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Non-critical dimensions for critical problems involving fractional Laplacians

Roberta Musina* and Alexander I. Nazarov†

Abstract

We study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving Dirichlet-Laplacian of arbitrary real order $m > 0$.

Keywords: Fractional Laplace operators, Sobolev inequality, Hardy inequality, critical dimensions.

1 Introduction

Let m, s be two given real numbers, with $0 \leq s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded and smooth domain in \mathbb{R}^n and put

$$2_m^* = \frac{2n}{n - 2m}.$$

We study equations

$$(-\Delta)^m u = \lambda(-\Delta)^s u + |u|^{2_m^* - 2} u \quad \text{in } \Omega, \quad (1.1)$$

$$(-\Delta)^m u = \lambda|x|^{-2s} u + |u|^{2_m^* - 2} u \quad \text{in } \Omega, \quad (1.2)$$

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under suitably defined Dirichlet boundary conditions. In dealing with equation (1.2) we always assume that Ω contains the origin. For the definition of fractional Dirichlet–Laplace operators $(-\Delta)^m, (-\Delta)^s$ and for the variational approach to (1.1), (1.2) we refer to the next section.

The celebrated paper [3] by Brezis and Nirenberg was the inspiration for a fruitful line of research about the effect of lower order perturbations in noncompact variational problems. They took as model the case $n > 2, m = 1, s = 0$, that is,

$$-\Delta u = \lambda u + |u|^{\frac{4}{n-2}}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.3)$$

Brezis and Nirenberg pointed out a remarkable phenomenon that appears for positive values of the parameter λ : they proved existence of a nontrivial solution for any small $\lambda > 0$ if $n \geq 4$; in contrast, in the lowest dimension $n = 3$ non-existence phenomena for sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical*¹ for problem (1.3).

Clearly, as larger s is, as stronger the effects of the lower order perturbations are expected in equations (1.1), (1.2). We are interested in the following question: *Given $m < \frac{n}{2}$, how large must be s in order to have the existence of a ground state solution, for any arbitrarily small $\lambda > 0$?* In case of an affirmative answer, we say that n is not a critical dimension.

We present our main result, that holds for any dimension $n \geq 1$ (see Section 4 for a more precise statement).

THEOREM. *If $s \geq 2m - \frac{n}{2}$ then n is not a critical dimension for the Dirichlet boundary value problems associated to equations (1.1) and (1.2).*

We point out some particular cases that are included in this result.

- If m is an integer and $s = m - 1$, then at most the lowest dimension $n = 2m + 1$ is critical.
- For any $n > 2m$ there always exist lower order perturbations of the type $|x|^{-2s}u$ and of the type $(-\Delta)^s u$ such that n is not a critical dimension.
- If $m < 1/4$ then no dimension is critical, for any choice of $s \in [0, m)$.

¹ compare with [13], [8].

After [3], a large number of papers have been focussed on studying the effect of linear perturbations in noncompact variational problems of the type (1.1). Most of these papers deal with $s = 0$, when the problems (1.1) and (1.2) coincide. Moreover, as far as we know, all of them consider either polyharmonic case $2 \leq m \in \mathbb{N}$, see for instance [13], [6], [2], [10], [7], or the case $m \in (0, 1)$, see [14], [15]. We cite also [4], where equation (1.1) is studied in case $m = 2$, $s = 1$. Thus, our Theorem 4.2 covers all earlier existence results.

Finally, we mention [1] (see also [16]) where equation (1.1) for the so-called Navier-Laplacian is studied in case $m \in (0, 1)$, $s = 0$. For a comparison between the Dirichlet and Navier Laplacians we refer to [12].

The paper is organized as follows. After introducing some notation and preliminary facts in Section 2, we provide the main estimates in Section 3. In Section 4 we prove Theorem 1 and point out an existence result for the case $s < 2m - \frac{n}{2}$.

