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Absorbing the Structural Rules in the Sequent Calculus with Additional Atomic Rules

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Abstract We show that if the structural rules are admissible over a set $\mathcal{R}$ of atomic rules, then they are admissible in the sequent calculus obtained by adding the rules in $\mathcal{R}$ to the multisuccedent minimal and intuitionistic **G3** calculi as well as to the classical one. Two applications to pure logic and to the sequent calculus with equality are presented.

Keywords Sequent Calculus · Structural Rules · Atomic Rules · Admissibility · Equality

Mathematics Subject Classification (2000) 03F05

1 Introduction

A multisuccedent sequent calculus for intuitionistic logic free of structural rules was presented in [2] and a detailed proof of their admissibility, based on [1], appeared in [4]. A single succedent version **G3i** of that calculus was adopted in [5]. In both cases the proof of the admissibility of the structural rules relies, as for the classical **G3c** system, on the height-preserving admissibility of the contraction rule. When additional atomic rules are added to the calculus the height-preserving admissibility of the contraction rule may fail. Such is the

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case for example for the following rules Ref and Rep for equality adopted in the second edition [6] of [5]

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Ref} \qquad\qquad \frac{s = r, P[x/s], P[x/r], \Gamma \Rightarrow \Delta}{s = r, P[x/s], \Gamma \Rightarrow \Delta}\ \mathrm{Rep}$$

For instance

$$a = f(a), a = f(a) \Rightarrow a = f(f(a))$$

has the following derivation of height equal one in the systems obtained by adding Ref and Rep to the **G3**[**mic**] calculi in [6]:

$$\frac{a = f(a), a = f(a), a = f(f(a)) \Rightarrow a = f(f(a))}{a = f(a), a = f(a) \Rightarrow a = f(f(a))}$$

but $a = f(a) \Rightarrow a = f(f(a))$ cannot have a derivation of height less than or equal one in such systems.

In such cases, to prove the admissibility of the structural rules we can follow a route different from the one used in [3] (see also [4]) and in [6] for extensions with rules of the **G3**[**mic**] calculi. The basic idea is to eliminate context-sharing cuts first, with the eliminability of contraction obtained as a consequence, due to its immediate derivability from context-sharing cut.

Actually we will show that we can proceed in that way for any set of rules whose active and principal formulae are all atomic, provided appropriate restrictions are placed in the intuitionistic and minimal case. More precisely, let m-**G3i** denote, as in [6], the multisuccedent **G3** calculus for intuitionistic logic, m-**G3m** the analogous calculus for minimal logic, m-**G3**[**mic**] any of the calculi m-**G3m**, m-**G3i** and **G3c** and m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ the calculi obtained by adding to them a set $\mathcal{R}$ of rules of the above form. We will show that any derivation in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}} + \mathrm{Cut}_{cs}$ can be transformed into a derivation in the same system in which the rules in $\mathcal{R}$ and the Cut_{cs} rule are applied before any logical rule different from the left introduction rule for $\perp$, namely $\overline{\perp, \Gamma \Rightarrow \Delta}$ ($L\perp$). From that it will follow that if the structural rules are admissible in the calculus that contains only the initial sequents, the rules in $\mathcal{R}$ and, in the intuitionistic and classical case, $L\perp$, then they are admissible in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ as well.

For $\mathcal{R} = \emptyset$ we have that the height preserving admissibility of the weakening rules and the height-preserving invertibility of the logical rules suffice for the eliminability of context-sharing cut, without having to obtain the admissibility of contraction first, and, as a consequence, for the admissibility of the cut rule in m-**G3**[**mic**]. For $\mathcal{R} = \{\mathrm{Ref}, \mathrm{Rep}\}$ we obtain that the structural rules are admissible in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$, thus extending the result proved in [3] in the case t, r and s are restricted to be constants.

2 Preliminaries

Adopting the notations in [6], the sequent calculus **G3c** has the following initial sequents and rules, where P is an atomic formula and A, B stand for any formula in a first order language (function symbols included) and Γ and Δ are finite multisets of formulae :

Initial sequents

$$P, \Gamma \Rightarrow \Delta, P$$

Logical rules

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\ R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\ R\to$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\ L\bot$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A}\ R\forall$$

$$\frac{A[x/a], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}\ R\exists$$

In m-**G3i** the rules $L \to$, $R \to$ and $R\forall$ are replaced by:

$$\frac{A \to B, \Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L^i \to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}\ R^i \to$$

$$\frac{\Gamma \Rightarrow A[x/a]}{\Gamma \Rightarrow \Delta, \forall x A}\ R^i\forall$$

In both **G3c** and m-**G3i**, a does not occur in the conclusion of $L\exists$ and $R\forall$.

Finally m-**G3m** is obtained from m-**G3i** by replacing $L\bot$ by the additional initial sequents $\bot, \Gamma \Rightarrow \Delta, \bot$.

m-**G3**[**mic**] denotes any of the systems m-**G3m**, m-**G3i** or **G3c**.

The left and right weakening rules, LW and RW, have the form:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

The left and right contraction rules, LC and RC have the form:

$$\frac{A.A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

The cut rule and the context-sharing cut rule, Cut and Cut$_{cs}$ have the form:

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta} \quad \text{Cut} \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \text{Cut}_{cs}$$

2.1 Separated derivations

We will deal with both context-sharing and context-independent atomic rules. They are defined as follows:

Definition 1 A context-independent atomic rule is a rule of the following form:

$$\frac{\mathbf{Q_1}, \Gamma_1 \Rightarrow \Delta_1, \mathbf{Q'_1} \quad \dots \quad \mathbf{Q_n}, \Gamma_n \Rightarrow \Delta_n, \mathbf{Q'_n}}{\mathbf{P}, \Gamma_1, \dots, \Gamma_n \Rightarrow \Delta_1, \dots, \Delta_n, \mathbf{P'}}$$

where $\mathbf{Q_1}, \mathbf{Q'_1}, \dots, \mathbf{Q_n}, \mathbf{Q'_n}, \mathbf{P}, \mathbf{P'}$ are sequences (possibly empty) of atomic formulae, $\Gamma_1, \dots, \Gamma_n, \Delta_1, \dots \Delta_n$ are finite multisets (possibly empty) of formulae that are not active in the rule. When $n = 0$ the rule has no premiss and a conclusion of the form $\mathbf{P}, \Gamma \Rightarrow \Delta, \mathbf{P'}$ and it will be denoted by $\overline{\mathbf{P}, \Gamma \Rightarrow \Delta, \mathbf{P'}}$.

A context-sharing atomic rule has the form above, where for $1 \leq i \leq n$, $\Gamma_i = \Gamma$, $\Delta_i = \Delta$ and $\Gamma_1, \dots, \Gamma_n$ and $\Delta_1, \dots, \Delta_n$ are replaced by Γ and Δ respectively.

Definition 2 For a set $\mathcal{R}$ of atomic rules, an $\mathcal{R}$-inference is an application of a rule in $\mathcal{R}$ and m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ is the sequent calculus obtained from m-$\mathbf{G3}[\mathbf{mic}]$ by adding the rules in $\mathcal{R}$; furthermore $\mathcal{R}[ic]$ is the calculus that contains only the initial sequents $P, \Gamma \Rightarrow \Delta, P$, the rule $L\bot$ and the rules in $\mathcal{R}$ and $\mathcal{R}[m]$ is the logic-free calculus that contains only the initial sequents including $\bot, \Gamma \Rightarrow \Delta, \bot$ and the rules in $\mathcal{R}$.

