On fractional Laplacians - 2

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On fractional Laplacians – 2

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Abstract. The present paper is the natural evolution of arXiv:1308.3606. For $s > -1$ we compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the "Navier" and the "Dirichlet" ones. As a main tool, we give the "dual" Caffarelli–Silvestre and Stinga–Torrea characterizations of these operators for $s \in (-1, 0)$.

1 Introduction

Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 \, d\xi,$$

see for instance Section 2.3.3 of the monograph [8]. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x) \, dx.$$

For arbitrary $s \in \mathbb{R}$ we define fractional Laplacian in $\mathbb{R}^n$ by the quadratic form

$$Q_s[u] = ((-\Delta)^s u, u) := \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

with domain

$$\mathrm{Dom}(Q_s) = \{u \in \mathcal{S}'(\mathbb{R}^n) \, : \, Q_s[u] < \infty\}.$$

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Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. We put

$$H^s(\Omega) = \left\{ u\big|_\Omega \,:\, u \in H^s(\mathbb{R}^n) \right\},$$

see [8, Sec. 4.2.1] and the extension theorem in [8, Sec. 4.2.3].

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{ u \in H^s(\mathbb{R}^n) \,:\, \operatorname{supp} u \subset \overline{\Omega} \}.$$

By Theorem 4.3.2/1 [8], for $s - \frac{1}{2} \notin \mathbb{Z}$ this space coincides with $H^s_0(\Omega)$ that is the closure of $\mathcal{C}^\infty_0(\Omega)$ in $H^s(\Omega)$ while for $s - \frac{1}{2} \in \mathbb{Z}$ one has $\widetilde{H}^s(\Omega) \subsetneq H^s_0(\Omega)$. Moreover, $\mathcal{C}^\infty_0(\Omega)$ is dense in $u \in \widetilde{H}^s(\Omega)$.

We introduce the "Dirichlet" fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)^s_D$) as the restriction of $(-\Delta)^s$. The domain of its quadratic form is

$$\operatorname{Dom}(Q^D_{s,\Omega}) = \{ u \in \operatorname{Dom}(Q_s) \,:\, \operatorname{supp} u \subset \overline{\Omega} \}.$$

Also we define the "Navier" fractional Laplacian as s-th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q^N_{s,\Omega}[u] = ((-\Delta_\Omega)^s_N u, u) := \sum\nolimits_j \lambda_j^s \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively, and $\operatorname{Dom}(Q^N_{s,\Omega})$ consists of distributions in Ω such that $Q^N_{s,\Omega}[u] < \infty$.

It is well known that for $s = 1$ these operators coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)^s_N$, the operator $(-\Delta_\Omega)^s_D$ is not the s-th power of the Dirichlet Laplacian for $s \neq 1$. In particular, $(-\Delta_\Omega)^{-s}_D$ is not inverse to $(-\Delta_\Omega)^s_D$.

The present paper is the natural evolution of [6], where we compared the operators $(-\Delta_\Omega)^s_D$ and $(-\Delta_\Omega)^s_N$ for $0 < s < 1$. In the first result we extend Theorem 2 of [6].

T:main1 **Theorem 1** *Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \operatorname{Dom}(Q^D_{s,\Omega})$, $u \not\equiv 0$, the following relations hold:*

$$Q^N_{s,\Omega}[u] > Q^D_{s,\Omega}[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{1}$$

$$Q^N_{s,\Omega}[u] < Q^D_{s,\Omega}[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{2}$$

Next, we take into account the role of dilations in $\mathbb{R}^n$. We denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. If $\Omega' \in F(\Omega)$, then any $u \in \operatorname{Dom}(Q^D_{s,\Omega})$ can be regarded as a function in $\operatorname{Dom}(Q^D_{s,\Omega'})$, and the corresponding form $Q^D_{s,\Omega'}[u]$ does not change. In contrast, the form $Q^N_{s,\Omega'}[u]$ does depend on $\Omega' \supset \Omega$. However, roughly speaking, the difference between these quadratic forms disappears as $\Omega' \to \mathbb{R}^n$.