2 Preliminaries

The fractional Laplacian $(-\Delta)^m u$ of a function $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is defined via the Fourier transform

$$\mathcal{F}[u](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx$$

by the identity

$$\mathcal{F}[(-\Delta)^m u](\xi) = |\xi|^{2m} \mathcal{F}[u](\xi). \quad (2.1)$$

In particular, Parseval's formula gives

$$\int_{\mathbb{R}^n} (-\Delta)^m u \cdot u dx = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 dx = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2 d\xi.$$

We recall the well known Sobolev inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 dx \geq \mathcal{S}_m \left(\int_{\mathbb{R}^n} |u|^{2^*_m} dx \right)^{2/2^*_m}, \quad (2.2)$$

that holds for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $m < \frac{n}{2}$, see for example [17, 2.8.1/15].

Let $\mathcal{D}^m(\mathbb{R}^n)$ be the Hilbert space obtained by completing $\mathcal{C}_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo norm

$$\|u\|_m^2 = \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 dx. \quad (2.3)$$

Thanks to (2.2), the space $\mathcal{D}^m(\mathbb{R}^n)$ is continuously embedded into $L^{2^*_m}(\mathbb{R}^n)$. The *best Sobolev constant* \mathcal{S}_m was explicitly computed in [5]. Moreover, it has been proved in [5] that \mathcal{S}_m is attained in $\mathcal{D}^m(\mathbb{R}^n)$ by a unique family of functions, all of them being obtained from

$$\phi(x) = (1 + |x|^2)^{\frac{2m-n}{2}} \quad (2.4)$$

by translations, dilations in \mathbb{R}^n and multiplication by constants.

Dilations play a crucial role in the problems under consideration. Notice that for any $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $R > 0$ it turns out that

$$\begin{aligned} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega](\xi)|^2 d\xi &= R^{n-2m} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\omega(R\cdot)](\xi)|^2 d\xi \\ \int_{\mathbb{R}^n} |\omega|^{2^*_m} dx &= R^n \int_{\mathbb{R}^n} |\omega(R\cdot)|^{2^*_m} dx. \end{aligned} \quad (2.5)$$

Finally, we point out that the Hardy inequality

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 dx \geq \mathcal{H}_m \int_{\mathbb{R}^n} |x|^{-2m} |u|^2 dx \quad (2.6)$$

holds for any function $u \in \mathcal{D}^m(\mathbb{R}^n)$. The *best Hardy constant* \mathcal{H}_m was explicitly computed in [11].

The natural ambient space to study the Dirichlet boundary value problems for (1.1), (1.2) is

$$\tilde{H}^m(\Omega) = \{u \in \mathcal{D}^m(\mathbb{R}^n) : \text{supp } u \subset \overline{\Omega}\},$$

endowed with the norm $\|u\|_m$. By Theorem 4.3.2/1 [17], for $m + \frac{1}{2} \notin \mathbb{N}$ this space coincides with $H_0^m(\Omega)$ (that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^m(\Omega)$), while for $m + \frac{1}{2} \in \mathbb{N}$ one has $\tilde{H}^m(\Omega) \subsetneq H_0^m(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\tilde{H}^m(\Omega)$. Clearly, if m is an integer then $\tilde{H}^m(\Omega)$ is the standard Sobolev space of functions $u \in H^m(\Omega)$ such that $D^\alpha u = 0$ for every multiindex $\alpha \in \mathbb{N}^n$ with $0 \leq |\alpha| < m$.

We agree that $(-\Delta)^0 u = u$, $\tilde{H}^0(\Omega) = L^2(\Omega)$, since (2.3) reduces to the standard L^2 norm in case $m = 0$.

We define (weak) solutions of the Dirichlet problems for (1.1), (1.2) as suitably normalized critical points of the functionals

$$\mathcal{R}_{\lambda,m,s}^\Omega[u] = \frac{\int_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2 dx - \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 dx}{\left(\int_{\Omega} |u|^{2_m^*} dx \right)^{2/2_m^*}} \quad (2.7)$$

$$\tilde{\mathcal{R}}_{\lambda,m,s}^\Omega[u] = \frac{\int_{\Omega} |(-\Delta)^{\frac{m}{2}} u|^2 dx - \lambda \int_{\Omega} |x|^{-2s} |u|^2 dx}{\left(\int_{\Omega} |u|^{2_m^*} dx \right)^{2/2_m^*}}, \quad (2.8)$$

respectively. It is easy to see that both functionals (2.7), (2.8) are well defined on $\tilde{H}^m(\Omega) \setminus \{0\}$.

