continuous. Notice also that $x \in P_3$ iff $\Phi(x) \in P_3$. On the other hand,

$$x \notin P_3 \iff (\exists k)(\forall m \ge k)(\exists^\infty n)(\Phi(x)(m,n) = 1).$$

Intuitively, we are modifying the $\mathbb{N}\times\mathbb{N}$ matrix x so that if there is a row of x that contains infinitely many 1s, then, from that row on, every row will contain infinitely many 1s.

We build a continuous map $f\colon 2^{\mathbb{N}\times\mathbb{N}} \to \mathbf{K}([0,1])$ s.t. $F := f\circ\Phi$ is the desired function.

For every n, let $T_n := [2^{-n-1}, 2^{-n}]$, $q_n := p(1-2^{-n-1})$ and consider the function $f_{q_n}\colon 2^{\mathbb{N}} \to \mathscr{S}_c([0,1])$ of Lemma 6.9. Fix also a similarity transformation $\tau_n\colon [0,1] \to T_n$ and define $g_n\colon 2^{\mathbb{N}} \to \mathscr{S}_c(T_n)$ as $g_n := \tau_n f_{q_n}$, so that, by Theorem 6.2,

$$\dim(g_n(y)) = \begin{cases} q_n & \text{if } y \in Q_2, \\ 0 & \text{if } y \notin Q_2. \end{cases}$$

Let x_m be the m-th row of $x \in 2^{\mathbb{N}\times\mathbb{N}}$. We define

$$f(x) := \{0\} \cup \bigcup_{m\in\mathbb{N}} g_m(x_m).$$

Intuitively, we are dividing the interval $[0,1]$ into countably many intervals and, on each interval, we are applying the construction we described in the proof of Lemma 6.9 (proportionally scaled down). The continuity of f follows from the continuity of each g_m. The accumulation point 0 is added to ensure that $f(x)$ is a closed set.

Recall that Hausdorff dimension is stable under countable unions, so

$$\dim_{\mathcal{H}}(f(x)) = \sup_{m\in\mathbb{N}} \dim_{\mathcal{H}}(g_m(x_m)).$$

Moreover, since the sets $\{T_m\}_{m\in\mathbb{N}}$ are closed, we can apply Theorem 6.1 and conclude that

$$\dim_{\mathrm{F}}(f(x)) = \sup_{m\in\mathbb{N}} \dim_{\mathrm{F}}(g_m(x_m)).$$

Since each $g_m(x_m)$ is Salem we have that $f(x)$ is Salem (and, in turn, $F(x)$ is Salem) and

$$\dim(f(x)) = \sup_{m\in\mathbb{N}} \dim_{\mathcal{H}}(g_m(x_m)) = \sup_{m\in\mathbb{N}} \dim_{\mathrm{F}}(g_m(x_m)).$$

If $x \in P_3$ then $\Phi(x) \in P_3$ and, for every m, $\Phi(x)_m \in Q_2$. This implies that $g_m(\Phi(x)_m)$ is a Salem set of dimension q_m and therefore

$$\dim(F(x)) = \sup_{m\in\mathbb{N}} q_m = p.$$

On the other hand, if $x \notin P_3$ then there is a $k > 0$ s.t. for every $m \geq k$, $\Phi(x)_m \notin Q_2$ and hence $\dim(g_m(\Phi(x)_m)) = 0$. This implies that

$$\dim(F(x)) \leq q_k < p,$$

and this completes the proof. ■

Theorem 6.12:
The set $\{A \in \mathbf{K}([0,1]) \,:\, A \in \mathscr{S}([0,1])\}$ *is* $\mathbf{\Pi}^0_3$*-complete.*

Proof: Let $K \in \mathbf{K}([0,1])$ be s.t. $\dim_{\mathcal{H}}(K) = p$ and $\dim_{\mathrm{F}}(K) = 0$. Let also F be the map provided by Theorem 6.11 and define the map $h\colon 2^{\mathbb{N}\times\mathbb{N}} \to \mathbf{K}([0,1])$ as

$$h(x) := F(x) \cup K.$$

Then h is continuous (see e.g. [62, Ex. 4.29(iv)]) and

$$\dim_{\mathcal{H}}(h(x)) = \max\{\dim(F(x)), p\},$$
$$\dim_{\mathrm{F}}(h(x)) = \dim(F(x)).$$

In particular, $h(x)$ is Salem iff $\dim(F(x)) \geq p$ iff $x \in P_3$. The claim follows by Theorem 6.7. ■

This shows that the upper bounds we obtained in Proposition 6.5, Proposition 6.6 and Theorem 6.7 are sharp. In particular, since $\mathbf{K}([0,1])$ is a Polish space, this implies that the hyperspace of closed Salem subsets of $[0,1]$ is not a Polish space (in the relative topology). This follows from [62, Thm. 3.11], as a subset of a Polish space is Polish iff it is $\boldsymbol{G}_\delta$.

Notice that, if we endow $\mathscr{S}_c([0,1])$ with the topology induced by $\mathbf{K}_U([0,1])$ then, by Proposition 6.5 (or, equivalently, by Proposition 6.6), we have that

$$\{(A,p) \in \mathscr{S}_c([0,1]) \times [0,1] \,:\, \dim(A) > p\} \text{ is } \mathbf{\Sigma}^0_2,$$
$$\{(A,p) \in \mathscr{S}_c([0,1]) \times [0,1] \,:\, \dim(A) \geq p\} \text{ is } \mathbf{\Pi}^0_3.$$

Moreover, the proofs of Proposition 6.10 and Theorem 6.11 show that, for every $p < 1$ and every $q > 0$,

$$Q_2 \leq_W \{A \in \mathscr{S}_c([0,1]) \,:\, \dim(A) > p\},$$
$$P_3 \leq_W \{A \in \mathscr{S}_c([0,1]) \,:\, \dim(A) \geq q\}.$$

However we cannot say that they are complete for their respective classes, because the definition of completeness requires the ambient space to be Polish, and $\mathscr{S}_c([0,1])$ is not.

Recall that the Fourier dimension of A is based on an estimate on the decay of the Fourier transform of a probability measure supported on A. In particular $\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}(\mu) \,:\, \mu \in \mathbb{P}(A)\}$. This is equivalent to let μ range over finite (non-trivial) Radon measures on A, as the estimate on the decay of the Fourier transform is only up to a multiplicative constant. One may wonder whether it is possible to strengthen this condition by defining the Fourier dimension of A as

$$\sup\{s \in [0,1] \,:\, (\exists \mu \in \mathbb{P}(A))(\forall x \in \mathbb{R})(|\widehat{\mu}(x)| \leq |x|^{-s/2})\}.$$

The $\mathbf{\Pi}^0_3$-completeness of $\mathscr{S}_c([0,1])$ implies that the notion of dimension we would obtain is different. Indeed, the space $\mathbb{P}([0,1])$ is a compact space (as already noticed in the proof of Proposition 6.5), while the space $[0,\infty) \times \mathbb{P}([0,1])$ is not. In particular, removing the constant c in the condition on the decay of the Fourier transform would imply that $\mathscr{S}_c([0,1])$ is $\mathbf{\Pi}^0_2$ (as the projection of a closed set along a compact space is closed, see the proofs of Proposition 6.5 and Proposition 6.6), and therefore not $\mathbf{\Pi}^0_3$-complete.

6.3 The complexity of closed Salem subsets of $[0,1]^d$

Let us now turn our attention to the family of closed Salem subsets of $[0,1]^d$.

Proposition 6.13:
For every $d \geq 1$:

1. $\{(A,p) \in \mathbf{K}_U([0,1]^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
2. $\{(A,p) \in \mathbf{K}_U([0,1]^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*;*
3. $\{(A,p) \in \mathbf{K}_U([0,1]^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
4. $\{(A,p) \in \mathbf{K}_U([0,1]^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*;*
5. $\{A \in \mathbf{K}_U([0,1]^d) \,:\, A \in \mathscr{S}([0,1]^d)\}$ *is* $\mathbf{\Pi}^0_3$.

Proof: For the first two points, the proof is a straightforward adaptation of the proof of Proposition 6.5. Indeed, recall that Frostman's lemma holds for Borel subsets of $\mathbb{R}^d$ ([79, Thm. 8.8]), hence we can characterize the Hausdorff dimension by means of the capacitary dimension. Moreover, since $[0,1]^d$ is compact, the condition $\mu(B(x,r)) \le cr^s$ is closed and the space $\mathbb{P}([0,1]^d)$ is compact. Therefore $\dim_c(A)$ is the supremum of a $\mathbf{\Sigma}^0_2$ set, from which the claim follows.

Similarly, points 3 and 4 follow by adapting the proof of Proposition 6.6. Indeed the map $F := (\mu, x) \mapsto |\widehat{\mu}(x)|$ is continuous and the condition $|\widehat{\mu}(x)| > c|x|^{-t/2}$ is open, therefore the Fourier dimension is the supremum of a $\mathbf{\Sigma}^0_2$ set.

Finally, the last point can be proved by following the proof of Theorem 6.7 and using points 1 and 3. ■

The fact that the lower bounds for the complexity of the above sets are tight does not come as a corollary of the results in the 1-dimensional case. Indeed, it is well known that the Fourier dimension is sensitive to the ambient space: any m-dimensional hyperplane has null Fourier dimension when seen as a subset of $\mathbb{R}^d$, with $d > m$ (in particular, the unit interval $[0,1]$ has full Fourier dimension if seen as a subset of itself or of $\mathbb{R}$, but it has null Fourier dimension if seen as a subset of $\mathbb{R}^2$).

We will instead prove a d-dimensional analogue of Lemma 6.9. In recent work, Fraser and Hambrook ([36]) presented a construction of a Salem subset of $[0,1]^d$ of dimension p, for every $p \in [0,d]$.

Definition 6.14 ([36]): Let K be a number field of degree d, i.e. K is a field extension of $\mathbb{Q}$ and $\dim_{\mathbb{Q}} K = d$. Let $B = \{\omega_0, \dots, \omega_{d-1}\}$ be an integral basis for K. We can identify $\mathbb{Q}^d$ with K by mapping a vector $q = (q_0, \dots, q_{n-1})$ to $\sum_{i<n} q_i \omega_i \in K$. Moreover, since B is an integral basis, we can also identify $\mathbb{Z}^d$ with the ring of integers $\mathcal{O}(K)$ for K. For every $\alpha \ge 0$ we define

$$E(K,B,\alpha) := \left\{ x \in [0,1]^d \,:\, (\exists^\infty (q,r) \in \mathbb{Z}^d \times \mathbb{Z}^d) \left(\left\| x - \frac{r}{q} \right\|_\infty \le \|q\|_\infty^{-2-\alpha} \right) \right\},$$

where $\|\cdot\|_\infty$ is the max-norm on $\mathbb{R}^d$.

The set $E(K,B,\alpha)$ is a higher dimensional analogue of the fractal $E(\alpha)$.

Theorem 6.15 ([36, Thm. 4.1]):
For every $\alpha \ge 0$, the set $E(K,B,\alpha)$ is a Salem set of dimension $2d/(2+\alpha)$.

The fact that $E(K,B,0)$ is Salem of dimension d is not explicitly mentioned in [36], but a simple proof was suggested by Hambrook (personal communication): indeed it is enough to notice that, for every α and every $\varepsilon > 0$, $E(K,B,\alpha+\varepsilon) \subset E(K,B,\alpha)$, and therefore the claim follows from the monotonicity of the Fourier dimension.

Notice that, in general, the set $E(K,B,\alpha)$ is not closed but $\mathbf{\Pi}^0_3$. Analogously to the one-dimensional case, the proof of Theorem 6.15 shows that there is a closed Salem subset $S(K,B,\alpha)$ of $E(K,B,\alpha)$ with dimension $2d/(2+\alpha)$. To prove the following Lemma 6.16 we cannot proceed as in the one-dimension case, as we do not know whether $E(K,B,\alpha)$ is dense in $[0,1]^d$.

Lemma 6.16:
Fix $d > 0$. For every $p \in [0,d]$ there exists a continuous map $f_p \colon 2^{\mathbb{N}} \to \mathbf{K}([0,1]^d)$ s.t. for every x, $f_p(x)$ is Salem and

$$\dim(f_p(x)) = \begin{cases} p & \text{if } x \in Q_2 \\ 0 & \text{if } x \notin Q_2 \end{cases}$$

Proof: The idea of the proof is similar to the one of Proposition 6.10: given $x \in 2^{\mathbb{N}}$, we define a closed set F_x by following the construction of the set $S(K,B,\alpha)$, having care of controlling the Hausdorff dimension whenever $x(k) = 1$.

Formally, let $p > 0$ (otherwise the claim follows trivially by considering the map $x \mapsto \emptyset$) and let α s.t. $2d/(2+\alpha) = p$.

Fix K and B as in Definition 6.14. For the sake of readability, let $S(\alpha) := S(K,B,\alpha)$. We can write $S(\alpha)$ as intersection of closed nested sets $S^{(k)}(\alpha)$ defined as

$$S^{(k)}(\alpha) := \{y \in [0,1]^d \,:\, d(y, S(\alpha)) \le 2^{-k}\}.$$

Clearly, $S^{(k)}(\alpha)$ is closed with non-empty interior.

For each non-degenerate hypercube C, define $S(\alpha, C) := \tau(S(\alpha))$, where τ is a similarity transformation that maps $[0,1]^d$ onto C, and $S^{(k)}(\alpha, C)$ accordingly.

We define $F_x^{(k)}$ recursively, ensuring that, for each k, $F_x^{(k)}$ is closed and has non-empty interior, and $F_x^{(k+1)} \subset F_x^{(k)}$:

Stage $k = 0$ $F_x^{(0)} := C_0 := [0,1]^d$, $P_0 := \emptyset$;

Stage $k+1$ If $x(k+1) = 1$, let $P_k := \{p_i^{(k)}\}_{i \le N_k}$ be a finite set of points in $F_x^{(k)}$ s.t. for each $t \in F_x^{(k)}$ there exists $i \le N_k$ s.t. $|t - p_i^{(k)}| \le 2^{-(k+1)}$. Let C_k be the largest (non-degenerate) hypercube contained in $F_x^{(k)}$. Define

$$F_x^{(k+1)} := S^{(0)}(\alpha, C_k) \cup P_k.$$

If $x(k+1) = 0$ then let $s < k$ be largest s.t. $x(s+1) = 1$ (or $s = 0$ if there is none). We define

$$F_x^{(k+1)} := S^{(k+1-s)}(\alpha, C_s) \cup P_s.$$

Define $f_p := x \mapsto F_x = \bigcap_{k \in \mathbb{N}} F_x^{(k)}$. Clearly F_x is closed, as intersection of closed sets. The continuity of the map f_p is guaranteed by the fact that for each k,

$$\mathrm{d}_{\mathcal{H}}(F_x^{(k)}, F_x^{(k+1)}) \le 2^{-(k+1)}.$$

This follows from our choice of P_k in the first case, and $\mathrm{d}_{\mathcal{H}}(S^{(k)}(\alpha), S^{(k+1)}(\alpha)) \le 2^{-(k+1)}$ in the second case.

Adapting the proof of Lemma 6.9, it is possible to show that F_x is Salem and that $\dim(F_x) = p$ iff $x \in Q_2$. ■

From Lemma 6.16 we can derive the following results, as we did with their analogs in the previous section.

