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# Fractional Hardy-Sobolev inequalities on half spaces

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## Abstract

We investigate the existence of extremals for Hardy–Sobolev inequalities involving the Dirichlet fractional Laplacian  $(-\Delta)^s$  of order  $s \in (0, 1)$  on half-spaces.

**Keywords:** Fractional Laplace operators, Sobolev inequality, Hardy inequality.

**2010 Mathematics Subject Classification:** 47A63; 35A23.

## 1 Introduction

We study Hardy-Sobolev type inequalities for the restricted Dirichlet fractional Laplacian  $(-\Delta)^s$  acting on functions that vanish outside an half-space, for instance outside

$$\mathbb{R}_+^n = \{ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 > 0 \}.$$

We always assume  $s \in (0, 1)$ ,  $n > 2s$  and we put

$$2_s^* := \frac{2n}{n - 2s}.$$

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We recall that the operator  $(-\Delta)^s$  is defined by

$$\mathcal{F}[(-\Delta)^s u] = |\xi|^{2s} \mathcal{F}[u], \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^n),$$

where  $\mathcal{F}$  is the Fourier transform  $\mathcal{F}[u](\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx$ . The corresponding quadratic form is given by

$$\langle (-\Delta)^s u, u \rangle = \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[u]|^2 d\xi.$$

Motivated by applications to variational fractional equations on half-spaces, in the present paper we study the inequality

$$\langle (-\Delta)^s u, u \rangle \geq \lambda \int_{\mathbb{R}_+^n} x_1^{-2s} |u|^2 dx + \mathcal{S}_s^{\lambda,p}(\mathbb{R}_+^n) \left( \int_{\mathbb{R}_+^n} x_1^{-pb} |u|^p dx \right)^{\frac{2}{p}}, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n) \quad (1.1)$$

under the following hypotheses on the data:

$$2 < p \leq 2_s^*, \quad \lambda < \mathcal{H}_s := \frac{1}{\pi} \Gamma\left(s + \frac{1}{2}\right)^2 \quad (1.2a)$$

$$\frac{b}{n} = \frac{1}{p} - \frac{1}{2_s^*}. \quad (1.2b)$$

The bounds on the exponent  $p$  are due to Sobolev embeddings; the relation (1.2b) is a necessary condition to have of (1.1) for some constant  $\mathcal{S}_s^{\lambda,p}(\mathbb{R}_+^n) > 0$ , use a rescaling argument.

Actually the assumptions (1.2a–1.2b) are sufficient to have that (1.1) holds with a positive best constant  $\mathcal{S}_s^{\lambda,p}(\mathbb{R}_+^n)$ . Here is the argument.

First, notice that for  $p = 2_s^*$ , that implies  $b = 0$ , we have

$$\mathcal{S}_s := \inf_{\substack{u \in \mathcal{C}_0^\infty(\mathbb{R}^n) \\ u \neq 0}} \frac{\langle (-\Delta)^s u, u \rangle}{\|u\|_{2_s^*}^2} = \inf_{\substack{u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n) \\ u \neq 0}} \frac{\langle (-\Delta)^s u, u \rangle}{\|u\|_{2_s^*}^2} = \mathcal{S}_s^{0,2_s^*}(\mathbb{R}_+^n) \quad (1.3)$$

because of the action of translations and dilations in  $\mathbb{R}^n$ . The explicit value of the Sobolev constant  $\mathcal{S}_s$  has been computed in [3].

Next, recall the Hardy-type inequality with cylindrical weights proved by Bogdan and Dyda in [2]. It turns out that

$$\langle (-\Delta)^s u, u \rangle \geq \mathcal{H}_s \int_{\mathbb{R}_+^n} x_1^{-2s} u^2 dx \quad \text{for any } u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n), \quad (1.4)$$

with a sharp constant in the right hand side. Thus  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) > 0$  for any  $\lambda < \mathcal{H}_s$ .

If  $p \in (2, 2_s^*)$  and (1.2a–1.2b) are satisfied, the existence of a positive constant  $\mathcal{S}_s^{\lambda, p}(\mathbb{R}_+^n)$  such that (1.1) holds is easily proved via Hölder interpolation between the Sobolev and the cylindrical Hardy inequalities.