We conclude this preliminary section with some embedding results.

Proposition 2.1 *Let m, s be given, with $0 \leq s < m < n/2$.*

i) The space $\tilde{H}^m(\Omega)$ is compactly embedded into $\tilde{H}^s(\Omega)$. In particular the infima

$$\Lambda_1(m, s) := \inf_{\substack{u \in \tilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\|u\|_s^2}, \quad \tilde{\Lambda}_1(m, s) := \inf_{\substack{u \in \tilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\| |x|^{-s} u \|_0^2} \quad (2.9)$$

are positive and achieved.

$$ii) \quad \inf_{\substack{u \in \tilde{H}^m(\Omega) \\ u \neq 0}} \frac{\|u\|_m^2}{\|u\|_{L^{2_m^*}}^2} = \mathcal{S}_m.$$

Statement *i)* is well known for $\Lambda_1(m, s)$ and follows from (2.6) for $\tilde{\Lambda}_1(m, s)$. To check *ii)*, use the inclusion $\tilde{H}^m(\Omega) \hookrightarrow \mathcal{D}^m(\mathbb{R}^n)$ and a rescaling argument. Clearly, the Sobolev constant \mathcal{S}_m is never achieved on $\tilde{H}^m(\Omega)$.

3 Main estimates

Let ϕ be the extremal of the Sobolev inequality (2.2) given by (2.4). In particular, it holds that

$$M := \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} \phi|^2 dx = \mathcal{S}_m \left(\int_{\mathbb{R}^n} |\phi|^{2^*} dx \right)^{2/2^*}. \quad (3.1)$$

Fix $\delta > 0$ and a cutoff function $\varphi \in \mathcal{C}_0^\infty(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside $\{|x| < 2\delta\}$. If δ is sufficiently small, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n} \varphi(x) \phi\left(\frac{x}{\varepsilon}\right) = \varphi(x) (\varepsilon^2 + |x|^2)^{\frac{2m-n}{2}}$$

has compact support in Ω . Next we define

$$\begin{aligned} A_m^\varepsilon &:= \int_{\Omega} |(-\Delta)^{\frac{m}{2}} u_\varepsilon|^2 dx & A_s^\varepsilon &:= \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_\varepsilon|^2 dx \\ \tilde{A}_s^\varepsilon &:= \int_{\Omega} |x|^{-2s} |u_\varepsilon|^2 dx & B^\varepsilon &:= \int_{\Omega} |u_\varepsilon|^{2^*} dx \end{aligned}$$

and we denote by c any universal positive constant.

Lemma 3.1 *It holds that*

$$\begin{cases} A_m^\varepsilon \leq \varepsilon^{2m-n} (M + c\varepsilon^{n-2m}) & (3.2a) \\ A_s^\varepsilon, \tilde{A}_s^\varepsilon \geq c\varepsilon^{4m-n-2s} & \text{if } s > 2m - \frac{n}{2} & (3.2b) \\ A_s^\varepsilon, \tilde{A}_s^\varepsilon \geq c|\log \varepsilon| & \text{if } s = 2m - \frac{n}{2} & (3.2c) \\ B^\varepsilon \geq \varepsilon^{-n} \left((M\mathcal{S}_m^{-1})^{2^*/2} - c\varepsilon^n \right). & (3.2d) \end{cases}$$

Proof of (3.2a). First of all, from (2.5) we get

$$A_m^\varepsilon = \varepsilon^{2m-n} \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\varphi(\varepsilon \cdot)\phi]|^2 d\xi. \quad (3.3)$$

Thus

$$\Gamma_m^\varepsilon := \varepsilon^{n-2m} A_m^\varepsilon - M = \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\varphi(\varepsilon \cdot)\phi]|^2 d\xi - \int_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[\phi]|^2 d\xi.$$