Proposition 6.17:
For every $p \in [0, d)$ *the sets*

$$\{A \in \mathbf{K}([0,1]^d) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{K}([0,1]^d) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are $\boldsymbol{\Sigma}^0_2$*-complete.*

Theorem 6.18:
For every $p \in (0, d]$ *there exists a continuous map* $F\colon 2^{\mathbb{N}\times\mathbb{N}} \to \mathbf{K}([0,1]^d)$ *s.t. for every* $x \in 2^{\mathbb{N}\times\mathbb{N}}$*,* $F(x)$ *is a Salem set and* $\dim(F(x)) \geq p$ *iff* $x \in P_3$*. Letting*

$$X_1 := \{A \in \mathbf{K}([0,1]^d) \,:\, \dim_{\mathcal{H}}(A) \geq p\},$$
$$X_2 := \{A \in \mathbf{K}([0,1]^d) \,:\, \dim_{\mathrm{F}}(A) \geq p\}$$

we have that every set X *s.t.* $X_2 \subset X \subset X_1$ *is* $\boldsymbol{\Pi}^0_3$*-hard. In particular,* X_1 *and* X_2 *are* $\boldsymbol{\Pi}^0_3$*-complete.*

Theorem 6.19:
The set $\{A \in \mathbf{K}([0,1]^d) \,:\, A \in \mathscr{S}([0,1]^d)\}$ *is* $\boldsymbol{\Pi}^0_3$*-complete.*

We now discuss an alternative proof for the $\boldsymbol{\Pi}^0_3$-completeness of the closed Salem subsets of $[0,1]^d$: as noticed in the introduction, using a theorem of Gatesoupe [40] we can show that if $A \subset [0,1]$ has at least two points[4] and is Salem with dimension α, then the rotationally invariant set $\tilde{A} := \{x \in [-1,1]^d \,:\, |x| \in A\}$ is a Salem set with dimension $d-1+\alpha$. Using Theorem 6.2, we can map $\tilde{A}$ to a Salem subset of C, for every d-dimensional cube C. Moreover, for each $p \in [d-1, d]$ there is a compact set $Y_p \subset [0,1]^d$ with null Fourier dimension and Hausdorff dimension p (e.g. consider the cartesian product of $[0,1]^{d-1}$ with a non-empty subset of $[0,1]$ with Hausdorff dimension $p-(d-1)$). Let $\{C_n\}_{n\in\mathbb{N}}$ be a family of mutually disjoint closed cubes s.t.

- $C_n \subset [0,1]^d$,
- $\overline{\bigcup_{n\in\mathbb{N}} C_n} = \{\mathbf{0}\} \cup \bigcup_{n\in\mathbb{N}} C_n$,

where $\mathbf{0}$ is the origin of the d-dimensional Euclidean space. We mimic the proof of Theorem 6.11 and construct a set X_n within each C_n, where each X_n is the image, under a similarity transformation, of either Y_p or a radial set of the type $\tilde{A}_n$ for some $A_n \subset [0,1]$. Then we define $X := \{\mathbf{0}\} \cup \bigcup_{n\in\mathbb{N}} X_n$. Since the cubes are disjoint we have that $\dim_{\mathrm{F}}(X) = \sup_n \dim_{\mathrm{F}}(X_n)$ and therefore, by carefully choosing the dimensions of each A_n, we obtain the results on the complexities.

[4] The requirement of having at least two points is just to avoid trivial counterexamples.

Notice however that each $\tilde{A}_n$ has Fourier (and hence Hausdorff) dimension at least $d-1$. Hence, while this argument suffices to show the $\mathbf{\Pi}^0_3$-completeness of $\{A \in \mathbf{K}([0,1]^d) \,:\, A \in \mathscr{S}([0,1]^d)\}$, it gives no information on the complexity of the first two sets listed in Theorem 6.18 when $p < d-1$. On the other hand, the construction presented in Lemma 6.16 has the advantage to work for every $p \in [0,d]$.

6.4 The complexity of closed Salem subsets of $\mathbb{R}^d$

Let us now turn our attention to the closed Salem subsets of $\mathbb{R}^d$. In this section, we determine the descriptive complexity of the conditions $\dim_{\mathcal{H}}(A) > p$, $\dim_{\mathcal{H}}(A) \geq p$, $\dim_{\mathrm{F}}(A) > p$, $\dim_{\mathrm{F}}(A) \geq p$, $A \in \mathscr{S}(\mathbb{R}^d)$, when A is a closed subset of $\mathbb{R}^d$ and $p \in \mathbb{R}$.

The hardness results lift easily from the compact cases.

Proposition 6.20:
For every $p \in (0,d]$ *and every* $q \in [0,d)$*, we have*

- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) > q\}$ *is* $\mathbf{\Sigma}^0_2$*-hard;*
- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*-hard;*
- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathrm{F}}(A) > q\}$ *is* $\mathbf{\Sigma}^0_2$*-hard;*
- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathrm{F}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*-hard;*
- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, A \in \mathscr{S}(\mathbb{R}^d)\}$ *is* $\mathbf{\Pi}^0_3$*-hard.*

Proof: This is a corollary of Proposition 6.17, Theorem 6.18 and Theorem 6.19. Indeed, since the Fourier and Hausdorff dimensions of $A \subset [0,1]^d$ do not change if we see A as a subset of $\mathbb{R}^d$, it is enough to notice that the inclusion map $\mathbf{K}([0,1]^d) \hookrightarrow \mathbf{F}(\mathbb{R}^d)$ is continuous. ■

Notice that, since the inclusion $\mathbf{K}([0,1]^d) \hookrightarrow \mathbf{V}(\mathbb{R}^d)$ is continuous as well, the same proof provides a lower bound for the above conditions when the hyperspace $\mathbf{F}(\mathbb{R}^d)$ is endowed with the Vietoris topology. However, since $\mathbf{V}(\mathbb{R}^d)$ is not Polish, we cannot say that the conditions are hard for their respective classes.

As in the previous sections, we obtain the upper bounds endowing $\mathbf{F}(\mathbb{R}^d)$ with the upper Fell topology. This will yield, as a corollary, that each of the above conditions is complete for its respective class when $\mathbf{F}(\mathbb{R}^d)$ is endowed with the Fell topology (in case of the upper Fell or Vietoris topology we only obtain a Wadge-equivalence).

Since the proofs of Proposition 6.5, Proposition 6.6 and Proposition 6.13 exploit the compactness of the ambient space, some extra care is needed when working in a non-compact environment.

Lemma 6.21:
- $\{(K,p) \in \mathbf{K}_U(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(K) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
- $\{(K,p) \in \mathbf{K}_U(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(K) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*.*

Proof: Define

$$D(K) := \{s \in [0,d] \,:\, (\exists \mu \in \mathbb{P}(K))(\exists c > 0)(\forall x \in \mathbb{R}^d)(\forall r > 0)(\mu(B\,(x,r)) \leq cr^s)\}$$

and recall that $\dim_{\mathcal{H}}(K) = \sup D(K)$. For every n, let $K_n := \overline{B\,(\mathbf{0}, n)}$. Observe that

$$\mu \in \mathbb{P}(K) \iff \mu \in \mathbb{P}(\mathbb{R}^d) \wedge \mu(K) \geq 1 \wedge (\exists n \in \mathbb{N})(\mu(K_n) \geq 1).$$

We can therefore rewrite $D(K)$ as follows

$$\begin{aligned} D(K) = \{s \in [0,d] \,:\, &(\exists \mu \in \mathbb{P}(\mathbb{R}^d))(\exists c > 0)(\exists n \in \mathbb{N}) \\ &(\mu(K) \geq 1 \wedge \mu(K_n) \geq 1 \wedge (\forall x \in \mathbb{R}^d)(\forall r > 0)(\mu(B\,(x,r)) \leq cr^s))\}. \end{aligned}$$

In particular $\mu(K_n) \geq 1$ implies that $\operatorname{spt} \mu \subset K_n$, hence

$$\mu(B\,(x,r)) \leq cr^s \iff \mu(H) \geq 1 - cr^s,$$

where $H := \overline{B\,(\mathbf{0}, n+x+r)} \setminus B\,(x,r)$. It is routine to prove that the function $\varphi\colon \mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbf{K}_U(\mathbb{R}^d)$ that sends (n,r,x) to the above-defined H is continuous. Notice that if we had set $H = K \setminus B\,(x,r)$ then the resulting map would not be continuous. This motivates the use of K_n in the above characterization of $D(K)$.

By Lemma 6.4 the set

$$\{(\mu, K, a) \in \mathbb{P}(\mathbb{R}^d) \times \mathbf{K}_U(\mathbb{R}^d) \times \mathbb{R} \,:\, \mu(K) \geq a\}$$

is closed. In particular the condition $\mu(B\,(x,r)) \leq cr^s$ is closed and the set

$$\begin{aligned} Q := \{(s,\mu) \in [0,d] \times \mathbb{P}(\mathbb{R}^d) \,:\, &(\exists c > 0)(\exists n \in \mathbb{N}) \\ &(\mu(K) \geq 1 \wedge \mu(K_n) \geq 1 \wedge (\forall x \in \mathbb{R}^d)(\forall r > 0)(\mu(B\,(x,r)) \leq cr^s))\} \end{aligned}$$

is $\mathbf{\Sigma}^0_2$.

Notice that we can equivalently consider Q as a subset of $[0,d] \times \bigcup_{n \in \mathbb{N}} \mathbb{P}(K_n)$. In particular, $D(K)$ is the projection of a $\mathbf{\Sigma}^0_2$ set along a metrizable and $\mathbf{K}_\sigma$ space (as $\mathbb{P}(X)$ is compact if X is). Therefore, using Lemma 6.3 we can conclude that $D(K)$ is $\mathbf{\Sigma}^0_2$ and that the conditions

$$\dim_{\mathcal{H}}(K) > p \iff (\exists s \in \mathbb{Q})(s > p \wedge s \in D(K)),$$

$$\dim_{\mathcal{H}}(K) \geq p \iff (\forall s \in \mathbb{Q})(s < p \to s \in D(K))$$

are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_3$ respectively. ■

Lemma 6.22:
The set $\{(A,B) \in \mathbf{F}_U(\mathbb{R}^d) \times \mathbf{F}(\mathbb{R}^d) \,:\, B \subset A\}$ *is* $\mathbf{\Pi}^0_1$.

PROOF: It suffices to show that the complement of the set is open. If $B \not\subset A$, fix $x \in B \setminus A$ and let $\varepsilon := d(x,A) > 0$. Let $\mathcal{U}_1 := \{F \in \mathbf{F}(\mathbb{R}^d) \,:\, F \cap \overline{B(x,\varepsilon/2)} = \emptyset\}$ and $\mathcal{U}_2 := \{F \in \mathbf{F}(\mathbb{R}^d) \,:\, F \cap B(x,\varepsilon/2) \neq \emptyset\}$. Clearly $\mathcal{U}_1 \times \mathcal{U}_2$ is open in $\mathbf{F}_U(\mathbb{R}^d) \times \mathbf{F}(\mathbb{R}^d)$, $(A,B) \in \mathcal{U}_1 \times \mathcal{U}_2$ and every $(A',B') \in \mathcal{U}_1 \times \mathcal{U}_2$ is s.t. $B' \not\subset A'$. ∎

Theorem 6.23:
The sets

$$X_1 := \{(A,p) \in \mathbf{F}_U(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$X_2 := \{(A,p) \in \mathbf{F}_U(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) \geq p\}$$

are $\mathbf{\Sigma}^0_2$ *and* $\mathbf{\Pi}^0_3$ *respectively. In particular, for every* $p \in [0,d)$ *and* $q \in (0,d]$*, the sets*

$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(A) \geq q\}$$

are $\mathbf{\Sigma}^0_2$*-complete and* $\mathbf{\Pi}^0_3$*-complete respectively.*

PROOF: Notice that, as a consequence of the countable stability of the Hausdorff dimension, we have

$$\dim_{\mathcal{H}}(A) = \sup\{\dim_{\mathcal{H}}(K) \,:\, K \subset A \text{ and } K \text{ is compact}\},$$

and therefore

$$\dim_{\mathcal{H}}(A) > p \iff (\exists K \in \mathbf{F}(\mathbb{R}^d))(K \subset A \wedge K \in \mathbf{K}(\mathbb{R}^d) \wedge \dim_{\mathcal{H}}(K) > p).$$

Notice that the condition $K \subset A$ is $\mathbf{\Pi}^0_1$ by Lemma 6.22. We claim that the conjunction $K \in \mathbf{K}(\mathbb{R}^d) \wedge \dim_{\mathcal{H}}(K) > p$ is $\mathbf{\Sigma}^0_2$. This follows from the fact $K \in \mathbf{K}(\mathbb{R}^d)$ is equivalent to $(\exists n)(K \subset \overline{B(\mathbf{0},n)})$, hence it is $\mathbf{\Sigma}^0_2$ using again Lemma 6.22; moreover, since the inclusion map $\mathbf{F}(X)|_{\mathbf{K}(X)} \hookrightarrow \mathbf{K}_U(X)$ is continuous, by Lemma 6.21 the condition $\dim_{\mathcal{H}}(K) > p$ is $\mathbf{\Sigma}^0_2$ when K is compact.

This shows that the set X_1 is the projection of a $\mathbf{\Sigma}^0_2$ set along $\mathbf{F}(\mathbb{R}^d)$. Since $\mathbf{F}(\mathbb{R}^d)$ is compact, we can use Lemma 6.3 and conclude that X_1 is $\mathbf{\Sigma}^0_2$.

Moreover, since $\dim_{\mathcal{H}}(A) \geq p$ iff $(\forall r \in \mathbb{Q})(r < p \to \dim_{\mathcal{H}}(A) > r)$, this also shows that X_2 is $\mathbf{\Pi}^0_3$. The completeness follows from Proposition 6.20. ∎

With a similar strategy, we can characterize the upper bounds for the Fourier dimension:

Theorem 6.24:
The sets

$$X_1 := \{(A,p) \in \mathbf{F}_U(\mathbb{R}^d) \times [0,d] \ : \ \dim_{\mathrm{F}}(A) > p\},$$
$$X_2 := \{(A,p) \in \mathbf{F}_U(\mathbb{R}^d) \times [0,d] \ : \ \dim_{\mathrm{F}}(A) \geq p\}$$

are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_3$ respectively. In particular, for every $p \in [0,d)$ and $q \in (0,d]$, the sets

$$\{A \in \mathbf{F}(\mathbb{R}^d) \ : \ \dim_{\mathrm{F}}(A) > p\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) \ : \ \dim_{\mathrm{F}}(A) \geq q\}$$

are $\mathbf{\Sigma}^0_2$-complete and $\mathbf{\Pi}^0_3$-complete respectively.

Proof: Notice that the condition

$$(\forall x \in \mathbb{R}^d)(|\widehat{\mu}(x)| \leq c|x|^{-s/2})$$

is closed, as the map $(\mu, x, s, c) \mapsto |\widehat{\mu}(x)| - c|x|^{-s/2}$ is continuous (see also the proof of Proposition 6.6). In particular, this implies that the sets

$$\{(K,p) \in \mathbf{K}_U(\mathbb{R}^d) \times [0,d] \ : \ \dim_{\mathrm{F}}(K) > p\},$$
$$\{(K,p) \in \mathbf{K}_U(\mathbb{R}^d) \times [0,d] \ : \ \dim_{\mathrm{F}}(K) \geq p\}$$

are $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_3$ respectively.

Since the Fourier dimension is inner regular for compact sets, we can write

$$\dim_{\mathrm{F}}(A) > p \iff (\exists K \in \mathbf{F}(\mathbb{R}^d))(K \subset A \land K \in \mathbf{K}(\mathbb{R}^d) \land \dim_{\mathrm{F}}(K) > p).$$

As in the proof of Theorem 6.23, using Lemma 6.22 and the fact that the inclusion map $\mathbf{F}(X)|_{\mathbf{K}(X)} \hookrightarrow \mathbf{K}_U(X)$ is continuous, we have that X_1 is the projection of a $\mathbf{\Sigma}^0_2$ set along $\mathbf{F}(\mathbb{R}^d)$. Lemma 6.3 implies that X_1 is $\mathbf{\Sigma}^0_2$ and X_2 is $\mathbf{\Pi}^0_3$.