We now set up an appropriate functional setting to study the existence of extremals for  $\mathcal{S}_s^{\lambda, p}(\mathbb{R}_+^n)$ . The quadratic form  $\langle (-\Delta)^s u, u \rangle$  induces an Hilbertian structure on the space

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{2_s^*}(\mathbb{R}^n) \mid \langle (-\Delta)^s u, u \rangle < \infty\},$$

and  $\mathcal{D}^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$  with a continuous embedding by the Sobolev inequality. Clearly  $\mathcal{D}^s(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  is the standard Sobolev space  $H^s(\mathbb{R}^n)$ , see [15] for basic results about  $H^s$ -spaces. In particular  $\mathcal{D}^s(\mathbb{R}^n) \supseteq H^s(\mathbb{R}^n)$  and  $\mathcal{D}^s(\mathbb{R}^n) \subset H_{\text{loc}}^s(\mathbb{R}^n)$ , that means  $\varphi u \in H^s(\mathbb{R}^n)$  for  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  and  $u \in \mathcal{D}^s(\mathbb{R}^n)$ . Therefore,  $\mathcal{C}_0^\infty(\mathbb{R}^n)$  is dense in  $\mathcal{D}^s(\mathbb{R}^n)$  and the Rellich-Kondrashov Theorem holds, that is,  $\mathcal{D}^s(\mathbb{R}^n)$  is compactly embedded into  $L_{\text{loc}}^q(\mathbb{R}^n)$  for any  $q < 2_s^*$ .

Next, let  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  be the closure of  $\mathcal{C}_0^\infty(\mathbb{R}_+^n)$  in  $\mathcal{D}^s(\mathbb{R}^n)$ . We have

$$\begin{aligned} \tilde{\mathcal{D}}^s(\mathbb{R}_+^n) &= \{u \in \mathcal{D}^s(\mathbb{R}^n) \mid u \equiv 0 \text{ on } \mathbb{R}_-^n := \mathbb{R}^n \setminus \overline{\mathbb{R}_+^n}\}, \\ \mathcal{S}_s^{\lambda, p}(\mathbb{R}_+^n) &= \inf_{\substack{u \in \tilde{\mathcal{D}}^s(\mathbb{R}_+^n) \\ u \neq 0}} \frac{\langle (-\Delta)^s u, u \rangle - \lambda \|x_1^{-s} u\|_2^2}{\|x_1^{-b} u\|_p^2}. \end{aligned} \quad (1.5)$$

The minimization problem in (1.5) is noncompact, due to the action of dilations in  $\mathbb{R}^n$ . If  $n \geq 2$  also translations in  $\mathbb{R}^{n-1}$  might generate noncompact minimizing sequences. Our first result about the existence of minimizers for  $\mathcal{S}_s^{\lambda, p}(\mathbb{R}_+^n)$  concerns the case  $p < 2_s^*$ .

**Theorem 1.1** *Let  $p \in (2, 2_s^*)$ ,  $-\infty < \lambda < \mathcal{H}_s$  and  $b \in (0, s)$  as in (1.2b). Then the infimum  $\mathcal{S}_s^{\lambda, p}(\mathbb{R}_+^n)$  is achieved in  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ .*

In the critical case  $p = 2_s^*$ , the noncompact group translations in the  $x_1$ -variable produces severe lack of compactness phenomena. Take for instance  $\lambda = 0$ . By the results in [3] we have that, up to dilations, translations and multiplications, the Sobolev constant  $\mathcal{S}_s$  is attained on  $\mathcal{D}^s(\mathbb{R}^n)$  only by the function  $U_s(x) = (1 + |x|^2)^{\frac{2s-n}{2}}$ . Therefore the infimum  $\mathcal{S}_s^{0,2_s^*}(\mathbb{R}_+^n) = \mathcal{S}_s$  is not achieved on  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ . In Section 2 we prove the next theorems.

**Theorem 1.2** *For  $\lambda < \mathcal{H}_s$  the following facts hold.*

- i)  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n) \leq \mathcal{S}_s$ ;
- ii) If  $-\infty < \lambda \leq 0$  then  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n) = \mathcal{S}_s$  and  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n)$  is not achieved;
- iii) If  $0 < \lambda < \mathcal{H}_s$  and  $n \geq 4s$  then  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n) < \mathcal{S}_s$ .