Note that the initial sequents $\bot, \Gamma \Rightarrow \Delta, \bot$ are present in $\mathcal{R}[ic]$ as instances of $L\bot$.

Proposition 1 *For a set $\mathcal{R}$ of atomic rules, the weakening rules are height-preserving admissible in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$.*

Proof For left weakening it suffices to add A to the antecedent of every sequent in a given derivation of $\Gamma \Rightarrow \Delta$, modulo a possible renaming of the proper variables in the $L\exists$ and $R\forall$ inferences, to obtain a derivation of the same height of $A, \Gamma \Rightarrow \Delta$. In the classical case one can proceed in the same way also for right weakening, while in the minimal or intuitionistic case one uses induction on the height of derivation, taking advantage of the possibility of adding an arbitrary context on the right in the applications of $R^i \rightarrow$ and $R^i\forall$. □

Definition 3 The logical rules different from $L\bot$ will be called *proper*.

Definition 4 A derivation in m-$\mathbf{G3[mic]}^{\mathcal{R}} + \mathrm{Cut}_{cs}$ is said to be *separated* if no proper logical inference precedes an $\mathcal{R}$ or Cut_{cs}-inference.

Our first goal is to show that every derivable sequent in m-$\mathbf{G3[mic]}^{\mathcal{R}} + \mathrm{Cut}_{cs}$ has a separated derivation in the same system, provided that in the minimal and intuitionistic case, the rules in $\mathcal{R}$ are suitably modified.

Derivations without proper logical inferences are trivially separated. For such derivations we have the following useful fact.

Lemma 1 *If $\Gamma \Rightarrow \Delta$ has a derivation $\mathcal{D}$ in $\mathcal{R}[m] + \mathrm{Cut}_{cs}$ or $\mathcal{R}[ic] + \mathrm{Cut}_{cs}$, then there is an atomic subsequent $\Gamma^\circ \Rightarrow \Delta^\circ$ of $\Gamma \Rightarrow \Delta$, namely a sequent with atomic formulae only, that has a derivation $\mathcal{D}^\circ$ in the same system, with only atomic sequents, such that the height of $\mathcal{D}^\circ$ is less than or equal to the height of $\mathcal{D}$.*

Proof The claim is proved by a straightforward induction on the height of derivations, thanks to the height-preserving admissibility of the weakening rules. □

Lemma 2 *If the conclusion of a classical logical rule has a derivation in $\mathcal{R}[m] + \mathrm{Cut}_{cs}$ or $\mathcal{R}[ic] + \mathrm{Cut}_{cs}$ of height bounded by h, then also its premisses have derivations in the same system of height bounded by h.*

Proof By the previous lemma, there is a derivation $\mathcal{D}^\circ$ of height bounded by h of an atomic subsequent $\Gamma^\circ \Rightarrow \Delta^\circ$ of the conclusion of the logical inference. Being atomic $\Gamma^\circ \Rightarrow \Delta^\circ$ does not contain the principal formula of the logical inference we are interested in and its premiss or premisses can be obtained from $\mathcal{D}^\circ$ by weakening. □

Proposition 2 *a) Height-preserving separated invertibility of the logical rules in $\mathbf{G3c}^{\mathcal{R}} + \mathrm{Cut}_{cs}$*
If the conclusion of a logical inference has a separated derivation of height bounded by h, then also its premisses have separated derivations of height bounded by h.

b) The same holds for m-$\mathbf{G3[mi]}^{\mathcal{R}} + \mathrm{Cut}_{cs}$, except for the rules $R^i \rightarrow$ and $R^i\forall$.

Proof If the given derivation $\mathcal{D}$ reduces to an initial sequent or to an instance of $L\bot$ or it ends with a logical inference that does not introduce the principal formula of the rule to be proved invertible, then the argument is the same as for the m-**G3**[**mic**] systems (see [2], [4]). For example if $\mathcal{D}$ has the form:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ E \to F, A \to B, \Gamma \Rightarrow \Delta, E\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ F, A \to B, \Gamma \Rightarrow \Delta\end{matrix}}{A \to B, E \to F, \Gamma \Rightarrow \Delta}$$

and the rule to be proved invertible is an $L^i \to$ with principal formula $A \to B$, then a derivation of the same height as $\mathcal{D}$ of its first premiss $A \to B, E \to F, \Gamma \Rightarrow \Delta, A$ is obtained by height preserving weakening applied to $\mathcal{D}$. As far as the second premiss, i.e. $B, E \to F, \Gamma \Rightarrow \Delta$ is concerned, by induction hypothesis, there is a separated derivation $\mathcal{D}'_0$ of height bounded by the height of $\mathcal{D}_0$, of $B, E \to F, \Gamma \Rightarrow \Delta, E$ and a separated derivation $\mathcal{D}'_1$, of height bounded by the height of $\mathcal{D}_1$, of $B, F, \Gamma \Rightarrow \Delta$, Then:

$$\dfrac{\begin{matrix}\mathcal{D}'_0 \\ B, E \to F, \Gamma \Rightarrow \Delta, E\end{matrix} \qquad \begin{matrix}\mathcal{D}'_1 \\ B, F, \Gamma \Rightarrow \Delta\end{matrix}}{B, E \to F, \Gamma \Rightarrow \Delta}$$

is a separated derivation of $B, E \to F, \Gamma \Rightarrow \Delta$ with height bounded by $h(\mathcal{D})$.

If the last inference does introduce the principal formula of the logical rule to be proved invertible, we only need, in addition, to note that the subderivations of a separated derivation are themselves separated.

If $\mathcal{D}$ ends with an $\mathcal{R}$ or Cut_{cs}-inference, then $\mathcal{D}$, being separated, does not contain any proper logical inference, and the previous Lemma applies. □

Proposition 3 *In the classical case, if the premisses of an atomic rule have a separated derivation, then its conclusion also has a separated derivation.*

Proof We proceed by induction on the sum of the heights of the derivations of the premisses of an atomic rule R. If all such derivations are free of proper logical inferences, then it suffices to apply to such premisses the rule R to obtain the desired separated derivation of its conclusion. Otherwise we can select a derivation of a premiss that ends with a proper logical inference and apply the induction hypothesis to the derivation(s) of the premisses of such an inference and to the derivations of the other premisses of R, if any. We distinguish two cases.

Case 1 R is context-independent. For example, suppose that R has the two premisses $Q_1, \Gamma_1 \Rightarrow \Delta'_1, E \to F, Q'_1$ and $Q_2, \Gamma_2 \Rightarrow \Delta_2, Q'_2$ and the conclusion $P, \Gamma_1, \Gamma_2 \Rightarrow \Delta'_1, E \to F, \Delta_2, P'$ and that $Q_1, \Gamma_1 \Rightarrow \Delta'_1, E \to F, Q'_1$ has a separated derivation $\mathcal{D}$ ending with a $R \to$-inference and that $Q_2, \Gamma_2 \Rightarrow \Delta_2, Q'_2$ has a separated derivation $\mathcal{E}$. $\mathcal{D}$ has the form:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ Q_1, E, \Gamma_1 \Rightarrow \Delta'_1, F, Q'_1\end{matrix}}{Q_1, \Gamma_1 \Rightarrow \Delta'_1, E \to F, Q'_1}$$

By induction hypothesis applied to $\mathcal{D}_0$ and $\mathcal{E}$, we have a separated derivation of:

$$P, E, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1', F, \Delta_2, P'$$

from which it suffices to apply the last $R \rightarrow$ inference of $\mathcal{D}$ to obtain the desired separated derivation of $P, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1', E \rightarrow F, \Delta_2, P'$.

