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Uniquely orderable interval graphs

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ABSTRACT. Interval graphs and interval orders are deeply linked. In fact, edges of an interval graphs represent the incomparability relation of an interval order, and in general, of different interval orders. The question about the conditions under which a given interval graph is associated to a unique interval order (up to duality) arises naturally. Fishburn provided a characterisation for uniquely orderable finite connected interval graphs. We show, by an entirely new proof, that the same characterisation holds also for infinite connected interval graphs. Using tools from reverse mathematics, we explain why the characterisation cannot be lifted from the finite to the infinite by compactness, as it often happens.

1. INTRODUCTION

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An *interval graph* is a graph whose vertices can be mapped (by an *interval representation*) to nonempty intervals of a linear order in such a way that two vertices are adjacent if and only if the intervals associated to them intersect (it is thus convenient to assume that the adjacency relation is reflexive). Consequently, if two vertices are incomparable in the graph, the corresponding intervals are placed one before the other in the linear order. The definition of interval graphs leads to an analogous concept for partial orders. In fact, a partial order $<_P$ is an *interval order* if its points can be mapped to nonempty intervals of a linear order in such a way that $x <_P y$ if and only if the interval associated to x completely precedes the interval associated to y. Thus interval graphs are the incomparability graphs of interval orders, i.e. two vertices are adjacent in the graph if and only if they are incomparable in the partial order.

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Norbert Wiener was probably the first to pay attention to interval orders, disguised under the less familiar name 'relations of complete sequence', in [Wie14]. Fifty years later interval graphs and interval orders were rediscovered and given the current name. There is now an extensive literature on the topic: [Tro97] provides a survey for many result in this area, focusing primarily on finite structures.

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Interval graphs and interval orders are extensively employed in diverse fields like psychology, archaeology and physics, just to mention a few. Wiener himself noticed that interval orders are useful for the analysis of temporal events and in the representation of measures subject to a margin of error. Interval orders actually occur in many digital calendars, where hours and days form a linear order and a rectangle covers the time assigned to an appointment: if two rectangles intersect, we better choose which event we will miss. Intervals are also suitable for representations of measurements of physical properties which are subject to error, since they can take into account the accuracy of the measuring device much better than a representation with points. In psychology and economics the overlap between two intervals often indicates that the corresponding stimuli or preferences are indistinguishable.

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In the first paragraph we described how to build an interval order from an interval representation of an interval graph. In general, an interval graph leads to many different interval orders on its vertices: an extreme example is a totally disconnected graph which is associated to any total order on its vertices. This paper deals with the situation were the interval graph is *uniquely orderable*, i.e. there is essentially only one interval order associated to the given interval graph. (The "essentially" in the previous sentence is due to the obvious observation that if an interval order is associated to a graph, then the same is true for the reverse partial order.) Here the extreme example is a complete graph, which is associated to a unique partial order, the antichain of its vertices.

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The question of which interval graphs are uniquely orderable is easily settled for non connected graphs. It is in fact immediate that a non connected interval graph is uniquely orderable if and only if it has at most two connected components each of which is complete.

We can thus restrict our attention to connected interval graphs. In this context, Fishburn [Fis85, §3.6], building on results proved in [Han82], provides two characterizations of unique orderability for finite graphs. Indeed, some steps of the proof heavily rely on the finiteness of the graph. This is in contrast with the rest of Fishburn's monograph, where results are systematically proved for arbitrary interval graphs and orders; we thus believe that Fishburn did not know whether his result held for infinite interval graphs as well. The main result of this paper solves this issue by extending Fishburn's characterizations to arbitrary interval graphs by an entirely different proof (for undefined notions see §2 below):

Theorem 1. *Let (V, E) be a (possibly infinite) connected interval graph. Let $W = \{(a, b) \in V \times V \mid \neg a\, E\, b\}$ and $(a, b)\, Q\, (c, d) \iff a\, E\, c \wedge b\, E\, d$. The following are equivalent:*

    (1) (V, E) is uniquely orderable;

    (2) (V, E) does not contain a buried subgraph;

    (3) the graph (W, Q) has two connected components.

Fishburn's statement is slightly different from ours, since it is formulated for connected interval graphs without universal vertices. Since universal vertices (i.e. those adjacent to all vertices of the graph) are incomparable to all other vertices in any partial order associated to an interval graph, removing all universal vertices does not change the unique orderability of the graph. We prefer our formulation of the result since it highlights the connectedness of the graph, which is the central property characterising the class of interval graphs for which Theorem 1 holds.

A typical method to lift a result from finite structures to arbitrary ones is compactness. Hence, once Theorem 1 is proved for finite interval graphs, the first attempt to generalise it to the infinite is to argue by compactness. This is not obvious and, using tools from mathematical logic, we are able to show that it is in fact impossible. To this end we work in the framework of reverse mathematics, a research program whose goal is to establish the minimal axioms needed to prove a theorem. In this framework compactness is embodied by the formal system WKL_0. We first indicate, with results which parallel those obtained in [Mar07] about interval orders, that all the basic aspects of the theory of interval orders can be developed in WKL_0. On the other hand we prove the following:

Theorem 2. *Over the base system RCA_0, the following are equivalent:*

    (1) ACA_0,

    (2) a countable connected interval graph (V, E) is uniquely orderable if and only if does not contain a buried subgraph.

Since ACA_0 is properly stronger than WKL_0 this shows that compactness does not suffice to prove Theorem 1.

Section 2 establishes notation and terminology, while Section 3 is devoted to the proof of Theorem 1. Section 4 gives an overview of the reverse mathematics of interval graphs: the first author's PhD thesis [FC19] includes full proofs. The last section is devoted to the proof of Theorem 2.

2. Preliminaries

In this section we establish the terminology used in the paper and underline some properties of interval graphs that turn out to be useful in the next section.

All the graphs (V, E) in this paper are such that $E \subseteq V \times V$ is a symmetric relation (we do not ask E to be irreflexive, as in some cases it is convenient to have reflexivity). As usual, we write $v\, E\, u$ to mean $(v, u) \in E$ and, if $V' \subseteq V$, we write (V', E) in place of $(V', E \cap (V' \times V'))$. We denote by $(V, \overline{E})$ the *complementary graph* of (V, E): for $u, v \in V$ we have $u\, \overline{E}\, v$ if and only if $u\, E\, v$ does not hold.

Paths and cycles are defined as usual, and their length is the number of their edges. A *simple cycle* $v_0\, E \ldots E\, v_n$ is a cycle such that the vertices in $v_0, \ldots, v_{n-1}$ are distinct. A *chord* of a cycle $v_0\, E\, v_1\, E \ldots E\, v_n$ is an edge (v_i, v_j) with $2 \leq j - i \leq n - 2$. The chord is *triangular* if either $j - i = 2$ or $j - i = n - 2$.

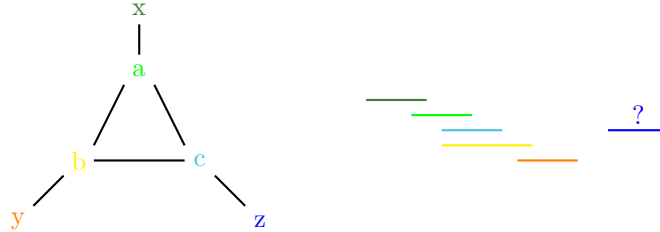FIGURE 1. An example of interval graph with its representation

FIGURE 2. A graph which is not an interval graph, with a partial representation

Definition 2.1. If $(V, \prec)$ is a strict partial order, the *comparability graph of* $(V, \prec)$ is the graph (V, E) such that for $v, u \in V$ it holds that $v \, E \, u$ if and only if either $v \prec u$ or $u \prec v$. The *incomparability graph of* $(V, \prec)$ is the complementary graph of the comparability graph, so that two distinct vertices are adjacent if and only if they coincide or are $\prec$-incomparable.