**Theorem 1.3** *Assume  $0 < \lambda < \mathcal{H}_s$ . If  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n) < \mathcal{S}_s$  then  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n)$  is achieved. In particular, if  $n \geq 4s$  then  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n)$  is achieved.*

Notice that  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n)$  is always achieved if  $n \geq 4$ , while the cases

$$n = 1 \text{ and } \frac{1}{4} < s < \frac{1}{2}, \quad n = 2 \text{ and } \frac{1}{2} < s < 1, \quad n = 3 \text{ and } \frac{3}{4} < s < 1$$

are not covered by Theorem 1.3.

All the proofs can be found in the next section. Our arguments to get the existence of minimizers are simple and self-contained. We construct an ad hoc bounded minimizing sequence that can neither concentrate at the origin nor vanish. In the locally compact case (see Theorem 1.1) the existence of a minimizer is readily obtained. In the critical case, concentration at points  $x \in \mathbb{R}_+^n$  is excluded by the assumption  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n) < \mathcal{S}_s$ , and the existence result in Theorem 1.3 follows.

Thanks to formula (3.2) below, an alternative proof can be obtained by adapting the arguments in the recent paper [5].

We conclude the paper with few additional remarks and open problems. In particular, in Section 3 we conjecture that Theorem 1.3 is sharp, that is,  $\mathcal{S}_s^{\lambda,2_s^*}(\mathbb{R}_+^n)$  is not attained if  $2s < n < 4s$ .

**Notation.**  $\Omega \subset \mathbb{R}^n$  is a domain, and  $\Omega^c = \mathbb{R}^n \setminus \Omega$  is its complement.

For  $q \in [1, \infty]$  we denote by  $\|\cdot\|_{q,\Omega}$  the norm in  $L^q(\Omega)$ . If  $\Omega = \mathbb{R}^n$  we simply write  $\|\cdot\|_q$ . Let  $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  and  $s \in (0, 1)$ . It is well known that

$$\langle (-\Delta)^s u, u \rangle = \frac{C_{n,s}}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy, \quad C_{n,s} = \frac{s 2^{2s} \Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}} \Gamma(1 - s)}. \quad (1.6)$$

By density, (1.6) holds for any  $u \in \mathcal{D}^s(\mathbb{R}^n)$ . Next, for  $\lambda < \mathcal{H}_s$  we put

$$\mathcal{E}_s^\lambda(u) = \langle (-\Delta)^s u - \lambda x_1^{-2s} u, u \rangle = \langle (-\Delta)^s u, u \rangle - \lambda \int_{\mathbb{R}_+^n} x_1^{-2s} |u|^2 dx,$$

that is the square of an equivalent norm in  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  by the Hardy inequality (1.4).

Through the paper, all constants depending only on  $n$  and  $s$  are denoted by  $c$ . To indicate that a constant depends on other quantities we list them in parentheses:  $c(\dots)$ .

## 2 Proofs

We start with a technical result that is essentially known, see for instance [11]. We provide its proof for the convenience of the reader.

**Lemma 2.1** *Let  $u \in \tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ ,  $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$  and let  $\Omega \subset \mathbb{R}^n$  be a bounded domain containing the support of  $\varphi$ . Then  $\varphi u \in \tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  and*

$$|\langle (-\Delta)^s \varphi u, \varphi u \rangle - \langle (-\Delta)^s u, \varphi^2 u \rangle| \leq c(\varphi, \Omega) \langle (-\Delta)^s u, u \rangle^{\frac{1}{2}} \cdot \|u\|_{2,\Omega}.$$

**Proof.** The first statement is evident. Further, we estimate

$$\Psi_\varphi(x, y) := \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2s}} \leq c(\varphi) \left( \frac{\chi_{\{|x-y|<1\}}}{|x - y|^{n-2(1-s)}} + \frac{\chi_{\{|x-y|>1\}}}{|x - y|^{n+2s}} \right)$$

to obtain

$$\int_{\mathbb{R}^n} \Psi_\varphi(x, y) dy \leq c(\varphi) \quad (2.1)$$

for all  $x \in \mathbb{R}^n$ . Taking (1.6) into account, by direct computation one finds

$$\langle (-\Delta)^s \varphi u, \varphi u \rangle - \langle (-\Delta)^s u, \varphi^2 u \rangle = c \iint_{\mathbb{R}^n \times \mathbb{R}^n} u(x) u(y) \Psi_\varphi(x, y) dx dy =: B_\varphi.$$

