Non-critical dimensions for critical problems involving fractional Laplacians

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Non-critical dimensions for critical problems involving fractional Laplacians

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Abstract

We study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m > 0$.

Keywords: Fractional Laplace operators, Sobolev inequality, Hardy inequality, critical dimensions.

1 Introduction

Let m, s be two given real numbers, with $0 \leq s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in $\mathbb{R}^n$ and put

$$2^*_m = \frac{2n}{n-2m}.$$

We study equations

$$(-\Delta)^m u = \lambda(-\Delta)^s u + |u|^{2^*_m - 2} u \quad \text{in } \Omega, \tag{1.1}$$

$$(-\Delta)^m u = \lambda|x|^{-2s} u + |u|^{2^*_m - 2} u \quad \text{in } \Omega, \tag{1.2}$$

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under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that Ω contains the origin. For the definition of fractional Dirichlet–Laplace operators $(-\Delta)^m, (-\Delta)^s$ and for the variational approach to (1.1), (1.2) we refer to the next section.

The celebrated paper [3] by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case $n > 2$, $m = 1$, $s = 0$, that is,

$$-\Delta u = \lambda u + |u|^{\frac{4}{n-2}} u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter λ: they proved existence of a nontrivial solution for any small $\lambda > 0$ if $n \geq 4$; in contrast, in the lowest dimension $n = 3$ non-existence phenomena for sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical*[1] for problem (1.3).

Clearly, as larger s is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: *Given $m < \frac{n}{2}$, how large must be s in order to have the existence of a ground state solution, for any arbitrarily small $\lambda > 0$?* In case of an affirmative answer, we say that n is <u>not</u> a critical dimension.

We present our main result, that holds for any dimension $n \geq 1$ (see Section 4 for a more precise statement).

THEOREM. *If $s \geq 2m - \frac{n}{2}$ then n is not a critical dimension for the Dirichlet boundary value problems associated to equations (1.1) and (1.2).*

We point out some particular cases that are included in this result.

- If m is an integer and $s = m - 1$, then at most the lowest dimension $n = 2m + 1$ is critical.
- For any $n > 2m$ there always exist lower order perturbations of the type $|x|^{-2s}u$ and of the type $(-\Delta)^s u$ such that n is not a critical dimension.
- If $m < 1/4$ then no dimension is critical, for any choice of $s \in [0, m)$.

[1] compare with [13], [8].

After [3], a large number of papers have been focussed on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with $s = 0$, when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case $2 \le m \in \mathbb{N}$, see for instance [13], [6], [2], [10], [7], or the case $m \in (0, 1)$, see [14], [15]. We cite also [4], where equation (1.1) is studied in case $m = 2$, $s = 1$. Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [16]) where equation (1.1) for the so-called Navier-Laplacian is studied in case $m \in (0, 1)$, $s = 0$. For a comparison between the Dirichlet and Navier Laplacians we refer to [12].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2, we provide the main estimates in Section 3. In Section 4 we prove Theorem 1 and point out an existence result for the case $s < 2m - \frac{n}{2}$.

2 Preliminaries

The fractional Laplacian $(-\Delta)^m u$ of a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined via the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx$$

by the identity

$$\mathcal{F}\left[(-\Delta)^m u\right](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi). \tag{2.1}$$

In particular, Parseval's formula gives

$$\int\limits_{\mathbb{R}^n} (-\Delta)^m u \cdot u\, dx = \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2\, dx = \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2\, d\xi\,.$$

We recall the well known Sobolev inequality

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2\, dx \ge \mathcal{S}_m \Big(\int\limits_{\mathbb{R}^n} |u|^{2^*_m}\, dx \Big)^{2/2^*_m}, \tag{2.2}$$

that holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $m < \frac{n}{2}$, see for example [17, 2.8.1/15].