While the comparability graph of a strict partial order is irreflexive, its incomparability graph is reflexive.

Notice that a graph (V, E) can be the incomparability graph of more than one partial order: we say that each such partial order is *associated to* (V, E). In particular, $\prec$ and the dual of $\prec$ (i.e. $\prec'$ such that $u \prec' v$ iff $v \prec u$) are associated to the same incomparability graph.

Definition 2.2. A graph (V, E) is *uniquely orderable* if it is the incomparability graph of a partial order $\prec$ and the only other partial order associated to (V, E) is the dual order of $\prec$; in other words, there exists a unique (up to duality) partial order $\prec$ such that for each $v, u \in V$ it holds that $\neg u \, E \, v$ if and only if $u \prec v$ or $v \prec u$.

The following definition formalises the intuitive idea of interval graph given in the previous pages.

Definition 2.3. A graph (V, E) is an *interval graph* if it is reflexive and there exist a linear order $(L, <_L)$ and a map $F \colon V \to \wp(L)$ such that for all $v, u \in V$, $F(v)$ is an interval in $(L, <_L)$ (i.e. if $\ell <_L \ell' <_L \ell''$ and $\ell, \ell'' \in F(v)$ then also $\ell' \in F(v)$) and

$$v \, E \, u \Leftrightarrow F(v) \cap F(u) \neq \emptyset.$$

It is well-known that we may in fact assume that there exist functions $f_L, f_R \colon V \to L$ such that $F(v) = \{\ell \in L \mid f_L(v) \leq_L \ell \leq_L f_R(v)\}$ for all $v \in V$ (this is the definition given in [Fis85]).

We say that $(L, <_L, f_L, f_R)$ (but often only (f_L, f_R) or just F) is a *representation* of (V, E).

To decide whether two vertices u and v are adjacent in an interval graph with representation (f_L, f_R) we can assume without loss of generality that $f_L(v) \leq_L f_L(u)$ and then simply check whether $f_L(u) \leq_L f_R(v)$.

In the context of a representation (f_L, f_R) of an interval graph, we write $F(v) <_L F(u)$ in place of $f_R(v) <_L f_L(u)$. Then $\neg v \, E \, u$ means that either $F(v) <_L F(u)$ or $F(u) <_L F(v)$.

Figure 1 provides an example of interval graph, while the graph in Figure 2 does not have an interval representation.

A classical characterization of interval graphs is the following ([LB62], see [Fis85, Theorem 3.6]).

Definition 2.4. A graph (V, E) is *triangulated* if every simple cycle of length four or more has a chord. An *asteroidal triple* in (V, E) is an independent set of three vertices (i.e. a set of pairwise non adjacent vertices) of V such that any two of them are connected by a path that avoids the vertices adjacent to the third.
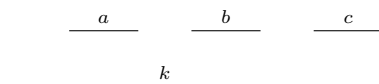
FIGURE 3. A non uniquely orderable connected interval graph

Theorem 2.5. *A graph (V, E) is an interval graph if and only it is triangulated and has no asteroidal triples.*

Proposition 2.6. *Let $v_0 E \ldots E v_n$ be a path in the interval graph (V, E) with representation F, and suppose $w \in V$ is such that $F(w) \not<_L F(v_0)$ and $F(v_n) \not<_L F(w)$. Then $v_i E w$ for some $i \leq n$, and hence $v_0 E \ldots E v_i E w$ and $w E v_i E \ldots E v_n$ are paths.*

Proof. Let $i \leq n$ be maximum such that $F(w) \not<_L F(v_i)$. If $i = n$ then $F(w) \not<_L F(v_n)$ and $F(v_n) \not<_L F(w)$ imply $v_n E w$. If $i < n$ then $F(w) <_L F(v_{i+1})$ and $F(v_i) \not<_L F(v_{i+1})$ (because $v_i E v_{i+1}$) imply $F(v_i) \not<_L F(w)$. This, together with $F(w) \not<_L F(v_i)$, yields $v_i E w$. $\square$

Definition 2.7. Let (V, E) be a graph. A path $v_0 E \ldots E v_n$ is a *minimal path* if $\neg v_i E v_j$ for every i, j such that $i + 1 < j \leq n$.

Notice that if $v_0 E \ldots E v_n$ is a path of minimal length among the paths connecting v_0 and v_n, then it is a minimal path, but the reverse implication does not hold.

Property 2.8. *Let (V, E) be a graph. Then each path can be refined to a minimal path.*

Proof. The statement follows immediately from the following observation: if $v_0 E \ldots E v_n$ is a path and $v_i E v_j$ with $i + 1 < j \leq n$, then $v_0 E \ldots E v_i E v_j E \ldots E v_n$ is still a path. $\square$

Property 2.9. *Let (V, E) be an interval graph with representation $(L, <_L, f_L, f_R)$ and suppose that $v_0 E \ldots E v_n$ is a minimal path with $F(v_0) <_L F(v_n)$.*

(i) Then $f_R(v_i) <_L f_R(v_{i+1})$ for each $i < n - 1$ and $f_L(v_i) <_L f_L(v_{i+1})$ for each $i > 0$;

(ii) if $F(v) <_L F(v_0)$ then $\neg v_i E v$ for every $i \neq 1$; symmetrically, if $F(v_n) <_L F(v)$ then $\neg v_i E v$ for every $i \neq n - 1$.

Proof. To check the first conjunct of (i), suppose $i < n - 1$ is least such that $f_R(v_{i+1}) \leq_L f_R(v_i)$. Since $i < n - 1$ it holds that $\neg v_j E v_n$ for each $j \leq i$ by definition of minimal path. An easy induction, starting with our assumption $F(v_0) <_L F(v_n)$, shows that $F(v_j) <_L F(v_n)$ for each $j \leq i$. It follows that there exists $m \leq n$ such that $f_R(v_i) <_L f_R(v_m)$. Let m be the least with this property and notice that $m > i + 1$ by choice of i. By choice of m it holds that $f_R(v_{m-1}) \leq_L f_R(v_i) <_L f_R(v_m)$, and so that $f_L(v_m) \leq_L f_R(v_{m-1})$ because $v_{m-1} E v_m$. To summarise we get that $f_L(v_m) \leq_L f_R(v_i) <_L f_R(v_m)$, namely that $v_i E v_m$ contrary to the definition of minimal path.

The second conjunct of (i) follows from the first considering the interval representation given by the linear order $(L, >_L)$ and by the maps f_L and f_R.