The completeness follows from Proposition 6.20. ■

Theorem 6.25:
The set $\{A \in \mathbf{F}(\mathbb{R}^d) \ : \ A \in \mathscr{S}(\mathbb{R}^d)\}$ is $\mathbf{\Pi}^0_3$-complete.

Proof: Using Theorem 6.23 and Theorem 6.24 we have that, for every p, the conditions $\dim_{\mathcal{H}}(A) > p$ and $\dim_{\mathrm{F}}(A) > p$ are $\mathbf{\Sigma}^0_2$. The fact that $\{A \in \mathbf{F}(\mathbb{R}^d) \ : \ A \in \mathscr{S}(\mathbb{R}^d)\}$ is $\mathbf{\Pi}^0_3$ follows as in the proof of Theorem 6.7, while the completeness follows from Proposition 6.20. ■

6.5 Further results

Let X be $[0,1]^d$ or $\mathbb{R}^d$, for some $d \geq 1$. Notice that the set $\mathscr{S}_c(X)$ is comeager in $\mathbf{V}(X)$. Indeed, the set $\{K \in \mathbf{V}(X) : \dim_{\mathcal{H}}(K) \leq 0\} \subset \mathscr{S}_c(X)$ is $\mathbf{\Pi}^0_2$ by Proposition 6.5 (and its higher-dimensional analogues), and dense because it contains the set $\{K \in \mathbf{V}(X) : K \text{ is finite}\}$, which is dense. The same argument also shows that for every p the sets $\{K \in \mathbf{V}(X) : \dim_{\mathcal{H}}(K) \leq p\}$ and $\{K \in \mathbf{V}(X) : \dim_{\mathrm{F}}(K) \leq p\}$ are comeager. The same argument shows that the above sets are comeager in $\mathbf{F}(X)$. This, in turn, implies that the sets $\{K \in \mathbf{F}(X) : \dim_{\mathcal{H}}(K) > 0\}$, $\{K \in \mathbf{F}(X) : \dim_{\mathrm{F}}(K) > 0\}$, and $\{K \in \mathbf{F}(X) : K \notin \mathscr{S}_c(X)\}$ are meager and not comeager (as $\mathbf{F}(X)$ is Polish, and hence it is a Baire space).

Notice that our results easily imply that the maps $\dim_{\mathcal{H}}$, $\dim_{\mathcal{H}}^{\mathbf{F}}$, $\dim_{\mathrm{F}}$, and $\dim_{\mathrm{F}}^{\mathbf{F}}$ are $\mathbf{\Sigma}^0_3$-measurable. Using [62, Thm. 24.3], this is equivalent to both $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$ being Baire class 2. We will prove a stronger result in Theorem 7.23.

An interesting question is whether the converse inequality in Gatesoupe's theorem holds. Precisely, fixed d, we denoted with $\tilde{A}$ the d-dimensional radial expansion of A. The question is whether $\dim_{\mathrm{F}}(\tilde{A}) = d-1+\dim_{\mathrm{F}}(A)$. In the following, we adapt a classical argument to answer affirmatively for $d = 1 + 4k$.

The proof of the following lemma was suggested by Betsy Stovall.

Lemma 6.26:
Let $A \subset \mathbb{R}^d$ be a radial set and let $\mu \in \mathcal{M}(A)$, where $\mathcal{M}(A)$ denotes the set of finite Radon measures supported on A. There is a radial measure $\nu \in \mathcal{M}(A)$ s.t. $\dim_{\mathrm{F}}(\mu) \leq \dim_{\mathrm{F}}(\nu)$.

Proof: Given μ we can define a measure ν on A as

$$\nu(E) := \int_{O(d)} \mu(g(E))\, d\theta_d(g)$$

where $O(d)$ is the orthonormal group in dimension d and θ_d is the d-dimensional Haar measure. Since $\operatorname{spt}(\mu) \subset A$ it is clear that $\operatorname{spt}(\nu) \subset A$. Moreover, it follows from the invariance of the Haar measure that the measure ν is radial, i.e. $\nu(E) = \nu(h(E))$ for every $h \in O(d)$. Notice that

$$\begin{aligned}
\widehat{\nu}(\xi) &= \int e^{-i\,x\xi}\, d\nu(x) = \\
&= \int e^{-i\,x\xi} \int_{O(d)} d\mu(g(x))\, d\theta_d(g) = \\
&= \int_{O(d)} \int e^{-i\,x\xi}\, d\mu(g(x))\, d\theta_d(g) = \\
&= \int_{O(d)} \int e^{-i\,g^{-1}(x)\xi}\, d\mu(x)\, d\theta_d(g) = \\
&= \int_{O(d)} \int e^{-i\,xg(\xi)}\, d\mu(x)\, d\theta_d(g) = \int_{O(d)} \widehat{\mu}(g(\xi))\, d\theta_d(g) \ .
\end{aligned}$$

In particular, since g preserves the norm,

$$|\widehat{\nu}(\xi)| \leq \int_{O(d)} |\widehat{\mu}(g(\xi))|\, d\theta_d(g) \leq \int_{O(d)} |g(\xi)|^{-\alpha}\, d\theta_d(g) = |\xi|^{-\alpha},$$

which concludes the proof. ∎

Proposition 6.27:
Let $A \subset [a,b]$ with $a > 0$ and let $\tilde{A}$ be the d-dimensional radial expansion of A, namely

$$\tilde{A} := \{x \in \mathbb{R}^d \, : \, |x| \in A\}$$

where $|\cdot|$ denotes the Euclidean norm. If $d = 1 + 4k$, for some $k \in \mathbb{N}$, then

$$\dim_{\mathrm{F}}(\tilde{A}) = d - 1 + \dim_{\mathrm{F}}(A)$$

Proof: The inequality $\dim_{\mathrm{F}}(\tilde{A}) \geq n-1+\dim_{\mathrm{F}}(A)$ follows from [40, Lem., p. 125]. To prove the converse inequality, let μ be a finite Radon measure on $\tilde{A}$ s.t. $\dim_{\mathrm{F}}(\mu) \geq d-1+\alpha$. By Lemma 6.26 we can assume that μ is radial. In particular, it can be written as the product measure of the $(d-1)$-dimensional surface measure σ^{d-1} and of a 1-dimensional measure supported on A. In particular, let μ_* be s.t.

$$d\mu = r^{-\frac{d-1}{2}}\, d\mu_* \otimes d\sigma^{d-1}\,.$$

Using the classical formula for integration in polar coordinates we have

$$\widehat{\mu}(\xi) = \int_0^\infty \int_{S^{d-1}} r^{\frac{d-1}{2}} e^{-i\,\xi r \alpha}\, d\sigma^{d-1}(\alpha)\, d\mu_*(r) \quad .$$

Following [80, Sec. 3.3], letting $\rho = |\xi|$, we have

$$\widehat{\mu}(\xi) = c_0 \rho^{1-\frac{d}{2}} \int_A r^{\frac{1}{2}} J_{\frac{d-2}{2}}(\rho r)\, d\mu_*(r)$$

where c_0 is a constant that only depends on the dimension of the space and $J_{\frac{d-2}{2}}$ is the Bessel function with index $\frac{d-2}{2}$. We use the following asymptotics for the Bessel function for $x \to \infty$

$$J_{\frac{d-2}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos\Big(x + \frac{\pi}{4}(d-1)\Big) + R(x),$$

where $R = O(x^{-\frac{3}{2}})$. Writing the cos as sum of complex exponentials, we have

$$\begin{aligned}
\widehat{\mu}(\xi) = c_0 \rho^{-\frac{d-1}{2}} \sqrt{\frac{2}{\pi}} &\left(c_1 \int_a^b e^{-i\,r\rho}\, d\mu_*(r) + \overline{c_1} \int_a^b e^{i\,r\rho}\, d\mu_*(r) \right) + \\
&+ c_2 \rho^{1-\frac{d}{2}} \int_a^b r^{\frac{1}{2}} R(\rho r)\, d\mu_*(r) = \\
&= c' \rho^{-\frac{d-1}{2}} \Big(c_1 \widehat{\mu_*}(\rho) + \overline{c_1 \widehat{\mu_*}(\rho)} \Big) + c_2 \rho^{1-\frac{d}{2}} \int_a^b r^{\frac{1}{2}} R(\rho r)\, d\mu_*(r) = \\
&= c' \rho^{-\frac{d-1}{2}} 2\,\mathrm{Re}(c_1 \widehat{\mu_*}(\rho)) + c_2 \rho^{1-\frac{d}{2}} \int_a^b r^{\frac{1}{2}} R(\rho r)\, d\mu_*(r)
\end{aligned}$$

where $c' := c_0\sqrt{2/\pi}$, and c_1, c_2 are constants that only depend on the dimension of the space. In particular $c_1 = e^{-i\beta}$, with $\beta := \frac{\pi}{4}(d-1)$. Therefore

$$|\widehat{\mu}(\xi)| \geq \left| c'\rho^{-\frac{d-1}{2}} 2\,\mathrm{Re}(c_1\widehat{\mu_*}(\rho))\right| - \left| c_2\rho^{1-\frac{1}{2}} \int_a^b r^{\frac{1}{2}} R(\rho r)\, d\mu_*(r)\right|$$

hence

$$\begin{aligned}
\left| c'\rho^{-\frac{d-1}{2}} 2\,\mathrm{Re}(c_1\widehat{\mu_*}(\rho))\right| &\leq |\widehat{\mu}(\xi)| + \left| c_2\rho^{1-\frac{d}{2}} \int_a^b r^{\frac{1}{2}} R(\rho r)\, d\mu_*(r)\right| \leq \\
&\leq |\widehat{\mu}(\xi)| + |c_2|\rho^{1-\frac{d}{2}} \int_a^b r^{\frac{1}{2}} |\rho r|^{-\frac{3}{2}}\, d\mu_*(r) \leq \\
&\leq \rho^{-\frac{d-1+\alpha}{2}} + |c_2|\rho^{-\frac{d+1}{2}} \int_a^b r^{-\frac{d-3}{2}}\, d\mu_*(r) \leq \\
&\leq \rho^{-\frac{d-1+\alpha}{2}} + |c_3|\rho^{-\frac{d+1}{2}}
\end{aligned}$$

for some constant c_3, where we used the fact that $a > 0$ and hence $r^{-\frac{d-3}{2}}$ attains a maximum in $[a, b]$. This shows that, for some constant $c > 0$

$$|2\,\mathrm{Re}(c_1\widehat{\mu_*}(\rho))| \leq c\,\rho^{-\frac{\alpha}{2}}\ .$$

Since $d = 1 + 4k$, we have $c_1 = e^{-ik\pi}$ and therefore

$$|2\,\mathrm{Re}(c_1\widehat{\mu_*}(\rho))| = |2\,\mathrm{Re}(\widehat{\mu_*}(\rho))|\,.$$

Define the measure ν as

$$\nu(E) := \mu_*(E) + \mu_*(-E).$$

Clearly $\mathrm{spt}(\nu) = \mathrm{spt}(\mu_*) \cup -\mathrm{spt}(\mu_*) \subset A \cup -A$. To conclude the proof, it suffices to show that $\dim_{\mathrm{F}}(\nu) \geq \alpha$. Indeed, $\dim_{\mathrm{F}}(A) = \dim_{\mathrm{F}}(-A)$ (as the Fourier dimension is invariant under invertible affine transformations) and $\dim_{\mathrm{F}}(A \cup -A) = \max\{\dim_{\mathrm{F}}(A), \dim_{\mathrm{F}}(-A)\}$ as A is closed[5]. In particular, $\dim_{\mathrm{F}}(\nu) \geq \alpha$ implies $\dim_{\mathrm{F}}(A \cup -A) = \dim_{\mathrm{F}}(A) \geq \alpha$.

Notice that

$$\begin{aligned}
\widehat{\nu}(t) &= \int e^{-i\,xt}\, d\nu(t) = \\
&= \int_A e^{-i\,xt}\, d\mu_*(x) + \int_{-A} e^{-i\,xt}\, d\mu_*(-x) = \\
&= \int_A e^{-i\,xt}\, d\mu_*(x) + \overline{\int_A e^{-i\,xt}\, d\mu_*(x)} = \\
&= \widehat{\mu_*}(t) + \overline{\widehat{\mu_*}(t)} = 2\,\mathrm{Re}(\widehat{\mu_*}(t))
\end{aligned}$$

The claim follows from the estimate we obtained on $|2\,\mathrm{Re}(\widehat{\mu_*}(t))|$. ■

[5]The closedness is not necessary in this case, as A and $-A$ are not only disjoint but also “nicely separated”.

7 Effective aspects of the Hausdorff and Fourier dimension

In this section, we study the effective counterparts of the results on the (boldface) complexity of the family of closed Salem sets. We first introduce a few notions from computable measure theory, and then, in Section 7.3 we will study the Hausdorff and Fourier dimension from the point of view of the lightface hierarchy.

7.1 Basic tools

Definition 7.1: Let (X, d, α) be a computable metric space and let $B_{\langle i,j \rangle} := B(\alpha(i), q_j)$. We say that a compact $K \subset X$ is *co-c.e. compact* if

$$\left\{ \sigma \in \mathbb{N}^{<\mathbb{N}} \,:\, K \subset \bigcup_{i<|\sigma|} B_{\sigma(i)} \right\} \in \Sigma_1^0 .$$

We say that K is *computably compact* if it is co-c.e. compact and there exists a computable dense sequence in K.

We say that a sequence $(K_n)_{n \in I}$ is *uniformly co-c.e. compact* if each K_n is co-c.e. compact in a computable metric space X_n and the set

$$\left\{ (n, \sigma) \in \mathbb{N} \times \mathbb{N}^{<\mathbb{N}} \,:\, K_n \subset \bigcup_{i<|\sigma|} B^n_{\sigma(i)} \right\}$$

is c.e., where B^n_k is the k-th basic open ball in X_n. In other words, the sequence $\{K_n\}_{n \in I}$ is uniformly co-c.e. compact if there is a unique computable function witnessing that each K_n is co-c.e. compact.

The notions of co-c.e. compact and computably compact are standard notions in computable analysis (see e.g. [17, Def. 2.10]). Notice that being co-c.e. compact implies being $\Pi^0_1(X)$ and that every $\Pi^0_1(X)$ subset of a co-c.e. compact space is co-c.e. compact. Clearly a computable metric space is co-c.e. compact iff it is computably compact. Moreover, if K is co-c.e. compact (resp. computably compact) and $f\colon K \to Y$ is computable and surjective, then Y is co-c.e. compact (resp. computably compact) as well (see [89, Prop. 5.3]). A list of equivalent conditions to being computably compact are listed in [89, Prop. 5.2]. The notion of co-c.e. compact can be extended in a straightforward way to effective spaces.

We also mention the following simple lemma:

Lemma 7.2:
If X is co-c.e. compact then so is $X^{\mathbb{N}}$.

Proof: The fact that the finite product of co-c.e. compact spaces is co-c.e. compact follows from [89, Prop. 5.4]. To prove that $X^{\mathbb{N}}$ is co-c.e. compact, recall that an open set in $X^{\mathbb{N}}$ is of the type $\mathcal{B} := \prod_{j\in\mathbb{N}} B_j$, where each B_j is open in X and $B_j \neq X$ only for finitely many indexes. Such an open set is canonically represented via (a name for) a finite sequence $(B_j)_{j<N}$ s.t. for every $j \geq N$, $B_j = X$.

