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### On fractional Laplacians - 2

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# On fractional Laplacians – 2

Roberta Musina\* and Alexander I. Nazarov†

**Abstract.** The present paper is the natural evolution of arXiv:1308.3606. For  $s > -1$  we compare two natural types of fractional Laplacians  $(-\Delta)^s$ , namely, the “Navier” and the “Dirichlet” ones. As a main tool, we give the “dual” Caffarelli–Silvestre and Stinga–Torrea characterizations of these operators for  $s \in (-1, 0)$ .

## 1 Introduction

Recall that the Sobolev space  $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$ ,  $s \in \mathbb{R}$ , is the space of distributions  $u \in \mathcal{S}'(\mathbb{R}^n)$  with finite norm

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi,$$

see for instance Section 2.3.3 of the monograph [8]. Here  $\mathcal{F}$  denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx.$$

For arbitrary  $s \in \mathbb{R}$  we define fractional Laplacian in  $\mathbb{R}^n$  by the quadratic form

$$Q_s[u] = ((-\Delta)^s u, u) := \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

with domain

$$\text{Dom}(Q_s) = \{u \in \mathcal{S}'(\mathbb{R}^n) : Q_s[u] < \infty\}.$$

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Let  $\Omega$  be a bounded and smooth domain in  $\mathbb{R}^n$ . We put

$$H^s(\Omega) = \{u|_{\Omega} : u \in H^s(\mathbb{R}^n)\},$$

see [8, Sec. 4.2.1] and the extension theorem in [8, Sec. 4.2.3].

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) : \text{supp } u \subset \overline{\Omega}\}.$$

By Theorem 4.3.2/1 [8], for  $s - \frac{1}{2} \notin \mathbb{Z}$  this space coincides with  $H_0^s(\Omega)$  that is the closure of  $C_0^\infty(\Omega)$  in  $H^s(\Omega)$  while for  $s - \frac{1}{2} \in \mathbb{Z}$  one has  $\widetilde{H}^s(\Omega) \subsetneq H_0^s(\Omega)$ . Moreover,  $C_0^\infty(\Omega)$  is dense in  $u \in \widetilde{H}^s(\Omega)$ .

We introduce the ‘‘Dirichlet’’ fractional Laplacian in  $\Omega$  (denoted by  $(-\Delta_\Omega)_D^s$ ) as the restriction of  $(-\Delta)^s$ . The domain of its quadratic form is

$$\text{Dom}(Q_{s,\Omega}^D) = \{u \in \text{Dom}(Q_s) : \text{supp } u \subset \overline{\Omega}\}.$$

Also we define the ‘‘Navier’’ fractional Laplacian as  $s$ -th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q_{s,\Omega}^N[u] = ((-\Delta_\Omega)_N^s u, u) := \sum_j \lambda_j^s \cdot |(u, \varphi_j)|^2.$$

Here,  $\lambda_j$  and  $\varphi_j$  are eigenvalues and eigenfunctions of the Dirichlet Laplacian in  $\Omega$ , respectively, and  $\text{Dom}(Q_{s,\Omega}^N)$  consists of distributions in  $\Omega$  such that  $Q_{s,\Omega}^N[u] < \infty$ .

It is well known that for  $s = 1$  these operators coincide:  $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$ . We emphasize that, in contrast to  $(-\Delta_\Omega)_N^s$ , the operator  $(-\Delta_\Omega)_D^s$  is not the  $s$ -th power of the Dirichlet Laplacian for  $s \neq 1$ . In particular,  $(-\Delta_\Omega)_D^{-s}$  is not inverse to  $(-\Delta_\Omega)_D^s$ .

The present paper is the natural evolution of [6], where we compared the operators  $(-\Delta_\Omega)_D^s$  and  $(-\Delta_\Omega)_N^s$  for  $0 < s < 1$ . In the first result we extend Theorem 2 of [6].