For (ii), let $v_0 E \ldots E v_n$ be a minimal path and $F(v) <_L F(v_0) <_L F(v_n)$. Assume $v E v_i$, for some $i > 1$ (notice that $\neg v E v_0$ by assumption). Since $f_R(v) <_L f_L(v_0)$ by assumption, $f_R(v_0) <_L f_R(v_i)$ by (i), and $f_L(v_i) <_L f_R(v)$ by $v E v_i$, it holds that $v_0 E v_i$, contrary to the definition of minimal path. $\square$

3. UNIQUELY ORDERABLE CONNECTED INTERVAL GRAPHS

In this section we prove Theorem 1. Suppose (V, E) is a connected incomparability graph. Saying that (V, E) is not uniquely orderable amounts to check that there are two partial orders $\prec$ and $\prec'$ associated to (V, E) and three vertices $a, b, c \in V$ such that $a \prec b \prec c$ and $b \prec' a \prec' c$. The vertices a and b can be reordered regardless, so to speak, the order of c. The connected graph pictured (by one of its interval representations) in Figure 3 is an example of a non uniquely orderable connected interval graph (in fact the intervals for a and b can be swapped without changing their relationship with the intervals c and k).

The first characterization of uniquely orderable interval graphs exploits the above observation to identify subgraphs which are forbidden in uniquely orderable interval graphs.

Definition 3.1. Let (V, E) be a graph. For $B \subseteq V$ let $K(B) = \{v \in V \mid \forall b \in B \, (v \, E \, b)\}$ and $R(B) = V \setminus (B \cup K(B))$. We say that B is a *buried subgraph* of (V, E) if the following hold:

   (i) there exist $a, b \in B$ such that $\neg a \, E \, b$,

   (ii) $K(B) \cap B = \emptyset$ and $R(B) \neq \emptyset$,

   (iii) if $b \in B$ and $r \in R(B)$, then $\neg b \, E \, r$.

The last point in the previous definition implies that any path between a vertex in B and a vertex outside B must go through a vertex in $K(B)$. The main consequence of (iii), which we use many times without mention, is that if $v \in V$ is such that there exist $a, b \in B$ such that $v \, E \, a$ and $\neg v \, E \, b$ then $v \in B$ (because $\neg v \, E \, b$ implies $v \notin K(B)$, while $v \, E \, a$ and (iii) imply $v \notin R(B)$).

Our definition of buried subgraph is slightly different from the one in [Fis85], but it is equivalent for the class of graphs studied by Fishburn, i.e. connected interval graphs without universal vertices. Since we allow universal vertices, in condition (ii) we substituted $K(B) \neq \emptyset$ with $R(B) \neq \emptyset$ (the former condition implies the latter if there are no universal vertices, the reverse implication holds if the graph is connected by (iii)). Moreover we restated condition (iii) in simpler, yet equivalent, terms.

The other main character of Theorem 1 is the graph (W, Q).

Definition 3.2. If (V, E) is a graph we let $W = \{(a, b) \in V \times V \mid \neg a \, E \, b\}$ and (writing ab in place of (a, b) for concision) $ab \, Q \, cd$ if and only if $a \, E \, c$ and $b \, E \, d$.

If ab and cd are elements of W which are connected by a path in (W, Q) we write $ab \, \bar{Q} \, cd$.

Proposition 3.3. *Let (V, E) be an interval graph and $\prec$ a partial order associated to (V, E). If $ab, cd \in W$ are such that $ab \, \bar{Q} \, cd$ and $a \prec b$ then $c \prec d$. In particular we have $\neg ab \, \bar{Q} \, ba$.*

Proof. Suppose first that $ab \, Q \, cd$, so that $a \, E \, c$ and $b \, E \, d$. Notice that $b \, E \, c$ and $a \, E \, d$ cannot both hold, otherwise $a \, E \, c \, E \, b \, E \, d \, E \, a$ would be a simple cycle of length four without chords, against Theorem 2.5. If $\neg b \, E \, c$, then $c \prec b$ because $a \prec b$ and $a \, E \, c$. From this we obtain $c \prec d$, since $d \, E \, b$. If instead $\neg a \, E \, d$ we obtain first $a \prec d$ and then again $c \prec d$.

To derive $c \prec d$ from $ab \, \bar{Q} \, cd$ it suffices to apply the transitivity of $\prec$ to a Q-path connecting ab with cd. $\qquad\qquad\square$

The last part of the previous proposition implies that if $W \neq \emptyset$ (which is equivalent to (V, E) being not complete) then (W, Q) has at least two connected components. Moreover, if (W, Q) has more than two (and so at least four) connected components then for every partial order $\prec$ associated to (V, E) there exist $ab, cd \in W$ such that $a \prec b$ and $c \prec d$, yet $ab \, \bar{Q} \, cd$ fails.

We split, as originally done by Fishburn, the proof of Theorem 1 in three steps corresponding to (1) implies (2) (Lemma 3.4), (3) implies (1) (Lemma 3.5), and (2) implies (3) (Theorem 3.12). The proof of the first implication in [Fis85] is not completely accurate, and we apply Fishburn's idea after a preliminary step which is necessary even when the graph is finite. The second implication is straightforward and applies to interval graphs of any cardinality. The proof of the last implication is completely new and requires more work.

The connectedness of the graph is not needed in the first two implications. Moreover, the hypotheses of Lemma 3.4 could be further relaxed, as the proof applies to arbitrary incomparability graphs.

Lemma 3.4. *Every uniquely orderable interval graph does not contain a buried subgraph.*

Proof. Let (V, E) be an interval graph with a buried subgraph B. Fix a partial order $\prec_0$ associated to (V, E) and some $b_0 \in B$. We define a new binary relation $\prec$ on V as follows: when either $u, v \in B$ or $u, v \notin B$ set $u \prec v$ if and only if $u \prec_0 v$; when $b \in B$ and $v \notin B$ set $b \prec v$ if and only if $b_0 \prec_0 v$, and $v \prec b$ if and only if $v \prec_0 b_0$. Thus the whole B is $\prec$-above the elements not in B which are $\prec_0$-below b_0 and $\prec$-below the elements not in B which are $\prec_0$-above b_0.

Using the fact that the vertices not in B are either $\prec_0$-incomparable to every vertex of B or $\prec_0$-comparable to every vertex of B, it is straightforward to check that $\prec$ is transitive, and hence a partial order. For the same reason $\prec$ is associated to (V, E). The key feature of $\prec$ (not necessarily shared by $\prec_0$) is that B is $\prec$-convex, i.e. if $b \prec v \prec b'$ with $b, b' \in B$ then $v \in B$ as well. Indeed, if $v \notin B$ then $b \prec v$ implies $b_0 \prec_0 v$ and $v \prec b'$ implies $v \prec_0 b_0$.

Following now [Fis85], let $\prec'$ be such that the restrictions of $\prec$ and $\prec'$ to B are dual, while $\prec'$ and $\prec$ coincide on $V \setminus B$ and between elements of B and $V \setminus B$. Formally, $u \prec' v$ if and only if either $u, v \in B$ and $v \prec u$, or if at least one of u and v does not belong to B and $u \prec v$. The

transitivity of $\prec'$ is a consequence of the $\prec$-convexity of B (an observation lacking in the proof given in [Fis85]) and hence $\prec'$ is a partial order associated to (V, E).

If $x, y \in B$ are such that $x \prec y$ and $v \in R(B)$ (these elements exist by Definition 3.1) we have either $x \prec y \prec v$ or $v \prec x \prec y$. In the first case $y \prec' x \prec' v$, in the second case $v \prec' y \prec' x$, witnessing that $\prec'$ is neither $\prec$ nor the dual order of $\prec$. $\qquad\square$

Lemma 3.5. *Let (V, E) be an interval graph. If (W, Q) has two connected components, then (V, E) is uniquely orderable.*

Proof. This follows easily from Proposition 3.3. $\qquad\square$

For the proof of Theorem 3.12 we describe a construction that, starting from a pair of non-adjacent vertices, attempts to build the minimal buried subgraph containing those two vertices. We then show that if this attempt always fails then for any $ab, cd \in W$ either $ab \, \bar{Q} \, cd$ or $ab \, \bar{Q} \, dc$.