Let $(\mathcal{B}_i)_{i<k}$ be a finite sequence of open subsets of $X^{\mathbb{N}}$, where $\mathcal{B}_i$ is represented by $(B^i_j)_{j<N_i}$. This sequence trivially induces a finite sequence $(\mathcal{C}_i)_{i<k}$ of open subsets of X^N, where $N := \max_{i<k} N_i$: for every i and every $j \in \{N_i, \dots, N-1\}$, let $B^i_j := X$ and define $\mathcal{C}_i := \prod_{j<N} B^i_j$. The sequence $(\mathcal{B}_i)_{i<k}$ covers $X^{\mathbb{N}}$ iff $(\mathcal{C}_i)_{i<k}$ covers X^N. The claim follows from the fact that the sets $(X^n)_{n\in\mathbb{N}}$ are uniformly co-c.e. compact. ■

We now give a brief introduction on how computable measure theory can be developed in the context of TTE. For a more thorough presentation we refer the reader to [24]. While the theory can be developed more generally for Borel measures on sequential topological spaces [99], for our purposes it is enough to focus on probability measures on X, where X is (computably homeomorphic to) either $[0,1]^d$ or $\mathbb{R}^d$. Notice that, since every represented space can be endowed with the final topology (which is sequential, as mentioned in Section 1.1.2), the theory can be developed for every represented space X.

Recall that, in general, if X is a separable metric space then so is $\mathbb{P}(X)$. A canonical choice for a dense subset of $\mathbb{P}(X)$ is the set $\mathcal{D}$ of probability measure concentrated on finitely many points of the dense subset of X, assigning rational mass to each of them (i.e. a weighted sum of Dirac deltas, where each weight is rational). Moreover, the Prokhorov metric on $\mathbb{P}(X)$ can be explicitly defined as

$$\rho(\mu,\nu) := \inf\{\varepsilon > 0 \,:\, (\forall A \in \mathcal{B}(X))(\mu(A) \leq \nu(A^\varepsilon) + \varepsilon)\},$$

with $A^\varepsilon := \{x \in X \,:\, d(x,A) < \varepsilon\}$. This metric induces the weak topology on $\mathbb{P}(X)$ defined in Section 6.1. It is known that the space $(\mathbb{P}(X), \rho, D)$ is a computable metric space ([55, Prop. 4.1.1]). As such, a canonical choice for a representation of a measure is the Cauchy representation.

From a computational point of view, it is often convenient to look at Borel (probability) measure from a different point of view. A *(probability) valuation* is a map $\nu\colon \boldsymbol{\Sigma}^0_1(X) \to [0,1]$ s.t.

- $\nu(\emptyset) = 0$;

- $\nu(X) = 1$;
- $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$.

Probability valuations can be defined in a slightly more general context as maps over a lattice ([99, Sec. 2.2]). Every Borel measure μ naturally induces a valuation $\nu := \mu|_{\mathbf{\Sigma}^0_1(X)}$. The induced valuation is lower semicontinuous[1], i.e. if $(A_i)_{i\in\mathbb{N}}$ are nested open sets then $\nu(\bigcup_i A_i) = \sup_i \nu(A_i)$. Since every finite Borel measure is uniquely identified by its restriction to the open sets (as every such measure on the Euclidean space is regular, and in particular outer regular, see e.g. [95, Thm. 2.18]), we can identify $\mathbb{P}(X)$ with the family of lower semicontinuous valuations on $\mathbf{\Sigma}^0_1(X)$.

The lower semicontinuity of the valuation functions can be naturally translated in the context of TTE. We define the represented space $(\mathbb{R}_<, \delta_<)$ as the set of real numbers, where $x \in \mathbb{R}$ is represented by a monotonically increasing sequence of rational numbers converging to x. Equivalently, we can think of a $\delta_<$-name for x as the list of all the rational numbers smaller than x. This is the so-called *left-cut representation* of the real numbers, and we say that a real is *left-c.e.* if it has a computable $\delta_<$-name (see [112, Sec. 4.1]). The final topology induced on $\mathbb{R}$ by $\delta_<$ is exactly the topology of lower semicontinuity (i.e. the topology whose open sets are of the form (x, ∞) for some $x \in \mathbb{R}$, see [112, Lem. 4.1.4]). Similarly, we can define the represented space $(\mathbb{R}_>, \delta_>)$ of the *right-c.e.* reals, where $\delta_>$ is the *right-cut representation* map, naming a real as a monotonically decreasing sequence of rationals converging to it. The final topology of $\delta_>$ is the topology of the upper-semicontinuity (again, see [112, Lem. 4.1.4]). It is straightforward to see that given a $\delta_<$-name for x we can computably find a $\delta_>$-name for $-x$ (and vice versa). Notice that $+\colon \mathbb{R}_< \times \mathbb{R}_< \to \mathbb{R}_<$ and $\sup\colon \mathbb{R}_<^{\mathbb{N}} \to \mathbb{R}_<$ are computable, but $-\colon \mathbb{R}_< \times \mathbb{R}_< \to \mathbb{R}_<$ is not. As a notational convenience, we can use $\mathbb{I}_<$ to denote the unit interval $[0, 1]$ represented using the left-cut representation.

With this in mind, we can define another representation on the space of Borel (probability) measures: the *canonical representation* δ_C for a (probability) measure names a measure μ using a name for the (realizer-)continuous function $\mu|_{\mathbf{\Sigma}^0_1(X)} : \mathbf{\Sigma}^0_1(X) \to \mathbb{I}_<$ ([99, Sec. 3.1]). The final topology on $\mathbb{P}(X)$ induced by δ_C coincides with the weak topology on $\mathbb{P}(X)$ ([99, Cor. 3.5]). Moreover, the canonical representation is equivalent to the Cauchy representation on $\mathbb{P}(X)$ ([99, Prop. 3.7]). In the development of the theory, it is often more convenient to think of a (probability) measure as being represented using the canonical representation, i.e. using a name for the induced valuation. We can therefore think of a name for a (probability) measure μ on X as a list of $\delta_<$-names for the measures of the basic open balls.

Theorem 7.3:
Let (X, δ_X) and (Y, δ_Y) be represented spaces, endowed with the final topology induced by their respective representation maps. Let also $\mathcal{C}(X, Y)$ be the set of continuous (equivalently, realizer-continuous) functions $X \to Y$. The following maps are computable:

1. $\mathbb{P}(X) \times \mathbf{\Sigma}^0_1(X) \to \mathbb{R}_< := (\mu, U) \mapsto \mu(U)$;
2. $\mathbb{P}(X) \times \mathbf{\Pi}^0_1(X) \to \mathbb{R}_> := (\mu, F) \mapsto \mu(F)$;
3. $\mathbb{P}(X) \times \mathbf{\Delta}^0_1(X) \to \mathbb{R} := (\mu, D) \mapsto \mu(D)$;
4. $\mathbb{P}(X) \times \mathcal{C}(X, Y) \to \mathbb{P}(Y) := (\mu, f) \mapsto \mu_f$, *where* $\mu_f(E) := \mu(f^{-1}(E))$ *is the push-forward measure;*

[1] Equivalently, it is continuous w.r.t. the Scott topology on $[0, 1]$, see e.g. [55, Rem. 3.1.1].

5. $\int : \mathcal{C}(X, \mathbb{R}_<) \times \mathbb{P}(X) \to \mathbb{R}_< := (f, \mu) \mapsto \int f \, d\mu$;

6. $\int : \mathcal{C}_{ebd}(X, \mathbb{R}) \times \mathbb{P}(X) \to \mathbb{R} := (f, \mu) \mapsto \int f \, d\mu$, *where* $\mathcal{C}_{ebd}(X, \mathbb{R})$ *denotes the space of effectively bounded continuous functions, i.e. a name for* $f \in \mathcal{C}_{ebd}(X, \mathbb{R})$ *is* $\langle p_f, p_a, p_b \rangle$, *where* p_f *is a* $\delta_{\mathcal{C}(X,\mathbb{R})}$*-name for* f *and* p_a, p_b *are computable names for* $a, b \in \mathbb{R}$ *s.t. for every* $x \in X$, $a < f(x) < b$.

Proof: Point 1. is straightforward from the definition of the canonical representation for $\mathbb{P}(X)$ (see also [55, Prop. 4.2.1]) and point 2. is a corollary of point 1. (as $\mu(F) = 1 - \mu(X \setminus F)$). Point 3. follows trivially from the points 1. and 2. as a $\delta_{\mathbb{R}}$-name for $x \in \mathbb{R}$ can be computably obtained from a $\delta_{\mathbb{R}_<}$-name and a $\delta_{\mathbb{R}_>}$-name of x. Point 4. is (essentially) a diagram-chasing exercise, see also [24, Prop. 49]. Points 5. and 6. are presented in [24, Sec. 3, in particular point 4. is prop. 7]. See also [99, Prop. 3.6] for a slightly more general version of point 5.. ■

For our purposes, we will also need an effective analog of the fact that if X is compact metrizable then so is $\mathbb{P}(X)$ ([62, Thm. 17.22]). The proof of the following theorem was suggested to us by Matthias Schröder.

Theorem 7.4:
For every computable metric space (X, d, α), *if* X *is computably compact then so is* $\mathbb{P}(X)$.

Proof: Since X is a computably compact computable metric space, there is a representation map $\delta : 2^{\mathbb{N}} \to X$ for X which is (computably) equivalent to the Cauchy representation on X ([11, Prop. 4.1]).

Every probability measure $\mu \in \mathbb{P}(2^{\mathbb{N}})$ can be identified with a function $\pi_\mu \in [0,1]^{\mathbb{N}}$ (identifying $\mathbb{N}$ with $2^{<\mathbb{N}}$) s.t.

$$(\forall \sigma \in \mathbb{N}^{<\mathbb{N}}) \, (\pi_\mu(()) = 1 \text{ and } \pi_\mu(\sigma) = \pi_\mu(\sigma^\frown(0)) + \pi_\mu(\sigma^\frown(1))) \, . \tag{$\star$}$$

The map $\Phi := \mu \mapsto \pi_\mu$ is a computable homeomorphism (i.e. a computable bijection with computable inverse) between $\mathbb{P}(\mathbb{N}^{\mathbb{N}})$ and a Π^0_1 subset of $[0,1]^{\mathbb{N}}$. Indeed, it is computable by Theorem 7.3(3) and its inverse Φ^{-1} is straightforwardly computable. Moreover, $\pi \in \operatorname{ran}(\Phi)$ iff it satisfies ($\star$), which is a Π^0_1 condition relative to π. This, in turn, implies that $\operatorname{ran}(\Phi)$ is co-c.e. compact, as $[0,1]^{\mathbb{N}}$ is computably compact (by Lemma 7.2). In particular, the fact that $\mathbb{P}(K)$ is computably homeomorphic to a co-c.e. compact space implies it is co-c.e. compact.

To conclude the proof, define $\psi : \mathbb{P}(2^{\mathbb{N}}) \to \mathbb{P}(X)$ as the pushforward operator $\psi(\mu) := \mu_\delta$ where $\mu_\delta(E) := \mu(\delta^{-1}(E))$. This map is computable (Theorem 7.3) and surjective ([101, Thm. 14]), and therefore $\mathbb{P}(X)$ is co-c.e. compact. ■

Corollary 7.5:
For every computable metric space X that admits an admissible representation $\delta :\subseteq k^{\mathbb{N}} \to X$ with co-c.e. compact domain, the space $\mathbb{P}(X)$ is co-c.e. compact.

PROOF: Trivial from Theorem 7.4 as if X admits an admissible representation $\delta :\subseteq k^{\mathbb{N}} \to X$ with co-c.e. compact domain then it is computably compact. ■

We conclude this section with the effective counterpart of Lemma 6.3:

Lemma 7.6:
Let X, Y be computable metric spaces. If Y is computably compact then, for every $F \in \Pi^0_1(X \times Y)$, $\operatorname{proj}_X F \in \Pi^0_1(X)$. If $Y = \bigcup_{n\in\mathbb{N}} Y_n$ where the sequence $(Y_n)_{n\in\mathbb{N}}$ is uniformly co-c.e. compact, then for every $F \in \Sigma^0_2(X \times Y)$, $\operatorname{proj}_X F \in \Sigma^0_2(X)$.

PROOF: Let us first assume that Y is computably compact and let $F \in \Pi^0_1(X \times Y)$. Let $p \in \mathbb{N}^{\mathbb{N}}$ be a computable map s.t. $F^{\mathrm{C}} = \bigcup_{n\in\mathbb{N}} B^X_{p(n)_0} \times B^Y_{p(n)_1}$. We can notice that $B^X_n \subset (\operatorname{proj}_X F)^{\mathrm{C}}$ iff the preimage $B^X_n \times Y$ of B^X_n via the projection map proj_X is contained in the complement of F.

Let $\varphi_p \colon \mathbb{N}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ be a computable function s.t., for all $\sigma \in \mathbb{N}^{<\mathbb{N}}$

$$\bigcap_{i\in\operatorname{ran}(\sigma)} B^X_{p(i)_0} = \bigcup_{k\in\mathbb{N}} B^X_{\varphi_p(\sigma,k)} .$$

Such a map exists because X is a computable metric space.

To show that $(\operatorname{proj}_X F)^{\mathrm{C}}$ is effectively open, notice that

$$\left\{ n \in \mathbb{N} \,:\, (\exists \sigma \in \mathbb{N}^{<\mathbb{N}}) \left(\bigcup_{i\in\operatorname{ran}(\sigma)} B^Y_{p(i)_1} = Y \text{ and } n \in \operatorname{ran}(\varphi_p(\sigma,\cdot)) \right) \right\} \in \Sigma^0_1.$$

This follows from the fact that φ_p is computable and that $\bigcup_{i\in\operatorname{ran}(\sigma)} B^Y_{p(i)_1} = Y$ is Σ^0_1 because Y is co-c.e. compact. This shows that we can computably enumerate a list of open sets exhausting the complement of $\operatorname{proj}_X F$, i.e. $\operatorname{proj}_X F \in \Pi^0_1(X)$.

The same argument shows (still assuming Y co-c.e. compact) that if $D \in \Pi^0_1(\mathbb{N} \times X \times Y)$ then $\operatorname{proj}_{\mathbb{N}\times X} D \in \Pi^0_1(\mathbb{N} \times X)$ (it is enough to replace X with $\mathbb{N} \times X$). If $F \in \Sigma^0_2(X \times Y)$ then it can be written as $F = \operatorname{proj}_{X\times Y} D$, for some $D \in \Pi^0_1(\mathbb{N} \times X \times Y)$. In particular,

$$\operatorname{proj}_X F = \operatorname{proj}_X \operatorname{proj}_{\mathbb{N}\times X} D ,$$

and therefore $\operatorname{proj}_X F \in \Sigma^0_2(X)$.

Finally, assume that $F \in \Sigma^0_2(X \times Y)$ and $Y = \bigcup_{n\in\mathbb{N}} Y_n$ where the $(Y_n)_{n\in\mathbb{N}}$ are uniformly co-c.e. compact. Let $D \in \Pi^0_1(\mathbb{N}\times X\times Y)$ be s.t. $F = \operatorname{proj}_{X\times Y} D$ (as above). Notice that, defining

$D_n := \{(k,x,y) \in D \,:\, y \in Y_n\}$, the sequence $(\mathrm{proj}_{\mathbb{N}\times X} D_n)_{n\in\mathbb{N}}$ is uniformly Π^0_1, as $(Y_n)_{n\in\mathbb{N}}$ is uniformly co-c.e. compact. In other words,

$$E := \{(n,k,x) \,:\, (\exists y)((k,x,y) \in D_n)\} \in \Pi^0_1(\mathbb{N}\times\mathbb{N}\times X),$$

and therefore $\mathrm{proj}_X F = \mathrm{proj}_X E \in \Sigma^0_2$ (using a canonical computable identification $\mathbb{N}^2 \to \mathbb{N}$). ■

7.2 The represented (hyper)spaces of closed and compact sets

In order to characterize the complexity of the Hausdorff and Fourier dimension from the point of view of the Kleene hierarchy, we first need to specify the lightface structure of the hyperspaces $\mathbf{F}(X)$ and $\mathbf{K}(X)$ are endowed with (recall that the lightface complexity is not uniquely determined by the topology, and, in fact, it is not invariant under homeomorphisms).