**T:main1**

**Theorem 1** *Let  $s > -1$ ,  $s \notin \mathbb{N}_0$ . Then for  $u \in \text{Dom}(Q_{s,\Omega}^D)$ ,  $u \neq 0$ , the following relations hold:*

$$Q_{s,\Omega}^N[u] > Q_{s,\Omega}^D[u], \quad \text{if } 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \quad (1)$$

$$Q_{s,\Omega}^N[u] < Q_{s,\Omega}^D[u], \quad \text{if } 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \quad (2)$$

Next, we take into account the role of dilations in  $\mathbb{R}^n$ . We denote by  $F(\Omega)$  the class of smooth and bounded domains containing  $\Omega$ . If  $\Omega' \in F(\Omega)$ , then any  $u \in \text{Dom}(Q_{s,\Omega}^D)$  can be regarded as a function in  $\text{Dom}(Q_{s,\Omega'}^D)$ , and the corresponding form  $Q_{s,\Omega'}^D[u]$  does not change. In contrast, the form  $Q_{s,\Omega'}^N[u]$  does depend on  $\Omega' \supset \Omega$ . However, roughly speaking, the difference between these quadratic forms disappears as  $\Omega' \rightarrow \mathbb{R}^n$ .

**T:new**

**Theorem 2** *Let  $s > -1$ . Then for  $u \in \text{Dom}(Q_{s,\Omega}^D)$  the following facts hold:*

$$Q_{s,\Omega}^D[u] = \inf_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if } 2k < s < 2k + 1, \quad k \in \mathbb{N}_0; \quad (3)$$

$$Q_{s,\Omega}^D[u] = \sup_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if } 2k - 1 < s < 2k, \quad k \in \mathbb{N}_0. \quad (4)$$

For  $-1 < s < 0$  we also obtain a pointwise comparison result reverse to the case  $0 < s < 1$  (compare with [6, Theorem 1]).

**T:main2**

**Theorem 3** *Let  $-1 < s < 0$ , and let  $f \in \text{Dom}(Q_{s,\Omega}^D)$ ,  $f \geq 0$  in the sense of distributions,  $f \not\equiv 0$ . Then the following relation holds:*

$$(-\Delta_\Omega)_N^s f < (-\Delta_\Omega)_D^s f. \quad (5)$$

**-pos\_pres**

Actually, fractional Laplacians of orders  $s \in (-1, 0)$  play a crucial role in our arguments. In Section 2 we give a variational characterization of these operators, “dual” to variational characterization of fractional Laplacians of orders  $s \in (0, 1)$  given in [4] and [7]. Theorems 1–3 are proved in Section 3.

Note that our statements hold in more general setting. Let  $\Omega$  be a bounded and smooth domain in a complete smooth Riemannian manifold  $\mathcal{M}$ . Denote by  $(-\Delta_\Omega)_N^s$  and  $(-\Delta_\Omega)_D^s$ , respectively, the  $s$ -th power of the Dirichlet Laplacian in  $\Omega$  and the restriction of  $s$ -th power of the Dirichlet Laplacian in  $\mathcal{M}$  to the set of functions supported in  $\Omega$ . Then proofs of Theorems 1–3 (and of Theorem 1 in [6] as well) run with minimal changes.

## 2 Fractional Laplacians of negative orders

**negat**

First, we recall some facts from the classical monograph [8] about the spaces  $H^s(\Omega)$  and  $\tilde{H}^s(\Omega)$ .

**Hs**

**Proposition 1** *(a particular case of [8, Theorem 4.3.2/1]).*

1. If  $0 < \sigma < \frac{1}{2}$  then  $\tilde{H}^\sigma(\Omega) = H_0^\sigma(\Omega) = H^\sigma(\Omega)$ ;
2. If  $\sigma = \frac{1}{2}$  then  $\tilde{H}^\sigma(\Omega)$  is dense in  $H^\sigma(\Omega) = H_0^\sigma(\Omega)$ ;
3. If  $\frac{1}{2} < \sigma < 1$  then  $\tilde{H}^\sigma(\Omega) = H_0^\sigma(\Omega)$  is a subspace of  $H^\sigma(\Omega)$ .

**duality**

**Proposition 2** *(a particular case of [8, Theorem 2.10.5/1]).*

For any  $\sigma \in \mathbb{R}$   $(\tilde{H}^\sigma(\Omega))' = H^{-\sigma}(\Omega)$ .