We need to prove that

$$|\Gamma_m^\varepsilon| \leq c\varepsilon^{n-2m}. \quad (3.4)$$

If $m \in \mathbb{N}$, the proof of (3.4) has been carried out in [3], [7]. Here we limit ourselves to the more difficult case, namely, when m is not an integer. We denote by $k := \lfloor m \rfloor \geq 0$ the integer part of m , so that $m - k > 0$. Then

$$\begin{aligned} \Gamma_m^\varepsilon &= \int_{\mathbb{R}^n} |\xi|^{2k} \mathcal{F}[U_-] \cdot |\xi|^{2(m-k)} \overline{\mathcal{F}[U_+]} d\xi \\ &= 2^{2(m-k)+\frac{n}{2}} \frac{\Gamma(m-k+\frac{n}{2})}{\Gamma(-(m-k))} \cdot \int_{\mathbb{R}^n} (-\Delta)^k U_-(x) \cdot V.P. \int_{\mathbb{R}^n} \underbrace{\frac{U_+(x) - U_+(y)}{|x-y|^{n+2(m-k)}}}_{\Psi(x,y)} dy dx, \end{aligned}$$

where $U_\pm = \varphi(\varepsilon \cdot) \phi \pm \phi$ (the last equality follows from [9, Ch. 2, Sec. 3]).

We split the interior integral as follows:

$$V.P. \int_{\mathbb{R}^n} \Psi dy = V.P. \underbrace{\int_{|y-x| \leq \frac{|x|}{2}} \Psi dy}_{I_1} + \underbrace{\int_{\substack{|y-x| \geq \frac{|x|}{2} \\ |y| \leq |x|}} \Psi dy}_{I_2} + \underbrace{\int_{\substack{|y-x| \geq \frac{|x|}{2} \\ |y| \geq |x|}} \Psi dy}_{I_3}.$$

We claim that $|I_j| \leq c|x|^{2k-n}$ for $j = 1, 2, 3$. Indeed, the Lagrange formula gives

$$\begin{aligned} |I_1| &\leq \max_{\substack{|y-x| \leq \frac{|x|}{2} \\ |z| \leq \frac{|x|}{2}}} |D^2 U_+(y)| \cdot \int_{|z| \leq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)-2}} \\ &\leq c|x|^{-(n-2m+2)} \cdot |x|^{2-2(m-k)} = c|x|^{2k-n}. \end{aligned}$$

As concerns the last two integrals we estimate

$$|I_2| \leq \int_{\substack{|y-x| \geq \frac{|x|}{2} \\ |y| \leq |x|}} \frac{c|y|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq |x|^{-(n+2(m-k))} \cdot c|x|^{2m} = c|x|^{2k-n}$$

and finally

$$\begin{aligned} |I_3| &\leq \int_{\substack{|y-x| \geq \frac{|x|}{2} \\ |y| \geq |x|}} \frac{c|x|^{-(n-2m)}}{|x-y|^{n+2(m-k)}} dy \leq c|x|^{-(n-2m)} \cdot \int_{|z| \geq \frac{|x|}{2}} \frac{dz}{|z|^{n+2(m-k)}} \\ &\leq c|x|^{-(n-2m)} \cdot |x|^{-2(m-k)} = c|x|^{2k-n}, \end{aligned}$$

and the claim follows. Now, since

$$|(-\Delta)^k U_-(x)| \leq \frac{c}{|x|^{n-2(m-k)}} \chi_{\{|x| \geq \delta/\varepsilon\}} + \frac{c\varepsilon^{2k}}{|x|^{n-2m}} \chi_{\{\delta/\varepsilon \leq |x| \leq 2\delta/\varepsilon\}},$$

we obtain

$$|\Gamma_m^\varepsilon| \leq c \int_{|x| \geq \delta/\varepsilon} \frac{dx}{|x|^{2n-2m}} + c \int_{\delta/\varepsilon \leq |x| \leq 2\delta/\varepsilon} \frac{\varepsilon^{2k} dx}{|x|^{2n-2(m+k)}} \leq c\varepsilon^{n-2m},$$

that completes the proof of (3.4) and of (3.2a).