In general, for a computable metric space (X,d,α), there is a canonical choice for an admissible representation on the hyperspaces $\mathbf{F}_U(X)$ and $\mathbf{F}(X)$. Indeed, the negative information representation we commonly use for the closed subsets of a represented space is an admissible representation for $\mathbf{F}_U(X)$, while the full information representation ψ is admissible for $\mathbf{F}(X)$ (see [12])[2].

If X is compact then $\mathbf{F}_U(X)$ and $\mathbf{K}_U(X)$ coincide, and, in turn, they are equal, as represented spaces, with the represented space we denoted as $\mathbf{\Pi}^0_1(X)$. Moreover, $\mathbf{F}(X)$ and $\mathbf{K}(X)$ coincide, and they are Polish spaces. In particular, a natural choice for a dense subset of $\mathbf{K}(X)$ is an enumeration β of the finite subsets of $\mathrm{ran}(\alpha)$, i.e.

$$\beta(\langle\sigma\rangle) := \bigcup_{i<|\sigma|} \{\alpha(\sigma(i))\}\ .$$

The space $(\mathbf{K}(X), \mathrm{d}_{\mathcal{H}}, \beta)$ is therefore a computable metric space, and hence, it is canonically endowed with the Cauchy representation. This, in turn, induces a representation on the pointclasses $\mathbf{\Gamma}(\mathbf{K}(X))$ as described in Section 1.2.2.

As an alternative representation for $\mathbf{K}(X)$ (when X is compact), we can consider the following: for every $\sigma \in \mathbb{N}^{<\mathbb{N}}$, define

$$D_\sigma := \bigcup_{i<|\sigma|-1} B\left(\alpha(\sigma(i+1)), q_{\sigma(0)}\right)\ ,$$

where $(q_i)_{i\in\mathbb{N}}$ is the standard enumeration of $\mathbb{Q}$. It is clear that the set $\{D_\sigma\}_{\sigma\in\mathbb{N}^{<\mathbb{N}}}$ is a countable basis for X (as it contains the basic open balls centered in $\mathrm{ran}(\alpha)$ and with rational radius).

A countable basis for $\mathbf{K}(X)$ is given by the sets $\{\mathcal{V}_{\langle\sigma,\tau\rangle}\}_{\sigma,\tau\in\mathbb{N}^{<\mathbb{N}}}$, with

$$\mathcal{V}_{\langle\sigma,\tau\rangle} := \{F \in \mathbf{K}(X) \,:\, F \subset D_\sigma \wedge (\forall i < |\tau|)(F \cap B_{\tau(i)} \neq \emptyset)\}.$$

Notice that, in general, we cannot replace D_σ with a basic open ball for X. Indeed, if B_1 and B_2 are basic disjoint open balls for X then

$$\{F \,:\, F \subset B_1\} \cup \{F \,:\, F \subset B_2\} \subsetneq \{F \,:\, F \subset B_1 \cup B_2\}\ .$$

[2] For the sake of completeness we should mention that the positive information representation is admissible for the space $\mathbf{F}_L(X)$, which also makes it (intuitively) clearer why the full information representation is admissible for $\mathbf{F}(X)$.

It is easy to see that the family $\{\mathcal{V}_{\langle\sigma,\tau\rangle}\}_{\sigma,\tau\in\mathbb{N}^{<\mathbb{N}}}$ is a basis for $\mathbf{K}(X)$. Indeed, if we denote with

$$B_{\mathcal{H}}(F,r) := \{G \in \mathbf{K}(X) \;:\; \mathrm{d}_{\mathcal{H}}(F,G) < r\}$$

the open ball with respect to the Hausdorff metric, then every basic open ball $B_{\mathcal{H}}(\beta(\langle\rho\rangle),q)$ can be written as $\mathcal{V}_{\langle\sigma,\tau\rangle}$, where σ,τ are s.t. $D_\sigma = \bigcup_{i<|\rho|} B(\alpha(\rho(i)),q)$ and, for every $j < |\tau|$, $B_{\tau(j)} = B(\alpha(\rho(j)),q)$ (this is just a definition-chasing problem).

In particular, a closed set F is equivalently represented with the Cauchy representation or with a list $(\mathcal{V}_i)_{i\in\mathbb{N}}$, where $\mathcal{V}_i$ is a basic open ball w.r.t. to the Hausdorff metric that contains F and has radius 2^{-i}.

If X is a (not necessarily compact) subspace of the Euclidean space, the Cauchy representation for the space $\mathbf{K}(X)\setminus\{\emptyset\}$ was studied in [22] under the name δ_{Haus}. The authors showed that it is (computably) equivalent to the representation map $\delta^{=}_{\mathscr{K}}|^{\mathscr{K}^*}$ that names a non-empty compact subset K of X via a full information representation name of K and an index of a basic open ball B s.t. $K \subset \overline{B}$ [22, Thm. 4.10]. The Cauchy representation for the hyperspace of non-empty compact subsets of a generic computable metric space was studied in [20] (under the name $\delta_{\mathrm{Hausdorff}}$).

We now explicitly show that, if X is a computably compact metric space, the representation δ_{Haus} and the full information representation ψ on $\mathbf{K}(X)$ are equivalent (i.e. the empty set is not problematic).

Proposition 7.7:
If (X,d,α) is a computably compact metric space the Cauchy representation δ_{Haus} and the full information representation ψ on $\mathbf{K}(X)$ are equivalent.

Proof: Recall that, given two representations δ and δ', $\delta \le \delta'$ if there is a computable map that translates δ-names into δ'-names. Recall also that for every non-empty closed G, $\mathrm{d}_{\mathcal{H}}(G,\emptyset) = \mathrm{diam}(X)$.

$\delta_{\mathrm{Haus}} \le \psi$: Let p be a δ_{Haus}-name for F, i.e. p is a list of (indexes for) basic open balls $(\mathcal{B}_n)_{n\in\mathbb{N}}$ w.r.t. the Hausdorff metric s.t. all the balls contain F and the radius of $\mathcal{B}_n$ is 2^{-n}. Since the empty set is isolated in $\mathbf{K}(X)$, it is enough to consider n sufficiently large so that $2^{-n} < \mathrm{diam}(X)$. Indeed, recall that we can assume that the enumeration of the dense subset of a metric space is injective (see the comments after Definition 1.9). In particular, we can computably tell whether the i-th ball is centered on $\emptyset$. If $\mathcal{B}_n$ is centered on $\emptyset$ then $F=\emptyset$ (as $\mathcal{B}_n = \{\emptyset\}$). Otherwise $F \neq \emptyset$, hence we can use the fact that $\delta_{\mathrm{Haus}} \le \delta^{=}_{\mathscr{K}}|^{\mathscr{K}^*} \equiv \psi$ (the equivalence $\delta^{=}_{\mathscr{K}}|^{\mathscr{K}^*} \equiv \psi$ follows from the fact that $F \subset B(\emptyset, \mathrm{diam}(X))$).

$\psi \le \delta_{\mathrm{Haus}}$: Let $\langle p,q\rangle$ be a ψ-name for F, where p is a negative information name and q is a positive information name for F. Notice that, if $F=\emptyset$ then p is a list of basic open balls that cover X. On the other hand, if $F\neq\emptyset$ then q eventually lists some basic open ball (in X) that intersects F (q is allowed to not produce any information at stage i). In other words, we wait for some sufficiently large n so that either $\bigcup_{i<n} B_i$ covers X or $B_{q(i)} \cap F \neq \emptyset$. This allows us to determine whether F is empty or not. If $F=\emptyset$ we can trivially compute a sequence of basic open balls (in $\mathbf{K}(X)$) centered on $\emptyset$ with rapidly decreasing radii. Otherwise, as in the previous reduction, we can use the fact that $\psi \equiv \delta^{=}_{\mathscr{K}}|^{\mathscr{K}^*} \le \delta_{\mathrm{Haus}}$ to produce a δ_{Haus}-name for F. ■

The following lemma is the effective counterpart of [62, Thm. 4.26].

Lemma 7.8:
If X is computably compact then so is $\mathbf{K}(X)$.

PROOF: Let $(\mathcal{V}_i)_{i<k}$ be a finite sequence of basic open sets in $\mathbf{K}(X)$, with $\mathcal{V}_i = \mathcal{V}_{\langle\sigma_i,\tau_i\rangle}$ defined as above.

We describe a c.e. procedure to determine if $(\mathcal{V}_i)_{i<k}$ is a cover for $\mathbf{K}(X)$. We first check whether there is i s.t. $\tau_i = ()$. If this is not the case then the procedure returns a negative answer (or, equivalently, enters an infinite loop). Otherwise, let $Y_0 := X$. At stage $s+1$, we wait for some unmarked $i < k$ s.t. D_{σ_i} covers Y_s. If such i is found, we mark it as visited and define $Y_{s+1} := Y_s \setminus (\bigcup_{j<|\tau_i|} B_{\tau_i(j)})$, then go to the next stage. The procedure ends if $Y_{s+1} = \emptyset$, in which case $(\mathcal{V}_i)_{i<k}$ is a cover for $\mathbf{K}(X)$.

Let us show that the procedure is c.e.: first of all, the search for some i s.t. $\tau_i = ()$ is computable (as there are only finitely many such τ_i). Notice that, at each stage, Y_s is co-c.e. compact (as so is X). In particular, determining whether D_{σ_i} covers Y_s is a c.e. condition. Moreover, determining whether $Y_{s+1} = \emptyset$ is also c.e. (as it is equivalent to being covered by the empty set). This shows that the procedure is c.e.. Notice also that if the procedure reaches the stage $s+1 = k$ then it halts iff $Y_{s+1} \neq \emptyset$ (as there are no more unmarked i).

To prove that it correctly determines if $(\mathcal{V}_i)_{i<k}$ is a cover, we first notice that the initial search for some i s.t. $\tau_i = ()$ is only a technical step to be sure that the empty set (which is isolated) is covered. If $i_1, \dots, i_s$ are the indexes selected by stage s, then

$$F \notin \bigcup_{n=1}^{s} \mathcal{V}_{i_n} \Rightarrow F \subset Y_{s+1}$$

(this just follows from the definitions). In particular, if $Y_{s+1} = \emptyset$ then every $F \in \mathbf{K}(X)$ belongs to some $\mathcal{V}_i$.

Finally, if the procedure does not halt then, for some $s \le k$, $Y_s \neq \emptyset$ and there are no unmarked i s.t. D_{σ_i} covers Y_s. In particular, Y_s witnesses the fact that $(\mathcal{V}_i)_{i<k}$ is not a cover. ■

Proposition 7.9:
$\mathbf{F}(\mathbb{R}^d)$ is computably compact.

PROOF: We show that there is a computable surjection $f\colon \mathbf{K}([0,1]^d) \to \mathbf{F}(\mathbb{R}^d)$, and the claim will follow using Lemma 7.8. Fix a computable homeomorphism $\varphi\colon (0,1)^d \to \mathbb{R}^d$ and define

$$f(K) := \varphi(K \cap (0,1)^d) \ .$$

It is easy to see that, for $K \in \mathbf{K}([0,1]^d)$, $f(K)$ is closed: indeed, if $x \notin f(K)$ then $\varphi^{-1}(x) \notin K \cap (0,1)^d$. Since $K \cap (0,1)^d$ is closed (in the relative topology on $(0,1)^d$), there is a open neighborhood $U \subset (0,1)^d$ of $\varphi^{-1}(x)$ s.t. $U \cap K = \emptyset$. Since φ is a homeomorphism, $\varphi(U)$ is a open neighborhood of x and $\varphi(U) \cap f(K) = \emptyset$.

Moreover, f is surjective: for every $F \in \mathbf{F}(\mathbb{R}^d)$, $\varphi^{-1}(F)$ is closed in the relative topology of $(0,1)^d$. If we denote with G its closure w.r.t. the relative topology of $[0,1]^d$, we have that $G \setminus \varphi^{-1}(F) \subset \partial([0,1]^d)$, hence, in particular, $f(G) = F$.

Finally, we show that f is computable. Recall that both $\mathbf{K}([0,1]^d)$ and $\mathbf{K}(\mathbb{R}^d)$ are admissibly represented with the full information representation ψ. Let $\langle p, q\rangle \in \mathbb{N}^\mathbb{N}$ be a name for $K \in \mathbf{K}([0,1]^d)$. A negative information name for $f(K)$ can be computed from p as φ is computable: in fact, for every basic open $B \subset K^\mathrm{C}$ we can computably list a sequence of basic open balls of $\mathbb{R}^d$ covering $\varphi(B)$. On the other hand, notice that a basic open ball B of $\mathbb{R}^d$ intersects $f(K)$ iff there is $i \in \mathbb{N}$ s.t. $B_{q(i)} \subset \varphi^{-1}(B)$ (this follows from the fact that φ is a homeomorphism). In particular, to produce a positive information name for $f(K)$, we list $B_n \subset \mathbb{R}^d$ whenever we find some i s.t. $B_{q(i)} \subset \varphi^{-1}(B)$ (which is a computable condition). ∎

Recall that, if X is not compact, then the hyperspace $\mathbf{V}(X)$ of closed subsets X endowed with the Vietoris topology is not metrizable. We now show that it is not even admissibly represented.

Proposition 7.10:
The space $\mathbf{F}_{UV}(\mathbb{R})$ (and hence $\mathbf{F}_{UV}(\mathbb{R}^d)$) does not have a countable pseudobase. In particular, it is not second-countable and it is not admissibly represented.

Proof: Recall that, by definition, a countable pseudobase for $\mathbf{F}_{UV}(\mathbb{R})$ is a family $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ s.t. for every open set $\mathcal{U} \subset \mathbf{F}_{UV}(\mathbb{R})$, every closed set $F \in \mathcal{U}$ and every sequence $(G_n)_{n\in\mathbb{N}}$ converging to F,

$$(\exists i)(\exists n_0)(\{F\} \cup \{G_n : n \geq n_0\} \subset \mathcal{P}_i \subset \mathcal{U}).$$

Fix a countable family $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$. We want to build a closed set F and an open set $\mathcal{U}$ which contains F s.t. for every i, either $F \notin \mathcal{P}_i$ or $\mathcal{P}_i \not\subset \mathcal{U}$. We define $F := \{x_i : i \in \mathbb{N}\}$, where $(x_i)_{i\in\mathbb{N}}$ is a strictly increasing sequence iteratively defined as follows: for each i, let n_i be the smallest integer greater than $x_j + 1$ for every $j < i$ (if $i = 0$ we let $n_i := 0$). Choose an unbounded $P_i \in \mathcal{P}_i$. If there is none we just define $x_i := n_i$. Let $y_i \in P_i \cap [n_i, \infty)$ and choose x_i s.t. $x_i > y_i + 1$. Notice that $d(y_i, \{x_j\}_{j\leq i}) > 1$.