Construction 3.6. *Let (V, E) be a connected interval graph and $v, u \in V$ be such that $\neg v \, E \, u$. We define recursively $B_n(v, u) \subseteq V$:*

$$B_0(v, u) = \{v, u\}$$
$$B_{n+1}(v, u) = \{w \in V \mid \exists z, z' \in B_n(v, u) \, (z \, E \, w \wedge \neg z' \, E \, w)\}$$

We then set $B(v, u) = \bigcup B_n(v, u)$. If $w \in B(v, u)$ let e_w be the least n such that $w \in B_n(v, u)$ (formally we should write $e_w^{v;u}$ but we omit the superscript as v and u will always be understood).

A straightforward induction shows that $B_n(v, u) \subseteq B_{n+1}(v, u)$, for each $n \in \mathbb{N}$ (for the base step recall that interval graphs are reflexive, so that v and u themselves witness that $v, u \in B_1(v, u)$).

We now show that $B(v, u)$ is close to being a buried subgraph.

Property 3.7. *In the situation of Construction 3.6, $B(v, u)$ is a buried subgraph if and only if $R(B(v, u)) \neq \emptyset$.*

Proof. Notice that Condition (i) of Definition 3.1 is witnessed by v and u. Condition (3) is obvious, because if $r \notin K(B(u, v))$ but $b \, E \, r$ for some $b \in B(u, v)$ then $r \in B(u, v)$. Moreover, if $k \in K(B(v, u))$ then $k \in K(B_n(u, v))$ for every n and hence $k \notin B(u, v)$; hence $B(v, u) \cap K(B(v, u)) = \emptyset$. Therefore, to verify that $B(v, u)$ is a buried subgraph it suffices that $R(B(v, u)) \neq \emptyset$. $\qquad\square$

In the next propositions, we will always consider a connected interval graph (V, E) with representation $(L, <_L, f_L, f_R)$, fix $v, u \in V$ with $\neg v \, E \, u$ and $F(v) <_L F(u)$ and define $B(v, u)$ as in Construction 3.6. For brevity, we call this set of hypotheses ($\maltese$) and indicate it next to the proposition number.

Proposition 3.8 ($\maltese$). *Let $x, y \in B(v, u)$. If $F(u) <_L F(x)$ and $f_L(x) \leq_L f_L(y)$ then $e_x \leq e_y$. Analogously, if $f_R(y) \leq_L f_R(x)$ and $F(x) <_L F(v)$ then $e_x \leq e_y$ as well.*

Proof. We prove the first half of the statement by induction on e_y. The base case is trivially satisfied since there is no $y \in B_0(v, u)$ satisfying the hypotheses. Assume $e_y > 0$ and let $z \in B_{e_y-1}(v, u)$ be such that $z \, E \, y$. If $f_L(x) \leq_L f_L(z)$, then $e_x \leq e_z < e_y$ by induction hypothesis. Otherwise, $f_L(z) <_L f_L(x) \leq f_L(y) \leq_L f_R(z)$ given that $z \, E \, y$. This means that $z \, E \, x$, which implies that $x \in B_{e_z+1}(v, u)$ since $u \in B_{e_z}(v, u)$ is such that $\neg u \, E \, x$. Hence $e_x \leq e_z + 1 \leq e_y$

The second half of the statement follows considering the representation $(L, >_L, f_L, f_R)$. $\qquad\square$

Proposition 3.9 ($\maltese$). *Let $w \in B(v, u)$. If $F(w) \not<_L F(u)$ then there exists a path $u \, E \, b_1 \, E \, \ldots b_k \, E \, w$ such that $e_{b_i} < e_w$ for all $i \leq k$.*

Analogously, if $F(v) \not<_L F(w)$ then there exists a path $v \, E \, b_1 \, E \, \ldots b_k \, E \, w$ such that $e_{b_i} < e_w$ for all $i \leq k$.

Proof. By definition of $B_{e_w}(v, u)$ there exists a path $b_0 \, E \, b_1 \, E \, \ldots \, E \, b_k \, E \, w$ where $b_i \in B(v, u)$ and $0 = e_{b_0} < e_{b_1} < \cdots < e_{b_k} < e_w$. Hence $b_0 \in \{u, v\}$ and, since $F(w) \not<_L F(u)$ and $F(u) \not<_L F(v)$, by Proposition 2.6 we can assume that $b_0 = u$.

The second half of the statement follows from the first one as usual. $\qquad\square$

Proposition 3.10 ($\maltese$). *Let $x, z \in B(v, u)$ and $m = \max\{e_x, e_z\}$. Assume $F(z) <_L F(x)$ and $F(v) \not<_L F(x)$ (this implies $m > 0$). Then there exists a minimal path $z \, E \, v_1 \, E \, \ldots \, E \, v_n \, E \, x$ and $s \in B(v, u)$ with $e_s < m$ such that $e_{v_i} < m$ and $F(v_i) <_L F(s)$ for each $i \leq n$.*

Analogously, if $F(x) <_L F(z)$ and $F(x) \not<_L F(u)$ there exists a minimal path $x \, E \, v_1 \, E \, \ldots \, E \, v_n \, E \, z$ and $s \in B(v, u)$ with $e_s < m$ such that $e_{v_i} < m$ and $F(s) <_L F(v_i)$ for each $i \leq n$.

Proof. Notice that once we find the minimal path $z\,E\,v_1\,E\,\dots\,E\,v_n\,E\,x$ and $s\in B(v,u)$ with $e_s<m$ such that $e_{v_i}<m$ for all $i\le n$ it suffices to prove that $F(v_n)<_L F(s)$, since then $F(v_i)<_L F(s)$ for $i<n$ follows from Property 2.9.i.

We can apply Proposition 3.9 to both x and z obtaining paths connecting v to x and v to z and with $e_b<m$ for all vertices b, distinct from x and z, occurring in the paths. Joining these paths and then using Property 2.8 we obtain a minimal path $z\,E\,v_1\,E\,\dots\,E\,v_n\,E\,x$ with $e_{v_i}<m$. Notice that $n>0$ as $\neg z\,E\,x$. Let $j<m$ be such that $e_{v_n}=j$: we may assume j is least for which such a minimal path exists.

If $j=0$ then we claim that we can assume $v_n=v$ and hence we can choose $s=u$. In fact if $v_n=u$ then $F(x)<_L F(v)$ is impossible and we have $v\,E\,x$. The hypotheses imply $F(v)\not<_L F(z)$ and, since $F(u)\not<_L F(v)$, by Proposition 2.6 we can find $i<n$ such that $v\,E\,v_i$ and consider the path $z\,E\,v_1\,E\,\dots\,E\,v_i\,E\,v\,E\,x$.

We now assume $j>0$: there exists $s\in B(v,u)$ such that $e_s<j$ and $\neg s\,E\,v_n$. We claim that $F(v_n)<_L F(s)$, completing the proof. Suppose on the contrary that $F(s)<_L F(v_n)$ ($F(v_n)\cap F(s)\ne\emptyset$ cannot hold because $\neg s\,E\,v_n$).