Since the support of  $\Psi_\varphi$  is contained in  $(\Omega \times \mathbb{R}^n) \cup (\mathbb{R}^n \times \Omega)$ , we have

$$\begin{aligned}
c |B_\varphi| &\leq \iint_{\Omega \times \Omega} |u(x)u(y)| \Psi_\varphi(x, y) \, dx dy + \int_{\Omega} |u(x)| \left( \int_{\Omega^c} \frac{|u(y)| |\varphi(x)|^2}{|x-y|^{n+2s}} dy \right) dx \\
&\stackrel{*}{\leq} \iint_{\Omega \times \Omega} |u(x)u(y)| \Psi_\varphi(x, y) \, dx dy + \int_{\Omega} |u(x) \varphi(x)|^2 \left( \int_{\Omega^c} \frac{dy}{|x-y|^{n+2s}} \right) dx \\
&\quad + \|\varphi\|_\infty \iint_{\Omega \times \Omega^c} \frac{|u(x) - u(y)|}{|x-y|^{\frac{n+2s}{2}}} \frac{|u(x) \varphi(x)|}{|x-y|^{\frac{n+2s}{2}}} \, dx dy =: I_1 + I_2 + \|\varphi\|_\infty I_3
\end{aligned}$$

(in (\*) we use the triangle inequality). By the Cauchy-Bunyakovsky-Schwarz inequality and (2.1) we obtain

$$\begin{aligned}
I_1 &= \iint_{\Omega \times \Omega} \left| |u(x)|^2 \Psi_\varphi(x, y) \right|^{\frac{1}{2}} \left| |u(y)|^2 \Psi_\varphi(x, y) \right|^{\frac{1}{2}} \, dx dy \\
&\leq \iint_{\Omega \times \Omega} |u(x)|^2 \Psi_\varphi(x, y) \, dx dy \leq c(\varphi) \int_{\Omega} |u(x)|^2 \, dx \leq c(\varphi, \Omega) \langle (-\Delta)^s u, u \rangle^{\frac{1}{2}} \cdot \|u\|_{2, \Omega}.
\end{aligned}$$

Since  $\text{supp}(\varphi)$  is compactly contained in  $\Omega$  we clearly have  $I_2 \leq c(\varphi, \Omega) \int_{\Omega} |u(x)|^2 \, dx$ .

To handle  $I_3$  we use the Cauchy-Bunyakovsky-Schwarz inequality in  $\Omega \times \Omega^c$  and the above estimate on  $I_2$  to get

$$I_3^2 \leq c \langle (-\Delta)^s u, u \rangle I_2 \leq c(\varphi, \Omega) \langle (-\Delta)^s u, u \rangle \int_{\Omega} |u(x)|^2 \, dx.$$

The proof is complete.  $\square$

**Proof of Theorem 1.1.** We follow the outline of the proof of Theorem 0.1 in [6]. Thanks to a standard convexity argument, we only need to construct a minimizing sequence that weakly converges in  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  to a nontrivial limit. For future convenience we notice that the assumption  $p < 2_s^*$  is only used in the last line of the proof.

In order to simplify notations we put

$$\mathcal{S}_\lambda = \mathcal{S}_s^{\lambda,p}(\mathbb{R}_+^n) = \inf_{\substack{u \in \tilde{\mathcal{D}}^s(\mathbb{R}_+^n) \\ u \neq 0}} \frac{\mathcal{E}_s^\lambda(u)}{\|x_1^{-b}u\|_p^2}.$$

We assume that  $n \geq 2$ . The proof for  $n = 1$  is similar, and simpler; only notation has to be adapted.