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the Hilbert space obtained by completing $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\|u\|_m^2 = \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx. \tag{2.3}$$

Thanks to (2.2), the space $\mathcal{D}^m(\mathbb{R}^n)$ is continuously embedded into $L^{2^*_m}(\mathbb{R}^n)$. The *best Sobolev constant* $\mathcal{S}_m$ was explicitly computed in [5]. Moreover, it has been proved in [5] that $\mathcal{S}_m$ is attained in $\mathcal{D}^m(\mathbb{R}^n)$ by a unique family of functions, all of them being obtained from

$$\phi(x) = (1+|x|^2)^{\frac{2m-n}{2}} \tag{2.4}$$

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $R > 0$ it turns out that

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega](\xi)|^2 \, d\xi &= R^{n-2m} \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega(R\cdot)](\xi)|^2 \, d\xi \\ \int\limits_{\mathbb{R}^n} |\omega|^{2^*_m} \, dx &= R^n \int\limits_{\mathbb{R}^n} |\omega(R\cdot)|^{2^*_m} \, dx\,. \end{aligned} \tag{2.5}$$

Finally, we point out that the Hardy inequality

$$\int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \, dx \ge \mathcal{H}_m \int\limits_{\mathbb{R}^n} |x|^{-2m} |u|^2 \, dx \tag{2.6}$$

holds for any function $u \in \mathcal{D}^m(\mathbb{R}^n)$. The *best Hardy constant* $\mathcal{H}_m$ was explicitly computed in [11].

The natural ambient space to study the Dirichlet boundary value problems for (1.1), (1.2) is

$$\widetilde{H}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n) \,:\, \operatorname{supp} u \subset \overline{\Omega}\},$$

endowed with the norm $\|u\|_m$. By Theorem 4.3.2/1 [17], for $m + \frac{1}{2} \notin \mathbb{N}$ this space coincides with $H_0^m(\Omega)$ (that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^m(\Omega)$), while for $m + \frac{1}{2} \in \mathbb{N}$ one has $\widetilde{H}^m(\Omega) \subsetneq H_0^m(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\widetilde{H}^m(\Omega)$. Clearly, if m is an integer then $\widetilde{H}^m(\Omega)$ is the standard Sobolev space of functions $u \in H^m(\Omega)$ such that $D^\alpha u = 0$ for every multiindex $\alpha \in \mathbb{N}^n$ with $0 \le |\alpha| < m$.

We agree that $(-\Delta)^0 u = u$, $\widetilde{H}^0(\Omega) = L^2(\Omega)$, since (2.3) reduces to the standard L^2 norm in case $m = 0$.

We define (weak) solutions of the Dirichlet problems for (1.1), (1.2) as suitably normalized critical points of the functionals

$$\mathcal{R}^{\Omega}_{\lambda,m,s}[u] = \frac{\int\limits_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2\, dx - \lambda \int\limits_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2\, dx}{\Big(\int\limits_{\Omega} |u|^{2^*_m} dx\Big)^{2/2^*_m}} \tag{2.7}$$

$$\widetilde{\mathcal{R}}^{\Omega}_{\lambda,m,s}[u] = \frac{\int\limits_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2\, dx - \lambda \int\limits_{\Omega} |x|^{-2s}|u|^2\, dx}{\Big(\int\limits_{\Omega} |u|^{2^*_m} dx\Big)^{2/2^*_m}}\,, \tag{2.8}$$

respectively. It is easy to see that both functionals (2.7), (2.8) are well defined on $\widetilde{H}^m(\Omega) \setminus \{0\}$.

We conclude this preliminary section with some embedding results.

Proposition 2.1 *Let m, s be given, with $0 \le s < m < n/2$.*

i) The space $\widetilde{H}^m(\Omega)$ is compactly embedded into $\widetilde{H}^s(\Omega)$. In particular the infima

$$\Lambda_1(m,s) := \inf_{\substack{u\in \widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\|u\|_s^2}\;, \qquad \widetilde{\Lambda}_1(m,s) := \inf_{\substack{u\in \widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\||x|^{-s}u\|_0^2} \tag{2.9}$$

are positive and achieved.

ii) $\displaystyle\inf_{\substack{u\in \widetilde{H}^m(\Omega)\\ u\neq 0}} \frac{\|u\|_m^2}{\|u\|_{L^{2^*_m}}^2} = \mathcal{S}_m.$

Statement *i)* is well known for $\Lambda_1(m,s)$ and follows from (2.6) for $\widetilde{\Lambda}_1(m,s)$. To check *ii)*, use the inclusion $\widetilde{H}^m(\Omega) \hookrightarrow \mathcal{D}^m(\mathbb{R}^n)$ and a rescaling argument. Clearly, the Sobolev constant $\mathcal{S}_m$ is never achieved on $\widetilde{H}^m(\Omega)$.