In this case we have $F(s)<_L F(x)$ because $f_L(v_n)<_L f_L(x)$ by Property 2.9.i. Hence $F(v)\not<_L F(s)$ and we can use Proposition 3.9 and Property 2.8 to obtain a minimal path $s\,E\,u_1\,E\,\dots\,E\,u_\ell$, with $u_\ell=v$ and $e_{u_i}<e_s$. Since $F(x)\not<_L F(s)$ and $F(v)\not<_L F(x)$ by Proposition 2.6 there exists $k\le\ell$ such that $u_k\,E\,x$. We distinguish two cases: $F(z)\not<_L F(s)$ and $F(z)<_L F(s)$.

In the first case we apply Proposition 2.6 to the path $s\,E\,u_1\,E\,\dots\,E\,u_k\,E\,x$: there exists $h\le k$ such that $z\,E\,u_h$. Since $z\,E\,u_h\,E\,\dots\,E\,u_k\,E\,x$ can be refined to a minimal path and $e_{u_i}<j$, the minimality of j is contradicted.

In the second case we apply Proposition 2.6 to the path $z\,E\,v_1\,E\,\dots\,E\,v_n\,E\,x$: there exists $h<n$ (recall that $\neg v_n\,E\,s$) such that $v_h\,E\,s$. Then $z\,E\,v_1\,E\,\dots\,E\,v_h\,E\,s\,E\,u_1\,E\,\dots\,E\,u_k\,E\,x$ is a path that can be refined to a minimal path $z\,E\,w_1\,E\,\dots\,E\,w_r\,E\,x$. Notice that $w_r=u_p$, for some $p\le k$ because $\neg v_i\,E\,x$ for every $i\le h<n$, by minimality of the path $z\,E\,v_1\,E\,\dots\,E\,v_n\,E\,x$, and $F(s)<_L F(x)$. Since $e_{w_r}<j$ we contradict again the minimality of j.

The second half of the statement follows from the first one as usual. $\qquad\square$

Lemma 3.11 (✠). *If $x,y\in B(v,u)$, $f_R(x)\le_L f_R(v)$ and $f_L(u)\le_L f_L(y)$ then $vu\,\bar Q\,xy$.*

Proof. The proof is by induction on e_x+e_y. If $e_x+e_y=0$ then $x=v$ and $y=u$, so that the conclusion is immediate (recall that Q is reflexive). Now assume that $e_x+e_y>0$ and suppose $e_x\le e_y$ (if $e_y<e_x$ we can employ the usual trick of reversing the representation) and hence $e_y>0$.

If $u\,E\,y$, then $xu\,Q\,xy$ and, since the induction hypothesis implies $vu\,\bar Q\,xu$ (because $e_u=0$), we obtain $vu\,\bar Q\,xy$. Thus we assume $\neg u\,E\,y$ and hence $F(u)<_L F(y)$. Let $w\in B_{e_y-1}(v,u)$ be such that $y\,E\,w$. If $f_L(u)\le_L f_L(w)$ then we can apply the induction hypothesis to xw obtaining $vu\,\bar Q\,xw$. Since $xw\,Q\,xy$, we are done.

We thus assume $f_L(w)<_L f_L(u)$ which, together with $F(u)<_L F(y)$ and $w\,E\,y$, implies $f_R(u)<_L f_R(w)$ and hence $w\,E\,u$. Notice moreover that $w\ne u$ and hence (since $w\ne v$ is obvious) $e_y>1$. If $\neg x\,E\,w$, then $xu\,Q\,xw\,Q\,xy$ and, since by induction hypothesis $vu\,\bar Q\,xu$, we have $vu\,\bar Q\,xy$. If instead $x\,E\,w$ we must have $f_L(w)\le_L f_R(x)\le_L f_R(v)$. Let $z\in B_{e_y-2}(v,u)$ be such that $\neg z\,E\,w$. If $F(w)<_L F(z)$ then $F(u)<_L F(z)$ and $f_L(y)<_L f_L(z)$, so that Proposition 3.8 implies $y\in B_{e_y-2}(v,u)$, which is impossible. Hence $F(z)<_L F(w)$. This implies $f_R(z)<_L f_R(x)$. It follows that $x\in B_{e_y-1}(v,u)$, either by Proposition 3.8, if $F(x)<_L F(v)$, or because $x\in B_1(v,u)$ if $x\,E\,v$, given that $\neg x\,E\,u$. Since $vu\,\bar Q\,zu$ holds by induction hypothesis and we have also $zu\,Q\,zw\,Q\,zy$ it suffices to show that $zy\,\bar Q\,xy$.

If $z\,E\,x$ the conclusion is immediate, otherwise $F(z)<_L F(x)$. Since $F(v)\not<_L F(x)$ we can apply Proposition 3.10 finding a minimal path $z\,E\,u_1\,E\,\dots\,E\,u_n\,E\,x$ and $s\in B_{e_y-2}(v,u)$ such that $u_i\in B_{e_y-2}(v,u)$ and $F(u_i)<_L F(s)$ for all $i\le n$. Since $F(z)<_L F(x)<_L F(y)$, by Property 2.9.ii we have $\neg u_i\,E\,y$ when $i<n$. We claim that $\neg u_n\,E\,y$ as well, so that $zy\,Q\,u_1y\,Q\,\dots\,Q\,u_ny\,Q\,xy$ witnesses $zy\,\bar Q\,xy$. Indeed, if $u_n\,E\,y$ we would have $f_L(y)<_L f_R(u_n)<_L f_L(s)$ and we could apply Proposition 3.8 to obtain $y\in B_{e_y-2}(v,u)$, which is impossible. $\qquad\square$

Theorem 3.12. *Let (V,E) be a connected interval graph. If (V,E) does not contain a buried subgraph, then (W,Q) has two connected components.*

Proof. Fix a representation $(L,<_L,f_L,f_R)$ of (V,E) and assume that (V,E) does not contain a buried subgraph. We show that if $ab,cd\in W$ are such that $F(a)<_L F(b)$ and $F(c)<_L F(d)$ then $ab\,\bar Q\,cd$. We can assume without loss of generality that $f_R(c)\le_L f_R(a)$. We consider three cases:

Case 1: $f_L(b) <_L f_L(d)$: $B(a,b)$ (which satisfies the hypotheses of (✠)) is not a buried subgraph and hence by Property 3.7 we must have $B(a,b) = V$. In particular $c, d \in B(a,b)$ and we are in the hypotheses of Lemma 3.11: we conclude that $ab\,\bar{Q}\,cd$.

Case 2: $f_R(a) <_L f_L(d) \leq_L f_L(b)$: $B(a,d)$ (which satisfies the hypotheses of (✠)) is not a buried subgraph and hence by Property 3.7 $b, c \in B(a,d)$. Lemma 3.11 implies both $ad\,\bar{Q}\,ab$ and $ad\,\bar{Q}\,cd$. It follows that $ab\,\bar{Q}\,cd$.