As an immediate consequence we obtain

**H-s**

**Corollary 1** 1. If  $0 < \sigma < \frac{1}{2}$  then  $\tilde{H}^{-\sigma}(\Omega) = H^{-\sigma}(\Omega)$ ;

2. If  $\sigma = \frac{1}{2}$  then  $\tilde{H}^{-\sigma}(\Omega)$  is dense in  $H^{-\sigma}(\Omega)$ ;
3. If  $\frac{1}{2} < \sigma < 1$  then  $H^{-\sigma}(\Omega)$  is a subspace of  $\tilde{H}^{-\sigma}(\Omega)$ .

**1D**

**Remark 1** *In the one-dimensional case, for  $\frac{1}{2} < \sigma < 1$  the codimension of  $H^{-\sigma}(\Omega)$  in  $\tilde{H}^{-\sigma}(\Omega)$  equals 2 since the same is codimension of  $\tilde{H}^\sigma(\Omega)$  in  $H^\sigma(\Omega)$ .*

The next statement gives explicit description of domains of quadratic forms under consideration.

**domain**

**Lemma 1** *Let  $0 < \sigma < 1$ . Then*

1.  $\text{Dom}(Q_{-\sigma,\Omega}^N) = H^{-\sigma}(\Omega)$ ;
2.  $\text{Dom}(Q_{-\sigma,\Omega}^D) = \tilde{H}^{-\sigma}(\Omega)$  if  $n \geq 2$  or  $\sigma < \frac{1}{2}$ ;
3.  $\text{Dom}(Q_{\sigma,\Omega}^D) = \{u \in \tilde{H}^{-\sigma}(\Omega) : \mathcal{F}u(0) = 0\}$  if  $n = 1$  and  $\sigma \geq \frac{1}{2}$ .

**Proof.** The first statement follows from the relation  $\text{Dom}(Q_{\sigma,\Omega}^N) = \tilde{H}^\sigma(\Omega)$ , see, e.g., [8, Theorems 1.15.3 and 4.3.2/2], and from Proposition 2.

The second and the third statements follow directly from definition of  $\tilde{H}^{-\sigma}(\Omega)$ , if we take into account that  $\mathcal{F}u$  is a smooth function.  $\square$

By Lemma 1 and Corollary 1, for  $0 < \sigma \leq \frac{1}{2}$  we have  $\text{Dom}(Q_{-\sigma,\Omega}^D) \subseteq \text{Dom}(Q_{-\sigma,\Omega}^N)$  (even  $\text{Dom}(Q_{-\sigma,\Omega}^D) = \text{Dom}(Q_{-\sigma,\Omega}^N)$  if  $0 < \sigma < \frac{1}{2}$ ). In the case  $\frac{1}{2} < \sigma < 1$ ,  $\text{Dom}(Q_{-\sigma,\Omega}^N)$  is a subspace of  $\text{Dom}(Q_{-\sigma,\Omega}^D)$  (for  $n = 1$  this follows from Remark 1). However, for arbitrary  $f \in \text{Dom}(Q_{-\sigma,\Omega}^D)$  we can consider  $f$  as a functional on  $H^\sigma(\Omega)$ , put  $\tilde{f} = f|_{\tilde{H}^\sigma(\Omega)} \in \text{Dom}(Q_{-\sigma,\Omega}^N)$  and define  $Q_{-\sigma,\Omega}^N[f] := Q_{-\sigma,\Omega}^N[\tilde{f}]$ .

Next, we recall that in the paper [4] the fractional Laplacian of order  $\sigma \in (0, 1)$  in  $\mathbb{R}^n$  was connected with the so-called *harmonic extension in  $n + 2 - 2\sigma$  dimensions* and with generalized Dirichlet-to-Neumann map (see also [3] for the case  $\sigma = \frac{1}{2}$ ). In particular, for any  $u \in \tilde{H}^\sigma(\Omega)$  the function  $w_\sigma^D(x, y)$  minimizing the weighted Dirichlet integral

$$\mathcal{E}_\sigma^D(w) = \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_\sigma^D(u) = \left\{ w(x, y) : \mathcal{E}_\sigma^D(w) < \infty, \quad w|_{y=0} = u \right\},$$