For  $\rho > 0$  and  $z \in \mathbb{R}^{n-1}$  we denote by  $B'_\rho(z)$  the  $(n-1)$ -dimensional ball

$$B'_\rho(z) = \{x' \in \mathbb{R}^{n-1} \mid |x' - z| < \rho\}.$$

Choose a finite number of points  $x'_1, \dots, x'_\tau \in \mathbb{R}^{n-1}$  such that

$$\overline{B'_2(0)} \subset \bigcup_{j=1}^{\tau} B'_1(x'_j). \quad (2.2)$$

Take a number  $\varepsilon_0$  such that  $0 < \varepsilon_0 < \frac{1}{2}\mathcal{S}_\lambda$ . Notice that the ratio in (1.5) is invariant with respect to translations in  $\mathbb{R}^{n-1}$  and with respect to the transforms  $u(x) \mapsto \alpha u(\beta x)$  for  $\alpha \neq 0, \beta > 0$ . Thus we can select a bounded minimizing sequence  $u_h$  for  $\mathcal{S}_\lambda$  satisfying the normalization condition

$$\|x_1^{-b}u_h\|_p^p = \mathcal{S}_\lambda^{\frac{p}{p-2}}, \quad \mathcal{E}_s^\lambda(u_h) = \mathcal{S}_\lambda^{\frac{p}{p-2}} + o(1) \quad (2.3)$$

and such that

$$\varepsilon_0^{\frac{p}{p-2}} \leq \max_j \int_0^2 \int_{B'_2(x'_j)} x_1^{-pb} |u_h|^p dx' dx_1 \leq \int_0^2 \int_{B'_2(0)} x_1^{-pb} |u_h|^p dx' dx_1 \leq (2\varepsilon_0)^{\frac{p}{p-2}}. \quad (2.4)$$

Up to a subsequence, we have that  $u_h \rightarrow u$  weakly in  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ . We claim that  $u \neq 0$ , that is enough to conclude the proof.

Assume by contradiction that  $u = 0$ . By Ekeland's variational principle we can assume that there exists a sequence  $f_h \rightarrow 0$  in  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n)'$ , such that

$$(-\Delta)^s u_h - \lambda x_1^{-2s} u_h = x_1^{-pb} |u_h|^{p-2} u_h + f_h \quad \text{in } \tilde{\mathcal{D}}^s(\mathbb{R}_+^n)'. \quad (2.5)$$

Take a cut-off function  $\varphi \in \mathcal{C}_0^\infty(-2, 2)$  such that  $\varphi \equiv 1$  on  $(-1, 1)$  and define  $\varphi_j(x') = \varphi(|x' - x'_j|)$ ,  $j = 1, \dots, \tau$ .



Note that the cut-off function  $\psi_j(x_1, x') := \varphi(x_1)\varphi_j(x')$  has compact support in  $(-2, 2) \times B'_2(x'_j)$  and that  $\psi_j^2 u_h$  is a bounded sequence in  $\widetilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  by Lemma 2.1. Use  $\psi_j^2 u_h$  as test function in (2.5) to find

$$\langle (-\Delta)^s u_h, \psi_j^2 u_h \rangle - \lambda \int_{\mathbb{R}_+^n} x_1^{-2s} |\psi_j u_h|^2 dx = \int_{\mathbb{R}^n} x_1^{-pb} |u_h|^{p-2} |\psi_j u_h|^2 dx + o(1). \quad (2.6)$$

Thanks to Hölder inequality and (2.4) we can estimate the right-hand side by

$$\begin{aligned} & \int_{\mathbb{R}^n} x_1^{-pb} |u_h|^{p-2} |\psi_j u_h|^2 dx \\ & \leq \left( \int_0^2 \int_{B'_2(x'_j)} x_1^{-pb} |u_h|^p dx' dx_1 \right)^{\frac{p-2}{p}} \|x_1^{-b} \psi_j u_h\|_p^2 \leq 2\varepsilon_0 \|x_1^{-b} \psi_j u_h\|_p^2. \end{aligned} \quad (2.7)$$

To handle the left-hand side of (2.6) we use Lemma 2.1, the compactness of embedding  $\widetilde{\mathcal{D}}^s(\mathbb{R}_+^n) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^n)$  and the definition of  $\mathcal{S}_\lambda = \mathcal{S}_s^{\lambda,p}(\mathbb{R}_+^n)$  to obtain

$$\langle (-\Delta)^s u_h, \psi_j^2 u_h \rangle - \lambda \int_{\mathbb{R}_+^n} x_1^{-2s} |\psi_j u_h|^2 dx = \mathcal{E}_s^\lambda(\psi_j u_h) + o(1) \geq \mathcal{S}_\lambda \|x_1^{-b} \psi_j u_h\|_p^2 + o(1).$$