Case 3: $f_L(d) \leq_L f_R(a)$: neither $B(a,b)$ nor $B(c,d)$ (which both satisfy the hypotheses of (✠)) is a buried subgraph. By Property 3.7 we have $c \in B(a,b)$, which implies $ab\,\bar{Q}\,cb$, and $b \in B(c,d)$, which together with $f_L(d) <_L f_L(b)$ yields $cd\,\bar{Q}\,cb$ (we use Lemma 3.11 in both cases). Thus $ab\,\bar{Q}\,cd$ also in this case. □

4. REVERSE MATHEMATICS AND INTERVAL GRAPHS

Reverse mathematics is a research program, which dates back to the Seventies, whose goal is to find the exact axiomatic strength of theorems from different areas of mathematics. It deals with statements about countable, or countably representable, structures, using the framework of the formal system of second order arithmetic Z_2. We do not introduce reverse mathematics here, but refer the reader to monographs such as [Sim09] and [Hir15].

The subsystems of second order arithmetic are obtained by limiting the comprehension and induction axioms of Z_2 to specific classes of formulas. We mention only the subsystems we are going to use in this paper: RCA_0 is the weak base theory corresponding to computable mathematics, WKL_0 extends RCA_0 by adding Weak König's Lemma (each infinite binary tree has an infinite path), and ACA_0 is even stronger allowing for definitions of sets by arithmetical comprehension. It is well-known that WKL_0 is equivalent to many compactness principles and thus we can claim that a theorem not provable in WKL_0 does not admit a proof by compactness. In particular this applies to Theorem 1, as Theorem 2 shows that it is not provable in WKL_0.

The second author studied the equivalence of different characterizations of interval orders from the reverse mathematics perspective in [Mar07]. A similar study can be carried out for interval graphs, and we summarize here the main results: full details and proofs are included in the first author's PhD thesis [FC19], which includes also results about the subclass of indifference graphs (corresponding to proper interval orders studied in [Mar07]).

As customary in reverse mathematics, the system in parenthesis indicates where the definition is given or the statement proved. Notice also that in this and in the next section we deal with countable graphs and orders, the only ones second order arithmetic and its subsystems can speak of.

In the literature it is possible to find slightly different definitions of interval graphs and orders, which depend on the notion of interval employed. For example intervals may be required to be closed or not. We thus have five conceptually distinct definitions of interval graphs:

Definition 4.1 (RCA_0)**.** Let (V, E) be a graph.

- (V, E) is an *interval graph* if there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ such that, abbreviating $\{x \in L \mid (p, x) \in F\}$ by $F(p)$, for all $p, q \in V$ the following hold:

  (i1) $F(p) \neq \emptyset$ and $\forall x, y \in F(p)\, \forall z \in L\, (x <_L z <_L y \rightarrow z \in F(p))$,

  (i2) $p\,E\,q \Leftrightarrow F(p) \cap F(q) \neq \emptyset$.

- (V, E) is a *1-1 interval graph* if it also satisfies

  (i3) $F(p) \neq F(q)$ whenever $p \neq q$.

- (V, E) is a *closed interval graph* if there exist a linear order $(L, <_L)$ and two functions $f_L, f_R \colon V \to L$ such that for all $p, q \in V$

  (c1) $f_L(p) <_L f_R(p)$,

  (c2) $p\,E\,q \Leftrightarrow f_L(p) \leq_L f_R(q) \leq_L f_R(p) \vee f_L(q) \leq_L f_R(p) \leq_L f_R(q)$

- A closed interval graph (V, E) is a *1-1 closed interval graph* if we also have

  (c3) $f_R(p) \neq f_R(q) \vee f_L(p) \neq f_L(q)$ whenever $p \neq q$.

- (V, E) is a *distinguishing interval graph* if (c1) and (c2) hold together with

  (c4) $f_i(p) \neq f_j(q)$ whenever $p \neq q \vee i \neq j$.

4.1. **Definitions and characterizations of interval graph.** In Definition 2.3 we mentioned that every interval graph is a closed interval graph: in fact all the notions introduced in Definition 4.1 are equivalent in a sufficiently strong theory. Our first results concern the systems where the implications between the notions introduced in Definition 4.1 can be proved. The same investigation for interval orders was carried out in [Mar07] and in this respect interval graphs and interval orders behave similarly. Indeed the proofs of the results we are going to state either mimic the corresponding proofs for interval orders or are easily derived from those results.

Definition 4.1 enumerates increasingly strong conditions, so that the implications from a later to an earlier notion are easily proved in RCA_0. Regarding the other implications we obtain that, as is the case for interval orders, there are three distinct notions of interval graphs in RCA_0, namely that of interval, 1-1 interval and closed interval graph.

Theorem 4.2 (RCA_0)**.** *Every closed interval graph is a distinguishing interval graph.*

Theorem 4.3 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0;
(2) every interval graph is a 1-1 interval graph;
(3) every 1-1 interval graph is a closed interval graph;
(4) every interval graph is a closed interval graph.

4.2. **Structural characterizations of interval graphs.** Since interval graphs are incomparability graphs (and Definition 2.1 can be given in RCA_0) we first look at the most important structural characterization of comparability graphs. The first result is due to Jeff Hirst ([Hir87, Theorem 3.20]).

Lemma 4.4 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0;
(2) every graph such that every cycle of odd length has a triangular chord is a comparability graph.

We then consider two structural characterizations of interval graphs (notice that Definition 2.4 can be given in RCA_0). The necessity of both conditions is provable in RCA_0, but the sufficiency of one of them requires WKL_0.

Theorem 4.5 (RCA_0)**.** *Every interval graph is an incomparability graph such that every simple cycle of length four has a chord. Moreover, every interval graph is triangulated and has no asteroidal triples.*
Every incomparability graph such that every simple cycle of length four has a chord is an interval graph.
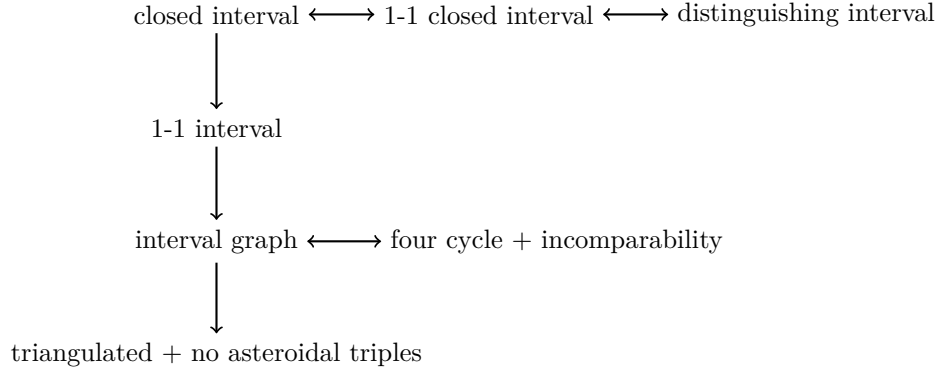
Theorem 4.6 (RCA_0)**.** *The following are equivalent:*

(1) WKL_0;
(2) if a graph is triangulated and has no asteroidal triples, then it is an interval graph.

Figure 4 summarizes the results about the different definitions and characterizations of interval graphs. The arrows correspond to provability in RCA_0, while every implication from a notion below another is equivalent to WKL_0.

Schmerl [Sch05] claimed that the statement "A graph is an interval graph if and only if each finite subgraph is representable by intervals" is equivalent to WKL_0. Theorem 4.6 confirms his claim and shows that compactness is necessary to prove the statement. On the other hand, the corresponding statement for interval orders, i.e. an order is an interval order if and only if each suborders is an interval order, is provable in RCA_0 because the structural characterization of interval orders (as the partial orders not containing $2 \oplus 2$) is provable in RCA_0 [Mar07, Theorem 2.1]. The different strengths of the structural characterizations of interval graphs and orders can be traced to the fact that an interval order carries full information about the relative position of the intervals in its representations, while an interval graph does not.