Case 2 R is context-sharing. Let us assume, for example, that R has the two premisses $Q_1, \Gamma \Rightarrow \Delta, E \rightarrow F, Q_1'$ and $Q_2, \Gamma \Rightarrow \Delta, E \rightarrow F, Q_2'$ and the conclusion $P, \Gamma \Rightarrow \Delta, E \rightarrow F, P'$, and that $Q_1, \Gamma \Rightarrow \Delta, E \rightarrow F, Q_1'$ has a separated derivation $\mathcal{D}$ ending with a $R \rightarrow$-inference with premiss $Q_1, E, \Gamma \Rightarrow \Delta, F, Q_1'$ and that $Q_2, \Gamma \Rightarrow \Delta, E \rightarrow F, Q_2'$ has a separated derivation $\mathcal{E}$. By Proposition 2 *a*), there is a separated derivation $\mathcal{E}'$ of $Q_2, E, \Gamma \Rightarrow \Delta, F, Q_2'$. Then we can apply the induction hypothesis to the immediate subderivation $\mathcal{D}_0$ of $\mathcal{D}$ and to $\mathcal{E}'$ to obtain a separated derivation of $P, E, \Gamma \Rightarrow \Delta, F, P'$, from which the desired separated derivation of $P, \Gamma \Rightarrow \Delta, P'$ can be obtained by an $R \rightarrow$-inference. □

In the minimal and intuitionistic case the above argument fails if all the derivations of the premisses of the rule R are either $R^i \rightarrow$ or $R^i\forall$-inferences. In order to extend Proposition 3 we adopt the following modification of the definition of atomic rule.

Definition 5 An *intuitionistic atomic rule* is a rule of one of the following forms:

$$a) \quad \frac{\mathbf{Q_1}, \Gamma_1 \Rightarrow Q_1' \ldots \mathbf{Q_n}, \Gamma_n \Rightarrow Q_n'}{\mathbf{P}, \Gamma_1, \ldots, \Gamma_n \Rightarrow \Delta, P'} \qquad b) \quad \frac{\mathbf{Q}, \Gamma \Rightarrow \Delta}{\mathbf{P}, \Gamma \Rightarrow \Delta}$$

$$c) \quad \frac{\mathbf{Q}, \Gamma \Rightarrow \Delta, Q'}{\mathbf{Q}, \Gamma \Rightarrow \Delta, P'} \qquad d) \quad \overline{\mathbf{Q}, \Gamma \Rightarrow \Delta, P'}$$

where Γ, Δ and, for $1 \leq i \leq n$, $\mathbf{Q}_i$, $\mathbf{P}_i$ Γ_i and Δ_i satisfy the same condition as in Definition 1, but Q_i and P_i' are single atomic formulae. In the context-sharing case $\Gamma_1, \ldots, \Gamma_n$ as well as $\Gamma_1 \ldots \Gamma_n$ are replaced by Γ.

The rules Ref and Rep in the Introduction are intuitionistic atomic rules under condition *b*). Some further examples are provided by the rules

$$\frac{r = s, \Gamma \Rightarrow \Delta, P[x/r]}{r = s, \Gamma \Rightarrow \Delta, P[x/s]} \qquad \frac{s = r, \Gamma \Rightarrow \Delta, P[x/r]}{s = r, \Gamma \Rightarrow \Delta, P[x/s]}$$

that satisfy condition *c*), and the rules

$$\frac{P[x/a], \Gamma \Rightarrow P[x/S(a)]}{P[x/0], \Gamma \Rightarrow \Delta, P[x/t]} \qquad \frac{\Gamma_1 \Rightarrow P[x/r] \qquad \Gamma_2 \Rightarrow r = s}{\Gamma_1, \Gamma_2 \Rightarrow \Delta, P[x/s]}$$

that satisfy condition *a*), as well as all the zero-premiss rules of the form $\overline{\Gamma \Rightarrow \Delta, P}$, that satisfy condition *d*). Note that the latter include all the rules of the form $\overline{\Gamma \Rightarrow \Delta, r = s}$, in particular the one expressing the reflexivity of equality, i.e. $\overline{\Gamma \Rightarrow \Delta, t = t}$.

The fact that in condition a) of Definition 5, an arbitrary context Δ can be present in the succedent of the conclusion of the rule, makes it immediate to extend Proposition 1 to set of intuitionistic atomic rules.

Proposition 4 *For a set of intuitionistic atomic rules $\mathcal{R}$, the weakening rules are height-preserving admissible in m-$\mathbf{G3}[mic]^{\mathcal{R}}$.*

For such set of rules, we have the following strengthening of Lemma 1, that is also proved by an easy induction on the height of derivations.

Lemma 3 *If $\mathcal{R}$ is a set of of intuitionistic atomic rules and $\Gamma \Rightarrow \Delta$ has a derivation $\mathcal{D}$ in $\mathcal{R}[m] + \mathrm{Cut}_{cs}$ or $\mathcal{R}[ic] + \mathrm{Cut}_{cs}$, then there is an atomic subsequent $\Gamma^\circ \Rightarrow \Delta^\circ$ of $\Gamma \Rightarrow \Delta$, such that $|\Delta^\circ| \leq 1$, that has a derivation $\mathcal{D}^\circ$ in the same system, with only atomic sequents with at most one formula in their succedent, such that the height of $\mathcal{D}^\circ$ is less than or equal to the height of $\mathcal{D}$.*

By Lemma 3, we have that Lemma 2 and Proposition 2 hold also for systems based on intuitionistic atomic rules. That allows for the extension to such systems of Proposition 3, as shown in the following:

Proposition 5 *If the premisses of an intuitionistic atomic rule have a separated derivation, then its conclusion also has a separated derivation.*

Proof We proceed by induction on the sum of the heights of the derivations of the premisses of an intuitionistic atomic rule R.

If R satisfies condition a) of Definition 5 the proof is the same as for Proposition 3, that works without exceptions since the given separated derivation of the premisses of R cannot conclude with a right inference, in particular not with a $R^i \to$ or $R^i\forall$-inference.

If R satisfies condition b) the derivation $\mathcal{D}$ of the premiss of R may end with a $R^i \to$ or $R^i\forall$- inference. Let us assume for example that it ends with a $R^i\forall$-inference, so that $\mathcal{D}$ is of the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathbf{Q}, \Gamma \Rightarrow E[x/a]\end{array}}{\mathbf{Q}, \Gamma \Rightarrow \Delta, \forall x E}$$

By induction hypothesis applied to $\mathcal{D}_0$ there is a separated derivation of $\mathbf{P}, \Gamma \Rightarrow E[x/a]$, from which the desired separated derivation of $\mathbf{P}, \Gamma \Rightarrow \Delta, \forall x E$ is obtained by the same last $R^i\forall$-inference of $\mathcal{D}$.