In this way, from (2.6) we infer

$$\mathcal{S}_\lambda \|x_1^{-b} \psi_j u_h\|_p^2 \leq 2\varepsilon_0 \|x_1^{-b} \psi_j u_h\|_p^2 + o(1). \quad (2.8)$$

Since  $2\varepsilon_0 < \mathcal{S}_\lambda$ , formula (2.8) implies that  $\|x_1^{-b} \psi_j u_h\|_p = o(1)$ . But then, using (2.2) and recalling that  $\psi_j \equiv 1$  on  $(0, 1) \times B'_1(x'_j)$ , we obtain

$$\begin{aligned} \int_0^1 \int_{B'_2(0)} x_1^{-pb} |u_h|^p dx' dx_1 & \leq \sum_{j=1}^\tau \int_0^1 \int_{B'_1(x'_j)} x_1^{-pb} |u_h|^p dx' dx_1 \\ & \leq \sum_{j=1}^\tau \int_{\mathbb{R}_+^n} x_1^{-pb} |\psi_j u_h|^p dx = o(1). \end{aligned}$$

Comparing with the first inequality in (2.4) we arrive at

$$2^{-pb} \int_1^2 \int_{B'_2(0)} |u_h|^p dx' dx_1 \geq \int_1^2 \int_{B'_2(0)} x_1^{-pb} |u_h|^p dx' dx_1 \geq \varepsilon_0^{\frac{p}{p-2}} + o(1), \quad (2.9)$$

that contradicts the compactness of embedding  $\widetilde{\mathcal{D}}^s(\mathbb{R}_+^n) \hookrightarrow L_{\text{loc}}^p(\mathbb{R}^n)$ , as  $p < 2_s^*$ .  $\square$

**Proof of Theorem 1.2.** Take any nontrivial function  $\varphi \in C_0^\infty(B)$ , where  $B$  is the unit ball about the origin. Let  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}_+^n$  and take  $h \geq 1$ . Testing  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n)$  with  $\varphi_h(x) = \varphi(h(x - e_1)) \in C_0^\infty(\mathbb{R}_+^n)$  we obtain

$$\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) \leq \frac{\langle (-\Delta)^s \varphi, \varphi \rangle - \lambda \|(x_1 + h)^{-s} \varphi\|_{2, B}^2}{\|\varphi\|_{2_s^*, B}^2}. \quad (2.10)$$

Letting  $h \rightarrow \infty$  we infer  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) \leq \frac{\langle (-\Delta)^s \varphi, \varphi \rangle}{\|\varphi\|_{2_s^*, B}^2}$ . Since  $\varphi$  was arbitrarily chosen we can conclude that

$$\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) \leq \inf_{\substack{\varphi \in C_0^\infty(B) \\ \varphi \neq 0}} \frac{\langle (-\Delta)^s \varphi, \varphi \rangle}{\|\varphi\|_{2_s^*, B}^2} = \mathcal{S}_s,$$

and *i*) is proved.

If  $\lambda \leq 0$  then trivially  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) \geq \mathcal{S}_s$ , because of (1.3) holds for  $u \in \widetilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ . Hence  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) = \mathcal{S}_s$  and is not attained.

If  $\lambda > 0$  we take  $h = 1$  in (2.10) to get

$$\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) \leq \inf_{\substack{\varphi \in C_0^\infty(B) \\ \varphi \neq 0}} \frac{\langle (-\Delta)^s \varphi, \varphi \rangle - 2^{-2s} \lambda \|\varphi\|_{2, B}^2}{\|\varphi\|_{2_s^*, B}^2}.$$

Therefore we can use Theorems 4.2, 4.3 in [10], see also [12], that give  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n) < \mathcal{S}_s$  if  $n \geq 4s$  or if  $\lambda > 0$  is large enough.  $\square$

**Proof of Theorem 1.3.** The first part of the proof goes as for Theorem 1.1. We assume that  $n \geq 2$  and use the same notation as in the proof of Theorem 1.1, with  $p = 2_s^*$  and  $b = 0$ .