T:new **Theorem 2** *Let $s > -1$. Then for $u \in \operatorname{Dom}(Q^D_{s,\Omega})$ the following facts hold:*

$$Q^D_{s,\Omega}[u] = \inf_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{3}$$

$$Q^D_{s,\Omega}[u] = \sup_{\Omega' \in F(\Omega)} Q^N_{s,\Omega'}[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{4}$$

For $-1 < s < 0$ we also obtain a pointwise comparison result reverse to the case $0 < s < 1$ (compare with [6, Theorem 1]).

T:main2 **Theorem 3** *Let $-1 < s < 0$, and let $f \in \mathrm{Dom}(Q^D_{s,\Omega})$, $f \geq 0$ in the sense of distributions, $f \not\equiv 0$. Then the following relation holds:*

$$(-\Delta_\Omega)^s_N f < (-\Delta_\Omega)^s_D f. \tag{5}$$ -pos_pres

Actually, fractional Laplacians of orders $s \in (-1, 0)$ play a crucial role in our arguments. In Section 2 we give a variational characterization of these operators, "dual" to variational characterization of fractional Laplacians of orders $s \in (0, 1)$ given in [4] and [7]. Theorems 1–3 are proved in Section 3.

Note that our statements hold in more general setting. Let Ω be a bounded and smooth domain in a complete smooth Riemannian manifold $\mathcal{M}$. Denote by $(-\Delta_\Omega)^s_N$ and $(-\Delta_\Omega)^s_D$, respectively, the s-th power of the Dirichlet Laplacian in Ω and the restriction of s-th power of the Dirichlet Laplacian in $\mathcal{M}$ to the set of functions supported in Ω. Then proofs of Theorems 1–3 (and of Theorem 1 in [6] as well) run with minimal changes.

2 Fractional Laplacians of negative orders

negat

First, we recall some facts from the classical monograph [8] about the spaces $H^s(\Omega)$ and $\widetilde{H}^s(\Omega)$.

Hs **Proposition 1** *(a particular case of [8, Theorem 4.3.2/1]).*

1. *If $0 < \sigma < \frac{1}{2}$ then $\widetilde{H}^\sigma(\Omega) = H^\sigma_0(\Omega) = H^\sigma(\Omega)$;*
2. *If $\sigma = \frac{1}{2}$ then $\widetilde{H}^\sigma(\Omega)$ is dense in $H^\sigma(\Omega) = H^\sigma_0(\Omega)$;*
3. *If $\frac{1}{2} < \sigma < 1$ then $\widetilde{H}^\sigma(\Omega) = H^\sigma_0(\Omega)$ is a subspace of $H^\sigma(\Omega)$.*

duality **Proposition 2** *(a particular case of [8, Theorem 2.10.5/1]).*
For any $\sigma \in \mathbb{R}$ $(\widetilde{H}^\sigma(\Omega))' = H^{-\sigma}(\Omega)$.

As an immediate consequence we obtain

H-s **Corollary 1** 1. *If $0 < \sigma < \frac{1}{2}$ then $\widetilde{H}^{-\sigma}(\Omega) = H^{-\sigma}(\Omega)$;*

2. *If $\sigma = \frac{1}{2}$ then $\widetilde{H}^{-\sigma}(\Omega)$ is dense in $H^{-\sigma}(\Omega)$;*
3. *If $\frac{1}{2} < \sigma < 1$ then $H^{-\sigma}(\Omega)$ is a subspace of $\widetilde{H}^{-\sigma}(\Omega)$.*

1D **Remark 1** *In the one-dimensional case, for $\frac{1}{2} < \sigma < 1$ the codimension of $H^{-\sigma}(\Omega)$ in $\widetilde{H}^{-\sigma}(\Omega)$ equals 2 since the same is codimension of $\widetilde{H}^\sigma(\Omega)$ in $H^\sigma(\Omega)$.*

The next statement gives explicit description of domains of quadratic forms under consideration.