3 Main estimates

Let ϕ be the extremal of the Sobolev inequality (2.2) given by (2.4). In particular, it holds that

$$M := \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}}\phi|^2\,dx = \mathcal{S}_m\Big(\int\limits_{\mathbb{R}^n} |\phi|^{2^*_m}\,dx\Big)^{2/2^*_m}. \tag{3.1}$$

Fix $\delta > 0$ and a cutoff function $\varphi \in \mathcal{C}^\infty_0(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside $\{|x| < 2\delta\}$. If δ is sufficiently small, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n}\varphi(x)\phi\left(\frac{x}{\varepsilon}\right) = \varphi(x)\left(\varepsilon^2 + |x|^2\right)^{\frac{2m-n}{2}}$$

has compact support in Ω. Next we define

$$A^\varepsilon_m := \int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}}u_\varepsilon|^2 dx \qquad A^\varepsilon_s := \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}}u_\varepsilon|^2 dx$$

$$\widetilde{A}^\varepsilon_s := \int\limits_\Omega |x|^{-2s}|u_\varepsilon|^2 dx \qquad B^\varepsilon := \int\limits_\Omega |u_\varepsilon|^{2^*_m} dx$$

and we denote by c any universal positive constant.

Lemma 3.1 *It holds that*

$$\begin{cases} A^\varepsilon_m \le \varepsilon^{2m-n}\left(M + c\varepsilon^{n-2m}\right) & & (3.2\text{a})\\ A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\varepsilon^{4m-n-2s} & \textit{if } s > 2m - \frac{n}{2} & (3.2\text{b})\\ A^\varepsilon_s, \widetilde{A}^\varepsilon_s \ge c\,|\log\varepsilon| & \textit{if } s = 2m - \frac{n}{2} & (3.2\text{c})\\ B^\varepsilon \ge \varepsilon^{-n}\left((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\varepsilon^n\right). & & (3.2\text{d}) \end{cases}$$

Proof of (3.2a). First of all, from (2.5) we get

$$A^\varepsilon_m = \varepsilon^{2m-n}\int\limits_{\mathbb{R}^n} |\xi|^{2m}\left|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]\right|^2\,d\xi. \tag{3.3}$$

Thus

$$\Gamma^\varepsilon_m := \varepsilon^{n-2m}A^\varepsilon_m - M = \int\limits_{\mathbb{R}^n} |\xi|^{2m}\left|\mathcal{F}\left[\varphi(\varepsilon\,\cdot)\phi\right]\right|^2\,d\xi - \int\limits_{\mathbb{R}^n} |\xi|^{2m}\left|\mathcal{F}[\phi]\right|^2\,d\xi.$$

We need to prove that

$$|\Gamma_m^\varepsilon| \le c\varepsilon^{n-2m}. \tag{3.4}$$

If $m \in \mathbb{N}$, the proof of (3.4) has been carried out in [3], [7]. Here we limit ourselves to the more difficult case, namely, when m is not an integer. We denote by $k := \lfloor m \rfloor \ge 0$ the integer part of m, so that $m-k>0$. Then

$$\begin{aligned}\Gamma_m^\varepsilon &= \int_{\mathbb{R}^n} |\xi|^{2k}\mathcal{F}[U_-]\cdot|\xi|^{2(m-k)}\overline{\mathcal{F}[U_+]}\,d\xi\\ &= 2^{2(m-k)+\frac{n}{2}}\,\frac{\Gamma(m-k+\frac{n}{2})}{\Gamma(-(m-k))}\cdot\int_{\mathbb{R}^n}(-\Delta)^k U_-(x)\cdot V.P.\int_{\mathbb{R}^n}\underbrace{\frac{U_+(x)-U_+(y)}{|x-y|^{n+2(m-k)}}}_{\Psi(x,y)}\,dy\,dx,\end{aligned}$$

where $U_\pm = \varphi(\varepsilon\,\cdot\,)\phi \pm \phi$ (the last equality follows from [9, Ch. 2, Sec. 3]).