satisfies

$$Q_{\sigma,\Omega}^D[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^D(w_\sigma^D), \tag{6} \quad \boxed{\text{quad\_D}}$$

where the constant  $C_\sigma$  is given by

$$C_\sigma := \frac{4^\sigma \Gamma(1 + \sigma)}{\Gamma(1 - \sigma)}.$$

Moreover,  $w_\sigma^D(x, y)$  is the solution of the BVP

$$-\text{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad w|_{y=0} = u,$$

and for sufficiently smooth  $u$

$$(-\Delta)^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^D(x, y), \quad x \in \mathbb{R}^n \tag{7} \quad \boxed{\text{extension\_D}}$$

(we recall that  $(-\Delta_\Omega)_D^\sigma u = (-\Delta)^\sigma u|_\Omega$ ).

In [7] this approach was developed in quite general situation. In particular, it was shown that for any  $u \in \tilde{H}^\sigma(\Omega)$  the function  $w_\sigma^N(x, y)$  minimizing the energy integral

$$\mathcal{E}_\sigma^N(w) = \int_0^\infty \int_\Omega y^{1-2\sigma} |\nabla w(x, y)|^2 dx dy$$

over the set

$$\mathcal{W}_{\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_\sigma^D(u) : w|_{x \in \partial\Omega} = 0\},$$

satisfies

$$Q_{\sigma, \Omega}^N[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \quad (8) \quad \boxed{\text{quad\_N}}$$

Moreover,  $w_\sigma^N(x, y)$  is the solution of the BVP

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad w|_{y=0} = u; \quad w|_{x \in \partial\Omega} = 0, \quad (9) \quad \boxed{\text{eq:ST}}$$

and for sufficiently smooth  $u$  it turns out that

$$(-\Delta_\Omega)_N^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w_\sigma^N(x, y). \quad (10) \quad \boxed{\text{extension\_N}}$$

In a similar way, negative fractional Laplacians are connected with generalized Neumann-to-Dirichlet map. Namely, let  $u \in \operatorname{Dom}(Q_{-\sigma, \Omega}^D)$ . We consider the problem of minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^D(w) = \mathcal{E}_{-\sigma}^D(w) - 2 \langle u, w|_{y=0} \rangle$$

over the set  $\mathcal{W}_{-\sigma}^D$ , that is closure of smooth functions on  $\mathbb{R}^n \times \bar{\mathbb{R}}_+$  with bounded support, with respect to  $\mathcal{E}_{-\sigma}^D(\cdot)$ . We recall that by Lemma 1  $u$  can be considered as a compactly supported functional on  $H^\sigma(\mathbb{R}^n)$ , and thus the duality  $\langle u, w|_{y=0} \rangle$  is well defined by the result of [4].

First, let  $n > 2\sigma$  (this is a restriction only for  $n = 1$ ). We claim that the Hardy type inequality

$$\mathcal{E}_\sigma^D(w) \geq \left(\frac{n-2\sigma}{2}\right)^2 \int_0^\infty \int_{\mathbb{R}^n} y^{1-2\sigma} \frac{w^2(x, y)}{r^2} dx dy \quad (11) \quad \boxed{\text{eq:H}}$$

holds for  $w \in \mathcal{W}_{-\sigma}^D$  (here  $r^2 = |x|^2 + y^2$ ). Indeed, for a smooth function  $w$  with bounded support we consider the restriction of  $w$  to arbitrary ray in  $\mathbb{R}^n \times \mathbb{R}_+$  and write down the classical Hardy inequality

$$\int_0^\infty r^{n+1-2\sigma} w_r^2 dr \geq \left(\frac{n-2\sigma}{2}\right)^2 \int_0^\infty r^{n-1-2\sigma} w^2 dr.$$

We multiply it by  $(\frac{y}{r})^{1-2\sigma}$ , integrate over unit hemisphere in  $\mathbb{R}^{n+1}$ , and the claim follows.