domain **Lemma 1** *Let $0<\sigma<1$. Then*

1. $\mathrm{Dom}(Q^N_{-\sigma,\Omega}) = H^{-\sigma}(\Omega)$;
2. $\mathrm{Dom}(Q^D_{-\sigma,\Omega}) = \widetilde{H}^{-\sigma}(\Omega)$ *if* $n\ge 2$ *or* $\sigma<\frac{1}{2}$;
3. $\mathrm{Dom}(Q^D_{\sigma,\Omega}) = \{u\in \widetilde{H}^{-\sigma}(\Omega)\,:\,\mathcal{F}u(0)=0\}$ *if* $n=1$ *and* $\sigma\ge\frac{1}{2}$.

Proof. The first statement follows from the relation $\mathrm{Dom}(Q^N_{\sigma,\Omega}) = \widetilde{H}^{\sigma}(\Omega)$, see, e.g., [8, Theorems 1.15.3 and 4.3.2/2], and from Proposition 2.

The second and the third statements follow directly from definition of $\widetilde{H}^{-\sigma}(\Omega)$, if we take into account that $\mathcal{F}u$ is a smooth function. $\square$

By Lemma 1 and Corollary 1, for $0<\sigma\le\frac{1}{2}$ we have $\mathrm{Dom}(Q^D_{-\sigma,\Omega})\subseteq \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ (even $\mathrm{Dom}(Q^D_{-\sigma,\Omega}) = \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ if $0<\sigma<\frac{1}{2}$). In the case $\frac{1}{2}<\sigma<1$, $\mathrm{Dom}(Q^N_{-\sigma,\Omega})$ is a subspace of $\mathrm{Dom}(Q^D_{-\sigma,\Omega})$ (for $n=1$ this follows from Remark 1). However, for arbitrary $f\in \mathrm{Dom}(Q^D_{-\sigma,\Omega})$ we can consider f as a functional on $H^\sigma(\Omega)$, put $\widetilde{f} = f|_{\widetilde{H}^\sigma(\Omega)} \in \mathrm{Dom}(Q^N_{-\sigma,\Omega})$ and define $Q^N_{-\sigma,\Omega}[f] := Q^N_{-\sigma,\Omega}[\widetilde{f}]$.

Next, we recall that in the paper [4] the fractional Laplacian of order $\sigma\in(0,1)$ in $\mathbb{R}^n$ was connected with the so-called *harmonic extension in* $n+2-2\sigma$ *dimensions* and with generalized Dirichlet-to-Neumann map (see also [3] for the case $\sigma=\frac{1}{2}$). In particular, for any $u\in\widetilde{H}^\sigma(\Omega)$ the function $w^D_\sigma(x,y)$ minimizing the weighted Dirichlet integral

$$\mathcal{E}^D_\sigma(w) = \int\limits_0^\infty\int\limits_{\mathbb{R}^n} y^{1-2\sigma}|\nabla w(x,y)|^2\,dxdy$$

over the set

$$\mathcal{W}^D_\sigma(u) = \Big\{w(x,y)\,:\,\mathcal{E}^D_\sigma(w)<\infty\,,\quad w\big|_{y=0}=u\Big\},$$

satisfies

$$Q^D_{\sigma,\Omega}[u] = \frac{C_\sigma}{2\sigma}\cdot\mathcal{E}^D_\sigma(w^D_\sigma), \tag{6}$$ quad_D

where the constant C_σ is given by

$$C_\sigma := \frac{4^\sigma\Gamma(1+\sigma)}{\Gamma(1-\sigma)}.$$

Moreover, $w^D_\sigma(x,y)$ is the solution of the BVP

$$-\mathrm{div}(y^{1-2\sigma}\nabla w)=0\quad\text{in}\quad \mathbb{R}^n\times\mathbb{R}_+;\qquad w\big|_{y=0}=u,$$

and for sufficiently smooth u

$$(-\Delta)^\sigma u(x) = -\frac{C_\sigma}{2\sigma}\cdot\lim_{y\to0^+} y^{1-2\sigma}\partial_y w^D_\sigma(x,y),\qquad x\in\mathbb{R}^n \tag{7}$$ extension_D

(we recall that $(-\Delta_\Omega)^\sigma_D u = (-\Delta)^\sigma u\big|_\Omega$).