If R satisfies condition c) and $\mathcal{D}$ does not end with an $R^i \to$ or $R^i\forall$ we apply the induction hypothesis to the immediate subderivation(s) of $\mathcal{D}$ and then the same R-inference. For example if the conclusion of R is $\mathbf{Q}, E \vee F, \Gamma \Rightarrow \Delta, P'$ and the derivation $\mathcal{D}$ of its premiss has the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathbf{Q}, E, \Gamma \Rightarrow \Delta, Q'\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \mathbf{Q}, F, \Gamma \Rightarrow \Delta, Q'\end{array}}{\mathbf{Q}, E \vee F, \Gamma \Rightarrow \Delta, Q'}$$

then we apply the induction hypothesis to $\mathcal{D}_0$ and $\mathcal{D}_1$ to obtain separated derivations of $\mathbf{Q}, E, \Gamma \Rightarrow \Delta, P'$ and $\mathbf{Q}, F, \Gamma \Rightarrow \Delta, P'$ and then the same last $L\vee$-inference of $\mathcal{D}$. If, instead, $\mathcal{D}$ ends with a $R^i \rightarrow$ or $R^i\forall$-inference, then the desired separated derivation of the conclusion of R can be obtained by applying directly the same kind of inference to the immediate subderivation $\mathcal{D}_0$ of $\mathcal{D}$. For example if $\mathcal{D}$ has the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathbf{Q}, E, \Gamma \Rightarrow F\end{array}}{\mathbf{Q}, \Gamma \Rightarrow \Delta, E \rightarrow F, Q'}$$

then the conclusion $\mathbf{Q}, \Gamma \Rightarrow \Delta, E \rightarrow F, P'$ of R, applied to the premiss
$\mathbf{Q}, \Gamma \Rightarrow \Delta, E \rightarrow F, Q'$, has the following separated derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \mathbf{Q}, E, \Gamma \Rightarrow F\end{array}}{\mathbf{Q}, \Gamma \Rightarrow \Delta, E \rightarrow F, P'}$$

If R satisfies condition d), the claim is trivial. □

Lemma 4 *If the premisses $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ of a* Cut$_{cs}$*-inference have separated derivations in m-$\mathbf{G3c}[\mathbf{mic}]^{\mathcal{R}}$ + Cut$_{cs}$, one of which is free of proper logical inferences, then its conclusion $\Gamma \Rightarrow \Delta$ has a separated derivation in the same system, provided that, in the intuitionistic and minimal case, $\mathcal{R}$ is a set of intuitionistic atomic rules.*

Proof Let $\mathcal{D}$ and $\mathcal{E}$ be separated derivations of $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ respectively. If both $\mathcal{D}$ and $\mathcal{E}$ are free of proper logical rules, to obtain the desired separated derivation of $\Gamma \Rightarrow \Delta$, actually a derivation free of proper logical rules, of $\Gamma \Rightarrow \Delta$, it suffices to apply a Cut$_{cs}$-inference to their endsequents. Otherwise we distinguish two cases.

Case 1. $\mathcal{D}$ is free of proper logical rules and $\mathcal{E}$ contains proper logical inferences. Since $\mathcal{E}$ is separated, $\mathcal{E}$ ends with a proper logical rule. To show that there is a separated derivation of $\Gamma \Rightarrow \Delta$, we proceed by induction on the height $h(\mathcal{E})$ of $\mathcal{E}$.

By Lemma 1 there is an atomic subsequent $\Gamma^\circ \Rightarrow \Delta^\circ$ of $\Gamma \Rightarrow \Delta, A$ such that $\Gamma^\circ \Rightarrow \Delta^\circ$ has a derivation $\mathcal{D}^\circ$ without proper logical inferences. If A does not occur in Δ°, a separated derivation, actually a derivation without proper logical inferences, of $\Gamma \Rightarrow \Delta$, can be obtained directly by weakening the conclusion of $\mathcal{D}^\circ$. Otherwise A is atomic, so that the principal formula of the last proper logical inference of $\mathcal{E}$ is different from A. Then we apply Lemma 2, the induction hypothesis and, finally, the same last inference of $\mathcal{E}$. For example, let us assume that the endsequent of $\mathcal{D}$ is $\Gamma \Rightarrow \Delta', E \rightarrow F, A$ and $\mathcal{E}$ has the form:

$$\dfrac{\begin{array}{c}\mathcal{E}_0\\ A, E, \Gamma \Rightarrow \Delta', F\end{array}}{A, \Gamma \Rightarrow \Delta', E \rightarrow F}$$

By Lemma 2 there is a separated derivation $\mathcal{D}'$ of $E, \Gamma \Rightarrow \Delta', F, A$. Then the induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{E}_0$ yields a separated derivation of $E, \Gamma \Rightarrow \Delta', F$, from which the desired separated derivation of $\Gamma \Rightarrow \Delta', E \to F$ is obtained by the same last $R \to$-inference of $\mathcal{E}$.

In the minimal and intuitionistic case, by Lemma 3, if A occurs in the succedent of the endsequent of $\mathcal{D}^\circ$, then, besides being atomic, A is the only formula in the succedent of the endsequent of $\mathcal{D}^\circ$. That allows to proceed as in the classical case, also when $\mathcal{E}$ ends with a $R^i \to$ or $R^i\forall$-inference. For example if $\mathcal{E}$ ends with a $R^i \to$-inference, namely it has the form

$$\dfrac{\begin{matrix}\mathcal{E}_0\\ A, E, \Gamma \Rightarrow F\end{matrix}}{A, \Gamma \Rightarrow \Delta', E \to F}$$

we note that the endsequent of $\mathcal{D}^\circ$ can be weakened to obtain a derivation $\mathcal{D}$' free of proper logical inferences of $E, \Gamma \Rightarrow A$, so that we can apply the induction hypothesis to $\mathcal{D}'$ and $\mathcal{E}_0$ to obtain a separated derivation of $E, \Gamma \Rightarrow F, A$, from which the desired separated derivations of $\Gamma \Rightarrow \Delta', E \to F$ can be obtained by the same last $R^i \to$-inference of $\mathcal{E}$.

Case 2. $\mathcal{D}$ contains proper logical inferences, so that it ends with a logical inference, and $\mathcal{E}$ is free of proper logical rules. Then we proceed by induction on the height of $\mathcal{D}$. By Lemma 1 there is an atomic subsequent of $\Gamma^\circ \Rightarrow \Delta^\circ$ of $A, \Gamma \Rightarrow \Delta$, with a derivation $\mathcal{E}^\circ$ without proper logical inferences. If A does not occur in Γ°, a separated derivation, actually a derivation free of proper logical inferences, of $\Gamma \Rightarrow \Delta$ can be obtained directly by weakening the conclusion of $\mathcal{E}^\circ$. Otherwise A is atomic, so that it is different from the principal formula of the last inference of $\mathcal{D}$. Then we distinguish two subcases.