By (11), a non-zero constant cannot be approximated by compactly supported functions. Thus, the minimizer of  $\tilde{\mathcal{E}}_{-\sigma}^D$  is determined uniquely. Denote it by  $w_{-\sigma}^D(x, y)$ . Then formulae (6) and (7) imply relations

$$Q_{-\sigma, \Omega}^D[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D); \quad (-\Delta_\Omega)_D^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma} w_{-\sigma}^D(x, 0), \quad x \in \Omega, \quad (12) \quad \boxed{-D}$$

that give the “dual” Caffarelli–Silvestre characterization of  $(-\Delta_\Omega)_D^{-\sigma}$ .

In case  $n = 1 \leq 2\sigma$  the above argument needs some modification. Namely, the minimizer  $w_{-\sigma}^D(x, y)$  in this case is defined up to an additive constant. However, by Lemma 1 we have

$$\mathcal{F}u(0) \equiv \langle u, \mathbf{1} \rangle = 0.$$

Therefore,  $\tilde{\mathcal{E}}_{-\sigma}^D(w_{-\sigma}^D)$  does not depend on the choice of the constant, and the first relation in (12) holds. The second equality in (12) also holds if we choose the constant such that  $w_{-\sigma}^D(x, 0) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

**Remark 2** Note that for sufficiently smooth  $u$  the function  $w_{-\sigma}^D$  solves the Neumann problem

$$-\operatorname{div}(y^{1-2\sigma} \nabla w) = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \lim_{y \rightarrow 0^+} y^{1-2\sigma} \partial_y w = -u. \quad (13) \quad \boxed{\text{eq: -CS}}$$

Analogously, formulae (8) and (10) imply the “dual” Stinga–Torrea characterization of  $(-\Delta_\Omega)_N^{-\sigma}$ . Namely, the function  $w_{-\sigma}^N(x, y)$  minimizing the functional

$$\tilde{\mathcal{E}}_{-\sigma}^N(w) = \mathcal{E}_\sigma^N(w) - 2 \langle u, w|_{y=0} \rangle$$

over the set

$$\mathcal{W}_{-\sigma, \Omega}^N(u) = \{w(x, y) \in \mathcal{W}_{-\sigma}^D : w|_{x \notin \Omega} = 0\},$$

satisfies

$$Q_{-\sigma, \Omega}^N[u] = -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_{-\sigma}^N(w_{-\sigma}^N); \quad (-\Delta_\Omega)_N^{-\sigma} u(x) = \frac{2\sigma}{C_\sigma} w_{-\sigma}^N(x, 0). \quad (14) \quad \boxed{-N}$$

**ext** **Remark 3** Formula (14) shows that  $w_{-\sigma}^N$  is the Stinga–Torrea extension of  $\frac{C_\sigma}{2\sigma} (-\Delta_\Omega)_N^{-\sigma} u$ . Similarly, from (12) we conclude that  $w_{-\sigma}^D$  is the Caffarelli–Silvestre extension of  $\frac{C_\sigma}{2\sigma} (-\Delta)^{-\sigma} u$  but not of  $\frac{C_\sigma}{2\sigma} (-\Delta_\Omega)_D^{-\sigma} u$ . This is due to the fact, already noticed in the introduction, that  $(-\Delta_\Omega)_D^{-\sigma}$  is not the inverse of  $(-\Delta_\Omega)_D^\sigma$ .

**Remark 4** The representation of  $(-\Delta)^{-\sigma} u$  via solution of the problem (13) was used in [2]. Similar representation of  $(-\Delta_\Omega)_N^{-\sigma} u$  via solution of corresponding mixed boundary value problem was used earlier in [5]. However, variational characterizations of negative fractional Laplacians (the first parts of formulae (12) and (14)) which play key role in what follows, are given for the first time.