Lekkerkerker and Boland [LB62] provide another characterization of interval graphs listing all the forbidden subgraphs. It is routine to check in RCA_0 that those graphs are a complete list of graphs whose cycles of length greater than four do not have chords or which contain an asteroidal triple.

$$\text{closed interval} \longleftrightarrow \text{1-1 closed interval} \longleftrightarrow \text{distinguishing interval}$$

$$\downarrow$$

$$\text{1-1 interval}$$

$$\downarrow$$

$$\text{interval graph} \longleftrightarrow \text{four cycle} + \text{incomparability}$$

$$\downarrow$$

$$\text{triangulated} + \text{no asteroidal triples}$$

FIGURE 4. Implications in RCA_0

4.3. **Interval graphs and interval orders.** Different definitions for interval orders, mirroring those of Definition 4.1, were given and studied in [Mar07]. We give here only the most basic one, as the others can be easily guessed from this.

Definition 4.7 (RCA_0)**.** A partial order $(V, \preceq)$ is an *interval order* if there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ such that, abbreviating $\{x \in L \mid (p, x) \in F\}$ by $F(p)$, for all $p, q \in V$ the following hold:

  (i1) $F(p) \neq \emptyset$ and $\forall x, y \in F(p)\, \forall z \in L\, (x <_L z <_L y \to z \in F(p))$,
  (i2) $p \preceq q \Leftrightarrow \forall x \in F(p)\, \forall y \in F(q)\, (x <_L y)$.

We explore the strength of the statements that allow moving from interval graphs to interval orders and back. By the previous results (and the corresponding ones in [Mar07]) it suffices to consider three different notions on each side, and we concentrate on the relationship between corresponding notions. In one direction everything goes through in RCA_0.

Theorem 4.8 (RCA_0)**.** *Let (V, E) be a graph and let $\mathcal{P}$ be any of "interval", "1-1 interval", "closed interval". (V, E) is a $\mathcal{P}$ graph if and only if there exists a $\mathcal{P}$ order $(V, \prec)$ such that $p\,E\,q \Leftrightarrow p \nprec q \land q \nprec p$ for all $p, q \in V$.*

The other direction is more interesting, as only in one case RCA_0 suffices. The proofs of the reversals to WKL_0 are modifications of the proof of [Mar07, Theorem 6.4].

Theorem 4.9 (RCA_0)**.** *Let $(V, \preceq)$ be a partial order. $(V, \preceq)$ is an interval order if and only if (V, E), where $p\,E\,q \Leftrightarrow p \nprec q \land q \nprec p$ for all $p, q \in V$, is an interval graph.*

Theorem 4.10 (RCA_0)**.** *The following are equivalent:*

  (1) WKL_0
  (2) Let $(V, \preceq)$ be a partial order. $(V, \preceq)$ is a 1-1 interval order if and only if (V, E), where $p\,E\,q \Leftrightarrow p \nprec q \land q \nprec p$ for all $p, q \in V$, is a 1-1 interval graph.
  (3) Let $(V, \preceq)$ be a partial order. $(V, \preceq)$ is a closed interval order if and only if (V, E), where $p\,E\,q \Leftrightarrow p \nprec q \land q \nprec p$ for all $p, q \in V$, is a closed interval graph.

5. WHY COMPACTNESS DOES NOT SUFFICE

It is immediate (using Theorem 4.5) that Lemmas 3.4 and 3.5 are provable in RCA_0. On the other hand, we now show that Theorem 3.12 is much stronger, and indeed equivalent to ACA_0. As mentioned in the introduction of the paper, this result explains why the attempts to prove it by compactness cannot succeed.

Lemma 5.1 (ACA_0)**.** *Let (V, E) be a connected graph which is triangulated and with no asteroidal triples. Suppose furthermore that $a, b, c, d \in V$ are such that $\neg ab\,\bar{Q}\,cd$ and $\neg ab\,\bar{Q}\,dc$. Then there exists a buried subgraph $B \subseteq V$ such that either $a, b \in B$ or $c, d \in B$, and no subgraph $A \subseteq B$, which contains either a, b or c, d respectively, is a buried subgraph.*

Proof. By Theorems 4.3 and 4.6 WKL_0, and a fortiori ACA_0, suffices to prove that any connected graph which is triangulated and with no asteroidal triples has a closed interval representation. We

then need to check that the proof of Theorem 3.12, which indeed provides a buried subgraph with the desired properties, goes through in ACA_0.

The first step is checking that, given $v, u \in V$ with $\neg v\,E\,u$, we can carry out Construction 3.6 and define $B(v, u)$ and the various $B_n(v, u)$'s in ACA_0. In fact the definition of each $B_n(v, u)$ in Construction 3.6 uses an instance of arithmetical comprehension and thus the whole construction, as presented there, appears to require the system known as ACA_0^+, which is properly stronger than ACA_0.

This problem can however be overcome in the following way. Given $v, u \in V$ as before, we can characterize $B(v, u)$ as the set of $w \in V$ such that there exists a finite tree $T \subseteq 2^{<\mathbb{N}}$ and a label function $\ell \colon T \to V$ with the following properties:

- $\ell(\emptyset) = w$ (here $\emptyset$ is the root of T);
- if $\sigma \in T$ is not a leaf of T, then $\sigma^\frown 0, \sigma^\frown 1 \in T$, $\ell(\sigma)\,E\,\ell(\sigma^\frown 0)$ and $\neg\ell(\sigma)\,E\,\ell(\sigma^\frown 1)$;
- if $\sigma \in T$ is a leaf of T, then $\ell(\sigma) \in \{v, u\}$.

In fact, the tree and its label function describe the 'steps' allowing w to enter $B(v, u)$. Moreover $B_n(v, u)$ is the set of $w \in V$ such that there exists $T \subseteq 2^{<n}$ and ℓ witnessing $w \in B(v, u)$. These characterizations of $B(v, u)$ and $B_n(v, u)$ use Σ_1^0-formulas, and show that ACA_0 suffices to prove the existence of the sets.

Once $B(v, u)$ and each $B_n(v, u)$ are defined, it is straightforward to check that all subsequent steps in the proof of Theorem 3.12 can be carried out in RCA_0. $\qquad\square$

To prove that Theorem 3.12 implies ACA_0 we use the following notions. Given an injective function $f \colon \mathbb{N} \to \mathbb{N}$ we say that i is true for f when $f(k) > f(i)$ for all $k > i$. It is easy to see that there exist infinitely many i which are true for f. If i is not true for f, i.e. if $f(k) < f(i)$ for some $k > i$, we say that i is false for f. Moreover, we say that i is true for f at stage s if $f(k) > f(i)$ whenever $i < k < s$, and that i is false for f at stage s if $f(k) < f(i)$ for some k with $i < k < s$. If the injective function f is fixed, we omit "for f" from this terminology.

The following Proposition is well-known (see e.g. the discussion after Definition 4.1 in [FHM$^+$16]).