Proof of (3.2b) and (3.2c). We use the Hardy inequality (2.6) to get

$$\begin{aligned} A_s^\varepsilon &\geq c\tilde{A}_s^\varepsilon \geq c\varepsilon^{4m-2s-n} \int_{\mathbb{R}^n} |x|^{-2s} |\varphi(\varepsilon \cdot) \phi|^2 dx \\ &\geq c\varepsilon^{4m-2s-n} \int_{|x| < \delta/\varepsilon} \frac{dx}{|x|^{2s}(1+|x|^2)^{n-2m}}. \end{aligned}$$

The last integral converges as $\varepsilon \rightarrow 0$ if $s > 2m - \frac{n}{2}$, and diverges with speed $|\log \varepsilon|$ if $s = 2m - \frac{n}{2}$.

Proof of (3.2d). For ε small enough we estimate by below

$$\begin{aligned} \int_{\mathbb{R}^n} |u_\varepsilon|^{2_m^*} &= \varepsilon^{-n} \int_{\mathbb{R}^n} |\varphi(\varepsilon \cdot) \phi|^{2_m^*} dx = \varepsilon^{-n} \left(\int_{\mathbb{R}^n} |\phi|^{2_m^*} dx - \int_{|x| > \delta/\varepsilon} |\varphi(\varepsilon \cdot) \phi|^{2_m^*} dx \right) \\ &\geq \varepsilon^{-n} \left((MS_m^{-1})^{2_m^*/2} - c \int_{|x| > \delta/\varepsilon} |x|^{-2n} dx \right) \\ &= \varepsilon^{-n} \left((MS_m^{-1})^{2_m^*/2} - c\varepsilon^n \right) \end{aligned}$$

and the Lemma is completely proved. \square

4 Two noncompact minimization problems

In this section we deal with the minimization problems

$$\mathcal{S}_\lambda^\Omega(m, s) = \inf_{\substack{u \in \tilde{H}^m(\Omega) \\ u \neq 0}} \mathcal{R}_{\lambda, m, s}^\Omega[u]; \quad \tilde{\mathcal{S}}_\lambda^\Omega(m, s) = \inf_{\substack{u \in \tilde{H}^m(\Omega) \\ u \neq 0}} \tilde{\mathcal{R}}_{\lambda, m, s}^\Omega[u],$$

where the functionals \mathcal{R} and $\tilde{\mathcal{R}}$ are introduced in (2.7) and (2.8), respectively.

Lemma 4.1 *The following facts hold for any $\lambda \in \mathbb{R}$:*

- i) $\mathcal{S}_\lambda^\Omega(m, s) \leq \mathcal{S}_m$;*
- ii) If $\lambda \leq 0$ then $\mathcal{S}_\lambda^\Omega(m, s) = \mathcal{S}_m$ and it is not achieved;*
- iii) If $0 < \mathcal{S}_\lambda^\Omega(m, s) < \mathcal{S}_m$, then $\mathcal{S}_\lambda^\Omega(m, s)$ is achieved.*

The same statements hold for $\tilde{\mathcal{S}}_\lambda^\Omega(m, s)$ instead of $\mathcal{S}_\lambda^\Omega(m, s)$.

Proof. The proof is nowadays standard, and is essentially due to Brezis and Nirenberg [3]. We sketch it for the infimum $\mathcal{S}_\lambda^\Omega(m, s)$, for the convenience of the reader.

Fix $\varepsilon > 0$ and take $u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \setminus \{0\}$ such that

$$(\mathcal{S}_m + \varepsilon) \left(\int_{\mathbb{R}^n} |u|^{2_m^*} dx \right)^{2/2_m^*} \geq \int_{\mathbb{R}^n} |(-\Delta)^{\frac{m}{2}} u|^2 dx. \quad (4.1)$$

Let $R > 0$ be large enough, so that $u_R(\cdot) := u(R\cdot) \in \mathcal{C}_0^\infty(\Omega)$. Using (2.5) we get

$$\mathcal{S}_\lambda^\Omega(m, s) \leq \frac{\|u\|_m^2 - \lambda R^{2(s-m)} \|u\|_s^2}{\|u\|_{L^{2_m^*}}^2} \leq (\mathcal{S}_m + \varepsilon) \left(1 + cR^{2(s-m)} \right),$$

where c depends only on u and λ . Letting $R \rightarrow \infty$ we get $\mathcal{S}_\lambda^\Omega(m, s) \leq (\mathcal{S}_m + \varepsilon)$ for any $\varepsilon > 0$, and *i*) is proved.