### 3 Proofs of main theorems

compar

We start by recalling an auxiliary result.

domain1

**Lemma 2** *Let  $s > 1$ . Then*

$$\begin{aligned} \text{Dom}(Q_{s,\Omega}^D) &= \tilde{H}^s(\Omega) = \text{Dom}(Q_{s,\Omega}^N) \quad \text{for } s < 3/2; \\ \text{Dom}(Q_{s,\Omega}^D) &= \tilde{H}^s(\Omega) \subsetneq \text{Dom}(Q_{s,\Omega}^N) \quad \text{for } s \geq 3/2. \end{aligned}$$

**Proof.** For  $Q_{s,\Omega}^D$  the conclusion follows directly from its definition. For  $Q_{s,\Omega}^N$  this fact is well known for  $s \in \mathbb{N}$ ; in general case it follows immediately from [8, Theorem 1.17.1/1] and [8, Theorem 4.3.2/1].  $\square$

**Proof of Theorem 1.** We split the proof in three parts.

1. Let  $0 < s < 1$ . Then the relation (1) is proved in [6, Theorem 2].

2. Let  $-1 < s < 0$ . We define  $\sigma = -s \in (0, 1)$  and construct extensions  $w_{-\sigma}^D$  and  $w_{-\sigma}^N$  as described in Section 2.

We evidently have  $\mathcal{W}_{-\sigma,\Omega}^N \subset \mathcal{W}_{-\sigma}^D$  and  $\tilde{\mathcal{E}}_{-\sigma}^N = \tilde{\mathcal{E}}_{-\sigma}^D|_{\mathcal{W}_{-\sigma,\Omega}^N}$ . Therefore, (12) and (14) provide

$$Q_{s,\Omega}^N[u] = -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}_{-\sigma,\Omega}^N} \tilde{\mathcal{E}}_{-\sigma}^N(w) \leq -\frac{2\sigma}{C_\sigma} \cdot \inf_{w \in \mathcal{W}_{-\sigma}^D} \tilde{\mathcal{E}}_{-\sigma}^D(w) = Q_{s,\Omega}^D[u].$$

To complete the proof, we observe that for  $u \neq 0$  the function  $w_{-\sigma}^N$  cannot be a solution of the homogeneous equation in (9) in the whole half-space, since such a solution is analytic in the half-space. Thus, it cannot provide  $\inf_{w \in \mathcal{W}_{-\sigma}^D} \tilde{\mathcal{E}}_{-\sigma}^D(w)$ , and (2) follows.

3. Now let  $s > 1$ ,  $s \notin \mathbb{N}$ . We put  $k = \lfloor \frac{s-1}{2} \rfloor$  and define for  $u \in \tilde{H}^s(\Omega)$

$$v = (-\Delta)^k u \in \tilde{H}^{s-2k}(\Omega), \quad s - 2k \in (-1, 0) \cup (0, 1).$$

Note that  $v \neq 0$  if  $u \neq 0$ . Then we have

$$Q_{s,\Omega}^N[u] = Q_{s-2k,\Omega}^N[v], \quad Q_{s,\Omega}^D[u] = Q_{s-2k,\Omega}^D[u],$$

and the conclusion follows from cases 1 and 2.  $\square$

**Proof of Theorem 2.** Here we again distinguish three cases.

1. Let  $0 < s < 1$ . Then the relation (3) is proved in [6, Theorem 3].

2. Let  $-1 < s < 0$ . We define  $\sigma = -s \in (0, 1)$  and proceed similarly to the proof of [6, Theorem 3]. It is sufficient to prove the statement for  $u \in C_0^\infty(\Omega)$ .

For  $\Omega' \supset \Omega$  we have  $\mathcal{W}_{-\sigma,\Omega}^N \subset \mathcal{W}_{-\sigma,\Omega'}^N$ . By (14), the quadratic form  $Q_{s,\Omega}^N[u]$  is monotone increasing with respect to the domain inclusion. Taking (2) into account, we obtain

$$Q_{s,\Omega}^D[u] > Q_{s,\Omega'}^N[u] \geq Q_{s,\Omega}^N[u]. \tag{15}$$

eq:monotone



Denote by  $w = w_{-\sigma}^D$  the Caffarelli–Silvestre extension of  $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma}u$ , described in Section 2. Next, for any  $y \geq 0$  let  $\phi_R(\cdot, y)$  be the harmonic extension of  $w(\cdot, y)$  on the ball  $B_R$ , that is,

$$-\Delta\phi_R(\cdot, y) = 0 \quad \text{in } B_R; \quad \phi_h(\cdot, y) = w(\cdot, y) \quad \text{on } \partial B_R.$$