Case 2.1 The last inference of $\mathcal{D}$ is not an $R^i \to$ or a $R^i\forall$-inference. In that case we can weaken the endsequent of $\mathcal{E}^\circ$, then apply the induction hypothesis to the immediate subderivation(s) of $\mathcal{D}$ and, finally, the last proper logical inference of $\mathcal{D}$. For example, let us assume that the endsequent of $\mathcal{E}^\circ$ is $A, \Gamma^{\circ'} \Rightarrow \Delta^\circ$ and $\mathcal{D}$ has the form:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \Delta', E, A\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma \Rightarrow \Delta', F, A\end{matrix}}{\Gamma \Rightarrow \Delta', E \wedge F, A}$$

The end sequent of $\mathcal{E}^\circ$ can be weakened to obtain derivations $\mathcal{E}'$ and $\mathcal{E}''$ free of proper logical inferences of $A, \Gamma \Rightarrow \Delta', E$ and $A, \Gamma \Rightarrow \Delta', F$. Then we can apply the induction hypothesis to $\mathcal{D}_0$ and $\mathcal{E}'$ as well as to $\mathcal{D}_1$ and $\mathcal{E}''$ to obtain separated derivations of $\Gamma \Rightarrow \Delta', E$ and $\Gamma \Rightarrow \Delta', F$, from which a separated derivation of $\Gamma \Rightarrow \Delta', E \wedge F$ can be obtained by means of the last $R\wedge$-inference of $\mathcal{D}$.

Case 2.2 The last inference of $\mathcal{D}$ is an $R^i \to$ or a $R^i\forall$-inference. In that case a separated derivation of $\Gamma \Rightarrow \Delta$ can be obtained directly from the immediate subderivation of $\mathcal{D}$ by means of its last inference. For example, if the end sequent of $\mathcal{E}$ is $A, \Gamma \Rightarrow \Delta', E \to F$ and $\mathcal{D}$ is of the form:

$$\cfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma, E \Rightarrow F\end{array}}{\Gamma \Rightarrow \Delta', E \to F, A}$$

then

$$\cfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma, E \Rightarrow F\end{array}}{\Gamma \Rightarrow \Delta', E \to F}$$

is a separated derivation of $\Gamma \Rightarrow \Delta', E \to F$. $\Box$

Proposition 6 *If the premisses $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ of a Cut_{cs}-inference have separated derivation in m-$\mathbf{G3c}[\mathbf{mic}]^{\mathcal{R}}+\mathrm{Cut}_{cs}$, then, provided in the minimal and intuitionistic case $\mathcal{R}$ is a set of atomic intuitionistic rules, its conclusion $\Gamma \Rightarrow \Delta$ has a separated derivation in the same system.*

Proof Let $\mathcal{D}$ and $\mathcal{E}$ be separated derivations of $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ respectively. We have to find a separated derivation of $\Gamma \Rightarrow \Delta$.

By the previous Lemma we can assume that both $\mathcal{D}$ and $\mathcal{E}$ end with a proper logical inference, and proceed by a principal induction on the height (of the formation tree) of A and a secondary induction on $h(\mathcal{D}) + h(\mathcal{E})$.

Classical case

Case 1 A is not principal in the last inference of $\mathcal{D}$.

Case 1. $L\wedge$. $\mathcal{D}$ is of the form

$$\cfrac{\begin{array}{c}\mathcal{D}_0\\ E, F, \Gamma' \Rightarrow \Delta, A\end{array}}{E \wedge F, \Gamma' \Rightarrow \Delta, A}$$

so that the endsequent of $\mathcal{E}$ has the form $A, E \wedge F, \Gamma' \Rightarrow \Delta$. By Proposition 2 there is a separated derivation $\mathcal{E}'$ of $A, E, F, \Gamma' \Rightarrow \Delta$ such that $h(\mathcal{E}') \leq h(\mathcal{E})$. By the (secondary) induction hypothesis applied to $\mathcal{D}_0$ and $\mathcal{E}'$ there is a separated derivation of $E, F, \Gamma' \Rightarrow \Delta$, from which the required separated derivation of $\Gamma \Rightarrow \Delta$ can be obtained by means of the last $L\wedge$- inference of $\mathcal{D}$. In the following we will express the argument as follows:

$$\cfrac{\cfrac{E, F, \Gamma' \Rightarrow \Delta, A}{E \wedge F, \Gamma' \Rightarrow \Delta, A} \quad A, E \wedge F, \Gamma' \Rightarrow \Delta}{E \wedge F, \Gamma' \Rightarrow \Delta}$$

is transformed into:

$$\cfrac{\cfrac{E, F, \Gamma' \Rightarrow \Delta, A \quad \cfrac{A, E \wedge F, \Gamma' \Rightarrow \Delta}{A, E, F, \Gamma' \Rightarrow \Delta}\,\mathrm{inv}}{E, F, \Gamma' \Rightarrow \Delta}\,\mathrm{ind}}{E \wedge F, \Gamma' \Rightarrow \Delta}$$

In this case the principal formula of the last logical inference of $\mathcal{D}$ occurs in the endsequent of $\mathcal{E}$, where it can be inverted, producing a separated derivation of a sequent identical to the premiss of the last inference of $\mathcal{D}$, except that the cut formula A is shifted from the succedent to the antecedent. We can then apply the secondary induction hypothesis to produce a sequent to which the last logical inference of $\mathcal{D}$ can be applied, yielding the required separated derivation. The same kind of argument applies to all the remaining cases. For example:

Case 1. $L\rightarrow$

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta, E, A \quad F, \Gamma' \Rightarrow \Delta, A}{E \rightarrow F, \Gamma' \Rightarrow \Delta, A} \quad A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{E \rightarrow F, \Gamma' \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma' \Rightarrow \Delta, E, A \quad \dfrac{A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{A, \Gamma' \Rightarrow \Delta, E}\,\text{inv}}{\Gamma' \Rightarrow \Delta, E}\,\text{ind} \qquad \dfrac{F, \Gamma' \Rightarrow \Delta, A \quad \dfrac{A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{A, F, \Gamma' \Rightarrow \Delta}\,\text{inv}}{F, \Gamma' \Rightarrow \Delta}\,\text{ind}}{E \rightarrow F, \Gamma' \Rightarrow \Delta}$$

Case1 $R\rightarrow$

$$\dfrac{\dfrac{\Gamma, E \Rightarrow \Delta', F, A}{\Gamma \Rightarrow \Delta', E \rightarrow F, A} \quad A, \Gamma \Rightarrow \Delta', E \rightarrow F}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma, E \Rightarrow \Delta', F, A \quad \dfrac{A, \Gamma \Rightarrow \Delta', E \rightarrow F}{A, \Gamma, E \Rightarrow \Delta', F}\,\text{inv}}{\Gamma, E \Rightarrow \Delta', F}\,\text{ind}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

Case 1 $R\forall$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E[x/a], A}{\Gamma \Rightarrow \Delta', \forall x E, A} \quad A, \Gamma \Rightarrow \Delta', \forall x E}{\Gamma \Rightarrow \Delta', \forall x E}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E[x/a], A \quad \dfrac{A, \Gamma \Rightarrow \Delta', \forall x E}{A, \Gamma \Rightarrow \Delta', E[x/a]}\,\text{inv}}{\Gamma \Rightarrow \Delta', E[x/a]}\,\text{ind}}{\Gamma \Rightarrow \Delta', \forall x E}$$