Proposition 5.2 (RCA_0). *The following are equivalent:*

(1) ACA_0*;*
(2) *if $f \colon \mathbb{N} \to \mathbb{N}$ is an injective function there exists an infinite set T such that every $i \in T$ is true for f.*

**Theorem 5.3** (RCA_0). *The following are equivalent:*

(1) ACA_0*;*
(2) *let (V, E) be a connected graph, triangulated and with no asteroidal triples; if $a, b, c, d \in V$ are such that $\neg ab\,\bar{Q}\,cd$ and $\neg ab\,\bar{Q}\,dc$, then there exists a buried subgraph $B \subseteq V$ such that either $a, b \in B$ or $c, d \in B$, and no subgraph $A \subseteq B$, which contains either a, b or c, d respectively, is a buried subgraph;*
(3) *let (V, E) be a connected closed interval graph; if (W, Q) has more than two connected components, then there exists a buried subgraph $B \subseteq V$;*
(4) *let (V, E) be a connected closed interval graph; if (V, E) is not uniquely orderable, then there exists a buried subgraph $B \subseteq V$.*

Proof. $(1 \Rightarrow 2)$ is Lemma 5.1. The implication $(2 \Rightarrow 3)$ is trivial, while $(3 \Rightarrow 4)$ follows directly from Lemma 3.5, which goes through in RCA_0.

To prove $(4 \Rightarrow 1)$ we fix an injective function $f \colon \mathbb{N} \to \mathbb{N}$ and we define (within RCA_0) a connected closed interval graph (V, E) such that (W, Q) has more than two connected components. We then prove, arguing in RCA_0, that the unique buried subgraph $B \subseteq V$ codes the (necessarily infinite) set of numbers which are true for f.

We let $V = \{a, b, k, s\} \cup \{x_i, y_i \mid i \in \mathbb{N}\}$. The definition of the edge relation is by stages: at stage s we define E on $V_s = \{a, b, k, s\} \cup \{x_i, y_i \mid i < s\}$. At stage 0 let k be adjacent to a, b and s (and add no other edges). At stage $s + 1$ we define the vertices adjacent to x_s and y_s by the following clauses:

- (a) $a\,E\,x_s\,E\,b\,E\,y_s$ and $x_s\,E\,k\,E\,y_s$,
- (b) $x_s\,E\,x_i$ and $y_s\,E\,y_i$ for each $i < s$,
- (c) $x_s\,E\,y_i$ for each $i \leq s$,
- (d) for $i \leq s$, $y_s\,E\,x_i$ if and only if i is true for f at stage $s + 1$.
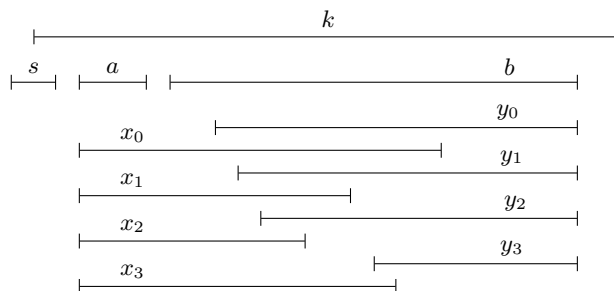
FIGURE 5. Interval representation of V_4 in case 1 (and so 2) becomes false at stage 3.

It is immediate that (V, E) is connected. To check that it is a closed interval graph we define a
closed interval representation $f_L, f_R : V \to L$ where $(L, <_L)$ is a dense linear order. The definition
of f_L and f_R reflects the construction of the graph by stages. At stage 0 assign to the members of
V_0 elements of L satisfying

$$f_L(s) <_L f_L(k) <_L f_R(s) <_L f_L(a) <_L f_R(a) <_L f_L(b) <_L f_R(b) <_L f_R(k).$$

This ensures that we are representing the restriction of the graph to V_0.

At stage $s + 1$, first let $f_L(x_s) = f_L(a)$ and $f_R(y_s) = f_R(b)$ (since this is done at every stage,
we are respecting conditions (a) and (b)). We thus still need to define $f_R(x_s)$ and $f_L(y_s)$; first of
all we make sure that $f_L(b) <_L f_L(y_i) <_L f_L(y_s) <_L f_R(x_s) <_L f_R(b)$ for every $i < s$, so that (c)
is also respected. To respect condition (d) as well we satisfy the following requirements:

- if $i < s$ is true at stage $s + 1$ then $f_R(x_s) <_L f_R(x_i)$ (which implies $f_L(y_s) <_L f_R(x_i)$);
- if $j < s$ is false at stage $s + 1$ then $f_R(x_j) <_L f_L(y_s)$.

The existence of $f_L(y_s) <_L f_R(x_s)$ with these properties follows from the density of L and from
the fact that if $i < s$ is true at stage $s + 1$ and $j < s$ is false at stage $s + 1$ then $f_R(x_j) <_L f_R(x_i)$.
To see this notice that:

- if $i < j$ then i was also true at stage $j + 1$ and we set $f_R(x_j) <_L f_R(x_i)$ then;
- if $j < i$ then j was already false at stage $i + 1$ (if j was true at stage $i + 1$ then $f(j) < f(i)$,
  and i would be false at stage $s + 1$ because j is false at that stage), and hence we set
  $f_R(x_j) <_L f_L(y_i) <_L f_R(x_i)$ at that stage.

Figure 5 depicts a sample interval representation following this construction.

To check that (V, E) is not uniquely orderable let $\prec_1$ be the partial order induced by the interval
representation we just described: $v \prec_1 u$ if and only if $f_R(v) <_L f_L(u)$. Define $\prec_2$ so that $\prec_1$ and
$\prec_2$ coincide on $V \setminus \{s\}$ and $u \prec_2 s$ for all $u \in V \setminus \{s, k\}$. It is immediate that both $\prec_1$ and $\prec_2$ are
associated to (V, E), and that $\prec_2$ is not dual of $\prec_1$.

By (4) there exists a buried subgraph $B \subseteq V$. First of all notice that $k \in K(B)$ and hence
$k \notin B$. Now observe that $s \in B$ implies, using Conditions (i) and (iii) of Definition 3.1, that either
some x_n or some y_n belongs to B. From there, using Condition (iii) again, it is easy to see that
$B = V \setminus \{k\}$ and hence $R(B) = \emptyset$, contradicting Condition (ii). Thus $s \notin B$. Then, in order to
satisfy Condition (i), we must have either $a, b \in B$ or $a, y_n \in B$ or $x_m, y_n \in B$, for some n and
some m which is false at stage $n + 1$. In any case we have $a \in B$: in the first two cases this is
obvious, and in the latter case this follows from Condition (iii) because $a E x_m$ and $\neg a E y_n$ for
every n and m. But then, using $b E y_n$ and $\neg b E a$ we obtain $b \in B$ even in the second and third
case. Thus we can conclude that $a, b \in B$. For each n we have $y_n E b$ and $\neg y_n E a$ and therefore
$y_n \in B$. Since $b E x_n E a$ for each $n \in \mathbb{N}$, then either $x_n \in K(B)$ or $x_n \in B$ depending whether x_n
is adjacent to every y_m or not, namely whether n is true or false for f. Therefore we showed

$$B = \{a, b\} \cup \{y_n \mid n \in \mathbb{N}\} \cup \{x_n \mid n \text{ is false}\},$$

so that $K(B) = \{k\} \cup \{x_n \mid n \text{ is true}\}$ and $R(B) = \{s\}$. Then $T = \{n \mid x_n \notin B\}$ is the (necessarily
infinite) set of all n which are true for f. $\qquad \square$

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