In [7] this approach was developed in quite general situation. In particular, it was shown that for any $u \in \widetilde{H}^\sigma(\Omega)$ the function $w_\sigma^N(x,y)$ minimizing the energy integral

$$\mathcal{E}_\sigma^N(w) = \int\limits_0^\infty \int\limits_\Omega y^{1-2\sigma} |\nabla w(x,y)|^2 \, dxdy$$

over the set

$$\mathcal{W}_{\sigma,\Omega}^N(u) = \{w(x,y) \in \mathcal{W}_\sigma^D(u) \, : \, w\big|_{x\in\partial\Omega} = 0\},$$

satisfies

$$Q_{\sigma,\Omega}^N[u] = \frac{C_\sigma}{2\sigma} \cdot \mathcal{E}_\sigma^N(w_\sigma^N). \tag{8}$$

quad_N

Moreover, $w_\sigma^N(x,y)$ is the solution of the BVP

$$-\operatorname{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad w\big|_{y=0} = u; \qquad w\big|_{x\in\partial\Omega} = 0, \tag{9}$$

eq:ST

and for sufficiently smooth u it turns out that

$$(-\Delta_\Omega)_N^\sigma u(x) = -\frac{C_\sigma}{2\sigma} \cdot \lim_{y\to 0^+} y^{1-2\sigma} \partial_y w_\sigma^N(x,y). \tag{10}$$

extension_N

In a similar way, negative fractional Laplacians are connected with generalized Neumann-to-Dirichlet map. Namely, let $u \in \mathrm{Dom}(Q_{-\sigma,\Omega}^D)$. We consider the problem of minimizing the functional

$$\widetilde{\mathcal{E}}_{-\sigma}^D(w) = \mathcal{E}_\sigma^D(w) \, - \, 2 \left\langle u, w\big|_{y=0} \right\rangle$$

over the set $\mathcal{W}_{-\sigma}^D$, that is closure of smooth functions on $\mathbb{R}^n \times \bar{\mathbb{R}}_+$ with bounded support, with respect to $\mathcal{E}_\sigma^D(\cdot)$. We recall that by Lemma 1 u can be considered as a compactly supported functional on $H^\sigma(\mathbb{R}^n)$, and thus the duality $\left\langle u, w\big|_{y=0} \right\rangle$ is well defined by the result of [4].

First, let $n > 2\sigma$ (this is a restriction only for $n = 1$). We claim that the Hardy type inequality

$$\mathcal{E}_\sigma^D(w) \geq \Big(\frac{n-2\sigma}{2}\Big)^2 \int\limits_0^\infty \int\limits_{\mathbb{R}^n} y^{1-2\sigma} \, \frac{w^2(x,y)}{r^2} \, dxdy \tag{11}$$

eq:H

holds for $w \in \mathcal{W}_{-\sigma}^D$ (here $r^2 = |x|^2 + y^2$). Indeed, for a smooth function w with bounded support we consider the restriction of w to arbitrary ray in $\mathbb{R}^n \times \mathbb{R}_+$ and write down the classical Hardy inequality

$$\int\limits_0^\infty r^{n+1-2\sigma} w_r^2 \, dr \geq \Big(\frac{n-2\sigma}{2}\Big)^2 \int\limits_0^\infty r^{n-1-2\sigma} w^2 \, dr.$$

We multiply it by $\left(\frac{y}{r}\right)^{1-2\sigma}$, integrate over unit hemisphere in $\mathbb{R}^{n+1}$, and the claim follows.