We split the interior integral as follows:

$$V.P.\int_{\mathbb{R}^n}\Psi dy = \underbrace{V.P.\int_{|y-x|\le\frac{|x|}{2}}\Psi dy}_{I_1} + \underbrace{\int_{\substack{|y-x|\ge\frac{|x|}{2}\\ |y|\le|x|}}\Psi dy}_{I_2} + \underbrace{\int_{\substack{|y-x|\ge\frac{|x|}{2}\\ |y|\ge|x|}}\Psi dy}_{I_3}.$$

We claim that $|I_j| \le c|x|^{2k-n}$ for $j=1,2,3$. Indeed, the Lagrange formula gives

$$\begin{aligned}|I_1| &\le \max_{|y-x|\le\frac{|x|}{2}}|D^2U_+(y)|\cdot\int_{|z|\le\frac{|x|}{2}}\frac{dz}{|z|^{n+2(m-k)-2}}\\ &\le c|x|^{-(n-2m+2)}\cdot|x|^{2-2(m-k)} = c|x|^{2k-n}.\end{aligned}$$

As concerns the last two integrals we estimate

$$|I_2| \le \int_{\substack{|y-x|\ge\frac{|x|}{2}\\ |y|\le|x|}}\frac{c|y|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\,dy \le |x|^{-(n+2(m-k))}\cdot c|x|^{2m} = c|x|^{2k-n}$$

and finally

$$\begin{aligned}|I_3| \le \int_{\substack{|y-x|\ge\frac{|x|}{2}\\ |y|\ge|x|}}\frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}}\,dy &\le c|x|^{-(n-2m)}\cdot\int_{|z|\ge\frac{|x|}{2}}\frac{dz}{|z|^{n+2(m-k)}}\\ &\le c|x|^{-(n-2m)}\cdot|x|^{-2(m-k)} = c|x|^{2k-n},\end{aligned}$$

and the claim follows. Now, since

$$|(-\Delta)^k U_-(x)| \le \frac{c}{|x|^{n-2(m-k)}}\,\chi_{\{|x|\ge\delta/\varepsilon\}} + \frac{c\varepsilon^{2k}}{|x|^{n-2m}}\,\chi_{\{\delta/\varepsilon\le|x|\le2\delta/\varepsilon\}},$$

we obtain

$$|\Gamma^\varepsilon_m| \le c \int\limits_{|x|\ge\delta/\varepsilon} \frac{dx}{|x|^{2n-2m}} + c \int\limits_{\delta/\varepsilon\le|x|\le2\delta/\varepsilon} \frac{\varepsilon^{2k}\,dx}{|x|^{2n-2(m+k)}} \le c\varepsilon^{n-2m},$$

that completes the proof of (3.4) and of (3.2a).

Proof of (3.2b) and (3.2c). We use the Hardy inequality (2.6) to get

$$\begin{aligned} A^\varepsilon_s &\ge c\widetilde{A}^\varepsilon_s \ge c\varepsilon^{4m-2s-n} \int\limits_{\mathbb{R}^n} |x|^{-2s}|\varphi(\varepsilon\,\cdot)\phi|^2 dx \\ &\ge c\varepsilon^{4m-2s-n} \int\limits_{|x|<\delta/\varepsilon} \frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}\,. \end{aligned}$$

The last integral converges as $\varepsilon \to 0$ if $s > 2m - \frac{n}{2}$, and diverges with speed $|\log\varepsilon|$ if $s = 2m - \frac{n}{2}$.