Notice that, for every $i \neq j$, $d(x_i, x_j) > 1$, hence the sequence $(x_i)_{i\in\mathbb{N}}$ is a non-convergent sequence with no accumulation points. In particular, the set $F := \{x_i : i \in \mathbb{N}\}$ is closed (as it is sequentially closed). Fix ε sufficiently small, e.g. $\varepsilon = 1/4$, and define $F_\varepsilon := \{x \in \mathbb{R} : d(x, F) < \varepsilon\}$ and $\mathcal{U} := \{G \in \mathbf{F}(\mathbb{R}) : G \subset F_\varepsilon\}$.

The set $\mathcal{U}$ and the closed set F witness the fact that $\{\mathcal{P}_i\}_{i\in\mathbb{N}}$ is not a pseudobase. Indeed, for every i, either every $P \in \mathcal{P}_i$ is bounded (and hence $F \notin \mathcal{P}_i$), or the set P_i defined above witnesses that $\mathcal{P}_i \not\subset \mathcal{U}$ (as by construction $d(y_i, F) > 1$).

This implies also that $\mathbf{F}_{UV}(\mathbb{R})$ is not second-countable, as every base is a pseudobase. The fact that it is not admissibly represented follows by Theorem 1.13. The claim generalizes to $\mathbf{F}_{UV}(\mathbb{R}^d)$ as every pseudobase of $\mathbf{F}_{UV}(\mathbb{R}^d)$ would induce a pseudobase on $\mathbf{F}_{UV}(\mathbb{R})$ by projection. ∎

Corollary 7.11:
The space $\mathbf{V}(\mathbb{R})$ (and hence $\mathbf{V}(\mathbb{R}^d)$) does not have a countable pseudobase. In particular, it is not second-countable and it is not admissibly represented.

Proof: This follows from the proof of Proposition 7.10. Indeed, the above proof only uses a the closed set $F \in \mathbf{F}(\mathbb{R})$ and an open set $\mathcal{U} \subset \mathbf{F}(\mathbb{R})$ to show that no countable subfamily of $\mathbf{F}(\mathbb{R})$ is a pseudobase. Since $\mathbf{F}_{UV}(\mathbb{R})$ is coarser than $\mathbf{V}(\mathbb{R})$ the same argument applies to $\mathbf{V}(\mathbb{R})$ (the claim would not follow immediately if, in the proof of Proposition 7.10 we would have exploited a convergent sequence to F, as convergence is a weaker notion in $\mathbf{F}_{UV}(\mathbb{R})$). ∎

7.3 The effective complexity of closed Salem sets

We are now ready to prove the effective counterparts of Proposition 6.5 and Proposition 6.6.

Proposition 7.12:
For every d and every compact $K \subset \mathbb{R}^d$,

- $\{(A,p) \in \mathbf{K}_U(K) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) > p\}$ *is* $\Sigma^{0,K}_2$;
- $\{(A,p) \in \mathbf{K}_U(K) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) \geq p\}$ *is* $\Pi^{0,K}_3$.

Proof: Following the proofs of Proposition 6.5 and Lemma 6.21, let

$$D(A) := \{s \in [0,d] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R}^d)(\forall r > 0)(\mu(B(x,r)) \leq cr^s)\}.$$

Notice that, if $a \in \mathbb{R}_<$ and $b \in \mathbb{R}$ (with the standard Cauchy representation) then the condition $a \leq b$ is a Π^0_1 predicate of a and b (as it is equivalent to $(\forall i)(p_a(i) \leq b)$, where $p_a \in \delta^{-1}_<(a)$), hence $\mu(B(x,r)) \leq cr^s$ is Π^0_1 as a predicate of μ, x, r, c, and s.

Moreover, $D(A)$ can be equivalently written as

$$\{s \in [0,d] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c \in \mathbb{Q}, c > 0)(\forall q_0 \in \mathbb{Q}^d)(\forall q_1 \in \mathbb{Q}, q_1 > 0)(\mu(B(q_0,q_1)) \leq cq_1^s)\}.$$

Indeed, since the measure μ is regular, for every $x \in \mathbb{R}^d$ and $r > 0$

$$\mu(B(x,r)) = \inf\{\mu(U) \,:\, U \in \mathbf{\Sigma}^0_1(\mathbb{R}^d) \text{ and } B(x,r) \subset U\}.$$

In particular, for every $\varepsilon > 0$ there are $q_0 \in \mathbb{Q}^d$ and $q_1 \in \mathbb{Q}$ s.t. $B(x,r) \subset B(q_0,q_1)$ and $q_1 < r+\varepsilon$. Hence

$$\begin{aligned}\mu(B(x,r)) &\leq \inf\{\mu(B(q_0,q_1)) \,:\, B(x,r) \subset B(q_0,q_1) \text{ and } q_1 < r+\varepsilon \text{ and } \varepsilon > 0\}\\ &\leq \inf\{cq_1^s \,:\, B(x,r) \subset B(q_0,q_1) \text{ and } q_1 < r+\varepsilon \text{ and } \varepsilon > 0\}\\ &\leq \inf\{c(r+\varepsilon)^s \,:\, \varepsilon > 0\} = cr^s\end{aligned}$$

Since the existential quantification on c can be trivially restricted to the rationals, we have

$$S := \{(s,\mu) \in [0,d] \times \mathbb{P}(A) \,:\, (\exists c > 0)(\forall x \in \mathbb{R}^d)(\forall r > 0)(\mu(B\,(x,r)) \le cr^s)\} \in \Sigma^{0,A}_2 \ .$$

Observe that $\mu \in \mathbb{P}(A)$ iff $\mu \in \mathbb{P}(K)$ and $\mu(A) \ge 1$. In particular, since $\mathbf{K}_U(K)$ is admissibly represented with the negative information representation, by Theorem 7.3.(2), given two names for μ and A, we can computably obtain a right-cut representation for $\mu(A)$, hence the the condition $\mu(A) \ge 1$ is a Π^0_1 predicate of μ and A (as if $x \in \mathbb{R}_>$ the condition $x \ge 1$ is x-co-c.e.). Since $\mathbb{P}(K)$ is computably compact (**??**), using (the relativized version of) Lemma 7.6, we have

$$D(A) = \mathrm{proj}_{[0,d]}\{(s,\mu) \in [0,d] \times \mathbb{P}(K) \,:\, \mu(A) \ge 1 \wedge (s,\mu) \in S\} \in \Sigma^{0,A}_2 \ .$$

To conclude the proof we notice that the conditions

$$\begin{aligned} \dim_{\mathcal{H}}(A) > p &\iff (\exists s \in \mathbb{Q})(s > p \wedge s \in D(A)), \\ \dim_{\mathcal{H}}(A) \ge p &\iff (\forall s \in \mathbb{Q})(s < p \to s \in D(A)) \end{aligned}$$

are Σ^0_2 and Π^0_3 respectively (as predicates of A and p), from which the claim follows. ∎

Proposition 7.13:
For every d and every compact $K \subset \mathbb{R}^d$,

- $\{(A,p) \in \mathbf{K}_U(K) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) > p\}$ *is* $\Sigma^{0,K}_2$*;*
- $\{(A,p) \in \mathbf{K}_U(K) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) \ge p\}$ *is* $\Pi^{0,K}_3$*.*

Proof: As in the proof of Proposition 6.6, consider the set

$$D(A) := \{s \in [0,d] \,:\, (\exists \mu \in \mathbb{P}(A))(\exists c > 0)(\forall x \in \mathbb{R}^d)(|\widehat{\mu}(x)| \le c|x|^{-s/2})\}.$$

Recall that, by definition,

$$\widehat{\mu}(x) = \int e^{-ix\cdot t}\, d\mu(t) = \int \cos(x \cdot t)\, d\mu(t) - i \int \sin(x\cdot t)\, d\mu(t)\ .$$

Since both cos and sin are effectively bounded, by Theorem 7.3.(5) the map

$$\mathbb{P}(\mathbb{R}^d) \times \mathbb{R} \to \mathbb{R} := (\mu, x) \mapsto |\widehat{\mu}(x)|$$

is computable. By the continuity of the Fourier transform, the universal quantification on $x \in \mathbb{R}^d$ can be restricted to $\mathbb{Q}^d$. Since the quantification on c can be trivially restricted to the rationals, we obtain that $D(A) = \mathrm{proj}_{[0,d]}\, Q$, with

$$Q := \{(\mu,s) \in \mathbb{P}(K) \times [0,d] \,:\, (\exists c \in \mathbb{Q}, c > 0)(\forall q \in \mathbb{Q}^d)(\mu \in \mathbb{P}(A) \wedge |\widehat{\mu}(q)| \le c|q|^{-s/2})\}.$$

The claim follows as in the proof of Proposition 7.12: since the condition $\mu \in \mathbb{P}(A)$ is a Π^0_1 predicate of μ and A and $\mathbb{P}(K)$ is computably compact, we have that $Q \in \Sigma^{0,A}_2$. Using (the relativized version of) Lemma 7.6 we conclude that $D(A) \in \Sigma_2^{0,A}$, and finally

$$\begin{aligned} \dim_{\mathrm{F}}(A) > p &\iff (\exists s \in \mathbb{Q})(s > p \land s \in D(A)), \\ \dim_{\mathrm{F}}(A) \geq p &\iff (\forall s \in \mathbb{Q})(s < p \to s \in D(A)) \end{aligned}$$

are, respectively, a Σ_2^0 and a Π_3^0 predicate of A and p. ■

Corollary 7.14:
For every compact $K \subset \mathbb{R}^d$, the set $\{A \in \mathbf{K}_U(K) : A \in \mathscr{S}([0,d])\}$ is $\Pi_3^{0,K}$.

Proof: As in the proof of Theorem 6.7, $\dim_{\mathcal{H}}(A) = \dim_{\mathrm{F}}(A)$ iff

$$(\forall r \in \mathbb{Q})(\dim_{\mathcal{H}}(A) > r \to \dim_{\mathrm{F}}(A) > r),$$

which is a $\Pi_3^{0,A}$ condition by Proposition 7.12 and Proposition 7.13. ■

We now show that, if we take $d = 1$ and $K = [0,1]$ then the above conditions are lightface complete for their respective classes. To do so, we prove an effective analogue of Lemma 6.9. Recall that, on p. 149, for every $\alpha \geq 0$ we introduced the set $S(\alpha)$ as a closed Salem subset of $E(\alpha)$. In Section 6.2, we wrote the set as

$$S(\alpha) = \bigcap_{k \in \mathbb{N}} \bigcup_{k' \leq n \leq k''} G_n(\alpha) \ ,$$

where $G_n(\alpha) = \{x \in [0,1] : \min_{m \in \mathbb{Z}} |nx - m| \leq n^{-1-\alpha}\}$, and k' and k'' are integers that depend on k and α.

We now show that the map $\alpha \mapsto S(\alpha)$ is computable. To do so, we need to be more precise on the relation between α, k and the integers k' and k''. This requires some tedious checking of the effectiveness of the propositions presented in [7]. We isolate these technicalities in the following lemma, while the result of the (relative) effectiveness of the set $S(\alpha)$ is stated Proposition 7.16.

Lemma 7.15:
The maps $(\alpha, k) \mapsto k'$ and $(\alpha, k) \mapsto k''$ are computable.

Proof: This proof assumes familiarity with [7]. Precisely, we now prove that [7, Lem. 3.2] is effective. First of all, since k'' is the greatest prime number smaller than $2k'$, we only need to show that the map $\varphi := (\alpha, k) \mapsto k'$ is computable. For the sake of readability, we adopt the same notation used in [7] and recall the relevant definitions: let $\mathbf{P}_M$ be the set of prime numbers between M and $2M$. Let also N be sufficiently large so that, for every $M > N$, $|\mathbf{P}_M| \geq \frac{M}{2\log(M)}$. The existence of such N follows from the asymptotic law of distribution of prime numbers (see [50, Sec. 22.19 and eq. (22.19.3)]).

Fix M s.t. $R := (4M)^{-1-\alpha} < 1/2$ and define $F_M : \mathbb{R} \to \mathbb{R}$ as the periodic extension with period 1 of the function on $[-1/2, 1/2]$ defined as

$$x \mapsto \begin{cases} \frac{15}{16} R^{-5}(R^2 - x^2)^2 & \text{if } |x| \le R \\ 0 & \text{if } R < |x| \le \frac{1}{2} \end{cases}$$

Let $\sum_{m\in\mathbb{Z}} a_m^{(M)} e^{2\pi i m x}$ be the Fourier series expansion of F_M. Define

$$q_M(x) := \sum_{p\in\mathbf{P}_M} F_M(px) = \sum_{p\in\mathbf{P}_M}\sum_{m\in\mathbb{Z}} a_m^{(M)} e^{2\pi i m p x} ,$$

and let $g_M(x) := q_M(x)/|\mathbf{P}_M|$, so that $\widehat{g}_M(0) = 1$. We stress that the choice of M (and hence the definition of F_M, q_M and g_M) depends on α.

Finally, define the function $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\theta(\alpha, x) := (1 + |x|)^{-\frac{1}{2+\alpha}} \log(e + |x|) \log\log(e + |x|) .$$

Let $\mathcal{C}^2_{cebd}(\mathbb{R})$ be the space of effectively bounded $\mathcal{C}^2(\mathbb{R})$ functions with compact support. In particular, a name for $f \in \mathcal{C}^2_{cebd}(\mathbb{R})$ is a name for a compact set $K \subset \mathbb{R}$ and three $\delta_{\mathcal{C}_{ebd}(\mathbb{R})}$-names p_0, p_1, p_2 respectively for f, f', f''.

The effective version of [7, Lem. 3.2] can be stated as follows: there is a computable functional Φ that, given $\alpha \ge 0$, $\psi \in \mathcal{C}^2_{cebd}(\mathbb{R})$, and $\delta > 0$, produces a positive integer M_0 s.t. for every $M \ge M_0$ and every $x \in \mathbb{R}$

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \le \delta\theta(\alpha, x) .$$

Before proving the claim, we show how it implies the computability of the map φ. Let $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be a computable function in $\mathcal{C}^2_{cebd}(\mathbb{R})$ with $\operatorname{spt}(\psi_0) = [0, 1]$ and $\int \psi_0(x)\, dx = 1$. For every $\alpha \ge 0$ and $k \ge 1$ define $M_{\alpha,k}$ to be the integer produced by $\Phi(\alpha, \psi_0 \cdot \prod_{j=1}^{k-1} g_{M_{\alpha,j}}, 2^{-k-2})$, and define $\varphi(\alpha, k) := M_{\alpha,k} = k'$.

To prove the effective version of [7, Lem. 3.2] we follow the steps of [7, Sec. 4]. Recall that we do not need to show that the proof is constructive, but only that the map Φ is computable. The proof of [7, Lem. 3.2] is divided into three steps.

Step 1: there exists $M_1 > 0$ and $A = A(\alpha)$ s.t. for every $M \ge M_1$

1. for every $k \in \mathbb{Z} \setminus \{0\}$, $|\widehat{g_M}(k)| \le A\frac{\log M}{M}$;

2. for every $k \in \mathbb{Z}$ with $|k| > (4M)^{2+\alpha}$, $|\widehat{g_M}(k)| \le A|k|^{-\frac{1}{2+\alpha}} \log |k|$.

The argument in [7] shows that, if we choose $A := 4(2 + \alpha)$ (which is, of course, computable in α), the second estimate always holds. Moreover, the first estimate holds for the same A, every $M > N$ and every k s.t. $|k| \le (4M)^{2+\alpha}$. To obtain a M_1 s.t. the first estimate holds for every $k \ne 0$, a simple observation is that

$$\frac{\log |k|}{|k|^{\frac{1}{2+\alpha}}} = (2 + \alpha)\frac{\log |k|^{\frac{1}{2+\alpha}}}{|k|^{\frac{1}{2+\alpha}}} \xrightarrow[|k|\to\infty]{} 0 .$$

In particular, since the function $\log(x)/x$ is strictly decreasing for $x > e$, if we choose $M_1 > N$ s.t. $M_1^{1/(2+\alpha)} > e$ then, for every $M > M_1$ we have

$$\frac{\log |k|}{|k|^{\frac{1}{2+\alpha}}} \le (2 + \alpha)\frac{\log M}{M} ,$$

hence the claim follows using the second estimate. In short, the estimates of step 1 hold if we choose $M_1 > \max\{N, e^{2+\alpha}\}$ and $A = 4(2+\alpha)$.