By (11), a non-zero constant cannot be approximated by compactly supported functions. Thus, the minimizer of $\widetilde{\mathcal{E}}^D_{-\sigma}$ is determined uniquely. Denote it by $w^D_{-\sigma}(x,y)$. Then formulae (6) and (7) imply relations

$$Q^D_{-\sigma,\Omega}[u] = -\frac{2\sigma}{C_\sigma}\cdot \widetilde{\mathcal{E}}^D_{-\sigma}(w^D_{-\sigma}); \quad (-\Delta_\Omega)^{-\sigma}_D u(x) = \frac{2\sigma}{C_\sigma}\, w^D_{-\sigma}(x,0), \quad x\in\Omega, \tag{12}$$

-D

that give the "dual" Caffarelli–Silvestre characterization of $(-\Delta_\Omega)^{-\sigma}_D$.

In case $n = 1 \le 2\sigma$ the above argument needs some modification. Namely, the minimizer $w^D_{-\sigma}(x,y)$ in this case is defined up to an additive constant. However, by Lemma 1 we have

$$\mathcal{F}u(0) \equiv \langle u, \mathbf{1}\rangle = 0.$$

Therefore, $\widetilde{\mathcal{E}}^D_{-\sigma}(w^D_{-\sigma})$ does not depend on the choice of the constant, and the first relation in (12) holds. The second equality in (12) also holds if we choose the constant such that $w^D_{-\sigma}(x,0)\to 0$ as $|x|\to\infty$.

Remark 2 *Note that for sufficiently smooth u the function $w^D_{-\sigma}$ solves the Neumann problem*

$$-\operatorname{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n\times\mathbb{R}_+; \qquad \lim_{y\to 0^+} y^{1-2\sigma}\partial_y w = -u. \tag{13}$$

eq:-CS

Analogously, formulae (8) and (10) imply the "dual" Stinga–Torrea characterization of $(-\Delta_\Omega)^{-\sigma}_N$. Namely, the function $w^N_{-\sigma}(x,y)$ minimizing the functional

$$\widetilde{\mathcal{E}}^N_{-\sigma}(w) = \mathcal{E}^N_\sigma(w) \,-\, 2\,\big\langle u, w\big|_{y=0}\big\rangle$$

over the set

$$\mathcal{W}^N_{-\sigma,\Omega}(u) = \{w(x,y)\in\mathcal{W}^D_{-\sigma} \,:\, w\big|_{x\notin\Omega} = 0\},$$

satisfies

$$Q^N_{-\sigma,\Omega}[u] = -\frac{2\sigma}{C_\sigma}\cdot\widetilde{\mathcal{E}}^N_{-\sigma}(w^N_{-\sigma}); \qquad (-\Delta_\Omega)^{-\sigma}_N u(x) = \frac{2\sigma}{C_\sigma}\, w^N_{-\sigma}(x,0). \tag{14}$$

-N

ext **Remark 3** *Formula (14) shows that $w^N_{-\sigma}$ is the Stinga–Torrea extension of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)^{-\sigma}_N u$. Similarly, from (12) we conclude that $w^D_{-\sigma}$ is the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma}u$ but not of $\frac{C_\sigma}{2\sigma}(-\Delta_\Omega)^{-\sigma}_D u$. This is due to the fact, already noticed in the introduction, that $(-\Delta_\Omega)^{-\sigma}_D$ is not the inverse of $(-\Delta_\Omega)^{\sigma}_D$.*

Remark 4 *The representation of $(-\Delta)^{-\sigma}u$ via solution of the problem (13) was used in [2]. Similar representation of $(-\Delta_\Omega)^{-\sigma}_N u$ via solution of corresponding mixed boundary value problem was used earlier in [5]. However, variational characterizations of negative fractional Laplacians (the first parts of formulae (12) and (14)) which play key role in what follows, are given for the first time.*

3 Proofs of main theorems

compar

We start by recalling an auxiliary result.

domain1 **Lemma 2** *Let $s>1$. Then*

$$\mathrm{Dom}(Q^D_{s,\Omega}) = \widetilde{H}^s(\Omega) = \mathrm{Dom}(Q^N_{s,\Omega}) \quad \text{for} \quad s<3/2;$$
$$\mathrm{Dom}(Q^D_{s,\Omega}) = \widetilde{H}^s(\Omega) \subsetneq \mathrm{Dom}(Q^N_{s,\Omega}) \quad \text{for} \quad s\ge 3/2.$$

Proof. For $Q^D_{s,\Omega}$ the conclusion follows directly from its definition. For $Q^N_{s,\Omega}$ this fact is well known for $s\in\mathbb{N}$; in general case it follows immediately from [8, Theorem 1.17.1/1] and [8, Theorem 4.3.2/1]. $\square$

Proof of Theorem 1. We split the proof in three parts.