Proof of (3.2d). For ε small enough we estimate by below

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |u_\varepsilon|^{2^*_m} &= \varepsilon^{-n} \int\limits_{\mathbb{R}^n} |\varphi(\varepsilon\,\cdot)\phi|^{2^*_m}\,dx = \varepsilon^{-n}\Big(\int\limits_{\mathbb{R}^n} |\phi|^{2^*_m}\,dx - \int\limits_{|x|>\delta/\varepsilon} |\varphi(\varepsilon\,\cdot)\phi|^{2^*_m}\,dx \Big) \\ &\ge \varepsilon^{-n}\Big((M\mathcal{S}_m^{-1})^{2^*_m/2} - c \int\limits_{|x|>\delta/\varepsilon} |x|^{-2n}\,dx \Big) \\ &= \varepsilon^{-n}((M\mathcal{S}_m^{-1})^{2^*_m/2} - c\varepsilon^n) \end{aligned}$$

and the Lemma is completely proved. □

4 Two noncompact minimization problems

In this section we deal with the minimization problems

$$\mathcal{S}^\Omega_\lambda(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne0}} \mathcal{R}^\Omega_{\lambda,m,s}[u]; \qquad \widetilde{\mathcal{S}}^\Omega_\lambda(m,s) = \inf_{\substack{u\in\widetilde{H}^m(\Omega)\\ u\ne0}} \widetilde{\mathcal{R}}^\Omega_{\lambda,m,s}[u]\,,$$

where the functionals $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are introduced in (2.7) and (2.8), respectively.

Lemma 4.1 *The following facts hold for any $\lambda \in \mathbb{R}$:*

i) $\mathcal{S}_\lambda^\Omega(m,s) \le \mathcal{S}_m$*;*

ii) If $\lambda \le 0$ *then* $\mathcal{S}_\lambda^\Omega(m,s) = \mathcal{S}_m$ *and it is not achieved;*

iii) If $0 < \mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$*, then* $\mathcal{S}_\lambda^\Omega(m,s)$ *is achieved.*

The same statements hold for $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ *instead of* $\mathcal{S}_\lambda^\Omega(m,s)$*.*

Proof. The proof is nowdays standard, and is essentially due to Brezis and Nirenberg [3]. We sketch it for the infimum $\mathcal{S}_\lambda^\Omega(m,s)$, for the convenience of the reader.

Fix $\varepsilon > 0$ and take $u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$(\mathcal{S}_m + \varepsilon)\Bigg(\int\limits_{\mathbb{R}^n} |u|^{2_m^*} dx\Bigg)^{2/2_m^*} \ge \int\limits_{\mathbb{R}^n} |\,(-\Delta)^{\frac{m}{2}} u|^2 \; dx. \tag{4.1}$$

Let $R > 0$ be large enough, so that $u_R(\cdot) := u(R\cdot) \in \mathcal{C}_0^\infty(\Omega)$. Using (2.5) we get

$$\mathcal{S}_\lambda^\Omega(m,s) \le \frac{\|u\|_m^2 - \lambda R^{2(s-m)}\|u\|_s^2}{\|u\|^2_{L^{2_m^*}}} \le (\mathcal{S}_m + \varepsilon)\left(1 + cR^{2(s-m)}\right),$$

where c depends only on u and λ. Letting $R \to \infty$ we get $\mathcal{S}_\lambda^\Omega(m,s) \le (\mathcal{S}_m + \varepsilon)$ for any $\varepsilon > 0$, and $i)$ is proved.

Next, if $\lambda \le 0$ then clearly $\mathcal{S}_\lambda^\Omega(m,s) = \mathcal{S}_m$. If $\lambda = 0$ then $\mathcal{S}_m$ is not achieved. The more it is not achieved for $\lambda < 0$, and $ii)$ holds.

Finally, to prove $iii)$ take a minimizing sequence u_h. It is convenient to normalize u_h with respect to the $L^{2_m^*}$-norm, so that

$$\int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}} u_h|^2 \, dx - \lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx = \mathcal{S}_\lambda^\Omega(m,s) + o(1).$$

We can assume that $u_h \to u$ weakly in $\widetilde{H}^m(\Omega)$ and strongly in $\widetilde{H}^s(\Omega)$ by Proposition 2.1. Since

$$\begin{aligned}
\lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u|^2 \, dx &= \lambda \int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}} u_h|^2 \, dx + o(1) \\
&= \int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}} u_h|^2 \, dx - \mathcal{S}_\lambda^\Omega(m,s) + o(1) \\
&\ge (\mathcal{S}_m - \mathcal{S}_\lambda^\Omega(m,s)) + o(1),
\end{aligned}$$

then $u \neq 0$. By the Brezis–Lieb lemma we get

$$1 = \|u_h\|_{L^{2^*_m}}^{2^*_m} = \|u_h - u\|_{L^{2^*_m}}^{2^*_m} + \|u\|_{L^{2^*_m}}^{2^*_m} + o(1).$$