Step 2: for every $M > M_1$ there exists $B > 0$ that depends on ψ and α s.t. for every $x \in \mathbb{R}$

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \le B\frac{\log M}{M} \ .$$

This inequality is verified choosing $B := 2AB_1$, where B_1 is a constant s.t.

$$|\widehat{\psi}(x)| \le \frac{B_1}{(1+|x|)^2} \ .$$

The existence of such B_1 follows by the classical formula that relates the Fourier transform of ψ with the Fourier transform of its derivative, namely $\widehat{\psi^{(n)}}(x) = (ix)^n\widehat{\psi}(x)$. To prove that such a constant can be found computably in ψ, notice that

$$|\widehat{\psi}(x)|(1+|x|)^2 = |\widehat{\psi}(x)| + 2|\widehat{\psi'}(x)| + |\widehat{\psi''}(x)| \le \| \psi \|_{L^1} + \| \psi' \|_{L^1} + \| \psi'' \|_{L^1} \ .$$

Since the L^1-norm of a continuous and effectively bounded function is computable (see Theorem 7.3.6), the constant B_1 is computable from a name of ψ.

Step 3: there exists $M_2 > 0$ s.t. for all $M \ge M_2$,

$$|\widehat{\psi g_M}(x) - \widehat{\psi}(x)| \le \delta\theta(\alpha, x) \ .$$

The constant M_2 can be found by unbounded search. Indeed, given two monotonically decreasing functions f and g s.t. $f = o(g)$, we can computably find x_0 s.t. for every $x \ge x_0$, $f(x) < g(x)$. In particular, the claim follows from the fact that the estimates needed to compute M_2 do not depend (directly) on ψ and g_M, but only on the mutual relation between the upper bounds obtained in step 1 and the function $c\theta(\alpha, x)$, where c is a constant that depends on B, α and δ.

The proof of the effectiveness of [7, Lem. 3.2], and hence of the current lemma, is concluded defining $\Phi(\alpha, \psi, \delta)$ as M_2. ■

Proposition 7.16:
The following maps are computable:

$$\mathbb{N} \times \mathbb{R} \to \mathbf{K}([0,1]) := (n, \alpha) \mapsto G_n(\alpha)$$
$$\mathbb{N} \times \mathbb{R} \to \mathbf{K}([0,1]) := (n, \alpha) \mapsto \bigcup_{k' \le n \le k''} G_n(\alpha)$$
$$\mathbb{R} \to \mathbf{\Pi}^0_1(\mathbb{R}) := \alpha \mapsto S(\alpha)$$

PROOF: Recall that

$$G_n(\alpha) = \{x \in [0,1] \ : \ \min_{m\in\mathbb{Z}} |nx - m| \le n^{-1-\alpha}\} = \bigcup_{m \le n} \overline{B\left(\frac{m}{n}, n^{-1-\alpha}\right)} \cap [0,1] \ .$$

It is straightforward to see that, given (n,α), we can computably produce a ψ-name (i.e. a name w.r.t. the full information representation) for $G_n(\alpha)$.

By Lemma 7.15, the maps $(\alpha,n)\mapsto k'$ and $(\alpha,n)\mapsto k''$ are computable, which implies that the second map in the statement of the proposition is computable.

The computability of the last map follows from the fact that $\bigcap\colon (\mathbf{\Pi}^0_1(\mathbb{R}))^{\mathbb{N}} \to \mathbf{\Pi}^0_1(\mathbb{R})$ is computable (see e.g. [10, Prop. 3.2(6)]). ■

In particular, if α is computable then $S(\alpha)$ is $\Pi^0_1(\mathbb{R})$. Notice however that, in the previous proposition, we only get a $\delta_{\mathbf{\Pi}^0_1}$-name for $S(\alpha)$. Indeed, in general, the map $\bigcap\colon (\mathbf{K}([0,1]))^{\mathbb{N}} \to \mathbf{K}([0,1])$ is not computable (it is not even continuous, see e.g. [62, Ex. 4.29(viii)]). In the construction used in proof of Lemma 6.9, we exploited instead a superset $R(\alpha)$ of $S(\alpha)$, defined specifically to deal with this problem. However, the construction of $R(\alpha)$ from $S(\alpha)$ is not computable: indeed, the set $R^{(k+1)}(\alpha)$ was obtained from $\tilde{R}^{(k+1)}(\alpha)$ and $R^{(k)}(\alpha)$ by removing the (finitely many) degenerate intervals in $\tilde{R}^{(k+1)}(\alpha)\cap R^{(k)}(\alpha)$, which is not a computable operation (as we cannot computably tell whether a given interval is a singleton or not). However, this step was only useful to make the presentation neater; indeed, retaining finitely many points at each stage does not affect the dimension of the final set.

Lemma 7.17:
There is a computable function $f\colon [0,1]_< \times 2^{\mathbb{N}} \to \mathbf{K}([0,1])$ s.t. for every p,x, $f(p,x)$ is Salem and

$$\dim(f(p,x)) = \begin{cases} p & \text{if } x\in Q_2 \\ 0 & \text{if } x\notin Q_2\end{cases}$$

Proof: Before proving the lemma, let us notice that a (possibly degenerate) closed interval $I=[a,b]$ can be equivalently represented via the full information representation ψ, or via a pair of Cauchy names for the endpoints a and b. In turn, a finite union $\bigcup_{i<k} I_i$ of closed intervals $I_i=[a_i,b_i]$ can be equivalently represented via a ψ-name or via a finite sequence of pairs $((p_i,q_i))_{i<k}$, s.t. p_i (resp. q_i) is a Cauchy name for a_i (resp. b_i).

The proof of the lemma is essentially based on the proof of Lemma 6.9, however, we change some of the details to ensure the computability of the map. As anticipated, we will not use the set $R(\alpha)$. To avoid ambiguities, for every $\alpha\ge 0$, we define a superset $P(\alpha)$ of $S(\alpha)$ as follows:

$$P(\alpha) = \bigcap_{k\in\mathbb{N}} P^{(k)}(\alpha) = \bigcap_{k\in\mathbb{N}}\bigcup_{j\le N_k} J_j(\alpha,k),$$

where each $J_j(\alpha,k)$ is a (possibly degenerate) closed interval. We define the levels $P^{(k)}(\alpha)$ of the construction so that $P^{(k+1)}(\alpha)\subset P^{(k)}(\alpha)$, and, moreover, for every $i\le N_k$ there exists $j\le N_{k+1}$ s.t. $J_j(\alpha,k+1)\subset J_i(\alpha,k)$. We define $P^{(k)}(\alpha)$ inductively as follows: $P^{(0)}(\alpha) := S^{(0)}(\alpha)$. At stage $k+1$, let

$$\tilde{P}^{(k+1)}(\alpha) := S^{(k+1)}(\alpha)\cup\bigcup_{n\in U_k} G_n(\alpha),$$

where $U_k\subset\mathbb{N}$ is a finite set of indexes s.t. for every interval $j\le N_k$,

$$\operatorname{Int}(J_j(\alpha,k))\cap\tilde{P}^{(k+1)}(\alpha)\neq\emptyset, \tag{$\star$}$$

where Int$(\cdot)$ denotes the interior. Such a choice of U_k is always possible by the density of $E(\alpha)$. Moreover, if we represent $\tilde{P}$ with the full information representation, then $(\star)$ is computable (straightforward from the definition of full information), hence we can computably find a sufficiently large set U_k that satisfies $(\star)$.

We obtain $P^{(k+1)}(\alpha)$ by considering the finitely many intervals whose union is $\tilde{P}^{(k+1)}(\alpha) \cap P^{(k)}(\alpha)$. In particular, since $(k, \alpha) \mapsto S^{(k)}(\alpha)$ is computable (Proposition 7.16), then so is the map $(k, \alpha) \mapsto P^{(k)}(\alpha)$. We also define $P^{(k)}(\infty) := \emptyset$.

Notice that, given two intervals $I, J \in \mathbf{K}([0,1])$, we can computably find two intervals I', J' s.t. $I \cup J = I' \cup J'$ and $|I' \cap J'| \le 1$ (i.e. their intersection contains at most a point). Hence we can always assume that, if $\alpha \neq \infty$, a name of $P^{(k)}(\alpha)$ is a finite sequence $(p_j)_{j<M_k}$ of names of mutually almost disjoint (their intersection contains at most one point) closed intervals whose union is $P^{(k)}$ (M_k is possibly larger than N_k).

As in the proof of Lemma 6.9, for every interval $I = [a, b]$ and every k let $P^{(k)}(\alpha, I)$ be the fractal obtained by scaling $P^{(k)}(\alpha)$ to the interval I. Notice that the mapping $x \mapsto a + (b-a)x$ computably sends $[0,1]$ onto I and is affine and invertible if I is non-degenerate. In particular, the map

$$\mathbb{N} \times \mathbb{R} \times \mathbf{K}([0,1]) \to \mathbf{K}([0,1]) := (k, \alpha, I) \mapsto P^{(k)}(\alpha, I)$$

is computable.

We first define a map $g :\subseteq \mathbb{Q} \times 2^{\mathbb{N}} \to \mathbf{K}([0,1])$ s.t. for every $q \in [0,1)$ and x, $g(q,x)$ is Salem and $\dim(g(q,x)) = q$ if $x \in Q_2$ and 0 otherwise. We will then obtain a function f with a similar strategy to the one we used in the proof of Theorem 6.11, namely considering countably many disjoint intervals $T_n := [2^{-2n-1}, 2^{-2n}]$ and building a separate set on each of them, so that the resulting set will have the prescribed dimension.

If $q = 0$ we just take $g(q,x) := \emptyset$. Assume $q \in (0,1) \cap \mathbb{Q}$ and let α be s.t. $2/(2+\alpha) = q$. We run the construction used in the proof of Lemma 6.9: we define $F_x^{(k)}$ recursively as

Stage $k = 0$: $F_x^{(0)} := [0,1]$;

Stage $k+1$: Let $J_0, \dots, J_{M_k}$ be the almost disjoint closed intervals s.t. $F_x^{(k)} = \bigcup_{i \le M_k} J_i$. If $x(k+1) = 1$ then, for each $i \le M_k$, let $H_i := \overline{B\left((a_i + b_i)/2, \varepsilon\right)}$, where $J_i = [a_i, b_i]$ and ε is sufficiently small so that

$$\sum_{i \le M_k} \operatorname{diam}(H_i)^{2^{-k}} \le 2^{-k}.$$

Define then $F_x^{(k+1)} := \bigcup_{i \le M_k} H_i \cap J_i$.

If $x(k+1) = 0$ then let $s \le k$ be largest s.t. $x(s) = 1$ (or $s = 0$ if there is none) and let $I_0, \dots, I_{M_s}$ be the intervals of $F_x^{(s)}$. For each $i \le M_s$, apply the $(k+1-s)$-th step of the construction of $T(\alpha, I_i)$. Define $F_x^{(k+1)} := \bigcup_{i \le M_s} P^{(k+1-s)}(\alpha, I_i)$.

We then define $g(q,x) := F_x := \bigcap_{k \in \mathbb{N}} F_x^k$. The facts that $g(q,x)$ is continuous and that F_x is Salem with dimension q follow as in the proof of Lemma 6.9. To conclude the proof we only need to show that a $\delta_{\mathbf{K}([0,1])}$-name for F_x can be uniformly computed from q and x.

Notice that, since the map $(k, \alpha, I) \mapsto P^{(k)}(\alpha, I)$ is computable, then so is the map $(k, p, x) \mapsto F_x^{(k)}$ (where the codomain is represented with the full information representation). Hence, a $\delta_{\mathbf{\Pi}_1^0([0,1])}$-name for F_x can be computed from a sequence $(r_k)_{k \in \mathbb{N}}$ where r_k is a $\delta_{\mathbf{\Pi}_1^0([0,1])}$-name for $F_x^{(k)}$. To compute a ψ_+-name for F_x (i.e. a positive information name), we use the fact that no interval is ever entirely removed and that no interval is entirely contained in F_x (as

$\dim(F_x) < 1$). In particular, a ψ_+-name for F_x is obtained by listing all the basic open balls U s.t. there are k and $i \le M_k$ s.t. U contains a k-th level interval J_i. Notice that, since no interval is entirely removed, $U \cap J_i \neq \emptyset$ implies $U \cap F_x \neq \emptyset$. Moreover, if $V \cap F_x \neq \emptyset$ for some basic open ball V, then for some k and $i \le M_k$, V contains the k-th level interval J_i.

To conclude the proof, let $p \in [0,1]_<$ and let $(q_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of rationals in $[0,1]$ that converge to p. W.l.o.g. we can assume that the sequence is strictly increasing (with the possible exception of a prefix of zeroes at the beginning of the sequence). Using a similar strategy as in the proof of Theorem 6.11, we define

$$f(p,x) := \{0\} \cup \bigcup_{n\in\mathbb{N}} \tau_n g(q_n, x),$$

where $\tau_n\colon [0,1] \to T_n$ is a computable similarity transformation. The fact that $f(p,x)$ has the prescribed dimension follows from the countable stability for closed sets of $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$. Notice that a $\delta_{\mathbf{K}([0,1])}$-name for $f(p,x)$ can be obtained by carefully merging the $\delta_{\mathbf{K}([0,1])}$-names of the sets $\tau_n g(q_n, x)$. We can briefly sketch the argument as follows: a basic open set intersects $f(p,x)$ iff it intersects $\tau_n g(q_n,x)$ for some n. On the other hand, to list the open sets that are contained in the complement of $f(p,x)$ it is enough to list all the open sets of the type $\bigcup_n B_n$ where B_n is a basic open ball contained in relative topology of $T_n \setminus \tau_n g(q_n,x)$. The claim follows from the fact that the intervals T_n are uniformly co-c.e. closed. ∎

A slightly different way to obtain the same result would be to explicitly define $g(q,x)$ also if $q = 1$. In this case, we skip the $k+1$ stage of the construction when $x(k+1) = 0$ so that the set F_x is either an interval or a singleton. With this modification, when defining the map $f(p,x)$ we could avoid assuming that the sequence $(q_n)_{n\in\mathbb{N}}$ is strictly increasing.

We can now state the effective counterpart of Proposition 6.10 and Theorem 6.11.

Proposition 7.18:
For every $p < 1$ the sets

$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are Σ^0_2-complete. For every $q \in (0,1]$, the sets

$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathcal{H}}(A) \ge q\},$$
$$\{A \in \mathbf{K}([0,1]) \,:\, \dim_{\mathrm{F}}(A) \ge q\},$$
$$\{A \in \mathbf{K}([0,1]) \,:\, A \in \mathscr{S}([0,1])\}$$

are Π^0_3-complete.