Next, if $\lambda \leq 0$ then clearly $\mathcal{S}_\lambda^\Omega(m, s) = \mathcal{S}_m$. If $\lambda = 0$ then \mathcal{S}_m is not achieved. The more it is not achieved for $\lambda < 0$, and *ii*) holds.

Finally, to prove *iii*) take a minimizing sequence u_h . It is convenient to normalize u_h with respect to the $L^{2_m^*}$ -norm, so that

$$\int_{\Omega} |(-\Delta)^{\frac{m}{2}} u_h|^2 dx - \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 dx = \mathcal{S}_\lambda^\Omega(m, s) + o(1).$$

We can assume that $u_h \rightarrow u$ weakly in $\tilde{H}^m(\Omega)$ and strongly in $\tilde{H}^s(\Omega)$ by Proposition 2.1. Since

$$\begin{aligned} \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u|^2 dx &= \lambda \int_{\Omega} |(-\Delta)^{\frac{s}{2}} u_h|^2 dx + o(1) \\ &= \int_{\Omega} |(-\Delta)^{\frac{m}{2}} u_h|^2 dx - \mathcal{S}_\lambda^\Omega(m, s) + o(1) \\ &\geq (\mathcal{S}_m - \mathcal{S}_\lambda^\Omega(m, s)) + o(1), \end{aligned}$$

then $u \neq 0$. By the Brezis–Lieb lemma we get

$$1 = \|u_h\|_{L^{2_m^*}}^{2_m^*} = \|u_h - u\|_{L^{2_m^*}}^{2_m^*} + \|u\|_{L^{2_m^*}}^{2_m^*} + o(1).$$

Thus

$$\begin{aligned} \mathcal{S}_\lambda^\Omega(m, s) &= \|u_h\|_m^2 - \lambda \|u_h\|_s^2 + o(1) \\ &= \left(\|u_h - u\|_m^2 + \|u\|_m^2 \right) - \lambda \left(\|u_h - u\|_s^2 + \|u\|_s^2 \right) + o(1) \\ &= \frac{\left(\|u_h - u\|_m^2 - \lambda \|u_h - u\|_s^2 \right) + \left(\|u\|_m^2 - \lambda \|u\|_s^2 \right)}{\left(\|u_h - u\|_{L^{2_m^*}}^{2_m^*} + \|u\|_{L^{2_m^*}}^{2_m^*} \right)^{2/2_m^*}} + o(1) \\ &\geq \mathcal{S}_\lambda^\Omega(m, s) \cdot \frac{\xi_h^2 + 1}{(\xi_h^{2_m^*} + 1)^{2/2_m^*}} + o(1), \end{aligned}$$

where we have set

$$\xi_h := \frac{\|u_h - u\|_{L^{2_m^*}}}{\|u\|_{L^{2_m^*}}}.$$

Since $2_m^* > 2$, this implies that $\xi_h \rightarrow 0$, that is, $u_h \rightarrow u$ in $L^{2_m^*}$ and hence u achieves $\mathcal{S}_\lambda^\Omega(m, s)$. \square

We are in position to prove our existence result, that includes the theorem already stated in the introduction.

Theorem 4.2 *Assume $s \geq 2m - \frac{n}{2}$.*

- i) If $0 < \lambda < \Lambda_1(m, s)$ then $\mathcal{S}_\lambda^\Omega(m, s)$ is achieved and (1.1) has a nontrivial solution in $\tilde{H}^m(\Omega)$.*
- ii) If $0 < \lambda < \tilde{\Lambda}_1(m, s)$ then $\tilde{\mathcal{S}}_\lambda^\Omega(m, s)$ is achieved and (1.2) has a nontrivial solution in $\tilde{H}^m(\Omega)$.*

Proof. Since $0 < \lambda < \Lambda_1(m, s)$ then $\mathcal{S}_\lambda^\Omega(m, s)$ is positive, by Proposition 2.1. The main estimates in Lemma 3.1 readily imply $\mathcal{S}_\lambda^\Omega(m, s) < \mathcal{S}_m$. By Lemma 4.1, $\mathcal{S}_\lambda^\Omega(m, s)$ is achieved by a nontrivial $u \in \tilde{H}^m(\Omega)$, that solves (1.1) after multiplication by a suitable constant. Thus *i)* is proved. For *ii)* argue in the same way. \square

In the case $s < 2m - \frac{n}{2}$ the situation is more complicated. We limit ourselves to point out the next simple existence result.