Thus

$$\begin{aligned}
\mathcal{S}_\lambda^\Omega(m,s) &= \|u_h\|_m^2 - \lambda\|u_h\|_s^2 + o(1) \\
&= \Big(\|u_h - u\|_m^2 + \|u\|_m^2\Big) - \lambda\Big(\|u_h - u\|_s^2 + \|u\|_s^2\Big) + o(1) \\
&= \frac{\Big(\|u_h - u\|_m^2 - \lambda\|u_h - u\|_s^2\Big) + \Big(\|u\|_m^2 - \lambda\|u\|_s^2\Big)}{\Big(\|u_h - u\|_{L^{2^*_m}}^{2^*_m} + \|u\|_{L^{2^*_m}}^{2^*_m}\Big)^{2/2^*_m}} + o(1) \\
&\geq \mathcal{S}_\lambda^\Omega(m,s) \cdot \frac{\xi_h^2 + 1}{(\xi_h^{2^*_m} + 1)^{2/2^*_m}} + o(1),
\end{aligned}$$

where we have set

$$\xi_h := \frac{\|u_h - u\|_{L^{2^*_m}}}{\|u\|_{L^{2^*_m}}}.$$

Since $2^*_m > 2$, this implies that $\xi_h \to 0$, that is, $u_h \to u$ in $L^{2^*_m}$ and hence u achieves $\mathcal{S}_\lambda^\Omega(m,s)$. □

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

Theorem 4.2 *Assume $s \geq 2m - \frac{n}{2}$.*

i) If $0 < \lambda < \Lambda_1(m,s)$ then $\mathcal{S}_\lambda^\Omega(m,s)$ is achieved and (1.1) has a nontrivial solution in $\widetilde{H}^m(\Omega)$.

ii) If $0 < \lambda < \widetilde{\Lambda}_1(m,s)$ then $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ is achieved and (1.2) has a nontrivial solution in $\widetilde{H}^m(\Omega)$.

Proof. Since $0 < \lambda < \Lambda_1(m,s)$ then $\mathcal{S}_\lambda^\Omega(m,s)$ is positive, by Proposition 2.1. The main estimates in Lemma 3.1 readily imply $\mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$. By Lemma 4.1, $\mathcal{S}_\lambda^\Omega(m,s)$ is achieved by a nontrivial $u \in \widetilde{H}^m(\Omega)$, that solves (1.1) after multiplication by a suitable constant. Thus i) is proved. For ii) argue in the same way. □

In the case $s < 2m - \frac{n}{2}$ the situation is more complicated. We limit ourselves to point out the next simple existence result.

Theorem 4.3 *Assume $s < 2m - \frac{n}{2}$.*

i) There exists $\lambda^* \in [0, \Lambda_1(m,s))$ such that the infimum $\mathcal{S}_\lambda^\Omega(m,s)$ is attained for any $\lambda \in (\lambda^*, \Lambda_1(m,s))$, and hence (1.1) has a nontrivial solution.

ii) There exists $\widetilde{\lambda}^* \in [0, \widetilde{\Lambda}_1(m,s))$ such that the infimum $\widetilde{\mathcal{S}}_\lambda^\Omega(m,s)$ is attained for any $\lambda \in (\widetilde{\lambda}^*, \widetilde{\Lambda}_1(m,s))$, and hence (1.2) has a nontrivial solution.

Proof. Use Proposition 2.1 to find $\varphi_1 \in \widetilde{H}^m(\Omega)$, $\varphi_1 \neq 0$, such that

$$\int\limits_\Omega |\,(-\Delta)^{\frac{m}{2}}\varphi_1|^2\,dx = \Lambda_1(m,s)\int\limits_\Omega |\,(-\Delta)^{\frac{s}{2}}\varphi_1|^2\,dx\,.$$

Then test $\mathcal{S}_\lambda^\Omega(m,s)$ with φ_1 to get the strict inequality $\mathcal{S}_\lambda^\Omega(m,s) < \mathcal{S}_m$. The first conclusion follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. □

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