We select a minimizing sequence  $u_h$  satisfying (2.3) and (2.4). Up to a subsequence, we have that  $u_h \rightarrow u$  weakly in  $\widetilde{\mathcal{D}}^s(\mathbb{R}_+^n)$ . If  $u = 0$  then we can assume that there exists a sequence  $f_h \rightarrow 0$  in  $\widetilde{\mathcal{D}}^s(\mathbb{R}_+^n)'$ , such that  $u_h$  solves

$$(-\Delta)^s u_h - \lambda x_1^{-2s} u_h = |u_h|^{2_s^* - 2} u_h + f_h \quad \text{in } \widetilde{\mathcal{D}}^s(\mathbb{R}_+^n)', \quad (2.11)$$

compare with (2.5). Arguing as in the proof of Theorem 1.1 one can prove that (2.9) holds with  $p = 2_s^*$  and  $b = 0$ .

Now we take a cut-off function  $\phi \in \mathcal{C}_0^\infty(\mathbb{R}_+^n)$  such that  $\phi \equiv 1$  on  $(1, 2) \times B_2'(0)$ . We test (2.11) with  $\phi^2 u_h \in \tilde{\mathcal{D}}^s(\mathbb{R}_+^n)$  to get

$$\langle (-\Delta)^s u_h, \phi^2 u_h \rangle - \lambda \int_{\mathbb{R}_+^n} x_1^{-2s} |\phi u_h|^2 dx = \int_{\mathbb{R}^n} |u_h|^{2_s^*-2} |\phi u_h|^2 dx + o(1). \quad (2.12)$$

Since  $\text{supp}(\phi) \subset \mathbb{R}_+^n$ , by compactness of embedding  $\tilde{\mathcal{D}}^s(\mathbb{R}_+^n) \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^n)$  we have  $\|x_1^{-s} \phi u_h\|_2 \rightarrow 0$ . Thus, we can use Lemma 2.1 and the Sobolev inequality to infer

$$\langle (-\Delta)^s u_h, \phi^2 u_h \rangle = \mathcal{E}_s^\lambda(\phi u_h) + o(1) = \langle (-\Delta)^s \phi u_h, \phi u_h \rangle + o(1) \geq \mathcal{S}_s \|\phi u_h\|_{2_s^*}^2 + o(1).$$

Therefore, estimating the right hand side of (2.12) via Hölder inequality we obtain

$$\mathcal{S}_s \|\phi u_h\|_{2_s^*}^2 \leq \|u_h\|_{2_s^*}^{2_s^*-2} \|\phi u_h\|_{2_s^*}^2 + o(1) = \mathcal{S}_\lambda \|\phi u_h\|_{2_s^*}^2 + o(1). \quad (2.13)$$

Now we recall that  $\mathcal{S}_\lambda < \mathcal{S}_s$  and  $\phi \equiv 1$  on  $(1, 2) \times B_2'(0)$ . Thus (2.13) gives

$$\int_{1 B_2'(0)} \int_{1 B_2'(0)} |u_h|^{2_s^*} dx_1 dx' = o(1).$$

We reached a contradiction with (2.9), that concludes the proof.  $\square$

### 3 Additional remarks and problems

In this section we compare the available results for  $s \in (0, 1)$  with some known results in the local case  $s = 1$ ,  $n \geq 2$ , when  $\mathcal{H}_1 = \frac{1}{4}$ ,  $(-\Delta)^s = -\Delta$  is the standard Laplace operator, and  $\langle -\Delta u, u \rangle = \|\nabla u\|_2^2$  for  $u \in \mathcal{D}^1(\mathbb{R}^n)$ .

Recall that Maz'ya proved in [8, 2.1.6, Corollary 3], that there exists a positive best constant  $\mathcal{S}_1^{\frac{1}{4}, p}(\mathbb{R}_+^n)$  such that

$$\langle -\Delta u - \mathcal{H}_1 x_1^{-2} u, u \rangle = \int_{\mathbb{R}_+^n} (|\nabla u|^2 - \frac{1}{4} x_1^{-2} |u|^2) dx \geq \mathcal{S}_1^{\frac{1}{4}, p}(\mathbb{R}_+^n) \left( \int_{\mathbb{R}_+^n} x_1^{-pb} |u|^p dx \right)^{\frac{2}{p}} \quad (3.1)$$

for any  $u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n)$ , where  $n \geq 3$ ,  $2 < p \leq 2_1^* = \frac{2n}{n-2}$  and  $\frac{b}{n} = \frac{1}{p} - \frac{n-2}{2n}$ , accordingly with (1.2b). Inequality (3.1) holds as well if  $n = 2$ , for any  $p > 2$  and for  $b = \frac{2}{p}$ , see [7, Appendix B].