Finally, for  $x \in B_R$  and  $y \geq 0$  we put

$$w_R(x, y) = w(x, y) - \phi_R(x, y).$$

It is shown in the proof of [6, Theorem 3] that there exists a sequence  $R_h \rightarrow \infty$  such that

$$\mathcal{E}_\sigma^N(w_{R_h}) \leq \mathcal{E}_\sigma^D(w) + o(1).$$

Further, since  $(-\Delta_\Omega)^{-\sigma}u$  vanishes at infinity, for any multi-index  $\beta$  we evidently have  $D^\beta\phi_{R_h}(\cdot, 0) \rightarrow 0$  locally uniformly as  $R_h \rightarrow \infty$ . This gives  $\langle u, \phi_{R_h}(\cdot, 0) \rangle = o(1)$ , and we obtain by (12) and (14)

$$\begin{aligned} Q_{s, B_{R_h}}^N[u] &\geq -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_\sigma^N(w_{R_h}) \\ &\geq -\frac{2\sigma}{C_\sigma} \cdot \tilde{\mathcal{E}}_\sigma^D(w) - o(1) = Q_{s, \Omega}^D[u] - o(1). \end{aligned} \tag{16} \quad \boxed{\text{eq:tesi}}$$

The relation (3) readily follows by comparing (15) and (16).

**3.** For  $s > 1$ ,  $s \notin \mathbb{N}$ , the conclusion follows from cases 1 and 2 just as in the proof of Theorem 1.  $\square$

**R:new**

**Remark 5** Assume that  $0 \in \Omega$  and put  $\alpha\Omega = \{\alpha x : x \in \Omega\}$ . Thanks to (15), the proof above shows indeed that

$$Q_{s, \Omega}^D[u] = \lim_{\alpha \rightarrow \infty} Q_{s, \alpha\Omega}^N[u] \quad \text{for any } u \in \tilde{H}^s(\Omega).$$

Now put  $u_\alpha(x) = \alpha^{\frac{n-2s}{2}}u(\alpha x)$ . Then the scaling shows that

$$Q_{s, \Omega}^D[u_\alpha] \equiv Q_{s, \Omega}^D[u] = \lim_{\alpha \rightarrow \infty} Q_{s, \Omega}^N[u_\alpha] \quad \text{for any } u \in \tilde{H}^s(\Omega).$$

**Proof of Theorem 3.** First, let  $f \in \mathcal{C}_0^\infty(\Omega)$ . We define  $\sigma = -s \in (0, 1)$  and construct extensions  $w_{-\sigma}^D$  and  $w_{-\sigma}^N$  described in Section 2. Making the change of the variable  $t = y^{2\sigma}$ , we rewrite the BVP (13) for  $w_{-\sigma}^D(x, t)$  as follows:

$$\Delta_x w_{-\sigma}^D + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial_{tt}^2 w_{-\sigma}^D = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+; \quad \partial_t w_{-\sigma}^D|_{t=0} = -\frac{f}{2\sigma}. \tag{17} \quad \boxed{\text{BVP}}$$

Since  $w_{-\sigma}^D$  vanishes at infinity,  $w_{-\sigma}^D(x, t) > 0$  for  $t > 0$  by the maximum principle. Moreover, by [1, Theorem 1.4] (the boundary point lemma) we have  $w_{-\sigma}^D(x, 0) > 0$ .

Further, the function  $w_{-\sigma}^N$  satisfies the equalities (17) in  $\Omega \times \mathbb{R}_+$ . Since  $w_{-\sigma}^N|_{x \notin \Omega} = 0$ , we infer that the function

$$W(x, t) := w_{-\sigma}^D(x, t) - w_{-\sigma}^N(x, t)$$

meets the following relations:

$$\Delta_x W + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial_{tt}^2 W = 0 \quad \text{in } \Omega \times \mathbb{R}_+; \quad \partial_t W|_{t=0} = 0; \quad W|_{x \in \partial\Omega} > 0.$$

Again, [1, Theorem 1.4] gives  $W(x, 0) > 0$ , which gives (5) in view of (12) and (14).

For  $f \in \tilde{H}^s(\Omega)$  the statement holds by approximation argument.  $\square$

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