1. Let $0<s<1$. Then the relation (1) is proved in [6, Theorem 2].

2. Let $-1<s<0$. We define $\sigma=-s\in(0,1)$ and construct extensions $w^D_{-\sigma}$ and $w^N_{-\sigma}$ as described in Section 2.

We evidently have $\mathcal{W}^N_{-\sigma,\Omega}\subset\mathcal{W}^D_{-\sigma}$ and $\widetilde{\mathcal{E}}^N_{-\sigma}=\widetilde{\mathcal{E}}^D_{-\sigma}\big|_{\mathcal{W}^N_{-\sigma,\Omega}}$. Therefore, (12) and (14) provide

$$Q^N_{s,\Omega}[u] = -\frac{2\sigma}{C_\sigma}\cdot\inf_{w\in\mathcal{W}^N_{-\sigma,\Omega}}\widetilde{\mathcal{E}}^N_{-\sigma}(w) \le -\frac{2\sigma}{C_\sigma}\cdot\inf_{w\in\mathcal{W}^D_{-\sigma}}\widetilde{\mathcal{E}}^D_{-\sigma}(w) = Q^D_{s,\Omega}[u].$$

To complete the proof, we observe that for $u\not\equiv 0$ the function $w^N_{-\sigma}$ cannot be a solution of the homogeneous equation in (9) in the whole half-space, since such a solution is analytic in the half-space. Thus, it cannot provide $\inf\limits_{w\in\mathcal{W}^D_{-\sigma}}\widetilde{\mathcal{E}}^D_{-\sigma}(w)$, and (2) follows.

3. Now let $s>1$, $s\notin\mathbb{N}$. We put $k=\lfloor\frac{s-1}{2}\rfloor$ and define for $u\in\widetilde{H}^s(\Omega)$

$$v=(-\Delta)^k u\in\widetilde{H}^{s-2k}(\Omega), \qquad s-2k\in(-1,0)\cup(0,1).$$

Note that $v\not\equiv 0$ if $u\not\equiv 0$. Then we have

$$Q^N_{s,\Omega}[u]=Q^N_{s-2k,\Omega}[v], \qquad Q^D_{s,\Omega}[u]=Q^D_{s-2k,\Omega}[u],$$

and the conclusion follows from cases 1 and 2. $\square$

Proof of Theorem 2. Here we again distinguish three cases.

1. Let $0<s<1$. Then the relation (3) is proved in [6, Theorem 3].

2. Let $-1<s<0$. We define $\sigma=-s\in(0,1)$ and proceed similarly to the proof of [6, Theorem 3]. It is sufficient to prove the statement for $u\in\mathcal{C}^\infty_0(\Omega)$.

For $\Omega'\supset\Omega$ we have $\mathcal{W}^N_{-\sigma,\Omega}\subset\mathcal{W}^N_{-\sigma,\Omega'}$. By (14), the quadratic form $Q^N_{s,\Omega}[u]$ is monotone increasing with respect to the domain inclusion. Taking (2) into account, we obtain

$$Q^D_{s,\Omega}[u] > Q^N_{s,\Omega'}[u] \ge Q^N_{s,\Omega}[u]. \qquad (15)$$

eq:monotone

Denote by $w = w^D_{-\sigma}$ the Caffarelli–Silvestre extension of $\frac{C_\sigma}{2\sigma}(-\Delta)^{-\sigma}u$, described in Section 2. Next, for any $y \geq 0$ let $\phi_R(\cdot, y)$ be the harmonic extension of $w(\cdot, y)$ on the ball B_R, that is,

$$-\Delta\phi_R(\cdot, y) = 0 \quad \text{in } B_R; \qquad \phi_h(\cdot, y) = w(\cdot, y) \quad \text{on } \partial B_R\,.$$