Case 2. A is not principal in the last inference of $\mathcal{E}$.
All the cases are treated dually to Case 1. For example:
Case 2. $R\wedge$

$$\dfrac{E \wedge F, \Gamma' \Rightarrow \Delta, A \quad \dfrac{A, E, F, \Gamma' \Rightarrow \Delta}{A, E \wedge F, \Gamma' \Rightarrow \Delta}}{E \wedge F, \Gamma' \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{E \wedge F, \Gamma' \Rightarrow \Delta, A}{E, F, \Gamma' \Rightarrow \Delta, A}\text{inv} \quad A, E, F, \Gamma' \Rightarrow \Delta}{\dfrac{E, F, \Gamma' \Rightarrow \Delta}{E \wedge F, \Gamma' \Rightarrow \Delta}}\text{ind}$$

Case 2. $R \rightarrow$

$$\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F \qquad \dfrac{A, \Gamma, E \Rightarrow \Delta', F}{A, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, A}{\Gamma, E \Rightarrow \Delta', F, A}\text{inv} \quad A, \Gamma, E \Rightarrow \Delta', F}{\dfrac{\Gamma, E \Rightarrow \Delta', F}{\Gamma \Rightarrow \Delta', E \rightarrow F}}\text{ind}$$

In the above cases all the applications of ind refer to the secondary induction hypothesis.

Case 3. A is principal in the last inferences of both $\mathcal{D}$ and $\mathcal{E}$.

Case 3. $\wedge$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B \qquad \Gamma \Rightarrow \Delta, C}{\Gamma \Rightarrow \Delta, B \wedge C} \qquad \dfrac{B, C, \Gamma \Rightarrow \Delta}{B \wedge C, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\Gamma \Rightarrow \Delta, C \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta, B}{C, \Gamma \Rightarrow \Delta, B}\text{w} \quad B, C, \Gamma \Rightarrow \Delta}{C, \Gamma \Rightarrow \Delta}\text{ind}}{\Gamma \Rightarrow \Delta}\text{ind}$$

In this case the second application of ind refers necessarily to the principal induction hypothesis, which is possible since $h(C) < h(B \wedge C)$, independently of the height of the separated derivation of the second premiss $C, \Gamma \Rightarrow \Delta$, previously obtained by the (secondary suffices) induction hypothesis. Of the remaining cases we deal with the Case 3 $\forall$ in which A is a universal formula, leaving the others to the reader.

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B[x/a]}{\Gamma \Rightarrow \Delta, \forall x B} \qquad \dfrac{\forall x B, B[x/t], \Gamma \Rightarrow \Delta}{\forall x B, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B[x/a]}{\Gamma \Rightarrow \Delta, B[x/t]}\,\mathrm{Sub}[a/t] \qquad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, B[x/a]}{\Gamma \Rightarrow \Delta, \forall x B}}{B[x/t], \Gamma \Rightarrow \Delta, \forall x B}\,\mathrm{w} \qquad \forall x B, B[x/t], \Gamma \Rightarrow \Delta}{B[x/t], \Gamma \Rightarrow \Delta}\,\mathrm{ind}}{\Gamma \Rightarrow \Delta}\,\mathrm{ind}$$

where $\mathrm{Sub}[a/t]$ yields the result of replacing a by t throughout the separated derivation of $\Gamma \Rightarrow \Delta, B[x/a]$. For the result of such a replacement to be a derivation it might be necessary that the parameters used as proper in the $L\exists$ and $R\forall$-inferences of the given derivation be renamed, so as not to occur in t.

Minimal and Intuitionistic case

The failure of separated invertibility of the rules $R^i \rightarrow$ and $R^i\forall$ in m-$\mathbf{G3}[\mathbf{mi}]^{\mathcal{R}}+$ Cut_{cs} (Proposition 2 *b*)) requires a substantial change in the proof with respect to the classical case. We can still deal first with the case in which A is not principal in the last inference of $\mathcal{D}$. Actually the treatment of the rule $R^i \rightarrow$ and $R^i\forall$ is even simpler than the treatment of the rules $R \rightarrow$ and $R\forall$ that they replace, as the induction hypothesis is not needed. But having disposed of that case, we cannot simply assume that A is not principal in the last inference of $\mathcal{E}$, disregarding whether A is principal in the last inference of $\mathcal{D}$ or not. Rather we have to assume also that A is principal in the last inference of $\mathcal{D}$ and proceed according to the form of A, when $\mathcal{E}$ ends with the non invertible rules $R^i \rightarrow$ and $R^i\forall$.

Case 1. A is not principal in the last inference of $\mathcal{D}$. We only need to replace Case1. $L \rightarrow$, $R \rightarrow$ and $R\forall$ with the following:

Case 1. $L^i \rightarrow$

$$\dfrac{\dfrac{E \rightarrow F, \Gamma' \Rightarrow \Delta, E, A \qquad F, \Gamma' \Rightarrow \Delta, A}{E \rightarrow F, \Gamma' \Rightarrow \Delta, A} \qquad A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{E \rightarrow F, \Gamma' \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{E \rightarrow F, \Gamma' \Rightarrow \Delta, E, A \qquad \dfrac{A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{A, E \rightarrow F, \Gamma' \Rightarrow \Delta, E}\,\mathrm{inv}}{E \rightarrow F, \Gamma' \Rightarrow \Delta, E}\,\mathrm{ind} \qquad \dfrac{F, \Gamma' \Rightarrow \Delta, A \qquad \dfrac{A, E \rightarrow F, \Gamma' \Rightarrow \Delta}{A, F, \Gamma' \Rightarrow \Delta}\,\mathrm{inv}}{F, \Gamma' \Rightarrow \Delta}\,\mathrm{ind}}{E \rightarrow F, \Gamma' \Rightarrow \Delta}$$

Case 1. $R^i \rightarrow$

$$\dfrac{\dfrac{E, \Gamma \Rightarrow F}{\Gamma \Rightarrow \Delta', E \rightarrow F, A} \qquad A, \Gamma \Rightarrow \Delta', E \rightarrow F}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{E, \Gamma \Rightarrow F}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

Case 1. $R^i\forall$.

$$\dfrac{\dfrac{\Gamma \Rightarrow E[x/a]}{\Gamma \Rightarrow \Delta', \forall x E, A} \quad A, \Gamma \Rightarrow \Delta', \forall x E}{\Gamma \Rightarrow \Delta', \forall x E}$$

is transformed into:

$$\dfrac{\Gamma \Rightarrow E[x/a]}{\Gamma \Rightarrow \Delta', \forall x E}$$

Case 2. A is not principal in the last inference of $\mathcal{E}$. If the principal formula of the last inference of $\mathcal{E}$ is invertible in $\mathcal{D}$, i.e. if it is not a $R^i \rightarrow$ or $R^i\forall$-inference then we proceed as in Case 2 of the classical case. If $\mathcal{E}$ ends with a $R^i \rightarrow$ or $R^i\forall$-inference, we distinguish cases according to the form of A.