Theorem 4.3 *Assume $s < 2m - \frac{n}{2}$.*

- i) There exists $\lambda^* \in [0, \Lambda_1(m, s))$ such that the infimum $\mathcal{S}_\lambda^\Omega(m, s)$ is attained for any $\lambda \in (\lambda^*, \Lambda_1(m, s))$, and hence (1.1) has a nontrivial solution.*
- ii) There exists $\tilde{\lambda}^* \in [0, \tilde{\Lambda}_1(m, s))$ such that the infimum $\tilde{\mathcal{S}}_\lambda^\Omega(m, s)$ is attained for any $\lambda \in (\tilde{\lambda}^*, \tilde{\Lambda}_1(m, s))$, and hence (1.2) has a nontrivial solution.*

Proof. Use Proposition 2.1 to find $\varphi_1 \in \tilde{H}^m(\Omega)$, $\varphi_1 \neq 0$, such that

$$\int_{\Omega} |(-\Delta)^{\frac{m}{2}} \varphi_1|^2 dx = \Lambda_1(m, s) \int_{\Omega} |(-\Delta)^{\frac{s}{2}} \varphi_1|^2 dx.$$

Then test $\mathcal{S}_\lambda^\Omega(m, s)$ with φ_1 to get the strict inequality $\mathcal{S}_\lambda^\Omega(m, s) < \mathcal{S}_m$. The first conclusion follows by Proposition 2.1 and Lemma 4.1. For (1.2) argue similarly. \square

References

- [1] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, On some critical problems for the fractional Laplacian operator, *J. Differential Equations* **252** (2012), no. 11, 6133–6162.
- [2] F. Bernis and H.-C. Grunau, Critical exponents and multiple critical dimensions for polyharmonic operators, *J. Differential Equations* **117** (1995), no. 2, 469–486.
- [3] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36** (1983), no. 4, 437–477.
- [4] P. Caldiroli and R. Musina, On Caffarelli-Kohn-Nirenberg-type inequalities for the weighted biharmonic operator in cones, *Milan J. Math.* **79** (2011), no. 2, 657–687.
- [5] A. Cotsiolis and N. K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.* **295** (2004), no. 1, 225–236.
- [6] D. E. Edmunds, D. Fortunato and E. Jannelli, Critical exponents, critical dimensions and the biharmonic operator, *Arch. Rational Mech. Anal.* **112** (1990), no. 3, 269–289.
- [7] F. Gazzola, Critical growth problems for polyharmonic operators, *Proc. Roy. Soc. Edinburgh Sect. A* **128** (1998), no. 2, 251–263.
- [8] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic boundary value problems*, Lecture Notes in Mathematics, 1991, Springer, Berlin, 2010.

- [9] I. M. Gelfand and G. E. Shilov, *Generalized functions. V. 1. Properties and operations*, Moscow, FML, 1958 (Russian); English transl.: Boston, MA: Academic press, 1964.
- [10] H.-C. Grunau, Critical exponents and multiple critical dimensions for polyharmonic operators. II, *Boll. Un. Mat. Ital. B (7)* **9** (1995), no. 4, 815–847.
- [11] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Comm. Math. Phys.* **53** (1977), no. 3, 285–294.
- [12] R. Musina and A. I. Nazarov, On fractional Laplacians, *Comm. Partial Differential Equations*, to appear. Preprint arXiv:1308.3606 (2013).
- [13] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators, *J. Math. Pures Appl. (9)* **69** (1990), no. 1, 55–83.
- [14] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.*, to appear (2012).
- [15] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.* **12** (2013), no. 6, 2445–2464.
- [16] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, *Calc. Var. Partial Differential Equations* **42** (2011), no. 1-2, 21–41.
- [17] H. Triebel, *Interpolation theory, function spaces, differential operators*, Deutscher Verlag Wissenschaft., Berlin, 1978.