Proof: The upper bounds have been shown in Proposition 7.12 and Proposition 7.13. A proof of the hardness is readily obtained by adapting the arguments used in the proofs of Proposition 6.10, Theorem 6.11, and Theorem 6.12 using Lemma 7.17 in place of Lemma 6.9. ∎

We now turn our attention to the closed Salem subsets of X, where X is $[0,1]^d$ or $\mathbb{R}^d$. We first notice the following result, which comes as a corollary of Proposition 7.12 and Proposition 7.13

Corollary 7.19:
- $\{(K,p) \in \mathbf{K}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(K) > p\}$ *is* Σ^0_2*;*
- $\{(K,p) \in \mathbf{K}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(K) \geq p\}$ *is* Π^0_3*;*
- $\{(K,p) \in \mathbf{K}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(K) > p\}$ *is* Σ^0_2*;*
- $\{(K,p) \in \mathbf{K}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(K) \geq p\}$ *is* Π^0_3*.*

PROOF: We only prove the statement for $\dim_{\mathcal{H}}(K) > p$, the proof of the complexity of the other sets is analogous. Notice that, if we define

$$X_n := \{(K,p) \in \mathbf{K}([0,n]^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(K) > p\},$$

then

$$\dim_{\mathcal{H}}(K) > p \iff (\exists n)(K \in X_n).$$

Hence, it is enough to show that the sets $(X_n)_{n\in\mathbb{N}}$ are uniformly Σ^0_2, i.e. that

$$\{(n,K,p) \,:\, K \in X_n\} \in \Sigma^0_2(\mathbb{N} \times \mathbf{K}(\mathbb{R}^d) \times [0,d]).$$

Notice that, since the sets $([0,n])_{n\in\mathbb{N}}$ are uniformly co-c.e. compact, then so are the sets $(\mathbf{K}([0,n]))_{n\in\mathbb{N}}$ (the argument of Lemma 7.8 can be run uniformly in n). This, in turn, implies that the set

$$\{(n,K,p) \,:\, (K,p) \in X_n\}$$

is Σ^0_2, as the argument in the proof of Proposition 7.12 can be run uniformly in n. ■

This corollary can be used to obtain the upper bounds in the non-compact case, i.e. the effective counterpart of the upper bounds obtained in Theorem 6.23 and Theorem 6.24.

Proposition 7.20:
- $\{(A,p) \in \mathbf{F}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) > p\}$ *is* Σ^0_2*;*
- $\{(A,p) \in \mathbf{F}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathcal{H}}(A) \geq p\}$ *is* Π^0_3*;*
- $\{(A,p) \in \mathbf{F}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) > p\}$ *is* Σ^0_2*;*
- $\{(A,p) \in \mathbf{F}(\mathbb{R}^d) \times [0,d] \,:\, \dim_{\mathrm{F}}(A) \geq p\}$ *is* Π^0_3*;*
- $\{A \in \mathbf{F}(\mathbb{R}^d) \,:\, A \in \mathscr{S}([0,d])\}$ *is* Π^0_3*.*

Proof: We only prove the statement for the Hausdorff dimension, the proof of the complexity of the Fourier dimension is analogous (as both are stable under countable union of closed sets), and the result on the complexity of the Salem sets is obtained in the usual way (see Corollary 7.14).

As in the proof of Theorem 6.23 we have

$$\dim_{\mathcal{H}}(A) > p \iff (\exists K \in \mathbf{F}(\mathbb{R}^d))(K \subset A \land K \in \mathbf{K}(\mathbb{R}^d) \land \dim_{\mathcal{H}}(K) > p) .$$

Notice that, if F, G are two closed sets represented with the full information representation, the predicate $F \subset G$ is Π^0_1 as a predicate of F and G. In fact we can prove something slightly stronger: if p_F is a positive information name for F and q_G is a negative information name for G then the condition $F \subset G$ is Π^0_1 in p_F and q_G. Indeed,

$$F \subset G \iff G^{\mathrm{C}} \cap F = \emptyset \iff (\forall i)(\forall j)(q_G(i) \neq p_F(j)) .$$

This shows that $K \subset A$ and $K \in \mathbf{K}(\mathbb{R}^d)$ are respectively Π^0_1 (as a predicate of K and A) and Σ^0_2 (as a predicate of K, as it is equivalent to $(\exists n)(K \subset \overline{B(\mathbf{0}, n)})$).

This implies that

$$\{(K, A, p) \in \mathbf{F}(\mathbb{R}^d) \times \mathbf{F}(\mathbb{R}^d) \times [0, d] \,:\, K \subset A \land K \in \mathbf{K}(\mathbb{R}^d) \land \dim_{\mathcal{H}}(K) > p\} \text{ is } \Sigma^0_2 .$$

Since $\mathbf{F}(\mathbb{R}^d)$ is computably compact (Proposition 7.9) we can apply Lemma 7.6 and conclude that

$$\{(A, p) \in \mathbf{F}(\mathbb{R}^d) \times [0, d] \,:\, \dim_{\mathcal{H}}(A) > p\} \text{ is } \Sigma^0_2 .$$

Since $\dim_{\mathcal{H}}(A) \geq p$ iff $(\forall r \in \mathbb{Q})(r < p \to \dim_{\mathcal{H}}(A) > r)$, this also shows that $\dim_{\mathcal{H}}(A) \geq p$ is a Π^0_3 predicate of A and p. ■

The arguments we used do not yield automatically the Σ^0_2-completeness of the conditions $\dim_{\mathcal{H}}(A) > p$ and $\dim_{\mathrm{F}}(A) > p$ when $p < d$ and $A \in \mathbf{K}([0,1]^d)$ or $A \in \mathbf{F}(\mathbb{R}^d)$. Similarly, we cannot conclude that the conditions $\dim_{\mathcal{H}}(A) \geq p$ and $\dim_{\mathrm{F}}(A) \geq p$ are Π^0_3-complete when $p > 0$ and $A \in \mathbf{K}([0,1]^d)$ or $A \in \mathbf{F}(\mathbb{R}^d)$.

In fact, we are not aware of any proof of the effectiveness of the construction of the closed subset $S(K, B, \alpha)$ of $E(K, B, \alpha)$ (Definition 6.14) that was used in the proof of Lemma 6.16. More generally, it would suffice any computable map $f\colon [0, d] \times 2^{\mathbb{N}} \to \mathbf{K}([0,1]^d)$ s.t. $f(p, x)$ is a closed Salem set of dimension p iff $x \in Q_2$, and dimension 0 otherwise.

However, we can follow the strategy mentioned after Theorem 6.18 and exploit a theorem of Gatesoupe to obtain a (slightly weaker) result, namely the completeness of the above conditions when p is sufficiently large. We briefly sketch the argument to stress that the proof is effective.

Theorem 7.21:
Let X be $[0,1]^d$ or $\mathbb{R}^d$. For every computable $p \in [d-1, d)$ the sets

$$\{A \in \mathbf{F}(X) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{F}(X) \,:\, \dim_{\mathrm{F}}(A) > p\}$$

are Σ^0_2-complete. For every computable $q \in (d-1, d]$, the sets

$$\{A \in \mathbf{F}(X) \,:\, \dim_{\mathcal{H}}(A) \ge q\},$$
$$\{A \in \mathbf{F}(X) \,:\, \dim_{\mathrm{F}}(A) \ge q\},$$
$$\{A \in \mathbf{F}(X) \,:\, A \in \mathscr{S}(X)\}$$

are Π^0_3-complete.

Proof: By Proposition 7.20, it is enough to show that the above sets are hard for their respective class.

Recall that, by a theorem of Gatesoupe [40], if $A \subset [0,1]$ has at least two points and is Salem with dimension α then the set $\tilde{A} := \{x \in [0,1]^d \,:\, |x| \in A\}$ is Salem with dimension $d-1+\alpha$. It is easy to see that the map $r\colon \mathbf{K}([0,1]) \to \mathbf{K}([0,1]^d) := A \mapsto \tilde{A}$ is computable.

To show that the first two sets are Σ^0_2-hard, observe that, for $x \in 2^{\mathbb{N}}$,

$$x \in Q_2 \iff \dim_{\mathcal{H}}(r(f(1,x))) > p \iff \dim_{\mathrm{F}}(r(f(1,x))) > p \ .$$

Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of mutually disjoint closed cubes s.t.

- $C_n \subset [0,1]^d$,
- $\overline{\bigcup_{n\in\mathbb{N}} C_n} = \{\mathbf{0}\} \cup \bigcup_{n\in\mathbb{N}} C_n$,
- the sets have uniformly computable ψ-names, i.e. there is a computable map that, given n, produces a ψ-name for C_n,

where $\mathbf{0}$ is the origin of the d-dimensional Euclidean space. It is easy to produce examples of sequences of closed sets that satisfy the above conditions. In particular, the last point guarantees that the similarity transformations $\tau_n\colon [0,1]^d \to C_n$ are uniformly computable.

To prove that the last three sets are Π^0_3-hard, we mimic the proof of Theorem 6.11. When we consider the family of closed Salem sets we also need a computable compact set $Y \subset C_0$ with null Fourier dimension and Hausdorff dimension d. ■

7.4 The Weihrauch degree of the Hausdorff and Fourier dimension

The results we obtained can be used to characterize the Weihrauch degree of the maps computing the Hausdorff and Fourier dimension of a closed subset of $\mathbb{R}^d$, for some fixed d. To avoid ambiguity, we write

$$\dim_{\mathcal{H}}, \dim_{\mathrm{F}}\colon \mathbf{F}_U(\mathbb{R}^d) \to \mathbb{R}$$
$$\dim_{\mathcal{H}}^{\mathbf{F}}, \dim_{\mathrm{F}}^{\mathbf{F}}\colon \mathbf{F}(\mathbb{R}^d) \to \mathbb{R}$$

to stress the fact that the closed sets are represented using the negative information representation in the first case, and the full information in the second one.

Let $\mathbf{\Gamma}$ be a Borel pointclass. We say that $f :\subseteq X \to Y$ is $\mathbf{\Gamma}$-measurable if, for every open $U \subset Y$, $f^{-1}(U) \in \mathbf{\Gamma}(\operatorname{dom}(f))$, i.e. there exists $V \in \mathbf{\Gamma}(X)$ s.t. $f^{-1}(U) = V \cap \operatorname{dom}(f)$. If X and Y are represented spaces, we say that f is *effectively* $\mathbf{\Gamma}$*-measurable* or $\mathbf{\Gamma}$*-computable* if the map

$$\mathbf{\Gamma}^{-1}(f)\colon \mathbf{\Sigma}^0_1(Y) \rightrightarrows \mathbf{\Gamma}(X) := U \mapsto \{V \subset X \,:\, f^{-1}(U) = V \cap \operatorname{dom}(f)\}$$

is computable. In particular, if f is total then $\mathbf{\Gamma}^{-1}(f)$ is single-valued. This notion can be generalized in a straightforward way to multi-valued functions (see [10, Def. 3.5]).

In the proof of Theorem 7.23 we will use the following important result:

Theorem 7.22 ([17, Thm. 6.5]):
f is effectively $\mathbf{\Sigma}^0_{k+1}$-measurable iff f is Weihrauch reducible to $\mathsf{lim}^{[k]}$.

This is a generalization of [10, Thm. 9.1], and (intuitively) says that $\mathsf{lim}^{[k]}$ is the hardest effectively $\mathbf{\Sigma}^0_{k+1}$-measurable problem.

Theorem 7.23:
$\mathsf{lim}^{[2]} \equiv_{\mathrm{W}} \dim^{\mathbf{F}}_{\mathcal{H}} \equiv_{\mathrm{W}} \dim_{\mathcal{H}} \equiv_{\mathrm{W}} \dim^{\mathbf{F}}_{\mathrm{F}} \equiv_{\mathrm{W}} \dim_{\mathrm{F}}$.

Proof: It is immediate to see that $\dim^{\mathbf{F}}_{\mathcal{H}} \le_{\mathrm{W}} \dim_{\mathcal{H}}$ and $\dim^{\mathbf{F}}_{\mathrm{F}} \le_{\mathrm{W}} \dim_{\mathrm{F}}$. To prove that $\dim_{\mathcal{H}} \le_{\mathrm{W}} \mathsf{lim}^{[2]}$ and $\dim_{\mathrm{F}} \le_{\mathrm{W}} \mathsf{lim}^{[2]}$, by Theorem 7.22 it suffices to prove that the maps $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$ are effectively $\mathbf{\Sigma}^0_3$-measurable. This follows by Proposition 7.20 as

$$\dim_{\mathcal{H}}^{-1}((a,b)) = \{F \in \mathbf{F}(\mathbb{R}^d) \,:\, \dim_{\mathcal{H}}(F) > a \land \dim_{\mathcal{H}}(F) < b\}.$$

In fact, given $a, b \in [0,d]$ we can uniformly compute a $(a \oplus b)$-computable $\delta_{\mathbf{\Sigma}^0_3}$-name for $\dim_{\mathcal{H}}^{-1}((a,b))$.

Finally, to show that $\mathsf{lim}^{[2]} \le_{\mathrm{W}} \dim_{\mathcal{H}}$ and $\mathsf{lim}^{[2]} \le_{\mathrm{W}} \dim_{\mathrm{F}}$ we prove that, given a sequence $(x_i)_{i\in\mathbb{N}}$ in $2^{\mathbb{N}}$, we can uniformly build a closed Salem subset A of $[0,1]^d$ s.t. $\dim(A)$ uniformly computes whether $x_i \in Q_2$, where Q_2 is the fixed Σ^0_2-complete set (as in the proof of Lemma 6.9).

Let f be the computable map provided by Lemma 7.17. Let also $r := F \mapsto \tilde{F}$, where $\tilde{F} = \{x \in \mathbb{R}^d \,:\, |x| \in F\}$ and define $g := r \circ f$. Recall that, by [40], if F is Salem with dimension α and has at least two points then $\tilde{F}$ is Salem with dimension $d - 1 + \alpha$. For every finite $I \subset \mathbb{N}$, let $p_I := \sum_{i\in I} 2^{-i}$. Define $y_I \in 2^{\mathbb{N}}$ by $y_I(n) := \max_{i\in I} x_i(n)$. Clearly

$$\begin{aligned}(\forall i \in I)(x_i \in Q_2) &\iff y_I \in Q_2 \\ &\iff \dim(f(p_I, y_I)) = p_I \iff \dim(g(p_I, y_I)) = d - 1 + p_I \ .\end{aligned}$$

As in the proof of Theorem 7.21, let $(C_n)_{n\in\mathbb{N}}$ be a sequence of mutually disjoint closed cubes s.t.

- $C_n \subset [0,1]^d$,
- $\overline{\bigcup_{n\in\mathbb{N}} C_n} = \{\mathbf{0}\} \cup \bigcup_{n\in\mathbb{N}} C_n$,
- the sets have uniformly computable ψ-names, i.e. there is a computable map that, given n, produces a ψ-name for C_n,

where $\mathbf{0}$ is the origin of the d-dimensional Euclidean space.

For every I as above, we can uniformly translate and scale the set $g(p_\sigma, y_\sigma)$ to a subset G_σ of $C_{\langle\sigma\rangle}$. Consider now the closed set $A := \{\mathbf{0}\} \cup \bigcup_\sigma G_\sigma$. It is easy to see that A is Salem and $\dim(A) = d - 1 + \sum_{i\in\mathbb{N}} 2^{-i}\chi_{Q_2}(x_i)$ and this concludes the proof. ■

The Weihrauch equivalence between $\mathsf{lim}^{[2]}$ and the map computing the Hausdorff dimension of a closed subset of $[0,1]$ (and, more generally, of a compact subset of $\mathbb{R}$) was already proved in [92, Thm. 48]. Our approach extends that result (since in the proof of $\mathsf{lim}^{[2]} \le_{\mathrm{W}} \dim_{\mathcal{H}}$ we always build a compact set) and, at the same time, characterizes the degree of the map computing the Fourier dimension. The same ideas can be used e.g. to show that $\chi_{\mathscr{S}_c(\mathbb{R}^d)} \equiv_{\mathrm{W}} \mathsf{LPO}^{(2)}$.

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