Finally, for $x \in B_R$ and $y \geq 0$ we put

$$w_R(x, y) = w(x, y) - \phi_R(x, y)\,.$$

It is shown in the proof of [6, Theorem 3] that there exists a sequence $R_h \to \infty$ such that

$$\mathcal{E}^N_\sigma(w_{R_h}) \leq \mathcal{E}^D_\sigma(w) + o(1).$$

Further, since $(-\Delta_\Omega)^{-\sigma}_D u$ vanishes at infinity, for any multi-index β we evidently have $D^\beta\phi_{R_h}(\cdot, 0) \to 0$ locally uniformly as $R_h \to \infty$. This gives $\big\langle u, \phi_{R_h}(\cdot, 0)\big\rangle = o(1)$, and we obtain by (12) and (14)

$$\begin{aligned} Q^N_{s,B_{R_h}}[u] &\geq -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}^N_{-\sigma}(w_{R_h}) \\ &\geq -\frac{2\sigma}{C_\sigma} \cdot \widetilde{\mathcal{E}}^D_\sigma(w) - o(1) = Q^D_{s,\Omega}[u] - o(1)\,. \end{aligned} \tag{16}$$ eq:tesi

The relation (3) readily follows by comparing (15) and (16).

3. For $s > 1$, $s \notin \mathbb{N}$, the conclusion follows from cases 1 and 2 just as in the proof of Theorem 1. □

R:new **Remark 5** *Assume that $0 \in \Omega$ and put $\alpha\Omega = \{\alpha x\,:\, x \in \Omega\}$. Thanks to (15), the proof above shows indeed that*

$$Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\alpha\Omega}[u] \quad \textit{for any} \quad u \in \widetilde{H}^s(\Omega).$$

Now put $u_\alpha(x) = \alpha^{\frac{n-2s}{2}}u(\alpha x)$. Then the scaling shows that

$$Q^D_{s,\Omega}[u_\alpha] \equiv Q^D_{s,\Omega}[u] = \lim_{\alpha\to\infty} Q^N_{s,\Omega}[u_\alpha] \quad \textit{for any} \quad u \in \widetilde{H}^s(\Omega).$$

Proof of Theorem 3. First, let $f \in \mathcal{C}^\infty_0(\Omega)$. We define $\sigma = -s \in (0, 1)$ and construct extensions $w^D_{-\sigma}$ and $w^N_{-\sigma}$ described in Section 2. Making the change of the variable $t = y^{2\sigma}$, we rewrite the BVP (13) for $w^D_{-\sigma}(x, t)$ as follows:

$$\Delta_x w^D_{-\sigma} + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}}\partial^2_{tt}w^D_{-\sigma} = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad \partial_t w^D_{-\sigma}\big|_{t=0} = -\frac{f}{2\sigma}. \tag{17}$$ BVP

Since $w^D_{-\sigma}$ vanishes at infinity, $w^D_{-\sigma}(x, t) > 0$ for $t > 0$ by the maximum principle. Moreover, by [1, Theorem 1.4] (the boundary point lemma) we have $w^D_{-\sigma}(x, 0) > 0$.

Further, the function $w^N_{-\sigma}$ satisfies the equalities (17) in $\Omega \times \mathbb{R}_+$. Since $w^N_{-\sigma}\big|_{x\notin\Omega} = 0$, we infer that the function

$$W(x,t) := w^D_{-\sigma}(x,t) - w^N_{-\sigma}(x,t)$$

meets the following relations:

$$\Delta_x W + 4\sigma^2 t^{\frac{2\sigma-1}{\sigma}} \partial^2_{tt} W = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+; \qquad \partial_t W\big|_{t=0} = 0; \qquad W\big|_{x\in\partial\Omega} > 0.$$

Again, [1, Theorem 1.4] gives $W(x,0) > 0$, which gives (5) in view of (12) and (14).

For $f \in \widetilde{H}^s(\Omega)$ the statement holds by approximation argument. □

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