Case 2. $R^i \rightarrow, \wedge$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, B \quad \Gamma \Rightarrow \Delta', E \rightarrow F, C}{\Gamma \Rightarrow \Delta', E \rightarrow F, B \wedge C} \quad \dfrac{B \wedge C, E, \Gamma \Rightarrow F}{B \wedge C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, C \quad \dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, B}{C, \Gamma \Rightarrow \Delta', E \rightarrow F, B}\,\mathrm{w} \quad \dfrac{\dfrac{B \wedge C, E, \Gamma \Rightarrow F}{B \wedge C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{B, C, \Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{inv}}{C, \Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{ind}}{\Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{ind}$$

Case 2. $R^i \rightarrow, \vee$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, B, C}{\Gamma \Rightarrow \Delta', E \rightarrow F, B \vee C} \quad \dfrac{B \vee C, E, \Gamma \Rightarrow F}{B \vee C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \rightarrow F, B, C \quad \dfrac{\dfrac{\dfrac{B \vee C, E, \Gamma \Rightarrow F}{B \vee C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{C, \Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{inv}}{C, \Gamma \Rightarrow \Delta', E \rightarrow F, B}\,\mathrm{w}}{\Gamma \Rightarrow \Delta', E \rightarrow F, B}\,\mathrm{ind} \quad \dfrac{\dfrac{B \vee C, E, \Gamma \Rightarrow F}{B \vee C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{B, \Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{inv}}{\Gamma \Rightarrow \Delta', E \rightarrow F}\,\mathrm{ind}$$

Case 2 $R^i \rightarrow, \rightarrow$

$$\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow \Delta', E \rightarrow F, B \rightarrow C} \quad \dfrac{B \rightarrow C, E, \Gamma \Rightarrow F}{B \rightarrow C, \Gamma \Rightarrow \Delta', E \rightarrow F}}{\Gamma \Rightarrow \Delta', E \rightarrow F}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow F, B \to C}}{E, \Gamma \Rightarrow F, B \to C}\,\mathrm{w} \qquad B \to C, E, \Gamma \Rightarrow F}{E, \Gamma \Rightarrow F}\,\mathrm{ind}}{\Gamma \Rightarrow \Delta', E \to F}$$

Case 2 $R^i \to, \forall$

$$\dfrac{\dfrac{\Gamma \Rightarrow B[x/a]}{\Gamma \Rightarrow \Delta', E \to F, \forall x B} \qquad \dfrac{\forall x B, E, \Gamma \Rightarrow F}{\forall x B, \Gamma \Rightarrow \Delta', E \to F}}{\Gamma \Rightarrow \Delta', E \to F}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow B[x/a]}{\Gamma \Rightarrow F, \forall x B}}{E, \Gamma \Rightarrow F, \forall x B}\,\mathrm{w} \qquad \forall x B, E, \Gamma \Rightarrow F}{E, \Gamma \Rightarrow F}\,\mathrm{ind}}{\Gamma \Rightarrow \Delta', E \to F}$$

Case 2 $R^i \to, \exists$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \to F, \exists x B, B[x/t]}{\Gamma \Rightarrow \Delta', E \to F, \exists x B} \qquad \dfrac{\exists x B, E, \Gamma \Rightarrow F}{\exists x B, \Gamma \Rightarrow \Delta', E \to F}}{\Gamma \Rightarrow \Delta', E \to F}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta', E \to F, \exists x B, B[x/t] \qquad \dfrac{\exists x B, E, \Gamma \Rightarrow F}{\exists x B, \Gamma \Rightarrow \Delta', E \to F, B[x/t]}}{\Gamma \Rightarrow \Delta', E \to F, B[x/t]}\,\mathrm{ind} \qquad \dfrac{\dfrac{\dfrac{\exists x B, E, \Gamma \Rightarrow F}{\exists x B, \Gamma \Rightarrow \Delta', E \to F}\,\mathrm{inv}}{B[x/a], \Gamma \Rightarrow \Delta', E \to F}\,\mathrm{Sub}[a/t]}{B[x/t], \Gamma \Rightarrow \Delta', E \to F}}{\Gamma \Rightarrow \Delta', E \to F}\,\mathrm{ind}$$

Case 2. $R^i\forall$. Similar to Case2. $R^i \to$. For example:
(Case 2 $R^i\forall, \to$)

$$\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow \Delta', \forall x E, B \to C} \qquad \dfrac{B \to C, \Gamma \Rightarrow E[x/a]}{B \to C, \Gamma \Rightarrow \Delta', \forall x E}}{\Gamma \Rightarrow \Delta', \forall x E}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow E[x/a], B \to C} \qquad B \to C, \Gamma \Rightarrow E[x/a]}{\Gamma \Rightarrow E[x/a]}\,\mathrm{ind}}{\Gamma \Rightarrow \Delta', \forall x E}$$

and (Case 2 $R^i\forall$, $\forall$)

$$\dfrac{\dfrac{\Gamma \Rightarrow B[x/b]}{\Gamma \Rightarrow \Delta', \forall x E, \forall x B} \qquad \dfrac{\forall x B, \Gamma \Rightarrow E[x/a]}{\forall x B, \Gamma \Rightarrow \Delta', \forall x E}}{\Gamma \Rightarrow \Delta', \forall x E}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow B[x/b]}{\Gamma \Rightarrow E[x/a], \forall x B} \qquad \forall x B, \Gamma \Rightarrow E[x/a]}{\Gamma \Rightarrow E[x/a]}\text{ind}}{\Gamma \Rightarrow \Delta', \forall x E}$$

Case 3. A is principal in both the last inference $\mathcal{D}$ and the last inference of $\mathcal{E}$. The only difference with respect to the classical case concern $\to$ and $\forall$.

Case 3i. $\to$

$$\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow \Delta, B \to C} \qquad \dfrac{B \to C, \Gamma \Rightarrow \Delta, B \qquad C, \Gamma \Rightarrow \Delta}{B \to C, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{B, \Gamma \Rightarrow C}{\Gamma \Rightarrow \Delta, B, B \to C} \qquad B \to C, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, B}\text{ind} \qquad \dfrac{\dfrac{B, \Gamma \Rightarrow C}{B, \Gamma \Rightarrow \Delta, C}\text{w} \qquad \dfrac{C, \Gamma \Rightarrow \Delta}{C, B, \Gamma \Rightarrow \Delta}\text{w}}{B, \Gamma \Rightarrow \Delta}\text{ind}}{\Gamma \Rightarrow \Delta}\text{ind}$$

Case 3i. $\forall$

$$\dfrac{\dfrac{\Gamma \Rightarrow B[x/a]}{\Gamma \Rightarrow \Delta, \forall x B} \qquad \dfrac{\forall x B, B[x/t], \Gamma \Rightarrow \Delta}{\forall x B, \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

is transformed into:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow B[x/a]}{\Gamma \Rightarrow B[x/t]}\text{Sub}[a/t]}{\Gamma \Rightarrow \Delta, B[x/t]}\text{w} \qquad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow B[x/a]}{\Gamma \Rightarrow \Delta, \forall x B}}{B[x/t], \Gamma \Rightarrow \Delta, \forall x B}\text{w} \qquad \forall x B, B[x/t], \Gamma \Rightarrow \Delta}{B[x/t], \Gamma \Rightarrow \Delta}\text{ind}}{\Gamma \Rightarrow \Delta}\text{ind}$$

□

From Proposition 3, 5 and 6, by a straightforward induction argument, we have the following separation property for m-$\mathbf{G3[mic]}^{\mathcal{R}} + \mathrm{Cut}_{cs}$, where, in the minimal and intuitionistic case, we assume that $\mathcal{R}$ is a set of intuitionistic atomic rules.