As concerns the attainability of  $\mathcal{S}_1^{\frac{1}{4},p}(\mathbb{R}_+^n)$  we refer to [14] for  $p = 2_1^*$  and  $n \geq 4$ , and to [7, Sec. 6] for  $p < 2_1^*$  and  $n \geq 2$ . Finally, it was proved in [9] that the best constant  $\mathcal{S}_1^{\lambda,p}(\mathbb{R}_+^n)$  is attained if  $2 < p < 2_1^*$  and  $-\infty < \lambda < \frac{1}{4}$ , and when  $p = 2_1^*$ ,  $n \geq 4$  and  $0 < \lambda < \frac{1}{4}$  (clearly,  $\mathcal{S}_1^{\lambda,2_1^*}(\mathbb{R}_+^n)$  is never achieved if  $\lambda \leq 0$ ).

Surprisingly, in the lower dimensional critical case  $n = 3, p = 6$  one has  $\mathcal{S}_1^{\lambda,6}(\mathbb{R}_+^n) = \mathcal{S}_1$  and the minimizer never exists, whatever  $\lambda \leq \frac{1}{4}$  is (see [1] and [7]).

Now take  $s \in (0, 1)$ ,  $u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n)$  and compute

$$\langle (-\Delta)^s u, u \rangle = \frac{C_{n,s}}{2} \iint_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy + \gamma_s \int_{\mathbb{R}_+^n} x_1^{-2s} u^2 dx, \quad (3.2)$$

where

$$\gamma_s = \frac{2^{2s-1} \Gamma\left(s + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(1 - s)}.$$

From the proof of [2, Lemma 2] one gets that  $\mathcal{H}_s > \gamma_s$  for  $s \neq \frac{1}{2}$ , while  $\mathcal{H}_{\frac{1}{2}} = \gamma_{\frac{1}{2}} = \frac{1}{\pi}$ .

The above computation and the inequality proved by C.A. Sloane in [13] readily imply the next result.

**Proposition 3.1** *Let  $n \geq 2$ ,  $s \in (\frac{1}{2}, 1)$ . There exists a best constant  $\mathcal{S}_s^{\mathcal{H}_s, 2_s^*}(\mathbb{R}_+^n) > 0$  such that*

$$\langle (-\Delta)^s u - \mathcal{H}_s x_1^{-2s} u, u \rangle \geq \mathcal{S}_s^{\mathcal{H}_s, 2_s^*}(\mathbb{R}_+^n) \left( \int_{\mathbb{R}_+^n} |u|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \quad \text{for any } u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n). \quad (3.3)$$

In the first version of the present paper the following question has been raised up.

**Problem 1.** *Let  $n \geq 2$ , and  $p \in (2, 2_s^*]$ . Find sharp conditions on  $s \in (0, 1)$  that guarantee the existence of a best constant  $\mathcal{S}_s^{\mathcal{H}_s, p}(\mathbb{R}_+^n) > 0$  such that for  $b = b(n, s, p)$  as in (1.2b) one has*

$$\langle (-\Delta)^s u - \mathcal{H}_s x_1^{-2s} u, u \rangle \geq \mathcal{S}_s^{\mathcal{H}_s, p}(\mathbb{R}_+^n) \left( \int_{\mathbb{R}_+^n} x_1^{-pb} |u|^p dx \right)^{\frac{2}{p}} \quad \text{for any } u \in \mathcal{C}_0^\infty(\mathbb{R}_+^n).$$

In the recent publication [4], Dyda, Leirbäck and Vähäkangas gave a complete answer to Problem 1. As far as we know, the next problem is still open.

**Problem 2** Assume  $\mathcal{S}_s^{\mathcal{H}_s,p}(\mathbb{R}_+^n) > 0$ . Is  $\mathcal{S}_s^{\mathcal{H}_s,p}(\mathbb{R}_+^n)$  attained?

Inspired by the result of [7], we formulate the following conjecture.

**Conjecture** Let  $s \in (0, 1)$ ,  $2s < n < 4s$  (hence,  $n \leq 3$ ). Then the best constant  $\mathcal{S}_s^{\lambda, 2_s^*}(\mathbb{R}_+^n)$  is never achieved.

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