Proposition 7 *Every derivation in m-$\mathbf{G3[mic]}^{\mathcal{R}} + \mathrm{Cut}_{cs}$ can be transformed into a separated derivation of its endsequent.*

Theorem 1 *If the structural rules are admissible in $\mathcal{R}$, then they are admissible in m-*$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ *as well. More precisely:*

a) If the structural rules are admissible in $\mathcal{R}[m]$, then they are admissible in m-$\mathbf{G3}[\mathbf{m}]^{\mathcal{R}}$
b) If the structural rules are admissible in $\mathcal{R}[ic]$, then they are admissible in m-$\mathbf{G3}[\mathbf{ic}]^{\mathcal{R}}$

Proof Let $\mathcal{D}$ be a derivation in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}+RLW+RLC+$Cut. We have to show that the applications of the RLW, RLC and Cut can be eliminated from $\mathcal{D}$. $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ + Cut_{cs} of the same endsequent. For, the application of the Cut-rule can be replaced by applications of the weakenings and the Cut_{cs} rule. Then the applications of the contraction rules can be replaced by derivations from their premiss by using the Cut_{cs}-rule. More precisely, as far as left contraction is concerned, the subderivations of $\mathcal{D}$ of the form

$$\frac{\begin{array}{c}\mathcal{E}\\ F, F, \Gamma \Rightarrow \Delta\end{array}}{F, \Gamma \Rightarrow \Delta}$$

can be replaced by:

$$\frac{\begin{array}{c}\mathcal{I}\\ F, \Gamma \Rightarrow \Delta, F\end{array} \qquad \begin{array}{c}\mathcal{E}\\ F, F, \Gamma \Rightarrow \Delta\end{array}}{F, \Gamma \Rightarrow \Delta}$$

where, in case F is not atomic, $\mathcal{I}$ is a derivation in m-**G3m** or in m-**G3i**. Similarly for right contraction. Finally the applications of the weakening rules can be eliminated by their (height-preserving) admissibiity in all the systems considered. Thus from $\mathcal{D}$ we obtain a derivation $\mathcal{D}'$ in m-$\mathbf{G3}[\mathbf{mic}]^{\mathcal{R}}$ + Cut_{cs}, that by Proposition 7, can be transformed into a separated derivation $\mathcal{D}''$ of the endsequent of $\mathcal{D}$. Therefore to obtain the desired derivation in m-$\mathbf{G3}[mic]^{\mathcal{R}}$ of the endsequent of $\mathcal{D}$, it suffices to eliminate the applications of Cut_{cs} in the initial subderivations of $\mathcal{D}''$ belonging to $\mathcal{R}[m]+\mathrm{Cut}_{cs}$ or $\mathcal{R}[ic]+\mathrm{Cut}_{cs}$, which is possible if the contraction and the cut rule are admissible in $\mathcal{R}[m]$ or $\mathcal{R}[ic]$. □

Remark In Theorem 1 *b*), $\mathcal{R}[ic]$ cannot be replaced by $\mathcal{R}[m]$. For example, if $\mathcal{R}$ consists of the single rule:

$$\frac{\Gamma \Rightarrow \Delta, P}{\Gamma \Rightarrow \Delta, \bot}$$

for a fixed atomic formula P, distinct from $\bot$, the structural rules are admissible in $\mathcal{R}[m]$ and the sequent $P \Rightarrow$ is derivable in $\mathcal{R}[ic]$, thus in m-$\mathbf{G3}[\mathbf{ic}]^{\mathcal{R}}$, but it has no cut-free derivation in the latter system.

2.2 Admissibility of the Structural Rules

Corollary 1 *The structural rules are admissible in m-***G3**[**mic**]

Proof By Theorem 1, with $\mathcal{R} = \emptyset$, it suffices to note that the sequents that can be derived from initial sequents or instances of $L\bot$ by means of the structural rules are themselves initial sequents or instances of $L\bot$. □

Let m-**G3**[**mic**]$^=$ denote m-**G3**[**mic**]$^{\mathcal{R}}$ for $\mathcal{R} = \{\text{Ref}, \text{Rep}\}$. As we have already noted, Ref and Rep are intuitionistic atomic rules, so that we can apply Theorem 1 also in the minimal and intuitionistic case.

Corollary 2 *The structural rules are admissible in m-***G3**[**mic**]$^=$

Proof For the admissibility of left contraction we proceed by induction on the height of $\mathcal{D}$ to show that a derivation $\mathcal{D}$ of $A, A, \Gamma \Rightarrow \Delta$ in $\mathcal{R}$ can be transformed into a derivation of $A, \Gamma \Rightarrow \Delta$ in $\mathcal{R}$. That is immediate if $\mathcal{D}$ reduces to an initial sequent or to an instance of $L\bot$. If $h(\mathcal{D}) > 0$, the conclusion is a straightforward consequence of the induction hypothesis, except when $\mathcal{D}$ has the form:

$$\frac{s = r, s = r, E[x/r], \Gamma' \Rightarrow \Delta}{s = r, s = r, \Gamma' \Rightarrow \Delta}$$

and A is $s = r$ which coincides with $E[x/s]$, We may assume that there is exactly one occurrence of x in E. Then E can have the form $x = r$ or r can have the form $r^\circ[x/s]$ and E the form $s = r^\circ$. In the former case the induction hypothesis applied to the immediate subderivation $\mathcal{D}_0$ of $\mathcal{D}$, whose endsequent is $s = r, s = r, r = r, \Gamma' \Rightarrow \Delta$, yields a derivation of $s = r, r = r, \Gamma' \Rightarrow \Delta$, from which we obtain $s = r, \Gamma' \Rightarrow \Delta$ by an application of Ref. In the latter case $\mathcal{D}$ has the form:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ s = r^\circ[x/s],\ s = r^\circ[x/s],\ s = r^\circ[x/r^\circ[x/s]],\ \Gamma' \Rightarrow \Delta\end{array}}{s = r^\circ[x/s],\ s = r^\circ[x/s],\ \Gamma' \Rightarrow \Delta}\ \text{Rep}$$

and can be transformed into:

$$\cfrac{\cfrac{\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_0\\ s = r^\circ[x/s],\ s = r^\circ[x/s],\ s = r^\circ[x/r^\circ[x/s]],\ \Gamma' \Rightarrow \Delta\end{array}}{s = r^\circ[x/s],\ s = r^\circ[x/s],\ \Gamma' \Rightarrow \Delta}\ \text{Rep}}{s = r^\circ[x/s],\ s = s,\ s = r^\circ[x/s],\ \Gamma' \Rightarrow \Delta}\ \text{w}}{s = r^\circ[x/s],\ s = s,\ \Gamma' \Rightarrow \Delta}\ \text{Rep}}{s = r^\circ[x/s],\ \Gamma' \Rightarrow \Delta}\ \text{Ref}$$

Since there are no active formulae in the succedent of the rules of $\mathcal{R}$, the admissibility of right contraction is immediate and the admissibility of the cut rule follows by a straightforward induction on the height of the derivation of its first premiss. □

As we plan to show in a sequel to the present work, many further applications of Theorem 1 can be given, in particular to several different sequent calculus formulations of first order logic with equality.

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